

ALGEBRAIC COMBINATORICS ON PARTIALLY ORDERED SETS AND GRAPHS

By

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ABSTRACT

This thesis considers three algebraically motivated combinatorics questions on partially ordered sets (posets) and graphs. In the process, we consider rooted tree posets, inflated rooted tree posets, shoelace posets, $(3 + 1)$ -free posets, as well as incomparability graphs of a given poset.

Rooted trees are posets whose Hasse diagram is a graph-theoretic tree having a unique minimal element. We study rowmotion on antichains and lower order ideals of rooted trees. Recently Elizalde, Roby, Plante, and Sagan considered rowmotion on fences which are posets whose Hasse diagram is a path (but permitting any number of minimal elements). They showed that in this case, the orbits could be described in terms of tilings of a cylinder. They also defined a new notion called homometry where a statistic takes a constant value on all orbits of the same size. This is a weaker condition than the well-studied concept of homomesy which requires a constant value for the average of the statistic over all orbits. Rowmotion on fences is often homometric for certain statistics, but not homomesic. We introduce a tiling model for rowmotion on rooted trees. We use it to study various specific types of trees and show that they exhibit homometry, although not homomesy, for certain statistics.

We also study Defant and Kravitz's generalization of Schützenberger's promotion operator to arbitrary labelings of finite posets. Defant and Kravitz showed that applying the promotion operator $n - 1$ times to a labeling of a poset on n elements always gives a natural labeling of the poset and called a labeling tangled if it requires the full $n - 1$ promotions to reach a natural labeling. They also conjectured that there are at most $(n - 1)!$ tangled labelings for any poset on n elements. We propose a strengthening of their conjecture by partitioning tangled labelings according to the element labeled $n - 1$ and prove that this stronger conjecture holds for inflated rooted forest posets and a new class of posets called shoelace posets. We also introduce sorting generating functions and cumulative generating functions for the number of labelings that require k applications of the promotion operator to give a natural labeling. We prove that the coefficients of the cumulative generating function of the ordinal sum of antichains are log-concave and obtain a refinement of the weak order on the symmetric group.

We also consider $(3 + 1)$ -free posets, motivated by a reduction of the Stanley-Stembridge conjecture posited by Foley, Hoàng, and Merkel (2019), stating that the twinning operation on graphs preserves e -positivity of the chromatic symmetric function. A counterexample to this general conjecture was given by Li, Li, Wang, and Yang (2021). We prove that e -positivity is preserved by the twinning operation on cycles, by giving an e -positive generating function for the chromatic symmetric function, as well as an e -positive recurrence. We derive similar e -positive generating functions and recurrences for twins of paths. Our methods make use of the important triple deletion formulas of Orellana and Scott (2014), as well as new symmetric function identities.

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CHAPTER 1

INTRODUCTION

This dissertation explores several attributes of enumerative and algebraic combinatorics, on both partially ordered sets (posets) and graphs. A brief outline of the chapters is as follows. We start by exploring posets through an algebraic lens, focusing on a group action called rowmotion and how the orbits of this action can demonstrate nice combinatorial properties when examining a particular family of posets. Then, we shift to studying an operation called extended promotion on labelings of particular families of posets, strengthening a conjecture posited last year. Finally, we transition to more questions related to graphs, motivated by incomparability graphs of posets. We evaluate how the twinning operation on graphs affects the e -positivity of a graph, determining several new results in a rich field of combinatorics. We now give a more detailed outline of each chapter.

In Chapter 2, we will start by investigating the group action rowmotion on rooted tree posets, analyzing the orbits for particular combinatorial properties known as homomesy and homometry. The action of rowmotion has been rediscovered and renamed many times, appearing in the literature as the Fon-der-Flaass action [42], the Panyushev action and complement [4, 7], among other references [1, 9, 21, 26, 52, 38]. We will follow the conventions of Striker and Williams in [56] and refer to the action as rowmotion due to its nature of working across “rows” of posets. This action relates to many other mathematical objects, such as flag simplicial complexes, representation finite algebras, trim lattices, Auslander–Reiten translation on certain quivers, Zamolodchikov periodicity, and totally symmetric self-complementary plane partitions, among many others [36] [57][56]. We will consider the action on both antichains and lower order ideals of rooted trees, specifically searching for examples of homomesy and homometry under different statistics.

Homomesy has been studied rather extensively in recent years; the term was coined in 2013 by Jim Propp and Tom Roby [39], but an example was conjectured in 2009 by Panyushev and proven in 2013 by Armstrong, Stump, and Thomas [38][4]. A statistic on a combinatorial object is *homomesic* if, for a given group action on these objects, the average value of the statistic is the same over all orbits. In this chapter, we use rowmotion as our group action and examine cardinality

statistics on rooted tree posets. Furthermore, we dive into the newer, more broad phenomenon known as homometry. Given a group action on a set of combinatorial objects, a statistic on these objects is considered *homometric* if its value is the same over all orbits of the same cardinality. Elizalde, Roby, Plante, and Sagan introduced this new concept in 2021 and found many examples of homometry demonstrated by the cardinality statistic on orbits of rowmotion on fence posets [23].

We present several new results concerning homomesy and homometry by restricting ourselves to a well-known family of posets. A *tree* can be defined as a graph in which any two vertices are connected by exactly one path. A *rooted tree* is a tree where one vertex is designated as the “root.” Rooted trees have been well-studied in graph theory, dating as far back as 1857 [11]. A rooted tree can be specialized further by assigning the edges of the rooted tree with a natural orientation, either towards or away from this root. More recently, these types of trees have heavily influenced data science methods through decision trees, tree data structures, and mathematical modeling, among other applications. Orienting the edges of a rooted tree naturally turns our graph into a poset, where the edges represent the covering relations of the partial order and the vertices represent the objects of that partial order. In Chapter 2, we will orient our rooted trees away from the root, where the root becomes the minimum element in our poset. This is called an arborescence, or out-tree. In our case, we will just call them rooted trees.

We conclude that for rooted tree posets, we can visualize the orbits of rowmotion by using a tiling model, and we can therefore count cardinality statistics using that tiling model. This tool allows us to present several natural homometry results concerning specific families of rooted tree posets, such as stars, trees with three leaves, combs, and zippers. We also provide an example of a rooted tree where the cardinality statistics do not exhibit any homometry or homomesy.

In Chapter 3, we transition to an investigation of extended promotion on labelings of posets. A *labeling* of a poset P with n elements is a bijection from P to $\{1, 2, \dots, n\}$. A labeling is considered *natural* if it respects the partial order of P . In 1972, Schützenberger introduced the promotion operator on natural labelings of posets [44]. The motivation for the promotion operator comes from an earlier paper of Schützenberger [43], in which he defines a related operator, evacuation, to study

the celebrated RSK algorithm. Promotion and evacuation were subsequently studied by Stanley in relation to Hecke algebra products [52], by Rhoades in relation to cyclic sieving phenomena [40], and by Striker and Williams in relation to rowmotion and alternating sign matrices [56]. Traditionally, promotion was only considered on the set of natural labelings of posets. In their 2023 paper, Defant and Kravitz introduced the notion of extended promotion, which acts on the set of all labelings of a poset [19]. They determined that extended promotion will eventually turn any labeling of a poset into a natural labeling, sorting the labeling with respect to the partial order. They showed that any labeling of an n element poset will become a natural after $n - 1$ applications of extended promotion, and we call labelings that take exactly that long *tangled*. They conjectured that for a poset P on n elements, there are no more than $(n - 1)!$ tangled labelings. They proved this conjecture for inflated rooted forests, which is a large class of posets related to rooted trees. In this case, we will orient the edges of a rooted tree towards the root. This orientation transforms the root into the maximum element in our poset, an example of an anti-arborescence, or in-tree [20]. We will also refer to these as rooted trees, with orientation made clear by context.

We refine this conjecture and prove our refinement for both inflated rooted forests as well as a new family of posets, called shoelaces. Additionally, we follow the lead of both [32] and [19] in investigating properties of the sorting time of various labelings. We count labelings by the number of extended promotion steps needed to yield a natural labeling, and we define two related generating functions on P in order to examine how these generating functions change if we attach some minimal elements to P . Our result provides a simple and unified proof of enumerating tangled labelings and quasi-tangled labelings in [19] and [32].

In Chapter 4, we will shift our attention to a different type of generating function related to a graph called its chromatic symmetric function, and examine the twinning operation in relation to this formal power series. To fully introduce the chromatic symmetric function, we must start by defining a proper coloring of a graph. A graph $G = (V, E)$ with vertex set V and edge set E is *colored* when one assigns labels (called colors) to each vertex in V . A coloring on G is *proper* if no two vertices connected by an edge share the same color. The *chromatic number* of G is the

smallest number of colors that can be used in a proper coloring. The chromatic number of a graph is one of the most well-studied invariants in graph theory. The Four Color Theorem states that if a graph G can be drawn in the plane without any edge crossings, then its chromatic number is at most four [3]. Famously, this theorem was a conjecture for over 100 years, and was one of the first theorems proven by using extensive computer assistance.

The chromatic polynomial of a graph is a closely related function that enumerates proper colorings of a graph. Birkhoff defined the *chromatic polynomial*, $P(G; t)$, to be the number of proper colorings of G with t colors [8]. This polynomial has various properties that seem a bit miraculous at first glance. For example, Stanley [50] proved that if G has n vertices, then

$$P(G, -1) = (-1)^n (\text{the number of acyclic orientations of } G)$$

It is not intuitively clear what it means to color a graph with -1 colors, but this result (among others) implies a deep mathematical significance to the polynomial.

Generalizing the chromatic polynomial further, Stanley defined the *chromatic symmetric function* of a graph $G = (V, E)$ to be

$$X_G(\mathbf{x}) = \sum_{\kappa} \prod_{v \in V} x_{\kappa(v)},$$

where $\mathbf{x} = \{x_1, x_2, \dots\}$ is a countably infinite set of variables, and the sum is over all proper colorings $\kappa : V \rightarrow \mathbb{Z}_{>0}$ of G by positive integers [47]. This made it possible to make new and unexpected connections between graph coloring, the theory of symmetric functions, and even algebraic geometry (Hessenberg varieties). Stanley proved that the chromatic symmetric functions of paths and cycles are *e-positive*, that is, their expansion in the basis of elementary symmetric functions has nonnegative coefficients. The result for paths is originally due to Carlitz, Scoville, and Vaughan in a different context [10, p.242]. More generally, much of the research on the chromatic symmetric function has centered around the incomparability graph $\text{Inc}(P)$ of a $(3+1)$ -free poset P , defined as a poset containing no induced subposet isomorphic to the disjoint union of a 3-chain and a 1-chain. This direction is motivated by the famous Stanley-Stembridge Conjecture, stating that if P is a $(3+1)$ -free poset, then $X_{\text{Inc}(P)}(\mathbf{x})$ is *e-positive*. This conjecture had been standing since

1993, though Hikita recently proved it in his preprint [31]. The work in Chapter 4 was completed prior to the appearance of this proof.

Given a graph G and a vertex v , the *twin of G at v* is the graph, denoted by G_v , obtained by adding a new vertex v' and connecting v' to v and to all of its neighbors. We refer to this operation as the *twinning* of a graph and to the resulting graph G_v as the *twinned graph*. Twinning is a natural operation considered often in graph theory, usually aiding in evaluating graph isomorphisms and subgraph inclusion. It is then reasonable to ask how twinning a graph might affect its chromatic symmetric function.

Specifically, we investigate the change in $X_G(\mathbf{x})$ when one twins a vertex v of a graph G . We determine explicit e -positive formulas for the generating function of the chromatic symmetric function of four types of twinned graphs, as well as e -positive recurrence relations for five different graph families.

CHAPTER 2

ROWMOTION ON ROOTED TREES

Let S be a set with $\#S$ finite where the hash symbol denotes cardinality. A *statistic* on S is a function $\text{st} : S \rightarrow \mathbb{Z}$ where $\mathbb{Z}$ is the integers. We extend st to subsets $R \subseteq S$ by letting

$$\text{st } R = \sum_{r \in R} \text{st } r.$$

Now suppose that G is a finite group acting on S . Statistic st is said to be *homomesic* if, for any orbit O of G , we have

$$\frac{\text{st } O}{\#O} = c$$

for some constant c . To be more specific, we say in this case that this statistic is *c-mesic*. Homomesy is a well-studied property; see the survey articles of Roby [41] or Striker [55]. Recently Elizalde, Roby, Plante, and Sagan [23] introduced a weaker notion which is exhibited by certain actions and statistics. We say that a statistic st is *homometric* if for any two orbits O_1 and O_2 of the same cardinality we have $\text{st } O_1 = \text{st } O_2$. We will see numerous examples of statistics which are homometric but not homomesic in the present work.

Now consider a finite partially ordered set, often abbreviated to *poset*, $(P, \leq)$. An *antichain* of P is a $A \subseteq P$ such that no two elements of A are comparable. We denote the set of all antichains as

$$\mathcal{A}(P) = \{A \subseteq P \mid A \text{ is an antichain}\}.$$

A *lower order ideal* of P is $L \subseteq P$ such that if $y \in L$ and $x \leq y$ then $x \in L$. We will use the notation

$$\mathcal{L}(P) = \{L \subseteq P \mid L \text{ is a lower order ideal}\}.$$

The lower order ideal *generated* by any $Q \subseteq P$ is

$$Q \downarrow = \{x \in P \mid x \leq y \text{ for some } y \in Q\}.$$

We also let $\min Q$ and $\max Q$ be the sets of minimal and maximal elements of Q , respectively. We now define *rowmotion on antichains* to be the action generated by $\rho : \mathcal{A}(P) \rightarrow \mathcal{A}(P)$ where

$$\rho(A) = \min\{x \notin (A \downarrow)\}.$$

Similarly, *rowmotion on ideals* has generator $\hat{\rho} : \mathcal{L}(P) \rightarrow \mathcal{L}(P)$ with

$$\hat{\rho}(L) = \rho(\max L) \downarrow .$$

We will usually use a hat to distinguish a notation on ideals from the corresponding one on antichains. More information about rowmotion can be found in the aforementioned survey articles.

The paper of Elizalde et al. was devoted to the study of rowmotion on fences. A fence is a poset whose Hasse diagram is a path. They showed that the antichain orbits can be modeled using certain tilings of a cylinder. This tool permitted them to prove a number of homometries which were not homomesies. In the present work we will consider rowmotion on rooted trees. In this chapter, we will orient our poset away from a minimum element. We consider a poset T as a *rooted tree* if its Hasse diagram is a tree in the graph theory sense of the term and it has a unique minimal element called the *root* and denoted $\hat{0}$. Note that these posets are more general than fences in that the tree need not be a path, but also more restricted in that fences can have any number of minimal elements. We will assume all our trees are rooted.

The rest of this chapter is structured as follows. In the next section we will show that rowmotion on antichains of a rooted tree can also be viewed in terms of certain cylindrical tilings. The following three sections will apply this tiling model to three different families of trees: stars, trees with three leaves, and finally combs and zippers. We end with a section with comments and open questions.

2.1 Tilings

We will show that rowmotion orbits on antichains can be more easily viewed as certain tilings of a cylinder. Given a rooted tree T we will fix an embedding of the Hasse diagram of T in the plane and label its leaves (maximal elements) as $1, 2, \dots, n$ from left to right. See the tree on the left of Figure 2.1 for an example where $n = 5$.

For nonnegative integers m, n we use interval notation

$$[m, n] = \{m, m + 1, \dots, n\}$$

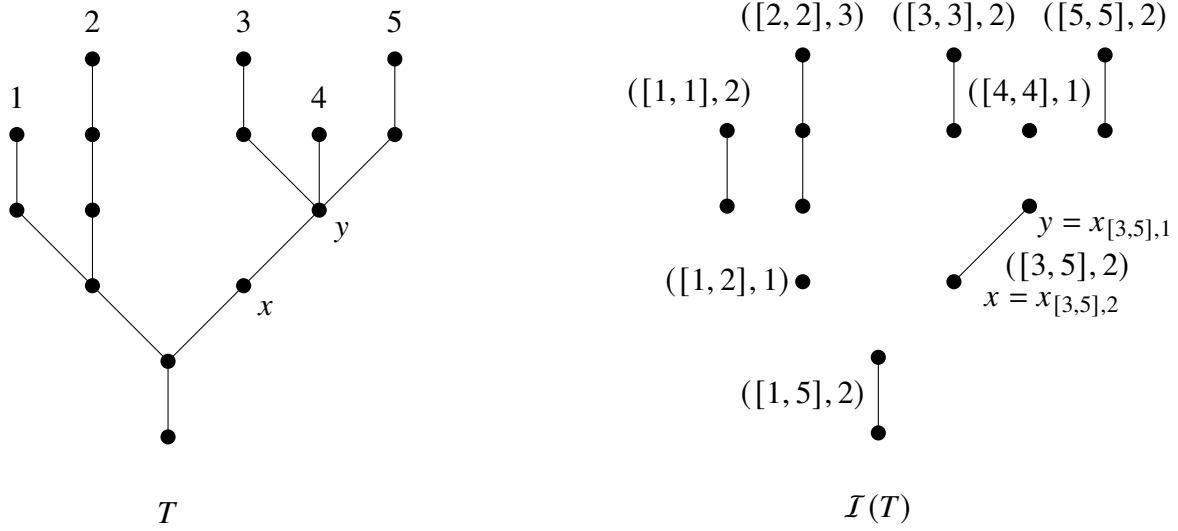


Figure 2.1 The intervals, branches, and β -values of a tree T

and abbreviate $[n] = [1, n]$. Associate with each vertex x in T the set of all labels of leaves z such that $z \geq x$. Note that by our choice of labeling, this set will be an interval I . And the set of all x with interval I form a path called the *branch* corresponding to I and denoted B_I . On the right in Figure 2.1, T has been decomposed into branches with each labeled by a pair where the first component is the interval I of B_I . For example, nodes x and y are exactly the ones below all three leaves 3, 4, 5. So their associated interval is $I = [3, 5]$ and $B_{[3,5]} = \{x, y\}$. We will also label the vertices on the branch for I as $x_{I,1}, x_{I,2}, \dots$ starting with the maximal element and working down. Returning to our example, $y = x_{[3,5],1}$ and $x = x_{[3,5],2}$. Note the following two simple but important properties of this family of intervals.

