ON 2-ADIC LOCAL AND INTEGRAL MODELS OF SHIMURA VARIETIES

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics—Doctor of Philosophy

2025

ABSTRACT

This thesis is about integral models of Shimura varieties with emphasis on the reduction at the prime $p = 2$.

In the first part of the thesis, we construct local models for wildly ramified unitary similitude groups of odd dimension $n \geq 3$ with special parahoric level structure and signature $(n-1, 1)$. We first give a lattice-theoretic description for parahoric subgroups using Bruhat-Tits theory in residue characteristic two, and apply them to define local models following the lead of Rapoport-Zink [RZ96] and Pappas-Rapoport [PR09]. In our case, there are two conjugacy classes of special parahoric subgroups. We show that the local models are smooth for the one class and normal, Cohen-Macaulay for the other class. We also prove that they represent the v-sheaf local models of Scholze-Weinstein. Under some additional assumptions, we obtain an explicit moduli interpretation of the local models.

The second part of the thesis focuses on constructing integral models over $p = 2$ for some Shimura varieties of abelian type with parahoric level structure, extending the previous work of Kim-Madapusi [KM16] and Kisin, Pappas, and Zhou [KP18; KZ24; KPZ24]. For Shimura varieties of Hodge type, we show that our integral models are canonical in the sense of Pappas-Rapoport [PR24].

ACKNOWLEDGEMENTS

Completing this thesis has been a long and challenging journey, and I could not have done it without the support, guidance, and encouragement of so many people along the way. I would like to take this opportunity to thank everyone who helped make this thesis possible.

First and foremost, I would like to express my profound gratitude to my advisor, Professor G. Pappas, for his unwavering support, patience, and guidance throughout my doctoral studies. His mentorship and constant encouragement have been invaluable in shaping this thesis and deeply influenced my growth as a mathematician.

I would also like to extend my sincere thanks to professors R. Kulkarni, A. Levin, I. Rapinchuk, P. Wake. Their courses and our discussion opened my eyes to many new ideas and inspired me to explore further.

To my peers at MSU, S. Bhutani, P. Čoupek, C. Guan, K. Huang, B. Kong, A. Oswal, P. Qi, A. Roy, J. Ruiter, A. Sathyanarayana, E. Sgallova, Y. Shen, Z. Xiao, S. Xu, thanks for creating a stimulating and supportive environment. The countless discussions, both formal and informal, have been a source of inspiration and motivation. Your friendship has been a source of joy and comfort. I also want to thank my roommate L. Shen for these years of friendship and support.

My heartfelt thanks go to my family for their unconditional love and encouragement throughout this journey. Thank you for always being there for me. Your constant support has been my anchor during the most challenging times.

This dissertation represents not only my efforts but also the collective support, guidance, and encouragement of many individuals. To everyone who has been part of my academic and personal growth, thank you from the bottom of my heart!

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CHAPTER 1

INTRODUCTION

1.1 Background

Shimura varieties, first reformulated in a modern framework by Deligne in his seminal papers [Del71; Del79], are higher dimensional generalizations of modular curves and play a central role in number theory.

Let p be a prime number. Let $\mathbb{A}_f$ denote the ring of finite adèles over $\mathbb{Q}$, and $\mathbb{A}_f^p$ denote the ring of prime-to-p finite adèles over $\mathbb{Q}$. Let $(\mathbf{G}, X)$ be a Shimura datum, i.e., $\mathbf{G}$ is a reductive group over $\mathbb{Q}$, X is a $\mathbf{G}(\mathbb{R})$-conjugacy class of an algebraic group homomorphism $h : \mathbb{S} \coloneqq \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathbf{G}_{\mathbb{R}}$, and $(\mathbf{G}, X)$ satisfies Deligne's axioms ([Del79, (2.1.1.1)-(2.1.1.3)]). It follows from these axioms that each connected component of X is a hermitian symmetric domain. For a sufficiently small open compact subgroup $\mathrm{K} \subset \mathbf{G}(\mathbb{A}_f)$, the associated Shimura variety is the double coset space

$$\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X) \coloneqq \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K},$$

which naturally carries the structure of a complex analytic space induced by X. By work of Baily and Borel, $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ is in fact a quasi-projective smooth projective variety over $\mathbb{C}$. Due to Shimura, Deligne, Borovoi, Milne, and others, the system

$$\mathrm{Sh}(\mathbf{G}, X) \coloneqq \varprojlim_{\mathrm{K}} \mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$$

has a *canonical model* defined over a number field $\mathbf{E}$, known as the *reflex field*, which only depends on the Shimura datum $(\mathbf{G}, X)$. The simplest Shimura varieties are the modular curves, which are given by the Shimura datum $(\mathrm{GL}_2, \mathcal{H}^{\pm})$. Here $\mathcal{H}^{\pm} = \mathbb{C} - \mathbb{R}$, the union of upper and lower half planes.

Let $v|p$ be a place of $\mathbf{E}$ and E be the completion of $\mathbf{E}$ at v. One active area of interest in the study of Shimura varieties is the construction of integral models. These are schemes over $\mathcal{O}_E$ with generic fiber $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)_E$. Integral models are useful in computing the Hasse-Weil zeta function of a Shimura variety in terms of automorphic L-functions, which is part of the Langlands program. The construction of integral models is also a starting point for Kudla's program relating special cycles on Shimura varieties and derivatives of Eisenstein series and L-functions.

Let $\mathcal{G}$ be a Bruhat-Tits stabilizer group scheme (see §3.4) over $\mathbb{Z}_p$ for $\mathbf{G}_{\mathbb{Q}_p}$ with neutral component $\mathcal{G}^\circ$. Set $\mathrm{K}_p \coloneqq \mathcal{G}(\mathbb{Z}_p)$ and $\mathrm{K}_p^\circ \coloneqq \mathcal{G}^\circ(\mathbb{Z}_p)$. Suppose $\mathrm{K}^\circ \subset \mathbf{G}(\mathbb{A}_f)$ is of the form $\mathrm{K}^\circ = \mathrm{K}_p^\circ \mathrm{K}^p$, where $\mathrm{K}^p \subset \mathbf{G}(\mathbb{A}_f^p)$ is a sufficiently small open compact subgroup. When $(\mathbf{G}, X)$ is of PEL type, the corresponding Shimura varieties $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ are (essentially) moduli spaces of abelian varieties with polarization, endomorphisms and level structure. Integral models of these Shimura varieties are studied in [Kot92, §5] and [RZ96, Chapter 6]. More generally, let $(\mathbf{G}, X)$ be a Shimura datum of *abelian type*, a large class that includes almost all cases where $\mathbf{G}$ is a classical group. Shimura varieties of abelian type are closely related to those of *Hodge type*, which can described as moduli spaces of abelian varieties equipped with families of Hodge tensors. If K_p is hyperspecial (which implies that $\mathbf{G}_{\mathbb{Q}_p}$ extends to a reductive group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ such that $\mathcal{G}(\mathbb{Z}_p) = \mathrm{K}_p$), Kisin [Kis10] and Kim-Madapusi [KM16] (when $p = 2$) constructed (smooth) canonical integral models over $\mathcal{O}_E$ of $\mathrm{Sh}_{\mathrm{K}^\circ}(\mathbf{G}, X)$, which are uniquely characterized by Milne's extension property. If $p > 2$, Kisin, Pappas and Zhou [KZ24; KPZ24], following earlier work of Kisin-Pappas [KP18], constructed normal flat integral models over $\mathcal{O}_E$ of $\mathrm{Sh}_{\mathrm{K}^\circ}(\mathbf{G}, X)$ with arbitrary parahoric level structure. Using Scholze's theory of p-adic shtukas, Pappas-Rapoport [PR24] and Daniels [Dan23] made the following conjecture about the existence of the *canonical integral model* of $\mathrm{Sh}_{\mathrm{K}^\circ}(\mathbf{G}, X)$ with parahoric level structure for any Shimura datum $(\mathbf{G}, X)$.

Conjecture 1.1.1 ([PR24, Conjecture 4.2.2],[Dan23, Conjecture 4.5])**.** *There exists a unique system $\{\mathscr{S}_{\mathrm{K}^\circ}\}_{\mathrm{K}^p}$ of normal flat schemes over $\mathcal{O}_E$, extending $\{\mathrm{Sh}_{\mathrm{K}^\circ}(\mathbf{G}, X)\}_{\mathrm{K}^p}$ and equipped with a p-adic shtuka satisfying the axioms in loc. cit..*

By [PR24, Theorem 1.3.2], Conjecture 1.1.1 holds when $(\mathbf{G}, X)$ is of Hodge type and K_p° is a stabilizer parahoric subgroup (i.e., $\mathrm{K}_p = \mathrm{K}_p^\circ$). Assuming the existence of $\mathscr{S}_{\mathrm{K}^\circ}$

as in Conjecture 1.1.1, Pappas and Rapoport also conjectured (at least when $\mathbf{G}$ satisfies the blanket assumption in [PR24, §4.1]) that $\mathscr{S}_{\mathrm{K}^\circ}$ fits into a scheme-theoretic local model diagram. Specifically, there should exist a diagram of $\mathcal{O}_E$-schemes

$$\mathscr{S}_{\mathrm{K}^\circ} \xleftarrow{\pi} \widetilde{\mathscr{S}}_{\mathrm{K}^\circ} \xrightarrow{q} \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}^\circ,\mu_h},$$

where μ_h denotes the geometric cocharacter of $\mathbf{G}_{\mathbb{Q}_p}$ corresponding to the Hodge cocharacter attached to $(\mathbf{G}, X)$, the $\mathcal{O}_E$-scheme $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}^\circ,\mu_h}$ denotes the *scheme local model* used in [PR24, §4.9.2] (see also Theorem 3.4.4), π is a $\mathcal{G}^\circ$-torsor, and q is $\mathcal{G}^\circ$-equivariant and smooth of relative dimension $\dim \mathbf{G}$, such that the compatibility conditions in [PR24, Definition 4.9.1] are satisfied. For the current status of Conjecture 1.1.1, we refer readers to Daniels-van Hoften-Kim-Zhang [DvHKZ24] and Daniels-Youcis [DY24], which build upon the work of Kisin, Pappas and Zhou [KP18; KZ24; KPZ24].

Local models are certain flat projective schemes over the p-adic integers which are expected to model the singularities of the integral models of Shimura varieties. Rapoport and Zink studied local models for Shimura varieties of PEL type with parahoric level structure at p in [RZ96]. Their local models were later called *naive local models*, since they are not always flat if the corresponding reductive group is ramified at p as pointed out in [Pap00, §4]. The construction of the naive local models relies on the lattice-theoretic description of parahoric subgroups, which is significantly more involved if $p = 2$ and the group is ramified. A more general approach is given in [PZ13] (see also a variant in [HPR20]) which constructs (flat) local models attached to a *local model triple* $(G, \{\mu\}, \mathcal{G})$, where G is a tamely ramified connected reductive group over a p-adic field L, $\{\mu\}$ is a geometric conjugacy class of cocharacters of G with reflex field E, and $\mathcal{G}$ is a parahoric group scheme over $\mathcal{O}_L$ with generic fiber G. Subsequent works [Lev16; Lou23; FHLR22] allow us to define local models for all triples $(G, \{\mu\}, \mathcal{G})$ excluding the case that $p = 2$ and G^{ad} contains, as an $\breve{L}$-factor, a wildly ramified unitary group of odd dimension. Here $\breve{L}$ denotes the completion of the maximal unramified extension of L in a fixed algebraic closure of L. These constructions a priori depend on certain auxiliary choices.

Another construction of local models is proposed in [SW20] using v-sheaves. The advantage is that this approach is canonical (without any auxiliary choices) and applies to arbitrary triples $(G, \{\mu\}, \mathcal{G})$, even for wildly ramified groups G and $p = 2$. It has been proven in [AGLR22; GL24] that when $\{\mu\}$ is minuscule, the v-sheaf local models are representable by flat normal projective schemes $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$ over $\mathcal{O}_E$ with reduced special fibers. Roughly, the local model $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$ is constructed as the weak normalization of certain orbit closure inside a Beilinson-Drinfeld type affine Grassmannian, extending the construction of Pappas and Zhu in [PZ13]. Excluding the case that $p = 2$ and G^{ad} contains, as an $\breve{L}$-factor, a wildly ramified unitary group of odd dimension, one can show that the corresponding scheme local models are Cohen-Macaulay with Frobenius split special fibers. We refer the readers to [FHLR22, Remark 2.2] for some explanation on this exceptional case. A key aspect of understanding the special fibers of local models is their identification with a union of (semi-normalizations of) Schubert varieties in affine flag varieties. It is worth noting that the theory of local models also has applications in the study of Galois deformation rings, leading to strong results in modularity lifting theorems, Breuil-Mézard conjecture, etc. See for example [Kis09; LLHLM23].

In the present thesis, we study the local and integral models of Shimura varieties over $p = 2$. Now we explain the main results of our work.

1.2 Main results

1.2.1 2-adic local models

The first part of the thesis focuses on the 2-adic local models for unitary similitude groups of odd dimension $n \geq 3$ with special parahoric level structure when the signature is $(n-1, 1)$.

Let $F_0/\mathbb{Q}_2$ be a finite extension and F be a (wildly) ramified quadratic extension of F_0. For any $x \in F$, we write $\overline{x}$ for the Galois conjugate of x in F. We can pick uniformizers $\pi \in F$ and $\pi_0 \in F_0$ such that F/F_0 falls into one of the following two distinct cases (see §3.1):

(R-U) $F = F_0(\sqrt{\theta})$, where θ is a unit in $\mathcal{O}_{F_0}$. The uniformizer π satisfies an Eisenstein equation

$$\pi^2 - t\pi + \pi_0 = 0,$$

where $t = \pi + \overline{\pi} \in \mathcal{O}_{F_0}$ satisfies $\pi_0 | t | 2$. We have $\sqrt{\theta} = 1 - 2\pi/t$ and $\theta = 1 - 4\pi_0/t^2$.

(R-P) $F = F_0(\sqrt{\pi_0})$, where $\pi^2 + \pi_0 = 0$.

Let (V, h) be a hermitian space, where V is an F-vector space of dimension $n = 2m+1 \geq 3$ and $h : V \times V \to F$ is a non-degenerate hermitian form. In this Introduction, we will assume that h is split, i.e., there exists an F-basis $(e_i)_{1 \leq i \leq n}$ of V such that $h(e_i, e_j) = \delta_{i,n+1-j}$ for $1 \leq i, j \leq n$. Let $G \coloneqq \mathrm{GU}(V, h)$ denote the unitary similitude group over F_0 attached to (V, h). Our first result is the lattice-theoretic description of parahoric subgroups of $G(F_0)$.

Theorem 1.2.1 (Proposition 2.4.1)**.** *Let I be a non-empty subset of $\{0, 1, \ldots, m\}$. Define*

$$\Lambda_i \coloneqq \mathcal{O}_F \langle \pi^{-1} e_1, \ldots, \pi^{-1} e_i, e_{i+1}, \ldots, e_{m+1}, \lambda e_{m+2}, \ldots, \lambda e_n \rangle, \ \textit{for } 0 \leq i \leq m,$$

where $\lambda = \overline{\pi}/t$ in the (R-U) case and $\lambda = 1/2$ in the (R-P) case. Then the subgroup

$$P_I \coloneqq \{ g \in G(F_0) \mid g\Lambda_i = \Lambda_i, \ \textit{for } i \in I \}$$

is a parahoric subgroup of $G(F_0)$. Furthermore, any parahoric subgroup of $G(F_0)$ is conjugate to P_I for a unique $I \subset \{0, 1, \ldots, m\}$. The conjugacy classes of special parahoric subgroups correspond to the sets $I = \{0\}$ and $\{m\}$.

The proof of Theorem 1.2.1 is based on Bruhat-Tits theory in (residue) characteristic two. Note that in our case, parahoric subgroups of $G(F_0)$ no longer correspond to self-dual lattice chains, which causes difficulties in the study of local models.

Given a special parahoric subgroup of $G(F_0)$ corresponding to $I = \{0\}$ or $\{m\}$, we define in §3.3 the *naive local model* $\mathrm{M}_I^{\mathrm{naive}}$ of signature $(n-1, 1)$, which is an analogue of the naive unitary local model considered in [RZ96]. To explain the construction, we start with a crucial but simple observation on the structure of the lattices Λ_i in Theorem 1.2.1. Set

$$\varepsilon := \begin{cases} t & \text{in the (R-U) case,} \\ 2 & \text{in the (R-P) case.} \end{cases} \tag{1.2.1}$$

The hermitian form h defines a symmetric F_0-bilinear form $s(-,-) : V \times V \to F_0$ and a quadratic form $q : V \to F_0$ via

$$s(x,y) := \varepsilon^{-1} \operatorname{Tr}_{F/F_0} h(x,y) \text{ and } q(x) := \frac{1}{2} s(x,x), \text{ for } x, y \in V. \tag{1.2.2}$$

Set $\mathscr{L} := \varepsilon^{-1}\mathcal{O}_{F_0}$, which is an invertible $\mathcal{O}_{F_0}$-module. Then for $0 \leq i \leq m$, the forms in (1.2.2) induce the $\mathscr{L}$-valued forms

$$s : \Lambda_i \times \Lambda_i \longrightarrow \mathscr{L} \text{ and } q : \Lambda_i \longrightarrow \mathscr{L}. \tag{1.2.3}$$

The triple $(\Lambda_i, q, \mathscr{L})$ is an $\mathscr{L}$-valued hermitian quadratic module over $\mathcal{O}_{F_0}$ in the sense of Definition 3.2.1, which roughly means that the quadratic form q is compatible with the $\mathcal{O}_F$-action.

For $I = \{0\}$ or $\{m\}$, denote $\Lambda_I := \Lambda_0$ or Λ_m respectively. Let

$$\Lambda_I^s := \{x \in V \mid s(x, \Lambda_I) \subset \mathcal{O}_{F_0}\}$$

be the dual lattice of Λ_I with respect to the pairing s in (1.2.2). Then we have a perfect $\mathcal{O}_{F_0}$-bilinear pairing

$$\Lambda_I \times \Lambda_I^s \longrightarrow \mathcal{O}_{F_0} \tag{1.2.4}$$

induced by the symmetric pairing in (1.2.2), and an inclusion of lattices

$$\Lambda_I \hookrightarrow \alpha\Lambda_I^s, \quad \text{where } \alpha := \begin{cases} \overline{\pi}/\varepsilon & \text{if } I = \{0\}, \\ 1/\varepsilon & \text{if } I = \{m\}. \end{cases}$$

We define the naive unitary local model $\mathrm{M}_I^{\mathrm{naive}}$ to be the functor

$$\mathrm{M}_I^{\mathrm{naive}} : (\mathrm{Sch}/\mathcal{O}_F)^{\mathrm{op}} \longrightarrow \mathrm{Sets}$$

which sends an $\mathcal{O}_F$-scheme S to the set of $\mathcal{O}_S$-modules $\mathcal{F}$ such that

(1) (π-stability condition) $\mathcal{F}$ is an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodule of $\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ and as an $\mathcal{O}_S$-module, it is a locally direct summand of rank n.

(2) (Kottwitz condition) The action of $\pi \otimes 1 \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ on $\mathcal{F}$ has characteristic polynomial

$$\det(T - \pi \otimes 1 \mid \mathcal{F}) = (T - \pi)(T - \overline{\pi})^{n-1}.$$

(3) Let $\mathcal{F}^{\perp}$ be the orthogonal complement of $\mathcal{F}$ in $\Lambda_I^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ with respect to the perfect pairing

$$(\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \times (\Lambda_I^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \to \mathcal{O}_S$$

induced by the perfect pairing in (1.2.4). We require that the map $\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to \alpha\Lambda_I^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ induced by $\Lambda_I \hookrightarrow \alpha\Lambda_I^s$ sends $\mathcal{F}$ to $\alpha\mathcal{F}^{\perp}$, where $\alpha\mathcal{F}^{\perp}$ denotes the image of $\mathcal{F}^{\perp}$ under the isomorphism $\alpha : \Lambda_I^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \xrightarrow{\sim} \alpha\Lambda_I^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$.

(4) $\mathcal{F}$ is totally isotropic with respect to the pairing

$$s : (\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \times (\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \to \mathscr{L} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$$

induced by s in (1.2.3), i.e., $s(\mathcal{F}, \mathcal{F}) = 0$ in $\mathscr{L} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$.

The moduli functor $\mathrm{M}_I^{\mathrm{naive}}$ is representable by a closed $\mathcal{O}_F$-subscheme of the Grassmannian $\mathrm{Gr}(n, \Lambda_I)_{\mathcal{O}_F}$. It turns out that $\mathrm{M}_I^{\mathrm{naive}}$ is not flat over $\mathcal{O}_F$. We define, as in [PR09], the *local model* $\mathrm{M}_I^{\mathrm{loc}}$ to be the flat closure of the generic fiber in $\mathrm{M}_I^{\mathrm{naive}}$. By construction, we have a closed immersion

$$\mathrm{M}_I^{\mathrm{loc}} \hookrightarrow \mathrm{M}_I^{\mathrm{naive}}$$

of projective schemes over $\mathcal{O}_F$ whose generic fibers are isomorphic to the $(n-1)$-dimensional projective space over F. We have the following results on further geometric properties of the scheme $\mathrm{M}_I^{\mathrm{loc}}$.

Theorem 1.2.2. *(1) If $I = \{0\}$, then $\mathrm{M}^{\mathrm{loc}}_{\{0\}}$ is flat projective of relative dimension $n-1$ over $\mathcal{O}_F$, normal and Cohen-Macaulay with geometrically integral special fiber. Moreover, $\mathrm{M}^{\mathrm{loc}}_{\{0\}}$ is smooth over $\mathcal{O}_F$ on the complement of a single closed point.*

(2) If $I = \{m\}$, then $\mathrm{M}^{\mathrm{loc}}_{\{m\}}$ is smooth projective of relative dimension $n-1$ over $\mathcal{O}_F$ with geometrically integral special fiber.

Let us explain the strategy of the proof of Theorem 1.2.2 in greater detail. For $I = \{0\}$ or $\{m\}$, let $\mathscr{H}_I$ denote the group scheme[1] of similitude automorphisms of the hermitian quadratic module $(\Lambda_m, q, \mathscr{L})$ (resp. $(\Lambda_0, q, \mathscr{L}, \phi)$), see Definition 3.2.2 and 3.2.3. Then $\mathscr{H}_I$ acts naturally on $\mathrm{M}^{\mathrm{naive}}_I$, and hence on $\mathrm{M}^{\mathrm{loc}}_I$. Let $\overline{k}$ denote the algebraic closure of the residue field of F. Using the results in Chapter 6, we can show that the (geometric) special fiber $\mathrm{M}^{\mathrm{loc}}_I \otimes_{\mathcal{O}_F} \overline{k}$ has two orbits under the action of $\mathscr{H}_I \otimes_{\mathcal{O}_{F_0}} \overline{k}$. One of the orbits consists of just one closed point. We call it the *worst point* of the local model. Using this, we are reduced to proving that there is an open affine subscheme of $\mathrm{M}^{\mathrm{loc}}_I$ containing the worst point and satisfying the geometric properties (normality, Cohen-Macaulayness, etc) as stated in Theorem 1.2.2.

To get the desired open affine subscheme of $\mathrm{M}^{\mathrm{loc}}_I$, we introduce a refinement M_I, as a closed subfunctor, of the moduli functor $\mathrm{M}^{\mathrm{naive}}_I$ such that

$$\mathrm{M}^{\mathrm{loc}}_I \subset \mathrm{M}_I \subset \mathrm{M}^{\mathrm{naive}}_I.$$

It turns out that the underlying topological space of M_I is equal to that of $\mathrm{M}^{\mathrm{loc}}_I$. For a matrix A, we will write $\mathcal{O}_F[A]$ for the polynomial ring over $\mathcal{O}_F$ whose variables are entries of the matrix A. Viewing M_I as a closed subscheme of the Grassmannian $\mathrm{Gr}(n, \Lambda_I)_{\mathcal{O}_F}$, we can find an open affine subscheme U_I of M_I which contains the worst point and which is isomorphic to a closed subscheme of $\mathrm{Spec}\,\mathcal{O}_F[Z]$, where Z is an $n \times n$ matrix, such that the worst point is defined by $Z = 0$ and $\pi = 0$. Then we explicitly write down the affine coordinate ring of

[1] In Chapter 6, we prove that $\mathscr{H}_I$ is smooth over $\mathcal{O}_{F_0}$ and isomorphic to the parahoric group scheme attached to Λ_I.

U_I defined by matrix identities. From this, we obtain the affine coordinate ring of $\mathrm{U}_I \cap \mathrm{M}_I^{\mathrm{loc}}$ by calculating the flat closure of U_I.

Theorem 1.2.3. *Let Y (resp. X) be a $2m \times 2m$ (resp. $2m \times 1$) matrix with variables as entries. Let H_{2m} denote the $2m \times 2m$ anti-diagonal unit matrix. There is an open affine subscheme $\mathrm{U}_I^{\mathrm{loc}}$ of $\mathrm{M}_I^{\mathrm{loc}}$ which contains the worst point and satisfies the following properties.*

(1) If $I = \{0\}$, then $\mathrm{U}_{\{0\}}^{\mathrm{loc}}$ is isomorphic to

$$\operatorname{Spec} \frac{\mathcal{O}_F[Y|X]}{\left(\wedge^2(Y|X), Y - Y^t, (\frac{\pi}{\overline{\pi}} \frac{\mathrm{tr}(H_{2m}Y)}{2} + \pi\sqrt{\theta})Y + XX^t\right)}, \quad \textit{in the (R-U) case,}$$
$$\operatorname{Spec} \frac{\mathcal{O}_F[Y|X]}{\left(\wedge^2(Y|X), Y - Y^t, (\frac{\mathrm{tr}(H_{2m}Y)}{2} - \pi)Y + XX^t\right)}, \quad \textit{in the (R-P) case.}$$

(We remark that under the relation $Y - Y^t = 0$, the polynomial $\mathrm{tr}(H_{2m}Y)$, which is the sum of the anti-diagonal entries of Y, is indeed divisible by 2 in $\mathcal{O}_F[Y]$.)

(2) If $I = \{m\}$, then $\mathrm{U}_{\{m\}}^{\mathrm{loc}}$ is isomorphic to

$$\operatorname{Spec} \frac{\mathcal{O}_F[Y|X]}{\left(\wedge^2(Y|X), Y - Y^t, (\frac{\mathrm{tr}(H_{2m}Y)}{t} + \sqrt{\theta})Y + XX^t\right)}, \quad \textit{in the (R-U) case,}$$

$$\operatorname{Spec} \mathcal{O}_F[X], \quad \textit{in the (R-P) case.}$$

Using the above result, we reduce the proof of Theorem 1.2.2 to a purely commutative algebra problem. We need to show that the affine coordinate rings in Theorem 1.2.3 satisfy the geometric properties stated in Theorem 1.2.2. The hardest part is to show the Cohen-Macaulayness when $I = \{0\}$, where we use a converse version of the miracle flatness theorem. We refer to Lemma 4.1.16 for more details.

We can also relate $\mathrm{M}_I^{\mathrm{loc}}$ to the v-sheaf local models considered in [SW20, §21.4] (see also §3.4). By the results in [AGLR22; FHLR22; GL24] (see Theorem 3.4.4), we already know that the v-sheaf local models in our case are representable by normal projective flat $\mathcal{O}_F$-schemes $\mathbb{M}_I$ (denoted by $\mathbb{M}_{\mathcal{G},\mu}^{\mathrm{loc}}$ in §3.4).

Theorem 1.2.4 (Theorem 3.4.5). *The local model* $\mathrm{M}_I^{\mathrm{loc}}$ *is isomorphic to* $\mathbb{M}_I$.

As a corollary, our result gives a very explicit construction of $\mathbb{M}_I$ and a more elementary proof of the representability of the v-sheaf local models in our setting.

Remark 1.2.5. If F/F_0 is of type (R-P), the arguments in [AGLR22] (see the paragraph after Theorem 1.1 in *loc. cit.*) also imply that $\mathbb{M}_I$ is Cohen-Macaulay. However, our methods can also deal with the (R-U) case, and we are able to give explicit local affine coordinate rings.

It should be pointed out that it could be useful to provide an explicit moduli interpretation of $\mathrm{M}_I^{\mathrm{loc}}$. As a by-product of our analysis of $\mathrm{U}_I^{\mathrm{loc}}$ (see Lemma 4.1.13), we obtain such a description in a special case.

Theorem 1.2.6. *Suppose F/F_0 is of type (R-U) and assume that the valuations of t and π_0 are equal*[2]*. Then* $\mathrm{M}_{\{0\}}^{\mathrm{loc}}$ *represents the functor*

$$(\mathrm{Sch}/\mathcal{O}_F)^{\mathrm{op}} \longrightarrow \mathrm{Sets}$$

which sends an $\mathcal{O}_F$-scheme S to the set of $\mathcal{O}_S$-modules $\mathcal{F}$ such that [3]

LM1 *(π-stability condition) $\mathcal{F}$ is an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodule of $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ and as an $\mathcal{O}_S$-module, it is a locally direct summand of rank n.*

LM2 *(Kottwitz condition) The action of $\pi \otimes 1 \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ on $\mathcal{F}$ has characteristic polynomial*

$$\det(T - \pi \otimes 1 \mid \mathcal{F}) = (T - \pi)(T - \overline{\pi})^{n-1}.$$

LM3 *Let $\mathcal{F}^{\perp}$ be the orthogonal complement in $\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ of $\mathcal{F}$ with respect to the perfect pairing*

$$(\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \times (\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \to \mathcal{O}_S$$

[2] This holds if F_0 is unramified over $\mathbb{Q}_2$, see some more discussion in Remark 4.1.14.
[3] As in [Smi15, Lemma 5.2, Remark 5.4], the conditions **LM2** and **LM5** are in fact implied by **LM6**.

induced by the perfect pairing in (1.2.4). *We require that the map* $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to \frac{\pi}{t}\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ *induced by* $\Lambda_0 \hookrightarrow \frac{\pi}{t}\Lambda_I^s$ *sends* $\mathcal{F}$ *to* $\frac{\pi}{t}\mathcal{F}^\perp$, *where* $\frac{\pi}{t}\mathcal{F}^\perp$ *denotes the image of* $\mathcal{F}^\perp$ *under the isomorphism* $\frac{\pi}{t} : \Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \xrightarrow{\sim} \frac{\pi}{t}\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$.

LM4 *(Hyperbolicity condition) The quadratic form* $q : \Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to \mathscr{L} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ *induced by* $q : \Lambda_0 \to \mathscr{L}$ *satisfies* $q(\mathcal{F}) = 0$.

LM5 *(Wedge condition) The action of* $\pi \otimes 1 - 1 \otimes \overline{\pi} \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ *on* $\mathcal{F}$ *satisfies*

$$\wedge^2(\pi \otimes 1 - 1 \otimes \overline{\pi} \mid \mathcal{F}) = 0.$$

LM6 *(Strengthened spin condition) The line* $\wedge^n \mathcal{F} \subset W(\Lambda_0) \otimes_{\mathcal{O}_F} \mathcal{O}_S$ *is contained in*

$$\mathrm{Im}\left(W(\Lambda_0)_{-1}^{n-1,1} \otimes_{\mathcal{O}_F} \mathcal{O}_S \to W(\Lambda_0) \otimes_{\mathcal{O}_F} \mathcal{O}_S\right).$$

(See §4.1.1.1 for the explanation of the notation in this condition.)

1.2.2 2-adic integral models

The second part of the thesis focuses on the 2-adic models of Shimura varieties.

Assume $p = 2$ and that $(\mathbf{G}, X)$ is a Shimura datum of abelian type. Let $v|p$ be a place of $\mathbf{E}$ and E be the completion of $\mathbf{E}$ at v. Denote by $\mathcal{O}_{\mathbf{E},(v)}$ the localization of $\mathcal{O}_{\mathbf{E}}$ at v. Denote by k_E the residue field of E and by k the algebraic closure of k_E. We will construct 2-adic integral models over $\mathcal{O}_{\mathbf{E},(v)}$ for $\mathrm{Sh}_{\mathrm{K}^\circ}(\mathbf{G}, X)$ under one of the following assumptions:

(A) $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ has no factor of type $D^{\mathbb{H}}$, $\mathbf{G}_{\mathbb{Q}_p}$ is unramified, and K_p° is contained in some hyperspecial subgroup;

(B) $\mathbf{G} = \mathrm{GU}(n-1, 1)$ is the unitary similitude group over $\mathbb{Q}$ of signature $(n-1, 1)$ for some odd integer $n \geq 3$, $\mathbf{G}_{\mathbb{Q}_p}$ is (wildly) ramified, and K_p° is a special parahoric subgroup.

Theorem 1.2.7. *Assume that either (A) or (B) holds.*

(1) The $\mathbf{E}$*-scheme*

$$\mathrm{Sh}_{\mathrm{K}_p^\circ}(\mathbf{G}, X) := \varprojlim_{\mathrm{K}^p} \mathrm{Sh}_{\mathrm{K}_p^\circ \mathrm{K}^p}(\mathbf{G}, X)$$

admits a $\mathbf{G}(\mathbb{A}_f^p)$*-equivariant extension to a flat normal* $\mathcal{O}_{\mathbf{E},(v)}$*-scheme* $\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G},X)$*. Any sufficiently small* $\mathrm{K}^p \subset \mathbf{G}(\mathbb{A}_f^p)$ *acts freely on* $\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G},X)$*, and the quotient*

$$\mathscr{S}_{\mathrm{K}^\circ}(\mathbf{G},X) := \mathscr{S}_{\mathrm{K}_p}(\mathbf{G},X)/\mathrm{K}^p$$

is a flat normal $\mathcal{O}_{\mathbf{E},(v)}$*-scheme extending* $\mathrm{Sh}_{\mathrm{K}^\circ}(\mathbf{G},X)$*.*

(2) For any discrete valuation ring R *of mixed characteristic* 0 *and* p*, the map*

$$\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G},X)(R) \to \mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G},X)(R[1/p])$$

is a bijection.

(3) There exists a diagram of $\mathcal{O}_E$*-schemes*

$$\begin{tikzcd} & \widetilde{\mathscr{S}}^{\mathrm{ad}}_{\mathrm{K}_p^\circ} \arrow[dl, "\pi"'] \arrow[dr, "q"] & \\ \mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G},X)_{\mathcal{O}_E} & & \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}^\circ,\mu_h}, \end{tikzcd}$$

where π *is a* $\mathbf{G}(\mathbb{A}_f^p)$*-equivariant* $G^{\mathrm{ad}\circ}_{\mathbb{Z}_p}$*-torsor,* q *is* $G^{\mathrm{ad}\circ}_{\mathbb{Z}_p}$*-equivariant, and for any sufficiently small* $\mathrm{K}^p \subset \mathbf{G}(\mathbb{A}_f^p)$*, the map* $\widetilde{\mathscr{S}}^{\mathrm{ad}}_{\mathrm{K}_p^\circ}/\mathrm{K}^p \to \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}^\circ,\mu_h}$ *induced by* q *is smooth of relative dimension* $\dim \mathbf{G}^{\mathrm{ad}}$*.*

(4) If κ *is a finite extension of* k_E *and* $y \in \mathscr{S}_{\mathrm{K}_p^\circ}(G,X)(\kappa)$*, then there exists* $z \in \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}^\circ,\mu_h}(\kappa)$ *such that we have an isomorphism of henselizations*

$$\mathcal{O}^h_{\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G},X),y} \simeq \mathcal{O}^h_{\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}^\circ,\mu_h},z}.$$

Here in (3), $G^{\mathrm{ad}\circ}_{\mathbb{Z}_p}$ denotes the parahoric group scheme over $\mathbb{Z}_p$ with generic fiber $\mathbf{G}^{\mathrm{ad}}_{\mathbb{Q}_p}$, defined by $\mathcal{G}^\circ$ using the map $\mathcal{B}(\mathbf{G}_{\mathbb{Q}_p},\mathbb{Q}_p) \to \mathcal{B}(\mathbf{G}^{\mathrm{ad}}_{\mathbb{Q}_p},\mathbb{Q}_p)$ between extended Bruhat-Tits buildings, see §7.3.2. The proof of Theorem 1.2.7 will be given in §7.3.2.2 and §7.3.3.3.

Remark 1.2.8. (1) When K_p° is hyperspecial, Theorem 1.2.7 has been proved by Kim-Madapusi [KM16]. In *loc. cit.*, $(\mathbf{G}^{\mathrm{ad}},X^{\mathrm{ad}})$ is allowed to have a factor of type $D^{\mathbb{H}}$.

(2) We expect that the results of van Hoften [vHof24] and Gleason-Lim-Xu [GLX22] can be extended to the 2-adic models constructed in this thesis.

Let us give two interesting cases in which Theorem 1.2.7 can be applied to obtain integral models over $\mathbb{Z}_{(2)}$ for $\mathrm{Sh}_{\mathrm{K}_2^\circ\mathrm{K}^2}(\mathbf{G}, X)$ when K_2° is a parahoric subgroup contained in some hyperspecial subgroup. Let F be a totally real number field which is unramified at primes over 2.

(i) $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSpin}(V, Q)$, where $\mathrm{GSpin}(V, Q)$ is the spin similitude group over F attached to a quadratic space (V, Q) of signature $(n, 2)$ at each real place (assume $\mathrm{GSpin}(V, Q)$ is unramified over F_v, $v|2$) and X is (a product of) the space of oriented negative definite planes;

(ii) $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GU}$, where GU is the unitary similitude group over F that is unramified over F_v, $v|2$. We note that this case is also known by [RSZ21, Appendix A].

As in [KP18, Corolary 0.3], Theorem 1.2.7 implies the following.

Corollary 1.2.9. *With the assumptions as in Theorem 1.2.7, the special fiber $\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G}, X) \otimes k_E$ is reduced, and the strict henselizations of the local rings on $\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G}, X) \otimes k_E$ have irreducible components which are normal and Cohen-Macaulay.*

If K_p° is associated to a point $x \in \mathcal{B}(\mathbf{G}_{\mathbb{Q}_p}, \mathbb{Q}_p)$ which is a special vertex in $\mathcal{B}(\mathbf{G}_{\mathbb{Q}_p}, \mathbb{Q}_p^{\mathrm{ur}})$, then the special fiber $\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G}, X) \otimes k_E$ is normal and Cohen-Macaulay.

We now explain the idea to prove Theorem 1.2.7. The overall strategy follows that of [KP18] and [KPZ24]. As in *loc. cit.*, the crucial case is when $(\mathbf{G}, X)$ is of Hodge type. A key step in this case involves identifying the formal neighborhood of $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ with that of the local model $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h}$. For $p > 2$, this identification is obtained in [KP18; KPZ24] by constructing a versal deformation of p-divisible groups (equipped with a family of crystalline tensors) over the formal neighborhood of the local model. The construction of this versal deformation uses Zink's theory of Dieudonné displays that classify p-divisible groups. For

$p = 2$, we modify Zink's theory by using Lau's results from [Lau14], and obtain a similar deformation theory for 2-divisible groups. A technical requirement arising in this step is that we need to find a Hodge embedding

$$\iota : (\mathbf{G}, X) \hookrightarrow (\mathbf{GSp}(V, \psi), S^{\pm}),$$

where V is a $\mathbb{Q}$-vector space of dimension $2g$ equipped with a perfect alternating pairing ψ, such that $\iota_{\mathbb{Q}_p}$ extends to a *very good* integral Hodge embedding $(\mathcal{G}, \mu_h) \hookrightarrow (\mathrm{GL}(\Lambda), \mu_g)$, where $\Lambda \subset V_{\mathbb{Q}_p}$ is a *self-dual* $\mathbb{Z}_p$-lattice with respect to ψ.

The concept of very good integral Hodge embeddings was introduced in [KPZ24, §5.2] for $p > 2$, refining the notion of *good* integral Hodge embeddings in [KZ24, Definition 3.1.6]. We generalize the concept to the case $p = 2$ (see Definition 7.2.13). Roughly speaking, a good integral integral Hodge embedding is an integral Hodge embedding

$$\widetilde{\iota} : (\mathcal{G}, \mu_h) \hookrightarrow (\mathrm{GL}(\Lambda), \mu_g)$$

extending $\iota_{\mathbb{Q}_p}$ such that $\widetilde{\iota}$ induces a closed immersion

$$\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h} \hookrightarrow \mathbb{M}^{\mathrm{loc}}_{\mathrm{GL}(\Lambda),\mu_g} \otimes_{\mathbb{Z}_p} \mathcal{O}_E = \mathrm{Gr}(g, \Lambda) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$$

of local models, where $\mathrm{Gr}(g, \Lambda)$ denotes the Grassmannian of rank g subspaces of Λ. The key idea behind very good Hodge embeddings is that certain collection of tensors (s_α) in the tensor algebra $\Lambda^{\otimes}$, cutting out $\mathcal{G}$ in $\mathrm{GL}(\Lambda)$, should satisfy a "horizontal" condition under the natural connection isomorphism. We refer to §7.2.2 for more details. For a good integral Hodge embedding $\widetilde{\iota}$, Kisin-Pappas-Zhou proved in [KPZ24, Proposition 5.3.1, Lemma 5.3.2] that this horizontality condition is satisfied in the following two cases (including for $p = 2$):

(1) For any $x \in \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h}(k)$, the image of the natural map

$$\left\{ f \in \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h}(k[[t]]) \mid f \bmod (t) = x \right\} \to T_x \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h}$$

spans, as a k-vector space, the tangent space $T_x \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h}$.

(2) The tensors $(s_\alpha) \subset \Lambda^{\otimes}$ are in $\Lambda \otimes_{\mathbb{Z}_p} \Lambda^\vee$.

Using this, they can produce sufficiently many very good Hodge embeddings when $p > 2$.

When $p = 2$, it is in general difficult to find a very good integral Hodge embedding $\widetilde{\iota}$ for a Shimura datum of Hodge type. In the present thesis, we establish the existence of very good Hodge embeddings under the assumption (A) or (B).

For Case (A), by applying [KPZ24, Proposition 5.3.1, Lemma 5.3.2], we are reduced to presenting the stabilizer group scheme $\mathcal{G}$ as $(\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\mathcal{H})^\Gamma$, where $F/\mathbb{Q}_p$ is a tame Galois extension with Galois group Γ and $\mathcal{H}$ is a reductive group over $\mathcal{O}_F$. For Case (B), we directly prove that the tangent space of the local model $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h}$ at any closed point is spanned by formal curves (see Lemma 7.3.17), using the explicit description of the (local) coordinate rings of the unitary local models in the first part of the thesis.

1.3 Organization

We now give an overview of the thesis.

In Chapter 2, we discuss Bruhat-Tits theory for (odd) unitary groups in residue characteristic two. In particular, we describe the maxi-minorant norms (*norme maximinorante* in French) used in [BT87] in terms of graded lattice chains, and thus obtain a lattice-theoretic description of the Bruhat-Tits buildings of unitary groups. As a corollary, we deduce Theorem 1.2.1.

In Chapter 3, we first discuss some basic facts about quadratic extensions of 2-adic fields. Then we equip the lattices Λ_i in Theorem 1.2.1 with the structure of hermitian quadratic modules. Using this, we define the naive local models $\mathrm{M}_I^{\mathrm{naive}}$ and local models $\mathrm{M}_I^{\mathrm{loc}}$. In §3.4, we review the Beilinson-Drinfeld Grassmannian (in mixed characteristic) and v-sheaf local models of Scholze-Weinstein. Assuming Theorem 1.2.2, we show that the local models in Theorem 1.2.2 represent the v-sheaf local models, thereby proving Theorem 1.2.4.

In Chapter 4 and 5, we prove Theorem 1.2.2, 1.2.3 and 1.2.6. We address the (R-U) and (R-P) case separately, although the techniques are very similar. In each chapter, we introduce the refinement M_I of $\mathrm{M}_I^{\mathrm{naive}}$ by imposing certain linear algebraic conditions and then explicitly write down the local affine coordinate rings. We then obtain Theorem 1.2.3 by computing the flat closure of these affine coordinate rings. Utilizing the group action on local models, we finish the proof of Theorem 1.2.2 and Theorem 1.2.6.

In Chapter 6, we show that, under certain conditions, hermitian quadratic modules étale locally have a normal form up to similitude. Along the way, we prove in Theorem 6.1.13 and Theorem 6.2.8 that the similitude automorphism group scheme of Λ_m (resp. (Λ_0, ϕ)) is affine smooth over $\mathcal{O}_{F_0}$ and is isomorphic to the parahoric group scheme attached to Λ_m (resp. Λ_0). The results in this chapter are used in Chapter 4 and 5.

In Chapter 7, we construct 2-adic integral models of Shimura varieties of abelian type and prove Theorem 1.2.7. Very often we will refer the readers to corresponding arguments in [KP18; KPZ24] that are similar or can be directly extended to the case $p = 2$ without repeating the proofs.

In §7.1, we review Lau's results in [Lau14], which generalizes Zink's theory of Dieudonné displays so that we can classify 2-divisible groups over 2-adic rings (see Theorem 7.1.14). A new feature of the theory of Dieudonné displays in the case $p = 2$ is the modified Verschiebung map for the Zink ring (see Lemma 7.1.2). In §7.1.2, we construct the natural "connection isomorphisms" for Dieudonné pairs when $p = 2$ (see Lemma 7.1.13), generalizing [KPZ24, Lemma 5.1.3] for $p > 2$. In §7.1.4, we compare Lau's classification of p-divisible groups with Breuil-Kisin's classification. This comparison is later used in §7.3.1.2 to construct $(\mathcal{G}_W, \mu_y)$-adapted deformations of p-divisible groups in the sense of Definition 7.2.17.

In §7.2, we apply Lau's theory to construct a versal deformation of p-divisible groups, extending results from [KP18, §3] to the case $p = 2$. We also generalize the concept of very good Hodge embeddings, introduced in [KPZ24], to $p = 2$. This is used to construct versal deformations of p-divisible groups with crystalline tensors (see Proposition 7.2.16). In Proposition 7.2.18, we establish a criterion for determining when a deformation is $(\mathcal{G}_W, \mu_y)$-adapted, extending [Zho20, Proposition 4.7] to $p = 2$.

In §7.3, we apply results in §7.2 to construct 2-adic integral models of Shimura varieties of abelian type under certain assumptions (see Theorem 7.3.9). The overall strategy follows that of [KP18; KPZ24]. We first treat the case of Shimura varieties of Hodge type and then extend to Shimura varieties of abelian type by finding suitable Hodge type lifts while closely following [KP18]. In §7.3.2.2 and §7.3.3.3, we complete the proof of Theorem 1.2.7 by verifying that the assumptions in Theorem 7.3.9 are satisfied in Case (A) or (B).

In §7.4, we show that, for an unramified group G over a 2-adic field F, if a stabilizer group scheme $\mathcal{G}$ satisfies $\mathcal{G}(\mathcal{O}_F) \subset H$ for some hyperspecial subgroup H of $G(F)$, then $\mathcal{G}$ can be written as the tame Galois fixed points of the Weil restriction of scalars of a reductive group scheme. This result is used in the construction of very good integral Hodge embeddings in Case (A).

CHAPTER 2

BRUHAT-TITS THEORY FOR UNITARY GROUPS IN RESIDUE CHARACTERISTIC TWO

In this chapter, we discuss Bruhat-Tits theory for (odd) unitary groups in residue characteristic two. In particular, we describe the maxi-minorant norms (*norme maximinorante* in French) used in [BT87] in terms of graded lattice chains, and thus obtain a lattice-theoretic description of the Bruhat-Tits buildings of unitary groups. As a corollary, we deduce Theorem 1.2.1 in the Introduction.

2.1 Notations

Let F_0 be a finite extension of $\mathbb{Q}_2$. Let $\omega : F_0 \to \mathbb{Z} \cup \{+\infty\}$ denote the normalized valuation on F_0. Let F/F_0 be a (wildly totally) ramified quadratic extension. The valuation ω uniquely extends to a valuation on F, which is still denoted by ω. Denote by σ the nontrivial element in $\mathrm{Gal}(F/F_0)$. For $x \in F$, we will write x^σ or $\overline{x}$ for the Galois conjugate of x in F. Let $\mathcal{O}_F$ (resp. $\mathcal{O}_{F_0}$) be the ring of integers of F (resp. F_0) with uniformizer π (resp. π_0). We assume $N_{F/F_0}(\pi) = \pi_0$. Let k be the common residue field of F and F_0. Let V be an F-vector space of dimension $n = 2m + 1 \geq 3$ with a non-degenerate hermitian form $h : V \times V \to F$. We assume that there exists an F-basis $(e_i)_{1 \leq i \leq n}$ of V such that $h(e_i, e_j) = \delta_{i,n+1-j}$ for $1 \leq i, j \leq n$. In this case, we will say that the hermitian form h is *split*, or (V, h) is a split hermitian space.

(We remark that all results in Chapter 2 have analogous (simpler) counterparts when F_0 is a finite extension of $\mathbb{Q}_p$ for $p > 2$, see Remark 2.2.6 and 2.2.10.)

2.2 Bruhat-Tits buildings in terms of norms

In this section, we would like to recall the description of Bruhat-Tits buildings of odd dimensional (quasi-split) unitary groups in residue characteristic two in terms of norms. The standard reference is [BT87]. There is a summary (in English) in [Lem09, §1]. See also [Tit79, Example 1.15, 2.10].

Let $G := \mathrm{U}(V, h)$ denote the unitary group over F_0 attached to (V, h). Then there is an embedding of (enlarged) buildings

$$\mathcal{B}(G, F_0) \hookrightarrow \mathcal{B}(\mathrm{GL}_F(V), F).$$

Definition 2.2.1. A *norm* on V is a map $\alpha : V \to \mathbb{R} \cup \{+\infty\}$ such that for $x, y \in V$ and $\lambda \in F$, we have

$$\alpha(x+y) \geq \inf\{\alpha(x), \alpha(y)\}, \ \alpha(\lambda x) = \omega(\lambda) + \alpha(x), \text{ and } x = 0 \Leftrightarrow \alpha(x) = +\infty.$$

Example 2.2.2. (1) Let V be a one dimensional F-vector space. Then any norm α on V is uniquely determined by its value of a non-zero element in V: for any $0 \neq x \in V$ and $\lambda \in F$, we have

$$\alpha(\lambda x) = \omega(\lambda) + \alpha(x).$$

(2) Let V_1 and V_2 be two finite dimensional F-vector spaces. Let α_i be a norm on V_i for $i = 1, 2$. The *direct sum* of α_1 and α_2 is defined as a norm $\alpha_1 \oplus \alpha_2 : V_1 \oplus V_2 \to \mathbb{R} \cup \{+\infty\}$ via

$$(\alpha_1 \oplus \alpha_2)(x_1 + x_2) := \inf\{\alpha_1(x_1), \alpha_2(x_2)\}, \text{ for } x_i \in V_i.$$

Proposition 2.2.3 ([KP23, 15.1.11])**.** *Let α be a norm on V. Then there exists a basis $(e_i)_{1 \leq i \leq n}$ of V and n real numbers c_i for $1 \leq i \leq n$ such that*

$$\alpha(\sum_{i=1}^{n} x_i e_i) = \inf_{1 \leq i \leq n} \{\omega(x_i) - c_i\}.$$

In this case, we say $(e_i)_{1 \leq i \leq n}$ is a splitting basis of α, or α is split by $(e_i)_{1 \leq i \leq n}$.

Denote by $\mathcal{N}$ the set of all norms on V. Then $\mathcal{N}$ carries a natural $\mathrm{GL}_F(V)(F)$-action via

$$(g\alpha)(x) := \alpha(g^{-1}x), \text{ for } g \in \mathrm{GL}_F(V)(F) \text{ and } x \in V. \tag{2.2.1}$$

For each F-basis $(e_i)_{1 \leq i \leq n}$ of V, we have a corresponding maximal F-split torus T of $\mathrm{GL}_F(V)$ whose F-points are diagonal matrices with respect to the basis $(e_i)_{1 \leq i \leq n}$. The cocharacter group $X_*(T)$ has a $\mathbb{Z}$-basis $(\mu_i)_{1\leq i\leq n}$, where $\mu_i : \mathbb{G}_{m,F} \to T$ is a cocharacter characterized by

$$\mu_i(t)e_j = t^{-\delta_{ij}}e_j, \text{ for } t \in F^\times \text{ and } 1 \leq i, j \leq n, \tag{2.2.2}$$

where δ_{ij} is the Kronecker symbol. Fixing an origin, we may identify the apartment $\mathcal{A} \subset \mathcal{B}(\mathrm{GL}_F(V), F)$ corresponding to T with $X_*(T)_\mathbb{R}$.

Proposition 2.2.4 ([BT84b, 2.8, 2.11])**.** *The map*

$$\begin{aligned} \mathcal{A} = X_*(T)_\mathbb{R} &\longrightarrow \mathcal{N} \\ \sum_{i=1}^n c_i\mu_i &\mapsto \left(\sum_{i=1}^n x_ie_i \mapsto \inf_{1\leq i\leq n}\{\omega(x_i) - c_i\}\right), \end{aligned} \tag{2.2.3}$$

where $c_i \in \mathbb{R}$, $x_i \in F$ and $\sum_{i=1}^n x_ie_i \in V$, extends uniquely to an isomorphism of $\mathrm{GL}_F(V)$-sets

$$\mathcal{B}(\mathrm{GL}_F(V), F) \xrightarrow{\sim} \mathcal{N}.$$

Moreover, the image of $X_*(T)_\mathbb{R}$ in $\mathcal{N}$ is the set of norms on V admitting $(e_i)_{1\leq i\leq n}$ as a splitting basis.

By Proposition 2.2.4, we can identify the building $\mathcal{B}(\mathrm{GL}_F(V), F)$ with the set $\mathcal{N}$ of norms on V. Next we will describe the image of the inclusion $\mathcal{B}(G, F_0) \hookrightarrow \mathcal{B}(\mathrm{GL}_F(V), F) = \mathcal{N}$ in terms of maxi-minorant norms (*norme maximinorante* in French).

Set $F_\sigma \coloneqq \{\lambda - \lambda^\sigma \mid \lambda \in F\}$. Then F_σ is an F_0-subspace of F and we denote by F/F_σ the quotient space. We can associate the hermitian form h with a map $\overline{q} : V \to F/F_\sigma$, called the *pseudo-quadratic form* in [BT87], defined by

$$\overline{q}(x) \coloneqq \frac{1}{2}h(x, x) + F_\sigma, \text{ for } x \in V.$$

The valuation ω induces a quotient norm $\overline{\omega}$ on the F_0-vector space F/F_σ:

$$\overline{\omega}(\lambda + F_\sigma) \coloneqq \sup\{\omega(\lambda + \mu - \mu^\sigma) \mid \mu \in F\}, \text{ for } \lambda \in F.$$

Definition 2.2.5. Let α be a norm on V. We say α *minorizes* (*minores* in French) $(h, \overline{q})$ if for all $x, y \in V$,

$$\alpha(x) + \alpha(y) \leq \omega(h(x,y)) \text{ and } \alpha(x) \leq \frac{1}{2}\overline{\omega}(\overline{q}(x)).$$

Following the terminology of [KP23, Remark 15.2.12], we say α is *maxi-minorant* (*maximinorante* in French) for $(h, \overline{q})$ if α minorizes $(h, \overline{q})$ and α is maximal for this property.

Denote by $\mathcal{N}_{mm}$ $(\subset \mathcal{N})$ the set of maxi-minorant norms for $(h, \overline{q})$ on V. One can easily check that $\mathcal{N}_{mm}$ carries a $G(F_0)$-action via (2.2.1). Here we view $G(F_0)$ as a subgroup of $\mathrm{GL}_F(V)$.

Remark 2.2.6. Let α be a norm on V. Set

$$\alpha^\vee(x) \coloneqq \inf_{y \in V} \{\omega(h(x,y)) - \alpha(y)\}, \text{ for } x \in V.$$

Then $\alpha^\vee$ is also a norm on V, called the *dual norm* of α. We say α is *self-dual* if $\alpha = \alpha^\vee$. If F has odd residue characteristic, then by [BT87, 2.16], the norm $\alpha \in \mathcal{N}_{mm}$ if and only if α is self-dual.

Note that for $x \in V$, we have

$$\begin{aligned}\overline{q}(x) &= \frac{1}{2}h(x,x) + F_\sigma = \{\frac{1}{2}h(x,x) + \mu - \mu^\sigma \mid \mu \in F\} \\ &= \{\lambda h(x,x) \mid \lambda \in F, \lambda + \lambda^\sigma = 1\} \in F/F_\sigma.\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{\omega}(\overline{q}(x)) &= \sup\{\omega(\lambda h(x,x)) \mid \lambda \in F, \lambda + \lambda^\sigma = 1\} \\ &= \omega(h(x,x)) + \sup\{\omega(\lambda) \mid \lambda \in F, \lambda + \lambda^\sigma = 1\}.\end{aligned}$$

Set

$$\delta \coloneqq \sup\{\omega(\lambda) \mid \lambda \in F, \lambda + \lambda^\sigma = 1\}. \tag{2.2.4}$$

We obtain that α minores $(h, \overline{q})$ if and only if for $x, y \in V$, we have

$$\alpha(x) + \alpha(y) \leq \omega(h(x,y)) \text{ and } \alpha(x) \leq \frac{1}{2}\omega(h(x,x)) + \frac{1}{2}\delta.$$

Definition 2.2.7. Let (V, h) be a (split) hermitian F-vector space of dimension n as in §2.1.

(1) A *Witt decomposition* of V is a decomposition $V = V_- \oplus V_0 \oplus V_+$ such that V_- and V_+ are two maximal isotropic subspaces of V, and V_0 is the orthogonal complement of $V_- \oplus V_+$ with respect to h. As we assume h is split, we have $\dim_F V_- = \dim_F V_+ = m$ and $\dim_F V_0 = 1$.

(2) For any F-basis $(e_i)_{1\le i\le n}$ of V, we put

$$V_- \coloneqq \operatorname{span}_F \{e_1, \ldots, e_m\}, V_0 \coloneqq \operatorname{span}_F \{e_{m+1}\}, V_+ \coloneqq \operatorname{span}_F \{e_{m+2}, \ldots, e_n\}.$$

We say $(e_i)_{1\le i\le n}$ induces a Witt decomposition of V if $V_- \oplus V_0 \oplus V_+$ is a Witt decomposition of V and $h(e_i, e_j) = \delta_{i,n+1-j}$ for $1 \le i, j \le n$.

Let $(e_i)_{1\le i\le n}$ be a basis of V inducing a Witt decomposition. Such a basis defines a maximal F_0-split torus S of G whose F_0-points are given by

$$\left\{ g \in G(F_0) \subset \mathrm{GL}_F(V)(F) \;\middle|\; \begin{array}{l} ge_i = x_i e_i \text{ and } x_i x_{n+1-i} = x_{m+1} = 1 \\ \text{for some } x_i \in F_0 \text{ and } 1 \le i \le n \end{array} \right\}.$$

The centralizer of S in $G \otimes_{F_0} F \simeq \mathrm{GL}_F(V)$ is T. For $m+2 \le i \le n$, let $\lambda_i : \mathbb{G}_{m,F_0} \to S$ be the cocharacter of S defined by

$$\lambda_i(t)e_i = t^{-1}e_i,\ \lambda_i(t)e_{n+1-i} = te_{n+1-i},\ \text{and } \lambda_i(t)e_j = e_j \text{ for } t \in F_0^\times \text{ and } j \ne i, n+1-i. \tag{2.2.5}$$

Then the set $(\lambda_i)_{m+2\le i\le n}$ forms a $\mathbb{Z}$-basis of $X_*(S)$. Fixing an origin, we may identify the apartment $\mathcal{A}(G, S)$ of $\mathcal{B}(G, F_0)$ corresponding to S with $X_*(S)_\mathbb{R}$. Then we have the following proposition.

Proposition 2.2.8. *The map*

$$\begin{aligned} X_*(S)_\mathbb{R} &\longrightarrow \mathcal{N}_{mm} \\ \sum_{i=m+2}^{n} c_i\lambda_i &\mapsto \left(\sum_{i=1}^{n} x_i e_i \mapsto \inf\{\omega(x_i) - c_i, \omega(x_{m+1}) + \frac{1}{2}\delta \mid 1 \le i \le n \textit{ and } i \ne m+1\} \right), \end{aligned} \tag{2.2.6}$$

where $c_i \coloneqq -c_{n+1-i}$ *if* $1 \le i \le m$*, extends uniquely to an isomorphism of* $G(F_0)$*-sets*

$$\mathcal{B}(G, F_0) \to \mathcal{N}_{mm}.$$

The image of $X_*(S)_\mathbb{R}$ *in* $\mathcal{N}_{mm}$ *is the set of maxi-minorant norms admitting* $(e_i)_{1\le i\le n}$ *as a splitting basis.*

Moreover, a norm $\alpha \in \mathcal{N}_{mm}$ *is special, i.e.,* α *corresponds to a special point in* $\mathcal{B}(G, F_0)$*, if and only if there is a basis* $(f_i)_{1\le i\le n}$ *of* V *inducing a Witt decomposition and a constant* $C \in \frac{1}{4}\mathbb{Z}$ *such that for* $x_i \in F$*, we have*

$$\alpha(\sum_{i=1}^{n} x_i f_i) = \inf\{\omega(x_i) - C, \omega(x_j) + C, \omega(x_{m+1}) + \frac{1}{2}\delta \mid 1 \le i < m+1 \text{ and } m+1 < j \le n\}.$$

Proof. See [BT87, 2.9, 2.12] and [Tit79, Example 2.10]. □

Corollary 2.2.9. *Let* $\alpha \in \mathcal{N}$*. Then* $\alpha \in \mathcal{N}_{mm}$ *if and only if there exists a basis* $(f_i)_{1\le i\le n}$ *of* V *inducing a Witt decomposition* $V = V_- \oplus V_0 \oplus V_+$ *such that* $\alpha = \alpha_\pm \oplus \alpha_0$*, where* $\alpha_\pm$ *is a self-dual norm on* $V_- \oplus V_+$ *split by the basis* $(f_i)_{i\ne m+1}$*, and* α_0 *is the unique norm on* V_0 *with* $\alpha(f_{m+1}) = \frac{1}{2}\delta$*.*

Proof. (⇒) We can view $X_*(S)_\mathbb{R}$ as a subset of $\mathcal{N}_{mm}$ via the map (2.2.6). Using the $G(F_0)$-action, we may assume α lies in $X_*(S)_\mathbb{R}$, say $\alpha = \sum_{i=m+2}^{n} c_i\lambda_i \in X_*(S)_\mathbb{R}$ for $c_i \in \mathbb{R}$. Then we take (f_i) to be (e_i), which induces a Witt decomposition $V = V_- \oplus V_0 \oplus V_+$. Define the norm $\alpha_\pm$ on $V_- \oplus V_+$ by

$$\begin{aligned} V_- \oplus V_+ &\longrightarrow \mathbb{R} \cup \{+\infty\} \\ \sum_{1\le i\le n, i\ne m+1} x_i f_i &\mapsto \inf\{\omega(x_i) - c_i \mid 1 \le i \le n \text{ and } i \ne m+1\}, \end{aligned} \tag{2.2.7}$$

where we define $c_i \coloneqq -c_{n+1-i}$ for $1 \le i \le m$. Clearly $\alpha_\pm$ is split by $(f_i)_{i\ne m+1}$. As $h(f_i, f_{n+1-j}) = \delta_{ij}$ and $c_i = -c_{n+1-i}$ for $1 \le i, j \le n$, we deduce that $\alpha_\pm$ is self-dual by [KP23, Remark 15.2.7]. Moreover, from the expression of (2.2.6), we immediately see that α decomposes as $\alpha = \alpha_\pm \oplus \alpha_0$.

($\Leftarrow$) Under the assumptions, there exist n real numbers c_i for $1 \leq i \leq n$ such that $c_{n+1-i} = -c_i$ and $\alpha_\pm$ is given by the norm as in (2.2.7). Let S' be the maximal F_0-split torus in G corresponding to the basis $(f_i)_{1\leq i\leq n}$. Let $(\lambda'_i)_{m+2\leq i\leq n}$ be a $\mathbb{Z}$-basis of $X_*(S')$ defined as in (2.2.5). Then α is the norm corresponding to the point $\sum_{i=m+2}^{n} c_i\lambda'_i \in X_*(S')_{\mathbb{R}}$ via a similar map as in (2.2.6). In particular, $\alpha \in \mathcal{N}_{mm}$. □

Remark 2.2.10. Assume F has odd residue characteristic. Then $\delta = 0$, and hence α_0 is self-dual. Then the norm $\alpha_\pm \oplus \alpha_0$ as in the Corollary 2.2.9 is self-dual. When F has odd residue characteristic, any self-dual norm admits a splitting basis inducing a Witt decomposition of V, see for example [KP23, Proposition 15.2.10]. Then we see again that $\alpha \in \mathcal{N}_{mm}$ if and only α is self-dual.

Remark 2.2.11. We can define a "twisted" Galois action of $\mathrm{Gal}(F/F_0)$ on $\mathrm{GL}_F(V)(F)$ as follows: for $g \in \mathrm{GL}_F(V)(F)$, define $\sigma(g)$ to be the element satisfying

$$h(g^{-1}x, y) = h(x, \sigma(g)y), \text{ for } x, y \in V.$$

Then we have $G(F_0) = \mathrm{GL}_F(V)(F)^{\sigma=1}$, the set of fixed points of σ. This twisted Galois action induces an involution on $\mathcal{N} = \mathcal{B}(\mathrm{GL}_F(V), F) = \mathcal{B}(G \otimes_{F_0} F, F)$, which is still denoted by σ. Next we give an explicit description of this involution.

Let $(e_i)_{1\leq i\leq n}$ be a basis inducing a Witt decomposition $V = V_- \oplus V_0 \oplus V_+$. Let T be the induced maximal torus of $\mathrm{GL}_F(V)$. Let $\mathcal{A}(T) \subset \mathcal{B}(\mathrm{GL}_F(V), F)$ be the apartment corresponding to T. We can identify $\mathcal{A}(T)$ with $X_*(T)_{\mathbb{R}}$ through the injection (cf. (2.2.3))

$$\begin{aligned} &X_*(T)_{\mathbb{R}} \longrightarrow \mathcal{N} \\ &\sum_{i=1}^{n} c_i\mu_i \mapsto \left(\sum_{i=1}^{n} x_ie_i \mapsto \inf\left\{\begin{matrix} \omega(x_{m+1}) - c_{m+1} + \frac{1}{2}\delta, \omega(x_i) - c_i \\ \text{for } 1 \leq i \leq n \text{ and } i \neq m+1 \end{matrix}\right\}\right), \end{aligned}$$

where μ_i is defined as in (2.2.2), $x_i \in F$ and $\sum_{i=1}^{n} x_ie_i \in V$. As G is quasi-split, we can pick a σ-stable point as the origin such that the twisted σ-action on $\mathcal{A}(T)$ is transported by the twisted σ-action on $X_*(T)_{\mathbb{R}}$. For $\alpha \in \mathcal{N}$, there is a $g \in \mathrm{GL}_F(V)(F)$ such that $g\alpha \in X_*(T)_{\mathbb{R}}$, since $\mathrm{GL}_F(V)(F)$ acts transitively on the apartments of $\mathcal{N}$. Then

$$g\alpha = \alpha_1 \oplus (\alpha_0 + C),$$

where α_1 is a norm on $V_- \oplus V_+$ admitting $(e_i)_{i\neq m+1}$ as a splitting basis, α_0 is the norm on V_0 as in the Corollary 2.2.9, and $C \in \mathbb{R}$ is a certain constant. The twisted σ-action on $X_*(T)_\mathbb{R}$ implies that $\sigma(\alpha_1 \oplus (\alpha_0 + C)) = \alpha_1^\vee \oplus (\alpha_0 - C)$. Hence, we see that σ acts on α as

$$\sigma(\alpha) = \sigma(g^{-1}) \left(\alpha_1^\vee \oplus (\alpha_0 - C)\right).$$

For $\alpha \in \mathcal{N}_{mm} = \mathcal{B}(G, F_0)$, we may take $g \in G(F_0)$ and $C = 0$. Thus, we get an inclusion

$$\mathcal{B}(G, F_0) \hookrightarrow \mathcal{B}(\mathrm{GL}_F(V), F)^{\sigma=1}.$$

The inclusion is strict: any norm of the form $\alpha_1 \oplus \alpha_0$, where α_1 is a self-dual norm on $V_- \oplus V_+$ but not split by any basis of $V_- \oplus V_+$ inducing a Witt decomposition, lies in $\mathcal{B}(\mathrm{GL}_F(V), F)^{\sigma=1}$ but not in $\mathcal{B}(G, F_0)$. Such a norm can only exist when the residue characteristic of F is two. For an explicit example, see Example 2.3.7.

2.3 Bruhat-Tits buildings in terms of lattices

In this section, we will translate the results in §2.2 into the language of lattices, which is more useful in the theory of local models.

Definition 2.3.1. Let V be a finite dimensional F-vector space.

(1) A *lattice* L in V is a finitely generated $\mathcal{O}_F$-submodule of V such that $L \otimes_{\mathcal{O}_F} F = V$.

(2) A *(periodic) lattice chain* of V is a non-empty set $L_\bullet$ of lattices in V such that lattices in $L_\bullet$ are totally ordered with respect to the inclusion relation, and $\lambda L \in L_\bullet$ for $\lambda \in F^\times$ and $L \in L_\bullet$.

(3) A *graded lattice chain* is a pair $(L_\bullet, c)$, where $L_\bullet$ is a lattice chain of V and $c : L_\bullet \to \mathbb{R}$ is a strictly decreasing function such that for any $\lambda \in F$ and $L \in L_\bullet$, we have

$$c(\lambda L) = \omega(\lambda) + c(L).$$

The function c is called a *grading* of $L_\bullet$.

(4) An F-basis $(e_i)_{1\le i\le n}$ of V is called *adapted to* a graded lattice chain $(L_\bullet, c)$ of V if for every $L \in L_\bullet$, there exist $x_1, \ldots, x_n \in F$ such that $(x_i e_i)_{1\le i\le n}$ is an $\mathcal{O}_F$-basis of L. In this case, we also say $(L_\bullet, c)$ is adapted to the basis $(e_i)_{1\le i\le n}$.

Remark 2.3.2. Since $L_\bullet$ is stable under homothety, the set $L_\bullet$ is determined by a finite number of lattices satisfying

$$\pi L_0 \subsetneq L_{r-1} \subsetneq L_{r-2} \subsetneq \cdots \subsetneq L_1 \subsetneq L_0.$$

We say $(L_0, L_1, \ldots, L_{r-1})$ is a *segment* of $L_\bullet$, and the integer r is the *rank* of $L_\bullet$.

Denote by $\mathcal{GLC}$ the set of graded lattice chains of V. There is a $\mathrm{GL}_F(V)(F)$-action on $\mathcal{GLC}$: for $(L_\bullet, c) \in \mathcal{GLC}$ and $g \in \mathrm{GL}_F(V)(F)$, define $g(L_\bullet, c) \coloneqq (gL_\bullet, gc)$, where $gL_\bullet$ consists of lattices of the form gL for $L \in L_\bullet$, and $(gc)(gL) \coloneqq c(L)$ for $L \in L_\bullet$.

Lemma 2.3.3. *(1) There is a one-to-one correspondence between $\mathcal{N}$ and $\mathcal{GLC}$. More precisely, given $\alpha \in \mathcal{N}$, we can associate a graded lattice chain (L_α, c_α), where L_α is the set of following lattices*

$$L_{\alpha,r} = \{x \in V \mid \alpha(x) \geq r\}, \text{ for } r \in \mathbb{R},$$

and the grading c_α is defined by

$$c_\alpha(L_{\alpha,r}) = \inf_{x\in L_{\alpha,r}} \alpha(x).$$

Conversely, given a graded lattice chain $(L_\bullet, c) \in \mathcal{GLC}$, we can associate a norm

$$\alpha_{(L_\bullet,c)}(x) \coloneqq \sup\{c(L) \mid x \in L \text{ and } L \in L_\bullet\}.$$

We say the norm α and the graded lattice chain (L_α, c_α) in the above bijection correspond to *each other.*

(2) The bijection in (1) is $\mathrm{GL}_F(V)(F)$-equivariant.

(3) *Let $(e_i)_{1\leq i\leq n}$ be a basis of V. Let $(L_\bullet, c)$ be the graded lattice chain corresponding to a norm α via (1). Then $(e_i)_{1\leq i\leq n}$ is adapted to $(L_\bullet, c)$ if and only if $(e_i)_{1\leq i\leq n}$ is a splitting basis of α.*

Proof. The proof of (1) and (3) can be found in [KP23, Proposition 15.1.21]. The assertion in (2) can be checked by direct computation. □

Using the above lemma, we can easily extend operations like direct sums or duality on norms to graded lattice chains.

Lemma 2.3.4. *(1) Let V and V' be two finite dimensional F-vector spaces. Let α and α' be two norms on V and V' respectively. Let $(L_\bullet, c)$ and $(L'_\bullet, c')$ be graded lattice chains corresponding to α and α' respectively. Then the graded lattice chain $(L_\bullet, c) \oplus (L'_\bullet, c')$ corresponding to $\alpha \oplus \alpha'$ is a pair $(L_\bullet \oplus L'_\bullet, c \oplus c')$, where $L_\bullet \oplus L'_\bullet$ is the set of lattices of the form $L_{\alpha,r} \oplus L_{\alpha',r}$ for $r \in \mathbb{R}$, and*

$$(c \oplus c')(L_{\alpha,r} \oplus L_{\alpha',r}) \coloneqq \inf\left\{c(L_{\alpha,r}), c'(L_{\alpha',r})\right\}.$$

(2) *Let $(L_\bullet, c)$ be the graded lattice chain corresponding to a norm α on V. Then the dual norm $\alpha^\vee$ corresponds to the graded lattice chain $(L^\vee_\bullet, c^\vee)$, where $L^\vee_\bullet$ is the set of the lattices of the form $L^\vee \coloneqq \{x \in V \mid h(x, L) \in \mathcal{O}_F\}$ for $L \in L_\bullet$, and*

$$c^\vee(L^\vee) \coloneqq -c(L^-) - 1,$$

where L^- is the smallest member of $L_\bullet$ that properly contains L.

Proof. The proof of (1) is straightforward. The proof of (2) can be found in [KP23, Fact 15.2.18]. □

We say $(L_\bullet, c)$ is *self-dual* if $(L_\bullet, c) = (L^\vee_\bullet, c^\vee)$.

Proposition 2.3.5. *Let $(L_\bullet, c) \in \mathcal{GLC}$. Then $(L_\bullet, c)$ corresponds to a norm in $\mathcal{N}_{mm}$ if and only if there exists a basis $(f_i)_{1\leq i\leq n}$ of V inducing a Witt decomposition $V = V_- \oplus V_0 \oplus V_+$*

and $(L_\bullet, c)$ decomposes as $(L_\bullet^\pm, c^\pm) \oplus (L_\bullet^0, c^0)$, such that $(L_\bullet^\pm, c^\pm)$ is a self-dual graded lattice chain of $V_- \oplus V_+$ adapted to the basis $(f_i)_{i \neq m+1}$, and $(L_\bullet^0, c^0)$ is the graded lattice chain corresponding to the norm α_0 on V_0.

Proof. This is a translation of Corollary 2.2.9 in view of the previous two lemmas. □

Remark 2.3.6. Let $(L_\bullet^\pm, c^\pm)$ be a self-dual graded lattice chain adapted to the basis $(f_i)_{i \neq m+1}$ as in Proposition 2.3.5. Then for any $L \in L_\bullet^\pm$, there exist $x_i \in F$ for $i \neq m+1$ such that $(x_i f_i)_{i \neq m+1}$ forms an $\mathcal{O}_F$-basis of L. As $h(f_i, f_j) = \delta_{i,n+1-j}$, we see that L is isomorphic to an orthogonal sum of "hyperbolic planes" of the form $H(i)$ ($i \in \mathbb{Z}$). Here $H(i)$ denotes a lattice in a two dimensional hermitian F-vector space (W, h) such that $H(i)$ is $\mathcal{O}_F\langle x, y\rangle$ spanned by some $x, y \in W$ with $h(x, x) = h(y, y) = 0$ and $h(x, y) = \pi^i$.

A lattice in W which is isomorphic to $H(i)$ for some $i \in \mathbb{Z}$ is also called a *hyperbolic lattice* in the sense of [Kir17, §2]. For any lattice K in W, define the *norm ideal* $n(K)$ of K to be the ideal in $\mathcal{O}_{F_0}$ generated by $h(x, x)$ for $x \in K$. Let $K^\vee$ denote the dual lattice of K with respect to the hermitian form h on W. Then by [Kir17, §2] (see also [Jac62, Proposition 9.2 (a)]), any lattice $K \subset W$ satisfying $K = \pi^i K^\vee$ (that is, K is π^i-modular) and $n(K) = n(H(i))$ is isomorphic to $H(i)$.

Example 2.3.7. Let $F_0 = \mathbb{Q}_2$ and $F = \mathbb{Q}_2(\sqrt{3})$. Pick uniformizers $\pi = \sqrt{3} - 1 \in F$ and $\pi_0 = -2 \in F_0$ so that $\pi^2 + 2\pi - 2 = 0$. We have

$$\delta = \sup\{\omega(\lambda) \mid \lambda \in F, \lambda + \lambda^\sigma = 1\} = \omega(\frac{\pi}{2}) = -\frac{1}{2}.$$

Let (V, h) be a 3-dimensional (split) hermitian F-vector space. Let $(e_i)_{1 \leq i \leq 3}$ be a basis of V inducing a Witt decomposition $V = V_- \oplus V_0 \oplus V_+$. Denote $V_\pm \coloneqq V_- \oplus V_+ = F\langle e_1, e_3\rangle$. Set

$$f_1 \coloneqq \pi^{-1}(e_1 + e_3),\ f_2 \coloneqq e_2,\ f_3 \coloneqq \pi^{-1}(e_1 - e_3).$$

Then $L_1 \coloneqq \mathcal{O}_F\langle f_1, f_3\rangle$ is a self-dual lattice in $(V_\pm, h)$. By [Jac62, Equation (9.1)], the self-dual hyperbolic plane $H(0)$ in $V_\pm$ has norm ideal $2\mathcal{O}_{F_0}$. On the other hand, we have

$n(L_1) = \mathcal{O}_{F_0}$ by direct computation. In particular, the self-dual lattice L_1 in $(V_\pm, h)$ is not isomorphic to $H(0)$, and hence L_1 is not adapted to any basis of $V_\pm$ induing a Witt decomposition.

Now define

$$L \coloneqq L_1 \oplus \mathcal{O}_F f_2.$$

Then the graded lattice chain $(L_\bullet, c)$, where $L_\bullet \coloneqq \{\pi^i L\}_{i\in\mathbb{Z}}$ and $c(\pi^i L) \coloneqq \frac{i}{2} + \frac{\delta}{2} = \frac{i}{2} - \frac{1}{4}$, defines a norm

$$\begin{aligned} \alpha : V &\longrightarrow \mathbb{R} \cup \{+\infty\} \\ \sum_{i=1}^{3} x_i f_i &\mapsto \inf_{1\le i\le 3} \{\omega(x_i) - \frac{1}{4}\}. \end{aligned}$$

Then we see α lies in the fixed point set $\mathcal{B}(\mathrm{GL}_F(V), F)^{\sigma=1} = \mathcal{N}^{\sigma=1}$, but does not lie in $\mathcal{N}_{mm}$.

2.4 Parahoric subgroups and lattices

Let us keep the notations as in §2.2. In particular, the set $(e_i)_{1\le i\le n}$ denotes a basis of V inducing a Witt decomposition $V = V_- \oplus V_0 \oplus V_+$ and S denotes the corresponding maximal F_0-split torus of $G = \mathrm{U}(V, h)$. Denote by $(a_i)_{m+2\le i\le n} \in X^*(S)$ the dual basis of $(\lambda_i)_{m+2\le i\le n} \in X_*(S)$.

By the calculations in [Tit79, Example 1.15], the relative root system $\Phi = \Phi(G, S)$ is

$$\{\pm a_i \pm a_j \mid m+2 \le i, j \le n, i \ne j\} \cup \{\pm a_i, \pm 2a_i \mid m+2 \le i \le n\},$$

and the affine root system Φ_a is

$$\begin{aligned} &\{\pm a_i \pm a_j + \frac{1}{2}\mathbb{Z} \mid m+2 \le i, j \le n, i \ne j\} \\ \cup &\{\pm a_i + \frac{1}{2}\delta + \frac{1}{2}\mathbb{Z} \mid m+2 \le i \le n\} \cup \{\pm 2a_i + \frac{1}{2} + \delta + \mathbb{Z} \mid m+2 \le i \le n\}. \end{aligned}$$

Here δ is defined as in (2.2.4). These affine roots endow $X_*(S)_\mathbb{R}$ with a simplicial structure. Following [Tit79, Example 3.11], we pick a chamber defined by the inequalities

$$\frac{1}{2}\delta < a_{m+2} < \cdots < a_n < \frac{1}{2}\delta + \frac{1}{4}.$$

Then we obtain $m+1$ vertices $v_0, \ldots, v_m$ in $X_*(S)_{\mathbb{R}}$ such that for $0 \leq i \leq m$,

$$a_j(v_i) = \begin{cases} \frac{1}{2}\delta & \text{if } m+2 \leq j \leq n-i, \\ \frac{1}{2}\delta + \frac{1}{4} & \text{if } n-i < j \leq n. \end{cases}$$

Now each v_i defines a (maxi-minorant) norm, and hence a graded lattice chain, by Proposition 2.2.8 and Lemma 2.3.3. Let $\lambda \in F$ be an element satisfying $\omega(\lambda) = \delta$. We shall see an explicit expression of λ in Lemma 3.2.4. Define

$$\Lambda_i := \mathcal{O}_F\langle \pi^{-1}e_1, \ldots, \pi^{-1}e_i, e_{i+1}, \ldots, e_{m+1}, \lambda e_{m+2}, \ldots, \lambda e_n \rangle, \tag{2.4.1}$$
$$\Lambda_i' = \mathcal{O}_F\langle e_1, \ldots, e_m, e_{m+1}, \lambda e_{m+2}, \ldots, \lambda e_{n-i}, \lambda\pi e_{n+1-i}, \ldots, \lambda\pi e_n \rangle.$$

Then the graded lattice chain corresponding to v_i is of rank 2 and has a segment

$$\pi\Lambda_i \subset \Lambda_i' \subset \Lambda_i.$$

Let $\widetilde{G} = \mathrm{GU}(V, h)$ be the unitary similitude group attached to the hermitian space (V, h). Let I be a non-empty subset of $\{0, 1, \ldots, m\}$. Define

$$P_I := \left\{ g \in \widetilde{G}(F_0) \mid g\Lambda_i = \Lambda_i, \text{ for } i \in I \right\}.$$

As in [PR09, 1.2.3], the Kottwitz map restricted to P_I is trivial. In particular, we obtain that the (maximal) parahoric subgroup of $\widetilde{G}(F_0)$ is the stabilizer of v_i in $\widetilde{G}(F_0)$, which also equals the stabilizer of Λ_i in $\widetilde{G}(F_0)$ (as the stabilizer of Λ_i' is larger). More generally, we have the following proposition.

Proposition 2.4.1. *Denote $\widetilde{G} = \mathrm{GU}(V, h)$. The subgroup P_I is a parahoric subgroup of $\widetilde{G}(F_0)$. Any parahoric subgroup of $\widetilde{G}(F_0)$ is conjugate to a subgroup P_I for a unique $I \subset \{0, 1, \ldots, m\}$. The conjugacy classes of special parahoric subgroups correspond to the sets $I = \{0\}$ and $\{m\}$.*

Proof. The results are similar to those in [PR08, §4] and [PR09, 1.2.3]. The first two assertions follow from the observation that $\widetilde{G}(F_0)$ acts transitively on the chambers in the building, and each I determines a (unique) facet in a chamber. The last assertion follows from the explicit expressions of the vertices v_i and Proposition 2.2.8. $\square$

CHAPTER 3

WILDLY RAMIFIED ODD UNITARY LOCAL MODELS

In this chapter, we construct local models for unitary similitude groups of odd dimension $n \geq 3$ with special parahoric level structure when the signature is $(n-1, 1)$.

3.1 Quadratic extensions of 2-adic fields

We start with some basic facts about quadratic extensions of 2-adic fields. The readers can find more details in [Jac62, §5] and [OMe00, §63].

Proposition 3.1.1. *Let E be a finite extension of $\mathbb{Q}_2$ of degree d with ring of integer $\mathcal{O}_E$. Let e (resp. f) be the ramification degree (resp. residue degree) of the field extension $E/\mathbb{Q}_2$. Note that $d = ef$.*

(1) The map sending a to $E(\sqrt{a})$ defines a bijection between $E^\times/(E^\times)^2$ and the set of isomorphism classes of field extensions of E of degree at most two. Furthermore, the cardinality of $E^\times/(E^\times)^2$ is 2^{2+d}. In particular, we have $2^{2+d} - 1$ quadratic extensions of E.

(2) Let U be the unit group of $\mathcal{O}_E$ and ϖ be a uniformizer of $\mathcal{O}_E$. For $i \geq 1$, let $U_i := 1 + \varpi^i \mathcal{O}_E$ be a subgroup of U. Then U_i is contained in U^2 for $i \geq 2e+1$ and the quotient $U_{2e}/(U_{2e} \cap U^2)$ has two elements corresponding to the trivial extension and the unramified quadratic extension of E. Note that $U_{2e} = 1 + 4\mathcal{O}_E$.

(3) Any non-trivial element in $E^\times/(E^\times)^2$ has a representative of the following three forms:

(i) a unit in $U_{2e} - U_{2e+1}$ (elements in U_{2e} but not in U_{2e+1}),

(ii) a prime element in E,

(iii) a unit in $U_{2i-1} - U_{2i}$ for some $1 \leq i \leq e$.

The corresponding quadratic extensions in (ii) and (iii) are ramified. Following [Jac62, §5], we will say the (ramified) quadratic extensions in (ii) and (iii) are of type (R-P) *and* (R-U) *respectively. There are* 2^{1+d} *quadratic extensions of* E *of type (R-P) and* $2^{1+d}-2$ *quadratic extensions of* E *of type (R-U).*

(4) *Let* $E(\sqrt{\theta})/E$ *be a quadratic extension of type (R-U) for some unit* $\theta \in U_{2i-1} - U_{2i}$ *for some* $1 \le i \le e$*. Then there exists a prime* π *in* $E(\sqrt{\theta})$ *and a prime* π_0 *in* E *satisfying*

$$\pi^2 - t\pi + \pi_0 = 0$$

for some $t \in \mathcal{O}_E$ *with* $\mathrm{ord}(t) = e+1-i$*, where* ord *denotes the normalized valuation on* E*.*

Proof. (1) The bijection is well-known from Kummer theory. The formula for the cardinality can be found in [OMe00, 63:9].

(2) See [OMe00, 63:1, 63:3].

(3) See [OMe00, 63:2]. The number of quadratic extensions of type (R-U) or (R-P) follows from the cardinality formula of $E^\times/(E^\times)^2$ in (1).

(4) Let ϖ be any prime in E. By assumption, $\theta = 1 + \varpi^{2i-1}u$ for some unit u. Set

$$\pi := \frac{1-\sqrt{\theta}}{\varpi^{i-1}} \in E(\sqrt{\theta}).$$

Let $\overline{\pi}$ be the Galois conjugate of π. Then

$$\pi + \overline{\pi} = \frac{2}{\varpi^{i-1}} \text{ and } \pi\overline{\pi} = -\varpi u.$$

Now take π_0 to be $-\varpi u$ and t to be $\frac{2}{\varpi^{i-1}}$. Then $t \in \mathcal{O}_E$, as $\mathrm{ord}(t) = e+1-i \ge 1$, and π satisfies

$$\pi^2 - t\pi + \pi_0 = 0.$$

In particular, π is a prime element in $E(\sqrt{\theta})$. □

Example 3.1.2. The (ramified) quadratic extension $\mathbb{Q}_2(\sqrt{3})/\mathbb{Q}_2$ is of type (R-U), while $\mathbb{Q}_2(\sqrt{2})/\mathbb{Q}_2$ is a quadratic extension of type (R-P).

Let us return to the setting in §2.1. By Proposition 3.1.1, we can find uniformizers $\pi \in F$ and $\pi_0 \in F_0$ such that the quadratic extension F/F_0 falls into one of the following two distinct cases[1]:

(R-U) $F = F_0(\sqrt{\theta})$, where θ is a unit in $\mathcal{O}_{F_0}$. The uniformizer π satisfies

$$\pi^2 - t\pi + \pi_0 = 0.$$

Here $t \in \mathcal{O}_{F_0}$ with $\pi_0 | t | 2$ and $\omega(t)$ depends only on F. We have $\sqrt{\theta} = 1 - \frac{2\pi}{t}$ and $\theta = 1 - \frac{4\pi_0}{t^2}$.

(R-P) $F = F_0(\sqrt{\pi_0})$, where $\pi^2 + \pi_0 = 0$.

Lemma 3.1.3. *Let F, F_0, π and π_0 be as above.*

(1) Suppose F/F_0 is of type (R-U). Then the inverse different of F/F_0 is $\frac{1}{t}\mathcal{O}_F$.

(2) Suppose F/F_0 is of type (R-P). Then the inverse different of F/F_0 is $\frac{1}{2\pi}\mathcal{O}_F$.

Proof. As π satisfies an Eisenstein polynomial f, by [Ser13, Chapter III, §6, Corollary 2] and [Ser13, Chapter I, §6, Proposition 18], we obtain that $\mathcal{O}_F = \mathcal{O}_{F_0}[\pi]$ and the inverse different of F/F_0 is given by

$$\delta_{F/F_0}^{-1} = \frac{1}{f'(\pi)}\mathcal{O}_F.$$

More precisely,

(1) when F/F_0 is of type (R-U), then $f(T) = T^2 - tT + \pi_0$ and $\delta_{F/F_0}^{-1} = \frac{1}{2\pi - t}\mathcal{O}_F = \frac{1}{t}\mathcal{O}_F$, as $t|2$.

(2) when F/F_0 is of type (R-P), then $f(T) = T^2 + \pi_0$ and $\delta_{F/F_0}^{-1} = \frac{1}{2\pi}\mathcal{O}_F$.

$\square$

[1] When $F_0/\mathbb{Q}_2$ is an unramified finite extension, there is a description in [Cho16, §2A] of these two cases in terms of the ramification groups of $\mathrm{Gal}(F/F_0)$.

3.2 Hermitian quadratic modules and parahoric group schemes

In this section, we define hermitian quadratic modules following [Ans18, §9] and relate them to parahoric group schemes.

Let R be an $\mathcal{O}_{F_0}$-algebra. The non-trivial Galois involution on $\mathcal{O}_F$ extends to a map

$$\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R \to \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R,\ x \otimes r \mapsto \overline{x} \otimes r$$

for $x \in \mathcal{O}_F$ and $r \in R$. We will also denote the map by $a \mapsto \overline{a}$ for $a \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$. The norm map on $\mathcal{O}_F$ induces the map

$$N_{F/F_0} : \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R \to R,\ a \mapsto a\overline{a}.$$

Definition 3.2.1 ([Ans18, Definition 9.1])**.** Let R be an $\mathcal{O}_{F_0}$-algebra. Let $d \geq 1$ be an integer. Consider a triple $(M, q, \mathscr{L})$, where M is a locally free $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-module of rank d, $\mathscr{L}$ is an invertible R-module, and $q : M \to \mathscr{L}$ is an $\mathscr{L}$-valued quadratic form. Let $f : M \times M \to \mathscr{L}$ denote the symmetric R-bilinear form sending $(x, y) \in M \times M$ to $f(x, y) \coloneqq q(x + y) - q(x) - q(y) \in \mathscr{L}$.

We say the triple $(M, q, \mathscr{L})$ is a *hermitian quadratic module of rank d* over R if for any $a \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$ and any $x, y \in M$, we have

$$q(ax) = N_{F/F_0}(a)q(x) \text{ and } f(ax, y) = f(x, \overline{a}y). \tag{3.2.1}$$

A quadratic form $q : M \to \mathscr{L}$ satisfying (3.2.1) is called an *$\mathscr{L}$-valued hermitian quadratic form* on M.

Definition 3.2.2. Let $(M_1, q_1, \mathscr{L}_1)$ and $(M_2, q_2, \mathscr{L}_2)$ be two hermitian quadratic modules over an $\mathcal{O}_{F_0}$-algebra R. A *similitude isomorphism* or simply *similitude* between $(M_i, q_i, \mathscr{L}_i)$ for $i = 1, 2$ is a pair (φ, γ) of isomorphisms, where $\varphi : M_1 \xrightarrow{\sim} M_2$ is an isomorphism of $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-modules and $\gamma : \mathscr{L}_1 \xrightarrow{\sim} \mathscr{L}_2$ is an isomorphism of R-modules such that

$$q_2(\varphi(m_1)) = \gamma(q_1(m_1)),\ \text{for any } m_1 \in M_1.$$

We will write

$$\underline{\mathrm{Sim}}\left((M_1, q_1, \mathscr{L}_1), (M_2, q_2, \mathscr{L}_2)\right), \text{ or simply } \underline{\mathrm{Sim}}\,(M_1, M_2), \tag{3.2.2}$$

for the functor over R which sends an R-algebra S to the set $\mathrm{Sim}(M_1 \otimes_R S, M_2 \otimes_R S)$ of similitude isomorphisms between $(M_i \otimes_R S, q_i \otimes_R S, \mathscr{L}_i \otimes_R S)$ for $i = 1, 2$. In the case $(M_1, q_1, \mathscr{L}_1) = (M_2, q_2, \mathscr{L}_2)$, we will write

$$\underline{\mathrm{Sim}}(M_1, q_1, \mathscr{L}_1), \text{ or simply } \underline{\mathrm{Sim}}(M_1), \tag{3.2.3}$$

for $\underline{\mathrm{Sim}}\left((M_1, q_1, \mathscr{L}_1), (M_2, q_2, \mathscr{L}_2)\right)$. This is in fact a group functor, and represented by an affine group scheme of finite type over R.

Definition 3.2.3. Let R be an $\mathcal{O}_{F_0}$-algebra. Denote by $\mathcal{C}_R$ the category of quadruples $(M, q, \mathscr{L}, \phi)$ such that $(M, q, \mathscr{L})$ is a hermitian quadratic module over R and ϕ is an R-bilinear form $\phi : M \times M \to \mathscr{L}$ such that for $x, y \in M$, we have

$$\begin{aligned} &\phi(x, \pi y) = q(x + y) - q(x) - q(y), \quad \phi(\pi x, y) = \phi(x, \overline{\pi} y), \\ &\phi(x, y) = \phi\left(\frac{\overline{\pi}}{\pi} y, x\right), \quad \phi(x, x) = \frac{t}{\pi_0} q(x). \end{aligned} \tag{3.2.4}$$

Here $t := \pi + \overline{\pi}$. In particular, $t = 0$ if F/F_0 is of type (R-P). We will say an object $(M, q, \mathscr{L}, \phi) \in \mathcal{C}_R$ is a hermitian quadratic module with ϕ, or simply a hermitian quadratic module.

Let $(M_i, q_i, \mathscr{L}_i, \phi_i) \in \mathcal{C}_R$ for $i = 1, 2$. A *similitude isomorphism preserving* ϕ between $(M_i, q_i, \mathscr{L}_i, \phi_i)$ is a pair (φ, γ) of isomorphisms such that (φ, γ) is a similitude between $(M_i, q_i, \mathscr{L}_i)$, and for $m_1, m_1' \in M_1$, we have

$$\phi_2(\varphi(m_1), \varphi(m_1')) = \gamma(\phi_1(m_1, m_1')).$$

We will use a similar notation as in (3.2.2) and (3.2.3) to denote the functor of similitudes preserving ϕ between two hermitian quadratic modules in $\mathcal{C}_R$.

Recall that we defined in §2.4 lattices Λ_i for $0 \le i \le m$ via

$$\Lambda_i = \mathcal{O}_F\langle \pi^{-1} e_1, \ldots, \pi^{-1} e_i, e_{i+1}, \ldots, e_{m+1}, \lambda e_{m+2}, \ldots, \lambda e_n\rangle,$$

where λ is an element in F such that

$$\omega(\lambda) = \delta = \sup_{x \in F} \{\omega(x) \mid x + \overline{x} = 1\}.$$

The stabilizer of Λ_i is a maximal parahoric subgroup of $\mathrm{GU}(V,h)$. We sometimes call these lattices Λ_i *standard lattices*. A more explicit expression of λ is given as follows.

Lemma 3.2.4. *(1) Suppose F/F_0 is of type (R-U). Then we may take $\lambda = \frac{\pi}{t}$.*

(2) Suppose F/F_0 is of type (R-P). Then we may take $\lambda = \frac{1}{2}$.

Proof. (1) By construction, we have $\omega(\lambda) \geq \omega(\frac{\overline{\pi}}{t}) > \omega(\frac{1}{2})$. Write $\lambda = a + b\sqrt{\theta} \in F$ for some $a, b \in F_0$. Then $\overline{\lambda} = a - b\sqrt{\theta}$. Since $\lambda + \overline{\lambda} = 1$, we get $a = \frac{1}{2}$ and

$$\omega(\lambda) = \omega(\frac{1}{2} + b\sqrt{\theta}).$$

If $\omega(\frac{1}{2}) \neq \omega(b\sqrt{\theta})$, then

$$\omega(\lambda) = \min\{\omega(\frac{1}{2}), \omega(b\sqrt{\theta})\} \leq \omega(\frac{1}{2}),$$

which is a contradiction. Therefore, we may assume $\omega(b) = \omega(b\sqrt{\theta}) = \omega(\frac{1}{2})$. Then we can write $b = \frac{1}{2}u$ for some unit u in $\mathcal{O}_{F_0}$. Then

$$\omega(\lambda) = \omega(\frac{1}{2} + \frac{1}{2}u(1 - \frac{2\pi}{t})) = \omega((\frac{1}{2} + u) - \frac{\pi}{t}u).$$

Since $\omega(\pi) = 1/2$, we have $\omega(\frac{1}{2} + u) \neq \omega(\frac{\pi}{t}u)$. It implies that

$$\omega(\lambda) = \min\{\omega(\frac{1}{2} + u), \omega(\frac{\pi}{t})\} \leq \omega(\frac{\overline{\pi}}{t})$$

Thus, we have $\omega(\lambda) = \omega(\frac{\overline{\pi}}{t})$.

(2) By construction, we have $\omega(\lambda) \geq \omega(\frac{1}{2})$. Write $\lambda = a + b\pi \in F$ for some $a, b \in F_0$. Then $\overline{\lambda} = a - b\pi$. Since $\lambda + \overline{\lambda} = 1$, we have $a = \frac{1}{2}$. As $\omega(\frac{1}{2})$ is even and $\omega(b\pi)$ is odd, they cannot be equal. We get

$$\omega(\lambda) = \omega(\frac{1}{2} + b\pi) = \min\{\omega(\frac{1}{2}), \omega(b\pi)\} \leq \omega(\frac{1}{2}).$$

Thus, we have $\omega(\lambda) = \omega(\frac{1}{2})$. $\square$

Set

$$\varepsilon := \begin{cases} t & \text{in the (R-U) case,} \\ 2 & \text{in the (R-P) case.} \end{cases}$$

The hermitian form h defines a symmetric F_0-bilinear form $s(-,-) : V \times V \to F_0$ and a quadratic form $q : V \to F_0$ via

$$s(x,y) := \varepsilon^{-1} \operatorname{Tr}_{F/F_0} h(x,y) \text{ and } q(x) := \frac{1}{2}s(x,x), \text{ for } x, y \in V.$$

Set $\mathscr{L} := \varepsilon^{-1}\mathcal{O}_{F_0}$, which is an invertible $\mathcal{O}_{F_0}$-module. Then for $0 \leq i \leq m$, we obtain induced forms

$$s : \Lambda_i \times \Lambda_i \longrightarrow \mathscr{L} \text{ and } q : \Lambda_i \longrightarrow \mathscr{L}. \tag{3.2.5}$$

It is straightforward to verify the following lemma.

Lemma 3.2.5. *(1) For $0 \leq i \leq m$, the triple $(\Lambda_i, q, \mathscr{L})$ forms an $\mathscr{L}$-valued hermitian quadratic module of rank n over $\mathcal{O}_{F_0}$ in the sense of Definition 3.2.1.*

(2) Define

$$\phi : \Lambda_0 \times \Lambda_0 \to \varepsilon^{-1}\mathcal{O}_{F_0}, \ (x,y) \mapsto \varepsilon^{-1} \operatorname{Tr}_{F/F_0} h(x, \pi^{-1}y).$$

Then $(\Lambda_0, q, \mathscr{L}, \phi)$ is a hermitian quadratic module with ϕ.

Now we state two theorems on hermitian quadratic modules. The proofs will be given in Chapter 6.

Theorem 3.2.6. *The functor $\underline{\mathrm{Sim}}(\Lambda_m)$ (resp. $\underline{\mathrm{Sim}}(\Lambda_0, \phi)$) is representable by an affine smooth group scheme over $\mathcal{O}_{F_0}$ with generic fiber $\mathrm{GU}(V, h)$. Moreover, the scheme $\underline{\mathrm{Sim}}(\Lambda)$ (resp. $\underline{\mathrm{Sim}}(\Lambda_0, \phi)$) is isomorphic to the parahoric group scheme attached to Λ_m (resp. Λ_0).*

Proof. See Theorem 6.1.13 and 6.2.8, Corollary 6.1.14 and 6.2.9. □

Theorem 3.2.7 (Theorem 6.1.12, 6.2.7)**.** *Let R be an $\mathcal{O}_{F_0}$-algebra. Let $(M, q, \mathscr{L})$ (resp. $(N, q, \mathscr{L}, \phi)$) be a hermitian quadratic module over R of rank n. Assume that $(M, q, \mathscr{L})$ (resp. $(N, q, \mathscr{L}, \phi)$) is of type Λ_m (resp. Λ_0) in the sense of Definition 6.1.8 (resp. Definition 6.2.4). Then the hermitian quadratic module $(M, q, \mathscr{L})$ is étale locally isomorphic to $(\Lambda_m, q, \varepsilon^{-1}\mathcal{O}_{F_0}) \otimes_{\mathcal{O}_{F_0}} R$ (resp. $(\Lambda_0, q, \varepsilon^{-1}\mathcal{O}_{F_0}, \phi) \otimes_{\mathcal{O}_{F_0}} R$) up to similitude.*

3.3 Construction of the unitary local models

3.3.1 Naive local models

Let $I = \{0\}$ or $\{m\}$. Then I corresponds to a special parahoric subgroup of $\mathrm{GU}(V, h)$ by Proposition 2.4.1. Let Λ_I denote the corresponding lattice, which is either Λ_0 or Λ_m. Set

$$\Lambda_I^h := \{x \in V \mid h(x, \Lambda_I) \subset \mathcal{O}_F\}, \ \Lambda_I^s := \{x \in V \mid s(x, \Lambda_I) \subset \mathcal{O}_{F_0}\}.$$

The symmetric pairing s on V induces a perfect $\mathcal{O}_{F_0}$-bilinear pairing

$$\Lambda_I \times \Lambda_I^s \to \mathcal{O}_{F_0}, \tag{3.3.1}$$

which is still denotes by $s(-,-)$. By Lemma 3.1.3, one can check that

$$\Lambda^s = \begin{cases} \Lambda^h & \text{in the (R-U) case,} \\ \pi^{-1}\Lambda^h & \text{in the (R-P) case.} \end{cases} \tag{3.3.2}$$

Note that

$$\Lambda_0^h = \mathcal{O}_F\langle \overline{\lambda}^{-1} e_1, \dots, \overline{\lambda}^{-1} e_m, e_{m+1}, e_{m+2}, \dots, e_n \rangle,$$
$$\Lambda_m^h = \mathcal{O}_F\langle \overline{\lambda}^{-1} e_1, \dots, \overline{\lambda}^{-1} e_m, e_{m+1}, \overline{\pi} e_{m+2}, \dots, \overline{\pi} e_n \rangle.$$

Using (3.3.2) and Lemma 3.2.4, we have

$$\Lambda_0^s \hookrightarrow \Lambda_0 \hookrightarrow \frac{\overline{\pi}}{t}\Lambda_0^s, \text{ in the (R-U) case, } \pi\Lambda_0^s \hookrightarrow \Lambda_0 \hookrightarrow \frac{\pi}{2}\Lambda_0^s, \text{ in the (R-P) case,}$$

and

$$\Lambda_m^s \hookrightarrow \Lambda_m \hookrightarrow \frac{1}{t}\Lambda_m^s, \text{ in the (R-U) case, } \pi\Lambda_m^s \hookrightarrow \Lambda_m \hookrightarrow \frac{1}{2}\Lambda_m^s, \text{ in the (R-P) case.}$$

In summary, we have an inclusion of lattices

$$\Lambda_I \hookrightarrow \alpha\Lambda_I^s, \quad \text{where } \alpha \coloneqq \begin{cases} \overline{\pi}/\varepsilon & \text{if } I = \{0\}, \\ 1/\varepsilon & \text{if } I = \{m\}. \end{cases}$$

We define the *naive unitary local model* of type I (and of signature $(n-1,1)$) as follows.

Definition 3.3.1. Let $\mathrm{M}_I^{\mathrm{naive}}$ be the functor

$$\mathrm{M}_I^{\mathrm{naive}} : (\mathrm{Sch}/\mathcal{O}_F)^{\mathrm{op}} \longrightarrow \mathrm{Sets}$$

which sends an $\mathcal{O}_F$-scheme S to the set of $\mathcal{O}_S$-modules $\mathcal{F}$ such that

(1) $\mathcal{F}$ is an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodule of $\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ and as an $\mathcal{O}_S$-module, it is a locally direct summand of rank n.

(2) (Kottwitz condition) The action of $\pi \otimes 1 \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ on $\mathcal{F}$ has characteristic polynomial

$$\det(T - \pi \otimes 1 \mid \mathcal{F}) = (T - \pi)(T - \overline{\pi})^{n-1}.$$

(3) Let $\mathcal{F}^\perp$ be the orthogonal complement of $\mathcal{F}$ in $\Lambda_I^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ with respect to the perfect pairing

$$(\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \times (\Lambda_I^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \to \mathcal{O}_S$$

induced by (3.3.1). We require that the map $\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to \alpha\Lambda_I^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ induced by the inclusion $\Lambda_I \hookrightarrow \alpha\Lambda_I^s$ sends $\mathcal{F}$ to $\alpha\mathcal{F}^\perp$, where $\alpha\mathcal{F}^\perp$ denotes the image of $\mathcal{F}^\perp$ under the isomorphism $\alpha : \Lambda_I^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \xrightarrow{\sim} \alpha\Lambda_I^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$.

(4) $\mathcal{F}$ is totally isotropic with respect to the form $(\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \times (\Lambda_I \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \to \mathscr{L} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ induced by s in (3.2.5), i.e., $s(\mathcal{F}, \mathcal{F}) = 0$ in $\mathscr{L} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$.

Lemma 3.3.2. *The functor $\mathrm{M}_I^{\mathrm{naive}}$ is representable by a projective scheme over $\mathcal{O}_F$ and the generic fiber is isomorphic to the $(n-1)$-dimensional projective space $\mathbb{P}_F^{n-1}$ over F.*

Proof. This is similar to [PR09, 1.5.3]. The representability follows by identifying $\mathrm{M}_I^{\mathrm{naive}}$ with a closed subscheme of the Grassmannian $\mathrm{Gr}(n, \Lambda_I)_{\mathcal{O}_F}$ classifying locally direct summands of rank n in Λ_I.

As $\pi \otimes 1$ is a semisimple operator on $V \otimes_{F_0} F$, we have

$$V \otimes_{F_0} F = V_\pi \oplus V_{\overline{\pi}},$$

where V_π (resp. $V_{\overline{\pi}}$) denotes the π-eigenspace (resp. $\overline{\pi}$-eigenspace) of $\pi \otimes 1$. Both eigenspaces V_π and $V_{\overline{\pi}}$ are n-dimensional F-vector spaces. We claim that V_π is totally isotropic for the induced symmetric pairing, which is still denoted by $s(-,-)$, on $V \otimes_{F_0} F$. Indeed, for any $x, y \in V_\pi$, we have $(\pi \otimes 1)x = \pi x$ and $(\pi \otimes 1)y = \pi y$. Then

$$s(x, y) = \pi^{-2} s(\pi x, \pi y) = \pi^{-2} s\left((\pi \otimes 1)x, (\pi \otimes 1)y\right) = (\pi_0/\pi^2) s(x, y).$$

So $s(x, y) = 0$. Similarly, we obtain that $V_{\overline{\pi}}$ is also totally isotropic. It implies that the induced pairing

$$s(-,-) : V_\pi \times V_{\overline{\pi}} \to F \tag{3.3.3}$$

is perfect.

Let $\mathbb{P}_F^{n-1}$ be the projective space associated with V_π. For any F-algebra R, define

$$\varphi : \mathrm{M}_I^{\mathrm{naive}}(R) \longrightarrow \mathbb{P}_F^{n-1}(R), \quad \mathcal{F} \mapsto \ker(\pi \otimes 1 - 1 \otimes \pi \mid \mathcal{F}).$$

By the Kottwitz condition for $\mathcal{F}$, this is a well-defined map. Conversely, let $\mathcal{G} \in \mathbb{P}_F^{n-1}(R)$, i.e., $\mathcal{G}$ is a direct summand of rank one of $V_\pi \otimes_F R$. The perfect pairing (3.3.3) gives a (unique) direct summand $\mathcal{G}'$ of rank $n-1$ of $V_{\overline{\pi}} \otimes_F R$ such that $s(\mathcal{G}, \mathcal{G}') = 0$. Set

$$\mathcal{F} \coloneqq \mathcal{G} \oplus \mathcal{G}' \subset V \otimes_{F_0} R.$$

Then by our construction, we have $\mathcal{F} \in \mathrm{M}_I^{\mathrm{naive}}(R)$. This process defines an inverse map of φ. In particular, φ is bijective, and hence the generic fiber of $\mathrm{M}_I^{\mathrm{naive}}$ is isomorphic to $\mathbb{P}_F^{n-1}$. □

Similar arguments as in [Pap00, Proposition 3.8] on the dimension of the special fiber of $\mathrm{M}_I^{\mathrm{naive}}$ show that $\mathrm{M}_I^{\mathrm{naive}}$ is not flat over $\mathcal{O}_F$.

3.3.2 Local models

Definition 3.3.3. The *local model* $\mathrm{M}_I^{\mathrm{loc}}$ is defined to be the (flat) Zariski closure of the generic fiber of $\mathrm{M}_I^{\mathrm{naive}}$ in $\mathrm{M}_I^{\mathrm{naive}}$.

By construction, the scheme $\mathrm{M}_I^{\mathrm{loc}}$ is a flat projective scheme of (relative) dimension $n-1$ over $\mathcal{O}_F$. In Chapter 4-5, we will prove Theorem 1.2.2-1.2.6 in the Introduction. The proof of Theorem 1.2.2 and 1.2.3 will be divided into four cases, depending on the index set I and the ramification types of F/F_0, see §4-5.2. In the course of the proof, we also establish Theorem 1.2.6.

3.4 Comparison with the v-sheaf local models

In this section, assuming Theorem 1.2.2 and 1.2.3, we relate the local model $\mathrm{M}_I^{\mathrm{loc}}$ for $I = \{0\}$ or $\{m\}$ to the v-sheaf local models considered in [SW20, §21.4] and [AGLR22]. We give a proof of Theorem 1.2.4.

We first briefly introduce the v-sheaf local models in the sense of Scholze-Weinstein. Let G be *any* connected reductive group over a complete discretely valued field $L/\mathbb{Q}_p$, where p is *any* prime. Let $\mathcal{B}(G,L)$ denote the associated (extended) Bruhat-Tits building, which carries an action of $G(L)$. For $x \in \mathcal{B}(G,L)$, the associated *Bruhat-Tits stabilizer group scheme* $\mathcal{G}_x$, in the sense of [BT84a], is a smooth affine group scheme over $\mathcal{O}_L$ such that the generic fiber of $\mathcal{G}_x$ is G and $\mathcal{G}_x(\mathcal{O}_L)$ is the stabilizer subgroup of x in $G(L)$. By definition, the neutral component $\mathcal{G}_x^\circ$ is the *parahoric group scheme* associated to x. Recall that a smooth affine group scheme $\mathcal{G}$ over $\mathcal{O}_L$ is *quasi-parahoric* if the neutral component of $\mathcal{G}$ is a parahoric group scheme and $\mathcal{G}_x^\circ(\mathcal{O}_{\breve{L}}) \subset \mathcal{G}(\mathcal{O}_{\breve{L}}) \subset \mathcal{G}_x(\mathcal{O}_{\breve{L}})$ for some Bruhat-Tits stabilizer group scheme $\mathcal{G}_x$. Here $\breve{L}$ denotes the completion of the maximal unramified extension of L in the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

Definition 3.4.1. A *local model triple* over L is a triple $(G, \{\mu\}, \mathcal{G})$, where G is a connected reductive group over L, $\{\mu\}$ is the $G(\overline{L})$-conjugacy class of a *minuscule* cocharacter $\mu : \mathbb{G}_{m,\overline{L}} \to G_{\overline{L}}$, and $\mathcal{G}$ is a quasi-parahoric group scheme for G.

We will often write $(\mathcal{G}, \mu)$ (resp. (G, μ)) for $(G, \{\mu\}, \mathcal{G})$ (resp. $(G, \{\mu\})$). A morphism of local model triples $(\mathcal{G}, \mu) \to (\mathcal{G}', \mu')$ is a group scheme homomorphism $\mathcal{G} \to \mathcal{G}'$ taking $\{\mu\}$ to $\{\mu'\}$.

Let $(G, \{\mu\}, \mathcal{G})$ be a local model triple over L. Denote by E the reflex field of $\{\mu\}$. Then we can form the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G}}$, which is a v-sheaf over $\mathcal{O}_L$. We have the following properties.

Theorem 3.4.2. *(1) The structure morphism $\mathrm{Gr}_{\mathcal{G}} \longrightarrow \mathrm{Spd}\, \mathcal{O}_L$ is ind-proper and ind-representable in spatial diamonds. The generic fiber of $\mathrm{Gr}_{\mathcal{G}}$ can be naturally identified with the B_{dR}^+-affine Grassmannian Gr_G.*

(2) If $\mathcal{G} \hookrightarrow \mathcal{H}$ is a closed immersion of parahoric group schemes, then the induced morphism $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{H}}$ is a closed immersion.

Proof. See [SW20, Proposition 20.3.6, Proposition 20.5.4, Theorem 21.2.1], or [AGLR22, Theorem 4.9, Lemma 4.10]. □

Recall that the B_{dR}^+-affine Grassmannian Gr_G is a union of (open) Schubert diamonds $\mathrm{Gr}^{\circ}_{G,\{\mu\}}$ indexed by geometric conjugacy classes $\{\mu\}$ of cocharacters of G. Let $\mathrm{Gr}_{G,\{\mu\}}$ denote the v-closure of $\mathrm{Gr}^{\circ}_{G,\{\mu\}}$. If $\{\mu\}$ is minuscule with reflex field E, then $\mathrm{Gr}_{G,\{\mu\}}$ is representable by a projective scheme over E (see [SW20, Proposition 19.4.2]). More precisely, $\mathrm{Gr}_{G,\{\mu\}}$ is the associated diamond of the flag variety $\mathscr{F}\!\ell_{G,\{\mu\}} := G/P_{\{\mu\}}$, where

$$P_{\{\mu\}} := \{g \in G \mid \lim_{t \to \infty} \mu(t) g \mu(t)^{-1} \text{ exists}\}.$$

is the parabolic subgroup associated to $\{\mu\}$. Sometimes, we will write μ for $\{\mu\}$ for simplicity.