(I1) The singleton intervals $[i, i]$ are in this family for all $i \in [n]$.

(I2) The family is *nested* in the sense that if I, J are in the family with $\#I \leq \#J$ then either $I \subseteq J$ or $I \cap J = \emptyset$.

Given an interval I , let

$$\beta_I = \beta_I(T) = \#B_I.$$

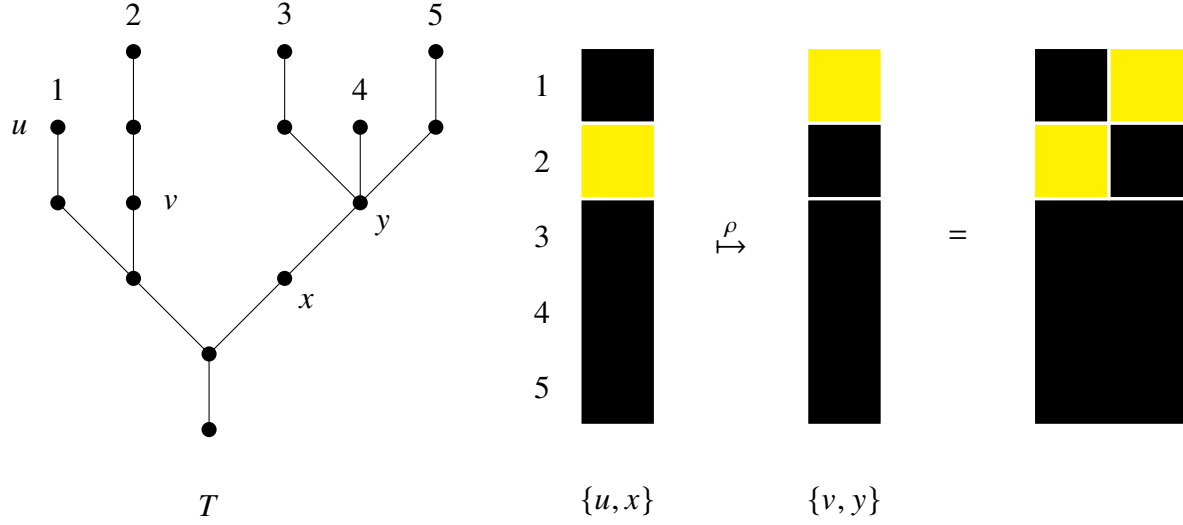


Figure 2.2 Rowmotion on antichains in terms of tilings

Returning to our usual example, for $I = [3, 5]$ we saw that $B_{[3,5]} = \{x, y\}$ which implies $\beta_{[3,5]} = 2$.

A crucial tool in defining the tilings will be the set

$$\mathcal{I}(T) = \{(I, \beta_I) \mid I \text{ is the interval of some branch of nodes in } T\}.$$

On the right in Figure 2.1, the elements of $\mathcal{I}(T)$ are displayed next to their corresponding branches.

We will abuse notation and write $I \in \mathcal{I}(T)$ to mean that $(I, \beta_I) \in \mathcal{I}(T)$.

We will need to consider partitions of intervals. A *partition* of an interval I is a collection of nonempty subintervals $I_1, \dots, I_k$ whose disjoint union is I . We say that another partition $J_1, \dots, J_l$ of I is a *refinement* of the first if for every J_j there is an I_i with $J_j \subseteq I_i$. The refinement is *proper* if the two collections of subintervals are not the same. Refinement is a partial order on partitions. If all the intervals of the partition come from $\mathcal{I}(T)$ then it is called an $\mathcal{I}(T)$ -*partition*.

We now describe the procedure to produce a tiling τ from an orbit $\mathcal{O}$ of rowmotion on antichains of a rooted tree T . Consider a column of n boxes where the i th box corresponds to the leaf labeled i in the embedding of T . The first column in Figure 2.2 is so labeled. Given an antichain A , we take each $x \in A$ and consider the interval I of its branch. The boxes labeled by the elements of I are then covered by a black tile. All other boxes are covered by a single yellow tile. Note that these boxes are exactly the ones in rows i such that there is no element of A below the leaf labeled i . Returning to Figure 2.2, consider the antichain $A = \{u, x\}$ and the leftmost column of tiles. Since

u has interval $[1, 1]$, the box in row 1 gets a black tile. Similarly, x 's interval is $[3, 5]$ so the rows for this interval also receive a black tile. The remaining square in row 2 then receives a yellow tile. The reader should now not find it hard to verify that $\rho(A) = \{v, y\}$ and that this antichain corresponds to the second column in the figure. We now paste the columns for all antichains in the orbit O together in the same order that they appear in the orbit to get a tiling $\tau = \tau(O)$ of a cylinder. Note that when pasting, if there are two consecutive columns with black tiles coming from the same interval I then these tiles are combined into one. Returning to our perennial example, the two tiles for $I = [3, 5]$ become one tile as seen in the final diagram. And if there were more elements on the branch corresponding to I , they would fatten the tile further. Three tilings corresponding to full orbits are shown in Figure 2.3. The vertical sides of these rectangles are identified to make them into cylinders. Also note that, in the middle tiling, a black tile in the second row stretches over this boundary as indicated by having it protrude beyond the sides of the rectangle.

We wish to characterize the possible $\tau(O)$. In the definition below, an $I \times b$ tile is a tile which covers the rows indexed by I and b columns. Also, the maximal partitions used are maximal with respect to the refinement order. They exist because property (I1) implies that any interval I has a partition using intervals in $\mathcal{I}(T)$ since all singletons are intervals. And property (I2) guarantees that among all such partitions of I there is a maximal one.

Definition 2.1.1. Given a rooted tree T , an $\mathcal{I}(T)$ -tiling is a tiling of a cylinder using $I \times \beta_I$ black tiles and $I \times 1$ yellow tiles if $\#I = 1$, satisfying the following two properties.

- (t1) An $I \times \beta_I$ black tile is followed by a yellow tile if $\#I = 1$, or by black tiles corresponding to the intervals in a maximal proper $\mathcal{I}(T)$ -partition of I if $\#I \geq 2$.
- (t2) If J is a maximal interval of yellow tiles in a column, then they are followed by black tiles corresponding to the intervals in a maximal $\mathcal{I}(T)$ -partition of J .

Theorem 2.1.2. Given a rooted tree, T , the map $O \mapsto \tau(O)$ is a bijection between the antichain rowmotion orbits of T and the possible $\mathcal{I}(T)$ -tilings.

Proof. We must first show that this map is well defined in that $\tau = \tau(O)$ has tiles satisfying (t1) and (t2) and of the correct shape. We will do this by studying how rowmotion affects various elements of T .

Consider $A \in O$ and any $x \in A$ which is not maximal in its branch and let I be the associated interval. Then there is a unique element y which covers x and it is in the same branch. Furthermore $y \in \rho(A)$. Since x and y correspond to the same interval I , it follows that the tile covering those rows in the column for A extends into the column for $\rho(A)$. By induction, this tile extends into a column for an antichain containing the maximal element on the branch.

Now suppose that $x \in A$ is maximal in its branch. If $\#I = 1$ then x is maximal in T . So in $\rho(A)$ the branch will be empty and the algorithm will place a yellow tile in the corresponding row and column. This proves the first case in (t1). On the other hand, if $\#I \geq 2$ then x is covered by at least two elements $y_1, \dots, y_k$. So the column for $\rho(A)$ will contain tiles in the corresponding intervals $I_1, \dots, I_k$ which is a proper $\mathcal{I}(T)$ -partition of I since $k \geq 2$. And it is maximal since if there is some $J \in \mathcal{I}(T)$ containing two or more of the I_i then there would have to be at least one element between x and the corresponding y_i 's. This completes the proof of (t1).

For (t2), we will assume for simplicity that $1, n \notin J$ where n is the number of leaves of T . The cases when J contains one or both of these special values is similar. Say $J = [m, n]$. Then by our assumption, there are black tiles covering rows $m - 1$ and $n + 1$ in the column for J . Let x and y be the corresponding elements of A . Removing the $\hat{0}$ - x and $\hat{0}$ - y paths from T breaks the lower order ideal generated by the leaves in J into rooted subtrees. Let $z_1, \dots, z_k$ be their roots with corresponding intervals $I_1, \dots, I_k$. Then $\rho(A)$ contains these z_i and so its column contains tiles for the intervals I_i which form a partition of J . Maximality is obtained by the same argument as in the previous paragraph.

To complete showing that τ is well defined, we must check the shape of the tiles. Yellow tiles are of the correct shape by definition of the algorithm. As far as the black tiles, they cover rows indexed by intervals in $\mathcal{I}(T)$ by definition. So it suffices to show that a tile in the rows indexed by I has the correct length. From the previous two paragraphs we see that the tiles in the partitions

following the maximal element of a black tile or following an interval of yellow tiles all begin with the minimal elements of their respective branches. And by the second paragraph, such a tile will extend to the maximal element on its branch. So the tile will have length β_I , the length of the branch.

To show that this map is a bijection, we construct its inverse. So given an $\mathcal{I}(T)$ -tiling τ , we must construct a corresponding orbit $\mathcal{O}$. For each column of τ we form an antichain A as follows. For each interval I covered by a black tile, suppose the given column is the i th in that tile. Then add the i th smallest element on the branch for I to A . Now arrange the antichains in the same order as the columns of the tiling to get an orbit. The demonstration that this map is well defined is similar to the one just given. And the two functions are inverses since the algorithms described are step-by-step reversals. This completes the proof. $\square$

We will often call the tiles of shape $I \times \beta_I$ simply I -tiles. As a first application of the tiling model, we will use it to compute various statistics on rowmotion orbits. It will also give us a simple proof of our first homomesy. Given $x \in T$ we have the statistic on antichains $A \in \mathcal{A}(T)$ given by

$$\chi_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

If we want to count the size of antichains we use the statistic

$$\chi(A) = \sum_{x \in T} \chi_x(A) = \#A.$$

The corresponding statistics for ideals are denoted $\hat{\chi}_x$ and $\hat{\chi}$. Given a $\mathcal{I}(T)$ -tiling τ we will use the notation

$$m_I = m_I(\tau) = \text{number of } I\text{-tiles in } \tau.$$

Corollary 2.1.3. *Let T be a rooted tree and τ be a $\mathcal{I}(T)$ -tiling corresponding to a rowmotion orbit $\mathcal{O}$ on T . The following hold.*

(a) *If $x \in T$ has interval I then*

$$\chi_x(\mathcal{O}) = m_I.$$

(b) We have

$$\chi(O) = \sum_{I \in I(T)} \beta_I m_I.$$

(c) If $x = x_{I,j}$ then

$$\hat{\chi}_x(O) = j \cdot m_I + c_I$$

where c_I is the number of columns of τ intersecting a J -tile for $J \subset I$.

(d) We have

$$\hat{\chi}(O) = \sum_{I \in I(T)} \left[\binom{\beta_I + 1}{2} m_I + \beta_I c_I \right]$$

(e) If x, y are in the same branch then $\chi_x - \chi_y$ is 0-mesic.

Proof. (a) This follows from the fact that v is represented by a single column in each I -tile of τ .

(b) Since I -tiles have length $\beta_I = \#B_I$ we get by summing (a)

$$\begin{aligned} \chi(O) &= \sum_{x \in T} \chi_x(O) \\ &= \sum_{I \in I(T)} \sum_{x \in B_I} m_I \\ &= \sum_{I \in I(T)} \beta_I m_I. \end{aligned}$$

(c) For a lower order ideal L we have that $x \in L$ if and only if $x \leq y$ for some $y \in A$ where $A = \max L$. Note also that if y has interval J then $y \geq x$ implies $J \subseteq I$. If $J = I$ then there are j choices for y and so $j \cdot m_I$ counts the total number of columns containing such an element. And c_I accounts for the columns intersection some J -tile where $J \subset I$.

(d) This result follows from (c) in much the same way that (b) followed from (a). So the proof is left to the reader.

(e) Let the common branch be B_I . Using (a) one last time we get

$$\chi_x(O) - \chi_y(O) = m_I - m_I = 0$$

which implies the homomesy. □

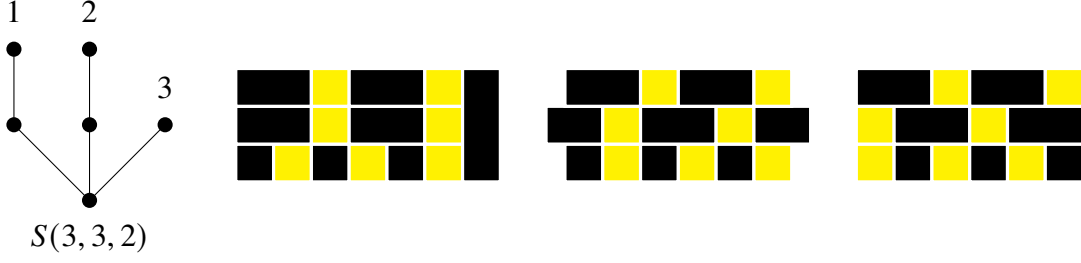


Figure 2.3 The star $S(3, 3, 2)$ and its tilings

We end this section with a recursive formula for the number of antichains in a rooted tree T which will be useful in the sequel. We use $T \setminus \{\hat{0}\}$ for the forest of rooted trees obtained by removing $\hat{0}$ from T .

Lemma 2.1.4. *Let T be a rooted tree. If $\#T = 1$ then $\#\mathcal{A}(T) = 2$. If $\#T \geq 2$ then let $T_1, \dots, T_k$ be the rooted tree components of $T \setminus \{\hat{0}\}$. In this case*

$$\#\mathcal{A}(T) = 1 + \prod_{i=1}^k \#\mathcal{A}(T_i).$$

Proof. If $\#T = 1$ then T has antichains $\emptyset$ and $\{\hat{0}\}$. When $\#T \geq 2$, let A be an antichain of T . Either $A = \{\hat{0}\}$, corresponding to the 1 in the sum, or $A \subseteq \cup_i T_i$. In the latter case the restriction A_i of A to T_i is an antichain and the product counts the possible A_i . $\square$

2.2 Stars

A *star*, S , is a rooted tree with n leaves and

$$\mathcal{I}(S) = \{([1, 1], \beta_1), \dots, ([n, n], \beta_n), ([n], 1)\}$$

where we are using the abbreviation $\beta_i = \beta_{[i, i]}$. We will use the same abbreviation for other notation involving a subscript $[i, i]$, for example $x_{i, j} = x_{[i, i], j}$. So S is the result of taking n chains of length $\beta_1, \dots, \beta_n$ and identifying their minimal elements. Note that all tiles in a corresponding tiling will only cover one row, except for the tile corresponding to $\hat{0}$. We denote this star by $S(\alpha_1, \dots, \alpha_n)$ where $\alpha_i = \beta_i + 1$ for $i \in [n]$. The reason for this change of variables is because it will make our results easier to state since α_i is the length of a black tile followed by a yellow tile in row i . The

star $S(3, 3, 2)$ and its tilings are found in Figure 2.3. Given an orbit O we will use the notation

$$\delta = \begin{cases} 1 & \text{if } \hat{0} \in O, \\ 0 & \text{if } \hat{0} \notin O. \end{cases}$$

Theorem 2.2.1. *Consider the star $S = S(\alpha_1, \dots, \alpha_n)$ and an orbit O of rowmotion on S . Let $l = \text{lcm}(\alpha_1, \dots, \alpha_n)$.*

(a) *We have*

$$\#O = l + \delta$$

and the number of orbits is $\alpha_1 \cdots \alpha_n / l$.

(b) *For any $x \in S$,*

$$\chi_x(O) = \begin{cases} l/\alpha_i & \text{if } x \in B_i, \\ \delta & \text{if } x = \hat{0}. \end{cases}$$

(c) *We have*

$$\chi(O) = \delta + \sum_{i=1}^n \frac{l}{\alpha_i} (\alpha_i - 1).$$

Thus χ is homometric but not homomesic.

(d) *For any $x \in S$*

$$\hat{\chi}_x(O) = \begin{cases} jl/\alpha_i & \text{if } x = x_{i,j} \\ l & \text{if } x = \hat{0}. \end{cases}$$

(e) *We have*

$$\hat{\chi}(O) = l + \sum_{i=1}^n \frac{l}{\alpha_i} \binom{\alpha_i}{2}.$$

Thus $\hat{\chi}$ is homometric but not homomesic.

Proof. (a) Consider the tiling $\tau = \tau(O)$. For all $i \in [n]$ the corresponding interval $I = [i, i]$ has $\#I = 1$. So, by condition (t1) in Definition 2.1.1, each black tile in that row is followed by a yellow tile. And this pair of tiles has length $\beta_i + 1 = \alpha_i$.

Now consider the case when $\hat{o} \notin O$. So no tile spans more than one row. Now the previous paragraph and (t2) imply that the black and yellow tiles alternate in row i . So the length of that row is divisible by α_i . Since this is true for all i we must have that l divides $\#O$. But since l is the least common multiple, a given column will recur after l steps. So we must have $\#O = l$. When $\hat{o} \in O$ then the same reasoning as above applies to the tiling once the column for $\hat{o}$ is removed. So in this case $\#O = l + 1$.

Now let k be the number of orbits. From what we have just proved, $\#\mathcal{A}(S) = 1 + kl$. Also, it follows easily from Lemma 2.1.4 that $\#\mathcal{A}(S) = 1 + \alpha_1 \cdots \alpha_n$. Equating the two expressions results in the desired count.

(b) We will consider the case $x \in B_i$ as the other is trivial. Consider the tiling $\tau = \tau(O)$. From the proof of (a), we see that row i has l columns which are tiled by a pair of consecutive black and yellow tiles of combined length α_i . So the number of black tiles in that row is

$$m_i = l/\alpha_i. \quad (2.1)$$

We are now done by Corollary 2.1.3 (a).