Definition 3.4.3. Let $\mathrm{Gr}_{\mathcal{G},\mathcal{O}_E}$ be the base change of $\mathrm{Gr}_{\mathcal{G}}$. The *v-sheaf local model* $\mathrm{M}^v_{\mathcal{G},\mu}$ is the v-closure of $\mathrm{Gr}_{G,\mu}$ inside $\mathrm{Gr}_{\mathcal{G},\mathcal{O}_E}$.

Recall that given a scheme X proper over $\mathcal{O}_E$, there is a functorially associated v-sheaf $X^{\diamondsuit}$ over $\mathrm{Spd}\, \mathcal{O}_E$. For details of the definition, we refer to [AGLR22, §2.2]. We have the following representability result of the v-sheaf local models.

Theorem 3.4.4 (Scholze-Weinstein Conjecture)**.** *Assume $\{\mu\}$ is minuscule. Then there exists a unique (up to unique isomorphism) flat, projective and normal $\mathcal{O}_E$-scheme $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$ with a closed immersion*

$$\mathbb{M}^{\mathrm{loc}\lozenge}_{\mathcal{G},\mu} \hookrightarrow \mathrm{Gr}_{\mathcal{G}} \otimes_{\mathcal{O}_L} \mathcal{O}_E$$

prolonging $\mathscr{F}\ell^{\lozenge}_{G,\mu} \xrightarrow{\sim} \mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_G \otimes_L E$. In particular, $\mathbb{M}^{\mathrm{loc}\lozenge}_{\mathcal{G},\mu} = \mathrm{M}^{v}_{\mathcal{G},\mu}$.

Proof. See [AGLR22, Theorem 1.1] and [GL24, Corollary 1.4]. □

We also have $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu} = \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}^\circ,\mu}$ by [SW20, Proposition 21.4.3]. By functoriality, any morphism $(\mathcal{G},\mu) \to (\mathcal{G}',\mu')$ of local model triples induces a natural morphism $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu} \to \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}',\mu'}$ of local models.

Now we return to the situation in §2.1. In particular, we let G denote the unitary similitude group $\mathrm{GU}(V,h)$ over F_0 attached to a split hermitian F/F_0-vector space (V,h) of dimension $n = 2m+1 \geq 3$, and there is an F-basis $(e_i)_{1\leq i\leq n}$ of V such that $h(e_i,e_j) = \delta_{i,n+1-j}$ for $1 \leq i,j \leq n$. Let $\mathcal{G}$ be the (special) parahoric group scheme corresponding to the index set $I = \{0\}$ or $\{m\}$. Let T be the maximal torus of G consisting of diagonal matrices with respect to the basis $(e_i)_{1\leq i\leq n}$. Under the isomorphism

$$G_F \simeq \mathrm{GL}_{n,F} \times \mathbb{G}_{m,F},$$

we can identify $X_*(T)$ with $\mathbb{Z}^n \times \mathbb{Z}$. Let $\mu \coloneqq \mu_{n-1,1} \in X_*(T)$ be the (minuscule) cocharacter corresponding to

$$(1, 0^{(n-1)}, 1) \in \mathbb{Z}^n \times \mathbb{Z}.$$

We write $0^{(n-1)}$ for a list of $n-1$ copies of 0. Then the reflex field E of $\{\mu\}$ equals F. Let $\mathrm{M}^{\mathrm{loc}}$ denote the local model $\mathrm{M}^{\mathrm{loc}}_I$ for $I = \{0\}$ or $\{m\}$ constructed in §3.3.2.

Theorem 3.4.5. *The scheme $\mathrm{M}^{\mathrm{loc}}$ is isomorphic to $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$ in Theorem 3.4.4.*

Proof. We have shown that the scheme $\mathrm{M}^{\mathrm{loc}}$ is normal, flat and projective over $\mathcal{O}_F$. By the uniqueness part of Theorem 3.4.4, it suffices to show that

$$\mathrm{M}^{v}_{\mathcal{G},\mu} = \mathrm{M}^{\mathrm{loc},\lozenge}.$$

By our concrete description of $\mathcal{G}$ in Corollary 6.1.14 and 6.2.9, we have a closed immersion

$$\mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda) \simeq \mathrm{GL}_{2n} \tag{3.4.1}$$

over $\mathcal{O}_{F_0}$, prolonging the closed immersion $G \hookrightarrow \mathrm{GL}_{F_0}(V) \simeq \mathrm{GL}_{2n,F_0}$, where Λ is either Λ_0 or Λ_m depending on what $\mathcal{G}$ is. Let T' be the maximal torus of GL_{2n,F_0} consisting of diagonal matrices. Then the map $G \hookrightarrow \mathrm{GL}_{F_0}(V)$ transports $\{\mu_{n-1,1}\}$ to the geometric conjugacy class $\{\mu_n\}$ of cocharacters of T'. Here, μ_n corresponds to $(1^{(n)}, 0^{(n)}) \in X_*(T') \simeq \mathbb{Z}^{2n}$. By Theorem 3.4.2 (2), the closed immersion (3.4.1) induces a closed immersion

$$\mathrm{M}^v_{\mathcal{G},\mu} \hookrightarrow \mathrm{M}^v_{\mathrm{GL}_{2n},\mu_n} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F = \mathrm{Gr}(n,2n)^{\lozenge}_{\mathcal{O}_F},$$

and we may identify $\mathrm{M}^v_{\mathcal{G},\mu}$ with the v-closure of $\mathscr{F}\!\ell^{\lozenge}_{G,\mu}$ inside $\mathrm{Gr}(n,2n)^{\lozenge}_{\mathcal{O}_F}$.

By Lemma 3.3.2, we can identify the generic fiber $\mathrm{M}^{\mathrm{loc}} \otimes_{\mathcal{O}_F} F$ with $\mathbb{P}^{n-1}_F \simeq \mathscr{F}\!\ell_{G,\mu}$, and there exists a closed immersion

$$\mathscr{F}\!\ell_{G,\mu} \hookrightarrow \mathscr{F}\!\ell_{\mathrm{GL}_{2n},\mu_n,F} = \mathrm{Gr}(n,2n)_F$$

induced by the embedding $G \hookrightarrow \mathrm{GL}_{F_0}(V)$. By our construction of $\mathrm{M}^{\mathrm{loc}}$, the scheme $\mathrm{M}^{\mathrm{loc}}$ is the Zariski closure of $\mathscr{F}\!\ell_{G,\{\mu\}}$ along $\mathscr{F}\!\ell_{G,\mu} \hookrightarrow \mathscr{F}\!\ell_{\mathrm{GL}_{2n},\mu_n,F} \hookrightarrow \mathrm{Gr}(n,2n)_{\mathcal{O}_F}$. Applying the diamond functor, we see that $\mathrm{M}^{\mathrm{loc},\lozenge}$ is the v-closure of $\mathscr{F}\!\ell^{\lozenge}_{G,\mu}$ inside $\mathrm{Gr}(n,2n)^{\lozenge}_{\mathcal{O}_F}$. Hence, we have $\mathrm{M}^v_{\mathcal{G},\mu} = \mathrm{M}^{\mathrm{loc},\lozenge}$. $\square$

This completes the proof of Theorem 1.2.4.

Remark 3.4.6. The proof of the above theorem also gives another proof of the representability of the v-sheaf local model $\mathrm{M}^v_{\mathcal{G},\mu}$ in our setting.

CHAPTER 4

THE CASE $I = \{0\}$

4.1 The case $I = \{0\}$ and (R-U)

In this section, we will prove Theorem 1.2.2 in the case when $I = \{0\}$ and the quadratic extension F/F_0 is of (R-U) type. In particular, we have

$$\pi^2 - t\pi + \pi_0 = 0,$$

where $t \in \mathcal{O}_{F_0}$ with $\pi_0 | t | 2$. Consider the following ordered $\mathcal{O}_{F_0}$-basis of Λ_0 and Λ_0^s:

$$\Lambda_0 : \frac{\overline{\pi}}{t} e_{m+2}, \ldots, \frac{\overline{\pi}}{t} e_n, e_1, \ldots, e_m, e_{m+1}, \frac{\pi_0}{t} e_{m+2}, \ldots, \frac{\pi_0}{t} e_n, \pi e_1, \ldots, \pi e_m, \pi e_{m+1}, \tag{4.1.1}$$

$$\Lambda_0^s : e_{m+2}, \ldots, e_n, \frac{t}{\pi} e_1, \ldots, \frac{t}{\pi} e_m, e_{m+1}, \pi e_{m+2}, \ldots, \pi e_n, t e_1, \ldots, t e_m, \pi e_{m+1}. \tag{4.1.2}$$

4.1.1 A refinement of $\mathrm{M}_{\{0\}}^{\mathrm{naive}}$ in the (R-U) case

In this subsection, we will propose a refinement of the functor $\mathrm{M}_{\{0\}}^{\mathrm{naive}}$. We first recall the "strengthened spin condition" raised by Smithling in [Smi15].

4.1.1.1 The strengthened spin condition

Take $g_1, \ldots, g_{2n}$ to be the ordered F-basis

$$e_1 \otimes 1 - \pi e_1 \otimes \pi^{-1}, \ldots, e_n \otimes 1 - \pi e_n \otimes \pi^{-1}, \pi e_1 \otimes \frac{\pi}{t} - e_1 \otimes \frac{\pi_0}{t}, \ldots, \pi e_n \otimes \frac{\pi}{t} - e_n \otimes \frac{\pi_0}{t}$$

of $V \otimes_{F_0} F$. Then with respect to the basis $(g_i)_{1 \le i \le 2n}$, the symmetric pairing $s(-,-) \otimes_{F_0} F$ on $V \otimes_{F_0} F$ is represented by the $2n \times 2n$ matrix anti-diag$(\theta, \ldots, \theta)$. Recall $\theta = 1 - \frac{4\pi_0}{t^2}$. One can easily check that

- $(g_i)_{1 \le i \le n}$ is a basis for $V_{\overline{\pi}}$ (the $\overline{\pi}$-eigenspace of the operator $\pi \otimes 1$ acting on $V \otimes_{F_0} F$),
- $(g_i)_{n+1 \le i \le 2n}$ is a basis for V_π (the π-eigenspace of the operator $\pi \otimes 1$ acting on $V \otimes_{F_0} F$).

Take $f_1, \ldots, f_{2n}$ to be the ordered $\mathcal{O}_F$-basis

$$e_1 \otimes 1, \ldots, e_{m+1} \otimes 1, \frac{\overline{\pi}}{t} e_{m+2} \otimes 1, \ldots, \frac{\overline{\pi}}{t} e_n \otimes 1,$$

$$\pi e_1 \otimes 1, \dots, \pi e_{m+1} \otimes 1, \frac{\pi_0}{t} e_{m+2} \otimes 1, \dots, \frac{\pi_0}{t} e_n \otimes 1$$

of $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F$. This is the base change of the basis in (4.1.1), but in different order. We have

$$(g_1, \dots, g_{2n}) = (f_1, \dots, f_{2n}) \begin{pmatrix} I_{m+1} & 0 & -\frac{\pi_0}{t} I_{m+1} & 0 \\ 0 & \frac{t}{\pi} I_m & 0 & -\pi I_m \\ -\frac{1}{\pi} I_{m+1} & 0 & \frac{\pi}{t} I_{m+1} & 0 \\ 0 & -\frac{t}{\pi^2} I_m & 0 & \frac{\pi^2}{\pi_0} I_m \end{pmatrix}. \tag{4.1.3}$$

As in [Smi15], we use the following convenient notations:

- For an integer i, we write

$$i^\vee := n + 1 - i, \quad i^* := 2n + 1 - i.$$

  For $S \subset \{1, \dots, 2n\}$ of cardinality n, we write

$$S^* := \{i^* \mid i \in S\}, \quad S^\perp := \{1, \dots, 2n\} \setminus S^*.$$

  Let σ_S be the permutation on $\{1, \dots, 2n\}$ sending $\{1, \dots, n\}$ to S in increasing order and sending $\{n+1, \dots, 2n\}$ to $\{1, \dots, 2n\} \setminus S$ in increasing order. Denote by $\operatorname{sgn}(\sigma_S) \in \{\pm 1\}$ the sign of σ_S.

- Set $W := \wedge^n (V \otimes_{F_0} F)$. For $S = \{i_1 < \dots < i_n\} \subset \{1, \dots, 2n\}$ of cardinality n, we write

$$e_S := f_{i_1} \wedge \dots \wedge f_{i_n} \in W, \text{ similarly, } g_S := g_{i_1} \wedge \dots \wedge g_{i_n} \in W.$$

  Note that $(e_S)_{\{\#S=n\}}$ (or $(g_S)_{\{\#S=n\}}$) is an F-basis of W.

- Set

$$W_{\pm 1} := \operatorname{span}_F \{g_S \pm \operatorname{sgn}(\sigma_S) g_{S^\perp} \mid S \subset \{1, \dots, 2n\} \text{ and } \#S = n\}.$$

This is a sub F-vector space of W. For any $\mathcal{O}_F$-lattice Λ in $V \otimes_{F_0} F$, set

$$W(\Lambda) := \wedge^n \left(\Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F\right),\ W(\Lambda)_{\pm 1} := W_{\pm 1} \cap W(\Lambda).$$

Then $W(\Lambda)$ (resp. $W(\Lambda)_{\pm 1}$) is an $\mathcal{O}_F$-lattice in W (resp. $W_{\pm 1}$).

- Set

$$W^{n-1,1} := \left(\wedge^{n-1} V_{\overline{\pi}}\right) \otimes_F (V_\pi),\ W^{n-1,1}_{\pm 1} := W^{n-1,1} \cap W_{\pm 1},\ W(\Lambda)^{n-1,1}_{\pm 1} := W^{n-1,1}_{\pm 1} \cap W(\Lambda).$$

Then the strengthened spin condition states that

For any $\mathcal{O}_F$-algebra R and $\mathcal{F} \in \mathrm{M}^{\mathrm{naive}}_{\{0\}}(R)$, the line $\wedge^n \mathcal{F} \subset W(\Lambda_0) \otimes_{\mathcal{O}_F} R$ is contained in the space

$$\mathrm{Im}\left(W(\Lambda_0)^{n-1,1}_{-1} \otimes_{\mathcal{O}_F} R \to W(\Lambda_0) \otimes_{\mathcal{O}_F} R\right).$$

4.1.1.2 The definition of the refinement

Definition 4.1.1. Let $\mathrm{M}_{\{0\}}$ be the functor

$$\mathrm{M}_{\{0\}} : (\mathrm{Sch}/\mathcal{O}_F)^{\mathrm{op}} \longrightarrow \mathrm{Sets}$$

which sends an $\mathcal{O}_F$-scheme S to the set of $\mathcal{O}_S$-modules $\mathcal{F}$ such that

LM1 (π-stability condition) $\mathcal{F}$ is an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodule of $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ and as an $\mathcal{O}_S$-module, it is a locally direct summand of rank n.

LM2 (Kottwitz condition) The action of $\pi \otimes 1 \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ on $\mathcal{F}$ has characteristic polynomial

$$\det(T - \pi \otimes 1 \mid \mathcal{F}) = (T - \pi)(T - \overline{\pi})^{n-1}.$$

LM3 Let $\mathcal{F}^\perp$ be the orthogonal complement in $\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ of $\mathcal{F}$ with respect to the perfect pairing

$$s(-,-) : (\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \times (\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \to \mathcal{O}_S.$$

We require the map $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to (\frac{\pi}{t}\Lambda_0^s) \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ induced by $\Lambda_0 \hookrightarrow \frac{\pi}{t}\Lambda_0^s$ sends $\mathcal{F}$ to $\frac{\pi}{t}\mathcal{F}^\perp$, where $\frac{\pi}{t}\mathcal{F}^\perp$ denotes the image of $\mathcal{F}^\perp$ under the isomorphism $\frac{\pi}{t} : \Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \xrightarrow{\sim} \frac{\pi}{t}\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$.

LM4 (Hyperbolicity condition) The quadratic form $q : \Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to \mathscr{L} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ induced by $q : \Lambda_0 \to \mathscr{L}$ satisfies $q(\mathcal{F}) = 0$.

LM5 (Wedge condition) The action of $\pi \otimes 1 - 1 \otimes \overline{\pi} \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ satisfies

$$\wedge^2(\pi \otimes 1 - 1 \otimes \overline{\pi} \mid \mathcal{F}) = 0.$$

LM6 (Strengthened spin condition) The line $\wedge^n \mathcal{F} \subset W(\Lambda_0) \otimes_{\mathcal{O}_F} \mathcal{O}_S$ is contained in

$$\operatorname{Im}\left(W(\Lambda_0)_{-1}^{n-1,1} \otimes_{\mathcal{O}_F} \mathcal{O}_S \to W(\Lambda_0) \otimes_{\mathcal{O}_F} \mathcal{O}_S\right).$$

Then $\mathrm{M}_{\{0\}}$ is representable by a projective $\mathcal{O}_F$-scheme, which is a closed subscheme of $\mathrm{M}_{\{0\}}^{\mathrm{naive}}$. Note that over the generic fiber of $\mathrm{M}_{\{0\}}$, the quadratic form q is determined by s via $q(x) = \frac{1}{2}s(x,x)$. So, over the generic fiber, the hyperbolicity condition **LM4** is implied by the Condition (3) in $\mathrm{M}_{\{0\}}^{\mathrm{naive}}$. Similarly as in [PR09, 1.5] and [Smi15, 2.5], we can deduce that the rest of the conditions of $\mathrm{M}_{\{0\}}$ do not affect the generic fiber of $\mathrm{M}_{\{0\}}^{\mathrm{naive}}$, and hence $\mathrm{M}_{\{0\}}$ and $\mathrm{M}_{\{0\}}^{\mathrm{naive}}$ have the same generic fiber.

Hence, we have closed immersions

$$\mathrm{M}_{\{0\}}^{\mathrm{loc}} \subset \mathrm{M}_{\{0\}} \subset \mathrm{M}_{\{0\}}^{\mathrm{naive}}$$

of projective schemes over $\mathcal{O}_F$, where all schemes have the same generic fiber.

4.1.2 An affine chart $\mathrm{U}_{\{0\}}$ around the worst point

Set

$$\mathcal{F}_0 := (\pi \otimes 1)(\Lambda_0 \otimes_{\mathcal{O}_{F_0}} k).$$

Then we can check that $\mathcal{F}_0 \in \mathrm{M}_{\{0\}}(k)$. We call it the *worst point* of $\mathrm{M}_{\{0\}}$.

With respect to the basis (4.1.1), the standard affine chart around $\mathcal{F}_0$ in $\mathrm{Gr}(n, \Lambda_0)_{\mathcal{O}_F}$ is the $\mathcal{O}_F$-scheme of $2n \times n$ matrices $\left(\begin{smallmatrix} X \\ I_n \end{smallmatrix}\right)$. We denote by $\mathrm{U}_{\{0\}}$ the intersection of $\mathrm{M}_{\{0\}}$ with the standard affine chart in $\mathrm{Gr}(n, \Lambda_0)_{\mathcal{O}_F}$. The worst point $\mathcal{F}_0$ of $\mathrm{M}_{\{0\}}$ is contained in $\mathrm{U}_{\{0\}}$ and corresponds to the closed point defined by $X = 0$ and $\pi = 0$. The conditions **LM1-6** yield the defining equations for $\mathrm{U}_{\{0\}}$. We will analyze each condition in detail. A reader who is only interested in the affine coordinate ring of $\mathrm{U}_{\{0\}}$ may proceed directly to Proposition 4.1.10.

4.1.2.1 Condition LM1

Let R be an $\mathcal{O}_F$-algebra. With respect to the basis (4.1.1), the operator $\pi \otimes 1$ acts on $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} R$ via the matrix

$$\begin{pmatrix} 0 & -\pi_0 I_n \\ I_n & tI_n \end{pmatrix}.$$

Then the π-stability condition **LM1** on $\mathcal{F}$ means there exists an $n \times n$ matrix $P \in M_n(R)$ such that

$$\begin{pmatrix} 0 & -\pi_0 I_n \\ I_n & tI_n \end{pmatrix} \begin{pmatrix} X \\ I_n \end{pmatrix} = \begin{pmatrix} X \\ I_n \end{pmatrix} P.$$

We obtain $P = X + tI_n$ and $X^2 + tX + \pi_0 I_n = 0$.

4.1.2.2 Condition LM2

We have already shown that $\pi \otimes 1$ acts on $\mathcal{F}$ via $X + tI_n$. Then the Kottwitz condition **LM2** translates to

$$\det(T - (X + tI_n)) = (T - \pi)(T - \overline{\pi})^{n-1}.$$

Equivalently,

$$\det(T - (X + \pi I_n)) = (T + \overline{\pi} - \pi)T^{n-1}.$$

Note that

$$\det(T - (X + \pi I_n)) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}(\wedge^i (X + \pi I_n))T^{n-i}.$$

Then by comparing the coefficients of T^{n-i}, the Kottwitz condition **LM2** becomes

$$\operatorname{tr}(X + \pi I_n) = \pi - \overline{\pi}, \ \operatorname{tr}\left(\wedge^i (X + \pi I_n)\right) = 0, \ \text{for } i \geq 2. \qquad (4.1.4)$$

4.1.2.3 Condition LM3

With respect to the bases (4.1.1) and (4.1.2), the perfect pairing

$$s(-,-) : (\Lambda_0 \otimes_{\mathcal{O}_{F_0}} R) \times (\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} R) \to R$$

and the map $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} R \to \frac{\overline{\pi}}{t}\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} R$ are represented respectively by the matrices

$$S = \begin{pmatrix} \frac{2}{t}H_{2m} & 0 & H_{2m} & 0 \\ 0 & \frac{2}{t} & 0 & 1 \\ H_{2m} & 0 & \frac{2\pi_0}{t}H_{2m} & 0 \\ 0 & 1 & 0 & \frac{2\pi_0}{t} \end{pmatrix} \text{ and } N = \begin{pmatrix} I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 & -tI_m & 0 \\ 0 & 0 & 0 & 0 & 0 & -t \\ 0 & 0 & 0 & I_m & 0 & 0 \\ 0 & \frac{t}{\pi_0}I_m & 0 & 0 & \frac{t^2-\pi_0}{\pi_0}I_m & 0 \\ 0 & 0 & \frac{t}{\pi_0} & 0 & 0 & \frac{t^2}{\pi_0} \end{pmatrix},$$

where H_{2m} denotes the $2m \times 2m$ anti-diagonal unit matrix, and I_m denotes the $m \times m$ identity matrix.

Then the Condition **LM3** translates to $\begin{pmatrix} X \\ I_n \end{pmatrix}^t S \left(N \begin{pmatrix} X \\ I_n \end{pmatrix} \right) = 0,$ or equivalently,

$$\begin{pmatrix} X \\ I_n \end{pmatrix}^t \begin{pmatrix} 0 & \frac{t^2-2\pi_0}{t\pi_0}H_m & 0 & 0 & \frac{t^2-3\pi_0}{\pi_0}H_m & 0 \\ \frac{2}{t}H_m & 0 & 0 & H_m & 0 & 0 \\ 0 & 0 & \frac{t}{\pi_0} & 0 & 0 & \frac{t^2-2\pi_0}{\pi_0} \\ 0 & H_m & 0 & 0 & \frac{t^2-2\pi_0}{t}H_m & 0 \\ H_m & 0 & 0 & \frac{2\pi_0}{t}H_m & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & t \end{pmatrix} \begin{pmatrix} X \\ I_n \end{pmatrix} = 0. \tag{4.1.5}$$

Write

$$X = \begin{pmatrix} A & B & E \\ C & D & F \\ G & H & x \end{pmatrix},$$

where $A, B, C, D \in M_m(R)$, $E, F \in M_{m\times 1}(R)$, $G, H \in M_{1\times m}(R)$ and $x \in R$. Then Equation (4.1.5) translates to

$$\frac{2}{t}C^tH_mA + \frac{t^2-2\pi_0}{t\pi_0}A^tH_mC + \frac{t}{\pi_0}G^tG + H_mC + C^tH_m = 0, \tag{\textbf{LM3}-1}$$

$$\frac{2}{t}C^tH_mB + \frac{t^2-2\pi_0}{t\pi_0}A^tH_mD + \frac{t}{\pi_0}G^tH + H_mD + \frac{t^2-3\pi_0}{\pi_0}A^tH_m + \frac{t^2-2\pi_0}{t}H_m = 0, \tag{\textbf{LM3}-2}$$

$$\frac{2}{t}C^tH_mE + \frac{t^2-2\pi_0}{t\pi_0}A^tH_mF + \frac{t}{\pi_0}G^tx + H_mF + \frac{t^2-2\pi_0}{\pi_0}G^t = 0, \tag{\textbf{LM3}-3}$$

$$\frac{2}{t}D^tH_mA + \frac{t^2-2\pi_0}{t\pi_0}B^tH_mC + \frac{t}{\pi_0}H^tG + H_mA + D^tH_m + \frac{2\pi_0}{t}H_m = 0, \tag{\textbf{LM3}-4}$$

$$\frac{2}{t}D^tH_mB + \frac{t^2-2\pi_0}{t\pi_0}B^tH_mD + \frac{t}{\pi_0}H^tH + H_mB + \frac{t^2-3\pi_0}{\pi_0}B^tH_m = 0, \tag{\textbf{LM3}-5}$$

$$\frac{2}{t}D^tH_mE + \frac{t^2-2\pi_0}{t\pi_0}B^tH_mF + \frac{t}{\pi_0}xH^t + H_mE + \frac{t^2-2\pi_0}{\pi_0}H^t = 0, \tag{\textbf{LM3}-6}$$

$$\frac{2}{t}F^tH_mA + \frac{t^2-2\pi_0}{t\pi_0}E^tH_mC + \frac{t}{\pi_0}xG + 2G + F^tH_m = 0, \tag{\textbf{LM3}-7}$$

$$\frac{2}{t}F^tH_mB + \frac{t^2-2\pi_0}{t\pi_0}E^tH_mD + \frac{t}{\pi_0}xH + 2H + \frac{t^2-3\pi_0}{\pi_0}E^tH_m = 0, \tag{\textbf{LM3}-8}$$

$$\frac{2}{t}F^tH_mE + \frac{t^2-\pi_0}{t\pi_0}E^tH_mF + \frac{t}{\pi_0}x^2 + 2x + \frac{t^2-2\pi_0}{\pi_0}x + t = 0. \tag{\textbf{LM3}-9}$$

4.1.2.4 Condition LM4

Recall $\mathscr{L} = t^{-1}\mathcal{O}_{F_0}$. With respect to the basis (4.1.1), the induced $(\mathscr{L} \otimes_{\mathcal{O}_{F_0}} R)$-valued symmetric pairing on $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} R$ is represented by the matrix

$$S_1 = \begin{pmatrix} 0 & H_m & 0 & 0 & \frac{t^2-2\pi_0}{t}H_m & 0 \\ H_m & 0 & 0 & \frac{2\pi_0}{t}H_m & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & t \\ 0 & \frac{2\pi_0}{t}H_m & 0 & 0 & \pi_0H_m & 0 \\ \frac{t^2-2\pi_0}{t}H_m & 0 & 0 & \pi_0H_m & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 2\pi_0 \end{pmatrix}.$$

Convention: Throughout the rest of the thesis, we often encounter a matrix $M = (M_{ij}) \in M_{\ell\times\ell}(R)$ whose diagonal entries are of the form $M_{ii} = 2a_{ii}$ for some $a_{ii} \in R$. We then use

$\frac{1}{2}M_{ii}$ to denote a_{ii}. When we refer to "half of the diagonal of M", we mean the row matrix consisting of the entries $\frac{1}{2}M_{ii}$ for $1 \leq i \leq \ell$.

The Condition **LM4** translates to

$$\begin{pmatrix} X \\ I_n \end{pmatrix}^t S_1 \begin{pmatrix} X \\ I_n \end{pmatrix} = 0 \text{ and half of the diagonal of } \begin{pmatrix} X \\ I_n \end{pmatrix}^t S_1 \begin{pmatrix} X \\ I_n \end{pmatrix} \text{ equals zero.}$$

One can check that the diagonal entries of $\left(\begin{smallmatrix} X \\ I_n \end{smallmatrix}\right)^t S_1 \left(\begin{smallmatrix} X \\ I_n \end{smallmatrix}\right)$ are indeed divisible by 2 in R. Equivalently, we obtain the following equations.

$$C^t H_m A + A^t H_m C + 2G^t G + \frac{2\pi_0}{t} H_m C + \frac{2\pi_0}{t} C^t H_m = 0, \tag{\textbf{LM4}-1}$$

$$C^t H_m B + A^t H_m D + 2G^t H + \frac{2\pi_0}{t} H_m D + \frac{t^2 - 2\pi_0}{t} A^t H_m + \pi_0 H_m = 0, \tag{\textbf{LM4}-2}$$

$$C^t H_m E + A^t H_m F + 2xG^t + \frac{2\pi_0}{t} H_m F + tG^t = 0, \tag{\textbf{LM4}-3}$$

$$D^t H_m A + B^t H_m C + 2H^t G + \frac{t^2 - 2\pi_0}{t} H_m A + \frac{2\pi_0}{t} D^t H_m + \pi_0 H_m = 0, \tag{\textbf{LM4}-4}$$

$$D^t H_m B + B^t H_m D + 2H^t H + \frac{t^2 - 2\pi_0}{t} H_m B + \frac{t^2 - 2\pi_0}{t} B^t H_m = 0, \tag{\textbf{LM4}-5}$$

$$D^t H_m E + B^t H_m F + 2xH^t + \frac{t^2 - 2\pi_0}{t} H_m E + tH^t = 0, \tag{\textbf{LM4}-6}$$

$$F^t H_m A + E^t H_m C + 2xG + tG + \frac{2\pi_0}{t} F^t H_m = 0, \tag{\textbf{LM4}-7}$$

$$F^t H_m B + E^t H_m D + 2xH + tH + \frac{t^2 - 2\pi_0}{t} E^t H_m = 0, \tag{\textbf{LM4}-8}$$

$$F^t H_m E + E^t H_m F + 2x^2 + 2tx + 2\pi_0 = 0, \tag{\textbf{LM4}-9}$$

half of the diagonal of matrices in **LM4**-1,5,9 equals 0. (**LM4**-10)

4.1.2.5 Condition LM5

We already know from §4.1.2.1 that $\pi \otimes 1$ acts as right multiplication by $X + tI_n$ on $\mathcal{F}$. Thus, the wedge condition **LM5** on $\mathcal{F}$ translates to

$$\wedge^2 (X + \pi I_n) = 0.$$

4.1.2.6 Condition LM6

We will use the same notations as in §4.1.1.1. To find the equations induced by the strengthened spin condition **LM6** on $\mathcal{F}$, we need to determine an $\mathcal{O}_F$-basis of $W(\Lambda_0)^{n-1,1}_{-1}$.

Definition 4.1.2. Let $S \subset \{1, \ldots, 2n\}$ be a subset of cardinality n.

(1) We say S is of type $(n-1, 1)$ if

$$\#(S \cap \{1, \ldots, n\}) = n - 1 \text{ and } \#(S \cap \{n+1, \ldots, 2n\}) = 1.$$

Such S necessarily has the form $\{1, \ldots, \widehat{j}, \ldots, n, n+i\}$ for some $i, j \in \{1, \ldots, n\}$.

(2) Let S be of type $(n-1, 1)$. Denote by i_S the unique element in $S \cap \{n+1, \ldots, 2n\}$. Define $S \preccurlyeq S^\perp$ if $i_S \leq i_{S^\perp}$.

Set

$$\mathcal{B} := \{S \subset \{1, \ldots, 2n\} \mid \#S = n\}, \quad \mathcal{B}^{n-1,1} := \{S \in \mathcal{B} \mid S \text{ is of type } (n-1,1)\},$$
$$\mathcal{B}_0 := \left\{S \in \mathcal{B}^{n-1,1} \mid S \preccurlyeq S^\perp\right\}.$$

By construction, the F-vector space $W(\Lambda_0)^{n-1,1}_{-1} \otimes_{\mathcal{O}_F} F$ equals $W^{n-1,1}_{-1}$, which is an F-subspace of W.

Lemma 4.1.3. *(1) The set $\{e_S \mid S \in \mathcal{B}\}$ (resp. $\{g_S \mid S \in \mathcal{B}\}$) is an F-basis of W.*

(2) For $S \in \mathcal{B}$, denote

$$h_S := g_S - \operatorname{sgn}(\sigma_S) g_{S^\perp}.$$

The set $\{h_S \mid S \in \mathcal{B}_0\}$ is an F-basis of $W^{n-1,1}_{-1}$.

Proof. (1) As $W = \wedge^n (V \otimes_{F_0} F)$ by definition, the statement is a standard fact about the wedge product of vector spaces.

(2) By [Smi15, Lemma 4.2], the F-space $W^{n-1,1}_{-1}$ is spanned by the set $\{h_S \mid S \in \mathcal{B}^{n-1,1}\}$. These h_S's are not linearly independent over F. Indeed, for $S \in \mathcal{B}^{n-1,1}$, we have $h_{S^\perp} = -\operatorname{sgn}(\sigma_S) h_S$ by using that $(S^\perp)^\perp = S$ and $\operatorname{sgn}(\sigma_S) = \operatorname{sgn}(\sigma_{S^\perp})$ (by [Smi15, Lemma 2.8]). However, the set $\{h_S \mid S \in \mathcal{B}_0\}$ is F-linearly independent, since $\{g_S \mid S \in \mathcal{B}\}$ is F-linearly independent. So the set $\{h_S \mid S \in \mathcal{B}_0\}$ is an F-basis of $W^{n-1,1}_{-1}$. $\square$

Definition 4.1.4. Let $w = \sum_{S \in \mathcal{B}} c_S e_S \in W$. The *worst term* of w is defined to be

$$WT(w) \coloneqq \sum_{S \in \mathcal{B}(w)} c_S e_S,$$

where $\mathcal{B}(w) \subset \mathcal{B}$ consists of elements $S \in \mathcal{B}$ such that $\omega(c_S) \leq \omega(c_T)$ for all $T \in \mathcal{B}$.

Recall $\sqrt{\theta} = 1 - 2\pi/t \in \mathcal{O}_F^\times$. Using (4.1.3), we immediately obtain the following.

Lemma 4.1.5. *Let $S \in \mathcal{B}^{n-1,1}$. Then exactly we have the following six cases.*

(1) If $S = \{1, \dots, \widehat{i}, \dots, n, n+i\}$ for some $i \leq m+1$, then

$$WT(g_S) = (-1)^{i-1} \frac{t^{m-1}}{\pi^{3m-1}} e_{\{n+1,\dots,2n\}}.$$

(2) If $S = \{1, \dots, \widehat{i}, \dots, n, n+i\}$ for some $m+1 < i \leq n$, then

$$WT(g_S) = (-1)^{i-1} \frac{t^{m-1}}{\pi^{3m-3}\pi_0} e_{\{n+1,\dots,2n\}}.$$

(3) If $S = \{1, \dots, \widehat{j}, \dots, n, n+i\}$ for some $i, j \leq m+1$ with $i \neq j$, then

$$WT(g_S) = -\sqrt{\theta} \frac{t^{m}}{\pi^{3m-1}} e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}}.$$

(4) If $S = \{1, \dots, \widehat{j}, \dots, n, n+i\}$ for some $i \leq m+1 < j$, then

$$WT(g_S) = -\sqrt{\theta} \frac{t^{m-1}}{\pi^{3m-2}} e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}}.$$

(5) If $S = \{1, \dots, \widehat{j}, \dots, n, n+i\}$ for some $j \leq m+1 < i$, then

$$WT(g_S) = -\sqrt{\theta} \frac{t^{m+1}}{\pi^{3m-2}\pi_0} e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}}.$$

(6) If $S = \{1, \dots, \widehat{j}, \dots, n, n+i\}$ for some $i, j > m+1$ with $i \neq j$, then

$$WT(g_S) = -\sqrt{\theta} \frac{t^{m}}{\pi^{3m-3}\pi_0} e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}}.$$

Definition 4.1.6. For $S \in \mathcal{B}^{n-1,1}$, the *weight vector* $\mathbf{w}_S \in \mathbb{Z}^n$ attached to S is defined to be an element of $\mathbb{Z}^n$ such that the i-th coordinate of $\mathbf{w}_S$ is $\#(S \cap \{i, n+i\})$.

Note that if $S \in \mathcal{B}^{n-1,1}$, then $S = \{1, \ldots, \widehat{j}, \ldots, n, n+i\}$ for some $1 \leq i, j \leq n$. Moreover, we have $\mathrm{sgn}(\sigma_S) = (-1)^{i+j+1}$ (see [Smi15, Remark 4.3]) and $S^\perp = \{1, \ldots, \widehat{i^\vee}, \ldots, n, j^*\}$. Similar arguments in [Smi15, Lemma 4.10] imply the following lemma.

Lemma 4.1.7. *Let $S \in \mathcal{B}_0$. Then exactly we have the following nine cases.*

(1) $S = \{1, \ldots, \widehat{m+1}, \ldots, n, n+m+1\}$. *Then* $S = S^\perp$, $\mathbf{w}_S = (1, \ldots, 1)$, *and*

$$WT(h_S) = WT(2g_S) = (-1)^m \frac{2t^{m-1}}{\pi^{3m-1}} e_{\{n+1,\ldots,2n\}}.$$

(2) $S = \{1, \ldots, \widehat{i^\vee}, \ldots, n, n+i\}$ *for some* $i < m+1$. *Then* $S = S^\perp$, $\mathbf{w}_S \neq (1, \ldots, 1)$, *and*

$$WT(h_S) = WT(2g_S) = -\sqrt{\theta}\frac{2t^{m-1}}{\pi^{3m-2}} e_{\{i,n+1,\ldots,\widehat{i^*},\ldots,2n\}}.$$

(3) $S = \{1, \ldots, \widehat{i^\vee}, \ldots, n, n+i\}$ *for some* $i > m+1$. *Then* $S = S^\perp$, $\mathbf{w}_S \neq (1, \ldots, 1)$, *and*

$$WT(h_S) = WT(2g_S) = -\sqrt{\theta}\frac{2t^{m+1}}{\pi^{3m-2}\pi_0} e_{\{i,n+1,\ldots,\widehat{i^*},\ldots,2n\}}.$$

(4) $S = \{1, \ldots, \widehat{i}, \ldots, n, n+i\}$ *for some* $i < m+1$. *Then* $S \neq S^\perp$, $\mathbf{w}_S = \mathbf{w}_{S^\perp} = (1, \ldots, 1)$, *and*

$$WT(h_S) = WT(g_{\{1,\ldots,\widehat{i},\ldots,n,n+i\}} + g_{\{1,\ldots,\widehat{i^\vee},\ldots,n,i^*\}}) = (-1)^{i-1}\frac{t^m}{\pi^{3m-2}\pi_0} e_{\{n+1,\ldots,2n\}}.$$

(5) $S = \{1, \ldots, \widehat{j}, \ldots, n, n+i\}$ *for some* $i < j^\vee < m+1$. *Then* $S \neq S^\perp$, $\mathbf{w}_S, \mathbf{w}_{S^\perp}$ *and* $(1, \ldots, 1)$ *are pairwise distinct and*

$$\begin{aligned} WT(h_S) &= WT(g_{\{1,\ldots,\widehat{j},\ldots,n,n+i\}} + (-1)^{i+j} g_{\{1,\ldots,\widehat{i^\vee},\ldots,n,j^*\}}) \\ &= -\sqrt{\theta}\frac{t^{m-1}}{\pi^{3m-2}} e_{\{i,n+1,\ldots,\widehat{n+j},\ldots,2n\}} + (-1)^{i+j+1}\sqrt{\theta}\frac{t^{m-1}}{\pi^{3m-2}} e_{\{j^\vee,n+1,\ldots,\widehat{i^*},\ldots,2n\}}. \end{aligned}$$

(6) $S = \{1, \ldots, \widehat{m+1}, \ldots, n, n+i\}$ *for some* $i < m+1$. *Then* $S \neq S^\perp$, $\mathbf{w}_S, \mathbf{w}_{S^\perp}$ *and* $(1, \ldots, 1)$ *are pairwise distinct and*

$$\begin{aligned} WT(h_S) &= WT(g_{\{1,\ldots,\widehat{m+1},\ldots,n,n+i\}} - (-1)^{m+i} g_{\{1,\ldots,\widehat{i^\vee},\ldots,n+m+1\}}) \\ &= (-1)^{m+i}\sqrt{\theta}\frac{t^{m-1}}{\pi^{3m-2}} e_{\{m+1,n+1,\ldots,\widehat{i^*},\ldots,2n\}}. \end{aligned}$$

(7) $S=\{1,\dots,\widehat{j},\dots,n,n+i\}$ *for some* $i<m+1<j^\vee$*. Then* $S\neq S^\perp$*,* $\mathbf{w}_S,\mathbf{w}_{S^\perp}$ *and* $(1,\dots,1)$ *are pairwise distinct and*

$$\begin{aligned}WT(h_S)&=WT(g_{\{1,\dots,\widehat{j},\dots,n,n+i\}}-(-1)^{i+j+1}g_{\{1,\dots,\widehat{i^\vee},\dots,n,j^*\}})\\&=-\sqrt{\theta}\frac{t^m}{\pi^{3m-1}}e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}}-(-1)^{i+j}\sqrt{\theta}\frac{t^m}{\pi^{3m-3}\pi_0}e_{\{j^\vee,n+1,\dots,\widehat{i^*},\dots,2n\}}.\end{aligned}$$

(8) $S=\{1,\dots,\widehat{j},\dots,n,n+m+1\}$ *for some* $j^\vee>m+1$*. Then* $S\neq S^\perp$*,* $\mathbf{w}_S,\mathbf{w}_{S^\perp}$ *and* $(1,\dots,1)$ *are pairwise distinct and*

$$\begin{aligned}WT(h_S)&=WT(g_{\{1,\dots,\widehat{j},\dots,n,n+m+1\}}-(-1)^{m+j+1}g_{\{1,\dots,\widehat{m+1},\dots,n,j^*\}})\\&=-\sqrt{\theta}\frac{t^m}{\pi^{3m-1}}e_{\{m+1,n+1,\dots,\widehat{n+j},\dots,2n\}}.\end{aligned}$$

(9) $S=\{1,\dots,\widehat{j},\dots,n,n+i\}$ *for some* $j^\vee>i>m+1$*. Then* $S\neq S^\perp$*,* $\mathbf{w}_S,\mathbf{w}_{S^\perp}$ *and* $(1,\dots,1)$ *are pairwise distinct and*

$$\begin{aligned}WT(h_S)&=WT(g_{\{1,\dots,\widehat{j},\dots,n,n+i\}}-(-1)^{i+j+1}g_{\{1,\dots,\widehat{i^\vee},\dots,j^*\}})\\&=-\sqrt{\theta}\frac{t^{m+1}}{\pi^{3m-2}\pi_0}e_{\{i,n+1,\dots,\widehat{n+j},\dots,2n\}}+(-1)^{i+j+1}\sqrt{\theta}\frac{t^{m+1}}{\pi^{3m-2}\pi_0}e_{\{j^\vee,n+1,\dots,\widehat{i^*},\dots,2n\}}.\end{aligned}$$

Let $w\in W_{-1}^{n-1,1}$. Recall that $\{h_S\mid S\in\mathcal{B}_0\}$ is an F-basis of $W_{-1}^{n-1,1}$ by Lemma 4.1.3. Write

$$w=\sum_{S\in\mathcal{B}_0}a_Sh_S=\sum_{\mathbf{w}\in\mathbb{Z}^n}\sum_{\substack{S\in\mathcal{B}_0\\ \text{and }\mathbf{w}_S=\mathbf{w}}}a_Sh_S,\quad a_S\in F.$$

Then as in the proof of [Smi15, Proposition 4.12], we have

$$w\in W(\Lambda_0)_{-1}^{n-1,1}\iff\sum_{\substack{S\in\mathcal{B}_0\\ \text{and }\mathbf{w}_S=\mathbf{w}}}a_Sh_S\in W(\Lambda_0)_{-1}^{n-1,1},\text{ for each }\mathbf{w}\in\mathbb{Z}^n$$

We have two distinct situations for $\mathbf{w}$:

Case 1: $\mathbf{w}\neq(1,\dots,1)$. Then there exists at most one $S\in\mathcal{B}_0$ such that $\mathbf{w}_S=\mathbf{w}$.

Case 2: $\mathbf{w}=(1,\dots,1)$. Then S is necessarily of the form

$$S_i:=\left\{1,\dots,\widehat{i},\dots,n,n+i\right\}$$

for some $1 \leq i \leq m+1$. For any $1 \leq i < m+1$, we have

$$
\begin{aligned}
h_{S_i} &= g_{S_i} + g_{S_{i^\vee}} \\
&= (-1)^i g_1 \wedge \cdots \wedge \widehat{g_i} \wedge \cdots \wedge \widehat{g_{i^\vee}} \wedge \cdots \wedge g_n \wedge (g_i \wedge g_{i^*} - g_{i^\vee} \wedge g_{n+i}) \\
&= (-1)^i g_1 \wedge \cdots \wedge \widehat{g_i} \wedge \cdots \wedge \widehat{g_{i^\vee}} \wedge \cdots \wedge g_n \\
&\quad \wedge (-t f_i \wedge f_{i^\vee} + \frac{t^2 - \pi_0}{\pi_0} f_i \wedge f_{i^*} - 2 f_{i^\vee} \wedge f_{n+i} - \frac{t}{\pi_0} f_{n+i} \wedge f_{i^*}),
\end{aligned}
$$

and

$$
\begin{aligned}
h_{S_{m+1}} &= 2 g_{S_{m+1}} \\
&= -2 \cdot g_1 \wedge \cdots \wedge \widehat{g_i} \wedge \cdots \wedge \widehat{g_{i^\vee}} \wedge \cdots \wedge g_n \wedge g_{n+m+1} \wedge (g_i \wedge g_{i^\vee}) \\
&= -2 \cdot g_1 \wedge \cdots \wedge \widehat{g_i} \wedge \cdots \wedge \widehat{g_{i^\vee}} \wedge \cdots \wedge g_n \wedge g_{n+m+1} \\
&\quad \wedge (\frac{t}{\pi} f_i \wedge f_{i^\vee} - \frac{t}{\pi^2} f_i \wedge f_{i^*} + \frac{t}{\pi^2} f_{i^\vee} \wedge f_{n+i} + \frac{t}{\pi^3} f_{n+i} \wedge f_{i^*}).
\end{aligned}
$$

Define

$$
\widetilde{h}_{S_i} \coloneqq \begin{cases} 2\pi h_{S_i} + (-1)^{m+i} t h_{S_{m+1}} & \text{if } i \neq m+1, \\ h_{S_{m+1}} & \text{if } i = m+1. \end{cases}
$$

Then for $1 \leq i < m+1$, terms of $\widetilde{h}_{S_i}$ do not contain (multiples of)

$$
WT(h_{S_{m+1}}) = (-1)^m \frac{2t^{m-1}}{\pi^{3m-1}} e_{\{n+1,\ldots,2n\}},
$$

and

$$
WT(\widetilde{h}_{S_i}) = -\sqrt{\theta} \frac{2t^m \pi_0}{\pi^{3m}} e_{\{i,n+1,\ldots,\widehat{n+i},\ldots,2n\}} - \sqrt{\theta} \frac{2t^m}{\pi^{3m-2}} e_{\{i^\vee,n+1,\ldots,\widehat{i^*},\ldots,2n\}}. \tag{4.1.6}
$$

For S with $\mathbf{w}_S \neq (1,\ldots,1)$, we set $\widetilde{h}_S \coloneqq h_S$. By Lemma 4.1.3, the set $\{\widetilde{h}_S \mid S \in \mathcal{B}_0\}$ forms an F-basis of $W_{-1}^{n-1,1}$. Previous analysis on $\mathbf{w}$ together with similar arguments in [Smi15, Proposition 4.12] imply the following lemma.

Lemma 4.1.8. *For each $S \in \mathcal{B}_0$, pick $b_S \in F$ such that the worst term $WT(b_S \widetilde{h}_S)$ is a sum of terms of the form $u_T e_T$ for some unit $u_T \in \mathcal{O}_F^\times$ and $T \in \mathcal{B}$. Then the set $\{b_S \widetilde{h}_S \mid S \in \mathcal{B}_0\}$ forms an $\mathcal{O}_F$-basis of the $\mathcal{O}_F$-module $W(\Lambda_0)_{-1}^{n-1,1}$.*

For the matrix $\left(\begin{smallmatrix} X \\ I_n \end{smallmatrix}\right)$ corresponding to $\mathcal{F}$, denote by $v \in \wedge^n \mathcal{F}$ the wedge product of n-columns of the matrix in the order from left to right. Then the strengthened spin condition **LM6** on $\mathcal{F}$ amounts to that

$$v \in \operatorname{Im}\left(W(\Lambda_0)^{n-1,1}_{-1} \otimes_{\mathcal{O}_F} R \to W(\Lambda_0) \otimes_{\mathcal{O}_F} R\right).$$

Write $v = \sum_{S \in \mathcal{B}} a_S e_S$ for some $a_S \in R$. By Lemma 4.1.8, we have

$$v = \sum_{S \in \mathcal{B}} a_S e_S = \sum_{S \in \mathcal{B}_0} c_S b_S \widetilde{h}_S \tag{4.1.7}$$

for some $c_S \in R$. By comparing the coefficients of both sides in Equation (4.1.7), we will obtain the defining equations of the condition **LM6** on the chart $\mathrm{U}_{\{0\}}$.

Recall

$$X = \begin{pmatrix} A & B & E \\ C & D & F \\ G & H & x \end{pmatrix},$$

where $A, B, C, D \in M_m(R)$, $E, F \in M_{m\times 1}(R)$, $G, H \in M_{1\times m}(R)$ and $x \in R$. In the following, we use a_{ij} to denote the (i,j)-entry of the matrix A. We use similar notations for other block matrices in X. For $1 \le i < m+1$, comparing the coefficients of $e_{\{n+1,\dots,2n\}}$ and $e_{S_i} = e_{\{1,\dots,\widehat{i},\dots,n,n+i\}}$ in (4.1.7), we obtain

$$\begin{aligned} c_{S_{m+1}}(-1)^m b_{S_{m+1}} \frac{2t^{m-1}}{\pi^{3m-1}} &= 1, \\ c_{S_{m+1}} b_{S_{m+1}}(-1)^{m+i} \frac{2t^{m-1}}{\pi^{3m-2}} + c_{S_i} b_{S_i} \left(-\sqrt{\theta}\frac{2t^m \pi_0}{\pi^{3m}}\right) &= (-1)^{1+i} d_{ii}, \\ c_{S_{m+1}} b_{S_{m+1}}(-1)^{m+i} \frac{2t^{m-1}}{\pi^{3m-2}} + c_{S_i} b_{S_i} \left(-\sqrt{\theta}\frac{2t^m}{\pi^{3m-2}}\right) &= (-1)^{1+i} a_{m+i-i,m+1-i}. \end{aligned}$$

Hence,

$$d_{ii} = \frac{\pi_0}{\pi^2} a_{m+1-i,m+1-i} + t\sqrt{\theta}. \tag{4.1.8}$$

For $1 \le i, j < m+1$ and $i \neq j$, by comparing the coefficients of $e_{\{1,\dots,\widehat{j},\dots,n,n+i\}}$ and $e_{\{j^\vee,n+1,\dots,\widehat{i^*},\dots,2n\}}$, we obtain

$$c_{\{1,\dots,\widehat{j},\dots,n,n+i\}} b_{\{1,\dots,\widehat{j},\dots,n,n+i\}} \left(-\sqrt{\theta}\frac{t^m}{\pi^{3m-1}}\right) = (-1)^{1+j} d_{ij},$$

$$c_{\left\{1,\dots,\widehat{j},\dots,n,n+i\right\}} b_{\left\{1,\dots,\widehat{j},\dots,n,n+i\right\}} \left((-1)^{1+i+j} \sqrt{\theta} \frac{t^m}{\pi^{3m-3}\pi_0} \right) = (-1)^{1+i} a_{m+1-j,m+1-i}.$$

Hence,

$$d_{ij} = \frac{\pi_0}{\pi^2} a_{m+1-j,m+1-i}. \tag{4.1.9}$$

Combining (4.1.8) and (4.1.9), we obtain

$$D = \frac{\pi_0}{\pi^2} H_m A^t H_m + t\sqrt{\theta} I_m.$$

Here the matrix $H_m A^t H_m$ is the reflection of A over its anti-diagonal. Equivalently,

$$D + \pi I_m = \frac{\overline{\pi}}{\pi} H_m (A + \pi I_m)^t H_m. \tag{4.1.10}$$

Similarly, we can obtain

$$B = H_m B^t H_m,\ C = H_m C^t H_m,\ E = \frac{t}{\overline{\pi}} H_m H^t,\ F = \frac{t}{\pi} H_m G^t. \tag{4.1.11}$$

Write

$$\widetilde{H}_{2m} := \begin{pmatrix} 0 & H_m \\ \frac{\overline{\pi}}{\pi} H_m & 0 \end{pmatrix},\ X_1 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Combining (4.1.10) and (4.1.11), we have

$$\widetilde{H}_{2m} \left(X_1 + \pi I_{2m} \right) = (X_1 + \pi I_{2m})^t \widetilde{H}_{2m}^t. \tag{4.1.12}$$

4.1.2.7 A simplification of equations

First we can see that under the wedge condition $\wedge^2(X + \pi I_n) = 0$, the Kottwitz condition (4.1.4) becomes

$$\mathrm{tr}(X + \pi I_n) = \pi - \overline{\pi}. \tag{4.1.13}$$

Next we claim that the equation

$$X^2 + tX + \pi_0 I_n = 0 \tag{4.1.14}$$

of Condition **LM1** is implied by the Kottwitz condition **LM2** and the wedge condition **LM5**. To justify the claim, we need an easy but useful lemma.

Lemma 4.1.9. *Let X be an $n \times n$ matrix. Then $X^2 \equiv (\operatorname{tr} X)X$ modulo $(\wedge^2 X)$.*

Proof. The (i,j)-entry of the matrix $X^2 - \operatorname{tr}(X)X$ is

$$\sum_{k=1}^{n} X_{ik}X_{kj} - \sum_{k=1}^{n} X_{kk}X_{ij} = \sum_{k=1}^{n} \left(X_{ik}X_{kj} - X_{kk}X_{ij}\right),$$

which is a sum of 2-minors of X. □

By Lemma 4.1.9 and the wedge condition **LM5**, the equation (4.1.14)

$$X^2 + tX + \pi_0 I_n = (X + \pi I_n)^2 + (t - 2\pi)(X + \pi I_n) = 0$$

is equivalent to

$$\operatorname{tr}(X + \pi I_n)(X + \pi I_n) + (t - 2\pi)(X + \pi I_n) = \left(\operatorname{tr}(X + \pi I_n) + \overline{\pi} - \pi\right)(X + \pi I_n) = 0,$$

which is implied by the Kottwitz condition (4.1.13).

Next, we examine the Condition **LM3**. For the equation (**LM3**-1), we have

$$\begin{aligned}
&\frac{2}{t}C^t H_m A + \frac{t^2 - 2\pi_0}{t\pi_0}A^t H_m C + \frac{t}{\pi_0}G^t G + H_m C + C^t H_m\\
=&\frac{2}{t}C^t H_m(A + \pi I_m) + \frac{t^2 - 2\pi_0}{t\pi_0}(A + \pi I_m)^t H_m C - \frac{2\pi}{t}C^t H_m - \frac{t^2 - 2\pi_0}{t\pi_0}\pi H_m C\\
&\quad + \frac{t}{\pi_0}G^t G + H_m C + C^t H_m\\
=&\frac{2}{t}C^t H_m(A + \pi I_m) + \frac{t^2 - 2\pi_0}{t\pi_0}(A + \pi I_m)^t H_m C + \frac{t}{\pi_0}G^t G + \sqrt{\theta}C^t H_m + \frac{\pi}{\overline{\pi}}\sqrt{\theta}H_m C.
\end{aligned}$$

A similar argument as in the proof of Lemma 4.1.9 implies that

$$C^t H_m(A + \pi I_m) \equiv (A + \pi I_m)^t H_m C \text{ modulo } (\wedge^2(X + \pi I_m)).$$

Hence, the equation (**LM3**-1) gives the same restriction on $\mathrm{U}_{\{0\}}$ as the equation

$$\frac{t}{\pi_0}(A + \pi I_m)^t H_m C + \frac{t}{\pi_0}G^t G + \sqrt{\theta}C^t H_m + \frac{\pi}{\overline{\pi}}\sqrt{\theta}H_m C = 0.$$

By (4.1.11), we further obtain

$$\frac{t}{\pi_0}(A + \pi I_m)^t H_m C + \frac{t}{\pi_0}G^t G + \frac{t}{\overline{\pi}}\sqrt{\theta}H_m C = 0, \tag{4.1.15}$$

$$(A+\pi I_m)^t H_m C = (C(A+\pi I_m))^t H_m.$$

Again, as in Lemma 4.1.9, the matrix $C(A+\pi I_m)$ is equivalent to $\operatorname{tr}(A+\pi I_m)C$. Thus, the equation (4.1.15) is equivalent to

$$\frac{t}{\pi_0}\operatorname{tr}(A+\pi I_m)C^t H_m + \frac{t}{\pi_0}G^t G + \frac{t}{\overline{\pi}}\sqrt{\theta}H_m C = 0.$$

Equivalently,

$$\frac{t}{\pi_0}\Big((\operatorname{tr}(A+\pi I_m)+\pi\sqrt{\theta})H_m C + G^t G\Big) = 0. \tag{4.1.16}$$

Similarly, under the wedge condition **LM5** and the strengthened spin condition **LM6**, one can verify that the equation (**LM3**-2) can be simplified to

$$\frac{t}{\pi_0}\Big((\operatorname{tr}(A+\pi I_m)+\pi\sqrt{\theta})H_m(D+\pi I_m) + G^t H\Big) = 0; \tag{4.1.17}$$

the equation (**LM3**-3) is trivial; the equation (**LM3**-4) is equivalent to (**LM3**-2); the equation (**LM3**-5) is equivalent to

$$\frac{t}{\pi_0}\left((\frac{\overline{\pi}}{\pi}\operatorname{tr}(A+\pi I_m)+\pi\sqrt{\theta})H_m B + H^t H\right) = 0; \tag{4.1.18}$$

the rest of the equations are trivial.