(c) Using part (b) and Corollary 2.1.3 (b) we obtain

$$\chi(O) = \beta_{[n]}m_{[n]} + \sum_{i=1}^n \beta_i m_i = \delta + \sum_{i=1}^n \frac{l}{\alpha_i}(\alpha_i - 1).$$

(d) Again, this is easy to see if $x = \hat{o}$. If $x = x_{i,j}$ then there is no $J \subset [i, i]$ in $\mathcal{I}(S)$. So by Corollary 2.1.3 (c) and equation (2.1)

$$\hat{\chi}_x(O) = j \cdot m_i = jl/\alpha_i.$$

(e) It suffices to calculate the terms in the sum of Corollary 2.1.3 (d). We will do the case when $\hat{o} \notin O$ as the unique orbit when $\hat{o} \in O$ is done similarly. We first look at the term for $I = [n]$. In this case $\beta_{[n]} = 1$ and $m_{[n]} = 0$ by the choice of O . Since $[i, i] \subset [n]$ for all i and there is no column for the empty antichain we have $c_{[n]} = l$, the number of columns of the tiling. So the term

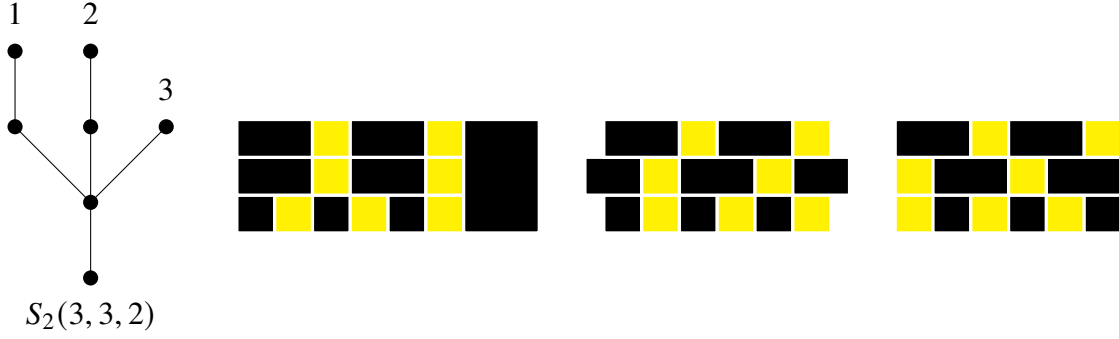


Figure 2.4 The extended star $S_2(3, 3, 2)$ and its tilings

for $I = [n]$ reduces to l . Now consider the summand for $[i, i]$. We have $\beta_i + 1 = \alpha_i$ and $m_i = l/\alpha_i$ by equation (2.1). Furthermore, there is no $J \subset [i, i]$ so $c_i = 0$. Thus the term for $I = [i, i]$ is the i th one in the sum given in (e), as desired. $\square$

Stars exhibit a number of homomesies. The following results are all gotten by simple manipulation of the formulas for χ and $\hat{\chi}$ in the previous theorem, so we suppress the demonstration.

Corollary 2.2.2. *Consider the star $S = S(\alpha_1, \dots, \alpha_n)$.*

- (a) *If $x \in B_i$, then $\alpha_i \chi_x + \chi_{\hat{0}}$ is 1-mesic.*
- (b) *If $x \in B_i$ and $y \in B_j$ then $\alpha_i \chi_x - \alpha_j \chi_y$ is 0-mesic.*
- (c) *If $x = x_{i,k}$ then $\alpha_i \hat{\chi}_x - k \hat{\chi}_{\hat{0}}$ is 0-mesic.*
- (d) *If $x = x_{i,k}$ and $y = x_{j,k}$, then $\alpha_i \hat{\chi}_x - \alpha_j \hat{\chi}_y$ is 0-mesic.* $\square$

It is easy to generalize Theorem 2.2.1 to the case where $b_{[n]} > 1$ so that one has a fatter $[n]$ -tile. More generally, we will describe what happens to any tree where $\hat{0}$ is covered by a single element. An example can be obtained by comparing Figures 2.3 and 2.4.

Proposition 2.2.3. *Suppose $T \setminus \{\hat{0}\} = T'$ is a rooted tree with n leaves. Let the $\mathcal{I}(T)$ -tilings be $\tau_1, \tau_2, \dots, \tau_k$ where τ_1 is the tiling for the orbit of $\hat{0}$. Then the $\mathcal{I}(T')$ -tilings are $\tau'_1, \tau_2, \dots, \tau_k$ where τ'_1 is obtained from τ_1 by widening the $[n]$ -tile by one column.*

Proof. Since $T \setminus \{\hat{0}\} = T'$, the intervals of T and T' are the same. Also

$$\beta_I(T') = \begin{cases} \beta_I(T) & \text{if } I \neq [n], \\ \beta_{[n]}(T) + 1 & \text{if } I = [n]. \end{cases}$$

Definition 2.1.1 now shows that the tilings transform as desired. $\square$

For a positive integer b the b -extended star, $S_b(\alpha_1, \dots, \alpha_n)$, is the rooted tree with

$$\mathcal{I}(S_b) = \{([1, 1], \beta_1), \dots, ([n, n], \beta_n), ([n], b)\}$$

and $\alpha_i = \beta_i + 1$ for $i \in [n]$. So we recover ordinary stars when $b = 1$. We see $S_2(3, 3, 2)$ in Figure 2.4. The next result follows easily from Theorem 2.2.1 and Proposition 2.2.3 and so the proof is omitted.

Corollary 2.2.4. *Consider the extended star $S_b = S_b(\alpha_1, \dots, \alpha_n)$ and an orbit $\mathcal{O}$ of rowmotion on S_b . Let $l = \text{lcm}(\alpha_1, \dots, \alpha_n)$.*

(a) *We have*

$$\#\mathcal{O} = l + \delta b$$

and the number of orbits is $\alpha_1 \cdots \alpha_n / l$.

(b) *For any $x \in S$,*

$$\chi_x(\mathcal{O}) = \begin{cases} l/\alpha_i & \text{if } x \in B_i, \\ \delta & \text{if } x \in B_{[n]}. \end{cases}$$

(c) *We have*

$$\chi(\mathcal{O}) = \delta b + \sum_{i=1}^n \frac{l}{\alpha_i} (\alpha_i - 1).$$

Thus χ is homometric but not homomesic.

(d) *For any $x \in S$*

$$\hat{\chi}_x(\mathcal{O}) = \begin{cases} jl/\alpha_i & \text{if } x = x_{i,j} \\ l + \delta(j - 1) & \text{if } x = x_{[n],j}. \end{cases}$$

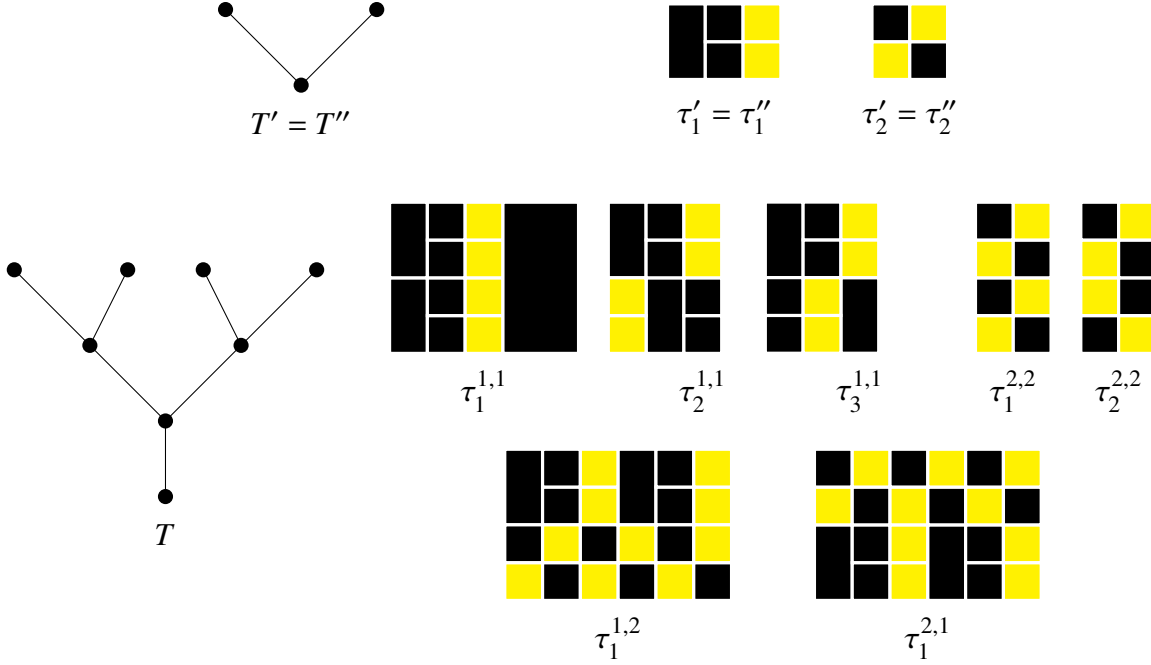


Figure 2.5 The trees T' , T'' , T and their tilings

(e) We have

$$\hat{\chi}(O) = lb + \delta \binom{b}{2} + \sum_{i=1}^n \frac{l}{\alpha_i} \binom{\alpha_i}{2}.$$

Thus $\hat{\chi}$ is homometric but not homomesic. □

2.3 Trees with Three Leaves

The special case $n = 3$ of Corollary 2.2.4 gives information about the rowmotion orbits on trees that have three leaves whose branches have minimal elements covering a single vertex of the tree. Up to isomorphism, there is only one other arrangement of branches in a tree with three leaves and this section is devoted to studying this case. First, we will prove a result about removing the branch containing $\hat{0}$ from a certain type of tree.

Proposition 2.2.3 describes the tilings of a tree T whose $\hat{0}$ is covered by a single element. We will determine what happens when it is covered by two elements or, more generally, when removing the branch of $\hat{0}$ leaves exactly two rooted trees remaining. It is possible to derive a similar result for any number of rooted subtrees, but the notation becomes cumbersome and we will only need the case of two subtrees in the sequel.

In order to state our result we will need some notation. Let T be a rooted tree such that $T \setminus B = T' \uplus T''$ where B is the branch of $\hat{0}$ and T', T'' are rooted trees. Suppose that T' has n' leaves and tilings $\tau'_1, \dots, \tau'_s$ where τ'_1 corresponds to the orbit containing $\hat{0}'$, the minimal element of T' . Further, let c'_i be the number of columns of τ'_i for $i \in [s]$. Notation used previously for T will be given a single prime when applied to T' . Similarly, let T'' have n'' leaves and tilings $\tau''_1, \dots, \tau''_{t'}$ with the same conventions about the tilings and other notation except with a double prime. An example of this construction can be found in Figure 2.5.

Theorem 2.3.1. *Let T be a rooted tree with $T \setminus B = T' \uplus T''$ as above.*

(a) *The tilings of T can be described as follows. For all $(i, j) \in [n'] \times [n'']$ there are tilings $\tau_m^{i,j}$ for $1 \leq m \leq g_{i,j} := \gcd(c'_i, c''_j)$. Unless $i = j = m = 1$, we have that $\tau_m^{i,j}$ consists of consecutive copies of τ'_i in the first n' rows, consecutive copies of τ''_j in the last n'' rows, and has $l_{i,j} := \text{lcm}(c'_i, c''_j)$ columns. Tiling $\tau_1^{1,1}$ is as in the previous sentence except that one copy of τ'_1 and one of τ''_1 align so that their columns of all yellow tiles coincide, and an $[n' + n''] \times b$ black tile is inserted directly after that column to make the total length of the orbit $l_{1,1} + b$ where $b = \#B$.*

(b) *Let O'_i, O''_j , and $O_m^{i,j}$ be the orbits corresponding to tilings τ'_i, τ''_j , and $\tau_m^{i,j}$, respectively. For any $x \in T$*

$$\chi_x(O_m^{i,j}) = \begin{cases} l_{i,j} \chi_x(O'_i) / c'_i & \text{if } x \in T', \\ l_{i,j} \chi_x(O''_j) / c''_j & \text{if } x \in T'', \\ \delta & \text{if } x \in B. \end{cases}$$

(c) *We have*

$$\chi(O_m^{i,j}) = \delta b + l_{i,j} \chi(O'_i) / c'_i + l_{i,j} \chi(O''_j) / c''_j.$$

(d) For any $x \in T$

$$\hat{\chi}_x(\mathcal{O}_m^{i,j}) = \begin{cases} l_{i,j} \hat{\chi}_x(\mathcal{O}'_i) / c'_i & \text{if } x \in T', \\ l_{i,j} \hat{\chi}_x(\mathcal{O}''_j) / c''_j & \text{if } x \in T'', \\ l_{i,j} + \delta(j-1) & \text{if } x = x_{[n'+n''],j}. \end{cases}$$

(e) We have

$$\hat{\chi}(\mathcal{O}_m^{i,j}) = l_{i,j} b + \delta \binom{b}{2} + l_{i,j} \hat{\chi}(\mathcal{O}'_i) / c'_i + l_{i,j} \hat{\chi}(\mathcal{O}''_j) / c''_j.$$

Proof. We will only prove (a), as once this is established then the other parts of the theorem follow from straight-forward computations similar to those already seen in Theorem 2.2.1. Let $\mathcal{O}$ be an antichain orbit of T . Pick an antichain A in $\mathcal{O}$ which does not contain an element of B , so that it can be written as $A = A' \uplus A''$ where $A' = A \cap T'$ and $A'' = A \cap T''$. Let $\mathcal{O}'$ and $\mathcal{O}''$ be the orbits of A' and A'' in T' and T'' , respectively.

First consider the case when (at least) one of $\mathcal{O}'$ and $\mathcal{O}''$ does not contain the empty antichain. It follows that as ρ is applied to A , the antichains A' and A'' will describe their respective orbits $\mathcal{O}'$ and $\mathcal{O}''$ in T' and T'' . If $c' = \#\mathcal{O}'$ and $c'' = \#\mathcal{O}''$ then, in order for both orbits to return to A' and A'' at the same time, we must have $\#\mathcal{O} = \text{lcm}(c', c'')$. And since there are $c'c''$ ways to pair an antichain in $\mathcal{O}'$ with one in $\mathcal{O}''$, the total number of orbits obtained from such pairs is $c'c'' / \text{lcm}(c', c'') = \text{gcd}(c', c'')$. This description matches the one given for the tilings $\tau_k^{i,j}$ for as long as we do not have $i = j = 1$.

In the case when both $\mathcal{O}'$ and $\mathcal{O}''$ contain the empty antichain, the argument of the previous paragraph goes through with one exception. Suppose the elements of $\mathcal{O}'$ and $\mathcal{O}''$ are repeated in $\mathcal{O}$ in such a way that at some point the empty antichain of T is reached. Then $\emptyset$ will be followed by the elements of B in increasing order. This, in turn, will be followed by the antichain $\{\hat{\mathcal{O}}', \hat{\mathcal{O}}''\}$ which will cause the orbits $\mathcal{O}'$ and $\mathcal{O}''$ to continue. This orbit corresponds to the tiling $\tau_1^{1,1}$ and completes our description of the orbits and their tilings. $\square$

Now consider a tree T with three leaves which is not an extended star. It follows that, using a

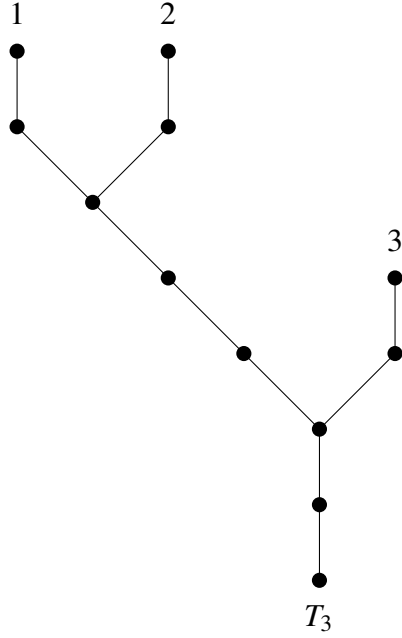


Figure 2.6 The tree T_3

suitable embedding, we will have

$$\mathcal{I}(T) = \{([3], a), ([2], b), ([1, 1], c), ([2, 2], d), ([3, 3], e)\}$$

for $a, b, c, d, e \geq 1$. A particular tree of this form is shown in Figure 2.6. Although we can use the previous theorem to calculate the orbits and their statistic values for arbitrary a, b, c, d, e the resulting formulas are not very enlightening. So we will concentrate on a specific tree of this type. Define the three-leaf tree T_k to be the one with

$$\mathcal{I}(T) = \{([3], k), ([2], k), ([1, 1], k - 1), ([2, 2], k - 1), ([3, 3], k - 1)\}.$$

The tree in Figure 2.6 is T_3 .

Theorem 2.3.2. *The orbits of rowmotion on T_k can be partitioned by length into three sets $\mathcal{S}$ (for small), $\mathcal{M}$ (for medium), and $\mathcal{L}$ (for large) with the following properties.*

(a) *We have*

$$\#\mathcal{S} = k(k - 1), \quad \#\mathcal{M} = k - 1, \quad \#\mathcal{L} = 1,$$

and

$$\#O = \begin{cases} k & \text{if } O \in \mathcal{S}, \\ 2k & \text{if } O \in \mathcal{M}, \\ 3k & \text{if } O \in \mathcal{L}. \end{cases}$$

(b) We have

$$\chi(O) = \begin{cases} 3k - 3 & \text{if } O \in \mathcal{S}, \\ 5k - 4 & \text{if } O \in \mathcal{M}, \\ 6k - 4 & \text{if } O \in \mathcal{L}. \end{cases}$$

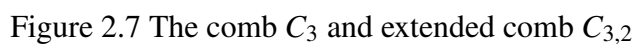
Thus χ is homometric but not homomesic.

(c) We have

$$\hat{\chi}(O) = \begin{cases} \frac{7}{2}k^2 - \frac{3}{2}k & \text{if } O \in \mathcal{S}, \\ \frac{11}{2}k^2 - \frac{5}{2}k & \text{if } O \in \mathcal{M}, \\ 6k^2 - 3k & \text{if } O \in \mathcal{L}. \end{cases}$$

Thus $\hat{\chi}$ is homometric but not homomesic.