Set

$$X_1 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, X_2 := \begin{pmatrix} E \\ F \end{pmatrix}, X_3 := \begin{pmatrix} G & H \end{pmatrix}, X_4 := x.$$

Then $X = \left(\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right)$, and equations (4.1.16), (4.1.17), (4.1.18) translate to

$$\frac{t}{\pi_0}\Big((\operatorname{tr}(A+\pi I_m)+\pi\sqrt{\theta})\widetilde{H}_{2m}(X_1+\pi I_{2m}) + X_3^t X_3\Big) = 0.$$

Using similar arguments, one can check that under the wedge condition **LM5** and the strengthened spin condition **LM6**, equations (**LM4**-1) to (**LM4**-9) are implied by the Condition **LM3**, and the equation (**LM4**-10) is equivalent to

the diagonal of $(\operatorname{tr}(A+\pi I_m)+\pi\sqrt{\theta})\widetilde{H}_{2m}(X_1+\pi I_{2m}) + X_3^t X_3$ equals 0.

Denote by $\mathcal{O}_F[X]$ the polynomial ring over $\mathcal{O}_F$ whose variables are entries of the matrix X. Then we can view the affine chart $\mathrm{U}_{\{0\}} \subset \mathrm{M}_{\{0\}}$ as a closed subscheme of $\operatorname{Spec} \mathcal{O}_F[X]$. In summary, we have shown the following.

Proposition 4.1.10. *The scheme* $\mathrm{U}_{\{0\}}$ *is a closed subscheme*[1] *of* $\mathrm{U}'_{\{0\}} := \operatorname{Spec} \mathcal{O}_F[X]/\mathcal{I}$, *where* $\mathcal{I}$ *is the ideal of* $\mathcal{O}_F[X]$ *generated by:*

$$\begin{aligned}
&\operatorname{tr}(X + \pi I_n) - \pi + \overline{\pi},\ \wedge^2(X + \pi I_n),\ \widetilde{H}_{2m}\,(X_1 + \pi I_{2m}) - (X_1 + \pi I_{2m})^t \widetilde{H}_{2m}^t,\\
&E - \frac{t}{\overline{\pi}} H_m H^t,\ F - \frac{t}{\pi} H_m G^t,\ \frac{t}{\pi_0}\Big((\operatorname{tr}(A + \pi I_m) + \pi\sqrt{\theta})\widetilde{H}_{2m}(X_1 + \pi I_{2m}) + X_3^t X_3\Big),\\
&\textit{the diagonal of } (\operatorname{tr}(A + \pi I_m) + \pi\sqrt{\theta})\widetilde{H}_{2m}(X_1 + \pi I_{2m}) + X_3^t X_3.
\end{aligned}$$

Set

$$\widetilde{X}_1 := X_1 + \pi I_{2m},\ \widetilde{A} := A + \pi I_m,\ \widetilde{X} := \begin{pmatrix} \widetilde{X}_1 \\ X_3 \end{pmatrix}.$$

As X_2 and X_4 are determined by X_1 and X_3 by relations in $\mathcal{I}$, we obtain the following proposition.

Proposition 4.1.11. *The scheme* $\mathrm{U}'_{\{0\}} = \operatorname{Spec} \mathcal{O}_F[X]/\mathcal{I}$ *is isomorphic to* $\operatorname{Spec} \mathcal{O}_F[\widetilde{X}]/\widetilde{\mathcal{I}}$, *where* $\widetilde{\mathcal{I}}$ *is the ideal of* $\mathcal{O}_F[\widetilde{X}]$ *generated by:*

$$\begin{aligned}
&\wedge^2(\widetilde{X}),\ \widetilde{H}_{2m}\widetilde{X}_1 - \widetilde{X}_1^t \widetilde{H}_{2m}^t,\ \frac{t}{\pi_0}\Big((\operatorname{tr}(\widetilde{A}) + \pi\sqrt{\theta})\widetilde{H}_{2m}\widetilde{X}_1 + X_3^t X_3\Big),\\
&\textit{the diagonal of } (\operatorname{tr}(\widetilde{A}) + \pi\sqrt{\theta})\widetilde{H}_{2m}\widetilde{X}_1 + X_3^t X_3.
\end{aligned}$$

Definition 4.1.12. Denote by $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ the closed subscheme of $\mathrm{U}'_{\{0\}} = \operatorname{Spec} \mathcal{O}_F[\widetilde{X}]/\mathcal{I}$ defined by the ideal $\widetilde{\mathcal{I}}^{\mathrm{fl}} \subset \mathcal{O}_F[\widetilde{X}]$ that is generated by:

$$\wedge^2(\widetilde{X}),\ \widetilde{H}_{2m}\widetilde{X}_1 - \widetilde{X}_1^t \widetilde{H}_{2m}^t,\ (\operatorname{tr}(\widetilde{A}) + \pi\sqrt{\theta})\widetilde{H}_{2m}\widetilde{X}_1 + X_3^t X_3.$$

Note that the ideal $\widetilde{\mathcal{I}}^{\mathrm{fl}}$ contains $\widetilde{\mathcal{I}}$.

[1] In fact, we expect that $\mathrm{U}_{\{0\}} = \mathrm{U}'_{\{0\}}$. This amounts to saying that the equations obtained by comparing coefficients of e_S in (4.1.7) for S not of type $(n-1, 1)$ are implied by relations in $\mathcal{I}$.

4.1.2.8 Geometric properties of $\mathrm{U}_{\{0\}}$ and $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$

In the following, we write $\mathcal{R}^{\mathrm{fl}}$ for the ring $\mathcal{O}_F[\widetilde{X}]/\widetilde{\mathcal{I}}^{\mathrm{fl}}$ and $\mathcal{R}$ for the ring $\mathcal{O}_F[\widetilde{X}]/\widetilde{\mathcal{I}}$.

Lemma 4.1.13. *If $\omega(\pi_0) = \omega(t)$, then $\mathcal{R} = \mathcal{R}^{\mathrm{fl}}$.*

Proof. Note that $\omega(\pi_0) = \omega(t)$ if and only if t/π_0 is a unit in $\mathcal{O}_F$. By comparing the lists of generators of $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{I}}^{\mathrm{fl}}$, we immediately see that $\widetilde{\mathcal{I}} = \widetilde{\mathcal{I}}^{\mathrm{fl}}$, and hence $\mathcal{R} = \mathcal{R}^{\mathrm{fl}}$. $\square$

Remark 4.1.14. Since $\pi_0|t|2$, the condition $\omega(t) = \omega(\pi_0)$ clearly holds if $F_0/\mathbb{Q}_2$ is unramified. More generally, by applying Proposition 3.1.1 (4) to F_0, we have $\omega(t) = \omega(\pi_0)$ if and only if $\theta \in U_{2e-1} - U_{2e}$. Namely, given a quadratic extension F of F_0 with a uniformizer π satisfying an Eisenstein equation $\pi^2 - t\pi + \pi_0 = 0$, the condition $\omega(t) = \omega(\pi_0)$ holds if and only if F is of the form $F_0(\sqrt{\theta})$ for some unit $\theta \in U_{2e-1} - U_{2e}$. We will count the number of such extensions F in the following.

We have a short exact sequence

$$0 \to \frac{U_{2e}}{U^2 \cap U_{2e}} \to \frac{U_{2e-1}}{U^2 \cap U_{2e-1}} \to \frac{U_{2e-1}}{U_{2e}(U^2 \cap U_{2e-1})} \to 0. \tag{4.1.19}$$

We claim that $U^2 \cap U_{2e-1} \subset U_{2e}$. For any $x \in U^2 \cap U_{2e-1}$, we can find $a \in \mathcal{O}_{F_0}$ and $u \in U$ such that $x = 1 + \pi_0^{2e-1}a = u^2$. We want to show $\omega(a) \geq 1$. Set $b = u - 1$. Then $b(b+2) = \pi_0^{2e-1}a$. If $\omega(b) < e = \omega(2)$, then $\omega(b+2) = \omega(b)$ and

$$\omega(\pi_0^{2e-1}a) = \omega\left(b(b+2)\right) = 2\omega(b).$$

As $2e-1$ is odd, this forces $\omega(a)$ to be odd and in particular $\omega(a) \geq 1$. If $\omega(b) \geq e$, then

$$\omega(\pi_0^{2e-1}a) = \omega\left(b(b+2)\right) \geq \omega(b) + \omega(2) \geq 2e.$$

Again we have $\omega(a) \geq 1$. This proves the claim.

Then we have $U_{2e}(U^2 \cap U_{2e-1}) = U_{2e}$ and by the short exact sequence (4.1.19),

$$\left|\frac{U_{2e-1}}{U^2 \cap U_{2e-1}}\right| = \left|\frac{U_{2e}}{U^2 \cap U_{2e}}\right| \left|\frac{U_{2e-1}}{U_{2e}}\right| = 2 \cdot 2^f = 2^{1+f},$$

where f denotes the residue degree of $F_0/\mathbb{Q}_2$. Note that there are two elements in $\frac{U_{2e-1}}{U^2\cap U_{2e-1}}$ defining the trivial extension and the unramified quadratic extension of F_0. Thus, we have $2^{1+f}-2$ ramified quadratic extensions of F_0 of type (R-U) with $\omega(t)=\omega(\pi_0)$.

By (4.1.10), we have

$$\operatorname{tr}(\widetilde{X}_1)=\operatorname{tr}(\widetilde{A})+\operatorname{tr}(\widetilde{D})=\frac{t}{\pi}\operatorname{tr}(\widetilde{A}).$$

So we can rewrite $\mathcal{R}^{\mathrm{fl}}$ as

$$\mathcal{R}^{\mathrm{fl}}=\frac{\mathcal{O}_F[\left(\begin{smallmatrix}\widetilde{X}_1\\ X_3\end{smallmatrix}\right)]}{\left(\wedge^2\left(\begin{smallmatrix}\widetilde{X}_1\\ X_3\end{smallmatrix}\right),\widetilde{H}_{2m}\widetilde{X}_1-\widetilde{X}_1^t\widetilde{H}_{2m}^t,(\frac{\pi}{t}\operatorname{tr}(\widetilde{X}_1)+\pi\sqrt{\theta})\widetilde{H}_{2m}\widetilde{X}_1+X_3^tX_3\right)}.$$

Let $Y\coloneqq\widetilde{H}_{2m}\widetilde{X}_1$. Then $\widetilde{X}_1=\frac{\pi}{\overline{\pi}}\widetilde{H}_{2m}Y$ and

$$\begin{aligned}\mathcal{R}^{\mathrm{fl}}&\simeq\frac{\mathcal{O}_F[\left(\begin{smallmatrix}Y\\ X_3\end{smallmatrix}\right)]}{\left(\wedge^2\left(\begin{smallmatrix}\frac{\pi}{\overline{\pi}}\widetilde{H}_{2m}Y\\ X_3\end{smallmatrix}\right),Y-Y^t,(\frac{\pi^2}{t\overline{\pi}}\operatorname{tr}(\widetilde{H}_{2m}Y)+\pi\sqrt{\theta})Y+X_3^tX_3\right)}\\&=\frac{\mathcal{O}_F[\left(\begin{smallmatrix}Y\\ X_3\end{smallmatrix}\right)]}{\left(\wedge^2\left(\begin{smallmatrix}Y\\ X_3\end{smallmatrix}\right),Y-Y^t,(\frac{\pi}{2\overline{\pi}}\operatorname{tr}(H_{2m}Y)+\pi\sqrt{\theta})Y+X_3^tX_3\right)}.\end{aligned}$$

For $1\leq i,j\leq 2m$, we denote by y_{ij} the (i,j)-entry of Y and by x_i the $(1,i)$-entry of X_3.

Lemma 4.1.15. *The scheme $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is irreducible of Krull dimension n and smooth over $\mathcal{O}_F$ on the complement of the worst point, which is the closed point defined by $Y=X_3=\pi=0$.*

Proof. For $1\leq\ell\leq 2m$, consider the principal open subscheme $D(y_{\ell\ell})$ of $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$, i.e., the locus where $y_{\ell\ell}$ is invertible. Then one can easily verify that $D(y_{\ell\ell})$ is isomorphic to the closed subscheme of

$$\operatorname{Spec}\mathcal{O}_F[y_{ij},x_i\mid 1\leq i,j\leq 2m]$$

defined by the ideal generated by the relations

$$y_{ij}=y_{ji},\ y_{ij}=y_{\ell\ell}^{-1}y_{\ell i}y_{\ell j},\ x_i=y_{\ell\ell}^{-1}x_\ell y_{\ell i},\ -x_\ell^2=(\frac{\pi}{\overline{\pi}}\sum_{i=1}^m y_{\ell i}y_{\ell,n-i})+\pi\sqrt{\theta}y_{\ell\ell}.$$

Hence, the scheme $D(y_{\ell\ell})$ is isomorphic to

$$\operatorname{Spec}\frac{\mathcal{O}_F[x_\ell,y_{\ell 1},\dots,y_{\ell\ell},\dots,y_{\ell,2m},y_{\ell\ell}^{-1}]}{(x_\ell^2+(\frac{\pi}{\overline{\pi}}\sum_{i=1}^m y_{\ell i}y_{\ell,n-i})+\pi\sqrt{\theta}y_{\ell\ell})}.$$

By the Jacobian criterion, $D(y_{\ell\ell})$ is smooth over $\mathcal{O}_F$ of Krull dimension n. Note that the worst point is defined (set-theoretically) by the ideal generated by π and $y_{\ell\ell}$ for $1 \leq \ell \leq 2m$. Since the generic fiber of $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is smooth, we obtain that $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is smooth over $\mathcal{O}_F$ on the complement of the worst point. As the generic fiber and all $D(y_{\ell\ell})$, for $1 \leq \ell \leq 2m$, are irreducible, we conclude that $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is irreducible. □

Lemma 4.1.16. *The scheme* $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ *is Cohen-Macaulay.*

Proof. Let $\mathcal{S}$ denote the polynomial ring $\mathcal{O}_F[y_{ii} \mid 1 \leq i \leq 2m]$. Then we have an obvious ring homomorphism $\mathcal{S} \to \mathcal{R}^{\mathrm{fl}}$. By the wedge condition **LM5** and $Y = Y^t$, for $1 \leq i, j \leq 2m$, we have

$$y_{ij}^2 = y_{ij}y_{ji} = y_{ii}y_{jj} \text{ and } x_i^2 = -(\frac{\pi}{\bar{\pi}}\sum_{\ell=1}^{m} y_{i\ell}y_{i,n-\ell}) - \pi\sqrt{\theta}y_{ii}$$

In particular, we deduce that $\mathcal{R}^{\mathrm{fl}}$ is integral (also of finite type) over $\mathcal{S}$, and hence $\mathcal{R}^{\mathrm{fl}}$ is a finitely generated $\mathcal{S}$-module. Since $\mathcal{S}$ is a domain of the same Krull dimension as $\mathcal{R}^{\mathrm{fl}}$, the map $\mathcal{S} \to \mathcal{R}^{\mathrm{fl}}$ is necessarily injective. By [Eis13, Corollary 18.17], to show $\mathcal{R}^{\mathrm{fl}}$ is Cohen-Macaulay, it suffices to show that $\mathcal{R}^{\mathrm{fl}}$ is a flat $\mathcal{S}$-module. Equivalently, we need to show that the induced morphism

$$\psi : \operatorname{Spec}\mathcal{R}^{\mathrm{fl}} \to \operatorname{Spec}\mathcal{S} \simeq \mathbb{A}^{2m}$$

is flat. Let P_0 be the closed point in $\operatorname{Spec}\mathcal{S}$ corresponding to the maximal ideal $\mathfrak{m}_0 := (\pi, y_{11}, \ldots, y_{2m,2m})$. Then ψ maps the worst point of $\operatorname{Spec}\mathcal{R}^{\mathrm{fl}}$ to P_0 and the preimage of $\operatorname{Spec}\mathcal{S}[y_{\ell\ell}^{-1}]$ is the scheme $D(y_{\ell\ell})$ considered in the proof of Lemma 4.1.15. As $D(y_{\ell\ell})$ is smooth over $\mathcal{O}_F$, by miracle flatness (see [Eis13, Theorem 18.16 b.]), the restriction $\psi|_{D(y_{\ell\ell})}$ is flat. Similarly, we obtain that ψ restricted to the generic fiber of $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is flat. It remains to show that ψ is flat at the worst point, i.e., the localization map $\mathcal{S}_{\mathfrak{m}_0} \to \mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0}$ is flat. The local ring $\mathcal{S}_{\mathfrak{m}_0}$ has residue field k. Let K denote the fraction field of $\mathcal{S}_{\mathfrak{m}_0}$. By an application of Nakayama's lemma (see [Har13, Chapter II, Lemma 8.9]), we are reduced to show that

$$\dim_K(\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K) = \dim_k(\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} k). \tag{4.1.20}$$

Note that K is the field $F(y_{11}, \ldots, y_{2m,2m})$ of rational functions. By the following Lemma 4.1.17, we have the desired equality (4.1.20) of dimensions. $\square$

Lemma 4.1.17. *The K-vector space (resp. k-vector space) $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K$ (resp. $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} k$) has a K-basis (resp. k-basis) consisting of (images of) monomials*

$$x_i^{\alpha} y_{i_1 j_1}^{\beta_1} y_{i_2 j_2}^{\beta_2} \cdots y_{i_\ell j_\ell}^{\beta_\ell},$$

where $\alpha, \beta_i \in \{0, 1\}$, $0 \le \ell \le m$, and $1 \le i < i_1 < j_1 < i_2 < j_2 < \cdots < i_\ell < j_\ell \le 2m$. Let S denote the set of these monomials. Then the cardinality $\#S$ equals 2^{2m}. In particular,

$$\dim_K(\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K) = \dim_k(\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} k) = 2^{2m}. \tag{4.1.21}$$

Proof. We first count the cardinality of S. For an integer $0 \le \ell \le m$, the number of monomials of the form $x_i y_{i_1 j_1}^{\beta_1} y_{i_2 j_2}^{\beta_2} \cdots y_{i_\ell j_\ell}^{\beta_\ell}$ in S is the number of tuples $(i, i_1, j_1, \ldots, i_\ell, j_\ell)$ such that $1 \le i < i_1 < j_1 < i_2 < j_2 < \cdots < i_\ell < j_\ell \le 2m$. It is well-known that the number is $\binom{2m}{2\ell+1}$. Here, we set $\binom{2m}{2\ell+1} = 0$ if $\ell = m$. Similarly, the number of monomials of the form $y_{i_1 j_1}^{\beta_1} y_{i_2 j_2}^{\beta_2} \cdots y_{i_\ell j_\ell}^{\beta_\ell}$ in S is $\binom{2m}{2\ell}$. Hence, we obtain that

$$\#S = \sum_{\ell=0}^{m} \binom{2m}{2\ell+1} + \sum_{\ell=0}^{m} \binom{2m}{2\ell} = \sum_{i=0}^{2m} \binom{2m}{i} = 2^{2m}.$$

Let $x_i^{\alpha} x_j^{\alpha'} y_{i_1 j_1}^{\beta_1} y_{i_2 j_2}^{\beta_2} \cdots y_{i_\ell j_\ell}^{\beta_\ell}$ be a general monomial in $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K$. As $y_{ij}^2 = y_{ij} y_{ji} = y_{ii} y_{jj}$ in $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0}$, we may assume β_i for $1 \le i \le \ell$ lies in $\{0, 1\}$. As

$$-X_3^t X_3 = (\frac{\pi}{2\overline{\pi}} \operatorname{tr}(H_{2m} Y) + \pi\sqrt{\theta}) Y$$

in $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0}$, we see $x_i x_j$ can be expressed by entries in Y. Hence, we may assume $\alpha' = 0$ and $\alpha \in \{0, 1\}$. We claim that the monomial $x_i^{\alpha} y_{i_1 j_1} y_{i_2 j_2} \cdots y_{i_\ell j_\ell}$ for $\alpha \in \{0, 1\}$ is generated by elements in S. By the wedge condition and $Y = Y^t$, it is straightforward to check that the product $x_r y_{ij} y_{pq}$ only depends on the indices $\{r, i, j, p, q\}$, namely, changing the order of indices gives the same product in $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0}$. Since $y_{ii} \in K$, we may assume $1 \le i < i_1 < j_1 < i_2 < j_2 < \cdots < i_\ell < j_\ell \le 2m$, and hence we may assume $0 \le \ell \le m$. Thus, the K-vector space $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K$ is generated by (images of) the elements in S. Now it suffices to show that these elements are K-linearly independent.

Note that the ring $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K$ corresponds to the generic point of $\operatorname{Spec} \mathcal{R}^{\mathrm{fl}}$. Since y_{11} is invertible over $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K$, the ring $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K$ is in fact the function field of $D(y_{11})$ in the proof of Lemma 4.1.15 (take $\ell = 1$), and the field

$$\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K = \frac{K[y_{12}, y_{13}, \dots, y_{1,2m}, x_1]}{\left(y_{12}^2 - y_{11}y_{22}, \dots, y_{1,2m}^2 - y_{11}y_{2m,2m}, x_1^2 + (\frac{\pi}{\overline{\pi}} \sum_{i=1}^m y_{1i}y_{1,n-i}) + \pi\sqrt{\theta}y_{11}\right)}.$$

is a compositum of successive quadratic extensions. In particular,

$$\dim_K(\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K) = 2^{2m}.$$

As $\#S = 2^{2m}$, elements in S are K-linearly independent, i.e., elements in S form a K-basis of $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K$.

Similar arguments (just note that now $y_{ii} = 0$ in k) as before imply that $\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} k$ is generated by (images of) elements in S. Hence,

$$\dim_k(\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} k) \leq \#S = \dim_K(\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K).$$

On the other hand, by Nakayama's lemma, we always have

$$\dim_k(\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} k) \geq \dim_K(\mathcal{R}^{\mathrm{fl}}_{\mathfrak{m}_0} \otimes_{\mathcal{S}_{\mathfrak{m}_0}} K).$$

This completes the proof of the lemma. □

Corollary 4.1.18. *The scheme $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is normal and flat over $\mathcal{O}_F$. The geometric special fiber $\mathrm{U}^{\mathrm{fl}}_{\{0\}} \otimes_{\mathcal{O}_F} \overline{k}$ is reduced and irreducible.*

Proof. As $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is smooth over $\mathcal{O}_F$ on the complement of a closed point, and Cohen-Macaulay by Lemma 4.1.15 and 4.1.16, the normality of $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ follows from the Serre's criterion for normality (see [Sta24, 031S]). By Lemma 4.1.15, the scheme $\mathrm{U}^{\mathrm{fl}}_{\{0\}} \otimes_{\mathcal{O}_F} \overline{k}$ is smooth over $\overline{k}$ on the complement of the worst point. The proof of Lemma 4.1.15 also implies that $\mathrm{U}^{\mathrm{fl}}_{\{0\}} \otimes_{\mathcal{O}_F} \overline{k}$

is irreducible of dimension $n-1$. As $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is Cohen-Macaulay and $\operatorname{Spec}\mathcal{O}_F$ is regular, then $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is flat over $\mathcal{O}_F$ by the miracle flatness (see [Eis13, Theorem 18.16 b.]).

Since $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is Cohen-Macaulay and π is not a zero divisor (follows from the flatness), the scheme $\mathrm{U}^{\mathrm{fl}}_{\{0\}}\otimes_{\mathcal{O}_F}\overline{k}$ is also Cohen-Macaulay. Then $\mathrm{U}^{\mathrm{fl}}_{\{0\}}\otimes_{\mathcal{O}_F}\overline{k}$ is reduced by the Serre's criterion for reducedness (see [Sta24, 031R]). □

Lemma 4.1.19. *The schemes* $\mathrm{U}_{\{0\}}$ *and* $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ *have the same underlying topological space.*

Proof. (1) Since $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is flat over $\mathcal{O}_F$, the scheme $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is the Zariski closure of its generic fiber. Then we have closed immersions

$$\mathrm{U}^{\mathrm{fl}}_{\{0\}}\hookrightarrow \mathrm{U}_{\{0\}}\hookrightarrow \mathrm{U}'_{\{0\}}$$

where all schemes have the same generic fiber. Then it suffices to prove that the special fibers of $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ and $\mathrm{U}'_{\{0\}}$ have the same underlying topological space. Since $\mathrm{U}^{\mathrm{fl}}_{\{0\}}\otimes_{\mathcal{O}_F}k$ is reduced, we are reduced to show that $\mathcal{I}^{\mathrm{fl}}\otimes_{\mathcal{O}_F}k$ is contained in the radical of $\mathcal{I}\otimes_{\mathcal{O}_F}k$.

If $\omega(\pi_0)=\omega(t)$, then the assertion follows from Lemma 4.1.13. We may assume t/π_0 is not a unit. In this case, we have

$$\begin{aligned}\mathcal{I}\otimes_{\mathcal{O}_F}k&=\Big(\wedge^2\big(\begin{smallmatrix}Y\\X_3\end{smallmatrix}\big),Y-Y^t,\text{the diagonal of }\big(\frac{\mathrm{tr}(H_{2m}Y)}{2}Y+X_3^tX_3\big)\Big),\\ \mathcal{I}^{\mathrm{fl}}\otimes_{\mathcal{O}_F}k&=\Big(\wedge^2\big(\begin{smallmatrix}Y\\X_3\end{smallmatrix}\big),Y-Y^t,\frac{\mathrm{tr}(H_{2m}Y)}{2}Y+X_3^tX_3\Big).\end{aligned}$$

Let M denote the matrix $\frac{\mathrm{tr}(H_{2m}Y)}{2}Y+X_3^tX_3$. Then for $1\le i,j\le 2m$, the (i,j)-entry M_{ij} of M is

$$\alpha y_{ij}+x_ix_j,\quad \alpha:=\mathrm{tr}(H_{2m}Y)/2.$$

Since $\mathrm{char}(k)=2$, we obtain $M_{ij}^2=\alpha^2y_{ij}^2+x_i^2x_j^2$. Therefore, we have

$$\begin{aligned}M_{ij}^2-M_{ii}M_{jj}&=\alpha^2(y_{ij}^2-y_{ii}y_{jj})-\alpha x_i^2y_{jj}-\alpha x_j^2y_{ii}\\&=\alpha^2(y_{ij}^2-y_{ii}y_{jj})-x_i^2M_{jj}-x_j^2M_{ii}+2x_i^2x_j^2\\&=\alpha^2(y_{ij}^2-y_{ii}y_{jj})-x_i^2M_{jj}-x_j^2M_{ii}\in\widetilde{\mathcal{I}}\otimes_{\mathcal{O}_F}k\end{aligned}$$

In particular, any M_{ij}^2 for $1 \le i, j \le 2m$ lies in $\widetilde{\mathcal{I}} \otimes_{\mathcal{O}_F} k$. Hence, $\mathcal{I}^{\mathrm{fl}} \otimes_{\mathcal{O}_F} k$ is contained in the radical of $\mathcal{I} \otimes_{\mathcal{O}_F} k$. This finishes the proof. $\square$

In summary, we have proven the following.

Proposition 4.1.20. *(1) The scheme $\mathrm{U}_{\{0\}}^{\mathrm{fl}}$ is flat over $\mathcal{O}_F$ of relative dimension $n-1$. In particular, $\mathrm{U}_{\{0\}}^{\mathrm{fl}}$ is isomorphic to an open subscheme of the local model $\mathrm{M}_{\{0\}}^{\mathrm{loc}}$ containing the worst point. Furthermore, $\mathrm{U}_{\{0\}}^{\mathrm{fl}}$ is normal, Cohen-Macaulay, and smooth over $\mathcal{O}_F$ on the complement of the worst point. The special fiber $\mathrm{U}_{\{0\}}^{\mathrm{fl}} \otimes_{\mathcal{O}_F} k$ is (geometrically) reduced and irreducible.*

(2) $\mathrm{U}_{\{0\}}$ and $\mathrm{U}_{\{0\}}^{\mathrm{fl}}$ have the same underlying topological space.

(3) If $\omega(\pi_0) = \omega(t)$, then $\mathrm{U}_{\{0\}} = \mathrm{U}_{\{0\}}^{\mathrm{fl}}$.

4.1.2.9 Global results

Recall that $(\Lambda_0, q, \mathscr{L}, \phi)$ is a hermitian quadratic module with ϕ over $\mathcal{O}_{F_0}$ by Lemma 3.2.5. Let $\mathscr{H}_{\{0\}} := \underline{\mathrm{Sim}}((\Lambda_0, q, \mathscr{L}, \phi))$ be the group scheme over $\mathcal{O}_{F_0}$ of similitudes preserving ϕ of $(\Lambda_0, q, \mathscr{L}, \phi)$. By Theorem 6.2.8, $\mathscr{H}_{\{0\}}$ is an affine smooth group scheme over $\mathcal{O}_{F_0}$.

Lemma 4.1.21. *The group scheme $\mathscr{H}_{\{0\}}$ acts on $\mathrm{M}_{\{0\}}^{\mathrm{naive}}$ and $\mathrm{M}_{\{0\}}$.*

Proof. It suffices to show the result for $\mathrm{M}_{\{0\}}$. Let R be an $\mathcal{O}_F$-algebra. Let $g = (\varphi, \gamma) \in \mathscr{H}_{\{0\}}(R)$ be a similitude preserving ϕ. For $\mathcal{F} \in \mathrm{M}_{\{0\}}$, we define $g\mathcal{F} := \varphi(\mathcal{F}) \subset \Lambda_0 \otimes_{\mathcal{O}_{F_0}} R$. We need to show that $g\mathcal{F} \in \mathrm{M}_{\{0\}}(R)$. It is clear that $g\mathcal{F}$ satisfies conditions **LM1,2,4**. Recall that $\phi : \Lambda_0 \times \Lambda_0 \to t^{-1}\mathcal{O}_{F_0}$ is defined by $(x, y) \mapsto t^{-1} \mathrm{Tr}_{F/F_0} h(x, \pi^{-1}y)$. We also use ϕ to denote the base change to $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} R$. Then we see that $\mathcal{F}$ satisfies **LM3** if and only if $\phi(\mathcal{F}, \mathcal{F}) = 0$. As g preserves ϕ, we have that

$$\phi(g\mathcal{F}, g\mathcal{F}) = \gamma\phi(\mathcal{F}, \mathcal{F}) = 0.$$

So $g\mathcal{F}$ satisfies **LM3**. As g is $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-linear by definition, we obtain that

$$(\pi \otimes 1 - 1 \otimes \overline{\pi}) \circ g = g \circ (\pi \otimes 1 - 1 \otimes \overline{\pi}).$$

By the functoriality of the wedge product of linear maps, we have

$$\wedge^2(\pi \otimes 1 - 1 \otimes \overline{\pi} \mid g\mathcal{F}) = \wedge^2(g \circ (\pi \otimes 1 - 1 \otimes \overline{\pi}) \mid \mathcal{F}) = \wedge^2(g) \circ \wedge^2(\pi \otimes 1 - 1 \otimes \overline{\pi} \mid \mathcal{F}) = 0.$$

Therefore, $g\mathcal{F}$ satisfies the wedge condition **LM5**. Since $\mathscr{H}_{\{0\}}$ is smooth over $\mathcal{O}_{F_0}$, using a similar argument of [RSZ18, Lemma 7.1], we can show that the R-submodule

$$\operatorname{Im}\left(W(\Lambda_0)^{n-1,1}_{-1} \otimes_{\mathcal{O}_F} R \to W(\Lambda_0) \otimes_{\mathcal{O}_F} R\right)$$

of $W(\Lambda_0) \otimes_{\mathcal{O}_F} R$ is stable under the natural action of $\mathscr{H}_{\{0\}}(R)$ on the space $W(\Lambda_0) \otimes_{\mathcal{O}_F} R = \wedge^n(\Lambda_0 \otimes_{\mathcal{O}_{F_0}} R)$. It follows that $g\mathcal{F}$ satisfies the strengthened spin condition **LM6**. $\square$

Lemma 4.1.22. *Let $\overline{k}$ be the algebraic closure of the residue field k. Then $\mathrm{M}_{\{0\}} \otimes_{\mathcal{O}_F} \overline{k}$ has two $\mathscr{H}_{\{0\}} \otimes_{\mathcal{O}_{F_0}} \overline{k}$-orbits, one of which consists of the worst point.*

Proof. By Lemma 4.1.21, the special fiber $\mathrm{M}_{\{0\}} \otimes_{\mathcal{O}_F} \overline{k}$ has an action of $\mathscr{H}_{\{0\}} \otimes_{\mathcal{O}_{F_0}} \overline{k}$. Let $\mathcal{F} \in \mathrm{M}_{\{0\}}(\overline{k})$. In particular, the subspace $\mathcal{F} \subset (\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k})$ is an n-dimensional $\overline{k}$-vector space. The wedge condition in this case becomes $\wedge^2(\pi \otimes 1 \mid \mathcal{F}) = 0$. Therefore, the image $(\pi \otimes 1)\mathcal{F}$ is at most one dimensional. We have the following two cases.

Suppose $(\pi \otimes 1)\mathcal{F} = 0$. Then $\mathcal{F} = (\pi \otimes 1)(\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k})$, namely, $\mathcal{F}$ is the worst point.

Suppose $(\pi \otimes 1)\mathcal{F}$ is one-dimensional. Then there exists a vector $v \in \mathcal{F}$ such that $(\pi \otimes 1)v$ generates $(\pi \otimes 1)\mathcal{F}$. For simplicity, write π for $\pi \otimes 1$. Recall the $\overline{k}$-bilinear form

$$\begin{aligned} \phi(-,-) : (\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k}) \times (\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k}) &\longrightarrow \mathscr{L} \otimes_{\mathcal{O}_{F_0}} \overline{k} \\ (x, y) &\mapsto s(x, \pi^{-1}y) = t^{-1} \operatorname{Tr} h(x, \pi^{-1}y), \end{aligned}$$

where π^{-1} is the induced isomorphism $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k} \xrightarrow{\sim} (\pi^{-1}\Lambda_0) \otimes_{\mathcal{O}_{F_0}} \overline{k}$. We can identify $\mathscr{L} \otimes_{\mathcal{O}_{F_0}} \overline{k}$ with $\overline{k}$ by sending $t^{-1} \otimes 1$ to 1. Denote by $N \coloneqq \overline{k}\langle e_{m+1}, \pi e_{m+1}\rangle$ the submodule of $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k}$. Then one can check that the radical of ϕ is contained in N. We claim that πv is not in N. Otherwise, after rescaling, we may assume $v = e_{m+1} \otimes 1 + \pi v_1$ for some $v_1 \in \Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k}$. Then for the quadratic form

$$q : \Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k} \longrightarrow \mathscr{L} \otimes_{\mathcal{O}_{F_0}} \overline{k} \simeq \overline{k},$$

we have

$$q(v) = q(e_{m+1} \otimes 1 + \pi v_1) = q(e_{m+1} \otimes 1) + s(e_{m+1} \otimes 1, \pi v_1) + q(\pi v_1).$$

One can check that $q(e_{m+1} \otimes 1) = 1$ and $s(e_{m+1} \otimes 1, \pi v_1) = q(\pi v_1) = 0$. Hence $q(v) \neq 0$. This contradicts the hyperbolicity condition **LM4** that $q(\mathcal{F}) = 0$. In particular, we obtain that πv is not in the radical of ϕ. Thus, we can find $w \in \Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k}$ such that $\phi(w, \pi v) \neq 0$ in $\overline{k}$. By rescaling, we may assume $\phi(w, \pi v) = 1$. Note that for $a \in \overline{k}$,

$$\begin{aligned} q(w + av) &= q(w) + as(w, v) + a^2 q(v) \\ &= q(w) + a\phi(w, \pi v) + 0, \quad \text{since } q(v) = 0, \\ &= q(w) + a. \end{aligned}$$

Replacing w by $w - q(w)v$, we may assume $q(w) = 0$. Put $b := -\phi(w, v)$. One can check that $\phi(w + b\overline{\pi}w) = 0$. Replacing w by $w + b\overline{\pi}w$, we have

$$q(w) = q(v) = 0, \ \phi(w, v) = 0 \text{ and } \phi(w, \pi v) = 1.$$

Denote $W_1 := \langle v, \pi v, w, \pi w \rangle$, the $\overline{k}$-subspace of $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k}$ generated by $v, \pi v, w, \pi w$. Then ϕ restricts to a perfect pairing on W_1. Now we can write

$$\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k} = W_1 \oplus W, \tag{4.1.22}$$

where W is the orthogonal complement of W_1 with respect to ϕ whose dimension is $2n - 4$ over $\overline{k}$. Note that the Condition **LM3** in Definition 4.1.1 of $\mathrm{M}_{\{0\}}$ implies that $\phi(\mathcal{F}, \mathcal{F}) = 0$, and hence $\mathcal{F} \cap \langle w, \pi w \rangle = 0$. Since $\langle v, \pi v \rangle \subset \mathcal{F}$ and $\phi(\mathcal{F}, \mathcal{F}) = 0$, we obtain that the $\overline{k}$-dimension of $\mathcal{F} \cap W$ is $n - 2$ and $\mathcal{F} \cap W$ is contained in $\pi W = \ker(\pi \mid W)$. Therefore, $\mathcal{F} \cap W = \pi W$ for dimension reasons. By (4.1.22), we have

$$\mathrm{disc}'(\phi) = \mathrm{disc}(\phi|_{W_1}) \mathrm{disc}'(\phi|_W).$$

Here, $\mathrm{disc}'(\phi)$ is the divided discriminant in the sense of Definition 6.2.4, and we view it as an element in $\overline{k}$ by using a basis of $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k}$. By Example 6.2.6, we have $\mathrm{disc}'(\phi) \in \overline{k}^{\times}$.

Since ϕ is perfect on W_1, we obtain that $\mathrm{disc}(\phi|_{W_1}) \in \overline{k}^\times$, and hence $\mathrm{disc}'(\phi|_W) \in \overline{k}^\times$. So W is a hermitian quadratic module of type Λ_0 over $\overline{k}$ in the sense of Definition 6.2.4. Set $v_1 := v$ and $v_n := w$. By applying Theorem 6.2.7 to W, we deduce that there is an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \overline{k}$-basis $\{v_i : 1 \leq i \leq n\}$ of $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \overline{k}$ with the property that $q(v_{m+1})$ generates R, $q(v_i) = 0$, $\phi(v_i, v_j) = 0$ and $\phi(v_i, \pi v_j) = \delta_{i,n+1-j}$ for all $1 \leq i < j \leq n$. With respect to this basis, we have

$$\mathcal{F} = \langle v, \pi v\rangle \oplus (\mathcal{F} \cap W) = \langle v, \pi v\rangle \oplus (\pi W) = \langle v_1, \pi v_1, \pi v_i, 2 \leq i \leq n-1\rangle.$$

This shows that points $\mathcal{F} \in \mathrm{M}_{\{0\}}(\overline{k})$ with $\dim_{\overline{k}} \pi\mathcal{F} = 1$ are in the same $\mathscr{H}_{\{0\}}(\overline{k})$-orbit. $\square$

As $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ is flat over $\mathcal{O}_F$, we may view $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ as an open subscheme of $\mathrm{M}^{\mathrm{loc}}_{\{0\}}$ containing the worst point. By Lemma 4.1.22, the $\mathscr{H}_{\{0\}}$-translation of $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ covers $\mathrm{M}^{\mathrm{loc}}_{\{0\}}$. By Proposition 4.1.20, we have shown Theorem 1.2.6, and Theorem 1.2.2, 1.2.3 in the case $I = \{0\}$ and (R-U).

4.2 The case $I = \{0\}$ and (R-P)

In this section, we consider the case when F/F_0 is of (R-P) type. In particular, we have

$$\pi^2 + \pi_0 = 0 \text{ and } \overline{\pi} = -\pi.$$

Consider the following ordered $\mathcal{O}_{F_0}$-basis of Λ_0 and Λ_0^s:

$$\Lambda_0 : \frac{1}{2}e_{m+2}, \ldots, \frac{1}{2}e_n, e_1, \ldots, e_m, e_{m+1}, \frac{\pi}{2}e_{m+2}, \ldots, \frac{\pi}{2}e_n, \pi e_1, \ldots, \pi e_m, \pi e_{m+1}, \tag{4.2.1}$$

$$\Lambda_0^s : \pi^{-1}e_{m+2}, \ldots, \pi^{-1}e_n, \frac{2}{\pi}e_1, \ldots, \frac{2}{\pi}e_m, \pi^{-1}e_{m+1}, e_{m+2}, \ldots, e_n, 2e_1, \ldots, 2e_m, e_{m+1}. \tag{4.2.2}$$

Recall that $(\Lambda_0, q, \mathscr{L})$ is a hermitian quadratic module for $\mathscr{L} = \frac{1}{2}\mathcal{O}_{F_0}$.

4.2.1 A refinement of $\mathrm{M}^{\mathrm{naive}}_{\{0\}}$ in the (R-P) case

Definition 4.2.1. Let $\mathrm{M}_{\{0\}}$ be the functor

$$\mathrm{M}_{\{0\}} : (\mathrm{Sch}/\mathcal{O}_F)^{\mathrm{op}} \longrightarrow \mathrm{Sets}$$

which sends an $\mathcal{O}_F$-scheme S to the set of $\mathcal{O}_S$-modules $\mathcal{F}$ such that

LM1 (π-stability condition) $\mathcal{F}$ is an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodule of $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ and as an $\mathcal{O}_S$-module, it is a locally direct summand of rank n.

LM2 (Kottwitz condition) The action of $\pi \otimes 1 \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ on $\mathcal{F}$ has characteristic polynomial

$$\det(T - \pi \otimes 1 \mid \mathcal{F}) = (T - \pi)(T - \overline{\pi})^{n-1}.$$

LM3 Let $\mathcal{F}^\perp$ be the orthogonal complement in $\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ of $\mathcal{F}$ with respect to the perfect pairing

$$s(-,-) : (\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \times (\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \to \mathcal{O}_S.$$

We require the map $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to (\frac{\pi}{2}\Lambda_0^s) \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ induced by $\Lambda_0 \hookrightarrow \frac{\pi}{2}\Lambda_0^s$ sends $\mathcal{F}$ to $\frac{\pi}{2}\mathcal{F}^\perp$, where $\frac{\pi}{2}\mathcal{F}^\perp$ is the image of $\mathcal{F}^\perp$ under the isomorphism $\frac{\pi}{2} : \Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \xrightarrow{\sim} \frac{\pi}{2}\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$.

LM4 (Hyperbolicity condition) The quadratic form $q : \Lambda_0 \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to \mathscr{L} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ induced by $q : \Lambda_0 \to \mathscr{L}$ satisfies $q(\mathcal{F}) = 0$.

LM5 (Wedge condition) The action of $\pi \otimes 1 - 1 \otimes \overline{\pi} \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ on $\mathcal{F}$ satisfies

$$\wedge^2(\pi \otimes 1 - 1 \otimes \overline{\pi} \mid \mathcal{F}) = 0.$$

Then as in the (R-U) case, the functor $\mathrm{M}_{\{0\}}$ is representable and we have closed immersions

$$\mathrm{M}_{\{0\}}^{\mathrm{loc}} \subset \mathrm{M}_{\{0\}} \subset \mathrm{M}_{\{0\}}^{\mathrm{naive}}$$

of projective schemes over $\mathcal{O}_F$, where all schemes have the same generic fiber.

4.2.2 An affine chart $\mathrm{U}_{\{0\}}$ around the worst point

Set

$$\mathcal{F}_0 \coloneqq (\pi \otimes 1)(\Lambda_0 \otimes_{\mathcal{O}_{F_0}} k).$$

Then we can check that $\mathcal{F}_0 \in \mathrm{M}_{\{0\}}(k)$. We call it the worst point of $\mathrm{M}_{\{0\}}$.

With respect to the basis (4.2.1), the standard affine chart around $\mathcal{F}_0$ in $\mathrm{Gr}(n,\Lambda_0)_{\mathcal{O}_F}$ is the $\mathcal{O}_F$-scheme of $2n \times n$ matrices $\left(\begin{smallmatrix} X \\ I_n \end{smallmatrix}\right)$. We denote by $\mathrm{U}_{\{0\}}$ the intersection of $\mathrm{M}_{\{0\}}$ with the standard affine chart in $\mathrm{Gr}(n,\Lambda_0)_{\mathcal{O}_F}$. The worst point $\mathcal{F}_0$ of $\mathrm{M}_{\{0\}}$ is contained in $\mathrm{U}_{\{0\}}$ and corresponds to the closed point defined by $X=0$ and $\pi=0$. The conditions **LM1-5** yield the defining equations for $\mathrm{U}_{\{0\}}$. We will analyze each condition as in the (R-U) case. A reader who is only interested in the affine coordinate ring of $\mathrm{U}_{\{0\}}$ may proceed directly to Proposition 4.2.2.

4.2.2.1 Condition LM1

Let R be an $\mathcal{O}_F$-algebra. With respect to the basis (4.2.1), the operator $\pi \otimes 1$ acts on $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} R$ via the matrix

$$\begin{pmatrix} 0 & -\pi_0 I_n \\ I_n & 0 \end{pmatrix}.$$

Then the π-stability condition **LM1** on $\mathcal{F}$ means there exists an $n\times n$ matrix $P \in M_n(R)$ such that

$$\begin{pmatrix} 0 & -\pi_0 I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} X \\ I_n \end{pmatrix} = \begin{pmatrix} X \\ I_n \end{pmatrix} P.$$

We obtain $P = X$ and $X^2 + \pi_0 I_n = 0$.

4.2.2.2 Condition LM2

We have already shown that $\pi\otimes 1$ acts on $\mathcal{F}$ via right multiplication of X. Then as in the (R-U) case, the Kottwitz condition **LM2** translates to

$$\mathrm{tr}(X+\pi I_n) = \pi - \overline{\pi} = 2\pi,\ \mathrm{tr}\left(\wedge^i(X+\pi I_n)\right) = 0,\ \text{for } i \geq 2. \tag{4.2.3}$$

4.2.2.3 Condition LM3

With respect to the bases (4.2.1) and (4.2.2), the perfect pairing

$$s(-,-) : (\Lambda_0 \otimes_{\mathcal{O}_{F_0}} R) \times (\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} R) \to R$$

and the map $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} R \to \frac{\pi}{2}\Lambda_0^s \otimes_{\mathcal{O}_{F_0}} R$ are represented respectively by the matrices

$$S = \begin{pmatrix} 0 & 0 & H_{2m} & 0 \\ 0 & 0 & 0 & 1 \\ -H_{2m} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \text{ and } N = \begin{pmatrix} I_{2m} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & I_{2m} & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Then the Condition **LM3** translates to $\begin{pmatrix} X \\ I_n \end{pmatrix}^t S \left(N \begin{pmatrix} X \\ I_n \end{pmatrix} \right) = 0$, or equivalently,

$$\begin{pmatrix} X \\ I_n \end{pmatrix}^t \begin{pmatrix} 0 & 0 & H_{2m} & 0 \\ 0 & 0 & 0 & 2 \\ -H_{2m} & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ I_n \end{pmatrix} = 0. \tag{4.2.4}$$

Write

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & x \end{pmatrix},$$

where $X_1 \in M_{2m}(R)$, $X_2 \in M_{2m\times 1}(R)$, $X_3 \in M_{1\times 2m}(R)$ and $x \in R$. Then (4.2.4) translates to

$$\begin{pmatrix} X_1^t H_{2m} - H_{2m} X_1 & 2X_3^t - H_{2m} X_2 \\ X_2^t H_{2m} - 2X_3 & 0 \end{pmatrix} = 0.$$

4.2.2.4 Condition LM4

Recall $\mathscr{L} = \frac{1}{2}\mathcal{O}_{F_0}$. With respect to the basis (4.2.1), the induced $\mathscr{L} \otimes_{\mathcal{O}_{F_0}} R$-valued symmetric pairing on $\Lambda_0 \otimes_{\mathcal{O}_{F_0}} R$ is represented by the matrix

$$S_1 = \begin{pmatrix} H_{2m} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \pi_0 H_{2m} & 0 \\ 0 & 0 & 0 & 2\pi_0 \end{pmatrix}.$$

The Condition **LM4** translates to

$$\begin{pmatrix} X \\ I_n \end{pmatrix}^t S_1 \begin{pmatrix} X \\ I_n \end{pmatrix} = 0 \text{ and half of the diagonal of } \begin{pmatrix} X \\ I_n \end{pmatrix}^t S_1 \begin{pmatrix} X \\ I_n \end{pmatrix} \text{ equals zero.}$$

One can check that the diagonal entries of $\left(\begin{smallmatrix} X \\ I_n \end{smallmatrix}\right)^t S_1 \left(\begin{smallmatrix} X \\ I_n \end{smallmatrix}\right)$ are indeed divisible by 2 in R. Equivalently, we obtain

$$\begin{pmatrix} X_1^t H_{2m} X_1 + 2X_3^t X_3 + \pi_0 H_{2m} & X_1^t H_{2m} X_2 + 2xX_3^t \\ X_2^t H_{2m} X_1 + 2xX_3 & X_2^t H_{2m} X_2 + 2x^2 + 2\pi_0 \end{pmatrix} = 0,$$

half of the diagonal of $X_1^t H_{2m} X_1 + 2X_3^t X_3 + \pi_0 H_{2m}$ equals 0,

$$\frac{1}{2}\left(X_2^t H_{2m} X_2 + 2x^2 + 2\pi_0\right) = 0.$$

4.2.2.5 Condition LM5

As $\pi \otimes 1$ acts as right multiplication by X on $\mathcal{F}$, the wedge condition on $\mathcal{F}$ translates to

$$\wedge^2(X + \pi I_n) = 0.$$

4.2.2.6 A simplification of equations

As in the (R-U) case, we can simplify the above equations and obtain the following proposition.

Proposition 4.2.2. *The scheme* $\mathrm{U}_{\{0\}} = \operatorname{Spec} \mathcal{O}_F[X]/\mathcal{I}$, *where* $\mathcal{I}$ *is the ideal generated by:*

$$\mathrm{tr}(X + \pi I_n) - 2\pi,\ \wedge^2(X + \pi I_n),\ X_1^t H_{2m} - H_{2m} X_1,\ 2X_3^t - H_{2m} X_2,$$

$$(\mathrm{tr}(X_1 + \pi I_{2m}) - 2\pi) H_{2m}(X_1 + \pi I_{2m}) + 2X_3^t X_3,$$

half of the diagonal of $(\mathrm{tr}(X_1 + \pi I_{2m}) - 2\pi) H_{2m}(X_1 + \pi I_{2m}) + 2X_3^t X_3$.

Set

$$\widetilde{X}_1 := X_1 + \pi I_{2m},\ \widetilde{X} := \begin{pmatrix} \widetilde{X}_1 \\ X_3 \end{pmatrix}.$$

Then we have the following proposition.

Proposition 4.2.3. *The scheme* $\mathrm{U}_{\{0\}}$ *is isomorphic to* $\operatorname{Spec} \mathcal{O}_F[\widetilde{X}]/\widetilde{\mathcal{I}}$, *where* $\widetilde{\mathcal{I}}$ *is the ideal in* $\mathcal{O}_F[\widetilde{X}]$ *generated by:*

$$\wedge^2(\widetilde{X}),\ H_{2m}\widetilde{X}_1 - \widetilde{X}_1^t H_{2m},\ (\operatorname{tr}(\widetilde{X}_1) - 2\pi)H_{2m}\widetilde{X}_1 + 2X_3^t X_3,$$

$$\textit{half of the diagonal of } (\operatorname{tr}(\widetilde{X}_1) - 2\pi)H_{2m}\widetilde{X}_1 + 2X_3^t X_3.$$

Definition 4.2.4. Denote by $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ the closed subscheme of $\mathrm{U}_{\{0\}} = \operatorname{Spec} \mathcal{O}_F[\widetilde{X}]/\widetilde{\mathcal{I}}$ defined by the ideal $\widetilde{\mathcal{I}}^{\mathrm{fl}} \subset \mathcal{O}_F[\widetilde{X}]$ generated by:

$$\wedge^2(\widetilde{X}),\ H_{2m}\widetilde{X}_1 - \widetilde{X}_1^t H_{2m},\ (\frac{1}{2}\operatorname{tr}(\widetilde{X}_1) - \pi)H_{2m}\widetilde{X}_1 + X_3^t X_3.$$

Note that $\operatorname{tr}(\widetilde{X}_1)$ is divisible by 2 by the relation $H_{2m}\widetilde{X}_1 = \widetilde{X}_1^t H_{2m}$.

4.2.2.7 Global results

As in the (R-U) case, we can prove the following proposition.

Proposition 4.2.5. *(1) The scheme* $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ *is flat over* $\mathcal{O}_F$ *of relative dimension* $n-1$. *In particular,* $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ *is isomorphic to an open subscheme of* $\mathrm{M}^{\mathrm{loc}}_{\{0\}}$ *containing the worst point. Furthermore,* $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ *is normal, Cohen-Macaulay, and smooth over* $\mathcal{O}_F$ *on the complement of the worst point. The special fiber* $\mathrm{U}^{\mathrm{fl}}_{\{0\}} \otimes_{\mathcal{O}_F} k$ *is (geometrically) reduced and irreducible.*

(2) $\mathrm{U}_{\{0\}}$ *and* $\mathrm{U}^{\mathrm{fl}}_{\{0\}}$ *have the same underlying topological space.*

Similar arguments as in the proof of Lemma 4.1.22 imply that the special fiber $\mathrm{M}_{\{0\}} \otimes_{\mathcal{O}_F} \overline{k}$ has only two $\mathscr{H}_{\{0\}}(\overline{k})$-orbits. Together with Proposition 4.2.5, we can deduce Theorem 1.2.2 and 1.2.3 in the case $I = \{0\}$ and (R-P).

CHAPTER 5

THE CASE $I = \{m\}$

5.1 The case $I = \{m\}$ and (R-U)

In this section, we will prove Theorem 1.2.2 in the case when F/F_0 is of (R-U) type and $I = \{m\}$. In particular, we have

$$\pi^2 - t\pi + \pi_0 = 0,$$

where $t \in \mathcal{O}_{F_0}$ with $\pi_0 | t | 2$. Consider the following ordered $\mathcal{O}_{F_0}$-basis of Λ_m and Λ_m^s:

$$\Lambda_m : \frac{\overline{\pi}}{t} e_{m+2}, \dots, \frac{\overline{\pi}}{t} e_n, \pi^{-1} e_1, \dots, \pi^{-1} e_m, e_{m+1}, \frac{\pi_0}{t} e_{m+2}, \dots, \frac{\pi_0}{t} e_n, e_1, \dots, e_m, \pi e_{m+1}, \quad (5.1.1)$$

$$\Lambda_m^s : \overline{\pi} e_{m+2}, \dots, \overline{\pi} e_n, \frac{t}{\pi} e_1, \dots, \frac{t}{\pi} e_m, e_{m+1}, \pi_0 e_{m+2}, \dots, \pi_0 e_n, t e_1, \dots, t e_m, \pi e_{m+1}. \quad (5.1.2)$$

Recall $(\Lambda_m, q, \mathscr{L})$ is a hermitian quadratic module for $\mathscr{L} = t^{-1}\mathcal{O}_{F_0}$.

5.1.1 A refinement of $\mathrm{M}^{\mathrm{naive}}_{\{m\}}$ in the (R-U) case

Definition 5.1.1. Let $\mathrm{M}_{\{m\}}$ be the functor

$$\mathrm{M}_{\{m\}} : (\mathrm{Sch}/\mathcal{O}_F)^{\mathrm{op}} \longrightarrow \mathrm{Sets}$$

which sends an $\mathcal{O}_F$-scheme S to the set of $\mathcal{O}_S$-modules $\mathcal{F}$ such that

LM1 (π-stability condition) $\mathcal{F}$ is an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodule of $\Lambda_m \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ and as an $\mathcal{O}_S$-module, it is a locally direct summand of rank n.

LM2 (Kottwitz condition) The action of $\pi \otimes 1 \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ on $\mathcal{F}$ has characteristic polynomial

$$\det(T - \pi \otimes 1 \mid \mathcal{F}) = (T - \pi)(T - \overline{\pi})^{n-1}.$$

LM3 Let $\mathcal{F}^\perp$ be the orthogonal complement in $\Lambda_m^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ of $\mathcal{F}$ with respect to the perfect pairing

$$s(-,-) : (\Lambda_m \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \times (\Lambda_m^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \to \mathcal{O}_S.$$

We require that the map $\Lambda_m \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to (t^{-1}\Lambda_m^s) \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ induced by the inclusion $\Lambda_m \hookrightarrow t^{-1}\Lambda_m^s$ sends $\mathcal{F}$ to $t^{-1}\mathcal{F}^\perp$, where $t^{-1}\mathcal{F}^\perp$ is the image of $\mathcal{F}^\perp$ under the isomorphism $t^{-1} : \Lambda_m^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \xrightarrow{\sim} t^{-1}\Lambda_m^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$.

LM4 (Hyperbolicity condition) The quadratic form $q : \Lambda_m \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to \mathscr{L} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ induced by $q : \Lambda_m \to \mathscr{L}$ satisfies $q(\mathcal{F}) = 0$.

LM5 (Wedge condition) The action of $\pi \otimes 1 - 1 \otimes \overline{\pi} \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ on $\mathcal{F}$ satisfies

$$\wedge^2(\pi \otimes 1 - 1 \otimes \overline{\pi} \mid \mathcal{F}) = 0.$$

Then $\mathrm{M}_{\{m\}}$ is representable and we have closed immersions

$$\mathrm{M}_{\{m\}}^{\mathrm{loc}} \subset \mathrm{M}_{\{m\}} \subset \mathrm{M}_{\{m\}}^{\mathrm{naive}}$$

of projective schemes over $\mathcal{O}_F$, where all schemes have the same generic fiber.

5.1.2 An affine chart $\mathrm{U}_{\{m\}}$ around the worst point

Set

$$\mathcal{F}_0 := (\pi \otimes 1)(\Lambda_m \otimes_{\mathcal{O}_{F_0}} k).$$

Then we can check that $\mathcal{F}_0 \in \mathrm{M}_{\{m\}}(k)$. We call it the worst point of $\mathrm{M}_{\{m\}}$.

With respect to the basis (5.1.1), the standard affine chart around $\mathcal{F}_0$ in $\mathrm{Gr}(n, \Lambda_m)_{\mathcal{O}_F}$ is the $\mathcal{O}_F$-scheme of $2n \times n$ matrices $\left(\begin{smallmatrix} X \\ I_n \end{smallmatrix}\right)$. We denote by $\mathrm{U}_{\{m\}}$ the intersection of $\mathrm{M}_{\{m\}}$ with the standard affine chart in $\mathrm{Gr}(n, \Lambda_m)_{\mathcal{O}_F}$. The worst point $\mathcal{F}_0$ of $\mathrm{M}_{\{m\}}$ is contained in $\mathrm{U}_{\{m\}}$ and corresponds to the point defined by $X = 0$ and $\pi = 0$. The conditions **LM1-5** yield the defining equations for $\mathrm{U}_{\{m\}}$. We will analyze each condition as in the case $I = \{0\}$. A reader who is only interested in the affine coordinate ring of $\mathrm{U}_{\{m\}}$ may proceed directly to Proposition 5.1.2.

5.1.2.1 Condition LM1

Let R be an $\mathcal{O}_F$-algebra. With respect to the basis (5.1.1), the operator $\pi \otimes 1$ acts on $\Lambda_m \otimes_{\mathcal{O}_{F_0}} R$ via the matrix

$$\begin{pmatrix} 0 & -\pi_0 I_n \\ I_n & tI_n \end{pmatrix}.$$

Then the π-stability condition **LM1** on $\mathcal{F}$ means there exists an $n \times n$ matrix $P \in M_n(R)$ such that

$$\begin{pmatrix} 0 & -\pi_0 I_n \\ I_n & tI_n \end{pmatrix} \begin{pmatrix} X \\ I_n \end{pmatrix} = \begin{pmatrix} X \\ I_n \end{pmatrix} P.$$

We obtain $P = X + tI_n$ and $X^2 + tX + \pi_0 I_n = 0$.

5.1.2.2 Condition LM2

We have already shown that $\pi \otimes 1$ acts on $\mathcal{F}$ via right multiplication of $X + tI_n$. Then the Kottwitz condition **LM2** translates to

$$\operatorname{tr}(X + \pi I_n) = \pi - \overline{\pi},\ \operatorname{tr}\left(\wedge^i (X + \pi I_n)\right) = 0,\ \text{for } i \geq 2. \tag{5.1.3}$$

5.1.2.3 Condition LM3

With respect to the bases (5.1.1) and (5.1.2), the perfect pairing

$$s(-,-) : (\Lambda_m \otimes_{\mathcal{O}_{F_0}} R) \times (\Lambda_m^s \otimes_{\mathcal{O}_{F_0}} R) \to R$$

and the map $\Lambda_m \otimes_{\mathcal{O}_{F_0}} R \to \frac{1}{t}\Lambda_m^s \otimes_{\mathcal{O}_{F_0}} R$ are represented respectively by the matrices

$$S = \begin{pmatrix} \frac{2}{t}H_{2m} & 0 & H_{2m} & 0 \\ 0 & \frac{2}{t} & 0 & 1 \\ H_{2m} & 0 & \frac{2\pi_0}{t}H_{2m} & 0 \\ 0 & 1 & 0 & \frac{2\pi_0}{t} \end{pmatrix} \text{ and } N = \begin{pmatrix} I_{2m} & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & I_{2m} & 0 \\ 0 & 0 & 0 & t \end{pmatrix}.$$

Then the Condition **LM3** translates to $\begin{pmatrix} X \\ I_n \end{pmatrix}^t S \left(N \begin{pmatrix} X \\ I_n \end{pmatrix} \right) = 0$, or equivalently,

$$\begin{pmatrix} X \\ I_n \end{pmatrix}^t \begin{pmatrix} \frac{2}{t}H_{2m} & 0 & H_{2m} & 0 \\ 0 & 2 & 0 & t \\ H_{2m} & 0 & \frac{2\pi_0}{t}H_{2m} & 0 \\ 0 & t & 0 & 2\pi_0 \end{pmatrix} \begin{pmatrix} X \\ I_n \end{pmatrix} = 0. \tag{5.1.4}$$

It amounts to the following equation.

$$(\frac{2}{t}X^t + I_n) \begin{pmatrix} H_{2m} & 0 \\ 0 & t \end{pmatrix} X + X^t \begin{pmatrix} H_{2m} & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} \frac{2\pi_0}{t}H_{2m} & 0 \\ 0 & 2\pi_0 \end{pmatrix} = 0. \tag{5.1.5}$$

Note that the π-stability condition **LM1** on $\mathcal{F}$ implies

$$\frac{2}{t}(X^t)^2 + 2X^t + \frac{2\pi_0}{t}I_n = 0, \text{ and hence } (\frac{2}{t}X^t + I_n)^2 = (1 - \frac{4\pi_0}{t^2})I_n = \theta I_n.$$

Multiplying $\frac{2}{t}X^t + I_n$ on both sides of (5.1.5), we can obtain

$$\begin{pmatrix} H_{2m} & 0 \\ 0 & t \end{pmatrix} X = X^t \begin{pmatrix} H_{2m} & 0 \\ 0 & t \end{pmatrix}.$$

Write

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & x \end{pmatrix},$$

where $X_1 \in M_{2m}(R)$, $X_2 \in M_{2m\times 1}(R)$, $X_3 \in M_{1\times 2m}(R)$ and $x \in R$. Equivalently, we obtain

$$H_{2m}X_1 = X_1^t H_{2m}, \ H_{2m}X_2 = tX_3^t.$$

5.1.2.4 Condition LM4

Recall $\mathscr{L} = \frac{1}{t}\mathcal{O}_{F_0}$. With respect to the basis (5.1.1), the induced $\mathscr{L} \otimes_{\mathcal{O}_{F_0}} R$-valued symmetric pairing on $\Lambda_m \otimes_{\mathcal{O}_{F_0}} R$ is represented by the matrix

$$S_1 = \begin{pmatrix} \frac{2}{t}H_{2m} & 0 & H_{2m} & 0 \\ 0 & 2 & 0 & t \\ H_{2m} & 0 & \frac{2\pi_0}{t}H_{2m} & 0 \\ 0 & t & 0 & 2\pi_0 \end{pmatrix}. \tag{5.1.6}$$

The Condition **LM4** translates to

$$\begin{pmatrix} X \\ I_n \end{pmatrix}^t S_1 \begin{pmatrix} X \\ I_n \end{pmatrix} = 0 \text{ and half of the diagonal of } \begin{pmatrix} X \\ I_n \end{pmatrix}^t S_1 \begin{pmatrix} X \\ I_n \end{pmatrix} \text{ equals zero.}$$

Equivalently, we obtain

$$\begin{pmatrix} \frac{2}{t}X_1^tH_{2m}X_1 + 2X_3^tX_3 + H_{2m}X_1 + X_1^tH_{2m} + \frac{2\pi_0}{t}H_{2m} & \frac{2}{t}X_1^tH_{2m}X_3 + 2xX_3^t + H_{2m}X_2 + tX_3^t \\ \frac{2}{t}X_2^tH_{2m}X_1 + 2xX_3 + tX_3 + X_2^tH_{2m} & \frac{2}{t}X_2^tH_{2m}X_2 + 2x^2 + 2tx + 2\pi_0 \end{pmatrix} = 0,$$

half of the diagonal of $\frac{2}{t}X_1^tH_{2m}X_1 + 2X_3^tX_3 + H_{2m}X_1 + X_1^tH_{2m} + \frac{2\pi_0}{t}H_{2m}$ equals 0,

$\frac{1}{2}(\frac{2}{t}X_2^tH_{2m}X_2 + 2x^2 + 2tx + 2\pi_0) = 0.$

5.1.2.5 Condition LM5

As $\pi \otimes 1$ acts as right multiplication by $X + tI_n$ on $\mathcal{F}$, the wedge condition **LM5** on $\mathcal{F}$ translates to

$$\wedge^2(X + \pi I_n) = 0.$$

5.1.2.6 A simplification of equations

As in the case $I = \{0\}$, we can simplify the above equations and obtain the following.

Proposition 5.1.2. *The scheme* $\mathrm{U}_{\{m\}} = \operatorname{Spec} \mathcal{O}_F[X]/\mathcal{I}$, *where* $\mathcal{I}$ *is the ideal generated by:*

$$\mathrm{tr}(X + \pi I_n) - \pi + \overline{\pi},\ \wedge^2(X + \pi I_n),\ X_1^tH_{2m} - H_{2m}X_1,\ tX_3^t - H_{2m}X_2,$$

half of the diagonal of $(\frac{2}{t}\,\mathrm{tr}(X_1 + \pi I_{2m}) + 2\sqrt{\theta})H_{2m}(X_1 + \pi I_{2m}) + 2X_3^tX_3.$

Set

$$\widetilde{X}_1 := X_1 + \pi I_{2m},\ \widetilde{X} := \begin{pmatrix} \widetilde{X}_1 \\ X_3 \end{pmatrix}.$$

Then we have the following proposition.

Proposition 5.1.3. *The scheme* $\mathrm{U}_{\{m\}}$ *is isomorphic to* $\operatorname{Spec} \mathcal{O}_F[\widetilde{X}]/\widetilde{\mathcal{I}}$*, where* $\widetilde{\mathcal{I}}$ *is the ideal generated by*

$$\wedge^2 (\widetilde{X}),\ H_{2m}\widetilde{X}_1 - \widetilde{X}_1^t H_{2m},\ \textit{half of the diagonal of } (\frac{2}{t}\operatorname{tr}(\widetilde{X}_1) + 2\sqrt{\theta})H_{2m}\widetilde{X}_1 + 2X_3^t X_3.$$

Definition 5.1.4. Denote by $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ the closed subscheme of $\mathrm{U}_{\{m\}} = \operatorname{Spec} \mathcal{O}_F[\widetilde{X}]/\widetilde{\mathcal{I}}$ defined by the ideal $\widetilde{\mathcal{I}}^{\mathrm{fl}} \subset \mathcal{O}_F[\widetilde{X}]$ generated by

$$\wedge^2 (\widetilde{X}),\ H_{2m}\widetilde{X}_1 - \widetilde{X}_1^t H_{2m},\ (\frac{\operatorname{tr}(\widetilde{X}_1)}{t} + \sqrt{\theta})H_{2m}\widetilde{X}_1 + X_3^t X_3.$$

Note that $\widetilde{\mathcal{I}} \subset \widetilde{\mathcal{I}}^{\mathrm{fl}}$.

5.1.2.7 Global results

We first give the results for the schemes $\mathrm{U}_{\{m\}}$ and $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$.

Proposition 5.1.5. *(1)* $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ *is smooth over* $\mathcal{O}_F$ *of relative dimension* $n-1$*. The special fiber is geometrically reduced and irreducible.*

(2) $\mathrm{U}_{\{m\}}$ *and* $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ *have the same underlying topological space.*

Proof. The proof of (2) is similar as that of Lemma 4.1.19. Now we prove the smoothness of $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$. We use the notation as in the proof of Lemma 4.1.15. In particular,

$$\mathcal{R}^{\mathrm{fl}} = \frac{\mathcal{O}_F[\left(\begin{smallmatrix} Y \\ X_3 \end{smallmatrix}\right)]}{\left(\wedge^2\left(\begin{smallmatrix} Y \\ X_3 \end{smallmatrix}\right), Y - Y^t, (\frac{1}{t}\operatorname{tr}(H_{2m}Y) + \sqrt{\theta})Y + X_3^t X_3\right)}.$$

Then one can similarly show that $D(y_{\ell\ell})$ for $1 \leq \ell \leq 2m$ is smooth over $\mathcal{O}_F$. Let $z := \frac{1}{t}\operatorname{tr}(H_{2m}Y) + \sqrt{\theta}$. Consider the principal open subscheme $D(z) = \operatorname{Spec} \mathcal{R}^{\mathrm{fl}}[z^{-1}]$. Then we have in $\mathcal{R}^{\mathrm{fl}}[z^{-1}]$ that

$$Y = -z^{-1} X_3^t X_3.$$

Thus, Y is determined by X_3 and $\mathcal{R}^{\mathrm{fl}}[z^{-1}] \simeq \mathcal{O}_F[X_3]$ is smooth over $\mathcal{O}_F$. Note that the scheme $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ is covered by $D(z)$ and $D(y_{\ell\ell})$ for $1 \leq \ell \leq 2m$. Hence, we conclude that $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ is smooth over $\mathcal{O}_F$. The special fiber is geometrically reduced by the smoothness. It is geometrically irreducible because the geometric special fibers of $D(z)$ and $D(y_{\ell\ell})$ for $1 \leq \ell \leq 2m$ are irreducible. $\square$

Recall $(\Lambda_m, q, \mathscr{L})$ is a hermitian quadratic module over $\mathcal{O}_{F_0}$ for $\mathscr{L} = \frac{1}{t}\mathcal{O}_{F_0}$. Let

$$\mathscr{H}_{\{m\}} \coloneqq \underline{\mathrm{Sim}}((\Lambda_m, q, \mathscr{L}))$$

be the group scheme over $\mathcal{O}_{F_0}$ of similitude automorphisms of $(\Lambda_m, q, \mathscr{L})$. By Theorem 6.1.13, $\mathscr{H}_{\{m\}}$ is an affine smooth group scheme over $\mathcal{O}_{F_0}$. As in Lemma 4.1.21, the group scheme $\mathscr{H}_{\{m\}}$ acts on $\mathrm{M}_{\{m\}}$.

Lemma 5.1.6. *Let $\overline{k}$ be the algebraic closure of the residue field k. Then $\mathrm{M}_{\{m\}} \otimes_{\mathcal{O}_F} \overline{k}$ has two $\mathscr{H}_{\{m\}} \otimes_{\mathcal{O}_{F_0}} \overline{k}$-orbits, one of which consists of the worst point.*

Proof. Let $\mathcal{F} \in \mathrm{M}_{\{m\}}(\overline{k})$. In particular, the subspace $\mathcal{F} \subset (\Lambda_m \otimes_{\mathcal{O}_{F_0}} \overline{k})$ is an n-dimensional $\overline{k}$-vector space. The wedge condition **LM5** in this case becomes $\wedge^2(\pi \otimes 1 \mid \mathcal{F}) = 0$. Therefore, the image $(\pi \otimes 1)\mathcal{F}$ is at most one dimensional. We have the following two cases.

Suppose $(\pi \otimes 1)\mathcal{F} = 0$. Then $\mathcal{F} = (\pi \otimes 1)(\Lambda_m \otimes_{\mathcal{O}_{F_0}} \overline{k})$, namely, $\mathcal{F}$ is the worst point.

Suppose $(\pi \otimes 1)\mathcal{F}$ is one-dimensional. Then there exists a vector $v \in \mathcal{F}$ such that $(\pi \otimes 1)v$ generates $(\pi \otimes 1)\mathcal{F}$. For simplicity, write π for $\pi \otimes 1$. Let $f : (\Lambda_m \otimes_{\mathcal{O}_{F_0}} \overline{k}) \times (\Lambda_m \otimes_{\mathcal{O}_{F_0}} \overline{k}) \to \mathscr{L} \simeq \overline{k}$ denote the associated symmetric pairing on $\Lambda_m \otimes_{\mathcal{O}_{F_0}} \overline{k}$. As in the proof of Lemma 4.1.22, we see that πv is not in the radical of the paring f, because $q(v) = 0$. Then we can find some $w \in \Lambda_m \otimes_{\mathcal{O}_{F_0}} \overline{k}$ such that $f(w, \pi v) \neq 0$ in $\overline{k}$. By rescaling, we may assume that $f(w, \pi v) = 1$. Similar arguments in Lemma 4.1.22 imply that after some linear transformations, we may assume

$$q(w) = q(v) = f(w, v) = 0 \text{ and } f(w, \pi v) = 1.$$

Let $W_1 \coloneqq \langle v, \pi v, w, \pi w \rangle$. Then f restricts to a perfect symmetric pairing on W_1. Now we can write

$$\Lambda_m \otimes_{\mathcal{O}_{F_0}} \overline{k} = W_1 \oplus W, \tag{5.1.7}$$

where W is the orthogonal complement of W_1 with respect to f whose dimension is $2n-4$ over $\overline{k}$. Since $q(\mathcal{F}) = 0$, we have $\mathcal{F} \cap \langle w, \pi w \rangle = 0$. Hence, we obtain that the $\overline{k}$-dimension of $\mathcal{F} \cap W$ is $n-2$ and $\mathcal{F} \cap W \subset \pi W = \ker(\pi \mid W)$. Therefore, $\mathcal{F} \cap W = \pi W$ for dimension reasons. Note that the space W carries a structure of hermitian quadratic module. By (5.1.7), we have

$$\mathrm{disc}'(q) = \mathrm{disc}(q|_{W_1})\mathrm{disc}'(q|_W).$$

Here $\mathrm{disc}'(q)$ is the divided discriminant in the sense of Definition 6.1.8, and we view it as an element in $\overline{k}$ by using a basis of $\Lambda_m \otimes_{\mathcal{O}_{F_0}} \overline{k}$. By Example 6.1.10, we have $\mathrm{disc}'(q) \in \overline{k}^\times$. Since ϕ is perfect on W_1, we obtain that $\mathrm{disc}(\phi|_{W_1}) \in \overline{k}^\times$, and hence $\mathrm{disc}'(q|_W) \in \overline{k}^\times$. In particular, W is a hermitian quadratic module of type Λ_m over $\overline{k}$ in the sense of Definition 6.1.8. Applying Theorem 6.1.12 to W and using similar arguments as in the proof of Lemma 4.1.22, we can conclude that points $\mathcal{F} \in \mathrm{M}_{\{m\}}(\overline{k})$ with $\dim_{\overline{k}} \pi\mathcal{F} = 1$ are in the same orbit under the action of $\mathscr{H}_{\{m\}} \otimes_{\mathcal{O}_{F_0}} \overline{k}$. □

As $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ is flat over $\mathcal{O}_F$, we may view $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ as an open subscheme of $\mathrm{M}^{\mathrm{loc}}_{\{m\}}$ containing the worst point. By Lemma 5.1.6, the $\mathscr{H}_{\{m\}}$-translation of $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ covers $\mathrm{M}^{\mathrm{loc}}_{\{m\}}$. Together with Proposition 5.1.5, we have proven Theorem 1.2.2 and 1.2.3 in the case $I = \{m\}$ and (R-U).

5.2 The case $I = \{m\}$ and (R-P)

In this section, we consider the case when F/F_0 is of (R-P) type and $I = \{m\}$. In particular, we have

$$\pi^2 + \pi_0 = 0 \text{ and } \pi + \overline{\pi} = 0.$$

Consider the following ordered $\mathcal{O}_{F_0}$-basis of Λ_m and Λ_m^s:

$$\Lambda_m : \frac{1}{2}e_{m+2}, \ldots, \frac{1}{2}e_n, \pi^{-1}e_1, \ldots, \pi^{-1}e_m, e_{m+1}, \frac{\pi}{2}e_{m+2}, \ldots, \frac{\pi}{2}e_n, e_1, \ldots, e_m, \pi e_{m+1}, \tag{5.2.1}$$

$$\Lambda_m^s : e_{m+2}, \dots, e_n, \frac{2}{\pi}e_1, \dots, \frac{2}{\pi}e_m, \pi^{-1}e_{m+1}, \pi e_{m+2}, \dots, \pi e_n, 2e_1, \dots, 2e_m, e_{m+1}. \tag{5.2.2}$$

Recall $(\Lambda_m, q, \mathscr{L})$ is a hermitian quadratic module for $\mathscr{L} = 2^{-1}\mathcal{O}_{F_0}$.

5.2.1 A refinement of $\mathrm{M}_{\{m\}}^{\mathrm{naive}}$ in the (R-P) case

Definition 5.2.1. Let $\mathrm{M}_{\{m\}}$ be the functor

$$\mathrm{M}_{\{m\}} : (\mathrm{Sch}/\mathcal{O}_F)^{\mathrm{op}} \longrightarrow \mathrm{Sets}$$

which sends an $\mathcal{O}_F$-scheme S to the set of $\mathcal{O}_S$-modules $\mathcal{F}$ such that

LM1 (π-stability condition) $\mathcal{F}$ is an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodule of $\Lambda_m \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ and as an $\mathcal{O}_S$-module, it is a locally direct summand of rank n.

LM2 (Kottwitz condition) The action of $\pi \otimes 1 \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ on $\mathcal{F}$ has characteristic polynomial

$$\det(T - \pi \otimes 1 \mid \mathcal{F}) = (T - \pi)(T - \overline{\pi})^{n-1}.$$

LM3 Let $\mathcal{F}^\perp$ be the orthogonal complement in $\Lambda_m^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ of $\mathcal{F}$ with respect to the perfect pairing

$$s(-,-) : (\Lambda_m \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \times (\Lambda_m^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \to \mathcal{O}_S.$$

We require the map $\Lambda_m \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to (2^{-1}\Lambda_m^s) \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ induced by $\Lambda_m \hookrightarrow 2^{-1}\Lambda_m^s$ sends $\mathcal{F}$ to $2^{-1}\mathcal{F}^\perp$, where $2^{-1}\mathcal{F}^\perp$ denotes the image of $\mathcal{F}^\perp$ under the isomorphism $2^{-1} : \Lambda_m^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \xrightarrow{\sim} 2^{-1}\Lambda_m^s \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$.

LM4 (Hyperbolicity condition) The quadratic form $q : \Lambda_m \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to \mathscr{L} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ induced by $q : \Lambda_m \to \mathscr{L}$ satisfies $q(\mathcal{F}) = 0$.

LM5 (Wedge condition) The action of $\pi \otimes 1 - 1 \otimes \overline{\pi} \in \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ satisfies

$$\wedge^2(\pi \otimes 1 - 1 \otimes \overline{\pi} \mid \mathcal{F}) = 0.$$

LM6 (Strengthened spin condition) The line $\wedge^n \mathcal{F} \subset W(\Lambda_m) \otimes_{\mathcal{O}_F} \mathcal{O}_S$ is contained in

$$\mathrm{Im}\left(W(\Lambda_m)_{-1}^{n-1,1} \otimes_{\mathcal{O}_F} \mathcal{O}_S \to W(\Lambda_m) \otimes_{\mathcal{O}_F} \mathcal{O}_S\right).$$

Here we use similar notations as in §4.1.1.1.

Then $\mathrm{M}_{\{m\}}$ is representable and we have closed immersions

$$\mathrm{M}^{\mathrm{loc}}_{\{m\}} \subset \mathrm{M}_{\{m\}} \subset \mathrm{M}^{\mathrm{naive}}_{\{m\}}$$

of projective schemes over $\mathcal{O}_F$, where all schemes have the same generic fiber.

5.2.2 An affine chart $\mathrm{U}_{\{m\}}$ around the worst point

Set

$$\mathcal{F}_0 := (\pi \otimes 1)(\Lambda_m \otimes_{\mathcal{O}_{F_0}} k).$$

Then we can check that $\mathcal{F}_0 \in \mathrm{M}_{\{m\}}(k)$. We call it the worst point of $\mathrm{M}_{\{m\}}$.

With respect to the basis (5.1.1), the standard affine chart around $\mathcal{F}_0$ in $\mathrm{Gr}(n, \Lambda_m)_{\mathcal{O}_F}$ is the $\mathcal{O}_F$-scheme of $2n \times n$ matrices $\left(\begin{smallmatrix} X \\ I_n \end{smallmatrix}\right)$. We denote by $\mathrm{U}_{\{m\}}$ the intersection of $\mathrm{M}_{\{m\}}$ with the standard affine chart in $\mathrm{Gr}(n, \Lambda_m)_{\mathcal{O}_F}$. The worst point $\mathcal{F}_0$ of $\mathrm{M}_{\{m\}}$ is contained in $\mathrm{U}_{\{m\}}$ and corresponds to the closed point defined by $X = 0$ and $\pi = 0$. The conditions **LM1-6** yield the defining equations for $\mathrm{U}_{\{m\}}$. We will analyze each condition as in the (R-U) case. A reader who is only interested in the affine coordinate ring of $\mathrm{U}_{\{m\}}$ may proceed directly to Proposition 5.2.2.

5.2.2.1 Condition LM1

Let R be an $\mathcal{O}_F$-algebra. With respect to the basis (5.2.1), the operator $\pi \otimes 1$ acts on $\Lambda_m \otimes_{\mathcal{O}_{F_0}} R$ via the matrix

$$\begin{pmatrix} 0 & -\pi_0 I_n \\ I_n & 0 \end{pmatrix}.$$

Then the π-stability condition **LM1** on $\mathcal{F}$ means there exists an $n \times n$ matrix $P \in M_n(R)$ such that

$$\begin{pmatrix} 0 & -\pi_0 I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} X \\ I_n \end{pmatrix} = \begin{pmatrix} X \\ I_n \end{pmatrix} P.$$

We obtain $P = X$ and $X^2 + \pi_0 I_n = 0$.

5.2.2.2 Condition LM2

We have already shown that $\pi \otimes 1$ acts on $\mathcal{F}$ via right multiplication by X. Then the Kottwitz condition **LM2** translates to

$$\operatorname{tr}(X + \pi I_n) = \pi - \overline{\pi} = 2\pi, \ \operatorname{tr}\left(\wedge^i (X + \pi I_n)\right) = 0, \ \text{for } i \geq 2. \tag{5.2.3}$$

5.2.2.3 Condition LM3

With respect to the bases (5.2.1) and (5.2.2), the perfect pairing

$$s(-,-) : (\Lambda_m \otimes_{\mathcal{O}_{F_0}} R) \times (\Lambda_m^s \otimes_{\mathcal{O}_{F_0}} R) \to R$$

and the map $\Lambda_m \otimes_{\mathcal{O}_{F_0}} \to \frac{1}{2}\Lambda_m^s \otimes_{\mathcal{O}_{F_0}} R$ are represented respectively by the matrices

$$S = \begin{pmatrix} 0 & 0 & J_{2m} & 0 \\ 0 & 0 & 0 & 1 \\ -J_{2m} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \text{ and } N = \begin{pmatrix} I_{2m} & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\pi_0 \\ 0 & 0 & I_{2m} & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix},$$

where $J_{2m} := \begin{pmatrix} 0 & H_m \\ -H_m & 0 \end{pmatrix}$.

Then the Condition **LM3** translates to $\begin{pmatrix} X \\ I_n \end{pmatrix}^t S \left(N \begin{pmatrix} X \\ I_n \end{pmatrix} \right) = 0$, or equivalently,

$$\begin{pmatrix} X \\ I_n \end{pmatrix}^t \begin{pmatrix} 0 & 0 & J_{2m} & 0 \\ 0 & 2 & 0 & 0 \\ -J_{2m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\pi_0 \end{pmatrix} \begin{pmatrix} X \\ I_n \end{pmatrix} = 0. \tag{5.2.4}$$

Write

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & x \end{pmatrix},$$

where $X_1 \in M_{2m}(R)$, $X_2 \in M_{2m\times 1}(R)$, $X_3 \in M_{1\times 2m}(R)$ and $x \in R$. The Equation (5.2.4) translates to

$$\begin{pmatrix} 2X_3^tX_3 + X_1^tJ_{2m} - J_{2m}X_1 & 2xX_3^t - J_{2m}X_2 \\ 2xX_3 + X_2^tJ_{2m} & 2x^2 + 2\pi_0 \end{pmatrix} = 0.$$

5.2.2.4 Condition LM4

Recall $\mathscr{L} = \frac{1}{2}\mathcal{O}_{F_0}$. With respect to the basis (5.2.1), the induced $\mathscr{L} \otimes_{\mathcal{O}_{F_0}} R$-valued symmetric pairing on $\Lambda_m \otimes_{\mathcal{O}_{F_0}} R$ is represented by the matrix

$$S_1 = \begin{pmatrix} 0 & 0 & J_{2m} & 0 \\ 0 & 2 & 0 & 0 \\ -J_{2m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\pi_0 \end{pmatrix}. \tag{5.2.5}$$

The Condition **LM4** translates to

$$\begin{pmatrix} X \\ I_n \end{pmatrix}^t S_1 \begin{pmatrix} X \\ I_n \end{pmatrix} = 0 \text{ and half of the diagonal of } \begin{pmatrix} X \\ I_n \end{pmatrix}^t S_1 \begin{pmatrix} X \\ I_n \end{pmatrix} \text{ equals zero.}$$

Equivalently, we obtain

$$\begin{pmatrix} 2X_3^tX_3 + X_1^tJ_{2m} - J_{2m}X_1 & 2xX_3^t - J_{2m}X_2 \\ 2xX_3 + X_2^tJ_{2m} & 2x^2 + 2\pi_0 \end{pmatrix} = 0,$$

$$x^2 + \pi_0 = 0,$$

half of the diagonal of $2X_3^tX_3 + X_1^tJ_{2m} - J_{2m}X_1$ equals zero.

5.2.2.5 Condition LM5

Since $\pi \otimes 1$ acts as right multiplication by X on $\mathcal{F}$, the wedge condition **LM5** on $\mathcal{F}$ translates to

$$\wedge^2(X + \pi I_n) = 0.$$

5.2.2.6 Condition LM6

As in §4.1.2.6, the strengthened spin condition **LM6** in this case implies that

$$X_1 = J_{2m} X_1^t J_{2m}, \ 2\pi X_3^t = J_{2m} X_2.$$

5.2.2.7 A simplification of equations

As in the case $I = \{0\}$, we can simplify the above equations and obtain the following.

Proposition 5.2.2. *The scheme* $\mathrm{U}_{\{m\}}$ *is a closed subscheme of* $\mathrm{U}'_{\{m\}} \coloneqq \operatorname{Spec} \mathcal{O}_F[X]/\mathcal{I}$, *where* $\mathcal{I}$ *is the ideal generated by:*

$$\operatorname{tr}(X + \pi I_n) - 2\pi, \ \wedge^2(X + \pi I_n), \ X_1^t J_{2m} + J_{2m} X_1, \ 2\pi X_3^t - J_{2m} X_2,$$
$$\textit{half of the diagonal of } 2X_3^t X_3 + X_1^t J_{2m} - J_{2m} X_1.$$

Set

$$\widetilde{X}_1 \coloneqq X_1 + \pi I_{2m}, \ \widetilde{X} \coloneqq \begin{pmatrix} \widetilde{X}_1 \\ X_3 \end{pmatrix}.$$

As X_2 and x are determined by X_1 and X_3 by relations in $\mathcal{I}$, we obtain the following proposition.

Proposition 5.2.3. *The scheme* $\mathrm{U}'_{\{m\}}$ *is isomorphic to* $\operatorname{Spec} \mathcal{O}_F[\widetilde{X}]/\widetilde{\mathcal{I}}$, *where* $\widetilde{\mathcal{I}}$ *is the ideal generated by:*

$$\wedge^2 (\widetilde{X}), \ J_{2m} \widetilde{X}_1 + \widetilde{X}_1^t J_{2m}, \ \textit{half of the diagonal of } 2X_3^t X_3 + \widetilde{X}_1^t J_{2m} - J_{2m} \widetilde{X}_1.$$

Definition 5.2.4. Denote by $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ the closed subscheme of $\mathrm{U}'_{\{m\}} = \operatorname{Spec} \mathcal{O}_F[\widetilde{X}]/\widetilde{\mathcal{I}}$ defined by the ideal $\mathcal{I}^{\mathrm{fl}} \subset \mathcal{O}_F[\widetilde{X}]$ generated by:

$$\wedge^2 (\widetilde{X}), \ J_{2m} \widetilde{X}_1 + \widetilde{X}_1^t J_{2m}, \ X_3^t X_3 + \widetilde{X}_1^t J_{2m}.$$

Note that $\widetilde{\mathcal{I}} \subset \widetilde{\mathcal{I}}^{\mathrm{fl}}$.

5.2.2.8 Global results

We first give results for the schemes $\mathrm{U}_{\{m\}}$ and $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$.

Proposition 5.2.5. *(1) $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ is smooth over $\mathcal{O}_F$ of relative dimension $n-1$ with geometrically integral special fiber.*

(2) $\mathrm{U}_{\{m\}}$ and $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ have the same underlying topological space.

Proof. The proof of (2) is similar as that of Lemma 4.1.19. Now we prove the smoothness of $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$. It is clear from the expression of $\widetilde{\mathcal{I}}^{\mathrm{fl}}$ that $\widetilde{X}_1$ is determined by X_3, and hence,

$$\mathcal{O}_F[\widetilde{X}]/\widetilde{\mathcal{I}}^{\mathrm{fl}} \simeq \operatorname{Spec} \mathcal{O}_F[X_3] \simeq \mathbb{A}^{n-1}_{\mathcal{O}_F},$$

which is smooth over $\mathcal{O}_F$ of relative dimension $n-1$. The special fiber of $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ is isomorphic to $\mathbb{A}^{n-1}_k$, which is geometrically integral. □

As $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ is flat over $\mathcal{O}_F$, we may view $\mathrm{U}^{\mathrm{fl}}_{\{m\}}$ as an open subscheme of $\mathrm{M}^{\mathrm{loc}}_{\{m\}}$ containing the worst point. Then as in Lemma 5.1.6, we can show that the special fiber $\mathrm{M}_{\{m\}} \otimes_{\mathcal{O}_F} \overline{k}$ has only two orbits under the action of $\mathscr{H}_{\{m\}} \otimes_{\mathcal{O}_{F_0}} \overline{k}$. Together with Proposition 5.2.5, we deduce Theorem 1.2.2 and 1.2.3 in the case $I = \{m\}$ and (R-P).

CHAPTER 6

NORMAL FORMS OF HERMITIAN QUADRATIC MODULES

Let us keep the notations as in §3.3. In this chapter, we will show that, under certain conditions, hermitian quadratic modules étale locally have a normal form up to similitude. This is a variant of [RZ96, Theorem 3.16] in our setting. This result will be important when we relate the local models to Shimura varieties.

In the following, we let

$$\mathrm{Nilp} := \mathrm{Nilp}_{\mathcal{O}_{F_0}}$$

denote the category of noetherian[1] $\mathcal{O}_{F_0}$-algebras such that π_0 is nilpotent. We set

$$t := \pi + \overline{\pi}.$$

In particular, $t = 0$ if F/F_0 is of (R-P) type. For an $\mathcal{O}_{F_0}$-algebra R and $a \in \mathcal{O}_F$, we will simply use a to denote the element $a \otimes 1$ in $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$, if there is no confusion. For a hermitian quadratic module $(M, q, \mathscr{L})$, we will use f to denote the associated symmetric pairing on M, as in Definition 3.2.1.

6.1 Hermitian quadratic modules of type Λ_m

The results in this subsection are essentially contained in [Ans18, §9], with some modifications to the proof.

Lemma 6.1.1 (cf. [Ans18, Lemma 9.6]). *Let $R \in \mathrm{Nilp}$. Let (M, q, R) be an R-valued hermitian quadratic module over R. Assume there exist $v, w \in M$ such that $f(v, \pi w) = 1$ in R. Then there exist v', w' in the R-submodule spanned by $\{v, w, \pi v, \pi w\}$ such that*

$$q(v') = q(w') = f(v', w') = 0 \text{ and } f(v', \pi w') = 1.$$

Proof. For $r \in R$, we have

$$q(v + r\pi w) = q(v) + rf(v, \pi w) + r^2\pi_0 q(w) = (\pi_0 q(w))r^2 + r + q(v),$$

[1] If R is noetherian, then a finitely generated R-module M is projective if and only if there exists a finite Zariski open cover $\{\mathrm{Spec}\, R_i\}_{i\in I}$ of $\mathrm{Spec}\, R$ such that M_{R_i} is free.

which can be viewed as a quadratic function of r. As $4\pi_0$ is nilpotent on R by assumption, there exists a sufficiently large integer N such that the sum

$$1 - 2\pi_0 q(v)q(w) + 2\pi_0^2 q(v)^2 q(w)^2 + \cdots + (-1)^N \binom{1/2}{N} 4^N \pi_0^N q(v)^N q(w)^N$$

in R is a square root of $1 - 4\pi_0 q(v)q(w)$. Note that $\binom{1/2}{N}4^N$ lies in R by a direct computation of the 2-adic valuation. In particular,

$$r_0 \coloneqq \frac{-1 + (1 - 4\pi_0 q(v)q(w))^{1/2}}{2\pi_0 q(w)} \in R,$$

and it is a solution for the quadratic equation $q(v + r\pi w) = 0$. Replacing v by $v + r_0\pi w$, we may assume $q(v) = 0$. Similarly, we may assume $q(w) = 0$ by replacing w by $w + r\overline{\pi}v$ for suitable r in R.

Set $r_1 \coloneqq (1 - f(x,y)f(v,\pi^2 w))^{-1}$ and $r_2 \coloneqq -r_1 f(v,w)$. Note that

$$f(v,\pi^2 w) = f(v,(t\pi - \pi_0)w) = tf(v,\pi w) - \pi_0 f(v,w) = t - \pi_0 f(v,w)$$

is nilpotent in R, so r_1 indeed exists in R. Set $v' \coloneqq r_1 v + r_2\overline{\pi}v$. Then the straightforward computation implies that

$$f(v',w) = r_1 f(v,w) + r_2 f(\overline{\pi}v,w) = r_1 f(v,w) + r_2 f(v,\pi w) = r_1 f(v,w) + r_2 = 0$$

and

$$f(v',\pi w) = r_1 f(v,\pi w) + r_2 f(\overline{\pi}v,\pi w) = r_1 + r_2 f(v,\pi^2 w) = 1.$$

□

Lemma 6.1.2. *Let R be an $\mathcal{O}_{F_0}$-algebra and M be a finite free $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-module of rank $d \geq 1$. Suppose $b : M \times M \to R$ is a perfect R-bilinear pairing. Then there exists $v, w \in M$ such that $b(v,\pi w) = 1$.*

Proof. By assumption, we may choose an R-basis $\{v_1, \ldots, v_{2d}\}$ of M such that $v_{d+i} = \pi v_i$ for $1 \leq i \leq d$. This basis yields a dual basis $\{v_1^\vee, \ldots, v_{2d}^\vee\}$ of $M^\vee \coloneqq \mathrm{Hom}_R(M,R)$ such that

$v_i^\vee(v_j) = b(v_i, v_j) = \delta_{ij}$. Since b is perfect, we can find elements $\{w_1, \ldots, w_{2d}\}$ in M such that

$$b(w_i, v_j) = v_i^\vee(v_j) = \delta_{ij}$$

for $1 \leq i, j \leq 2d$. Set $v \coloneqq w_{d+1}$ and $w \coloneqq v_1$. Then we have

$$b(v, \pi w) = b(w_{d+1}, v_{d+1}) = v_{d+1}^\vee(v_d) = 1.$$

$\square$

Lemma 6.1.3. *Let R be an $\mathcal{O}_{F_0}$-algebra and M be a finite free $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-module of rank $d \geq 1$. Suppose $b : M \times M \to R$ is an R-bilinear pairing on M such that*

$$b(\pi m_1, m_2) = b(m_1, \overline{\pi} m_2) \tag{6.1.1}$$

for any m_1 and m_2 in M. Let N be a free $(\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R)$-submodule of M such that b restricts to a perfect pairing on N. Denote by

$$N^\perp \coloneqq \{m \in M \mid b(m, n) = 0 \textit{ for any } n \in N\}$$

the (left) orthogonal complement of N with respect to b.

Then $N^\perp$ is a projective $(\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R)$-module and $M = N \oplus N^\perp$ as $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-modules.

Proof. By construction, we have an exact sequence of R-modules

$$0 \to N^\perp \xrightarrow{\alpha} M \xrightarrow{\beta} \mathrm{Hom}_R(N, R), \tag{6.1.2}$$

where α denotes the inclusion map and β denotes the map $m \mapsto (n \mapsto b(m, n))$ for $m \in M$ and $n \in N$. By (6.1.1), the R-submodule $N^\perp$ is also an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-submodule. For any $\varphi \in \mathrm{Hom}_R(N, R)$, define $\pi\varphi \in \mathrm{Hom}_R(N, R)$ by setting $(\pi\varphi)(n) \coloneqq \varphi(\overline{\pi} n)$ for $n \in N$. This endows $\mathrm{Hom}_R(N, R)$ with the structure of an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-module, and the exact sequence (6.1.2) becomes an exact sequence of $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-modules. Since b is perfect on N, the map β is surjective with a section $\mathrm{Hom}_R(N, R) \to N \subset M$. It follows that $M = N \oplus N^\perp$ as $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-modules and $N^\perp$ is projective. $\square$

Lemma 6.1.4 (cf. [Ans18, Lemma 9.2])**.** *Let R be an $\mathcal{O}_{F_0}$-algebra and let M be a free $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-module of rank d. Then the functor*

$$\begin{aligned} HQF(M) : (\mathrm{Sch}/R)^{\mathrm{op}} &\longrightarrow \mathrm{Sets} \\ S &\mapsto \{\mathcal{O}_S\text{-valued hermitian quadratic forms on } M \otimes_R \mathcal{O}_S\} \end{aligned}$$

is represented by the affine space $\mathbb{A}_R^{d^2}$ of dimension d^2 over R.

Proof. Choose a basis $e_1, \ldots, e_d$ of M over $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$. This is also a basis of $M \otimes_R \mathcal{O}_S$. By the properties of hermitian quadratic forms, we can see that any hermitian quadratic form $q : M \otimes_R \mathcal{O}_S \to \mathcal{O}_S$ is determined by values $q(e_i)$ for $1 \leq i \leq d$ and $f(e_i, e_j)$, $f(e_i, \pi e_j)$ for $1 \leq i < j \leq d$. More precisely, for any element $m = \sum_{i=1}^d (a_i e_i + b_i \pi e_i) \in M \otimes_R \mathcal{O}_S$ for $a_i, b_i \in \mathcal{O}_S$, we have

$$\begin{aligned} q(m) &= q(\sum_{i=1}^d a_i e_i) + f(\sum_{i=1}^d a_i e_i, \sum_{i=1}^d b_i \pi e_i) + q(\sum_{i=1}^d b_i \pi e_i) \\ &= \sum_{i=1}^d a_i^2 q(e_i) + \sum_{1 \leq i < j \leq d} a_i a_j f(e_i, e_j) + \sum_{1 \leq i, j \leq d} a_i b_j f(e_i, \pi e_j) \\ &\quad + \sum_{i=1}^d \pi_0 b_i^2 q(e_i) + \sum_{i \leq i < j \leq d} \pi_0 b_i b_j f(e_i, e_j). \end{aligned} \tag{6.1.3}$$

Note also that for $1 \leq i, j \leq d$, we have

$$f(e_i, \pi e_j) = f(\pi e_j, e_i) = f(e_j, \overline{\pi} e_i) = f(e_j, (t - \pi) e_i) = t f(e_j, e_i) - f(e_j, \pi e_i).$$

Conversely, given d^2 elements in $\mathcal{O}_S$ denoted as A_{ii} for $1 \leq i \leq d$ and A_{ij}, B_{ij} for $1 \leq i < j \leq d$, we can define a hermitian quadratic form on $M \otimes_R \mathcal{O}_S$ as follows. We first define two $d \times d$ matrices A and B via setting $B_{ii} \coloneqq t A_{ii}$ for $1 \leq i \leq d$, $A_{ij} \coloneqq A_{ji}$ and $B_{ij} \coloneqq t A_{ij} - B_{ji}$ for $i > j$. Then we define a map q as in (6.1.3). We can check that q is an $\mathcal{O}_S$-valued hermitian quadratic form. $\square$

The proof of Lemma 6.1.4 also implies that the scheme $HQF(M)$ is (non-canonically) isomorphic to $\operatorname{Spec} R[A, B]/I$, where A, B are two $d \times d$ matrices, and I is the ideal generated by

$$A_{ij} - A_{ji}, B_{k\ell} + B_{\ell k} - tA_{k\ell}, B_{ii} - tA_{ii}$$

for $1 \leq i, j \leq d$ and $1 \leq k < \ell \leq d$.

Definition 6.1.5. Let $(M, q, \mathscr{L})$ be an $\mathscr{L}$-valued hermitian quadratic module of rank d over some $\mathcal{O}_{F_0}$-algebra R. Then as an R-module, the rank of M is $2d$. We define the *discriminant* as the morphism

$$\mathrm{disc}(q) : \wedge_R^{2d} M \to \wedge_R^{2d}(M^\vee \otimes_R \mathscr{L}) \simeq \wedge_R^{2d}(M^\vee) \otimes_R \mathscr{L}^{2d}$$

induced by the morphism $M \to M^\vee \otimes_R \mathscr{L}$, $m \mapsto f(m, -)$. Here $M^\vee$ denotes the R-dual module $\mathrm{Hom}_R(M, R)$.

Example 6.1.6. Assume $d = 1$. Let $x \in M$ be a generator of M over $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$. Then with respect to the basis $\{x, \pi x\}$, the symmetric pairing $f : M \times M \to \mathscr{L}$ associated with q is given by the matrix

$$\begin{pmatrix} 2q(x) & tq(x) \\ tq(x) & 2\pi_0 q(x) \end{pmatrix}.$$

Using the above basis, the discriminant map can be identified with the determinant of the previous matrix, as an element in $\mathscr{L}^2$. Therefore,

$$\mathrm{disc}(q) = (4\pi_0 - t^2)q(x)^2.$$

We find that when $d = 1$, the discriminant is "divisible" by $4\pi_0 - t^2$. More generally, we have the following lemma.

Lemma 6.1.7 (cf. [Ans18, Lemma 9.4])**.** *Assume $d \geq 1$ is odd. Then there exists a functorial factorization*

$$\begin{tikzcd} \wedge_R^{2d} M \arrow[rr, "\mathrm{disc}(q)"] \arrow[d, "\mathrm{disc}'(q)"'] & & \wedge_R^{2d} M^\vee \otimes_R \mathscr{L}^{2d} \\ \wedge_R^{2d} M^\vee \otimes_R \mathscr{L}^{2d} \otimes_{\mathcal{O}_{F_0}} (4\pi_0 - t^2) \arrow[urr, "j"'] & & \end{tikzcd}$$

Here the map j is induced by the natural inclusion of the ideal $(4\pi_0 - t^2)$ in $\mathcal{O}_{F_0}$.

Proof. It suffices to prove this in the universal case, i.e., R is the ring

$$R = \mathcal{O}_{F_0}[A, B]/I,$$

where I is the ideal generated by

$$A_{ij} - A_{ji}, B_{k\ell} + B_{\ell k} - tA_{k\ell}, B_{ii} - tA_{ii}$$

for $1 \leq i, j \leq d$ and $1 \leq k < \ell \leq d$, and M is equipped with the universal quadratic form $q : M \to R$ given by

$$q(\sum_{i=1}^{d}(a_i e_i + b_i \pi e_i)) \coloneqq \sum_{1\leq i,j\leq d} A_{ij} a_i a_j + \sum_{1\leq i,j\leq d} B_{ij} a_i b_j + \pi_0 \sum_{1\leq i,j\leq d} A_{ij} b_i b_j,$$

for some R-basis $(e_i, \pi e_i)_{1\leq i\leq d}$ of M. Under the chosen basis, the associated symmetric bilinear form f is given by the matrix

$$C \coloneqq \begin{pmatrix} \widetilde{A} & B \\ B^t & \pi_0 \widetilde{A} \end{pmatrix} \in M_{2d,2d}(R), \tag{6.1.4}$$

where $\widetilde{A}_{ii} \coloneqq 2A_{ii}$ for $1 \leq i \leq d$, $\widetilde{A}_{ij} \coloneqq A_{ij}$ for $i \neq j$, and the transpose matrix B^t of B equals $t\widetilde{A} - B$. We may identify $\mathrm{disc}(q)$ with the determinant of the above matrix C. To finish the proof, we need to show that the ideal $(\mathrm{disc}(q))$ is contained in the ideal $(4\pi_0 - t^2)$ in R. As $(4\pi_0 - t^2)$ becomes the unit ideal in $R[1/\pi_0]$, it suffices to show that the ideal $(\mathrm{disc}(q))$ is contained in $(4\pi_0 - t^2)$ in the localization $R_{\mathfrak{m}}$, where $\mathfrak{m}$ is the ideal (π_0). Equivalently, we need to show that $\mathrm{disc}(q)$ is divisible by $4\pi_0 - t^2$ in $R_{\mathfrak{m}}/\mathfrak{m}^k$ for all $k \geq 1$.

We will argue by induction on the rank d. If $d = 1$, this follows by the computation in Example 6.1.6. Note that in the ring $R_{\mathfrak{m}}/\mathfrak{m}^k$, the element $B_{ij} = f(e_i, \pi e_j)$ is a unit for $i \neq j$ and π_0 is nilpotent. In particular, we may assume $f(e_1, \pi e_2) = 1$. Then by Lemma 6.1.1, we may assume f restricting to the submodule $R\langle e_1, e_2, \pi e_1, \pi e_2\rangle$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The determinant of the above matrix is one. In particular, f is perfect on $R\langle e_1, e_2, \pi e_1, \pi e_2\rangle$. Then we can write $M = R\langle e_1, e_2, \pi e_1, \pi e_2\rangle \oplus M'$, where M' is the orthogonal complement of $R\langle e_1, e_2, \pi e_1, \pi e_2\rangle$ in M with respect to f. The rank of M' over $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$ is $d-2$, which is odd. By induction, $\mathrm{disc}(q|_{M'})$ is divisible by $4\pi_0 - t^2$. Hence, $\mathrm{disc}(q) = \mathrm{disc}(q|_{M'})$ is also divisible by $4\pi_0 - t^2$. $\square$

Definition 6.1.8. We call the morphism $\mathrm{disc}'(q)$ in Lemma 6.1.7 the *divided discriminant* of q. If $\mathrm{disc}'(q)$ is an isomorphism, then we say $(M, q, \mathscr{L})$ is a hermitian quadratic module of type Λ_m.

Example 6.1.9 (cf. [Ans18, Definition 9.7])**.** Let R be an $\mathcal{O}_{F_0}$-algebra. Define

$$M_{std,2} \coloneqq (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R)\langle e_1, e_2\rangle$$

with hermitian quadratic form $q_{std,2} : M_{std,2} \to R$ determined by

$$q_{std,2}(e_1) = q_{std,2}(e_2) = 0, f_{std,2}(e_1, e_2) = 0, f_{std,2}(e_1, \pi e_2) = 1.$$

For an odd integer $n = 2m + 1$, we define

$$M_{std,n} \coloneqq M_{std,2}^{\oplus m} \oplus (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R)e_n$$

as an orthogonal direct sum and $q_{std,n}(e_n) \coloneqq 1$. Viewing $\mathrm{disc}'(q_{std,n})$ as an element in R, then we have

$$\mathrm{disc}'(q_{std,n}) = 1.$$

Hence, $(M_{std,n}, q_{std,n}, R)$ is a hermitian quadratic module over R of type Λ_m.

Example 6.1.10. By direct computation of the determinants of matrices (5.1.6) and (5.2.5), the hermitian quadratic module $(\Lambda_m, q, \varepsilon^{-1}\mathcal{O}_{F_0})$ is of type Λ_m.

Lemma 6.1.11. *Let S be a scheme. Let $\mathscr{G}$ be a smooth group scheme over S. Let X be a scheme over S equipped with a $\mathscr{G}$-action $\rho : \mathscr{G} \times_S X \to X$. Assume ρ is simply transitive in the sense that for any S-scheme T, the set $X(T)$ is either empty or the action of $\mathscr{G}(T)$*

on $X(T)$ is simply transitive. If the structure morphism $X \to S$ is surjective, then X is an étale $\mathscr{G}$-torsor over S.

Proof. As ρ is simply transitive, we have an isomorphism $\Phi : \mathscr{G} \times_S X \xrightarrow{\sim} X \times_S X$, $(g, x) \mapsto (\rho((g, x)), x)$ by [Sta24, 0499]. As $\mathscr{G} \to S$ is a smooth cover of S and smoothness is an fpqc local property on the target, the isomorphism Φ implies that $X \to S$ is smooth. If $X \to S$ is surjective, then $X \to S$ is a smooth cover of S. Let $s : X \to \mathscr{G} \times_S X$ be the morphism induced by the identity section of $\mathscr{G}$. Then the composite $\Phi \circ s$ gives a section of $X \times_S X \to X$. By [Sta24, 055V], we can find an étale cover $\{U_i\}_{i \in I}$ of S such that $X \times_S U_i \to U_i$ has a section for each $i \in I$. Hence, we deduce that X is an étale $\mathscr{G}$-torsor over S. □

Theorem 6.1.12 (cf. [Ans18, Theorem 9.10]). *Let $(M, q, \mathscr{L})$ be a hermitian quadratic module of type Λ_m of rank $n = 2m + 1$ over R. Then $(M, q, \mathscr{L})$ is étale locally isomorphic to $(M_{std,n}, q_{std,n}, R)$ up to similitude. In particular, $(M, q, \mathscr{L})$ is étale locally isomorphic to $(\Lambda_m, q, \varepsilon^{-1}\mathcal{O}_{F_0}) \otimes_{\mathcal{O}_{F_0}} R$ up to similitude.*

Proof. Denote $\mathscr{G}_m \coloneqq \underline{\mathrm{Sim}}(M_{std,n})$. It suffices to show that the sheaf

$$\mathcal{F} \coloneqq \underline{\mathrm{Sim}}((M_{std,n}, q_{std,n}, R), (M, q, \mathscr{L}))$$

of similitudes is an étale $\mathscr{G}_m$-torsor over R.

Clearly, $\mathcal{F}$ is represented by an affine scheme of finite type over R. We next prove that $\mathcal{F}$ is smooth over R. Over $R[1/\pi_0]$, the quadratic form is determined by the associated symmetric pairing, and both M_{std} and M are self-dual with respect to the symmetric pairing. Then by the arguments in [RZ96, Appendix to Chapter 3], we see that $\mathcal{F}$ is smooth and surjective over $R[1/\pi_0]$. Hence, to show the smoothness of $\mathcal{F}$ over R, it suffices to prove that the morphism $\mathcal{F} \to \operatorname{Spec} \mathcal{O}_F$ is (formally) smooth at points over $\operatorname{Spec} R/\pi_0 R$. For any surjection $S \to \overline{S}$ in Nilp_R with nilpotent kernel J and a similitude $(\overline{\varphi}, \overline{\gamma}) \in \mathcal{F}(\overline{S})$, we need to show that there exists a lift of $(\overline{\varphi}, \overline{\gamma})$ to S. We argue by induction on the rank n. We denote by $e_1, \dots, e_n$ the standard basis of $M_{std,n}$. We reorder the basis such that $q(e_{m+1}) = 1$ and $(\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R)\langle e_i, e_{n+1-i}\rangle \simeq M_{std,2}$. We claim that there exist elements $v_1, \dots, v_n$ in $M \otimes_R S$ and a generator $u \in \mathscr{L} \otimes_R S$ such that $\overline{v_i} = \overline{\varphi}(\overline{e_i})$ in $M \otimes_R \overline{S}$ and

$$q(v_{m+1}) = u,\ q(v_i) = f(v_i, v_j) = 0 \text{ and } f(v_i, \pi v_j) = u\delta_{i,n+1-j}$$

for $1 \leq i < j \leq n$ and $i, j \neq m+1$. Then the maps $\varphi : e_i \mapsto v_i$ and $\gamma : 1 \mapsto u$ define a lift of $(\overline{\varphi}, \overline{\gamma})$. Thus, it suffices to prove the claim.

Suppose $n = 1$. Set $\overline{v}_1 := \overline{\varphi}(\overline{e_1}) \in M \otimes_R \overline{S}$. Then $\overline{v}_1$ is a generator of $M \otimes_R \overline{S}$. Pick any lift $v_1 \in M$ of $\overline{v}_1$. As $\mathrm{disc}'(q)$ is an isomorphism, $q(v_1)$ is a generator of $\mathscr{L}$. Let $u = q(v_1)$. This proves the claim for $n = 1$. For $n \geq 3$, pick lifts $v_1, \dots, v_n$ in $M \otimes_R S$ such that $\overline{v_i} = \overline{\varphi}(\overline{e_i})$. Let f be the associated symmetric pairing of M. Then $f(v_1, \pi v_n)$ is a generator in $\mathscr{L} \otimes_R S$, as its reduction in $\mathscr{L} \otimes_R \overline{S}$ is a generator. Set $u = f(v_1, \pi v_n)$. Using the generator u, we may identify $\mathscr{L} \otimes_R S$ with S, and we may assume that $f(v_1, \pi v_2) = 1$ in $\mathscr{L} \otimes_R S \simeq S$. Note that as elements $q(v_1), q(v_2)$ and $f(v_1, v_2)$ reduce to zero in $\overline{S}$ by properties of $\overline{v_1}$ and $\overline{v_2}$, they lie in the kernel J. Then the linear transformation in Lemma 6.1.1 does not change the reduction of v_1 and v_2, and hence, we may assume that

$$q(v_1) = q(v_n) = f(v_1, v_n) = 0 \text{ and } f(v_1, \pi v_n) = 1.$$

Then f is perfect on the S-submodule N generated by $v_1, v_n, \pi v_1, \pi v_n$. Let $N^\perp$ be the orthogonal complement of N in $M \otimes_R S$. Then $N^\perp \otimes_R \overline{S}$ is the $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \overline{S}$-submodule in $M \otimes_R \overline{S}$ generated by $\overline{v_2}, \dots, \overline{v_{n-1}}$. For $2 \leq i \leq n-1$, we can write $v_i = w' + w$, where $w' \in N^\perp$ and $w \in N$. As $\overline{v_i}$ is orthogonal to $\overline{N}$, we have $\overline{w}$ is orthogonal to $\overline{N}$. Since f is perfect on N, we obtain $\overline{w} = 0$. In particular, we may choose v_i in $N^\perp$ as a lift of $\overline{v_i}$ for $2 \leq i \leq n-1$. Now the claim follows by induction on the rank of M, and we deduce the (formal) smoothness of $\mathcal{F}$ over R.