Proof. (a) Let $B = B_{[3]}$ and $b = \#B = k$. Then $T_k \setminus B = S_k(k, k) \uplus S(k)$ is a disjoint union of two (extended) stars. Clearly $T'' = S(k)$ has only one orbit which contains $\hat{0}$. By Corollary 2.2.4, $T' = S_k(k, k)$ has orbits of size $\text{lcm}(k, k) = k$ and the total number of orbits is $(k \cdot k)/k = k$. So one of these orbits contains $\hat{0}$ and the other $k - 1$ do not, and they will have lengths given by $k + \delta k$. It follows that the latter will be of length k while the former is of length $2k$. Applying Theorem 2.3.1, T_k will have $k(k - 1)$ orbits $O_m^{i,1}$ with $i \neq 1$ and these will have length $\text{lcm}(k, k) = k$. These are the orbits in $\mathcal{S}$. There will also be the orbits $O_m^{1,1}$ for $m \in [2, k]$ which gives $k - 1$ possible values for m . Here the length is $\text{lcm}(2k, k) = 2k$. These are the orbits in $\mathcal{M}$. Finally, the unique orbit $O_1^{1,1}$ is of length $2k + b = 2k + k = 3k$ and this describes $\mathcal{L}$.


$$\chi(O'_1) = k + \sum_{i=1}^2 \frac{k}{k} (k-1) = 3k-2$$
$$\chi(\mathcal{O}_1'') = k - 1.$$
$$\chi(O_1^{1,1}) = k + 2k(3k - 2)/(2k) + 2k(k - 1)/k = 6k - 4.$$

2.4 Combs and Zippers

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It will be convenient to consider combs which have $n + 1$ leaves. Specifically, the *comb*, C_n , is the rooted tree with

$$\mathcal{I}(C_n) = \{([n + 1], 1), ([n], 1), \dots, ([2], 1), ([1, 1], 1), ([2, 2], 1), \dots, ([n + 1, n + 1], 1)\}.$$

The comb C_3 is shown on the left in Figure 2.7.

Theorem 2.4.1. *The orbits of rowmotion on C_n can be partitioned into two sets $\mathcal{S}$ and $\mathcal{L}$ having the following properties.*

(a) *We have*

$$\#\mathcal{S} = 2^{n-1}, \quad \#\mathcal{L} = 1,$$

and

$$\#O = \begin{cases} 2 & \text{if } O \in \mathcal{S}, \\ 2^{n+1} - 1 & \text{if } O \in \mathcal{L}. \end{cases}$$

(b) *We have*

$$\chi(O) = \begin{cases} n + 1 & \text{if } O \in \mathcal{S}, \\ (2n + 1)2^{n-1} & \text{if } O \in \mathcal{L}. \end{cases}$$

Thus χ is homometric but not homomesic.

(c) *We have*

$$\hat{\chi}(O) = \begin{cases} 3n + 1 & \text{if } O \in \mathcal{S}, \\ 2^{n-1}(6n - 5) + 3 & \text{if } O \in \mathcal{L}. \end{cases}$$

Thus $\hat{\chi}$ is homometric but not homomesic.

Proof. (a) We induct on n where the result is easy to check if $n = 1$. Assume the orbits are as stated for C_n and that the unique orbit in $\mathcal{L}$ is the one containing $\hat{0}$. We see that $C_{n+1} \setminus \{\hat{0}\} = C_n \uplus \{v\}$ where v is the leaf labeled $n + 2$. We will subscript notation with n or $n + 1$ to make it clear which comb is meant.

Now $T'' = \{v\}$ has only one orbit of length 2. By Theorem 2.3.1, this combines with each of the orbits in $\mathcal{S}_n$ to give orbits of length $\text{lcm}(2, 2) = 2$. Also, there will be $\text{gcd}(2, 2) = 2$ orbits in $\mathcal{S}_{n+1}$ for every one in $\mathcal{S}_n$ for a total of $2 \cdot 2^{n-1} = 2^n$ orbits. Thus the information about $\mathcal{S}_{n+1}$ is as desired.

The one orbit in $\mathcal{L}_n$ will combine with the one for $\{v\}$ to give $\text{gcd}(2, 2^{n+1} - 1) = 1$ orbit which must be the one containing $\hat{0}$. So its length will be $\text{lcm}(2, 2^{n+1} - 1) + 1 = 2^{n+2} - 1$, which finishes the induction.

(b) Again we induct, only providing details for the orbit of $\hat{0}$ in $\mathcal{L}$. Using the notation for Theorem 2.3.1 we have $c'_1 = 2^{n+1} - 1$ and $c''_1 = 2$. So $l_{1,1} = c'_1 c''_1$ and the formula in part (c) of that theorem becomes

$$\chi(\mathcal{O}) = 1 + 2(2n + 1)2^{n-1} + (2^{n+1} - 1) \cdot 1 = (2n + 3)2^n$$

as it should be.

(c) This demonstration is similar to that of (b) above using Theorem 2.3.1 (d) and so is omitted. $\square$

We can generalize these comb results as follows. The *backbone* of a comb is the set of elements which are not leaves. So C_n has an n -element backbone and each element is an interval in $\mathcal{I}(C_n)$. We will extend each of these intervals, except for the one corresponding to $\hat{0}$, so that they have k elements. Formally, the *extended comb*, $C_{n,k}$, is defined as the tree with

$$\mathcal{I}(C_n) = \{([n+1], 1), ([n], k), \dots, ([2], k), ([1, 1], 1), ([2, 2], 1), \dots, ([n+1, n+1], 1)\}.$$

On the right in Figure 2.7 is the extended comb $C_{3,2}$. Note that $C_{n,1} = C_n$.

Theorem 2.4.2. *The orbits of the extended comb $C_{n,k}$ can be partitioned into two sets $\mathcal{S}$ and $\mathcal{L}$ when k is odd, and into $n+1$ sets $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$ and $\mathcal{L}$ when k is even. The orbits have properties given by the following tables for k odd:*

k odd	$\mathcal{S}$	$\mathcal{L}$
$\#O$	2	$(k+1)2^n - 2k + 1$
number of O	2^{n-1}	1
$\chi(O)$	$n+1$	$((k+1)n+1)2^{n-1} - k + 1$
$\hat{\chi}(O)$	$(2k+1)n - 2k + 3$	$(2k+1)(k+1)n2^{n-1} - (5k^2 + 3k - 3)2^{n-1} + 3k^2$

and for k even:

k even	$\mathcal{S}_i$ for $i \in [n]$	$\mathcal{L}$
$\#O$	$k(i-1) + 2$	$k(n-1) + 3$
number of O	2^{n-i}	1
$\chi(O)$	$\frac{k(i-1)+2}{2}n - \frac{k}{4}(i^2 - 5i + 4) + 1$	$\frac{k}{4}n^2 + \frac{3k+4}{4}n - k + 2$
$\hat{\chi}(O)$	$\frac{(2k+1)(k(i-1)+2)}{2}n - \frac{k(2k+1)}{4}i^2 + \frac{3k}{4}i + \binom{k-2}{2}$	$\frac{k(2k+1)}{4}n^2 - \frac{4k^2-9k-4}{4}n + \binom{k-2}{2}$

Thus χ and $\hat{\chi}$ are homometric on $C_{n,k}$.

Proof. We will just verify the orbit structure as, once that is done, the calculation of χ and $\hat{\chi}$ are routine using Proposition 2.2.3 and Theorem 2.3.1. We will induct on n where the base case is easy. Note that $C_{n+1,k} \setminus \{\hat{0}\} = C'_{n,k} \uplus \{v\}$ where v is the leaf labeled $n+2$ and $C'_{n,k}$ is $C_{n,k}$ with its $\hat{0}$ -interval replaced by one with k elements. It follows from Proposition 2.2.3 that the orbits of these two posets are identical except for the orbit of $\hat{0}$ whose $[n+1]$ -tile has been widened by adding $k-1$ columns.

We now consider what happens when k is odd. The orbits of length 2 for $C'_{n,k}$ combine with the orbit of length 2 for $\{v\}$ in exactly the same way as in the proof of Theorem 2.4.1. As far as the orbit containing $\hat{0}$ in $C'_{n,k}$, by induction and the last sentence of the previous paragraph it has length

$$[(k+1)2^n - 2k + 1] + k - 1 = (k+1)2^n - k$$

which is odd by the parity of k . So, by Theorem 2.3.1, the orbit containing $\hat{0}$ in $C_{n+1,k}$ has length

$$\text{lcm}((k+1)2^n - k, 2) + 1 = 2[(k+1)2^n - k] + 1 = (k+1)2^{n+1} - 2k + 1$$

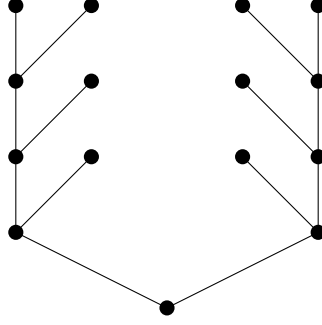


Figure 2.8 The zipper Z_3

which is the desired quantity.

When k is even we have, by induction, that all the orbits of $C_{n,k}$ have even length except for the orbit of $\hat{0}$ whose length is odd. It follows that all the orbits of $C'_{n,k}$ are of even length. So, when each non- $\hat{0}$ is combined with v 's orbit of length 2, this will result in two orbits of the same length. This accounts for the orbits in $\mathcal{S}_i$ of $C_{n+1,k}$ for $i < n$. The $\hat{0}$ -orbit of $C'_{n,k}$ will have length

$$[k(n-1) + 3] + k - 1 = kn + 2.$$

Since this is even, when it combines with v 's orbit it will produce $\gcd(kn + 2, 2) = 2$ orbits for $C_{n+1,k}$. One of these will be of size $\text{lcm}(kn + 2, 2) = kn + 2$ and that one will take care of $\mathcal{S}_n$. The other will have length one more and will be the orbit in $\mathcal{L}$. $\square$

Another way to modify combs is by combining them together. If T is a rooted tree and $T \setminus \{\hat{0}\} = T' \uplus T''$ then we will also write $T = T' \oplus T''$. Define the *zipper*, Z_n , to be

$$Z_n = C_n \oplus C_n$$

A picture of Z_3 will be found in Figure 2.8.

Theorem 2.4.3. *The orbits of Z_n can be partitioned into four sets $\mathcal{S}$, $\mathcal{M}$, $\mathcal{L}$, and $\mathcal{G}$ (for gigantic). The properties of the orbits is summarized in the following table:*

	$\mathcal{S}$	$\mathcal{M}$	$\mathcal{L}$	$\mathcal{G}$
$\#O$	2	$2^{n+1} - 1$	2^{n+1}	$2^{n+2} - 2$
number of O	2^{2n-1}	$2^{n+1} - 2$	1	2^n
$\chi(O)$	$2n + 2$	$2^n(2n + 1)$	$2^n(2n + 1) + 1$	$2^n(4n + 3) - n - 1$
$\hat{\chi}(O)$	$6n + 4$	$3 \cdot 2^n(2n - 1) + 5$	$3 \cdot 2^n(2n - 1) + 5$	$2^{n-1}(51n - 25) + 3$

Thus χ and $\hat{\chi}$ are homometric on Z_n .

Proof. As usual, we will just give details about the orbit structure. Since $Z_n \setminus \{\hat{0}\}$ is a disjoint union of two copies of C_n , we use Theorems 2.4.1 and 2.3.1. Let $\mathcal{S}'$ and $\mathcal{L}'$ refer to the orbit partition of C_n and use unprimed notation for Z_n .

Combining two orbits from $\mathcal{S}'$ gives $\gcd(2, 2) = 2$ orbits of Z_n of length $\text{lcm}(2, 2) = 2$. Since $\#\mathcal{S}' = 2^{n-1}$, the total number of orbits formed in this way is

$$2 \cdot 2^{n-1} \cdot 2^{n-1} = 2^{2n-1}.$$

These are the orbits of $\mathcal{S}$.

Putting together an orbit from $\mathcal{S}'$ with the unique orbit in $\mathcal{L}'$ results in $\gcd(2, 2^{n+1} - 1) = 1$ orbit of size $\text{lcm}(2, 2^{n+1} - 1) = 2^{n+2} - 2$. Now the total number of orbits is

$$2 \cdot 2^{n-1} \cdot 1 = 2^n$$

and they are the orbits in $\mathcal{G}$.

Finally, the combination of the orbit in $\mathcal{L}'$ with itself gives $\gcd(2^{n+1} - 1, 2^{n+1} - 1) = 2^{n+1} - 1$ orbits. All of these orbits will have length $\text{lcm}(2^{n+1} - 1, 2^{n+1} - 1) = 2^{n+1} - 1$ except for the one containing $\hat{0}$ which will have one more element. These orbits are precisely the ones in $\mathcal{M} \uplus \mathcal{L}$, and so we are done. $\square$

2.5 Comments and Open Questions

2.5.1 Other Trees

The trees considered in the previous section had such nice homometry properties that one might ask if the same is true for other binary trees. In particular, one could consider the complete

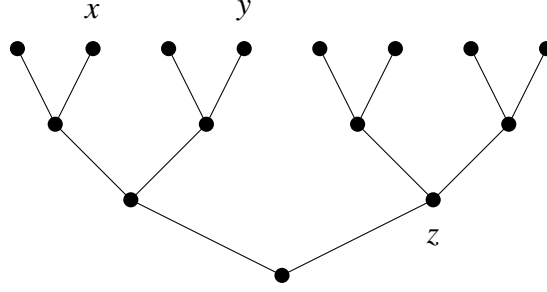


Figure 2.9 A complete binary tree

binary trees which are those all of whose leaves are at the same rank. Such a tree is displayed in Figure 2.9. Unfortunately, homometry fails for this example tree. Consider the orbit $\mathcal{O}$ which contains the antichain $\{x, y\}$ as well as the one $\mathcal{O}'$ which contains $\{z\}$. Then it is easy to verify that $\#\mathcal{O} = \#\mathcal{O}' = 4$. But

$$\chi(\mathcal{O}) = 15 \neq 14 = \chi(\mathcal{O}') \text{ and } \hat{\chi}(\mathcal{O}) = 35 \neq 26 = \hat{\chi}(\mathcal{O}').$$

As mentioned in the introduction, Elizalde et al. [23] considered fences whose Hasse diagrams are paths with any number of minimal elements. Here we have concentrated on arbitrary trees, but insisted that there be a unique minimal element. In Chapter 3, we define a family of posets called *shoelaces*, which are a generalization of fences. It would be interesting to study homomesy in shoelaces, or other general poset structures.

2.5.2 Piecewise-linear and Birational Rowmotion

There are two generalizations of rowmotion which have also been studied and which have consequences for trees. We first need to describe rowmotion in terms of toggles. In the discussion which follows we will just write ideal for lower order ideal.

If $(P, \leq_P)$ is a finite poset and $x \in P$ then the corresponding *toggle map* is $t_x : \mathcal{L}(P) \rightarrow \mathcal{L}(P)$ defined by

$$t_x(L) = \begin{cases} L \Delta \{x\} & \text{if } L \Delta \{x\} \in \mathcal{L}(P), \\ L & \text{else} \end{cases}$$

where Δ denotes symmetric difference of sets. A *linear extension* of P is a listing of P 's elements $x_1, x_2, \dots, x_p$ such that $x_i \leq_P x_j$ implies x_i is weakly left of x_j in the sequence, that is, $i \leq j$.

Cameron and Fon-Der-Flaass showed that rowmotion on ideals can be broken into a sequence of toggles. In what follows we compose functions right to left.

Theorem 2.5.1 ([9]). *For any finite poset P and any linear extension $x_1, x_2, \dots, x_p$ of P we have*

$$\hat{\rho} = t_{x_1} t_{x_2} \cdots t_{x_p}. \quad \square$$

Stanley [51] introduced the order polytope as a way to use geometry to study posets. Poset $P = \{x_1, \dots, x_p\}$ has *order polytope*

$$\Pi(P) = \{(f(x_1), \dots, f(x_p)) \in [0, 1]^p \mid x_i \leq_P x_j \text{ implies } f(x_i) \leq f(x_j)\}.$$

So $\Pi(P)$ is a subpolytope of the p -dimensional unit cube. Also note that every ideal L of P has a corresponding point of $\Pi(P)$ defined by the function

$$f(x) = \begin{cases} 0 & \text{if } x \notin P, \\ 1 & \text{if } x \in P. \end{cases}$$

Einstein and Propp [22] extended rowmotion to $\Pi(P)$. Write $x < y$ if x is covered by y in P , that is $x <_P y$ and there is no z with $x <_P z <_P y$. If $f \in \Pi(P)$ and $x \in P$ then define the *piecewise-linear toggle* σ_x of f at x to be $g = \sigma_x f \in \Pi(P)$ where

$$g(v) = \begin{cases} M + m - f(x) & \text{if } v = x, \\ f(v) & \text{if } v \neq x \end{cases} \quad (2.2)$$

using the notation

$$M = \max_{y < x} f(y) \text{ and } m = \min_{z > x} f(z). \quad (2.3)$$

It is not hard to verify from the definitions that $g \in \Pi(P)$. One can also show that σ_x is an involution just like t_x , and σ_x is also piecewise-linear as a function. Finally, one defines *piecewise-linear rowmotion*, $\rho_{\text{PL}} : \Pi(P) \rightarrow \Pi(P)$, by

$$\rho_{\text{PL}} = \sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_p}$$

where $x_1, x_2, \dots, x_p$ is a linear extension of P . It is true, but not obvious from the equation just given, that ρ_{PL} is well defined in that it does not depend on the chosen linear extension. Since $\Pi(P)$

has an infinite number of points, it is very possible for orbits of ρ_{PL} to be infinite. However, in certain cases the orbits are nice. Take, for example, the poset $[p] \times [q]$ which is the poset product of a p -element chain and a q -element chain.

Theorem 2.5.2 ([22]). *The order of ρ_{PL} on $[p] \times [q]$ is $p + q$.* □

One can extend piecewise-linear rowmotion even further to the birational realm by detropicalizing as done by Grinberg and Roby [29, 28]. This means that in equations (2.2) and (2.3) sum becomes product, difference becomes quotient, and maximum become sum. To take care of the minimum, we use the previous dictionary and the fact that for any set S of real numbers $\min S = -\max(-S)$ where $-S = \{-s \mid s \in S\}$. Now let P be a finite poset and let $\hat{P}$ be P with a minimum element $\hat{0}$ and a maximum element $\hat{1}$ added. Let $\mathbb{F}$ be a field and consider a function $f : \hat{P} \rightarrow \mathbb{F}$. The *birational toggle* of f at $x \in P$ is $g = T_x f$ where

$$g(v) = \begin{cases} \frac{\sum_{y \leq x} f(y)}{f(x) \sum_{z \geq x} f(z)^{-1}} & \text{if } v = x, \\ f(v) & \text{if } v \neq x. \end{cases}$$

One can verify that T_x is an involution, is a birational function, and that the following is well defined.