Note that the same proof implies that the group scheme $\mathscr{G}_m$ is smooth over R. As the $\mathscr{G}_m$-action on $\mathcal{F}$ is simply transitive by construction, by Lemma 6.1.11, it remains to show that $\mathcal{F}$ is a surjective scheme over R. Since we have already shown that $\mathcal{F}$ is surjective over

$R[1/\pi_0]$, it suffices to prove the surjectivity of $\mathcal{F}$ over $R/\pi_0 R$. Then we may assume $R = \overline{k}$ is the algebraic closure of the residue field k of $\mathcal{O}_{F_0}$ and $\mathscr{L} = \overline{k}$. We need to show that there exists a similitude isomorphism (φ, γ) between $(M_{std,n}, q_{std,n}, \overline{k})$ and $(M, q, \overline{k})$. For the case $n = 1$, we can construct a similitude as in the previous paragraph. For $n \geq 3$ odd, we first claim that there exist v and w in M such that $f(v, \pi w) = 1$. Otherwise, under a basis of the form $(v_1, \dots, v_n, \pi v_1, \dots, \pi v_n)$, the pairing f corresponds to the $2n \times 2n$ matrix

$$\begin{pmatrix} \widetilde{A} & 0 \\ 0 & 0 \end{pmatrix}$$

for some $n \times n$ matrix $\widetilde{A}$, where $\widetilde{A}_{ii} = 2q(v_i) = 0$ for $1 \leq i \leq n$ and $\widetilde{A}_{ij} = f(v_i, v_j)$ for $i \neq j$. Suppose for some indices $i_0 \neq j_0$, we have $f(v_{i_0}, v_{j_0}) \neq 0$. We may assume $f(v_1, v_2) \neq 0$. Then by a suitable linear transformation of the basis $v_1, \dots, v_n$, we may assume that $\widetilde{A}$ is of the form

$$\left(\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{$\mathbf{0}$} \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \widetilde{A}_1 \end{array}\right)$$

In particular, $M_1 := (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \overline{k})\langle v_1, v_2 \rangle$ and $M_2 := (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \overline{k})\langle v_3, \dots, v_n \rangle$ are orthogonal complement of each other. Then

$$\mathrm{disc}'(q) = \mathrm{disc}(q|_{M_1})\mathrm{disc}'(q|_{M_2}).$$

However,

$$\mathrm{disc}(q|_{M_1}) = \det \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$$

This contradicts the assumption that $\mathrm{disc}'(q)$ is a unit. Then we see $f(v_i, v_j) = 0$ for any $i \neq j$, i.e., $\widetilde{A}$ is a diagonal matrix. Hence, M is an orthogonal direct sum of rank one hermitian quadratic modules. This also contradicts $\mathrm{disc}'(q) \neq 0$. Then we conclude that there exist v and w in M such that $f(v, \pi w) = 1$. Then as in Lemma 6.1.1, we may assume that f restricting to $(\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \overline{k})\langle v, w\rangle$ corresponds to the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, $(\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \overline{k})\langle v, w\rangle$ is isomorphic to $M_{std,2}$. Its orthogonal complement is a hermitian quadratic module of type Λ_m of rank $n-2$. Now we can finish the proof by induction on the rank of M. □

Theorem 6.1.13 (cf. [Ans18, Proposition 9.9])**.** *The group functor $\underline{\mathrm{Sim}}(\Lambda_m)$ is representable by an affine smooth group scheme over $\mathcal{O}_{F_0}$ whose generic fiber is* $\mathrm{GU}(V, h)$.

Proof. By the proof of Theorem 6.1.12, the functor $\underline{\mathrm{Sim}}(\Lambda_m)$ is representable by an affine smooth group scheme of finite type over $\mathcal{O}_{F_0}$. It remains to prove the assertion for the generic fiber. Following the notations in §3.2, we denote by s the symmetric pairing on Λ_m. For any F_0-algebra R, we have

$$\begin{aligned}
\underline{\mathrm{Sim}}(\Lambda_m)(R) &= \left\{ (\varphi, \gamma) \;\middle|\; \begin{array}{l} \varphi \text{ is an automorphism of the } \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R\text{-module } \Lambda_m \otimes_{\mathcal{O}_{F_0}} R \\ \gamma : \mathscr{L} \otimes_{\mathcal{O}_{F_0}} R \xrightarrow{\sim} \mathscr{L} \otimes_{\mathcal{O}_{F_0}} R \\ q(\varphi(x)) = \gamma(q(x)) \text{ for } x \in \Lambda_m \otimes_{\mathcal{O}_{F_0}} R = V \otimes_{F_0} R \end{array} \right\} \\
&= \left\{ \varphi \in \mathrm{GL}_{F \otimes_{F_0} R}(V \otimes_{F_0} R) \;\middle|\; \begin{array}{l} \gamma : R \xrightarrow{\sim} R \\ s(\varphi(x), \varphi(y)) = \gamma\left(s(x, y)\right) \text{ for } x, y \in V \otimes_{F_0} R \end{array} \right\} \\
&= \left\{ \varphi \in \mathrm{GL}_{F \otimes_{F_0} R}(V \otimes_{F_0} F) \;\middle|\; \begin{array}{l} s(\varphi(x), \varphi(y)) = c(\varphi) s(x, y) \\ \text{for } x, y \in V \otimes_{F_0} R \text{ and some } c(\varphi) \in R^\times \end{array} \right\} \\
&= \left\{ \varphi \in \mathrm{GL}_{F \otimes_{F_0} R}(V \otimes_{F_0} F) \;\middle|\; \begin{array}{l} h(\varphi(x), \varphi(y)) = c(\varphi) h(x, y) \\ \text{for } x, y \in V \otimes_{F_0} R \text{ and some } c(\varphi) \in R^\times \end{array} \right\}
\end{aligned}$$

$= \mathrm{GU}(V,h)(R).$

Therefore, the generic fiber of $\underline{\mathrm{Sim}}(\Lambda_m)$ is $\mathrm{GU}(V,h)$. $\square$

Corollary 6.1.14. *The scheme $\underline{\mathrm{Sim}}(\Lambda_m)$ is isomorphic to the parahoric group scheme attached to Λ_m.*

Proof. Let $\breve{F}_0$ denote the completion of the maximal unramified extension of F_0. By construction, we know that $\underline{\mathrm{Sim}}(\Lambda)(\mathcal{O}_{\breve{F}_0})$ is the stabilizer of Λ_m in $\mathrm{GU}(V,h)(\breve{F}_0)$, which is a parahoric subgroup by Proposition 2.4.1. As $\underline{\mathrm{Sim}}(\Lambda)$ is smooth over $\mathcal{O}_{F_0}$ by Theorem 6.1.13, the corollary follows by [BT84a, 1.7.6]. $\square$

6.2 Hermitian quadratic modules of type Λ_0

Let R be an $\mathcal{O}_{F_0}$-algebra. Recall that in Definition 3.2.3, we have defined the category $\mathcal{C}_R$ of hermitian quadratic modules with ϕ. By a similar proof as in Lemma 6.1.4, we can show that for a fixed free $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-module M of rank d, the moduli functor of all bilinear forms ϕ and quadratic forms q on M satisfying (3.2.4) in Definition 3.2.3 is representable by the affine space of dimension d^2 over R.

Let $(M,q,\mathscr{L},\phi) \in \mathcal{C}_R$. Choose a basis $(e_1,\dots,e_d,\pi e_1,\dots,\pi e_d)$ of M. The pairing ϕ is then given by the matrix

$$\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ t\widetilde{A} - \widetilde{B} & \pi_0\widetilde{A} \end{pmatrix},$$

where $\widetilde{A}_{ii} = (t/\pi_0)q(e_i)$ and $\widetilde{B}_{ii} = 2q(e_i)$ for $1 \le i \le d$, $\widetilde{A}_{ij} = \phi(e_i,e_j)$ and $\widetilde{B}_{ij} = \phi(e_i,\pi e_j)$ for $1 \le i,j \le d$ and $i \ne j$, and they satisfy $\widetilde{A} = -\widetilde{A}^t + (t/\pi_0)\widetilde{B}$ and $\widetilde{B} = \widetilde{B}^t$.

Definition 6.2.1. Let $(M,q,\mathscr{L},\phi) \in \mathcal{C}_R$ and the rank of M over R is $2d$. We define the *discriminant* as the morphism

$$\mathrm{disc}(\phi) : \wedge_R^{2d} M \to \wedge_R^{2d}(M^\vee \otimes_R \mathscr{L}) \simeq \wedge_R^{2d}(M^\vee) \otimes_R \mathscr{L}^{2d}$$

induced by the morphism $M \to M^\vee \otimes_R \mathscr{L}$, $m \mapsto \phi(m,-)$.

Example 6.2.2. Assume $d = 1$. Let $x \in M$ be a generator of M over $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$. Suppose $(M, q, \mathscr{L})$ is a hermitian quadratic module. Then we can define a bilinear form $\phi : M \times M \to \mathscr{L}$ given by the matrix

$$\begin{pmatrix} t/\pi_0 q(x) & 2q(x) \\ (t^2 - 2\pi_0)/\pi_0 q(x) & tq(x) \end{pmatrix}$$

with respect to the basis $\{x, \pi x\}$. Equipped with such ϕ, we have $(M, q, \mathscr{L}, \phi) \in \mathcal{C}_R$. Using the basis $\{x, \pi x\}$, we may view the discriminant map $\mathrm{disc}(\phi)$ as the determinant of the above matrix. We have

$$\mathrm{disc}(\phi) = \frac{4\pi_0 - t^2}{\pi_0} q(x)^2.$$

Arguing similarly as in Lemma 6.1.7, we can show the following result.

Lemma 6.2.3. *Assume $d \geq 1$ is odd. Then there exists a functorial factorization*

$$\begin{tikzcd} \wedge_R^{2d} M \arrow[r, "\mathrm{disc}(\phi)"] \arrow[d, "\mathrm{disc}'(\phi)"'] & \wedge_R^{2d} M^\vee \otimes_R \mathscr{L}^{2d} \\ \wedge_R^{2d} M^\vee \otimes_R \mathscr{L}^{2d} \otimes_{\mathcal{O}_{F_0}} \left(\frac{4\pi_0 - t^2}{\pi_0}\right) \arrow[ur, "j"'] & \end{tikzcd}$$

Here the map j is induced by the natural inclusion of the ideal $(\frac{4\pi_0 - t^2}{\pi_0})$ in $\mathcal{O}_{F_0}$.

Proof. As in the proof of Lemma 6.1.7, we can reduce to show that the determinant, which equals $\mathrm{disc}(\phi)$, of a matrix of the form

$$\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ t\widetilde{A} - \widetilde{B} & \pi_0 \widetilde{A} \end{pmatrix} \in M_{2d,2d}(R),$$

is divisible by $(4\pi_0 - t^2)/\pi_0$ in R, where $\widetilde{A}_{ii} = (t/\pi_0)q(e_i)$ and $\widetilde{B}_{ii} = 2q(e_i)$ for $1 \leq i \leq d$, $\widetilde{A}_{ij} = \phi(e_i, e_j)$ and $\widetilde{B}_{ij} = \phi(e_i, \pi e_j)$ for $1 \leq i, j \leq d$ and $i \neq j$, and they satisfy $\widetilde{A} = -\widetilde{A}^t + (t/\pi_0)\widetilde{B}$ and $\widetilde{B} = \widetilde{B}^t$.

If $d = 1$, then the lemma follows by Example 6.2.2. Suppose $d \geq 3$. We may assume π_0 is nilpotent in R and $B_{12} = \phi(e_1, \pi e_2) = 1$ as in the proof of Lemma 6.1.7. As in Lemma

6.1.1, replacing e_1 by $r_1e_1 + r_2\overline{\pi}e_1$ for suitable r_1 and r_2 in R, we may assume further that $\phi(e_1, e_2) = 0$. Then restricting to the submodule $\langle e_1, e_2, \pi e_1, \pi e_2\rangle$, the pairing ϕ is given by the matrix

$$\begin{pmatrix} \frac{t}{\pi_0}q(e_1) & 0 & 2q(e_1) & 1 \\ \frac{t}{\pi_0} & \frac{t}{\pi_0}q(e_2) & 1 & 2q(e_2) \\ \frac{t^2-2\pi_0}{\pi_0}q(e_1) & -1 & tq(e_1) & 0 \\ \frac{t^2-\pi_0}{\pi_0} & \frac{t^2-2\pi_0}{\pi_0}q(e_2) & t & tq(e_2) \end{pmatrix}.$$

By direct computation, the above is an invertible matrix, and hence the pairing ϕ is perfect on the module $\langle e_1, e_2, \pi e_2, \pi e_2\rangle$. Therefore, the orthogonal complement M' of $\langle e_1, e_2, \pi e_2, \pi e_2\rangle$ in M has rank $n-2$ over $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$, and $M' \in \mathcal{C}_R$. Then we finish the proof by induction on the rank of M. $\square$

Definition 6.2.4. Let R be an $\mathcal{O}_{F_0}$-algebra. We say that a hermitian quadratic module $(M, q, \mathscr{L}, \phi) \in \mathcal{C}_R$ over R is of type Λ_0 if $\mathrm{disc}'(\phi)$ is an isomorphism.

Example 6.2.5. Let R be an $\mathcal{O}_{F_0}$-algebra.

(1) Suppose (M, q, R) is a hermitian quadratic module of rank one. Let $x \in M$ be a generator and assume $q(x) = 1$. We can define a bilinear form $\phi_{std,1} : M \times M \to R$ as in Example 6.2.2. Then $(M, q, \mathscr{L}, \phi_{std,1}) \in \mathcal{C}_R$. Viewing $\mathrm{disc}'(\phi_{std,1})$ as an element in R, we have $\mathrm{disc}'(\phi_{std,1}) = 1$.

(2) Define

$$N_{std,2} \coloneqq (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R)\langle e_1, e_2\rangle$$

with hermitian quadratic form $q_{std,2} : N_{std,2} \to R$ determined by

$$q_{std,2}(e_1) = q_{std,2}(e_2) = 0, \phi_{std,2}(e_1, e_2) = 0, \phi_{std,2}(e_1, \pi e_2) = 1.$$

For an odd integer $n = 2m + 1$, we define

$$N_{std,n} \coloneqq N_{std,2}^{\oplus m} \oplus (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R)e_n.$$

Here $(\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R)e_n$ is a hermitian quadratic module of rank one as in (1), and the direct sum is an orthogonal direct sum with respect to $\phi_{std,n} := \phi_{std,2}^{\oplus m} \oplus \phi_{std,1}$. Viewing $\mathrm{disc}'(\phi_{std,n})$ as an element in R, we have

$$\mathrm{disc}'(\phi_{std,n}) = 1.$$

Hence, $(N_{std,n}, q_{std,n}, R, \phi_{std,n})$ is a hermitian quadratic module over R of type Λ_0.

Example 6.2.6. Equipped with the following bilinear form

$$\phi(-,-) : \Lambda_0 \times \Lambda_0 \longrightarrow \mathscr{L} = \varepsilon^{-1}\mathcal{O}_{F_0}, \quad (x,y) \mapsto s(x, \pi^{-1}y) = \varepsilon^{-1} \operatorname{Tr}_{F/F_0} h(x, \pi^{-1}y),$$

the hermitian quadratic module $(\Lambda_0, q, \varepsilon^{-1}\mathcal{O}_{F_0}, \phi)$ is of type Λ_0.

Theorem 6.2.7. *Let $(M, q, \mathscr{L}, \phi)$ be a hermitian quadratic module of type Λ_0 of rank $n = 2m+1$ over R. Then $(M, q, \mathscr{L}, \phi)$ is étale locally isomorphic to $(N_{std,n}, q_{std,n}, R, \phi_{std,n})$ up to similitude. In particular, $(M, q, \mathscr{L}, \phi)$ is étale locally isomorphic to $(\Lambda_0, q, \varepsilon^{-1}\mathcal{O}_{F_0}, \phi) \otimes_{\mathcal{O}_{F_0}} R$ up to similitude.*

Proof. As in the proof of Theorem 6.1.12, it suffices to show that the representable sheaf

$$\mathcal{F} := \underline{\mathrm{Sim}}((N_{std,n}, q_{std,n}, R, \phi_{std,n}), (M, q, \mathscr{L}, \phi))$$

of similitudes is surjective over R and smooth at points over $\operatorname{Spec} R/\pi_0 R$.

We first check that for any surjection $S \to \overline{S}$ in Nilp_R with nilpotent kernel J and a similitude $(\overline{\varphi}, \overline{\gamma}) \in \mathcal{F}(\overline{S})$, there exists a lift of $(\overline{\varphi}, \overline{\gamma})$ to S. We denote by $e_1, \ldots, e_n$ the standard basis of $N_{std,n}$. We reorder the basis such that $q(e_{m+1}) = 1$ and $(\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R)\langle e_i, e_{n+1-i}\rangle \simeq N_{std,2}$. We claim that there exist lifts $v_i \in M \otimes_R S$ of $\overline{v_i} := \overline{\varphi}(\overline{e_i})$ for $1 \leq i \leq n$ and a generator $u \in \mathscr{L} \otimes_R S$ such that

$$q(v_{m+1}) = u,\ q(v_i) = \phi(v_i, v_j) = 0 \text{ and } \phi(v_i, \pi v_j) = u\delta_{i,n+1-j}$$

for $1 \leq i < j \leq n$ and $i, j \neq m+1$. The the maps $\varphi : e_i \mapsto v_i$ and $\gamma : 1 \mapsto u$ defines a lift of $(\overline{\varphi}, \overline{\gamma})$ and (φ, γ) preserves ϕ. Thus it suffices to prove the claim.

Suppose $n = 1$. Pick any lift v_1 of $\overline{v_1}$. As $\mathrm{disc}'(\phi)$ is an isomorphism, $q(v_1)$ is a generator of $\mathscr{L} \otimes_R S$. Set $u = q(v_1)$. This proves the claim for $n = 1$. For $n \geq 3$, pick any lifts $v_1, \ldots, v_n$ in $M \otimes_R S$ of $\overline{v_1}, \ldots, \overline{v_n}$. As in the proof of Theorem 6.1.12, we may assume that $\mathscr{L} \otimes_R S \simeq S$ and $\phi(v_1, \pi v_n) = 1$ in S. Let $r_0 \in R$ be a solution of the quadratic equation $q(v_n)r^2 + r + q(v_1) = 0$, which exists by arguments in Lemma 6.1.1. Since $q(v_1)$ and $q(v_n)$ lie in J, we have $r_0 \in J$. Then $v_1' := v_1 + r_0 v_n$ and $\overline{v_1'} = \overline{v_1}$. So we may find a lift v_n' such that $\phi(v_1', v_n') = 1$. Set $v_n'' := v_n' - q(v_n')v_1'$. Then $q(v_n'') = 0$ and $\overline{v_n''} = \overline{v_n}$. Set

$$r_1 := (1 - \phi(v_1', v_n'')\phi(v_1', \pi^2 v_n''))^{-1} \text{ and } r_2 := -r_1\phi(v_1', v_n'').$$

Since $(\overline{\varphi}, \overline{\gamma})$ preserves ϕ, we have $\phi(\overline{v_1'}, \overline{v_n''}) = \overline{\gamma}(\phi_{std,n}(e_1, e_n)) = 0$. Thus, $\phi(v_1', v_n'')$ and r_2 are in J. Set $v_1'' := r_1 v_1' + r_2 \overline{\pi} v_1'$. Then $\overline{v_1''} = \overline{v}$. As in Lemma 6.1.1, we have $\phi(v_1'', \pi v_n'') = 1$ and $\phi(v_1'', v_n'') = 0$. By replacing v_1 by v_1'' and v_n by v_n'', we may assume that

$$q(v_1) = q(v_n) = \phi(v_1, v_n) = 0 \text{ and } \phi(v_1, \pi v_n) = 1.$$

Then ϕ is perfect on the S-submodule N generated by $v_1, v_2, \pi v_1, \pi v_2$. Let $N^{\perp}$ be the orthogonal complement (with respect to ϕ) of N in $M \otimes_R S$. As in the proof of Theorem 6.1.12, we may assume that lifts v_i for $2 \leq i \leq n-1$ lie in $N^{\perp}$. The claim follows by induction on the rank of M, and hence, we deduce the smoothness of $\mathcal{F}$ over R.

Next we prove the surjectivity of $\mathcal{F}$ over R. It suffices to prove that $\mathcal{F}$ has non-empty fibers over $R/\pi_0 R$. Then we may assume $R = \overline{k}$ is the algebraic closure of the residue field of $\mathcal{O}_{F_0}$ and $\mathscr{L} = \overline{k}$. We need to show that there exists a similitude isomorphism (φ, γ) preserving ϕ between $(N_{std,n}, q_{std,n}, \overline{k}, \phi_{std,n})$ and $(M, q, \overline{k}, \phi)$. Suppose $n = 1$. Then $M \otimes_R S = (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} S)v$ for some v. Define

$$\begin{aligned} \varphi : N_{std} \otimes_R S &\longrightarrow M \otimes_R S = (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} S)v, & \gamma : S &\longrightarrow \mathscr{L} \otimes_R S \\ e_1 &\mapsto v, & 1 &\mapsto q(v). \end{aligned}$$

As $\mathrm{disc}'(\phi)$ is an isomorphism, $q(v)$ is a generator. Since ϕ is determined by q in this case by computation in Example 6.2.2, the similitude (φ, γ) preserves ϕ. For $n \geq 3$ odd, we claim that there exist v and w in $M \otimes_R S$ such that $\phi(v, \pi w) = 1$. This can be done using proof by contradiction as in Theorem 6.1.12. Set $v' := v + r_0 w$, where $r_0 \in \overline{k}$ is a solution for the quadratic equation $q(v') = q(w)r^2 + r + q(v)$. Then

$$\phi(v', \pi w) = \phi(v, \pi w) + r_0 \phi(w, \pi w) = 1 + 2r_0 q(w) = 1.$$

The last equality holds since char $\overline{k} = 2$. Set $w' := w - q(w)v'$. Then $q(w') = 0$. As in the previous paragraph, we may find suitable r_1 and r_2 such that $v'' := r_1 v' + r_2 \overline{\pi} v'$ satisfies $\phi(v'', \pi w) = 1$ and $\phi(v'', w') = 0$. Replacing v by v'' and w by w', we see that ϕ restricting to $(\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \overline{k})\langle v, w\rangle$ acts the same as $\phi_{std,2}$. In particular, the subspace $(\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \overline{k})\langle v, w\rangle$ is isomorphic to $N_{std,2}$. Its orthogonal complement is a hermitian quadratic module of type Λ_0 of rank $n-2$. Now we can finish the proof by induction on the rank of M. □

Theorem 6.2.8. *The group functor $\underline{\mathrm{Sim}}((\Lambda_0, \phi))$ of similitudes preserving ϕ is representable by an affine smooth group scheme over $\mathcal{O}_{F_0}$ whose generic fiber is* $\mathrm{GU}(V, h)$.

Proof. By the proof of Theorem 6.2.7, the functor $\underline{\mathrm{Sim}}((\Lambda_0, \phi))$ is representable by an affine smooth group scheme over $\mathcal{O}_{F_0}$. It remains to show the assertion for the generic fiber. Let R be an F-algebra. For any similitude $(\varphi, \gamma) \in \mathrm{Sim}(\Lambda_0)$ and $x, y \in \Lambda_0 \otimes_{\mathcal{O}_{F_0}} R = V \otimes_{F_0} R$, we have

$$\begin{aligned}\phi(\varphi(x), \varphi(y)) &= \phi(\varphi(x), \pi(\pi^{-1}\varphi(y))) = q(\varphi(x) + \varphi(\pi^{-1}y)) - q(\varphi(x)) - q(\varphi(\pi^{-1}y))\\ &= \gamma(q(x + \pi^{-1}y) - q(x) - q(\pi^{-1}y)) = \gamma(\phi(x, y)).\end{aligned}$$

Hence, over the generic fiber, any similitude of Λ_0 preserves ϕ. Then as in the proof of Theorem 6.1.13, we see that the generic fiber of $\underline{\mathrm{Sim}}((\Lambda_0, \phi))$ is $\mathrm{GU}(V, h)$. □

The same argument as in the proof of Corollary 6.1.14 implies the following.

Corollary 6.2.9. *The scheme $\underline{\mathrm{Sim}}((\Lambda_0, \phi))$ is isomorphic to the parahoric group scheme attached to Λ_0.*

CHAPTER 7

2-ADIC INTEGRAL MODELS OF SHIMURA VARIETIES

In this chapter, we will constuct 2-adic integral models of Shimura varieties of abelian type with parahoric level structure. Our goal is to prove Theorem 1.2.7 in the Introduction.

7.1 p-divisible groups and Lau's classification

In this section, we review Lau's work [Lau14] on the classification of 2-divisible groups in terms of Dieudonné displays. We generalize the construction of the natural "connection isomorphisms" for Dieudonné pairs in [KPZ24] to the case $p = 2$. We also compare Lau's classification of p-divisible groups with Breuil-Kisin's classfication.

7.1.1 Zink rings, frames and windows

Let $(R, \mathfrak{m}_R, k)$ be an artinian local ring (or more generally an *admissible ring* in the sense of [Lau14, §1]) with residue field k. Denote by $W(R)$ its associated Witt ring equipped with Frobenius φ and Verschiebung V. By [Lau14, §1B], the exact sequence

$$0 \to W(\mathfrak{m}_R) \to W(R) \to W(k) \to 0$$

has a unique ring homomorphism section $s : W(k) \to W(R)$, which is φ-equivariant.

Definition 7.1.1 ([Zin01])**.** The *Zink ring* of R is $\mathbb{W}(R) = sW(k) \oplus \widehat{W}(\mathfrak{m}_R)$, where $\widehat{W}(\mathfrak{m}_R) \subset W(\mathfrak{m}_R)$ consists of elements $(x_0, x_1, \dots) \in W(\mathfrak{m}_R)$ such that $x_i = 0$ for almost all i.

The Zink ring $\mathbb{W}(R)$ is a φ-stable subring of $W(R)$. If $p = 2$, $\mathbb{W}(R)$ is in general not stable under the Verschiebung V. We need to modify V as follows. The element $p-[p] \in W(\mathbb{Z}_p)$ lies in the image of V because it maps to zero in $\mathbb{Z}_p$. Moreover, the element $V^{-1}(p-[p]) \in W(\mathbb{Z}_p)$ is a unit, since it maps to 1 in $W(\mathbb{F}_p)$. Define

$$u_0 \coloneqq \begin{cases} V^{-1}(2-[2]) & \text{if } p = 2, \\ 1 & \text{if } p \geq 3. \end{cases} \tag{7.1.1}$$

The image of $u_0 \in W(\mathbb{Z}_p)^\times$ in $W(R)^\times$ is also denoted by u_0. For $x \in W(R)$, set

$$\mathbb{V}(x) \coloneqq V(u_0 x).$$

Lemma 7.1.2 ([Lau14, Lemma 1.7]). *The map* $\mathbb{V} : W(R) \to W(R)$ *satisfies* $\mathbb{V}(\mathbb{W}(R)) \subset \mathbb{W}(R)$*. Moreover, there is an exact sequence*

$$0 \to \mathbb{W}(R) \xrightarrow{\mathbb{V}} \mathbb{W}(R) \xrightarrow{w_0} R \to 0.$$

Remark 7.1.3. We will call the map

$$\mathbb{V} : \mathbb{W}(R) \to \mathbb{W}(R)$$

the *modified Verschiebung* for $\mathbb{W}(R)$. Many statements about $\mathbb{W}(R)$ in the case $p = 2$ are proven by adapting the corresponding proofs for $p > 2$, with adjustments for the modified Verschiebung map.

Now we recall the logarithm coordinates of the Witt ring, see [Lau14, §1C]. Let $(S \to R, \delta)$ be a divided power extension of rings with kernel $\mathfrak{a} \subset S$. Denote by $\mathfrak{a}^{\mathbb{N}}$ the additive group $\prod_{i\in\mathbb{N}} \mathfrak{a}$, equipped with a $W(S)$-module structure

$$x[a_0, a_1, \ldots] := [w_0(x)a_0, w_1(x)a_1, \ldots]$$

for $x \in W(S)$ and $[a_0, a_1, \ldots] \in \prod_{i\in\mathbb{N}} \mathfrak{a}$. Then the δ-divided Witt polynomials w'_n define an isomorphism of $W(S)$-modules

$$\begin{gathered} \mathrm{Log} : W(\mathfrak{a}) \xrightarrow{\sim} \mathfrak{a}^{\mathbb{N}} \\ \underline{a} = (a_0, a_1, \ldots) \mapsto [w'_0(\underline{a}), w'_1(\underline{a}), \ldots] \end{gathered}$$

where $w'_n(X_0, \ldots, X_n) = (p^n - 1)!\delta_{p^n}(X_0) + (p^{n-1} - 1)!\delta_{p^{n-1}}(X_1) + \cdots + X_n$. For $x \in W(\mathfrak{a})$, we call $\mathrm{Log}(x)$ the *logarithmic coordinate* of x. In terms of logarithmic coordinates, the Frobenius and Verschiebung of $W(\mathfrak{a})$ act on $\mathfrak{a}^{\mathbb{N}}$ as

$$\varphi([a_0, a_1, \ldots]) = [pa_1, pa_2, \ldots], \quad V([a_0, a_1, \ldots]) = [0, a_0, a_1, \ldots]. \tag{7.1.2}$$

Moreover, Log induces an injective map

$$\mathrm{Log} : \widehat{W}(\mathfrak{a}) \hookrightarrow \mathfrak{a}^{(\mathbb{N})},$$

which is bijective when the divided powers δ are nilpotent. Here, the group $\widehat{W}(\mathfrak{a})$ denotes the set of elements $(a_0, a_1, \ldots) \in W(\mathfrak{a})$ such that $a_i = 0$ for almost all i, and $\mathfrak{a}^{(\mathbb{N})} \subset \mathfrak{a}^{\mathbb{N}}$ denotes $\bigoplus_{i \in \mathbb{N}} \mathfrak{a}$. The ideal $\mathfrak{a} \subset W(S)$ is by definition the set of elements whose logarithmic coordinates are of the form $[a, 0, 0, \ldots]$, $a \in \mathfrak{a}$.

Definition 7.1.4. For a (Noetherian) complete local ring R with residue field k, we set

$$\mathbb{W}(R) \coloneqq \varprojlim_n \mathbb{W}(R/\mathfrak{m}_R^n).$$

For a complete local ring R, we can define the modified Verschibung $\mathbb{V}$ on $\mathbb{W}(R)$ by passing to the limit. Then $\mathbb{W}(R)$ is a subring of $W(R) \coloneqq \varprojlim_n W(R/\mathfrak{m}_R^n)$, which is stable under φ and $\mathbb{V}$. We also have $\mathbb{W}(R)/\mathbb{V}(\mathbb{W}(R)) \simeq R$, see [Lau14, §1E]. Note that $\mathbb{W}(R)$ is p-adically complete by [Lau14, Proposition 1.14].

Here, we introduce notions of frames and windows following [Lau10, §2] and [Lau14, §2].

Definition 7.1.5. (1) A *frame* is a quintuple $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$, where S and $R = S/I$ are rings, $\sigma : S \to S$ is a ring endomorphism with $\sigma(a) \equiv a^p \bmod pS$, $\sigma_1 : I \to S$ is a σ-linear map of S-modules whose image generates S as an S-module, and $I + pS$ lies in the Jacobson radical of S. A frame is called a *lifting frame* if all projective R-modules of finite type can be lifted to projective S-modules.

(2) A homomorphism of frames

$$\alpha : \mathcal{F} \longrightarrow \mathcal{F}' = (S', I', R', \sigma', \sigma_1')$$

is a ring homomorphism $\alpha : S \to S'$ with $\alpha(I) \subset I'$ such that $\sigma'\alpha = \alpha\sigma$ and $\sigma_1'\alpha = u \cdot \alpha\sigma_1$ for a unit $u \in S'$, which is then determined by α. We say that α is a frame u-homomorphism. If $u = 1$, then α is called *strict*.

(3) Let $\mathcal{F}$ be a frame. A *window over* $\mathcal{F}$ (or $\mathcal{F}$-window) is a quadruple

$$\mathcal{P} = (M, M_1, F, F_1),$$

where M is a projective S-module of finite type with a submodule M_1 such that there exists a decomposition of S-modules $M = L \oplus T$ with $M_1 = L \oplus IT$, called a *normal decomposition*, and where $F : M \to M$ and $F_1 : M_1 \to M$ are σ-linear maps of S-modules with

$$F_1(ax) = \sigma_1(a)F(x)$$

for $a \in I$ and $x \in M$, and $F_1(M_1)$ generates M as an S-module.

Remark 7.1.6. If $\mathcal{F}$ is a lifting frame, then the existence of a normal decomposition in (3) of the above definition is equivalent to that M/M_1 is a projective R-module. A frame is a lifting frame if S is local or I-adic.

A u-homomorphism $\alpha : \mathcal{F} \to \mathcal{F}'$ induces a base change functor

$$\alpha_* : (\text{windows over } \mathcal{F}) \longrightarrow (\text{windows over } \mathcal{F}') \tag{7.1.3}$$

from the category of windows over $\mathcal{F}$ to the category of windows over $\mathcal{F}'$. In terms of normal representations, the functor α_* is given by

$$(L, T, \Psi) \mapsto (S' \otimes_S L, S' \otimes_S T, \Psi')$$

with $\Psi'(s' \otimes l) = u\sigma'(s') \otimes \Psi(l)$ and $\Psi'(s' \otimes t) = \sigma'(s') \otimes \Psi(t)$.

Definition 7.1.7. A frame homomorphism $\alpha : \mathcal{F} \to \mathcal{F}'$ is called *crystalline* if the functor α_* is an equivalence of categories.

Note that for a frame $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$, there is a unique element $\theta \in S$ such that $\sigma(a) = \theta\sigma_1(a)$ for all $a \in I$. For an S-module M, we write $M^{(\sigma)} = S \otimes_{\sigma,S} M$. Then for a window $\mathcal{P} = (M, M_1, F, F_1)$ over $\mathcal{F}$, by [Lau14, Lemma 2.3], there exists a unique S-linear map

$$V^\sharp : M \longrightarrow M^{(\sigma)} \tag{7.1.4}$$

such that $V^\sharp(F_1(x)) = 1 \otimes x$ for $x \in M_1$. It satisfies $F^\# V^\sharp = \theta$ and $V^\sharp F^\# = \theta$, where $F^\# : M^{(\sigma)} \to M$ is the linearization of F.

Example 7.1.8. For a complete local ring R with perfect residue field, we will be interested in the following (lifting) frames:

(1) the *Dieudonné frame*

$$\mathcal{D}_R \coloneqq (\mathbb{W}(R), \mathbb{I}_R, R, \varphi, \varphi_1),$$

where $\mathbb{I}_R = \ker(w_0 : \mathbb{W}(R) \to R)$ and $\varphi_1 : \mathbb{I}_R \to \mathbb{W}(R)$ is the inverse of $\mathbb{V}$;

(2) assume $R = \mathcal{O}_K$ for some finite extension K of $\mathbb{Q}_p$ with residue field k, choose a presentation $R = \mathfrak{S}/E\mathfrak{S}$, where $\mathfrak{S} = W(k)[[u]]$ and $E \in \mathfrak{S}$ is an Eisenstein polynomial with constant term p. Define the *Breuil-Kisin frame*

$$\mathcal{B} \coloneqq (\mathfrak{S}, E\mathfrak{S}, R, \varphi, \varphi_1),$$

where $\varphi : \mathfrak{S} \to \mathfrak{S}$ acts on $W(k)$ as usual Frobenius and sends u to u^p, and $\varphi_1(Ex) \coloneqq \varphi(x)$ for $x \in \mathfrak{S}$.

7.1.2 Dieudonné displays and Dieudonné pairs

Let R be a complete local ring with perfect residue field of characteristic p. By Remark 7.1.6, a window over $\mathcal{D}_R$ (also called a *Dieudonné display over* R later) is a tuple (M, M_1, F, F_1), where

(i) M is a finite free $\mathbb{W}(R)$-module,

(ii) $M_1 \subset M$ is a $\mathbb{W}(R)$-submodule such that $\mathbb{I}_R M \subset M_1 \subset M$ and M/M_1 is a projective R-module,

(iii) $F : M \to M$ is a φ-linear map,

(iv) $F_1 : M_1 \to M$ is a φ-linear map, whose image generates M as a $\mathbb{W}(R)$-module, and which satisfies

$$F_1(\mathbb{V}(w)m) = wF(m) \tag{7.1.5}$$

for any $w \in \mathbb{W}(R)$ and $m \in M_1$.

Remark 7.1.9. For $p > 2$, windows over $\mathcal{D}_R$ are the same as the Dieudonné displays over R used in [KP18, 3.1.3], and the ring $\mathbb{W}(R)$ here is denoted by $\widehat{W}(R)$ in *loc. cit.*.

For a Dieudonné display (M, M_1, F, F_1), by taking $w = 1$ and $m \in M_1$ in the equation (7.1.5), we get

$$F(m) = F_1(\mathbb{V}(1)m) = \varphi\mathbb{V}(1)F_1(m) = pu_0F_1(m).$$

Recall that $u_0 \in W(R)^\times$ is defined by (7.1.1). In particular, we can consider the condition

(iv′) $F_1 : M_1 \to M$ is a φ-linear map, whose image generates M as a $\mathbb{W}(R)$-module, and which satisfies

$$F_1(\mathbb{V}(w)m) = wpu_0F_1(m)$$

for any $w \in \mathbb{W}(R)$ and $m \in M_1$.

Let $\widetilde{M_1}$ be the image of the homomorphism

$$\varphi^*(i) : \varphi^*M_1 = \mathbb{W}(R) \otimes_{\varphi,\mathbb{W}(R)} M_1 \to \varphi^*M = \mathbb{W}(R) \otimes_{\varphi,\mathbb{W}(R)} M$$

induced by the inclusion $i : M_1 \hookrightarrow M$. Note that $\widetilde{M_1}$ and the notion of a normal decomposition depend only on M and M_1, not on F and F_1.

Lemma 7.1.10. *Suppose $\mathbb{W}(R)$ is p-torsion free (e.g. if R is p-torsion free, or $pR = 0$ and R is reduced).*

(1) Giving a Dieudonné display (M, M_1, F, F_1) over R is the same as giving (M, M_1, F_1) satisfying (i), (ii) and (iv′). In this case, we also refer to the tuple (M, M_1, F_1) as a Dieudonné display over R.

(2) For a Dieudonné display (M, M_1, F_1) over R, the linearization $F_1^\#$ of F_1 factors as

$$\varphi^*M_1 \to \widetilde{M_1} \xrightarrow{\Psi} M$$

with Ψ a $\mathbb{W}(R)$-module isomorphism.

(3) *Given an isomorphism $\Psi : \widetilde{M_1} \to M$ of $\mathbb{W}(R)$-modules, there exists a unique Dieudonné display (M, M_1, F_1) over R, which produces the given (M, M_1, Ψ) via the construction in* (2).

Proof. The proof closely follows [KP18, §3.1.3, Lemma 3.1.5], with adjustments for the modified Verschiebung $\mathbb{V}$. We take this lemma as an example to illustrate how we modify the arguments concerning Dieudonné displays in [KP18] to deal with the case $p = 2$.

(1) Given the tuple (M, M_1, F_1), set $F(m) \coloneqq F_1(\mathbb{V}(1)m)$ for $m \in M$. Clearly $F : M \to M$ is φ-linear. Then for $w \in \mathbb{W}(R)$ and $m \in M$, we have

$$pu_0F_1(\mathbb{V}(w)m) = F_1(\mathbb{V}(1)\mathbb{V}(w)m) = F(\mathbb{V}(w)m) = \varphi\mathbb{V}(w)F(m) = pu_0wF(m).$$

Since $u_0 \in W(R)^\times$ and $\mathbb{W}(R)$ is p-torsion free, we obtain that $\mathbb{W}(R)$ is (pu_0)-torsion free, and hence

$$F_1(\mathbb{V}(w)m) = wF(m).$$

In particular, (M, M_1, F, F_1) is a Dieudonné display.

(2) Let $M = L \oplus T$ be a normal decomposition for M. Since $\varphi(\mathbb{I}_R) = pu_0\mathbb{W}(R)$ and $\mathbb{W}(R)$ is pu_0-torsion free, we have

$$\widetilde{M_1} = \varphi^*(L) \oplus pu_0\varphi^*(T) \simeq \mathbb{W}(R)^d,$$

where $d = \mathrm{rk}_{\mathbb{W}(R)} M$. Firstly, we show that $F_1^\#$ factors through $\widetilde{M_1}$. Let K denote the kernel of $\varphi^*(i) : \varphi^*M_1 \to \varphi^*M$. Note that $F|_{M_1} = pu_0F_1$, and so $pu_0F_1^\# = F^\# \circ \varphi^*(i)$. In particular, $pu_0F_1^\#$ vanishes on K. Since $\mathbb{W}(R)$ is pu_0-torsion free, we conclude that $F_1^\#$ vanishes on K, and hence $F_1^\#$ factors through $\widetilde{M_1}$. Since $F_1^\#$ is surjective by definition, we obtain a surjective map $\Psi : \widetilde{M_1} \to M$ between free $\mathbb{W}(R)$-modules of the same rank. Hence, Ψ is an isomorphism.

(3) Define $F_1 : M_1 \to M$ by

$$F_1(m_1) \coloneqq \Psi(1 \otimes m_1),$$

where $1 \otimes m_1$ denotes the image of $1 \otimes m_1 \in \mathbb{W}(R) \otimes_{\varphi, \mathbb{W}(R)} M_1 = \varphi^* M_1$ in $\varphi^* M$. Then F_1 is clearly φ-linear and its linearization $F_1^{\#}$ is surjective. Thus, we obtain a Dieudonné display (M, M_1, F_1). □

Definition 7.1.11 ([Hof23, §1.1])**.** Let R be a complete local ring.

(1) A *Dieudonné pair of type* (n, d) over R is a pair (M, M_1) of $\mathbb{W}(R)$-modules such that M is a finite free $\mathbb{W}(R)$-module of rank n, M_1 is a $\mathbb{W}(R)$-submodule of M and M/M_1 is a finite free R-module of rank d. Sometimes, we simply say that (M, M_1) is a Dieudonné pair.

(2) A morphism between two Dieudonné pairs (M, M_1) and (M', M_1') is a homomorphism of $\mathbb{W}(R)$-modules $f : M \to M'$ such that $f(M_1) \subset M_1'$.

Lemma 7.1.12. *There exists a functor* $\mathcal{F} : (M, M_1) \mapsto \widetilde{M_1}$*, from the category of Dieudonné pairs over* R *of type* (n, d) *to the category of finite free* $\mathbb{W}(R)$*-modules of rank* n*, such that* $\mathcal{F}$ *is compatible with base change in* R *and there is a natural isomorphism* $\widetilde{M_1}[1/p] = (\varphi^* M)[1/p]$*. If* $\mathbb{W}(R)$ *is* p*-torsion free, then* $\widetilde{M_1}$ *is given by the construction in Lemma 7.1.10.*

Proof. (cf. [KPZ24, §5.1.1].) Let (M, M_1) be a Dieudonné pair of type (n, d). Choose a normal decomposition $M = L \oplus T$ and a basis $\mathscr{B} = (e_1, \ldots, e_n)$ of M such that $(e_1, \ldots, e_d)$ is a basis of L and $(e_{d+1}, \ldots, e_n)$ is a basis of T. Such a basis $\mathscr{B}$ is said to be *adapted* to the normal decomposition $M = L \oplus T$. Set

$$\mathcal{F}((M, M_1)) = \widetilde{M_1} = (\varphi^* L) \oplus (\varphi^* T),$$

which is a free $\mathbb{W}(R)$-module of rank n. We denote by $\widetilde{\mathscr{B}} = (\varphi^* e_1, \ldots, \varphi^* e_n)$ the basis of $\widetilde{M_1}$.

Let (M', M_1') be a second Dieudonné pair with a normal decomposition $M' = L' \oplus T'$ and an adapted basis $\mathscr{B}' = (e_1', \ldots, e_n')$. Let f be a morphism between (M, M_1) and (M', M_1').

Using the normal decompositions, we may express f as a block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_n(\mathbb{W}(R))$$

with respect to the bases $\mathscr{B}$ and $\mathscr{B}'$, where the entries of C are in $\mathbb{I}_R$. Then we define $\mathcal{F}(f)$ to be the morphism $\widetilde{f} : \widetilde{M_1} \to \widetilde{M_1'}$ given by the block matrix

$$\begin{pmatrix} \varphi(A) & pu_0\varphi(B) \\ \mathbb{V}^{-1}(C) & \varphi(D) \end{pmatrix}$$

in terms of the bases $\widetilde{\mathscr{B}}$ and $\widetilde{\mathscr{B}'}$. Using $pu_0\mathbb{V}^{-1} = \varphi$, it is straightforward to check that $\mathcal{F}$ is a well-defined functor. By construction, $\mathcal{F}$ is compatible with base change in R.

There is a natural isomorphism

$$\begin{gathered} \widetilde{M_1}[1/p] = (\varphi^*L)[1/p] \oplus (\varphi^*T)[1/p] \xrightarrow{\sim} (\varphi^*M)[1/p] = (\varphi^*L)[1/p] \oplus (\varphi^*T)[1/p] \\ l + t \mapsto l + pu_0 t. \end{gathered}$$

When $\mathbb{W}(R)$ is p-torsion free, the above isomorphism restricts to an injective map

$$\widetilde{M_1} \hookrightarrow \varphi^*M,$$

and we recover the construction of $\widetilde{M_1}$ in Lemma 7.1.10. □

Lemma 7.1.13 (cf. [KPZ24, Lemma 5.1.3]). *Let R be a complete local ring with residue field k. Suppose that $\mathbb{W}(R)$ is p-torsion free. Let (M, M_1) be a Dieudonné pair over R with reduction $(M_0, M_{0,1})$ over k. Set $\mathfrak{a}_R := \mathfrak{m}_R^2 + pR$. Then there exists a natural isomorphism*

$$c : \widetilde{M_{0,1}} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{a}_R) \xrightarrow{\sim} \widetilde{M_1} \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{a}_R),$$

which is called the "connection isomorphism", fitting into a canonical commutative diagram

$$\begin{tikzcd} \widetilde{M_1} \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{a}_R) \arrow[r] & \varphi^*(M_{R/\mathfrak{a}_R}) \arrow[d, equal] \\ \widetilde{M_{0,1}} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{a}_R) \arrow[r] \arrow[u, "c", "\simeq"'] & \varphi^*(M_0) \otimes_{W(k)} \mathbb{W}(R/\mathfrak{a}_R), \end{tikzcd}$$

where $M_{R/\mathfrak{a}_R} := M \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{a}_R)$ *and horizontal maps are induced by taking the base change of the natural maps* $\widetilde{M}_{0,1} \to \varphi^*(M_0)$ *and* $\widetilde{M}_1 \to \varphi^*(M)$.

Proof. Using Lemma 7.1.12, the construction of c and the proof of [KPZ24, Lemma 5.1.3] (replacing V^{-1} by $\mathbb{V}^{-1}$) also work for $p = 2$. $\square$

7.1.3 Lau's classification of p-divisible groups

One of the main results in [Lau14] is the following.

Theorem 7.1.14. *Let R be a complete local ring with perfect residue field of characteristic p.*

(1) There is an anti-equivalence of exact categories

$$\Theta_R : (p\text{-divisible groups over } R) \xrightarrow{\sim} (\text{Dieudonné displays over } R)\ ,$$

which is compatible with base change in R.

(2) For any p-divisible group $\mathscr{G}$ over R, there is a natural isomorphism

$$\Theta_R(\mathscr{G})/\mathbb{I}_R\Theta_R(\mathscr{G}) \simeq \mathbb{D}(\mathscr{G})(R),$$

where $\mathbb{D}(\mathscr{G})$ denotes the contravariant Dieudonné crystal of $\mathscr{G}$.

(3) Let $\mathscr{G}$ be a p-divisible group over R. Write $\Theta_R(\mathscr{G}) = (M, M_1, F, F_1)$. The Hodge filtration of $\Theta_R(\mathscr{G})$ is defined as

$$M_1/\mathbb{I}_R M \subset M/\mathbb{I}_R M.$$

Then the isomorphism in (2) respects the Hodge filtrations on both sides.

Remark 7.1.15. For $p > 2$, the functor Θ_R recovers the anti-equivalence used in [KP18, 3.1.7] by sending a p-divisible group $\mathscr{G}$ over R to $\mathbb{D}(\mathscr{G})(\mathbb{W}(R))$. Note that when $p > 2$, $\mathbb{W}(R) \to R$ has divided powers on $\mathbb{I}_R$ by [Lau14, Lemma 1.16]. For $p = 2$, Θ_R is not as explicit as in the case $p > 2$, but see the case when R is a ring of p-adic integers in §7.1.4.

Proof. (1) For any p-divisible group $\mathscr{G}$ over R, set

$$\Theta_R(\mathscr{G}) \coloneqq \Phi_R(\mathscr{G}^*),$$

where $\mathscr{G}^*$ denotes the Cartier dual of $\mathscr{G}$ and Φ_R denotes the equivalence in [Lau14, Corollary 5.4]. Then we see that Θ_R is an anti-equivalence of exact categories. It commutes with base change in R by [Lau14, Theorem 3.9, 4.9].

(2) and (3) follow from [Lau14, Corollary 3.22, 4.10]. Note that we use *contravariant* Dieudonné crystals following [KP18], while Lau uses *covariant* Dieudonné crystals in [Lau14]. One can switch between contravariant and covariant Dieudonné crystals by taking Cartier duals.

□

7.1.4 Comparison with Breuil-Kisin's classification

Here the notation is as in Example 7.1.8 (2). In particular, we denote by $\mathcal{O}_K$ the ring of integers for some finite extension K of $\mathbb{Q}_p$ with residue field k. Let π be a uniformizer of $\mathcal{O}_K$ satisfying $E(\pi) = 0$. Then there is a Frobenius-equivariant ring homomorphism

$$\kappa : \mathfrak{S} = W(k)[[u]] \to W(\mathcal{O}_K)$$

sending u to $[\pi]$, lifting the quotient map $\mathfrak{S} \to \mathcal{O}_K$. Here $[\cdot]$ denotes the Teichmüller map $\mathcal{O}_K \to W(\mathcal{O}_K)$. Moreover, the image of κ lies in $\mathbb{W}(\mathcal{O}_K)$, see [Lau14, Remark 6.3]. Recall that $\mathcal{B}$ denotes the Breuil-Kisin frame in Example 7.1.8 (2). By [Lau14, Theorem 6.6], κ induces a crystalline homomorphism

$$\kappa : \mathcal{B} \to \mathcal{D}_{\mathcal{O}_K}.$$

That is, the induced functor κ_* as in (7.1.3) gives an equivalence

$$\kappa_* : (\text{windows over } \mathcal{B}) \xrightarrow{\sim} (\text{windows over } \mathcal{D}_{\mathcal{O}_K}) = (\text{Dieudonné displays over } \mathcal{O}_K).$$

Using the anti-equivalence $\Theta_{\mathcal{O}_K}$ in Theorem 7.1.14, we obtain the anti-equivalence

$$\mathcal{B}(-) \coloneqq \kappa_*^{-1} \circ \Theta_{\mathcal{O}_K} : (p\text{-divisible groups over } \mathcal{O}_K) \xrightarrow{\sim} (\text{windows over } \mathcal{B}). \tag{7.1.6}$$

On the other hand, we have, by [Kis10, Theorem 1.4.2], a fully faithful contravariant functor

$$\mathfrak{M}(-) : (p\text{-divisible groups over } \mathcal{O}_K) \longrightarrow \mathrm{BT}^{\varphi}_{\mathfrak{S}},$$

where $\mathrm{BT}^{\varphi}_{\mathfrak{S}}$ denotes the category of Breuil-Kisin modules $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ of E-height one, i.e., $\mathfrak{M}$ is a finite free $\mathfrak{S}$-module and $\varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is an $\mathfrak{S}$-module homomorphism whose cokernel is killed by E.

Proposition 7.1.16. *There is an equivalence*

$$\mathcal{F} : \mathrm{BT}^{\varphi}_{\mathfrak{S}} \longrightarrow \textit{(windows over } \mathcal{B}\textit{)}$$

such that $\mathcal{F} \circ \mathfrak{M}(-)$ *is the equivalence* $\mathcal{B}(-)$ *in* (7.1.6)*. In particular,* $\mathfrak{M}(-)$ *is an anti-equivalence.*

Proof. The proposition is implicitly contained in [Lau10, §6, 7] (see also [KM16, §2]). To a Breuil-Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ in $\mathrm{BT}^{\varphi}_{\mathfrak{S}}$, we can associate a triple (M, M_1, F_1), where $M := \varphi^*\mathfrak{M}$; $M_1 := \mathfrak{M}$, viewed as a submodule of M via the unique map $V_{\mathfrak{M}} : \mathfrak{M} \to \varphi^*\mathfrak{M}$ whose composition with $\varphi_{\mathfrak{M}}$ is the multiplication by $E(u)$; and $F_1 : M_1 \to M$ is given by $x \in \mathfrak{M} \mapsto 1 \otimes x \in \varphi^*\mathfrak{M}$. Then we see

$$E(u)M \subset M_1 \subset M. \tag{7.1.7}$$

Define $F : M \to M$ by sending $m \in M$ to $F_1(E(u)m)$. Then (M, M_1, F, F_1) defines a window over $\mathcal{B}$. Hence, we obtain a functor

$$\mathcal{F} : \mathrm{BT}^{\varphi}_{\mathfrak{S}} \longrightarrow (\text{windows over } \mathcal{B}).$$

The functor $\mathcal{F}$ is an equivalence (cf. [Lau10, Lemma 8.2, 8.6]). Its inverse can be described as follows. Let (M, M_1, F, F_1) be a window over $\mathcal{B}$. The $\mathfrak{S}$-module M_1 is necessarily free, and hence the surjection $F_1^{\#} : \varphi^*M_1 \to M$ is an isomorphism. Let $\phi : M_1 \hookrightarrow \varphi^*M_1$ denote the composition of the inclusion $M_1 \hookrightarrow M$ with the inverse of $F_1^{\#}$. There is a unique $\mathfrak{S}$-linear map $\psi : \varphi^*M_1 \to M_1$ such that $\psi\phi = E(u)$. Then (M_1, ψ) defines an object in $\mathrm{BT}^{\varphi}_{\mathfrak{S}}$ and the functor $(M, M_1, F, F_1) \mapsto (M_1, \psi)$ is the inverse of $\mathcal{F}$. Going through the proof of [KM16, Theorem 2.12], we have

$$\mathcal{F} \circ \mathfrak{M}(-) = \mathcal{B}(-).$$

In particular, $\mathfrak{M}(-)$ is also an equivalence. $\square$

Definition 7.1.17. For $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in \mathrm{BT}_{\mathfrak{S}}^{\varphi}$, the Hodge filtration of $\varphi^*\mathfrak{M}$ is defined as

$$\mathfrak{M}/E(u)\varphi^*\mathfrak{M} \subset \varphi^*\mathfrak{M}/E(u)\varphi^*\mathfrak{M},$$

where the inclusion is induced by (7.1.7).

Corollary 7.1.18. *Let $\mathscr{G}$ be a p-divisible group over $\mathcal{O}_K$.*

(1) There exists a natural isomorphism

$$\Theta_{\mathcal{O}_K}(\mathscr{G}) \simeq \varphi^*\mathfrak{M}(\mathscr{G}) \otimes_{\mathfrak{S},\kappa} \mathbb{W}(\mathcal{O}_K)$$

as Dieudonné displays over $\mathcal{O}_K$.

(2) There exists a natural isomorphism

$$\mathbb{D}(\mathscr{G})(\mathcal{O}_K) \simeq \varphi^*\mathfrak{M}(\mathscr{G}) \otimes_{\mathfrak{S}} \mathcal{O}_K = \varphi^*\mathfrak{M}/E(u)\varphi^*\mathfrak{M},$$

which respects the Hodge filtrations on both sides.

Proof. (1) It follows from the equality $\mathcal{F} \circ \mathfrak{M}(-) = \mathcal{B}(-)$ in Proposition 7.1.16 and the definition of base change of Dieudonné displays.

(2) Denote by ψ the isomorphism in (1). By base change of ψ along the natural surjection $\mathbb{W}(\mathcal{O}_K) \to \mathcal{O}_K$, we obtain an isomorphism

$$\Theta_{\mathcal{O}_K}(\mathscr{G})/\mathbb{I}_{\mathcal{O}_K}\Theta_{\mathcal{O}_K}(\mathscr{G}) \simeq \varphi^*\mathfrak{M}(\mathscr{G})/E(u)\varphi^*\mathfrak{M}.$$

Since ψ is an isomorphism of Dieudonné displays, the above isomorphism respects the Hodge filtrations. By Theorem 7.1.14 (2) and (3), we obtain an isomorphism

$$\mathbb{D}(\mathscr{G})(\mathcal{O}_K) \simeq \varphi^*\mathfrak{M}(\mathscr{G}) \otimes_{\mathfrak{S}} \mathcal{O}_K = \varphi^*\mathfrak{M}/E(u)\varphi^*\mathfrak{M}$$

respecting the Hodge filtrations. $\square$

7.2 Deformation theory

In this section, we extend the deformation theory of p-divisible groups in [KP18, §3] to the case $p = 2$. We also generalize the notion of very good Hodge embeddings for $p = 2$, allowing us to construct versal deformation of p-divisible groups with crystalline tensors (see Proposition 7.2.16). In Proposition 7.2.18, we establish a criterion for determining when a deformation is $(\mathcal{G}_W, \mu_y)$-adapted in the sense of Definition 7.2.17.

7.2.1 Versal deformations of p-divisible groups

The notations are as in §7.1. In this subsection, we aim to extend the construction of the versal deformation space of p-divisible groups in [KP18, §3.1] to the case $p = 2$.

Firstly we generalize [Zin01, Theorem 3, 4], which deals with the case when R has residue characteristic $p > 2$ or $2R = 0$.