Define *birational rowmotion* on functions $f : \hat{P} \rightarrow \mathbb{F}$ as

$$\rho_{\text{B}} = T_{x_1} T_{x_2} \cdots T_{x_p}$$

where, as usual, $x_1, x_2, \dots, x_p$ is a linear extension of P . It is even more surprising when birational orbits are finite. Indeed, ρ_{B} being of finite order implies this is true for ρ_{PL} . Again, everything works well for rectangular posets.

Theorem 2.5.3 ([28]). *The order of ρ_{B} on $[p] \times [q]$ is $p + q$.* □

Call a poset P *graded* if all chains from a minimal element of P to a maximal element have the same length. Grinberg and Roby consider a class of inductively defined posets which they call *skeletal* and includes graded rooted forests, that is, disjoint unions of rooted trees such that all leaves have the same rank. In this context, they prove the following result.

Theorem 2.5.4 ([29]). *If P is a skeletal poset then ρ_B has finite order.*

They also give a formula for order of ρ_B in the case that P is a graded rooted forest which agrees with the results in Corollary 2.2.4 for graded extended stars. A natural question is whether ρ_B has finite order for any rooted trees which are not graded. Computer experiments suggest that this is not the case, although we have not been able to provide a proof. Specifically, 200 trials were run on 16 posets, and in all but one case the orbit had not repeated after 1,000,000 iterations of rowmotion.

CHAPTER 3

EXTENDED PROMOTION

A *labeling* of a poset P with n elements is a bijection from P to $[n]$. P is *naturally labeled* if the labeling respects the ordering on elements of P . In 1972, Schützenberger introduced the promotion operator on natural labelings of posets [44].

As originally defined, promotion applies only to natural labelings of posets. Defant and Kravitz generalized the notion of promotion to operate on arbitrary poset labelings and referred to their generalization as extended promotion [19]. Given a labeling L of a poset, the extended promotion of L is denoted ∂L . A key property of extended promotion is that applying it to a labeling yields a new labeling that is closer to a natural labeling. This property is quantified precisely in the following theorem.

Theorem 3.0.1 ([19, Theorem 2.8]). *For any labeling L of an n -element poset, the labeling $\partial^{n-1} L$ is a natural labeling.*

When applied to an arbitrary poset labeling, extended promotion will always result in a natural labeling after a maximum of $n - 1$ applications. Applied to a natural labeling of a poset, the extended promotion will always produce another natural labeling. Defant and Kravitz [19] define a tangled labeling of an n -element poset as a labeling that requires $n - 1$ promotions to give a natural labeling. Intuitively, the tangled labelings of a poset are those that are furthest from being sorted by extended promotion; they require the full $n - 1$ applications of extended promotion in theorem 3.0.1. Defant and Kravitz studied the number of tangled labelings of a poset and conjectured the following upper bound on the number of tangled labelings.

Conjecture 3.0.2 ([19, Conjecture 5.1]). *An n -element poset has at most $(n - 1)!$ tangled labelings.*

Defant and Kravitz proved an enumerative formula for a large class of posets known as inflated rooted forest posets (see section 3.2 for details). This formula was used by Hodges to show

conjecture 3.0.2 holds for all inflated rooted forest posets. Furthermore, Hodges conjectured a stronger version of conjecture 3.0.2.

Conjecture 3.0.3 ([32, Conjecture 31]). *An n -element poset with m minimal elements has at most $(n - m)(n - 2)!$ tangled labelings.*

Both [19] and [32] also considered counting labelings by the number of extended promotion steps needed to yield a natural labeling. In the preprint [19], Defant and Kravitz proposed the following, listed as Conjecture 5.2. Hodges further examined this conjecture.

Conjecture 3.0.4 ([32, Conjecture 29]). *Let P be an n -element poset, and let $a_k(P)$ denote the number of labelings of P requiring exactly k applications of the extended promotion to be a natural labeling. Then the sequence $a_0(P), \dots, a_{n-1}(P)$ is unimodal.*

In this chapter, we study the number of tangled labelings of posets by partitioning tangled labelings according to which poset element has label $n - 1$. We propose the following new conjecture.

Conjecture 3.0.5 (The $(n - 2)!$ Conjecture). *Let P be an n -element poset with $n \geq 2$. For all $x \in P$, let $|\mathcal{T}_x(P)|$ denote the number of tangled labelings of P such that x is labeled $n - 1$. Then $|\mathcal{T}_x(P)| \leq (n - 2)!$ with equality if and only if there is a unique minimal element $y \in P$ such that $y <_P x$.*

By results in section 3.1, both conjecture 3.0.2 and conjecture 3.0.3 follow from the $(n - 2)!$ conjecture. In theorem 3.2.14 and theorem 3.3.4, we prove that the $(n - 2)!$ conjecture holds for inflated rooted forest posets and for a new class of posets that we call shoelace posets. Furthermore, the conjecture has been computationally verified on all posets with nine or fewer elements.

Following [32], we also consider the sorting time for labelings that are not tangled and introduce associated generating functions. In remark 3.5.4, we give a poset on six elements that is a counterexample to conjecture 3.0.4. Our results completely determine the generating functions for ordinal sums of antichains. We introduce a related generating function called the cumulative

generating function and prove log-concavity of the cumulative generating function for ordinal sums of antichains.

In section 3.1 we review the basic properties of extended promotion. In section 3.2 we prove that inflated rooted forest posets satisfy the $(n-2)!$ conjecture. In section 3.3 we prove that inflated shoelace posets satisfy the $(n-2)!$ conjecture and give an exact enumeration for the number of tangled labelings of a particular type of shoelace poset called a W -poset. In section 3.4 we study the generating function of the sorting time of labelings of the ordinal sum of a poset P with the antichain T_k on k elements. In section 3.5 we show that the cumulative generating function for ordinal sums of antichains are log-concave and use the cumulative generating functions to introduce a new partial order on the symmetric group $\mathfrak{S}_n$. In section 3.6 we propose future directions to explore.

3.1 Definitions and Properties of Extended Promotion

In this section, we review and prove some properties of the extended promotion operator that will be used in later sections. Many of the definitions and results in this section come from [19] and are cited appropriately.

3.1.1 Notation and Terminology

Let $[n] = \{1, 2, \dots, n\}$. For a partially ordered set (or poset) P , the partial order on P will be denoted $\leq_P$. An element $y \in P$ is said to *cover* $x \in P$, denoted $x <_P y$, if $x <_P y$ and there does not exist an element $z \in P$ such that $x <_P z <_P y$. A *lower (resp. upper) order ideal* of P is a set $X \subseteq P$ with the property that if $y \in X$ and $x <_P y$ (resp. $x >_P y$) then $x \in X$ also. For an element $y \in P$, the *principal lower order ideal* of y is denoted $\downarrow y = \{x \in P : x \leq_P y\}$. A poset P is said to be *connected* if its Hasse diagram is a connected graph. In this chapter, we only consider finite posets and assume the reader is familiar with standard results on posets as can be found in [53, Chapter 3].

A *labeling* of a poset P with n elements is a bijection from P to $[n]$. A labeling L of P is a *natural labeling* if the sequence $L^{-1}(1), L^{-1}(2), \dots, L^{-1}(n)$ is a linear extension of P . Equivalently, for any elements $x, y \in P$, if $x <_P y$ then $L(x) < L(y)$. Given a poset P , the set of all labelings of P

will be denoted $\Lambda(P)$. The set of all natural labelings (equivalently, linear extensions) of P will be denoted $\mathcal{L}(P)$.

Definition 3.1.1 ([19, Definition 2.1]). Let P be an n -element poset and $L \in \Lambda(P)$. The *extended promotion* of L , denoted ∂L , is obtained from L by the following algorithm:

1. Repeat until the element labeled 1 is maximal: Let x be the element labeled 1 and let y be the element with the smallest label such that $y >_P x$. Swap the labels of x and y .
2. Simultaneously replace the label 1 with n and replace the label i with $i - 1$ for all $i > 1$.

In what follows, we will refer to extended promotion simply as *promotion*. For $i \geq 0$, the notations L_i and $\partial^i L$ are used interchangeably to denote the i th promotion of L . By convention, L_0 and $\partial^0 L$ denote the original labeling L . Promotion can be loosely thought of as “sorting” a labeling L so that ∂L is closer to being a natural labeling.

Definition 3.1.2 ([19, Section 2]). Let $L \in \Lambda(P)$. The *promotion chain* of L is the ordered set of elements of P whose labels are swapped in the first step of definition 3.1.1. The order of the promotion chain is the order in which the labels were swapped in the first step of definition 3.1.1.

Example 3.1.3. fig. 3.1 shows the promotion algorithm applied to a labeling L of a 6-element poset P . The promotion chain of L is the ordered set $\{L^{-1}(1), L^{-1}(2), L^{-1}(5)\}$. A sequence of five promotions of L is shown in fig. 3.2. Observe that L_i is not a natural labeling for $0 \leq i < 5$ but L_5 is a natural labeling. Since the poset P has six elements and it takes five promotions to reach a natural labeling, the labeling L is tangled.

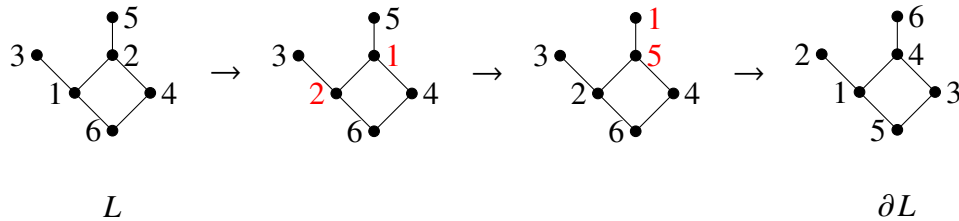


Figure 3.1 One promotion of the labeling L on poset P . Swapped labels are shown in red

Definition 3.1.4 ([19, Section 1.1]). Let P be an n -element poset and $L \in \Lambda(P)$. The *order* or *sorting time* of L , denoted $\text{or}(L)$, is the smallest integer $k \geq 0$ such that $L_k \in \mathcal{L}(P)$. If $\text{or}(L) = n - 1$, then L is a *tangled labeling*. The set of all tangled labelings of P is denoted $\mathcal{T}(P)$.

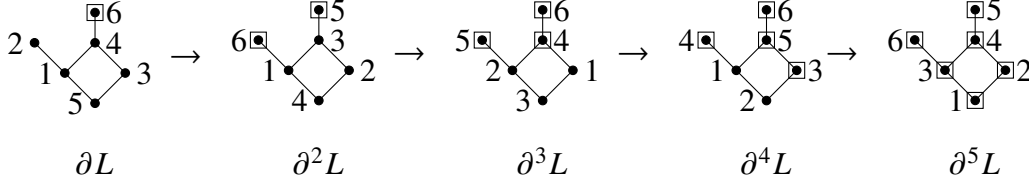


Figure 3.2 Promotions of the labeling L in fig. 3.1. Elements enclosed in a box are frozen

Definition 3.1.5. Let P be an n -element poset and $x \in P$. A labeling L of P is said to be an *x -labeling* if $L(x) = n - 1$. The set of all tangled x -labelings of P is denoted $\mathcal{T}_x(P)$.

For a poset P , the set of tangled labelings $\mathcal{T}(P)$ is the disjoint union of $\mathcal{T}_x(P)$ as x ranges over elements in P . Thus, the number of tangled labelings of P is equal to the sum

$$|\mathcal{T}(P)| = \sum_{x \in P} |\mathcal{T}_x(P)|. \quad (3.1)$$

We shall see that no tangled labeling has label $n - 1$ on a minimal element of P . Thus, it follows that the $(n - 2)!$ conjecture implies conjecture 3.0.2. While investigating this conjecture, we will occasionally want to consider a labeling restricted to a subposet.

Definition 3.1.6 ([19, Section 1.3]). Let P be an n -element poset, Q be an m -element subposet of P , and $L \in \Lambda(P)$. The *standardization* of L on Q is the unique labeling $\text{st}(L) : Q \rightarrow [m]$ such that $\text{st}(L)(x) < \text{st}(L)(y)$ if and only if $L(x) < L(y)$ for all $x, y \in Q$.

Definition 3.1.7 ([19, Section 2]). Let P be an n -element poset and $x \in P$. The element x is said to be *frozen* with respect to a labeling $L \in \Lambda(P)$ if $L^{-1}(\{a, a + 1, \dots, n\})$ is an upper order ideal for every a such that $L(x) \leq a \leq n$. The set of frozen elements of L will be denoted $\mathcal{F}(L)$.

Equivalently, if x is frozen, then the standardization of L on the subposet $L^{-1}(\{L(x), L(x) + 1, \dots, n\})$ is a natural labeling. Thus, L is a natural labeling of P if and only if $\mathcal{F}(L) = P$. Observe

that by definition 3.1.1, for any labeling L of an n -element poset P , the element labeled n in $L_1 = \partial L$ is a maximal element of P . More generally, by [19, Lemma 2.7], $L_{j+1}^{-1}(n-j)$ is frozen, so the elements of P with labels $\{n-j, n-j+1, \dots, n\}$ are “sorted.” The standardization of L_{j+1} on the subposet of P whose elements have L_{j+1} -labels in $\{n-j, n-j+1, \dots, n\}$ is a natural labeling.

Example 3.1.8. In fig. 3.2, the frozen elements of each labeling are enclosed in boxes. Observe that once an element is frozen, it remains frozen in subsequent promotions. Figure 3.3 shows a subposet Q and the standardization of the labeling L in fig. 3.1 on Q .

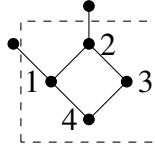


Figure 3.3 The standardization of the labeling L in fig. 3.1 on the subposet in the dotted box

We conclude this subsection by introducing funnels and basins. The basin elements of a poset are a subset of its minimal elements. In proposition 3.1.17, we will see that for tangled labelings, basins are the appropriate subset of minimal elements to pay attention to.

Definition 3.1.9. Let $x \in P$ be a minimal element. The *funnel* of x is

$$\text{fun}(x) = \{y \in P : x <_P y \text{ and } x \text{ is the unique minimal element in } \downarrow y\}.$$

Definition 3.1.10. A minimal element $x \in P$ is a *basin* if $\text{fun}(x) \neq \emptyset$.

Example 3.1.11. Let P be the poset with Hasse diagram in fig. 3.4. The basin elements in P are g and i . Their funnels are $\text{fun}(g) = \{d\}$ and $\text{fun}(i) = \{f, c\}$, respectively. There are two basins g, i in the lower order ideal $\downarrow a$ and a single basin i in the lower order ideal $\downarrow c$.

In the terminology of this section, Defant’s and Kravitz’s characterization of tangled labelings is as follows.

Theorem 3.1.12 ([19, Theorem 2.10]). *A poset P has a tangled labeling if and only if P has a basin.*

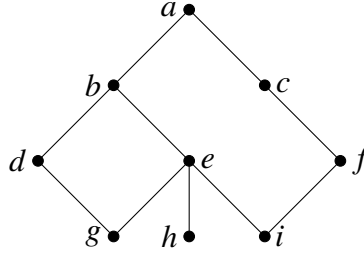


Figure 3.4 A poset with two basin elements g and i

3.1.2 Properties of Extended Promotion

In this subsection, we provide some general lemmas on extended promotion and tangled labelings. We begin with a lemma implicit in [19] that gives a useful criterion for checking whether or not a labeling is tangled.

Lemma 3.1.13. *Let P be a poset on n elements and $L \in \Lambda(P)$. The labeling L is tangled if and only if both of the following conditions are met:*

1. $L^{-1}(n)$ is minimal in P ,
2. $L^{-1}(n) <_P L_{n-2}^{-1}(1)$.

Proof. First, we will prove that conditions (1) and (2) together are sufficient for L to be tangled. Let x denote $L^{-1}(n)$. By condition (1), x is minimal so $L_{i+r}(x) = L_i(x) - r$ whenever $L_i(x) > r$. Since $L(x) = n$, it follows that $L_{n-2}(x) = 2$ and hence $L_{n-2}^{-1}(2) = L^{-1}(n)$. Substituting into condition (2) yields $L_{n-2}^{-1}(2) <_P L_{n-2}^{-1}(1)$. Thus, L_{n-2} is not yet sorted, and so L is tangled.

By [19, Lemma 3.8], condition (1) is necessary for L to be tangled. Thus, it remains to show that condition (2) follows from assuming that L is tangled and that condition (1) holds. By [19, Lemma 2.7], $L_{n-2}^{-1}(3), \dots, L_{n-2}^{-1}(n)$ are frozen with respect to L_{n-2} . Since L is tangled, L_{n-2} is not sorted, which may occur only if $L_{n-2}^{-1}(2) <_P L_{n-2}^{-1}(1)$. Because $L^{-1}(n)$ is minimal, we may substitute $L^{-1}(n) = L_{n-2}^{-1}(2)$ to yield condition (2). $\square$

As a consequence of condition (2), the element labeled $n - 1$ cannot be minimal in a tangled labeling of P . If an n -element poset P has m minimal elements, then conjecture 3.0.5 would imply

that the number of tangled labelings of P is at most $(n - m)(n - 2)!$. Therefore, conjecture 3.0.5 also implies conjecture 3.0.3.

Lemma 3.1.14. *Let P be a poset on n elements and $L \in \Lambda(P)$. Then for all $2 \leq i \leq n$ and $0 \leq j \leq n - 1$,*

$$L_{j+1}^{-1}(i - 1) \leq_P L_j^{-1}(i).$$

Proof. If i is not the label of an element in the promotion chain of L_j , then the element $L_j^{-1}(i)$ will be labeled $i - 1$ in L_{j+1} , so $L_{j+1}^{-1}(i - 1) = L_j^{-1}(i)$. If $L_j^{-1}(i)$ is in the promotion chain of L_j , let x denote the element immediately preceding $L_j^{-1}(i)$ in the promotion chain of L_j . Such an element exists since $i \geq 2$ so $L_j^{-1}(i)$ cannot be the first element in the promotion chain. It follows that $L_{j+1}^{-1}(i - 1) = x \leq_P L_j^{-1}(i)$. $\square$

A consequence of lemma 3.1.14 is that for all $2 \leq i \leq n$,

$$L_{i-1}^{-1}(1) \leq_P \dots \leq_P L_1^{-1}(i - 1) \leq_P L^{-1}(i). \quad (3.2)$$

Setting $i = n - 1$ gives, in particular,

$$L_{n-2}^{-1}(1) \leq_P L_{n-3}^{-1}(2) \leq_P \dots \leq_P L_1^{-1}(n - 2) \leq_P L^{-1}(n - 1). \quad (3.3)$$

Corollary 3.1.15. *Let P be a poset on n elements and let $L \in \mathcal{L}(P)$ be a tangled labeling. For $r = 0, 1, \dots, n - 2$,*

$$L_r^{-1}(n - r) <_P L_r^{-1}(n - 1 - r).$$

In particular, $L^{-1}(n) <_P L^{-1}(n - 1)$.

Proof. By lemma 3.1.14, $L_r^{-1}(n - r) \leq_P L^{-1}(n)$, and by (2) in Lemma 3.1.13, $L^{-1}(n) <_P L_{n-2}^{-1}(1)$. Additionally, by eq. (3.3), $L_{n-2}^{-1}(1) \leq_P L_{n-3}^{-1}(2) \leq_P \dots \leq_P L_r^{-1}(n - 1 - r)$. Combining these inequalities yields the desired result $L_r^{-1}(n - r) <_P L_r^{-1}(n - 1 - r)$. If we set $r = 0$, then we see that $L^{-1}(n) <_P L^{-1}(n - 1)$. $\square$

In [19, Corollary 3.7], Defant and Kravitz showed that any poset with a unique minimal element satisfies conjecture 3.0.2. We strengthen this result to show that posets with any number of minimal elements—but only one basin—also satisfy conjecture 3.0.2. We will need the following lemma that is the key tool in Defant and Kravitz’s proof of theorem 3.0.1.

Lemma 3.1.16 ([19, Lemma 2.6]). *Let P be an n -element poset and let $L \in \Lambda(P) \setminus \mathcal{L}(P)$. Then $\mathcal{F}(L) \subsetneq \mathcal{F}(\partial L)$.*

Proposition 3.1.17. *If L is a tangled labeling of P , then $L^{-1}(n)$ is a basin. In particular, if P has exactly one basin, then $|\mathcal{T}(P)| \leq (n-1)!$.*

Proof. We first show that for any minimal element $x \in P$ that is not a basin, there is no tangled labeling L with $L(x) = n$. Suppose to the contrary that there exists such a tangled labeling L . Let $w = L_{n-2}^{-1}(1)$. By lemma 3.1.13, $x <_P w$. Since x is not a basin, $\text{fun}(x) = \emptyset$. Hence, there exists a minimal element $z \neq x$ such that $z <_P w$.

Since $w = L_{n-2}^{-1}(1)$ and $x = L^{-1}(n) = L_{n-2}^{-1}(2)$, it follows that $z = L_{n-2}^{-1}(m)$ for some $m \geq 3$. The elements $L_{n-2}^{-1}(3), \dots, L_{n-2}^{-1}(n)$ are frozen as a consequence of lemma 3.1.16. Recall that the set of frozen elements is an upper order ideal. Since z is a frozen element and $z <_P w$, w must also be a frozen element, which is a contradiction since L_{n-2} is not a natural labeling. Therefore if L is a tangled labeling and $L^{-1}(n)$ is a minimal element of P , then $L^{-1}(n)$ must be a basin.

Finally, suppose P has a unique basin x . Then any tangled labeling L of P must satisfy $L(x) = n$. There are $(n-1)!$ labelings L that satisfy $L(x) = n$, so $|\mathcal{T}(P)| \leq (n-1)!$. $\square$

The following two lemmas relate tangled labelings and funnels of posets. They will be used in section 3.3 to prove that shoelace posets satisfy the $(n-2)!$ conjecture.

Lemma 3.1.18. *Let x be a basin of P and let L be a labeling such that $L^{-1}(n) = x$ and $L^{-1}(n-1) \in \text{fun}(x)$. Then L is tangled.*

Proof. It is clear from the definition of basins that condition (1) of lemma 3.1.13 is satisfied. So it suffices to show that $L^{-1}(n) <_P (L_{n-2})^{-1}(1)$. From eq. (3.3) and the condition that $L^{-1}(n-1) \in$

$\text{fun}(x)$,

$$x \leq_P (L_{n-2})^{-1}(1) \leq_P L^{-1}(n-1).$$

Furthermore,

$$x = L^{-1}(n) = (L_{n-2})^{-1}(2) \neq (L_{n-2})^{-1}(1).$$

Thus, we have the strict inequality $x = L^{-1}(n) <_P (L_{n-2})^{-1}(1)$, which is precisely condition (2) of lemma 3.1.13. $\square$

Lemma 3.1.19. *Let P be a poset on n elements and L a tangled labeling of P . Let $x, y \in P$ such that x is a minimal element and $x <_P y$. If $L(x) = n$ and $L(y) = n-1$, then there exists $z \in \text{fun}(x)$ such that $z \leq_P y$.*

Proof. Let $z = L_{n-2}^{-1}(1)$. By eq. (3.3), $z = L_{n-2}^{-1}(1) \leq_P L^{-1}(n-1) = y$. Thus, $z \leq_P y$. Since L is a tangled labeling, lemma 3.1.13 implies that $x = L_{n-2}^{-1}(2) <_P L_{n-2}^{-1}(1) = z$. There are at least $n-2$ frozen elements with respect to L_{n-2} , but x and z are not frozen with respect to L_{n-2} . Since the set of frozen elements with respect to a labeling form an upper order ideal, it follows that z covers x and no other elements. Hence, $z \in \text{fun}(x)$. $\square$

Lemma 3.1.20. *Let P_1 be a poset with n_1 elements and P_2 a poset with n_2 elements. If conjecture 3.0.5 holds for P_1 and P_2 , then conjecture 3.0.5 also holds for the disjoint union $P_1 \sqcup P_2$.*

Proof. Let $x \in P_1 \sqcup P_2$ and L be an x -labeling of P (i.e., $L(x) = n-1$). If $x \in P_1$ and $n_1 \geq 2$, then by [19, Theorem 3.4], L is tangled if and only if $L^{-1}(n) \in P_1$ and $\text{st}(L|_{P_1}) \in \mathcal{T}(P_1)$. Thus, the tangled x -labelings of $P_1 \sqcup P_2$ are enumerated by a choice of one of the $|\mathcal{T}_x(P_1)|$ tangled x -labelings of P_1 , one of the $\binom{n_1+n_2-2}{n_1-2}$ assignments of the labels $L^{-1}(P_1) \setminus \{n, n-1\}$, and one of the $n_2!$ labelings on P_2 . Since P_1 satisfies conjecture 3.0.5, $|\mathcal{T}_x(P_1)| \leq (n_1-2)!$. Therefore,

$$\begin{aligned} |\mathcal{T}_x(P_1 \sqcup P_2)| &= |\mathcal{T}_x(P_1)| \cdot n_2! \cdot \binom{n_1+n_2-2}{n_1-2} \\ &\leq (n_1-2)! \cdot n_2! \cdot \binom{n_1+n_2-2}{n_1-2} \\ &= (n_1+n_2-2)!. \end{aligned} \tag{3.4}$$

If $x \in P_1$ and $n_1 < 2$, then by the contrapositive of corollary 3.1.15, L is not tangled. In this case, tangled x -labelings of $P_1 \sqcup P_2$ do not exist, so $|\mathcal{T}_x(P_1 \sqcup P_2)| \leq (n_1 + n_2 - 2)!$ clearly. Equality in eq. (3.4) holds if and only if $|\mathcal{T}_x(P_1)| = (n_1 - 2)!$. Since P_1 satisfies conjecture 3.0.5, $|\mathcal{T}_x(P_1)| = (n_1 - 2)!$ if and only if there is a unique minimal element $z \in P_1$ such that $z <_{P_1} x$. It follows that equality in eq. (3.4) holds if and only if there is a unique minimal element $z \in P_1 \sqcup P_2$ such that $z <_{P_1 \sqcup P_2} x$. If $x \in P_2$, then by an identical argument, $|\mathcal{T}_x(P_1 \sqcup P_2)| \leq (n_1 + n_2 - 2)!$, with equality if and only if there is a unique minimal element $z \in P_2$ such that $z <_{P_1 \sqcup P_2} x$. Therefore, $P_1 \sqcup P_2$ satisfies conjecture 3.0.5. $\square$

By lemma 3.1.20, it suffices to show the $(n - 2)!$ conjecture for connected posets. Thus, for the remainder of the paper, we will assume our posets are connected.

3.2 Inflated Rooted Forest Posets

In [19], a large class of posets known as *inflated rooted forest posets* was introduced and it was shown in [32] that conjecture 3.0.2 holds for inflated rooted forest posets. In this section, we strengthen this result by showing that conjecture 3.0.5 holds for inflated rooted forest posets.

Definition 3.2.1 ([19, Definition 3.2]). Let P, Q be finite posets. The poset P is an *inflation* of Q if there exists a surjective map $\varphi : P \rightarrow Q$ that satisfies the following two properties:

1. For any $x \in Q$, the preimage $\varphi^{-1}(x)$ has a unique minimal element in P .
2. For any $x, y \in P$ such that $\varphi(x) \neq \varphi(y)$, $x <_P y$ if and only if $\varphi(x) <_Q \varphi(y)$.

Such a map φ is called an *inflation map*.

Example 3.2.2. In fig. 3.5 the poset P is an inflation of the poset Q . The inflation map φ is constant on each colored box in P and maps to the corresponding element in Q pointed to by the arrow. For example, the element labeled $u_{1,1}$ in Q corresponds to the subposet $\varphi^{-1}(u_{1,1})$ in P outlined in green. In general, the preimage of an element in Q must have a unique minimal element, by definition, but may have multiple maximal elements.

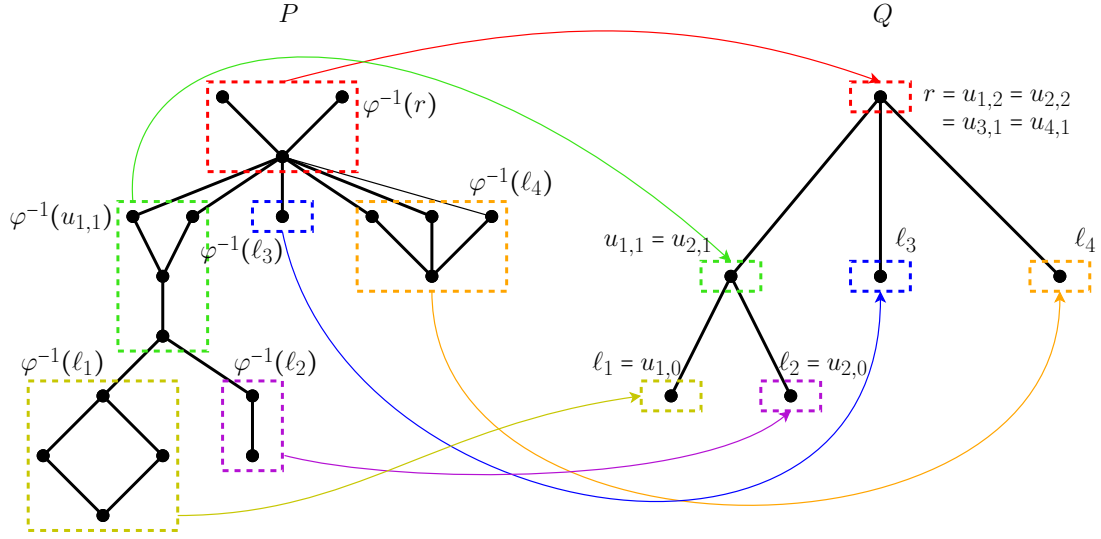


Figure 3.5 A rooted tree Q and its inflation P . The inflation map φ is represented by the arrows from P to Q

Definition 3.2.3 ([19, Definition 3.1]). A *rooted tree poset* Q is a finite poset satisfying the following two properties:

1. There is a unique maximal element of Q called the *root* of Q .
2. Every non-root element in Q is covered by exactly one element.

Notice here that we are taking the convention of an anti-arborescence, where the root is a *maximal* element, the opposite orientation of a rooted tree in Chapter 2. A *rooted forest poset* is defined to be a finite poset that can be written as the disjoint union of rooted tree posets. The posets P and Q in fig. 3.5 are examples of an inflated rooted tree poset and a rooted tree poset, respectively. Throughout the rest of this section, unless otherwise specified, Q will denote a rooted tree poset and P will denote an inflation of Q with inflation map φ .

The following definitions on inflated rooted tree posets can be found in [19]. We reproduce them here for the reader's convenience and to state lemma 3.2.5 precisely. Let r be the root of Q and let x be a non-root element of Q . The unique element y that covers x in Q is called the *parent* of x . The minimal elements of Q are called *leaves*. A rooted tree poset is said to be *reduced* if every non-leaf element covers at least 2 elements. By [19, Remark 3.3], every inflated rooted tree

poset can be obtained as an inflation of a reduced rooted tree poset, so in the following we will generally restrict ourselves to reduced rooted tree posets.

Let $\ell_1, \dots, \ell_m$ denote the leaves of Q , where m is the number of leaves. For each $i \in [m]$, we have a unique maximal chain from ℓ_i to r

$$\ell_i = u_{i,0} \leq_Q u_{i,1} \leq_Q \cdots \leq_Q u_{i,\omega_i} = r, \quad (3.5)$$

where ω_i denotes the length of the chain. Recall that $u_{i,0} \leq_Q u_{i,1}$ means that $u_{i,1}$ covers $u_{i,0}$ in Q .

For $i \in [m]$ and $j \in [\omega_i]$, define the two quantities

$$\begin{aligned} b_{i,j} &= \sum_{v \leq_Q u_{i,j-1}} |\varphi^{-1}(v)|, \\ c_{i,j} &= \sum_{v \leq_Q u_{i,j}} |\varphi^{-1}(v)|. \end{aligned} \quad (3.6)$$

The fraction $\frac{b_{i,j}}{c_{i,j}}$ therefore represents the fraction of elements in P below the minimal element of $\varphi^{-1}(u_{i,j})$ that lie on the preimage of the maximal chain from ℓ_i to r . When it is necessary to specify the rooted tree poset Q , we shall do so by indicating Q in parentheses. For example, we will write $u_{i,j}(Q)$ instead of $u_{i,j}$ or $\omega_i(Q)$ instead of ω_i .

Example 3.2.4. The vertices of Q in fig. 3.5 are labeled in accordance with our definitions above. For example, the maximal chain from ℓ_1 to r is $\ell_1 = u_{1,0} \leq u_{1,1} \leq u_{1,2} = r$. The length of this maximal chain is $\omega_1 = 2$. As another example, the maximal chain from ℓ_2 to r is $\ell_2 = u_{2,0} \leq u_{2,1} \leq u_{2,2} = r$. Notice that $u_{i,j}$ may refer to the same element in Q for distinct i and j . For example, $u_{2,1} = u_{1,1}$ in fig. 3.5, and the root r is equal to $u_{1,2}$, $u_{2,2}$, $u_{3,1}$, and $u_{4,1}$.

The quantity $b_{1,1}$ can be computed by

$$b_{1,1} = \sum_{v \leq_Q u_{1,0}} |\varphi^{-1}(v)| = |\varphi^{-1}(u_{1,0})| = 4.$$

Similarly, the quantity $c_{1,1}$ can be computed by

$$c_{1,1} = \sum_{v \leq_Q u_{1,1}} |\varphi^{-1}(v)| = |\varphi^{-1}(u_{1,0})| + |\varphi^{-1}(u_{2,0})| = 6.$$

Therefore $\frac{b_{1,1}}{c_{1,1}} = \frac{4}{6}$ of the elements in P below the minimal element of $\varphi^{-1}(u_{1,1})$ lie in the direction of $\varphi^{-1}(\ell_1)$.

The following technical lemma provides a useful bound for the formula in theorem 3.2.13. The left side of eq. (3.7) appears in [19, Theorem 3.5], and a similar term also appears in [32, Theorem 9].

Lemma 3.2.5. *Let Q be a reduced rooted tree poset with m leaves and let P be an inflation of Q with n elements. Then*

$$\sum_{i=1}^m \prod_{j=1}^{\omega_i(Q)} \frac{b_{i,j}(Q) - 1}{c_{i,j}(Q) - 1} \leq \begin{cases} 1 & \text{if } n = 1, \\ \frac{n-m}{n-1} & \text{otherwise.} \end{cases} \quad (3.7)$$

Proof. We will prove the bound by inducting on $h(Q) = \max\{\omega_1(Q), \dots, \omega_m(Q)\}$. The base case is when $h(Q) = 0$. In this case, there is a single leaf so $m = 1$ and $\omega_1(Q) = 0$. Thus, the left side of the inequality is the sum of a single empty product which is equal to 1. The right side is 1 regardless of whether $n = 1$ or $n > 1$, so the inequality holds when $h(Q) = 0$.

Now, suppose $h(Q) > 0$ and that the lemma holds for all rooted tree posets Q' with $h(Q') < h(Q)$. Since $h(Q) > 0$, $n > 1$. Now, let r denote the root of Q and let $q_1, \dots, q_t$ be the elements covered by r . Recall that for an element x in a poset, $\downarrow x$ denotes the set of elements less than or equal to x . The subposets $Q_k = \downarrow q_k$ are all rooted tree posets with $h(Q_k) \leq h(Q) - 1$, and $P_k = \varphi^{-1}(Q_k)$ is an inflation of Q_k . Let $n_k = |\varphi^{-1}(Q_k)|$ so that $n - |\varphi^{-1}(r)| = n_1 + \dots + n_t$, and let m_k denote the number of leaves of Q_k so that $m = m_1 + \dots + m_t$. For convenience, let M_k denote the k th partial sum $m_1 + \dots + m_k$ and let $M_0 = 0$. Without loss of generality, order the leaves $\ell_1, \dots, \ell_m$ of Q such that the leaves of Q_k are $\ell_{M_{k-1}+1}, \dots, \ell_{M_k}$.