Theorem 7.2.1. *Let k be a perfect field of characteristic p. Let $(S \to R, \delta)$ be a nilpotent divided power extension of artinian local rings of residue field k, i.e., the kernel $\mathfrak{a}$ of the surjection $S \to R$ is equipped with nilpotent divided powers δ.*

(1) Let $\mathcal{P} = (M, M_1, F, F_1)$ be a Dieudonné display over S and $\overline{\mathcal{P}} = (\overline{M}, \overline{M}_1, F, F_1)$ be the reduction of $\mathcal{P}$ over R. Denote by $\widehat{M_1}$ the inverse image of $\overline{M}_1$ under the homomorphism

$$M \to \overline{M} = \mathbb{W}(R) \otimes_{\mathbb{W}(S)} M.$$

Then $F_1 : M_1 \to M$ extends uniquely to a $\mathbb{W}(S)$-module homomorphism

$$\widehat{F}_1 : \widehat{M_1} \to M$$

such that $\widehat{F}_1(\mathfrak{a}M) = 0$. Therefore, $\widehat{F}_1$ restricted to $\widehat{W}(\mathfrak{a})M$ is given by

$$\widehat{F}_1([a_0, a_1, \dots]x) = [w_0(u_0^{-1})a_1, w_1(u_0^{-1})a_2, \dots]F(x)$$

in logarithmic coordinates.

(2) *Let* $\mathcal{P} = (M, M_1, F, F_1)$ *(resp.* $\mathcal{P}' = (M', M_1', F', F_1')$*) be a Dieudonné display over* S*. Let* $\overline{\mathcal{P}}$ *(resp.* $\overline{\mathcal{P}}'$*) be the reduction over* R*. Assume that* $\overline{u} : \overline{\mathcal{P}} \to \overline{\mathcal{P}}'$ *is a morphism of Dieudonné displays over* R*. Then there exists a unique morphism of quadruples*

$$u : (M, \widehat{M}_1, F, \widehat{F}_1) \to (M', \widehat{M}_1', F', \widehat{F}_1')$$

lifting $\overline{u}$*. Hence, we can associate a crystal to a Dieudonné display as follows: Let* $\mathcal{P} = (M, M_1, F, F_1)$ *be a Dieudonné display over* R*,* $(T \to R, \delta)$ *be a divided power extension, then define the* Dieudonné crystal $\mathbb{D}(\mathcal{P})$ *evaluated at* $(T \to R, \delta)$ *as*

$$\mathbb{D}(\mathcal{P})(T) := T \otimes_{w_0, \mathbb{W}(T)} \widetilde{M},$$

where $\widetilde{\mathcal{P}} = (\widetilde{M}, \widetilde{M}_1, \widetilde{F}, \widetilde{F}_1)$ *is any lifting of* $\mathcal{P}$ *over* T*.*

(3) *Let* $\mathcal{C}$ *be the category of all pairs* $(\mathcal{P}, Fil)$*, where* $\mathcal{P}$ *is a Dieudonné display over* R *and* $Fil \subset \mathbb{D}(\mathcal{P})(S)$ *is a direct summand lifting the Hodge filtration* $M_1/\mathbb{I}_R M \hookrightarrow M/\mathbb{I}_R M$ *of* $\mathbb{D}(\mathcal{P})(R)$*. Then the category* $\mathcal{C}$ *is canonically isomorphic to the category of Dieudonné displays over* S*.*

Remark 7.2.2. The above theorem has a reformulation in terms of *relative Dieudonné displays* as in [Lau14, §2D, 2F]: the quadruple $(M, \widehat{M}_1, F, \widehat{F}_1)$ defines a window over the *relative Dieudonné frame* $\mathcal{D}_{S/R}$.

Proof. The proof adapts arguments in [Zin01, Theorem 3, 4] and [Zin02, Lemma 38, 42], with adjustments for $\mathbb{V}$.

(1) Choose a normal decomposition $M = L \oplus T$. Then

$$\widehat{M}_1 = \widehat{W}(\mathfrak{a})M + M_1 = \mathfrak{a}T \oplus L \oplus \mathbb{I}_S T.$$

Using this decomposition, we can extend F_1 by setting $\widehat{F}_1(\mathfrak{a}T) = 0$. We claim that $\widehat{F}_1(\mathfrak{a}L) = 0$. Note that by formula (7.1.2), we have $\varphi(\mathfrak{a}) = 0$. Since F_1 is φ-linear, we have $\widehat{F}_1(\mathfrak{a}L) = \varphi(\mathfrak{a})\widehat{F}_1(L) = 0$. Thus, the extension $\widehat{F}_1$ satisfies $\widehat{F}_1(\mathfrak{a}M) = 0$. It is unique since $\widehat{M}_1 =$

$\widehat{W}(\mathfrak{a})M + M_1 = \mathfrak{a}M + M_1$. For any $[a_0, a_1, \ldots] \in \widehat{W}(\mathfrak{a})$ and $x \in M$, we have

$$\begin{aligned}\widehat{F}_1([a_0, a_1, \ldots]x) &= \widehat{F}_1([a_0, 0, 0, \ldots]x) + \widehat{F}_1(V[a_1, a_2, \ldots]x) \\ &= 0 + F_1(\mathbb{V}(u_0^{-1}[a_1, \ldots])x) = F_1(\mathbb{V}([w_0(u_0^{-1})a_1, w_1(u_0^{-1})a_2, \ldots])x) \\ &= [w_0(u_0^{-1})a_1, w_1(u_0^{-1})a_2, \ldots]F(x).\end{aligned}$$

(2) For the uniqueness of u, it is enough to consider the case $\overline{u} = 0$. Recall that for a Dieudonné display (M, M_1, F, F_1) over S, we have defined the map $V^\sharp : M \to \mathbb{W}(S) \otimes_{\varphi, \mathbb{W}(S)} M$ in (7.1.4). For any integer $N \geq 1$, we define $(V^N)^\sharp : M \to M \otimes_{\varphi^N, \mathbb{W}(S)} M$ as the composite

$$M \xrightarrow{V^\sharp} \mathbb{W}(S) \otimes_{\varphi, \mathbb{W}(S)} M \xrightarrow{1 \otimes V^\sharp} \mathbb{W}(S) \otimes_{\varphi^2, \mathbb{W}(S)} M \to \cdots \to \mathbb{W}(S) \otimes_{\varphi^N, \mathbb{W}(S)} M.$$

Similarly, we can define maps $(F_1^N)^\#$ and $(\widehat{F}_1^N)^\#$. As in the proof of [Zin01, Theorem 3], we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "(V^N)^\sharp"'] & \widehat{W}(\mathfrak{a})M' \\ \mathbb{W}(S) \otimes_{\varphi^N, \mathbb{W}(S)} M \arrow[r, "1 \otimes u"] & \mathbb{W}(S) \otimes_{\varphi^N, \mathbb{W}(S)} \widehat{W}(\mathfrak{a})M' \arrow[u, "(\widehat{F}_1'^N)^\#"'] \end{tikzcd}$$

By (1), for $[a_0, a_1, \ldots] \in \widehat{W}(\mathfrak{a})$ and $x \in M'$, we have

$$\widehat{F}_1'^N([a_0, \ldots]x) = [\prod_{i=0}^{N-1} w_i(u_0^{-1})a_N, \prod_{i=1}^{N} w_i(u_0^{-1})a_{N+1}, \ldots]F'^N(x).$$

Since $a_i = 0$ for almost all i, $\widehat{W}(\mathfrak{a})M'$ is annihilated by $\widehat{F}_1'^N$ for sufficiently large N. This shows $u = 0$ as desired.

For the existence of u, we can repeat the proof of [Zin01, Theorem 3].

(3) Clearly we can get a lifting of the Hodge filtration of $\mathbb{D}(\mathcal{P})(R)$ from a Dieudonné display over S. On the other hand, given $(\mathcal{P}, Fil) \in \mathcal{C}$, any lifting of $\mathcal{P}$ to S gives a unique quadruple $(M, \widehat{M_1}, F, \widehat{F}_1)$ by (2). Let $M_1 \subset \widehat{M_1}$ be the inverse image of $Fil \subset M/\mathbb{I}_S M$ under the projection $M \to M/\mathbb{I}_S M$, then we obtain a Dieudonné display $(M, M_1, F, \widehat{F}_1|_{M_1})$ over S. By (2), these two constructions are mutually inverse. $\square$

Now we fix a p-divisible group $\mathscr{G}_0$ over k, and let $(\mathbb{D}, \mathbb{D}_1, F, F_1)$ be the corresponding Dieudonné display. Note that $\mathbb{D}$ is given by $\mathbb{D}(\mathscr{G}_0)(W)$, see [Lau14, Corollary 2.34]. By Lemma 7.1.10, the Dieudonné display $(\mathbb{D}, \mathbb{D}_1, F, F_1)$ corresponds to a triple $(\mathbb{D}, \mathbb{D}_1, \Psi_0)$ for an isomorphism $\Psi_0 : \widetilde{\mathbb{D}}_1 \xrightarrow{\sim} \mathbb{D}$. Next we will construct a versal deformation space of $\mathscr{G}_0$, equivalently a versal deformation space of the Dieudonné display $(\mathbb{D}, \mathbb{D}_1, \Psi_0)$.

Recall there is a canonical Hodge filtration on $\mathbb{D} \otimes_W k = \mathbb{D}(\mathscr{G}_0)(k)$:

$$0 \to \mathrm{Hom}_k(\mathrm{Lie}\,\mathscr{G}_0, k) \to \mathbb{D} \otimes_W k \to \mathrm{Lie}\,\mathscr{G}_0^* \to 0.$$

We think of $\mathbb{D} \otimes_W k$ as a filtered k-module by setting $\mathrm{Fil}^0(\mathbb{D} \otimes_W k) = \mathbb{D} \otimes_W k$, $\mathrm{Fil}^1(\mathbb{D} \otimes_W k) = \mathrm{Hom}_k(\mathrm{Lie}\,\mathscr{G}_0, k)$. This filtration corresponds to a parabolic subgroup $P_0 \subset \mathrm{GL}(\mathbb{D} \otimes_W k)$. Fix a lifting of P_0 to a parabolic subgroup $P \subset \mathrm{GL}(\mathbb{D})$. Write

$$M^{\mathrm{loc}} = \mathrm{GL}(\mathbb{D})/P \text{ and } \widehat{M}^{\mathrm{loc}} = \mathrm{Spf}\, R, \tag{7.2.1}$$

where $\widehat{M}^{\mathrm{loc}}$ is the completion of $\mathrm{GL}(\mathbb{D})/P$ along the image of the identity in $\mathrm{GL}(\mathbb{D} \otimes_W k)$. Then R is a power series ring over W.

Set $M = \mathbb{D} \otimes_W \mathbb{W}(R)$, and let $\overline{M}_1 \subset M/\mathbb{I}_R M$ be the direct summand corresponding to the parabolic subgroup $gPg^{-1} \subset \mathrm{GL}(\mathbb{D})$ over $\widehat{M}^{\mathrm{loc}}$, where $g \in (\mathrm{GL}(\mathbb{D})/P)(R)$ is the universal point. Let $M_1 \subset M$ be the preimage of $\overline{M}_1$ in M and $\Psi : \widetilde{M}_1 \xrightarrow{\sim} M$ be a $\mathbb{W}(R)$-module isomorphism reducing to Ψ_0 modulo $\mathfrak{m}_R$, where $\widetilde{M}_1$ is defined as in Lemma 7.1.10. Then the triple

$$(M, M_1, \Psi)$$

gives a Dieudonné display over R reducing to $(\mathbb{D}, \mathbb{D}_1, \Psi_0)$. By Theorem 7.1.14, the Dieudonné display (M, M_1, Ψ) corresponds to a p-divisible group $\mathscr{G}_R$ over R, which is a deformation of $\mathscr{G}_0$.

Set $\mathfrak{a}_R \coloneqq \mathfrak{m}_R^2 + pR$. By Lemma 7.1.13, there exists a natural connection isomorphism

$$c : \widetilde{\mathbb{D}}_1 \otimes_W \mathbb{W}(R/\mathfrak{a}_R) \xrightarrow{\sim} \widetilde{M}_1 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{a}_R).$$

Definition 7.2.3. The map Ψ is said to be *constant modulo* $\mathfrak{a}_R$ if the composite map

$$\widetilde{\mathbb{D}}_1 \otimes_W \mathbb{W}(R/\mathfrak{a}_R) \xrightarrow{c} \widetilde{M}_1 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{a}_R) \xrightarrow{\Psi\otimes 1} M_{R/\mathfrak{a}_R} \simeq \mathbb{D} \otimes_W \mathbb{W}(R/\mathfrak{a}_R)$$

is equal to $\Psi_0 \otimes 1$.

Lemma 7.2.4. *If Ψ is constant modulo $\mathfrak{a}_R$, then the deformation $\mathscr{G}_R$ of $\mathscr{G}_0$ is versal.*

Proof. Recall that there exists a versal deformation ring R^{univ} for $\mathscr{G}_0$, which is a power series ring over W of the same dimension as R. The deformation $\mathscr{G}_R$ is induced by a map $R^{\text{univ}} \to R$. We want to show this is an isomorphism. It suffices to prove that the induced map on tangent spaces is an isomorphism.

We have two Dieudonné displays over $R/\mathfrak{a}_R$. One is obtained from $(M, M_1, F_R, F_{R,1})$ (the Dieudonné display corresponding to (M, M_1, Ψ)) by the base change along $R \to R/\mathfrak{a}_R$, the other is obtained from $(\mathbb{D}, \mathbb{D}_1, F, F_1)$ by the base change along $k \to R/\mathfrak{a}_R$.

If Ψ is constant modulo $\mathfrak{a}_R$, then as in the proof of [KP18, Lemma 3.1.12], we know $\widehat{F}_{R,1} = \widehat{F}_1$ on $\widehat{M}_{R/\mathfrak{a}_R,1}$, see the notation in Theorem 7.2.1. Hence, these two Dieudonné displays give rise to the same quadruple

$$(M_{R/\mathfrak{a}_R}, \widehat{M}_{R/\mathfrak{a}_R,1}, F_{R/\mathfrak{a}_R}, \widehat{F}_{R/\mathfrak{a}_R,1}).$$

Let $\mathscr{G}$ be a deformation over the ring $k[\epsilon]$ of dual numbers. Since $k[\epsilon] \to k$ has trivial divided powers, it is a nilpotent divided power extension, then by Theorem 7.2.1 (1) and (2), the base change of $(\mathbb{D}, \mathbb{D}_1, F, F_1)$ along the natural map $k \to k[\epsilon]$ gives rise to a quadruple $(M_{k[\epsilon]}, \widehat{M}_{k[\epsilon],1}, F_{k[\epsilon]}, \widehat{F}_{k[\epsilon],1})$. By the proof of Theorem 7.2.1 (3), the Dieudonné display corresponding to $\mathscr{G}$ is of the form

$$(M_{k[\epsilon]}, \widetilde{\mathrm{Fil}}, F_{k[\epsilon]}, \widehat{F}_{k[\epsilon],1}),$$

where $\widetilde{\mathrm{Fil}} \subset \widehat{M}_{k[\epsilon],1}$ is the preimage of certain lifting $\mathrm{Fil} \subset (\mathbb{D} \otimes_W k) \otimes_k k[\epsilon]$ of the Hodge filtration of $\mathbb{D}$. From the versality of the filtration $\overline{M}_1 \subset \mathbb{D} \otimes_W R$, there is a map $\alpha : R \to k[\epsilon]$ (necessarily factors through $R/\mathfrak{a}_R$) such that the induced map $\mathbb{D} \otimes_W R \to \mathbb{D} \otimes_W k[\epsilon]$ sends

$\overline{M}_1$ to Fil. Then by the discussion in the previous paragraph, $(M_{k[\epsilon]}, \widetilde{\mathrm{Fil}}, F_{k[\epsilon]}, \widehat{F}_{k[\epsilon],1})$ is the base change of $(M, M_1, F_R, F_{R,1})$ along α. Thus, $\mathscr{G}$ is the base change of $\mathscr{G}_R$ along α. In particular, $R^{\mathrm{univ}} \to R$ induces an isomorphism of tangent spaces. Hence, we proved that $\mathscr{G}_R$ is versal. □

Remark 7.2.5. Note that the functor $\mathcal{F} := \underline{\mathrm{Isom}}(\widetilde{M}_1, M)$ of isomorphisms of finite free $\mathbb{W}(R)$-modules between $\widetilde{M}_1$ and M is a $\mathrm{GL}(M)$-torsor over $\mathbb{W}(R)$. Hence, the surjection $\mathbb{W}(R) \twoheadrightarrow \mathbb{W}(R/\mathfrak{a}_R)$ induces a surjection $\mathcal{F}(\mathbb{W}(R)) \twoheadrightarrow \mathcal{F}(\mathbb{W}(R/\mathfrak{a}_R))$. This implies that an isomorphism Ψ, which is constant modulo $\mathfrak{a}_R$, always exists.

7.2.2 Local models and local Hodge embeddings

Before discussing the deformation of p-divisible groups with crystalline tensors, we will make a digression into local models and local Hodge embeddings in this subsection.

Definition 7.2.6 ([KPZ24, Definition 3.1.2])**.** Let $F/\mathbb{Q}_p$ be a complete discrete valued field. Let $(G, \{\mu\}, \mathcal{G})$ be a local model triple over F (see §3.4).

(1) A pair (G, μ) is of *(local) Hodge type* if there is a closed immersion $\rho : G \hookrightarrow \mathrm{GL}(V)$, where V is an F-vector space of dimension h, such that

  (i) ρ is a minuscule representation in the sense of [KP18, §1.2.9].

  (ii) $\rho \circ \mu$ is conjugate to the standard minuscule cocharacter μ_d of $\mathrm{GL}(V_{\overline{F}})$, where

$$\mu_d(t) := \mathrm{diag}(t^{(d)}, 1^{(h-d)}),\ t \in \overline{F}.$$

  (iii) $\rho(G)$ contains the scalars.

  Such a ρ will be said to give a (local) Hodge embedding $\rho : (G, \mu) \hookrightarrow (\mathrm{GL}(V), \mu_d)$.

(2) An *integral Hodge embedding* for $(\mathcal{G}, \mu)$ is a closed immersion $\rho : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda)$ over $\mathcal{O}_F$, where Λ is a finite free $\mathcal{O}_F$-module, such that the base change $\rho \otimes_{\mathcal{O}_F} F$ is a Hodge embedding for (G, μ).

Lemma 7.2.7. *Let $(G, \{\mu\}, \mathcal{G})$ be a local model triple over F. Suppose $\rho : (\mathcal{G}, \mu) \hookrightarrow (\mathrm{GL}(\Lambda), \mu_d)$ is an integral Hodge embedding. Then ρ induces a closed immersion*

$$X_{G,\mu} = G/P_\mu \hookrightarrow X_{\mathrm{GL}(V),\mu_d} \otimes_F E = \mathrm{Gr}(d, V)_E,$$

where $\mathrm{Gr}(d, V)$ denotes the Grassmannian classifying subspaces of V of rank d. Let $\overline{X}_{G,\mu}$ be the (reduced) Zariski closure of $X_{G,\mu} \subset \mathrm{Gr}(d, V)_E$ in $\mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}$.

If $\overline{X}_{G,\mu}$ is normal, then $\overline{X}_{G,\mu}$ is isomorphic to $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$, and the closed immersion $\overline{X}_{G,\mu} \hookrightarrow \mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}$ is identified with the natural morphism $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu} \to \mathbb{M}^{\mathrm{loc}}_{\mathrm{GL}(\Lambda),\mu_d} \otimes_{\mathcal{O}_F} \mathcal{O}_E$ induced by ρ.

Proof. See [KPZ24, Lemma 3.4.1]. Note that by [GL24], the condition in *loc. cit.* requiring the special fiber of $\overline{X}_{G,\mu}$ to be reduced is in fact implied by the remaining conditions. □

Definition 7.2.8 ([KPZ24, Definition 3.4.4])**.** Let $\rho : (\mathcal{G}, \mu) \hookrightarrow (\mathrm{GL}(\Lambda), \mu_d)$ be an integral Hodge embedding over $\mathcal{O}_F$. We say that ρ is a *good* Hodge embedding, if the morphism

$$\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu} \longrightarrow \mathbb{M}^{\mathrm{loc}}_{\mathrm{GL}(\Lambda),\mu_d} \otimes_{\mathcal{O}_F} \mathcal{O}_E$$

induced by ρ is a closed immersion.

By Lemma 7.2.7, ρ is good if the Zariski closure of $X_{G,\mu}$ in $\mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}$ is normal.

From now on, we suppose that $F/\mathbb{Q}_p$ is unramified and $\rho : (\mathcal{G}, \mu) \hookrightarrow (\mathrm{GL}(\Lambda), \mu_d)$ is a good integral Hodge embedding over $\mathcal{O}_F$. In particular, we have a closed immersion $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu} \hookrightarrow \mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}$.

For any $x \in \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu}(k)$, where $k = \overline{\mathbb{F}}_p$, we let $R_G = R_{G,x}$ (resp. R_E) denote the completion of $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$ (resp. $\mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}$) at x. By our assumptions, R_E is isomorphic to a power series ring over $\mathcal{O}_E W(k)$ and R_G is a (normal) quotient ring of R_E. Then $\mathbb{W}(R_E)$ and $\mathbb{W}(R_G)$ are p-torsion free rings. Set

$$M \coloneqq \Lambda \otimes_{\mathcal{O}_F} \mathbb{W}(R_E).$$

Let $\overline{M}_1 \subset M/\mathbb{I}_{R_E} M = \Lambda \otimes_{\mathcal{O}_F} R_E$ be the direct summand corresponding to the universal R_E-valued point of $\mathrm{Gr}(d, \Lambda)$. Set

$$M_1 \coloneqq \text{ the preimage of } \overline{M}_1 \text{ in } M.$$

Then (M, M_1) is a Dieudonné pair over R_E. By the base change along $R_E \twoheadrightarrow R_G$, we obtain a Dieudonné pair $(M_{R_G}, M_{R_G,1})$ over R_G. By Lemma 7.1.12, we can associate a free $\mathbb{W}(R_G)$-module $\widetilde{M}_{R_G,1}$ with

$$\widetilde{M}_{R_G,1}[1/p] = (\varphi^* M_{R_G})[1/p].$$

Definition 7.2.9. For any ring A and a finite free A-module N, we denote by $N^{\otimes}$ the direct sum of all A-modules which can be formed from N by using the operations of taking tensor products, duals, symmetric and exterior powers. If N is equipped with a filtration, then $N^{\otimes}$ is equipped with a filtration accordingly.

If $(s_\alpha) \subset N^{\otimes}$ and $G \subset \mathrm{GL}(N)$ is the pointwise stabilizer of s_α, we say G is the group scheme *cut out* by the tensors s_α.

Lemma 7.2.10 ([Kis10, Proposition 1.3.2])**.** *Suppose that A is a discrete valuation ring of mixed characteristic and N is a finite free A-module. If $G \subset \mathrm{GL}(N)$ is a closed A-flat subgroup whose generic fiber is reductive, then G is cut by a finite collection of tensors in $N^{\otimes}$.*

Remark 7.2.11. By an argument of Deligne, the tensors in Lemma 7.2.10 can be taken in the submodule $\oplus_{m,n\geq 0} N^{\otimes m} \otimes_A (N^{\vee})^{\otimes n}$. Here, $N^{\vee}$ denotes the A-dual module $\mathrm{Hom}_A(N, A)$.

Let $\rho : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda)$ be a Hodge embedding. Then $\mathcal{G} \subset \mathrm{GL}(\Lambda)$ (via ρ) is cut out by a set of tensors $(s_\alpha) \subset \Lambda^{\otimes}$ by Lemma 7.2.10. Set

$$\widetilde{s}_\alpha \coloneqq s_\alpha \otimes 1 = \varphi^*(s_\alpha \otimes 1) \in \Lambda^{\otimes} \otimes_{\mathcal{O}_F} \mathbb{W}(R_G) = \varphi^* M_{R_G}^{\otimes}.$$

We may view $(\widetilde{s}_\alpha)$ as tensors in $(\varphi^* M_{R_G})^{\otimes}[1/p] = \widetilde{M}_{R_G,1}^{\otimes}[1/p]$. By [KPZ24, §5.2] (and [KP18, Corollary 3.2.11]), we have the following proposition.

Proposition 7.2.12. *Suppose that $F/\mathbb{Q}_p$ is unramified and $\rho : (\mathcal{G}, \mu) \hookrightarrow (\mathrm{GL}(\Lambda), \mu_d)$ is a good integral Hodge embedding over $\mathcal{O}_F$. Then $\widetilde{s}_\alpha \in \widetilde{M}_{R_G,1}^{\otimes}$.*

Denote by $\widetilde{s}_{\alpha,0}$ the reduction of $\widetilde{s}_\alpha$ in $\widetilde{M}_{0,1}^\otimes$, where $\widetilde{M}_{0,1} = \widetilde{M}_{R_G,1} \otimes_{\mathbb{W}(R_G)} W(k)$. By Lemma 7.1.13, we have a connection isomorphism

$$c_{\mathcal{G}} : \widetilde{M}_{0,1} \otimes_{W(k)} \mathbb{W}(R_G/\mathfrak{a}_{R_G}) \xrightarrow{\sim} \widetilde{M}_{R_G,1} \otimes_{\mathbb{W}(R_G)} \mathbb{W}(R_G/\mathfrak{a}_{R_G}).$$

Definition 7.2.13. Under the assumptions in Proposition 7.2.12, we say that ρ is *very good* at $x \in \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu}(k)$, if $c_{\mathcal{G}}(\widetilde{s}_{\alpha,0} \otimes 1) = \widetilde{s}_\alpha \otimes 1$. In this case, we say that the tensors $(\widetilde{s}_\alpha)$ are *horizontal* at x.

We say ρ is a very good (integral) Hodge embedding if ρ is very good at every $x \in \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu}(k)$.

Definition 7.2.14 ([KPZ24, Definition 4.1.4])**.** For a scheme X over k and $x \in X(k)$, we say that the tangent space T_xX of X at x is *spanned by smooth formal curves* if the images of the tangent spaces by k-morphisms $\operatorname{Spec} k[[t]] \to X$ with the closed point mapping to x generate the k-vector space T_xX.

We will use the following lemma in §7.3.3.3.

Lemma 7.2.15 ([KPZ24, Proposition 5.3.10])**.** *Assume $\rho : (\mathcal{G}, \mu) \hookrightarrow (\mathrm{GL}(\Lambda), \mu_d)$ is a good integral Hodge embedding over $\mathbb{Z}_p$. Let $x \in \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu}(k)$ be a closed point. If the tangent space of the special fiber $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu} \otimes_{\mathcal{O}_E} k$ at x is spanned by smooth formal curves, then ρ is very good at x.*

We refer to [KPZ24, §5.3] for more properties of very good Hodge embeddings.

7.2.3 Deformations with crystalline tensors

We continue to use the notation in §7.2.1, and as in [KP18, §3.2, 3.3], we may assume k is algebraically closed for simplicity.

Let $\mathscr{G}_0$ be a p-divisible group over k. Denote $\mathbb{D} = \mathbb{D}(\mathscr{G}_0)(W)$. Let $(s_{\alpha,0}) \subset \mathbb{D}^\otimes$ be a collection of φ-invariant tensors whose images in $\mathbb{D}(\mathscr{G}_0)(k)^\otimes$ lie in $\mathrm{Fil}^0\, \mathbb{D}(\mathscr{G}_0)(k)^\otimes$. In this subsection, we assume the following conditions:

(A1) there is an isomorphism $\Lambda \otimes_{\mathbb{Z}_p} W \simeq \mathbb{D}$ for some free $\mathbb{Z}_p$-module Λ such that $s_{\alpha,0} \in \Lambda^\otimes$;

(A2) the stabilizer group scheme $\mathcal{G} \subset \mathrm{GL}(\Lambda)$ cut out by $(s_{\alpha,0}) \subset \Lambda^{\otimes}$ has reductive generic fiber G and $\mathcal{G}^{\circ}$ is a parahoric group scheme over $\mathbb{Z}_p$.

Note that the base change $\mathcal{G}_W := \mathcal{G} \otimes_{\mathbb{Z}_p} W \subset \mathrm{GL}(\mathbb{D})$ is cut out by $(s_{\alpha,0}) \subset \mathbb{D}^{\otimes}$. In (7.2.1) of §7.2.1, we have defined M^{loc} and $\widehat{M}^{\mathrm{loc}} = \operatorname{Spf} R$. Let K'/K_0 be a finite extension and $y : R \to K'$ be a map such that $s_{\alpha,0} \in \mathrm{Fil}^0(\mathbb{D} \otimes_W K')^{\otimes}$ for the filtration induced by y on $\mathbb{D} \otimes_W K'$. By [Kis10, Lemma 1.4.5], the filtration is induced by a G-valued cocoharacter μ_y. We further impose the following assumption:

(A3) there is a very good Hodge embedding $(\mathcal{G}, \mu_y^{-1}) \hookrightarrow (\mathrm{GL}(\Lambda), \mu_d)$ for $d = \dim_k \mathrm{Lie}\,\mathscr{G}_0$.

Denote by $E \subset K'$ the local reflex field of the G-conjugacy class of cocharacters $\{\mu_y\}$. Write $M^{\mathrm{loc}}_{G,y}$ for the closure of the G-orbit $G.y \subset M^{\mathrm{loc}} \otimes_W E$ in $M^{\mathrm{loc}} \otimes_W \mathcal{O}_E$. By assumption (A3) and Lemma 7.2.7, the scheme $M^{\mathrm{loc}}_{G,y}$ is isomorphic to the local model $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_y^{-1}}$ attached to the local model triple $(G, \{\mu_y^{-1}\}, \mathcal{G})$, and hence $M^{\mathrm{loc}}_{G,y}$ is normal and only depends on the G-conjugacy class $\{\mu_y\}$ (not on y). We denote by $\widehat{M}^{\mathrm{loc}}_{G,y} = \operatorname{Spf} R_G$ the completion of $M^{\mathrm{loc}}_{G,y}$ along (the image of) the identity in $\mathrm{GL}(\mathbb{D} \otimes_W k)$. Then R_G is a normal quotient ring of $R \otimes_W \mathcal{O}_E$.

Recall in §7.2.1, we constructed a versal deformation $\mathscr{G}_R$ over R corresponding to a Dieudonné display (M, M_1, Ψ), where Ψ is constant modulo $\mathfrak{a}_R$. Set

$$M_{R_E} := M \otimes_{\mathbb{W}(R)} \mathbb{W}(R_E), \quad M_{R_G} := M_{R_E} \otimes_{\mathbb{W}(R_E)} \mathbb{W}(R_G).$$

The tensors $s_{\alpha,0} \in \mathbb{D}^{\otimes}$ induce tensors in $M_{R_G}^{\otimes}$, still denoted as $s_{\alpha,0}$. Notice that $\widetilde{M}_{R_G,1} \subset \varphi^* M_{R_G}$ and $(s_{\alpha,0})$ are φ-invariant. By [KP18, Corollary 3.2.11], we have $(s_{\alpha,0}) \subset \widetilde{M}_{R_G,1}$. (Here we uses [Ans22, Proposition 10.3] to remove the condition (3.2.3) in [KP18].) Recall that the p-divisible group $\mathscr{G}_0$ over k corresponds to a Dieudonné display $(\mathbb{D}, \mathbb{D}_1, \Psi_0 : \widetilde{\mathbb{D}}_1 \xrightarrow{\sim} \mathbb{D})$. Since $\widetilde{\mathbb{D}}_1 = \varphi^*(\mathbb{D})$ and $(s_{\alpha,0})$ are φ-invariant, we have $(s_{\alpha,0}) \subset \widetilde{\mathbb{D}}_1^{\otimes}$. Set

$$\mathfrak{a}_{R_E} := \mathfrak{m}_{R_E}^2 + \pi_E R_E,$$

where $\pi_E \in \mathcal{O}_E$ is a uniformizer. In particular $R_E/\mathfrak{a}_{R_E} \simeq R/\mathfrak{a}_R$. Set

$$\mathfrak{a}_{R_G} \coloneqq \mathfrak{m}_{R_G}^2 + \pi_E R_G,$$

Proposition 7.2.16 (cf. [KP18, §3.2.12])**.** *Assume (A1) to (A3).*

(1) The scheme

$$\mathcal{T} \coloneqq \underline{\mathrm{Isom}}_{(s_{\alpha,0})}(\widetilde{M}_{R_G,1}, M_{R_G})$$

consisting of isomorphisms respecting tensors $s_{\alpha,0}$ is a trivial $\mathcal{G}$-torsor over $\mathbb{W}(R_G)$.

(2) There exists an isomorphism $\Psi_{R_G} : \widetilde{M}_{R_G,1} \xrightarrow{\sim} M_{R_G}$ respecting $s_{\alpha,0}$ which lifts to an isomorphism $\Psi_{R_E} : \widetilde{M}_{R_E,1} \to M_{R_E}$ that is constant modulo $\mathfrak{a}_{R_E}$. Moreover, the p-divisible group $\mathscr{G}_{R_E}$ over R_E corresponding to the Dieudonné display $(M_{R_E}, M_{R_E,1}, \Psi_{R_E})$ is a versal deformation of $\mathscr{G}_0$.

Proof. (1) This follows from [KP18, Corollary 3.2.11] and [Ans22, Proposition 10.3].

(2) By assumption (A3), the isomorphism $\Psi_{R_G/\mathfrak{a}_{R_G}}$

$$\widetilde{M}_{R_G,1} \otimes_{\mathbb{W}(R_G)} \mathbb{W}(R_G/\mathfrak{a}_{R_G}) \xrightarrow{c_{\mathcal{G}}^{-1}} \widetilde{\mathbb{D}}_1 \otimes_W \mathbb{W}(R_G/\mathfrak{a}_{R_G}) \xrightarrow{\Psi_0 \otimes 1} \mathbb{D} \otimes_W \mathbb{W}(R_G/\mathfrak{a}_{R_G}) = M_{R_G/\mathfrak{a}_{R_G}}$$

preserves the tensors $s_{\alpha,0}$, and hence defines a point in $\mathcal{T}(\mathbb{W}(R_G/\mathfrak{a}_{R_G}))$. Since $\mathcal{T}$ is a $\mathcal{G}$-torsor, we can lift the point to a point in $\mathcal{T}(\mathbb{W}(R_G))$, which corresponds to an isomorphism $\Psi_{R_G} : \widetilde{M}_{R_G,1} \xrightarrow{\sim} M_{R_G}$ respecting $s_{\alpha,0}$. By construction, $\Psi_{R_G/\mathfrak{a}_{R_G}}$ is the reduction of the isomorphism $\Psi_{R_E/\mathfrak{a}_{R_E}}$

$$\widetilde{M}_{R_E,1} \otimes_{\mathbb{W}(R_E)} \mathbb{W}(R_E/\mathfrak{a}_{R_E}) \xrightarrow{c^{-1}} \widetilde{\mathbb{D}}_1 \otimes_W \mathbb{W}(R_E/\mathfrak{a}_{R_E}) \xrightarrow{\Psi_0 \otimes 1} \mathbb{D} \otimes_W \mathbb{W}(R_E/\mathfrak{a}_{R_E}) = M_{R_E/\mathfrak{a}_{R_E}}.$$

Denote by $\mathcal{F}$ the $\mathrm{GL}(M_{R_E})$-torsor $\underline{\mathrm{Isom}}(\widetilde{M}_{R_E,1}, M_{R_E})$ over $\mathbb{W}(R_E)$. Then Ψ_{R_G} and $\Psi_{R_E/\mathfrak{a}_{R_E}}$ define a point of $\mathcal{F}$ valued in $\mathbb{W}(R_G) \times_{\mathbb{W}(R_G/\mathfrak{a}_{R_G})} \mathbb{W}(R_E/\mathfrak{a}_{R_E}) = \mathbb{W}(R_G \times_{R_G/\mathfrak{a}_{R_G}} R_E/\mathfrak{a}_{R_E})$. We can lift this point to an $\mathbb{W}(R_E)$-valued point of $\mathcal{F}$, which corresponds to an isomorphism $\Psi_{R_E} : \widetilde{M}_{R_E,1} \xrightarrow{\sim} M_{R_E}$. Hence, Ψ_{R_E} is constant modulo $\mathfrak{a}_{R_E}$. By Lemma 7.2.4 and the discussion in [KP18, §3.2.12], the Dieudonné display $(M_{R_E}, M_{R_E,1}, \Psi_{R_E})$ is versal. $\square$

Following [Zho20, §4], we make the following definition.

Definition 7.2.17. Let $\mathscr{G}$ be a p-divisible group over $\mathcal{O}_K$ deforming $\mathscr{G}_0$. We say that $\mathscr{G}$ is $(\mathcal{G}_W, \mu_y)$-*adapted* if the tensors $s_{\alpha,0}$ lift to Frobenius invariant tensors $\widetilde{s}_\alpha \in \Theta_{\mathcal{O}_K}(\mathscr{G})^{\otimes}$ such that the following two conditions hold:

(1) There is an isomorphism $\Theta_{\mathcal{O}_K}(\mathscr{G}) \simeq \mathbb{D} \otimes_W \mathbb{W}(\mathcal{O}_K)$ sending $\widetilde{s}_\alpha$ to $s_{\alpha,0} \otimes 1$.

(2) Under the canonical isomorphism $\mathbb{D}(\mathscr{G})(\mathcal{O}_K) \otimes_{\mathcal{O}_K} K \simeq \mathbb{D} \otimes_W K$, the filtration on $\mathbb{D} \otimes_W K$ is induced by a G-valued cocharacter G-conjugate to μ_y.

Proposition 7.2.18. *Assume (A1) to (A3). View* $\operatorname{Spf} R_E$ *as the versal deformation space of* $\mathscr{G}_0$ *by the construction in Proposition 7.2.16 (2). Then for any finite extension* K/E*, a map* $\xi : R_E \to \mathcal{O}_K$ *factors through* R_G *if and only if the* p*-divisible group* $\mathscr{G}_\xi = \xi^* \mathscr{G}_{R_E}$ *is* $(\mathcal{G}_W, \mu_y)$*-adapted.*

Proof. ($\Rightarrow$) See [Zho20, Proposition 4.7] and [KZ24, Proposition 3.2.7].

($\Leftarrow$) The proof goes as in [KP18, Proposition 3.2.17]. For completeness, we recall the arguments here. Suppose $\mathscr{G}_\xi$ is $(\mathcal{G}_W, \mu_y)$-adapted. Denote by $s_\alpha \in \mathbb{D}(\mathscr{G})(\mathcal{O}_K)^{\otimes}$ the image of $\widetilde{s}_\alpha$ modulo $\mathbb{I}_{\mathcal{O}_K}$. Then the isomorphism in (1) of Definition 7.2.17 gives an isomorphism $\mathbb{D}_{\mathcal{O}_K} \coloneqq \mathbb{D} \otimes_W \mathcal{O}_K \xrightarrow{\sim} \mathbb{D}(\mathscr{G})(\mathcal{O}_K)$ taking $s_{\alpha,0}$ to s_α. Hence, by (2) in Definition 7.2.17, this isomorphism induces a filtration on $\mathbb{D}_{\mathcal{O}_K}$ corresponding to a map $y' : R_G \to \mathcal{O}_K$ and $s_{\alpha,0} \in \mathrm{Fil}^0 \mathbb{D}_{\mathcal{O}_K}^{\otimes}$. As R_G depends only on the reduction of y and the conjugacy class of μ_y, we may assume $y = y'$ (and $K' = K$).

The map $y : R_G \to \mathcal{O}_K$ induces a Dieudonné display $(M_{\mathcal{O}_K}, M_{\mathcal{O}_K,1}, \Psi)$, and by the construction of Ψ_{R_G}, the isomorphism $\Psi : \widetilde{M}_{\mathcal{O}_K,1} \xrightarrow{\sim} M_{\mathcal{O}_K}$ takes $s_{\alpha,0}$ to $s_{\alpha,0}$. Since $y = y'$, the p-divisible group $\mathscr{G}_\xi$ corresponds to a Dieudonné display $(M_{\mathcal{O}_K}, M_{\mathcal{O}_K,1}, \Psi')$. As $\widetilde{s}_\alpha$ is Frobenius invariant and Ψ' differs from the Frobenius a scalar (contained in G by assumption), then Ψ' takes $s_{\alpha,0}$ to $s_{\alpha,0}$, and reduces to $\Psi_0 : \widetilde{\mathbb{D}}_1 \xrightarrow{\sim} \mathbb{D}$.

Now we construct a Dieudonné display over $S \coloneqq \mathcal{O}_K[[T]]$. First consider the Dieudonné display $(M_S, M_{S,1}, \Psi)$, the base change of $(M_{\mathcal{O}_K}, M_{\mathcal{O}_K,1}, \Psi)$ to S. The map $S \to \mathcal{O}_K \times_k \mathcal{O}_K$

given by $T \mapsto (0, \pi)$ is surjective, and hence so is $\mathbb{W}(S) \to \mathbb{W}(\mathcal{O}_K) \times_W \mathbb{W}(\mathcal{O}_K)$. Note that by Proposition 7.2.16, $\mathcal{T}$ is a (trivial) $\mathcal{G}$-torsor. Since $\mathcal{G}$ is smooth, we have a surjection

$$\mathcal{T}(\mathbb{W}(S)) \twoheadrightarrow \mathcal{T}(\mathbb{W}(\mathcal{O}_K) \times_W \mathbb{W}(\mathcal{O}_K)).$$

That is, there exists an isomorphism $\Psi_S : \widetilde{M}_{S,1} \xrightarrow{\sim} M_S$ which takes $s_{\alpha,0}$ to $s_{\alpha,0}$, and specializes to (Ψ, Ψ') under $T \mapsto (0, \pi)$. We take $\mathcal{M}_S$ to be the Dieudonné display associated to $(M_S, M_{S,1}, \Psi_S)$.

By versality, we may lift the map $(y, \xi) : R_E \to \mathcal{O}_K \times_k \mathcal{O}_K$ to a map $\widetilde{\xi} : R_E \to S$ which induces the Dieudonné display $\mathcal{M}_S$ and $\mathcal{M}_S$ is the base change of $\mathcal{M}_{R_E}$ by $\widetilde{\xi}$. Now the rest of the proof is similar as in [KP18, Proposition 3.2.17]. Then we conclude that $\widetilde{\xi}$ factors though R_G, and hence ξ does as well. □

In §7.3, we will construct $(\mathcal{G}_W, \mu_y)$-adapted deformations of p-divisible groups associated to closed points in integral models of Shimura varieties, and apply Proposition 7.2.18 to describe the local structure of integral models of Shimura varieties.

7.3 Integral models of Shimura varieties of abelian type

In this section, we will prove Theorem 1.2.7 in the Introduction. Following the strategy of [KP18; KPZ24], we first consider Shimura varieties $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ of Hodge type. We construct their integral models $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ by using the Hodge embeddings into Siegel modular varieties, as in *loc. cit.*. Under certain assumptions (see Theorem 7.3.4), we apply the deformation theory developed in §7.2 to identify the formal neighborhood of $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ with that of the local model. Then we extend this construction of integral models to the case of Shimura varieties of abelian type by choosing suitable Hodge type lifts under certain conditions (see Theorem 7.3.9). We complete the proof of Theorem 1.2.7 by showing that these conditions are satisfied in Case (A) or (B).

7.3.1 Shimura varieties of Hodge type

Let $(\mathbf{G}, X)$ be a Shimura datum, that is, $\mathbf{G}$ is a reductive group over $\mathbb{Q}$ and X is a $\mathbf{G}(\mathbb{R})$-conjugacy class of

$$h : \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathbf{G}_{\mathbb{R}}$$

satisfying axioms 2.1.1.1-2.1.1.3 in [Del79, §2.1]. Denote by $\mu_h : \mathbb{G}_{m\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$ the associated Hodge cocharacter, defined by $\mu_h(z) = h_{\mathbb{C}}(z, 1)$. Set $w_h := \mu_h^{-1}\mu_h^{c-1}$ (the weight homomorphism), where c denotes the complex conjugation.

Fix a $\mathbb{Q}$-vector space V of dimension $2g$ with a perfect alternating pairing $\psi : V \times V \to \mathbb{Q}$. Let $\mathbf{GSp} = \mathbf{GSp}(V, \psi)$ be the corresponding symplectic similitude group over $\mathbb{Q}$, and let $S^{\pm} = S^{\pm}(V, \psi)$ be the Siegel double space consisting of maps $h : \mathbb{S} \to \mathbf{GSp}_{\mathbb{R}}$ such that

(1) The map $\mathbb{S} \xrightarrow{h} \mathbf{GSp}_{\mathbb{R}} \hookrightarrow \mathrm{GL}(V_{\mathbb{R}})$ gives rise to a Hodge structure of type $(-1, 0), (0, -1)$ on $V_{\mathbb{R}}$, i.e., $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$.

(2) The pairing $(x, y) \mapsto \psi(x, h(i)y)$ is (positive or negative) definite on $V_{\mathbb{R}}$.

Then $(\mathbf{GSp}, S^{\pm})$ is a Shimura datum, which is called a *Siegel Shimura datum*.

For the rest of the subsection, we assume $(\mathbf{G}, X)$ is of Hodge type, i.e., there exists an embedding of Shimura data

$$\iota : (\mathbf{G}, X) \hookrightarrow (\mathbf{GSp}(V, \psi), S^{\pm}).$$

Sometimes we will write G for $\mathbf{G}_{\mathbb{Q}_p}$ for simplicity.

Let $\mathbf{E} = \mathbf{E}(\mathbf{G}, X)$ be the reflex field with ring of integers $\mathcal{O}_{\mathbf{E}}$. Let p be a prime number. Let $\mathbb{A}_f$ denote the ring of finite adèles over $\mathbb{Q}$, and $\mathbb{A}_f^p$ denote the ring of prime-to-p finite adèles, which we consider as the subgroup of $\mathbb{A}_f$ with trivial component at p. Fix a place $v|p$ of $\mathbf{E}$, and let E denote the completion of $\mathbf{E}$ at v. Denote by $\mathcal{O}_{\mathbf{E},(v)}$ (resp. $\mathcal{O}_E$) the localization (resp. completion) of $\mathcal{O}_{\mathbf{E}}$ at v. We write G for the base change $\mathbf{G}_{\mathbb{Q}_p}$. Let $\mathcal{G}$ be the Bruhat-Tits group scheme over $\mathbb{Z}_p$ associated with some $x \in \mathcal{B}(G, \mathbb{Q}_p)$, whose neutral component $\mathcal{G}^\circ$ is parahoric. Set $\mathrm{K}_p = \mathcal{G}(\mathbb{Z}_p)$ or $\mathcal{G}^\circ(\mathbb{Z}_p)$ and $\mathrm{K} = \mathrm{K}_p\mathrm{K}^p$ with $\mathrm{K}^p \subset \mathbf{G}(\mathbb{A}_f^p)$

sufficiently small open compact subgroup. By general theory of Shimura varieties, these data yield a quasi-projective smooth algebraic variety $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ canonically defined over $\mathbf{E}$, whose $\mathbb{C}$-points are given by

$$\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)(\mathbb{C}) = \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K}.$$

We can also consider the projective limit of $\mathbf{E}$-schemes

$$\mathrm{Sh}(\mathbf{G}, X) = \varprojlim_{\mathrm{K}} \mathrm{Sh}_{\mathbf{K}}(\mathbf{G}, X), \text{ resp. } \mathrm{Sh}_{\mathrm{K}_p}(\mathbf{G}, X) = \varprojlim_{\mathrm{K}^p} \mathrm{Sh}_{\mathrm{K}_p\mathrm{K}^p}(\mathbf{G}, X),$$

which carries a natural action of $\mathbf{G}(\mathbb{A}_f)$ (resp. $\mathbf{G}(\mathbb{A}_f^p)$). The projective limit exists since the transition maps are finite, hence affine.

7.3.1.1 Integral models for level $\mathcal{G}(\mathbb{Z}_p)$: construction

Assume that

(i) $\mathrm{K}_p = \mathcal{G}(\mathbb{Z}_p)$;

(ii) $\iota_{\mathbb{Q}_p}$ extends to a very good integral Hodge embedding $\widetilde{\iota} : (\mathcal{G}, \mu_h) \hookrightarrow (\mathrm{GL}(V_{\mathbb{Z}_p}), \mu_g)$, where $V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$ is a *self-dual* $\mathbb{Z}_p$-lattice with respect to ψ.

We let $\mathcal{GSP}$ denote the parahoric group scheme associated to the self-dual lattice $V_{\mathbb{Z}_p}$. Set $V_{\mathbb{Z}_{(p)}} := V \cap V_{\mathbb{Z}_p}$. Denote by $G_{\mathbb{Z}_{(p)}}$ the Zariski closure of G in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$, then $\mathcal{G}$ is isomorphic to $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$. Set $\mathrm{K}'_p := \mathcal{GSP}(\mathbb{Z}_p)$. Let K'^p be a small enough open compact subgroup of $\mathbf{GSp}(\mathbb{A}_f^p)$ containing K^p, which leaves $V_{\widehat{\mathbb{Z}}^p}$ stable. Here $\widehat{\mathbb{Z}}^p := \prod_{\ell \neq p} \mathbb{Z}_\ell$. Set $\mathrm{K}' = \mathrm{K}'_p\mathrm{K}'^p$. Then the embedding ι induces a closed immersion

$$\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X) \hookrightarrow \mathrm{Sh}_{\mathrm{K}'}(\mathbf{GSp}, S^{\pm}) \otimes_{\mathbb{Q}} \mathbf{E}$$

over $\mathbf{E}$. The choice of $V_{\mathbb{Z}_{(p)}}$ gives rise to an interpretation of $\mathrm{Sh}_{\mathrm{K}'}(\mathbf{GSp}, S^{\pm})$ as a moduli space of polarized abelian varieties, and hence to a natural integral model $\mathscr{S}_{\mathrm{K}'}(\mathbf{GSp}, S^{\pm})$ over $\mathbb{Z}_{(p)}$ (cf. [Zho20, §6.3]).

Definition 7.3.1. The integral model $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ over $\mathcal{O}_{\mathbf{E},(v)}$ of $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ is the normalization of the (reduced) Zariski closure $\mathscr{S}_{\mathrm{K}}^{-}(\mathbf{G}, X)$ of $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ in $\mathscr{S}_{K'}(\mathbf{GSp}, S^{\pm})_{\mathcal{O}_{\mathbf{E},(v)}}$. We set

$$\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X) \coloneqq \varprojlim_{\mathrm{K}^p} \mathscr{S}_{\mathrm{K}_p\mathrm{K}^p}(\mathbf{G}, X).$$

The $\mathbf{G}(\mathbb{A}_f^p)$-action on $\mathrm{Sh}_{\mathrm{K}_p}(\mathbf{G}, X)$ extends to $\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)$.

7.3.1.2 Hodge tensors and deformation theory

Since $G_{\mathbb{Z}_{(p)}}$ has reductive generic fiber, by Lemma 7.2.10, we can find a finite collection of tensors

$$(s_\alpha) \subset V_{\mathbb{Z}_{(p)}}^{\otimes} = (V_{\mathbb{Z}_{(p)}}^{\vee})^{\otimes}$$

whose scheme-theoretic stabilizer in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ is $G_{\mathbb{Z}_{(p)}}$. Let $h : \mathcal{A} \to \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ denote the pullback of the universal abelian scheme over $\mathscr{S}_{\mathrm{K}'}(\mathbf{GSp}, S^{\pm})$. Denote by $\mathcal{V} = R^1h_*\Omega^\bullet$ the (relative) algebraic de Rham cohomology of $\mathcal{A}$. Then the tensors (s_α), by the de Rham isomorphism, give rise to a collection of (absolute) Hodge cycles $s_{\alpha,\mathrm{dR}} \in \mathcal{V}_{\mathbb{C}}^{\otimes}$, where $\mathcal{V}_{\mathbb{C}}$ is the complex analytic vector bundle attached to $\mathcal{V}$, and $s_{\alpha,\mathrm{dR}}$ descends to $\mathcal{V}^{\otimes}$ by [KP18, Proposition 4.2.6] (i.e., $s_{\alpha,\mathrm{dR}}$ can be defined over $\mathcal{O}_{\mathbf{E},(v)}$).

Recall that $\breve{E}$ denotes the completion of the maximal unramified extension of E in $\overline{\mathbb{Q}}_p$ with residue field k. Let $L/\breve{E}$ be a finite extension. For a point $x \in \mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)(L)$ specializing to $\overline{x} \in \mathscr{S}_{\mathrm{K}}^{-}(\mathbf{G}, X)(k)$, we write $\mathcal{A}_x$ for the pullback of $\mathcal{A}$ to x and write $\mathscr{G}_x$ for the p-divisible group associated with $\mathcal{A}_x$. Then $s_{\alpha,\mathrm{dR}}$ pullbacks to $s_{\alpha,\mathrm{dR},x} \in H^1_{\mathrm{dR}}(\mathcal{A}_x)^{\otimes}$. We can also obtain corresponding tensors $s_{\alpha,\text{ét},x}$ in $T_p\mathscr{G}_x^{\vee\otimes}$ by the Betti-étale comparison theorem. Here $T_p\mathscr{G}_x^{\vee} \coloneqq \mathrm{Hom}_{\mathbb{Z}_p}(T_p\mathscr{G}_x, \mathbb{Z}_p)$. The tensors $s_{\alpha,\text{ét},x}$ are Galois invariant and their scheme-theoretic stabilizer is isomorphic to $\mathcal{G}$. Write $\mathscr{G}_{\overline{x}}$ for the p-divisible group corresponding to $\overline{x}$ and $\mathbb{D}_{\overline{x}}$ for $\mathbb{D}(\mathscr{G}_{\overline{x}})(W)$. Set $V \coloneqq T_p\mathscr{G}_x^{\vee} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then V is a crystalline representation of $\Gamma_L \coloneqq \mathrm{Gal}(\overline{L}/L)$. The p-adic comparison isomorphism

$$B_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T_p\mathscr{G}_x^{\vee} \simeq B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V), \quad D_{\mathrm{cris}}(V) \coloneqq (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\Gamma_L},$$

takes the Galois invariant tensors $s_{\alpha,\text{ét},x}$ to the φ-invariant tensors $s_{\alpha,0} \in D_{\mathrm{cris}}(V)^{\otimes}$.

Proposition 7.3.2. *We have $s_{\alpha,0} \in \mathbb{D}_{\overline{x}}^{\otimes}$, where we view $\mathbb{D}_{\overline{x}}^{\otimes}$ as a W-submodule of the K_0-vector space $D_{\mathrm{cris}}(V)^{\otimes}$. Moreover, we have the following properties.*

(1) The tensors $s_{\alpha,0} \in \mathbb{D}_{\overline{x}}^{\otimes}$ lift to φ-invariant tensors $\widetilde{s}_{\alpha,x} \in \Theta_{\mathcal{O}_L}(\mathscr{G}_x)^{\otimes}$, which map into $\mathrm{Fil}^0\,\mathbb{D}(\mathscr{G}_x)(\mathcal{O}_L)^{\otimes}$ along the natural projection $\Theta_{\mathcal{O}_L}(\mathscr{G}_x) \to \mathbb{D}(\mathscr{G}_x)(\mathcal{O}_L)$ given by Theorem 7.1.14 (2). Denote by $s_{\alpha,x}$ the image of $\widetilde{s}_{\alpha,x}$.

(2) There exists an isomorphism $\Theta_{\mathcal{O}_L}(\mathscr{G}_x) \simeq \mathbb{W}(\mathcal{O}_L) \otimes_{\mathbb{Z}_p} T_p\mathscr{G}_x^{\vee}$ taking $\widetilde{s}_{\alpha,x}$ to $s_{\alpha,\acute{e}t,x}$. In particular, there exists an isomorphism

$$\mathbb{D}_{\overline{x}} \simeq W \otimes_{\mathbb{Z}_p} T_p\mathscr{G}_x^{\vee}$$

taking $s_{\alpha,0}$ to $s_{\alpha,\acute{e}t,x}$, and an isomorphism

$$\mathbb{D}(\mathscr{G}_x)(\mathcal{O}_L) \simeq \mathbb{D}(\mathscr{G}_{\overline{x}})(W) \otimes_W \mathcal{O}_L$$

taking $s_{\alpha,x}$ to $s_{\alpha,0}$. Therefore, we can identify the group scheme $\mathcal{G}_W \subset \mathrm{GL}(\mathbb{D}_{\overline{x}})$ defined by $s_{\alpha,0}$ with $\mathcal{G} \otimes_{\mathbb{Z}_p} W$, and there exists a $G_{K_0}(= \mathcal{G}_W \otimes_W K_0)$-valued cocharacter μ_y such that

a) The filtration on $\mathbb{D}_{\overline{x}} \otimes_W L$ induced by the canonical isomorphism

$$\mathbb{D}_{\overline{x}} \otimes_W L \simeq \mathbb{D}(\mathscr{G}_x)(\mathcal{O}_L) \otimes_{\mathcal{O}_L} L$$

is given by a G_{K_0}-valued cocharacter G_{K_0}-conjugate to μ_y.

b) μ_y induces a filtration on $\mathbb{D}_{\overline{x}}$ which lifts the Hodge filtration on $\mathbb{D}_{\overline{x}} \otimes_W k = \mathbb{D}(\mathscr{G}_{\overline{x}})(k)$.

Proof. As in [KP18, Proposition 3.3.8], the tensors $(s_{\alpha,\text{ét},x}) \subset T_p\mathscr{G}_x^{\vee\otimes}$ give rise to φ-invariant tensors $s_{\alpha,x}^{\mathfrak{M}} \subset \mathfrak{M}(\mathscr{G}_x)^{\otimes}$. The tensors $s_{\alpha,x}^{\mathfrak{M}}$ map to tensors $\widetilde{s}_{\alpha,x}$ in $\Theta_{\mathcal{O}_L}(\mathscr{G}_x)^{\otimes}$ via the isomorphism

$$\Theta_{\mathcal{O}_K}(\mathscr{G}_x) \simeq \varphi^*\mathfrak{M}(\mathscr{G}_x) \otimes_{\mathfrak{S},\kappa} \mathbb{W}(\mathcal{O}_K)$$

in Corollary 7.1.18 (1). Since the above isomorphism respects the Hodge filtrations by Corollary 7.1.18 (2), the tensors $\widetilde{s}_{\alpha,x}$ map into $\mathrm{Fil}^0\,\mathbb{D}(\mathscr{G}_x)(\mathcal{O}_L)^{\otimes}$. The rest of the proof proceeds as in [KP18, Proposition 3.3.8, Corollary 3.3.10]. $\square$

The above proposition implies that $\mathscr{G}_x$ is a $(\mathcal{G}_W, \mu_y)$-adapted deformation of $\mathscr{G}_{\overline{x}}$ in the sense of Definition 7.2.17.