Observe that for $M_{k-1} + 1 \leq i \leq M_k$, $\omega_i(Q_k) = \omega_i(Q) - 1$, and for $1 \leq j \leq \omega_i(Q_k)$, $b_{i,j}(Q_k) = b_{i,j}(Q)$ and $c_{i,j}(Q_k) = c_{i,j}(Q)$. Additionally, $b_{i,\omega_i(Q)}(Q) = n_k$ and $c_{i,\omega_i(Q)}(Q) = n - |\varphi^{-1}(r)|$.

Thus,

$$\begin{aligned}
\sum_{i=1}^m \prod_{j=1}^{\omega_i(Q)} \frac{b_{i,j}(Q) - 1}{c_{i,j}(Q) - 1} &= \sum_{k=1}^t \left(\sum_{i=M_{k-1}+1}^{M_k} \prod_{j=1}^{\omega_i(Q)} \frac{b_{i,j}(Q) - 1}{c_{i,j}(Q) - 1} \right) \\
&= \sum_{k=1}^t \left(\sum_{i=M_{k-1}+1}^{M_k} \frac{n_k - 1}{n - |\varphi^{-1}(r)| - 1} \cdot \prod_{j=1}^{\omega_i(Q)-1} \frac{b_{i,j}(Q) - 1}{c_{i,j}(Q) - 1} \right) \\
&= \sum_{k=1}^t \frac{n_k - 1}{n - |\varphi^{-1}(r)| - 1} \cdot \left(\sum_{i=M_{k-1}+1}^{M_k} \prod_{j=1}^{\omega_i(Q_k)} \frac{b_{i,j}(Q_k) - 1}{c_{i,j}(Q_k) - 1} \right).
\end{aligned}$$

For each $1 \leq k \leq t$, if $n_k = m_k = 1$, then we clearly have

$$\frac{n_k - 1}{n - |\varphi^{-1}(r)| - 1} \cdot \left(\sum_{i=M_{k-1}+1}^{M_k} \prod_{j=1}^{\omega_i(Q_k)} \frac{b_{i,j}(Q_k) - 1}{c_{i,j}(Q_k) - 1} \right) \leq \frac{n_k - m_k}{n - |\varphi^{-1}(r)| - 1},$$

as both sides of the inequality are 0. If $n_k > 1$, then by the inductive hypothesis we also have

$$\begin{aligned}
\frac{n_k - 1}{n - |\varphi^{-1}(r)| - 1} \cdot \left(\sum_{i=M_{k-1}+1}^{M_k} \prod_{j=1}^{\omega_i(Q_k)} \frac{b_{i,j}(Q_k) - 1}{c_{i,j}(Q_k) - 1} \right) &\leq \frac{n_k - 1}{n - |\varphi^{-1}(r)| - 1} \cdot \frac{n_k - m_k}{n_k - 1} \\
&= \frac{n_k - m_k}{n - |\varphi^{-1}(r)| - 1}.
\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}
\sum_{i=1}^m \prod_{j=1}^{\omega_i(Q)} \frac{b_{i,j}(Q) - 1}{c_{i,j}(Q) - 1} &\leq \sum_{k=1}^t \frac{n_k - m_k}{n - |\varphi^{-1}(r)| - 1} \\
&= \frac{n - |\varphi^{-1}(r)| - m}{n - |\varphi^{-1}(r)| - 1} \\
&\leq \frac{n - m}{n - 1}.
\end{aligned} \tag{3.8}$$

□

Remark 3.2.6. Since $|\varphi^{-1}(r)| > 0$, the final inequality in eq. (3.8) is strict for $m > 1$. If $m = 1$, then there is only one leaf in Q , so $b_{1,j}(Q) = c_{1,j}(Q)$ for $1 \leq j \leq \omega_1(Q)$ and equality holds. In particular, the upper bound in lemma 3.2.5 is never sharp for $m > 1$.

Definition 3.2.7. Let P be an n element poset and $X \subseteq P$. A *partial labeling* of P is an injective map $M : X \rightarrow [n]$. A labeling $L : P \rightarrow [n]$ is an *extension* of M if $L|_X = M$. The set of extensions of M is denoted $\Lambda(P, M)$.

Definition 3.2.8. Let P be a poset and $x \in P$. The element x is *lower order ideal complete* (LOI-complete) if any element that is comparable to some element in $\downarrow x$ is also comparable to x itself.

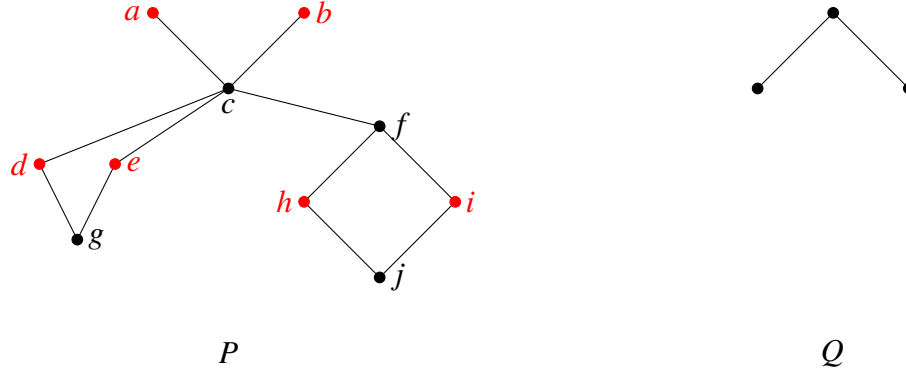


Figure 3.6 A rooted tree poset Q and an inflation P of Q . The LOI-complete elements in P are colored black

Example 3.2.9. Consider the rooted tree poset Q and its inflation P in fig. 3.6. In P , the elements c, f, g , and j are all LOI-complete, since for each of those elements, all elements comparable to $\downarrow c, \downarrow f, \downarrow g$, and $\downarrow j$ are also comparable to c, f, g , and j , respectively. The elements a, b, d, e, h , and i , colored in red, are *not* LOI-complete. For example, b is not LOI-complete because the element a is comparable to $c \in \downarrow b$ but a is not comparable to b .

Lemma 3.2.10. Let Q be a rooted tree poset and let P be an inflation of Q with inflation map $\varphi : P \rightarrow Q$. For any $q \in Q$, the unique minimal element of $\varphi^{-1}(q)$ is LOI-complete in P .

Proof. Denote the unique minimal element of $\varphi^{-1}(q)$ by x . Let $y \in \downarrow \varphi^{-1}(q)$ and suppose $z \in P$ is comparable to y . If $y = x$ then z and x are comparable by definition. Otherwise, if $y \neq x$, then $\varphi(y) <_Q \varphi(x) = q$ since x is the unique minimal element of $\varphi^{-1}(q)$. Since $z \in P$ is comparable to y and P is a rooted tree poset, $\varphi(z)$ and $\varphi(y)$ are comparable and hence $\varphi(z)$ and q is comparable.

If $\varphi(z) <_Q q$, then $z <_P x$ by definition 3.2.1. If $\varphi(z) = q$, then $x \leq_P z$ since x is the unique minimal element of $\varphi^{-1}(q)$. If $q <_Q \varphi(z)$, then $x <_P z$. In each case, z is comparable to x . Therefore x is LOI-complete in P . □

We will need the following probability lemmas from [19], so we have reproduced them for convenience.

Lemma 3.2.11 ([19, Lemma 3.10]). *Let P be an n element poset and $x \in P$ be LOI-complete. Let $X = \downarrow x \setminus \{x\}$. For $L \in \Lambda(P)$ and $k \geq 0$, the set $L_k(X)$ depends only on the set $L(X)$ and the restriction $L|_{P \setminus X}$. It does not depend on the way in which labels in $L(X)$ are distributed among the elements of X .*

Lemma 3.2.12 ([19, Lemma 3.11]). *Let P be an n element poset and $x \in P$ be LOI-complete. Let $X = \downarrow x \setminus \{x\}$ and suppose that $X \neq \emptyset$. Let $A \subseteq X$ have the property that no element of A is comparable with any element in $X \setminus A$ and let $M : P \setminus X \rightarrow [n]$ be a partial labeling such that $L_{n-1}^{-1}(1) \in X$ for every extension L of M . If a labeling L is chosen uniformly at random from the extensions in $\Lambda(P, M)$, then the probability that $L_{n-1}^{-1}(1) \in A$ is $\frac{|A|}{|X|}$.*

By suitably modifying the proof of [19, Theorem 3.5], one can strengthen it to obtain theorem 3.2.13. The following proof is self-contained, but the interested reader may wish to refer to [19, Section 3] for further details.

Theorem 3.2.13. *Let Q be a reduced rooted tree poset with m leaves and let P be an inflation of Q with n elements, with inflation map $\varphi : P \rightarrow Q$. For a nonminimal element $x \in P$, let $\ell(x) = \{i \in [m] : \ell_i \leq_Q \varphi(x)\}$ and $\omega_{i,x} = \max\{j : u_{i,j} \leq_Q \varphi(x)\}$. Then the number of tangled x -labelings of P is given by*

$$|\mathcal{T}_x(P)| = (n-2)! \sum_{i \in \ell(x)} \prod_{j=1}^{\omega_{i,x}} \frac{b_{i,j}(Q) - 1}{c_{i,j}(Q) - 1}.$$

Proof. Fix a leaf ℓ_i of Q and let x_0 be the unique minimal element of $\varphi^{-1}(\ell_i)$. We will count the number of tangled labelings L such that $L^{-1}(n) = x_0$ and $L^{-1}(n-1) = x$. By corollary 3.1.15, if L is tangled, then $x_0 = L^{-1}(n) <_P L^{-1}(n-1) = x$. Thus, we need only consider leaves ℓ_i such that $\ell_i \leq_Q \varphi(x)$. Furthermore, since Q is reduced, L is tangled if and only if $L_{n-2}^{-1}(1) \in \varphi^{-1}(\ell_i)$.

If $\omega_{i,x} = 0$, then the product $\prod_{j=1}^{\omega_{i,x}} \frac{b_{i,j}(Q)-1}{c_{i,j}(Q)-1}$ is the empty product 1. In this case, $x \in \varphi^{-1}(\ell_i)$ so all x -labelings L such that $L^{-1}(n) = x_0$ are tangled.

Now, assume $\omega_{i,x} \geq 1$ and choose a labeling $L \in \Lambda(P)$ uniformly at random among the $(n-2)!$ labelings that satisfy $L^{-1}(n) = x_0$ and $L^{-1}(n-1) = x$. We will proceed to compute the probability that L is tangled. Let $\tilde{P} = P \setminus \{x_0\}$ and $\tilde{\varphi} = \varphi|_{\tilde{P}}$. For $1 \leq j \leq \omega_{i,x}$, let x_j be the unique minimal element of $\tilde{\varphi}^{-1}(u_{i,j})$, and define the sets

$$X_j = \downarrow x_j \setminus \{x_j\} \text{ and } A_j = \bigcup_{v \leq_Q u_{i,j-1}} \tilde{\varphi}^{-1}(v).$$

The sizes of the sets are $|X_j| = b_{i,j}(Q) - 1$ and $|A_j| = c_{i,j}(Q) - 1$.

For any partial labeling $M : \tilde{P} \setminus X_{\omega_{i,x}} \rightarrow [n-1]$ such that $M(x) = n-1$ and any extension L of M , the condition $L_{n-2}^{-1}(1) \in X_{\omega_{i,x}}$ holds since $x \in \varphi^{-1}(u_{i,\omega_{i,x}})$. Furthermore, since P is an inflated rooted forest poset and x_j is the unique minimal element of $\tilde{\varphi}^{-1}(u_{i,j})$, x_j is LOI-complete, and no element of A_j is comparable with any element of $X_j \setminus A_j$. Thus, the poset $\tilde{P}$, the subsets $X_{\omega_{i,x}}$ and $A_{\omega_{i,x}}$, and the partial labeling M satisfy the conditions in lemma 3.2.12. Applying the lemma tells us that the probability that $L_{n-2}^{-1}(1) \in A_{\omega_{i,x}}$ is

$$\frac{|A_{\omega_{i,x}}|}{|X_{\omega_{i,x}}|} = \frac{b_{i,\omega_{i,x}}(Q) - 1}{c_{i,\omega_{i,x}}(Q) - 1}.$$

Furthermore, lemma 3.2.11 tells us that the occurrence of this event only depends on $L|_{\tilde{P} \setminus A_{\omega_{i,x}}}$.

This process can be continued for $j = \omega_{i,x} - 1, \dots, 1$ to deduce that the probability that $L_{n-2}^{-1}(1) \in \varphi^{-1}(u_{i,0})$ is the product

$$\prod_{j=1}^{\omega_{i,x}} \frac{b_{i,j}(Q) - 1}{c_{i,j}(Q) - 1}.$$

Summing over all the leaves such that $\ell_i \leq_Q \varphi(x)$ yields the result. $\square$

Theorem 3.2.14. *If P is an inflated rooted forest poset on n elements and $x \in P$, then $|\mathcal{T}_x(P)| \leq (n-2)!$. Equality holds if and only if there is a unique minimal element $z \in P$ such that $z <_P x$.*

Proof. We first consider the case of an inflated rooted tree poset. Let Q be a reduced rooted tree poset and P an inflation of Q with $|P| = n$. For an element x of P , theorem 3.2.13 implies that

$$|\mathcal{T}_x(P)| = (n-2)! \sum_{i \in \ell(x)} \prod_{j=1}^{\omega_{i,x}} \frac{b_{i,j}(Q) - 1}{c_{i,j}(Q) - 1}.$$

The subposet $\tilde{Q} := \downarrow \varphi(x)$ is also a rooted tree poset. Let $\tilde{P} := \varphi^{-1}(\tilde{Q})$ and $\tilde{\varphi}$ be the restriction $\varphi|_{\tilde{P}}$. Then $\tilde{P}$ is an inflated rooted tree poset, so lemma 3.2.5 gives the upper bound

$$\sum_{i \in \ell(x)} \prod_{j=1}^{\omega_{i,x}} \frac{b_{i,j}(Q) - 1}{c_{i,j}(Q) - 1} \leq 1. \quad (3.9)$$

Therefore, $|\mathcal{T}_x(P)| \leq (n - 2)!$ in the case of an inflated rooted tree poset.

Let m denote the number of leaves in the subposet $\tilde{Q}$. By remark 3.2.6, the inequality in eq. (3.9) is strict if and only if $m > 1$. The number of leaves in the subposet $\tilde{Q}$ is precisely the number of minimal elements in $\tilde{P}$. By definition of $\tilde{P}$, the minimal elements in $\tilde{P}$ are precisely the minimal elements $z \in P$ that satisfy $z <_P x$. Thus, equality in eq. (3.9) holds if and only if there is a unique minimal element $z \in P$ that satisfies $z <_P x$.

The general case of an inflated rooted forest poset follows from lemma 3.1.20, since an inflated rooted forest poset is a disjoint union of inflated rooted tree posets. $\square$

3.3 Shoelace Posets

In this section, we will study tangled labelings on a new family of posets called *shoelace posets* and show that the $(n - 2)!$ conjecture holds for them. The proof involves a careful analysis of the number of tangled labelings where a fixed element in the poset is labeled $n - 1$. We note that in general, shoelace posets are not the inflation of any rooted forest poset. We will also examine a specific subset of shoelace posets called *W-posets*, and enumerate the exact number of tangled labelings of these posets.

Definition 3.3.1. A *shoelace poset* P is a connected poset defined by a set of minimal elements $\{x_1, \dots, x_\ell\}$, a set of maximal elements $\{y_1, \dots, y_m\}$, and a set $\mathcal{S}(P) \subseteq \{x_1, \dots, x_\ell\} \times \{y_1, \dots, y_m\}$ such that the following three conditions hold:

1. For every $(i, j) \in [\ell] \times [m]$, the elements x_i and y_j are comparable in P if and only if $(x_i, y_j) \in \mathcal{S}(P)$.
2. For every $(x_i, y_j) \in \mathcal{S}(P)$, the open interval $(x_i, y_j)_P$ is a (possibly empty) chain, denoted C_i^j .

3. For distinct pairs $(x_i, y_j), (x_{i'}, y_{j'}) \in \mathcal{S}(P)$, the chains C_i^j and $C_{i'}^{j'}$ are disjoint.

We will use the following notation

$$\mathcal{S}^j(P) = \{x_i : (x_i, y_j) \in \mathcal{S}(P)\}, \quad \mathcal{S}_i(P) = \{y_j : (x_i, y_j) \in \mathcal{S}(P)\}.$$

The funnels of a shoelace poset can be described fairly simply. The funnel of a minimal element x_i consists of the elements in C_i^j for $y_j \in \mathcal{S}_i(P)$, along with the maximal elements y_j for $y_j \in \mathcal{S}_i(P)$ that satisfy $\mathcal{S}^j(P) = \{x_i\}$.

Example 3.3.2. fig. 3.7 depicts a shoelace poset P with 3 minimal elements and 4 maximal elements. In this example,

$$\mathcal{S}(P) = \{(x_1, y_2), (x_1, y_3), (x_1, y_4), (x_2, y_1), (x_2, y_3), (x_3, y_3), (x_3, y_4)\}.$$

The elements of the chain C_3^4 are highlighted in red and the chain C_1^3 is empty. Notice also that $\mathcal{S}_1(P) = \{y_2, y_3, y_4\}$ and $\mathcal{S}^2(P) = \{x_1\}$.

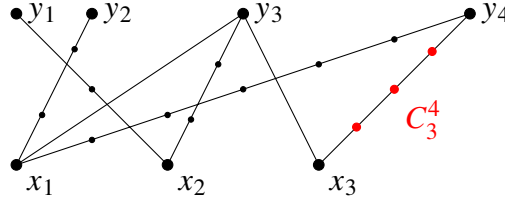


Figure 3.7 An example of a shoelace poset

In order to prove that shoelace posets satisfy the $(n-2)!$ conjecture, we will partition labelings according to the location of the label $n-1$, and bound $|\mathcal{T}_x(P)|$ for the various elements x .