7.3.1.3 Integral models for level $\mathcal{G}(\mathbb{Z}_p)$: properties

Fix a parabolic subgroup $P \subset \mathrm{GL}(\mathbb{D}_{\overline{x}})$ lifting P_0 corresponding to the Hodge filtration of $\mathbb{D}(\mathscr{G}_{\overline{x}})(k) = \mathbb{D}_{\overline{x}} \otimes_W k$. Let $y = y(x) \in (\mathrm{GL}(\mathbb{D}_{\overline{x}})/P))\,(L)$ correspond to the cocharacter μ_y as in Proposition 7.3.2 (2). Then as in §7.2.3, we obtain from y a closed subscheme $M^{\mathrm{loc}}_{G,y} \subset (\mathrm{GL}(\mathbb{D}_{\overline{x}})/P)_{\mathcal{O}_{\breve{E}}}$ and formal local models

$$\widehat{M}^{\mathrm{loc}} = \mathrm{Spf}\, R, \quad \widehat{M}^{\mathrm{loc}}_{G,y} = \mathrm{Spf}\, R_G.$$

Note that R_G is a quotient of $R_E = R \otimes_W \mathcal{O}_{\breve{E}}$. By Proposition 7.3.2 (2) and the Betti-étale comparison theorem, the scheme $\underline{\mathrm{Isom}}_{(s_\alpha, s_{\alpha,0})}(V^\vee_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W, \mathbb{D}_{\overline{x}})$ of tensor-preserving isomorphisms is a trivial $\mathcal{G}$-torsor. Then we may choose an isomorphism $V^\vee_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W \simeq \mathbb{D}_{\overline{x}}$ preserving tensors such that the very good Hodge embedding (by our assumption on $\widetilde{\iota}$)

$$(\mathcal{G} \otimes_{\mathbb{Z}_p} W, \mu_h) \overset{\widetilde{\iota}}{\hookrightarrow} (\mathrm{GL}(V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W), \mu_g) \simeq (\mathrm{GL}(V^\vee_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W), \mu_g) \simeq (\mathrm{GL}(\mathbb{D}_{\overline{x}}), \mu_g)$$

induces a closed immersion $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h} \otimes_{\mathcal{O}_E} \mathcal{O}_{\breve{E}} \hookrightarrow (\mathrm{GL}(\mathbb{D}_{\overline{x}})/P)_{\mathcal{O}_{\breve{E}}} \xrightarrow{\sim} \mathrm{Gr}(g, \mathbb{D}_{\overline{x}})_{\mathcal{O}_{\breve{E}}}$. Note that the Hodge filtration on $\mathbb{D}_{\overline{x}} \otimes_W L$ is induced by a G-valued cocharacter conjugate to μ_h^{-1}. Hence, we can identify $M^{\mathrm{loc}}_{G,y}$ with $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h} \otimes_{\mathcal{O}_E} \otimes \mathcal{O}_{\breve{E}}$ by Lemma 7.2.7, and so R_G is normal.

Proposition 7.3.3. *Suppose that conditions (i) and (ii) in the beginning of §7.3.1.1 are satisfied. Let $\widehat{U}_{\overline{x}}$ be the completion of $\mathscr{S}^-_{\mathrm{K}}(\mathbf{G}, X)_{\mathcal{O}_{\breve{E}}}$ at $\overline{x}$. Then the irreducible component of $\widehat{U}_{\overline{x}}$ containing x is isomorphic to $\widehat{M}^{\mathrm{loc}}_{G,y} = \mathrm{Spf}\, R_G$ as formal schemes over $\mathcal{O}_{\breve{E}}$.*

Proof. We follow the arguments of [KP18, Proposition 4.2.2].

Note that $G_{K_0} \subset \mathrm{GL}(\mathbb{D}_{\overline{x}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ contains scalars, since $\mathbf{G} \subset \mathrm{GL}(V_{\mathbb{Q}})$ contains the image of the weight homomorphism w_h. As $\iota_{\mathbb{Q}_p}$ extend to a very good Hodge embedding, the constructions and results in §7.2 can apply. In particular, by Proposition 7.2.16 (2), we can view Spf R_E as a versal deformation space of $\mathscr{G}_{\overline{x}}$. Then the p-divisible group over $\widehat{U}_{\overline{x}}$ arising from the universal abelian scheme $\mathcal{A}$ gives rise to a natural map $\Phi : \widehat{U}_{\overline{x}} \to \operatorname{Spf} R_E$, which is a closed embedding by Serre-Tate theorem.

Let $Z \subset \widehat{U}_{\overline{x}}$ be the irreducible component containing x. Let $x' \in Z(L')$ for some finite field extension L' of $\breve{E}$. Then we can argue as in [KP18, Proposition 4.2.2] to show: $s_{\alpha,\text{ét},x'}$ corresponds to $s_{\alpha,0}$ under the p-adic comparison isomorphism for the p-divisible group $\mathscr{G}_{x'}$. Since the filtration on $\mathbb{D}_{\overline{x}} \otimes_W K'$ corresponding to $\mathscr{G}_{x'}$ is given by a G-valued cocharacter which is conjugate to μ_y, by Proposition 7.3.2, $\mathscr{G}_{x'}$ is $(\mathcal{G}_W, \mu_y)$-adapted. By our assumption on the integral Hodge embedding $\widetilde{\iota}$ and Proposition 7.3.2, the assumptions in Proposition 7.2.18 are satisfied. Hence, x' is induced by a point of $\widehat{M}^{\text{loc}}_{G,y}$ by Proposition 7.2.18. Since x' is arbitrary, it follows that $\Phi(Z) \subset \widehat{M}^{\text{loc}}_{G,y}$. They are equal, as Z and $\widehat{M}^{\text{loc}}_{G,y}$ are of the same dimension. $\square$

Theorem 7.3.4. *Assume the following conditions:*

(i) $\mathrm{K}_p = \mathcal{G}(\mathbb{Z}_p)$*;*

(ii) $\iota_{\mathbb{Q}_p}$ *extends to a very good integral Hodge embedding* $\widetilde{\iota} : (\mathcal{G}, \mu_h) \hookrightarrow (\mathrm{GL}(V_{\mathbb{Z}_p}), \mu_g)$*, where* $V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$ *is a* self-dual $\mathbb{Z}_p$*-lattice with respect to* ψ*.*

Then the $\mathcal{O}_{\mathbf{E},(v)}$*-schemes* $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ *and* $\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)$ *constructed in Definition 7.3.1 satisfy the following properties.*

(1) $\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)$ *is an* $\mathcal{O}_{\mathbf{E},(v)}$*-flat,* $\mathbf{G}(\mathbb{A}_f^p)$*-equivariant extension of* $\mathrm{Sh}_{\mathrm{K}_p}(\mathbf{G}, X)$*. The integral model* $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ *is canonical in the sense of [PR24].*

(2) For any discrete valuation ring R *of mixed characteristic* 0 *and* p*, the natural map*

$$\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)(R) \to \mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)(R[1/p])$$

is a bijection.

(3) $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ *fits into a local model diagram*

$$\begin{tikzcd} & \widetilde{\mathscr{S}}_{\mathrm{K}}(\mathbf{G}, X)_{\mathcal{O}_E} \arrow[dl, "\pi"'] \arrow[dr, "q"] & \\ \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)_{\mathcal{O}_E} & & \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h} \end{tikzcd}$$

of $\mathcal{O}_E$-schemes, in which π is a $\mathcal{G}$-torsor and q is $\mathcal{G}$-equivariant and smooth of relative dimension $\dim G$*.*

(4) If in addition, we have $\mathcal{G} = \mathcal{G}^\circ$, then for each $x \in \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)(k')$ with k'/k_E finite, there is a point $y \in \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h}(k')$ such that we have an isomorphism of henselizations

$$\mathcal{O}^h_{\mathscr{S}_{\mathrm{K}}(\mathbf{G},X),x} \simeq \mathcal{O}^h_{\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h},y}.$$

Proof. Note that under the assumptions of the above theorem, we have Proposition 7.3.3, which extends [KP18, Proposition 4.2.2] to the case $p = 2$. Then the proofs of [KP18, Proposition 4.2.2, 4.2.7] and [KPZ24, Theorem 7.1.3] go through, and we obtain the theorem. We note that the assumption (B) in [KPZ24, Theorem 7.1.3] is not used in the proof.

The integral model $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ is canonical by the construction in [PR24]. □

7.3.1.4 Integral models for parahoric level $\mathcal{G}^\circ(\mathbb{Z}_p)$

Now we use previous results to study integral models with parahoric level structure. That is, the level at p is given by $\mathcal{G}^\circ(\mathbb{Z}_p)$. Write $\mathrm{K}_p^\circ = \mathcal{G}^\circ(\mathbb{Z}_p)$ and $\mathrm{K}^\circ = \mathrm{K}_p^\circ \mathrm{K}^p$. Note that there is a natural finite morphism of Shimura varieties $\mathrm{Sh}_{\mathrm{K}^\circ}(\mathbf{G}, X) \to \mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$.

Definition 7.3.5. The integral model $\mathscr{S}_{\mathrm{K}^\circ}(\mathbf{G}, X)$ for parahoric level K° is the normalization of $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ in $\mathrm{Sh}_{\mathrm{K}^\circ}(\mathbf{G}, X)$. We also set

$$\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G}, X) \coloneqq \varprojlim_{\mathrm{K}^p} \mathscr{S}_{\mathrm{K}_p^\circ \mathrm{K}^p}(\mathbf{G}, X).$$

Let $\mathbf{G}^{\mathrm{sc}}$ denote the simply connected cover of $\mathbf{G}^{\mathrm{der}}$ and set $\mathbf{C} = \ker(\mathbf{G}^{\mathrm{sc}} \to \mathbf{G}^{\mathrm{der}})$. For a finite prime ℓ and $c \in H^1(\mathbb{Q}, \mathbf{C})$, we write c_ℓ for the image of c in $H^1(\mathbb{Q}_\ell, \mathbf{C})$. We introduce the following assumption:

$$\text{If } c \in H^1(\mathbb{Q}, \mathbf{C}) \text{ satisfies } c_\ell = 0 \text{ for all } \ell \neq p, \text{ then } c_p = 0. \tag{7.3.1}$$

Proposition 7.3.6. *Assume that conditions (i) and (ii) in Theorem 7.3.4 and condition (7.3.1) are satisfied.*

(1) Assume K^p is sufficiently small. Then the covering $\mathscr{S}_{\mathrm{K}^\circ}(\mathbf{G}, X) \to \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ is étale, and splits over an unramified extension of $\mathcal{O}_E$.

(2) The geometrically connected components of $\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G}, X)$ are defined over the maximal extension of $\mathbf{E}$ that is unramified at primes above p.

Proof. The proof follows the same argument as in [KP18, Proposition 4.3.7, 4.3.9]. □

7.3.2 Shimura varieties of abelian type

Let $(\mathbf{G}, X)$ be a Shimura datum of Hodge type with a Hodge embedding $\iota : (\mathbf{G}, X) \hookrightarrow (\mathbf{GSp}(V, \psi), S^\pm)$. Denote by G the base change $\mathbf{G}_{\mathbb{Q}_p}$. Let $\mathcal{G}^\circ$ be the parahoric group scheme associated to some point $x \in \mathcal{B}(G, \mathbb{Q}_p)$. Assume

(i) $\mathrm{K}_p = \mathcal{G}(\mathbb{Z}_p)$;

(ii) $\iota_{\mathbb{Q}_p}$ extends to a very good integral Hodge embedding $\widetilde{\iota} : (\mathcal{G}, \mu_h) \hookrightarrow (\mathrm{GL}(V_{\mathbb{Z}_p}), \mu_g)$, where $V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$ is a *self-dual* $\mathbb{Z}_p$-lattice with respect to ψ;

(iii) $\mathbf{G}$ satisfies condition (7.3.1);

(iv) The center Z_G of G is an R-smooth torus (see [KZ24, §2.4]).

Assume $(\mathbf{G}_2, X_2)$ is a Shimura datum of abelian type such that there is a central isogeny $\mathbf{G}^{\mathrm{der}} \to \mathbf{G}_2^{\mathrm{der}}$ inducing an isomorphism of Shimura data $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}}) \xrightarrow{\sim} (\mathbf{G}_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$. Here, X^{ad} denotes the $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$-conjugacy class of $h^{\mathrm{ad}} : \mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}} \to \mathbf{G}_{\mathbb{R}}^{\mathrm{ad}}$ for some $h \in X$; X_2^{ad} is similar.

As usual, we denote $\mathrm{K}_p^\circ \coloneqq \mathcal{G}^\circ(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ and $G_2 \coloneqq \mathbf{G}_{2,\mathbb{Q}_p}$. Let $x_2 \in \mathcal{B}(G_2, \mathbb{Q}_p)$ be a lift of $x_2^{\mathrm{ad}} = x^{\mathrm{ad}}$ in the identification $\mathcal{B}(G_2^{\mathrm{ad}}, \mathbb{Q}_p) = \mathcal{B}(G^{\mathrm{ad}}, \mathbb{Q}_p)$. Let $\mathcal{G}_2^\circ$ be the parahoric group scheme associated to x_2. Write $\mathrm{K}_{2,p}^\circ = \mathcal{G}_2^\circ(\mathbb{Z}_p)$. Denote by $\mathbf{E}_2$ the reflex field of $(\mathbf{G}_2, X_2)$ and set $\mathbf{E}' \coloneqq \mathbf{E} \cdot \mathbf{E}_2$, recall $\mathbf{E}$ denotes the reflex field of $(\mathbf{G}, X)$. We fix a place v' of $\mathbf{E}'$ above v. Denote by E' the completion of $\mathbf{E}'$ at v'.

Fix a connected component $X^+ \subset X$. Denote by $\mathrm{Sh}_{\mathrm{K}_p^\circ}(\mathbf{G}, X)^+$ the geometrically connected component containing the image of $X^+ \times 1$ in

$$\varprojlim_{\mathrm{K}^p} \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K}_p^\circ \mathrm{K}^p.$$

By Proposition 7.3.6 (2), $\mathrm{Sh}_{\mathrm{K}_p^\circ}(\mathbf{G}, X)^+$ is defined over the maximal extension $\mathbf{E}^p$ of $\mathbf{E}$ that is unramified at primes above p. We denote by $\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G}, X)^+$ the component of $\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G}, X)$ extending $\mathrm{Sh}_{\mathrm{K}_p^\circ}(\mathbf{G}, X)^+$, which is defined over $\mathcal{O}_{\mathbf{E}^p,(v)}$.

7.3.2.1 Integral models of Shimura varieties of abelian type

We recall the notation of [Del79]. Let H be a group equipped with an action of a group Δ, and let $\Gamma \subset H$ be a Δ-stable subgroup. Suppose we are given a Δ-equivariant map $\varphi : \Gamma \to \Delta$ where Δ acts on itself by inner automorphisms, and suppose that for $\gamma \in \Gamma$, $\varphi(\gamma)$ acts on H as conjugation by γ. Then the elements of the form $(\gamma, \varphi(\gamma)^{-1})$ form a normal subgroup of the semi-direct product $H \rtimes \Delta$. We denote by

$$H *_\Gamma \Delta$$

the quotient of $H \rtimes \Delta$ by this normal subgroup.

For a subgroup $H \subset \mathbf{G}(\mathbb{R})$, denote by H_+ the preimage in H of the connected component $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$ of the identity in $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$. We write $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q})^+ = \mathbf{G}^{\mathrm{ad}}(\mathbb{Q}) \cap \mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$.

Lemma 7.3.7. *Suppose S is an affine $\mathbb{Q}$-scheme, and let $S_{\mathbb{Z}_p}$ be a flat affine $\mathbb{Z}_p$-scheme with generic fiber $S \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Then there exists a $\mathbb{Z}_{(p)}$-scheme $S_{\mathbb{Z}_{(p)}}$, which is unique up to isomorphism, with generic fiber S and $S_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p = S_{\mathbb{Z}_p}$.*

Proof. Let A (resp. B) be the affine coordinate ring of $S_{\mathbb{Z}_p}$ (resp. S). By assumption, we have $A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Then we can take $S_{\mathbb{Z}_{(p)}}$ to be $\operatorname{Spec} A \cap B$, where the intersection happens in $A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Any $\mathbb{Z}_{(p)}$-scheme T with generic fiber S and $T \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p = S_{\mathbb{Z}_p}$ is necessarily isomorphic to $\operatorname{Spec} A \cap B$. $\square$

By applying the above lemma to the group schemes $\mathcal{G}$ and $\mathcal{G}^\circ$ over $\mathbb{Z}_p$, we obtain $\mathbb{Z}_{(p)}$-smooth affine group schemes $G_{\mathbb{Z}_{(p)}}$ and $G^\circ \coloneqq G^\circ_{\mathbb{Z}_{(p)}}$. Similarly, let $G^{\mathrm{ad}\circ} = G^{\mathrm{ad}\circ}_{\mathbb{Z}_{(p)}}$ be the $\mathbb{Z}_{(p)}$-model of the parahoric group scheme associated to $x^{\mathrm{ad}} \in \mathcal{B}(G^{\mathrm{ad}}, \mathbb{Q}_p)$. Let $G^{\mathrm{ad}} = G_{\mathbb{Z}_{(p)}}/Z$, where Z denotes the Zariski closure in $G_{\mathbb{Z}_{(p)}}$ of the center $\mathbf{Z}$ of the $\mathbb{Q}$-group $\mathbf{G}$. As we assume that the center Z_G of G is an R-smooth torus, we have $G^{\mathrm{ad}\circ}$ is the neutral component of G^{ad}, see [KP18, Lemma 4.6.2] and [KZ24, Proposition 2.4.14].

Following [KP18, §4.6.3], we set

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \coloneqq \mathbf{G}(\mathbb{A}_f^p)/Z(\mathbb{Z}_{(p)})^- *_{G^\circ(\mathbb{Z}_{(p)})_+/Z^\circ(\mathbb{Z}_{(p)})} G^{\mathrm{ad}\circ}(\mathbb{Z}_{(p)})^+,$$
$$\mathscr{A}(\mathbf{G}) \coloneqq \mathbf{G}(\mathbb{A}_f)/\mathbf{Z}(\mathbb{Q})^- *_{\mathbf{G}(\mathbb{Q})_+/\mathbf{Z}(\mathbb{Q})} \mathbf{G}^{\mathrm{ad}}(\mathbb{Q})^+,$$

and

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ \coloneqq G^\circ(\mathbb{Z}_{(p)})_+^-/Z^\circ(\mathbb{Z}_{(p)})^- *_{G^\circ(\mathbb{Z}_{(p)})_+/Z^\circ(\mathbb{Z}_{(p)})} G^{\mathrm{ad}\circ}(\mathbb{Z}_{(p)})^+,$$
$$\mathscr{A}(\mathbf{G})^\circ \coloneqq \mathbf{G}(\mathbb{Q})_+^-/\mathbf{Z}(\mathbb{Q})^- *_{\mathbf{G}(\mathbb{Q})_+/\mathbf{Z}(\mathbb{Q})} \mathbf{G}^{\mathrm{ad}}(\mathbb{Q})^+.$$

Here, $G^\circ(\mathbb{Z}_{(p)})_+^-$ is the closure of $G^\circ(\mathbb{Z}_{(p)})_+$ in $\mathbf{G}(\mathbb{A}_f^p)$, and Z° is the Zariski closure of $\mathbf{Z}$ in G°. Similarly, we have $\mathscr{A}(G_{2,\mathbb{Z}_{(p)}})$ and $\mathscr{A}(G_{2,\mathbb{Z}_{(p)}})$. Since $G^{\mathrm{ad}\circ}$ is the neutral component of $G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$ (we assume Z_G is an R-smooth torus), the action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ on $\mathrm{Sh}_{\mathrm{K}_p^\circ}(\mathbf{G}, X)$ extends to $\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G}, X)$. There is an injection by [KP18, Lemma 4.6.10],

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ \backslash \mathscr{A}(G_{2,\mathbb{Z}_{(p)}}) \hookrightarrow \mathscr{A}(\mathbf{G})^\circ \backslash \mathscr{A}(\mathbf{G}_2)/\mathrm{K}_{2,p}^\circ.$$

Let $J \subset G_2(\mathbb{Q}_p)$ be a set of coset representatives for the image of the above injection.

Definition 7.3.8. The integral model $\mathscr{S}_{\mathrm{K}_{2,p}^\circ}(\mathbf{G}_2, X_2)$ for $\mathrm{Sh}_{\mathrm{K}_{2,p}^\circ}(\mathbf{G}_2, X_2)$ is

$$[[\mathscr{S}_{\mathrm{K}_p^\circ}(\mathbf{G}, X)^+ \times \mathscr{A}(G_{2,\mathbb{Z}_{(p)}})]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ]^{|J|}.$$

The scheme $\mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)$ is priori defined over $\mathcal{O}_{\mathbf{E}'^p,(v)}$, but it descends to an $\mathcal{O}_{\mathbf{E}',(v')}$-scheme with a $\mathbf{G}_2(\mathbb{A}_f^p)$-action, see [KP18, Corollary 4.6.15].

Theorem 7.3.9. *Assume that conditions (i) to (iv) in the beginning of §7.3.2 are satisfied.*

(1) The $\mathbf{E}$-scheme $\mathrm{Sh}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)$ admits a $\mathbf{G}_2(\mathbb{A}_f^p)$-equivariant extension to a flat normal $\mathcal{O}_{\mathbf{E}',(v')}$-scheme $\mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)$. Any sufficiently small $\mathrm{K}_2^p \subset \mathbf{G}_2(\mathbb{A}_f^p)$ acts freely on $\mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)$, and the quotient

$$\mathscr{S}_{\mathrm{K}^\circ_2}(\mathbf{G}_2, X_2) \coloneqq \mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)/\mathrm{K}_2^p$$

is a flat normal $\mathcal{O}_{\mathbf{E}',(v')}$-scheme extending $\mathrm{Sh}_{\mathbf{K}^\circ_2}(\mathbf{G}_2, X_2)$.

(2) For any discrete valuation ring R of mixed characteristic 0 and p, the map

$$\mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)(R) \to \mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)(R[1/p])$$

is a bijection.

(3) There is a diagram of $\mathcal{O}_{E'}$-schemes

$$\begin{tikzcd}[column sep=small] & \widetilde{\mathscr{S}}^{\mathrm{ad}}_{\mathrm{K}^\circ_{2,p}} \arrow[dl, "\pi"'] \arrow[dr, "q"] & \\ \mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)_{\mathcal{O}_{E'}} & & \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_2^\circ,\mu_{h_2}} \otimes_{\mathcal{O}_{E_2}} \mathcal{O}_{E'}, \end{tikzcd}$$

where π is a $\mathbf{G}_2(\mathbb{A}_f^p)$-equivariant $G^{\mathrm{ad}}_{2,\mathbb{Z}_p}$-torsor, q is $G^{\mathrm{ad}}_{2,\mathbb{Z}_p}$-equivariant, and for any sufficiently small $\mathrm{K}_2^p \subset \mathbf{G}_2(\mathbb{A}_f^p)$, the map $\widetilde{\mathscr{S}}^{\mathrm{ad}}_{\mathrm{K}^\circ_{2,p}}/\mathrm{K}_2^p \to \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_2^\circ,\mu_{h_2}} \otimes_{\mathcal{O}_{E_2}} \mathcal{O}_{E'}$ induced by q is smooth of relative dimension $\dim \mathbf{G}_2^{\mathrm{ad}}$. If in addition, we have $\mathcal{G} = \mathcal{G}^\circ$, then π reduces to a $G^{\mathrm{ad}\circ}_{2,\mathbb{Z}_p}$-torsor.

Proof. Under the assumptions of the above theorem, we can construct the integral model $\mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)$ as in Definition 7.3.8. The properties of $\mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)$ are deduced from Theorem 7.3.4 by following the arguments in [KPZ24, Proposition 7.1.14] (cf. [KP18, §4.4-4.6]). Note that arguments in [KP18, §4.4-4.6] also work for $p = 2$. ∎

Remark 7.3.10. For a Shimura datum $(\mathbf{G}_2, X_2)$ of abelian type as in Theorem 7.3.9, we expect that the integral model $\mathscr{S}_{\mathrm{K}_2^\circ}(\mathbf{G}_2, X_2)$ is canonical in the sense of [PR24], which would imply that $\mathscr{S}_{\mathrm{K}_2^\circ}(\mathbf{G}_2, X_2)$ is independent of the choice of a Shimura datum $(\mathbf{G}, X)$, as well as the choice of a symplectic embedding $(\mathbf{G}, X) \hookrightarrow (\mathbf{GSp}, S^\pm)$.

7.3.2.2 Proof of Theorem 1.2.7 in Case (A)

Now we start with a Shimura datum $(\mathbf{G}_2, X_2)$ of abelian type with reflex field $\mathbf{E}_2$, and denote by $\mathrm{K}_{2,p}^\circ \subset G_2(\mathbb{Q}_p)$ the parahoric subgroup associated to some $x_2 \in \mathcal{B}(G_2, \mathbb{Q}_p)$.

Lemma 7.3.11. *Suppose that $(\mathbf{G}_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ has no factor of type $D^{\mathbb{H}}$, G_2 is unramified over $\mathbb{Q}_p$, and $\mathrm{K}_{2,p}^\circ$ is contained in some hyperspecial subgroup. Then there exists a Shimura datum $(\mathbf{G}, X)$ of Hodge type, together with a central isogeny $\mathbf{G}^{\mathrm{der}} \to \mathbf{G}_2^{\mathrm{der}}$ inducing an isomorphism $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}}) \simeq (\mathbf{G}_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$, such that the following conditions hold.*

(1) $\pi_1(G^{\mathrm{der}})$ is trivial.

(2) Any prime $v_2|p$ of $\mathbf{E}_2$ splits completely in $\mathbf{E}' = \mathbf{E} \cdot \mathbf{E}_2$.

(3) $X_*(G^{\mathrm{ab}})_{I_{\mathbb{Q}_p}}$ is torsion free, where G^{ab} denotes the quotient G/G^{der} and $I_{\mathbb{Q}_p}$ denotes the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

(4) Conditions (i) to (iv) in the beginning of §7.3.2 are satisfied.

Proof. As discussed in [KZ24, 2.4.5], the proof of [Edi92, Theorem 4.2] implies that a tamely ramified torus is R-smooth. As we assume G_2 is unramified (in particular, G_2 is tamely ramified), by [KP18, Lemma 4.6.22], it remains to show that there exists a Hodge embedding $\iota : (\mathbf{G}, X) \hookrightarrow (\mathbf{GSp}(V, \psi), S^\pm)$ satisfying condition (ii) in the beginning of §7.3.2. Since $\pi_1(G^{\mathrm{der}})$ is trivial by our choice of $(\mathbf{G}, X)$, we may assume that, by Zarhin's trick and [KP18, Corollary 2.3.16], there exists a good integral Hodge embedding $\widetilde{\iota} : (\mathcal{G}, \mu_h) \hookrightarrow (\mathrm{GL}(\Lambda), \mu_g)$ extending $\iota_{\mathbb{Q}_p}$, where $\Lambda \subset V_{\mathbb{Q}_p}$ is a self-dual $\mathbb{Z}_p$-lattice with respect to $\psi_{\mathbb{Q}_p}$. Denote $\mathrm{GSp} \coloneqq \mathbf{GSp}(V, \psi)_{\mathbb{Q}_p}$. By our assumptions and Theorem 7.4.1, there is a tame Galois extension

$F/\mathbb{Q}_p$ with Galois group Γ such that in the diagram

$$\begin{tikzcd} \mathcal{B}(G,\mathbb{Q}_p) \arrow[r, "\iota"] \arrow[d] & \mathcal{B}(\mathrm{GSp},\mathbb{Q}_p) \arrow[d] \\ \mathcal{B}(G_F,F) \arrow[r] & \mathcal{B}(\mathrm{GSp}_F,F) \end{tikzcd}$$

of Bruhat-Tits buildings, we have

- the image of $x \in \mathcal{B}(G,\mathbb{Q}_p)$ in $\mathcal{B}(G_F,F)$ is hyperspecial, and determines a reductive group $\mathcal{H}$ over $\mathcal{O}_F$ satisfying $\mathcal{G} \simeq (\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\mathcal{H})^\Gamma$;
- the point $\iota(x)$ is hyperspecial corresponding to the self-dual lattice Λ, and its image in $\mathcal{B}(\mathrm{GSp}_F,F)$ is hyperspecial corresponding to the lattice $\widetilde{\Lambda} \coloneqq \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_F$, which is self-dual with respect to the pairing ψ_F.

By [DD11, Lemma 3.1], there exist a totally real number field $\mathbb{F}/\mathbb{Q}$ and a place w above p such that $\mathbb{F}_w \simeq F$. Let $\widetilde{V}$ denote the $\mathbb{Q}$-vector space $V \otimes_{\mathbb{Q}} \mathbb{F}$. We pick an element $a \in \mathbb{F}$ such that its image in F generates the different ideal $\delta_{F/\mathbb{Q}_p}$. Then $\widetilde{V}$ is equipped with a perfect alternating pairing given by

$$\widetilde{\psi}(x,y) \coloneqq \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}(a^{-1}\psi_F(x,y))$$

for $x,y \in \widetilde{V}$. Then $\widetilde{\Lambda}$ is self-dual with respect to $\widetilde{\psi}$, and the closed immersion

$$\widetilde{\iota}: \mathcal{G} \hookrightarrow \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\mathcal{H} \hookrightarrow \mathrm{GL}(\widetilde{\Lambda})$$

extends the Hodge embedding $G \hookrightarrow \mathrm{GSp} \hookrightarrow \mathbf{GSp}(\widetilde{V},\widetilde{\psi})_{\mathbb{Q}_p} \subset \mathrm{GL}(\widetilde{V}_{\mathbb{Q}_p})$. As $\pi_1(G^{\mathrm{der}})$ is trivial and G is unramified over $\mathbb{Q}_p$, the Pappas-Zhu local model for $(\mathcal{G},\mu_h)$ is isomorphic to $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\mu_h}$, and $\widetilde{\iota}$ is a good integral Hodge embedding by [KP18, Proposition 2.3.7]. As $(\mathbf{G}_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ has no factor of type $D^{\mathbb{H}}$, the closed immersion $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\mathcal{H} \hookrightarrow \mathrm{GL}(\widetilde{\Lambda})$ gives a very good integral Hodge embedding by [KPZ24, Proposition 5.3.10, Theorem 1.2.3]. Since $\mathcal{G} = (\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\mathcal{H})^\Gamma$, we obtain that $\widetilde{\iota}$ is also very good by [KPZ24, Corollary 5.3.4]. We then obtain a desired Hodge embedding by replacing ι by the Hodge embedding $(\mathbf{G},X) \hookrightarrow (\mathbf{GSp}(\widetilde{V},\widetilde{\psi}), S^{\pm}(\widetilde{V},\widetilde{\psi}))$. $\square$

Corollary 7.3.12. *Under the same assumptions as in Lemma 7.3.11, the integral model $\mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)$ constructed in Definition 7.3.8 is defined over $\mathcal{O}_{\mathbf{E}_2,(v_2)}$ for some fixed prime $v_2|p$ of $\mathbf{E}_2$. Moreover, we have $\mathcal{G} = \mathcal{G}^\circ$, and the conclusions of Theorem 7.3.9 hold. In particular, if κ is a finite extension of $\kappa(v_2)$ and $y \in \mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)(\kappa)$, then there exists $z \in \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}^\circ_2,\mu_{h_2}}(\kappa)$ such that we have an isomorphism of henselizations*

$$\mathcal{O}^h_{\mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2,X_2),y} \simeq \mathcal{O}^h_{\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}^\circ_2,\mu_{h_2}},z}.$$

Proof. By Theorem 7.3.9 and Lemma 7.3.11 (4), the integer model $\mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)$ is constructed using the Shimura datum $(\mathbf{G}, X)$ chosen in Lemma 7.3.11. By Lemma 7.3.11 (2), there exists a prime $v_2|p$ of $\mathbf{E}_2$ extending to the prime v' of $\mathbf{E}'$, and we have $\mathcal{O}_{\mathbf{E}_2,(v_2)} \simeq \mathcal{O}_{\mathbf{E}',(v')}$. Hence, the scheme $\mathscr{S}_{\mathrm{K}^\circ_{2,p}}(\mathbf{G}_2, X_2)$ is defined over $\mathcal{O}_{\mathbf{E}_2,(v_2)}$. Since $\pi_1(G^{\mathrm{der}})$ is trivial by Lemma 7.3.11 (1), we have $\pi_1(G) = X_*(G^{\mathrm{ab}})$, and $\pi_1(G)_{I_{\mathbb{Q}_p}}$ is torsion-free by Lemma 7.3.11 (3). In particular, we have $\mathcal{G} = \mathcal{G}^\circ$. □

By Theorem 7.3.9 and Corollary 7.3.12, we obtain Theorem 1.2.7 in Case (A). Note that the group $\mathbf{G}$ in Theorem 1.2.7 is denoted by $\mathbf{G}_2$ here.

7.3.3 Integral models of unitary Shimura varieties

In this subsection, we consider Shimura varieties in Case (B) of §1.2.2. We show that, in this case, the assumptions in Theorem 7.3.9 are satisfied, allowing us to construct integral models of Shimura varieties for which the conclusions of Theorem 7.3.9 hold.

7.3.3.1

Let $n = 2m + 1 \geq 3$ be an odd integer. Let $\mathbf{F}/\mathbb{Q}$ be an imaginary quadratic extension such that 2 is ramified in $\mathbf{F}$. Then $F \coloneqq \mathbf{F} \otimes_{\mathbb{Q}} \mathbb{Q}_2$ is a ramified quadratic extension of $\mathbb{Q}_2$ with residue field $\mathbb{F}_2$. Let $(\mathbf{V}, \mathbf{h})$ be an n-dimensional non-degenerate $\mathbf{F}/\mathbb{Q}$-hermitian space of signature $(n-1, 1)$. Denote by

$$\mathbf{G} \coloneqq \mathrm{GU}(\mathbf{V}, \mathbf{h})$$

the unitary similitude group over $\mathbb{Q}$ attached to $(\mathbf{V}, \mathbf{h})$. Suppose that

$$\mathrm{K}_2 \subset \mathbf{G}(\mathbb{Q}_2)$$

is a special parahoric subgroup in the sense of Bruhat-Tits theory. For an open compact subgroup of the form $\mathrm{K} = \mathrm{K}_2\mathrm{K}^2 \subset \mathbf{G}(\mathbb{A}_f)$, where $\mathrm{K}^2 \subset \mathbf{G}(\mathbb{A}_f^2)$ is open compact and sufficiently small, we can associate a Shimura variety $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ of level K as in [PR09, §1.1]. Then $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ is a quasi-projective smooth variety of dimension $n-1$ over the reflex field $\mathbf{F}$. Denote by $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)_F$ the base change of $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ to F.

7.3.3.2 Unitary local models

Note that the vector space $V := \mathbf{V} \otimes_{\mathbf{F}} F$ equipped with the $F/\mathbb{Q}_2$-hermitian form $h := \mathbf{h}_{\mathbb{Q}_2}$ defines a unitary similitude group $G = \mathbf{G}_{\mathbb{Q}_2}$ over $\mathbb{Q}_2$.

Lemma 7.3.13. *For any non-degenerate hermitian form h' on V, we have $G \simeq \mathrm{GU}(V, h')$.*

Proof. By the classification of hermitian spaces over local fields (see, for example, [Jac62, Theorem 3.1]), there are two isomorphism classes of n-dimensional non-degenerate hermitian spaces over $\mathbb{Q}_2$, classified by discriminants in $\mathbb{Q}_2^\times / N_{F/\mathbb{Q}_2}(F^\times)$. Let $a \in \mathbb{Q}_2^\times$ be an element not in $N_{F/\mathbb{Q}_2}(F^\times)$. Define a hermitian form h_a on V by setting $h_a(x, y) := ah(x, y)$ for $x, y \in V$. Since $\mathrm{disc}(h_a) = a^n \mathrm{disc}(h)$ and n is odd, the hermitian spaces (V, h) and (V, h_a) represent the two isomorphism classes of n-dimensional non-degenerate hermitian spaces over $\mathbb{Q}_2$. Moreover, multiplication by a induces an isomorphism between $\mathrm{GU}(V, h)$ and $\mathrm{GU}(V, h_a)$. Hence, the lemma follows. $\square$

By Lemma 7.3.13, we may assume that the hermitian form h is split, that is, there exists an F-basis $e_1, \ldots, e_n$ of V such that $h(e_i, e_j) = \delta_{i,n+1-j}$. Then we are in the situation of the first part of the thesis. Up to conjugation, we may assume that the special parahoric subgroup $\mathrm{K}_2 \subset \mathbf{G}(\mathbb{Q}_2)$ corresponds to $I = \{0\}$ or $\{m\}$ by Theorem 1.2.1. Let $\mathcal{G}_I$ denote the special parahoric group scheme corresponding to $I = \{0\}$ or $\{m\}$. By [PR09, 1.2.3], $\mathscr{G}_I$ is a

Bruhat-Tits stabilizer group scheme. Let μ denote the geometric cocharacter

$$\mathbb{G}_{m,\overline{F}} \to G_{\overline{F}} \simeq \mathrm{GL}_{n,\overline{F}} \times \mathbb{G}_{m,\overline{F}}$$

given by $z \mapsto (\mathrm{diag}(z, 1^{(n-1)}), z)$. Let $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_I,\mu}$ be the local model attached to $(\mathcal{G}_I, \mu)$ by Theorem 3.4.4. By Proposition 3.4.5, this is isomorphic to the unitary local model $\mathrm{M}^{\mathrm{loc}}_I$ in Theorem 1.2.2.

Lemma 7.3.14. *Let $\Lambda_I \subset V$ be the lattice as in Theorem 1.2.1 corresponding to the special parahoric subgroup $\mathrm{K}_2 \subset G(\mathbb{Q}_2)$. Then there exists a good integral Hodge embedding $(\mathcal{G}_I, \mu) \hookrightarrow (\mathrm{GL}(\Lambda_I), \mu_n)$.*

Proof. By the concrete description of the parahoric group scheme $\mathcal{G}_I$ in Chapter 6, there is a closed immersion $\iota : \mathcal{G}_I \hookrightarrow \mathrm{GL}(\Lambda_I)$. The base change $\iota_{\mathbb{Q}_p}$ is the standard Hodge embedding $G = \mathrm{GU}(V, h) \hookrightarrow \mathrm{GL}(V)$, which sends the conjugacy class $\{\mu\}$ to $\{\mu_n\}$. As G contains the scalars, ι is an integral Hodge embedding. Moreover, ι is good, since it induces a closed immersion $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_I,\mu} \simeq \mathrm{M}^{\mathrm{loc}}_I \hookrightarrow \mathrm{Gr}(n, \Lambda_I)_{\mathcal{O}_F}$ by our construction of $\mathrm{M}^{\mathrm{loc}}_I$. $\square$

The following theorem is a key ingredient in the construction of very good Hodge embeddings for $(\mathcal{G}_I, \mu)$.

Theorem 7.3.15. *For any closed point $x \in \mathrm{M}^{\mathrm{loc}}_I(k)$, the tangent space of the special fiber $\mathrm{M}^{\mathrm{loc}}_I \otimes_{\mathcal{O}_F} k$ at x is spanned by smooth formal curves (see Definition 7.2.14).*

The proof of Theorem 7.3.15 is divided into the following two cases.

The case $I = \{m\}$

By Theorem 1.2.2 (2), the local model $\mathrm{M}^{\mathrm{loc}}_{\{m\}}$ is smooth over $\mathcal{O}_F$. Clearly Theorem 7.3.15 holds in this case by the infinitesimal lifting property of smooth morphisms.

The case $I = \{0\}$

By Theorem 1.2.2 (1), $\mathrm{M}^{\mathrm{loc}}_{\{0\}}$ is $\mathcal{O}_F$-smooth on the complement of a single closed point, which we will call the *worst point.* To prove Theorem 7.3.15 in this case, it suffices to prove the tangent space of $\mathrm{M}^{\mathrm{loc}}_{\{0\}} \otimes_{\mathcal{O}_F} k$ at the worst point is spanned by smooth formal curves.

Definition 7.3.16. Let X be an affine scheme of finite type over k. Let $x \in X(k)$ be a k-point. We may express X as a closed subscheme of $\mathbb{A}^d = \operatorname{Spec} k[T_1, \dots, T_d]$ defined by an ideal $\mathfrak{a} \subset k[T_1, \dots, T_d]$ such that x is the origin of $\mathbb{A}^d$.

(1) For a polynomial $f \in k[T_1, \dots, T_d]$, write $f = \sum_{i=r}^{N} f_i$ as a decomposition into homogeneous polynomials with $f_r \neq 0$. Denote by f^* (resp. $f^{(1)}$) the lowest degree term f_r (resp. f_1). If $r \geq 2$, set $f_1 = 0$.

(2) Denote by $\mathfrak{a}^*$ (resp. $\mathfrak{a}^{(1)}$) the ideal in $k[T_1, \dots, T_d]$ generated by f^* (resp. $f^{(1)}$), for all $f \in \mathfrak{a}$. The *tangent cone* TC_xX (resp. *schematic tangent space* $T_x^{sch}X$) of X at x is the scheme $\operatorname{Spec} k[T_1, \dots, T_d]/\mathfrak{a}^*$ (resp. $\operatorname{Spec} k[T_1, \dots, T_d]/\mathfrak{a}^{(1)}$).

Note that the definition of TC_xX (resp. $T_x^{sch}X$) is independent of the embeddings of X in affine spaces. See [Mum99, Chapter III, §3, 4]. Clearly $T_x^{sch}X$ is a linear subspace of $\mathbb{A}^d$ and there is a closed immersion $TC_xX \hookrightarrow T_x^{sch}X$. Note that there is a natural bijection between the k-points $T_x^{sch}X(k)$ and the tangent space T_xX, see [Mum99, §4]. Concretely, for any $z \in T_x^{sch}X(k)$ corresponding to a k-algebra homomorphism $z : k[T_1, \dots, T_d]/\mathfrak{a}^{(1)} \to k$, we can associate a k-algebra homomorphism $t_z : k[T_1, \dots, T_d]/\mathfrak{a} \to k[t]/(t^2)$ via $T_i \mapsto z(T_i)t$. The morphism t_z defines a tangent vector of X at x.

Lemma 7.3.17. *Let X be a reduced affine scheme of finite type over k. Let $x \in X(k)$. Assume that there exists a closed immersion $i : X \hookrightarrow \mathbb{A}^d$ such that X is defined by a homogeneous ideal $\mathfrak{a}$ and $i(x)$ is the origin O of $\mathbb{A}^d$. Then the set $TC_xX(k)$ spans the k-vector space T_xX.*

Proof. Without loss of generality, we may assume that i does not factor through any (proper) linear subspace of $\mathbb{A}^d$. As X is reduced, the image $i(X)$ is not contained in any (proper) linear subspace of $\mathbb{A}^d$. Since $\mathfrak{a}$ is homogeneous, X is isomorphic to the tangent cone TC_xX and i is identified with the embedding $TC_xX \hookrightarrow T_x^{sch}X \hookrightarrow T_O^{sch}\mathbb{A}^d$. Let W denote the subspace in T_xX spanned by $TC_xX(k)$. Then we have a linear subspace $W^{sch} \subset \mathbb{A}^d$ such that $W^{sch}(k) = W$. We obtain a factorization

$$i : X \hookrightarrow W^{sch} \hookrightarrow T_x^{sch} X \hookrightarrow \mathbb{A}^d.$$

Since $i : X \hookrightarrow \mathbb{A}^d$ does not factor through any proper linear subspace of $\mathbb{A}^d$, it forces that $W^{sch} = T_x^{sch} X = \mathbb{A}^d$, and hence, $W = T_x X$. $\square$

Corollary 7.3.18. *Under the same assumptions as in Lemma 7.3.17, the tangent space $T_x X$ is spanned by smooth formal curves.*

Proof. Denote $X = \operatorname{Spec} R = \operatorname{Spec} k[T_1, \dots, T_d]/\mathfrak{a}$. By assumption, the tangent cone $TC_x X$ is isomorphic to X. Recall that for a k-point $z \in TC_x X(k)$ corresponding to $z : R = k[T_1, \dots, T_d]/\mathfrak{a} \to k$, the associated tangent vector $t_z \in X(k[t]/(t^2))$ is given by the k-algebra homomorphism $R \to k[t]/(t^2)$ sending $T_i \mapsto z(T_i)t$. Define a k-algebra homomorphism $\widetilde{t}_z : k[T_1, \dots, T_d] \to k[[t]]$ via $T_i \mapsto z(T_i)t$. For any homogeneous polynomial $f \in \mathfrak{a}$, we have

$$\widetilde{t}_z(f) = f(z(T_1)t, \dots, z(T_d)t) = t^{\deg f} f(z(T_1), \dots, z(T_d)) = 0.$$

Hence, the map $\widetilde{t}_z$ factors through $R/\mathfrak{a}$. In other words, the tangent vector t_z lifts to the smooth formal curve $\widetilde{t}_z \in X(k[[t]])$. Now the corollary follows from Lemma 7.3.17 immediately. $\square$

Recall that, by Theorem 1.2.3 (1), there is an open affine neighborhood $\mathrm{U}^{\mathrm{loc}}_{\{0\}}$ of $\mathrm{M}^{\mathrm{loc}}_{\{0\}}$ containing the worst point such that $\mathrm{U}^{\mathrm{loc}}_{\{0\}} \otimes_{\mathcal{O}_F} k$ is defined by a homogeneous ideal under the obvious closed embedding $\mathrm{U}^{\mathrm{loc}}_{\{0\}} \otimes_{\mathcal{O}_F} k \hookrightarrow \operatorname{Spec} k[A|B]$, which sends the worst point to the origin. By Corollary 7.3.18, we obtain the following.

Corollary 7.3.19. *The tangent space of $\mathrm{M}^{\mathrm{loc}}_{\{0\}} \otimes_{\mathcal{O}_F} k$ at the worst point is spanned by smooth formal curves.*

This proves Theorem 7.3.15 in the case $I = \{0\}$.

7.3.3.3 Proof of Theorem 1.2.7 in Case (B)

Let us keep the notation as in §7.3.3.1. Let $a \in \mathbf{F}^\times$ be an element such that $a = -\overline{a}$. Then the hermitian form $\mathbf{h}$ on $\mathbf{V}$ induces a perfect alternating $\mathbb{Q}$-bilinear form ψ on $\mathbf{V}$ by setting

$$\psi(x, y) \coloneqq \mathrm{Tr}_{\mathbf{F}/\mathbb{Q}}(a^{-1}\mathbf{h}(x, y)), \text{ for } x, y \in \mathbf{V}.$$

Denote by $\mathbf{GSp}(\mathbf{V}, \psi)$ the symplectic similitude group over $\mathbb{Q}$ associated with the above pairing. Then we obtain an embedding $\iota_1 : \mathbf{G} \hookrightarrow \mathbf{GSp}(\mathbf{V}, \psi)$, which also induces an embedding of Shimura data

$$\iota_1 : (\mathbf{G}, X) \hookrightarrow (\mathbf{GSp}(\mathbf{V}, \psi), S^\pm(\mathbf{V}, \psi)).$$

By Lemma 7.3.14, there exists a good integral Hodge embedding

$$\widetilde{\iota}_1 : (\mathcal{G}_I, \mu) \hookrightarrow (\mathrm{GL}(\Lambda_I), \mu_n)$$

extending $\iota_{1,\mathbb{Q}_2}$. By Theorem 7.3.15 and Lemma 7.2.15, $\widetilde{\iota}_1$ is very good. Denote by $\Lambda_I^\# \subset V$ the dual lattice of Λ_I with respect to ψ. Set $\Lambda \coloneqq (\Lambda_I)^4 \oplus (\Lambda_I^\#)^4 \subset V^8$. Using Zarhin's trick as in the proof of [KPZ24, Proposition 7.2.10 (3)], there exists a non-degenerate alternating pairing ψ' on $\mathbf{V}^8$ such that Λ is self-dual with respect to $\psi'_{\mathbb{Q}_2}$, and an embedding of Shimura data

$$\iota : (\mathbf{G}, X) \hookrightarrow (\mathbf{GSp}(\mathbf{V}^8, \psi'), S^\pm(\mathbf{V}^8, \psi'))$$

such that ι extends to a very good integral Hodge embedding $(\mathcal{G}_I, \mu) \hookrightarrow (\mathrm{GL}(\Lambda), \mu_{8n})$.

Denote $(\mathbf{GSp}, S^\pm) \coloneqq (\mathrm{GSp}(\mathbf{V}^8, \psi'), S^\pm(\mathbf{V}^8, \psi'))$. Then we obtain an embedding of Shimura data

$$\iota : (\mathbf{G}, X) \hookrightarrow (\mathbf{GSp}, S^\pm).$$

Moreover, the embedding $\iota_{\mathbb{Q}_2}$ extends to a very good integral Hodge embedding by previous discussion. Note that for odd unitary similitude groups, the parahoric group scheme corresponding to K_2 is connected by [PR09, 1.2.3]. In particular, the assumptions in Theorem 7.3.4 are satisfied and we obtain the following theorem.

Theorem 7.3.20. *There exists a normal flat $\mathcal{O}_F$-scheme $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ extending $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ such that the conclusions of Theorem 7.3.4 hold for $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$.*

This finishes the proof of Theorem 1.2.7 in Case (B).

7.4 Bruhat-Tits group schemes and tame Galois fixed points

In this section, we show that, for an unramified group G over a 2-adic field F, if a stabilizer group scheme $\mathcal{G}$ satisfies $\mathcal{G}(\mathcal{O}_F) \subset H$ for some hyperspecial subgroup H of $G(F)$, then $\mathcal{G}$ can be written as the tame Galois fixed points of the Weil restriction of scalars of a reductive group scheme. This result is used in the proof of Lemma 7.3.11 to construct very good Hodge embeddings in Case (A).

Let F be a complete discrete valued field with residue characteristic $p = 2$. Let G be a connected reductive group over F. Denote by $\mathcal{B}(G, F)$ (resp. $\overline{\mathcal{B}}(G, F)$) the extended (resp. "classical") Bruhat-Tits building. Recall that for a finite tame Galois extension K/F with Galois group Γ, the inclusion

$$\mathcal{B}(G, F) \hookrightarrow \mathcal{B}(G, K)$$

of buildings identifies the image with the fixed point set $\mathcal{B}(G, K)^{\Gamma}$. For $x \in \mathcal{B}(G, F)$, we use $\mathcal{G}_x^K$ to denote the Bruhat-Tits group scheme over $\mathcal{O}_K$ attached to the image of x in $\mathcal{B}(G, K)$.

Theorem 7.4.1. *Assume G is unramified. Let $\mathcal{G} = \mathcal{G}_{\mathbf{f}}$ be the Bruhat-Tits group scheme attached to some facet $\mathbf{f}$ in $\mathcal{B}(G, F)$ whose closure contains a hyperspecial point.*

Then there exist a point $x \in \mathcal{B}(G, F)$ and a finite tame Galois extension K/F with Galois group Γ such that $G \otimes_F K$ is split, $\mathcal{G} = \mathcal{G}_x$, and (the image of) x is hyperspecial in $\mathcal{B}(G, K)$. Moreover, we have an isomorphism of (smooth) $\mathcal{O}_F$-group schemes

$$\mathcal{G} \simeq (\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathcal{G}_x^K)^{\Gamma}$$

extending the isomorphism $G \simeq (\mathrm{Res}_{K/F} G_K)^{\Gamma}$.

The proof of Theorem 7.4.1

We first consider the case when G is split, absolutely simple, and simply connected. Fix a maximal torus T and a Borel subgroup B containing T. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the subset of simple roots with respect to (T, B) in the root system $\Phi = \Phi(T, B)$. Denote by $\Phi^+ = \Phi \cap \mathbb{Z}_{\geq 0}\Delta$ the set of positive roots. Note that there is a perfect pairing

$$\langle -, - \rangle : X_*(T) \times X^*(T) \to \mathbb{Z}$$

between the cocharacter group $X_*(T)$ and the character group $X^*(T)$ of T. There is an isomorphism between the apartment $\mathcal{A}$ of $\mathcal{B}(G, F)$ corresponding to T and $V \coloneqq X_*(T)_{\mathbb{R}}$ such that the origin in V corresponds to a special vertex, which is also hyperspecial, in $\mathcal{A}$. Moreover, a chamber C of $\mathcal{A}$ is given by

$$C = \left\{ x \in V \mid 0 < \langle x, \alpha \rangle < 1 \text{ for all } \alpha \in \Phi^+ \right\}.$$

For $1 \leq i \leq n$, denote by $\omega_i \in V$ the fundamental coroot corresponding to $\alpha_i \in \Delta$. Then the chamber C has $n+1$ vertices $v_0, \dots, v_n$, where $v_0 = 0$ and $v_i = \omega_i / c_i$ for $1 \leq i \leq n$, where c_i is a positive integer such that $\sum_{i=1}^n c_i \alpha_i$ is the highest root in Φ. Since $G(F)$ acts transitively on the set of chambers in $\mathcal{A}$ (see, for example, [Tit79, §1.8]), we may assume that $\mathbf{f}$ is contained in the closure of C. By assumption, the closure $\overline{\mathbf{f}}$ of $\mathbf{f}$ contains a hyperspecial vertex $v_{\mathbf{f}}$. Note that $v_{\mathbf{f}}$ is some vertex v_i for which $c_i = 1$. If $\overline{\mathbf{f}}$ consists of only a single point, there is nothing to prove. Hence, we may assume that $\overline{\mathbf{f}}$ strictly contains $v_{\mathbf{f}}$. Let $y \in V$ be the barycenter of the (sub)facet determined by the vertices in $\overline{\mathbf{f}}$ except $v_{\mathbf{f}}$. Then y is of the form

$$y = \frac{1}{m2^d} y_1,$$

where m is an odd integer, $d \geq 0$ is an integer, and $y_1 \in \mathbb{Z}\Delta$. Set

$$x \coloneqq \frac{1}{m2^{d+1}+1} v_{\mathbf{f}} + \frac{m2^{d+1}}{m2^{d+1}+1} y = \frac{1}{m2^{d+1}+1} v_{\mathbf{f}} + \frac{2}{m2^{d+1}+1} y_1$$

Then x lies in the line segment between $v_{\mathbf{f}}$ and y, and hence in $\mathbf{f}$. Since G is simply connected, we have

$$\mathcal{G}_x = \mathcal{G}_{\mathbf{f}}.$$

Let F_1 be a finite extension of F with ramification index $m2^{d+1}+1$. Denote by

$$\rho : \mathcal{B}(G,F) \hookrightarrow \mathcal{B}(G,F_1),$$

the natural inclusion of buildings. Then we see that

$$\rho(x) = v_{\mathbf{f}} + 2y_1 \in v_{\mathbf{f}} + X_*(T).$$

Thus, $\rho(x)$ is a hyperspecial point in $\mathcal{B}(G,F_1)$. As $p=2$, the extension F_1/F is tame. Let K be the Galois closure of F_1/F. Then K is a tame Galois extension of F. Note that the image of $\rho(x)$ in $\mathcal{B}(G,K)$ is also hyperspecial. The pair (K,x) satisfies the conclusion of Theorem 7.4.1.

Next we consider the case when G is unramified, absolutely simple and simply connected. Let F_1/F be an unramified Galois extension over which G is split. Denote by Γ_1 the Galois group of F_1/F. Then the facets in $\mathcal{B}(G,F)$ correspond to Γ_1-invariant facets in $\mathcal{B}(G,F_1)$. Let $\mathbf{f}_1$ be the Γ_1-invariant facet in $\mathcal{B}(G,F_1)$ corresponding to $\mathbf{f}$. The closure of the facet $\mathbf{f}_1$ contains a hyperspecial point, which is the image of $v_{\mathbf{f}}$ in $\mathcal{B}(G,F_1)$. Let y_1 be the barycenter of $\mathbf{f}_1$. Then y_1 is a fixed point of Γ_1 and we have

$$\mathcal{G} = (\mathrm{Res}_{\mathcal{O}_{F_1}/\mathcal{O}_F}\mathcal{G}_{y_1}^{F_1})^{\Gamma_1}.$$

Note that y_1 is of the form

$$y_1 = \frac{1}{m2^d}(v_{\mathbf{f}} + y_2),$$

where m is odd and $y_2 \in X_*(T)$ for a maximal torus T in the split group G_{F_1}. Since y_1 and $v_{\mathbf{f}}$ are fixed by Γ_1, so is any point in the line segment of y_1 and $v_{\mathbf{f}}$. Set

$$x \coloneqq \frac{1}{m2^{d+1}+1}v_{\mathbf{f}} + \frac{m2^{d+1}}{m2^{d+1}+1}y_1 = \frac{3}{m2^{d+1}+1}v_{\mathbf{f}} + \frac{2}{m2^{d+1}+1}y_2.$$

Then x lies in the line segment between y_1 and $v_{\mathbf{f}}$, and hence is fixed by Γ_1. We obtain that x corresponds to a point in $\mathcal{B}(G,F)$ and $\mathcal{G}_x = \mathcal{G}_{\mathbf{f}}$. Let F_2 be a finite (tame) extension of F_1 with ramification index $m2^{d+1}+1$. Then the image of x in $\mathcal{B}(G,F_2)$ is of the form

$3v_{\mathbf{f}} + 2y_2 \in 3v_{\mathbf{f}} + X_*(T)$. Since $3v_{\mathbf{f}}$ is hyperspecial, x is hyperspecial in $\mathcal{B}(G, F_2)$. Let K be the Galois closure of F_2/F. Then K is a tame Galois extension of F and the pair (K, x) satisfies the conclusion of Theorem 7.4.1. In particular, Theorem 7.4.1 holds when G is unramified, absolutely simple and simply connected.

Following the proof of [KPZ24, Proposition 2.2.2], we see that Theorem 7.4.1 holds when G is any unramified group over F.

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