For the following lemma, we use the following notation: for S a set and f a function whose codomain is well-ordered, $\operatorname{argmin}_S f$ is the element $x \in S$ such that $f(x)$ is minimal.

Lemma 3.3.3. *Let P be a shoelace poset with minimal elements $x_1, \dots, x_\ell$ and maximal elements $y_1, \dots, y_m$. Let $L \in \mathcal{T}(P)$, $i \in [\ell]$, and $j \in [m]$ such that $L(y_j) = n-1$ and $L(x_i) = n$. If $|\mathcal{S}^j(P)| \geq 2$, then $x_i \in \mathcal{S}^j(P)$, $C_i^j \neq \emptyset$, and*

$$\operatorname{argmin}_{\downarrow y_j \setminus \mathcal{S}^j(P)} L \in C_i^j.$$

Proof. Since L is tangled, $L^{-1}(n) <_P L^{-1}(n-1)$ by corollary 3.1.15. Therefore, $x_i <_P y_j$, which implies $i \in \mathcal{S}^j(P)$. By lemma 3.1.19, there exists $z \in \text{fun}(x_i)$ such that $z \leq_P y_j$. By the assumption that $|\mathcal{S}^j(P)| \geq 2$, we observe that $y_j \notin \text{fun}(x_i)$. Therefore, $x_i <_P z <_P y_j$, so $C_i^j \neq \emptyset$.

Next, let r be the smallest positive integer such that the r th promotion chain ends in y_j . Denote the r th promotion chain by $(z_1, \dots, z_n, y_j)$. Since r is the smallest such positive integer, y_j does not lie on the q th promotion chain for $q < r$, and hence $L_{r-1}^{-1}(n-1-(r-1)) = y_j$. Then, after the r th promotion, $L_r^{-1}(n-1-r) = z_n$. Since L is a tangled labeling, corollary 3.1.15 implies that

$$x_i = L_r^{-1}(n-r) <_P L_r^{-1}(n-1-r) = z_n.$$

Therefore, $z_n \in C_i^j$. Since $z_1 <_P \dots <_P z_{n-1}$, the remaining elements $z_1, \dots, z_{n-1}$ in the r th promotion chain are also on C_i^j .

Now, let $z \in \downarrow y_j \setminus \mathcal{S}^j(P)$ and let $t = L(z)$. Then either $L_{t-1}^{-1}(1) = z$ and the t th promotion chain ends in y_j , or $L_{t-1}^{-1}(1) <_P z$ and the t' -th promotion chain ends in y_j for some $t' < t$. In either case, it follows that $r \leq t$. Since the starting element of the r th promotion chain lies in C_i^j , we conclude that $\argmin_{\downarrow y_j \setminus \mathcal{S}^j(P)} L \in C_i^j$. $\square$

Essentially, if a labeling on a shoelace poset is tangled, and $L(y_j) = n-1$, then the element with smallest label in $\downarrow y_j \setminus \mathcal{S}^j(P)$ must be above the element labeled n . This is therefore a necessary condition for a labeling on a shoelace poset to be tangled. This will be instrumental in proving the following theorem.

Theorem 3.3.4. *If P is a shoelace poset on n elements and $z \in P$, then $|\mathcal{T}_z(P)| \leq (n-2)!$. Equality holds if and only if there is a unique minimal element $x <_P z$.*

Proof. Let $x_1, \dots, x_\ell$ be minimal elements of P , and $y_1, \dots, y_m$ be maximal elements of P . The element z can either be a minimal element, an element on a chain C_i^j for some i and j , or a maximal element. There is a unique minimal element $x <_P z$ only if $z \in C_i^j$ or if z is one of the maximal elements y_j and $|\mathcal{S}^j(P)| = 1$. For convenience, we set $s := |\mathcal{S}^j(P)|$. Below, we separate the cases mentioned above and claim that equality holds only in Case 2 and Case 3.

Case 1: Suppose z is a minimal element. In this case, it is impossible to find an element labeled n such that $L^{-1}(n) <_P L^{-1}(n-1) = z$. So by corollary 3.1.15, $|\mathcal{T}_z(P)| = 0$.

Case 2: Suppose z lies on some chain C_i^j . In this case there is a unique basin x_i that $\text{in } \downarrow z$. Any tangled labeling $L \in \mathcal{T}_z(P)$ must satisfy $L(z) = n-1$ and $L(x_i) = n$. There are at most $(n-2)!$ such labelings, and by lemma 3.1.18 all such labelings are tangled so $|\mathcal{T}_z(P)| = (n-2)!$.

Case 3: Suppose z is a maximal element y_j and $s = 1$. Since $s = 1$, for any tangled labeling L , $L^{-1}(n)$ must be the unique x_i satisfying $x_i <_P y_j = z$. There are $(n-2)!$ such labelings, and by lemma 3.1.18 all such labelings are tangled. Thus, $|\mathcal{T}_z(P)| = (n-2)!$.

Case 4: Suppose z is a maximal element y_j and $s \geq 2$. Partition $\Lambda(P)$ into equivalence classes, where two labelings L and L' belong to the same equivalence class if and only if they restrict to the same labeling on $P \setminus \mathcal{S}^j(P)$. Labelings in $\mathcal{T}_z(P)$ require y_j to be labeled $n-1$ and some element in $\mathcal{S}^j(P)$ to be labeled n . The number of equivalence classes where this is possible is $(n-2)(n-3) \cdots s$. In each such equivalence class, the tangled labelings L have only one choice of $L^{-1}(n)$ according to lemma 3.3.3. Therefore, at most $(s-1)!$ labelings in each equivalence class are tangled. Consequently, $|\mathcal{T}_z(P)| \leq (n-2)(n-3) \cdots s(s-1)! = (n-2)!$.

With a little more careful analysis, one can conclude that at least one of the equivalence classes has strictly fewer than $(s-1)!$ labelings. Consider an equivalence class where the label 1 is in $\mathcal{S}^j(P)$ and the label 2 is in $\downarrow y_j \setminus \mathcal{S}^j(P)$. Then in this equivalence class, there is the additional restriction $L^{-1}(1) \not<_P L^{-1}(2)$. Thus, there are strictly fewer than $(s-1)!$ tangled labelings, so $|\mathcal{T}_z(P)| < (n-2)!$. $\square$

Notice that theorem 3.3.4 shows that shoelaces satisfy conjecture 3.0.5, and therefore also satisfy conjecture 3.0.3 and conjecture 3.0.2.

We have proven an upper bound on the number of tangled labelings of shoelaces, but we are also able to enumerate the exact number of tangled labelings for a specific subfamily of shoelace posets called *W-posets*. In general, few explicit formulas for tangled labelings are known. The proof of this formula will also involve counting the number of tangled labelings by fixing the label

$n - 1$.

Definition 3.3.5. Given $a, b, c, d \in \mathbb{Z}_{\geq 0}$, the W -poset $W_{a,b,c,d}$ is a poset on $a + b + c + d + 3$ elements: $\alpha_1, \dots, \alpha_a, \beta_1, \dots, \beta_b, \gamma_1, \dots, \gamma_c, \delta_1, \dots, \delta_d, x, y, z$. The partial order has covering relations $\alpha_i \leq_W \alpha_{i+1}, \beta_i \leq_W \beta_{i+1}, \gamma_i \leq_W \gamma_{i+1}, \delta_i \leq_W \delta_{i+1}, x \leq_W \alpha_1, x \leq_W \beta_1, \beta_b \leq_W y, \gamma_c \leq_W y, z \leq_W \gamma_1$, and $z \leq_W \delta_1$.

The poset $W_{a,b,c,d}$ can be viewed as the shoelace poset with the set of minimal elements $\{x, z\}$, the set of maximal elements $\{\alpha_a, y, \delta_d\}$ and the relations $\mathcal{S}(P) = \{(x, \alpha_a), (x, y), (z, y), (z, \delta_d)\}$.

Example 3.3.6. The Hasse diagram for $W_{2,2,1,1}$ is shown in fig. 3.8. There are 34,412 tangled labelings of this poset.

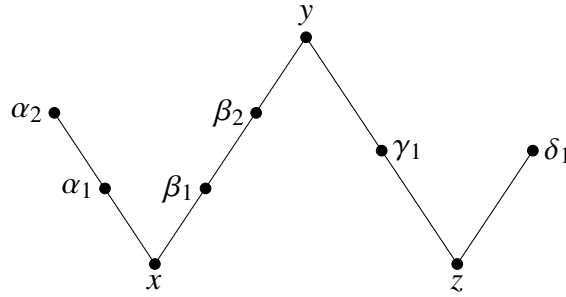


Figure 3.8 The poset $W_{2,2,1,1}$

Theorem 3.3.7. Let a, b, c, d be four positive integers and $n = a + b + c + d + 3$. Let

$$X = \binom{n-2}{a} \sum_{i=0}^{b-1} \sum_{j=0}^d (d-j+1) \binom{i+j+c-1}{i, j, c-1}, \text{ and}$$

$$Z = \binom{n-2}{d} \sum_{i=0}^{c-1} \sum_{j=0}^a (a-j+1) \binom{i+j+b-1}{i, j, b-1}.$$

Then the number of tangled labelings of $W_{a,b,c,d}$ is given by $(n-2)(n-2)! - a!b!c!d!(X+Z)$.

Proof. Fix a, b, c, d and write $W = W_{a,b,c,d}$. By eq. (3.1), it suffices to compute $|\mathcal{T}_p(W)|$ as p ranges over elements of W . If $p = x$ or $p = z$, then $|\mathcal{T}_p(W)| = 0$ due to Case 1 in the proof of theorem 3.3.4. If $p = \alpha_i$ or $p = \beta_i$, then this belongs to Cases 2 and 3 in the proof of theorem 3.3.4,

and so $|\mathcal{T}_p(W)| = (n-2)!$. Similarly, if $p = \gamma_i$ or $p = \delta_i$, then $|\mathcal{T}_p(W)| = (n-2)!$. With the exception of $p = y$, we have counted $(a+b+c+d)(n-2)! = (n-3)(n-2)!$ tangled labelings.

Let us now count the number of tangled labelings L that satisfy $L(y) = n-1$. Observe that permuting the labels $L(\alpha_1), \dots, L(\alpha_a)$ does not change whether or not L is tangled. Similarly, permuting the labels $L(\beta_1), \dots, L(\beta_b)$, the labels $L(\gamma_1), \dots, L(\gamma_c)$, and the labels $L(\delta_1), \dots, L(\delta_d)$ among themselves does not change whether or not L is tangled. Thus, we will additionally impose the conditions $L(\alpha_1) < \dots < L(\alpha_a)$, $L(\beta_1) < \dots < L(\beta_b)$, $L(\gamma_1) < \dots < L(\gamma_c)$, and $L(\delta_1) < \dots < L(\delta_d)$. To obtain the total number of tangled labelings, we will count the number of such tangled labelings L satisfying these conditions and then multiply by $a!b!c!d!$.

We split into two cases. The first case is where $L(\beta_1) < L(\gamma_1)$. Let $m_\beta = L(\beta_1)$. In this case, a necessary condition for L to be tangled is that $L(x) = n$. To see this, suppose otherwise that $L(z) = n$. Then note that $L_{m_\beta}^{-1}(n-1-m_\beta) \in [x, y)$. This is because for the first m_β promotions, the only promotion chains ending in y are those that begin with some element in $[x, y)$ and furthermore, there exists at least one promotion chain ending in y , namely the m_β -th one. It follows that $L_{n-2}^{-1}(1) \not\prec_W z$ so L cannot be tangled if $L(z) = n$ (lemma 3.1.13).

Now, the total number of labelings that satisfy all these conditions is given by $\frac{1}{2} \binom{n-2}{a, b, c, d, 1}$, since it amounts to choosing a of the labels in $[n-2]$ for $\alpha_1, \dots, \alpha_a$, b of the labels for the β s and so on. To account for the condition $L(\beta_1) < L(\gamma_1)$, we divide by 2 because there is an involution swapping $L(\beta_1)$ and $L(\gamma_1)$. We will now subtract the number of labelings satisfying these conditions that are *not* tangled.

Given that L satisfies all the conditions above, L is *not* tangled if and only if $L(z) < L(\beta_1)$ and there do not exist δ_i such that $L(\beta_1) < L(\delta_i) < L(\gamma_1)$. To see this, observe that L is not tangled if and only if there is some $j < m_\beta$ where the j th promotion chain begins with an element in $[z, y)$ and ends in y . Since $L(\beta_1) < L(\gamma_1)$, this can occur only if $L(z) < L(\beta_1)$. Now, let $\delta_i <_W \delta_{i+1} <_W \dots <_W \delta_j$ be all the δ 's with labels in between $L(z)$ and $L(\gamma_1)$. Then the $L(z), L(\delta_i), \dots, L(\delta_{j-1})$ th promotion chains would all begin with z and end with some δ_k , and the $L(\delta_j)$ th promotion chain would begin with z and end with y . Thus, in order for L to not be tangled

we must have $L(\delta_j) < L(\beta_1)$. And conversely, if we do have $L(\delta_j) < L(\beta_1)$ then L is not tangled since the $L(\delta_j)$ th promotion chain would start with z and end with y .

Now, we wish to count the number of such labelings L . To do so, observe that the labels of the α 's are subject to no constraints. We will suppose that $L(\delta_1) < \cdots < L(\delta_{d-j}) < L(\beta_1) < \cdots < L(\beta_{b-i}) < L(\gamma_1)$ and sum over $0 \leq i \leq b-1$ and $0 \leq j \leq d$.

For each i, j there are $(d-j+1)$ choices of what $L(z)$ could be and $\binom{i+j+c-1}{i, j, c-1}$ choices for the labels greater than $L(\gamma_1)$. This yields

$$X = \binom{n-2}{a} \sum_{i=0}^{b-1} \sum_{j=0}^d (d-j+1) \binom{i+j+c-1}{i, j, c-1}.$$

By a similar argument, if $L(\gamma_1) < L(\beta_1)$ then a necessary condition for L to be tangled is $L(z) = n$. The number of labelings satisfying these conditions is $\frac{1}{2} \binom{n-2}{a, b, c, d, 1}$ and the number of these labelings that are not tangled is

$$Z = \binom{n-2}{d} \sum_{i=0}^{c-1} \sum_{j=0}^a (a-j+1) \binom{i+j+b-1}{i, j, b-1}.$$

Let $E = a!b!c!d!$. Then, the number of tangled labelings L that satisfy $L(y) = n-1$ is

$$\begin{aligned} E \left(\frac{1}{2} \binom{n-2}{a, b, c, d, 1} - X + \frac{1}{2} \binom{n-2}{a, b, c, d, 1} - Z \right) &= E \binom{n-2}{a, b, c, d, 1} - E(X + Z) \\ &= (n-2)! - E(X + Z). \end{aligned}$$

Adding this to the $(n-3)(n-2)!$ tangled labelings where $L^{-1}(n-1) \neq y$ yields the desired formula. $\square$

In principle, one could compute the exact number of tangled labelings for various subsets of shoelace posets in this way. Even for the class of W -posets, however, the computations appear rather unwieldy.

3.4 Generating Functions

In the previous sections, we focused on counting the number of tangled labelings of various posets and analyzed their upper bounds. In this section, we are interested in exploring the number

of labelings of a poset P on n elements that have a fixed order k . Recall that the order of a labeling L is the minimal integer $k \geq 0$ such that L_k is sorted. Such labelings we will call k -sorted; see definition 3.4.1. Dual to k -sorted labelings are k -tangled labelings that have order $n - k - 1$. We define two kinds of generating functions (definition 3.4.2) on P and investigate how these generating functions change if we attach some minimal elements to P . Our result provides a simple and unified proof of enumerating tangled labelings and quasi-tangled labelings in [19] and [32] (see remark 3.4.11).

Definition 3.4.1. Let P be an n -element poset. A labeling $L \in \Lambda(P)$ is said to be k -sorted if $\text{or}(L) = k$ and is said to be k -tangled if $\text{or}(L) = n - k - 1$.

Observe that natural labelings are synonymous with 0-sorted labelings and tangled labelings are synonymous with 0-tangled labelings. Quasi-tangled labelings introduced in [32] correspond exactly to 1-tangled labelings.

Definition 3.4.2. Let P be an n -element poset. The *sorting generating function* of P is defined to be

$$f_P(q) := \sum_{L \in \Lambda(P)} q^{\text{or}(L)} = \sum_{i=0}^{n-1} a_i q^i,$$

where a_i counts the number of i -sorted labelings of P . The *cumulative generating function* of P is defined to be

$$g_P(q) := \sum_{i=0}^{n-1} b_i q^i,$$

where $b_i := a_0 + a_1 + \cdots + a_i$ is the partial sum of a_i 's. In particular, $b_{n-1} = n!$.

Example 3.4.3. We list all the six labelings and their orders of the Λ -shaped poset P in table 3.1. The sorting generating function and cumulative generating function of P are given by $f_P(q) = 2 + 4q$ and $g_P(q) = 2 + 6q + 6q^2$.

We now define precisely what it means to attach k minimal elements to a poset. The operation we need is the ordinal sum of two posets P and Q .

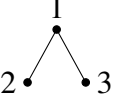
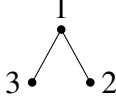
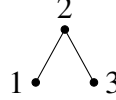
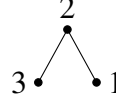
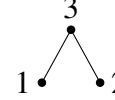
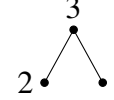
Labeling						
Order	1	1	1	1	0	0

Table 3.1 The six labelings and their corresponding orders for the Λ -shaped poset

Definition 3.4.4. Let P and Q be two posets. The *ordinal sum* of P and Q is the poset $P \oplus Q$ on the elements of the disjoint union $P \sqcup Q$ such that $s \leq t$ in $P \oplus Q$ if and only if at least one of the following conditions hold:

1. $s, t \in P$ and $s \leq_P t$, or
2. $s, t \in Q$ and $s \leq_Q t$, or
3. $s \in P$ and $t \in Q$.

The n -element chain will be denoted C_n and the k -element antichain will be denoted T_k . In the language of ordinal sums, we can view C_n as the ordinal sum of n copies of C_1 's and we can view attaching k minimal elements to a poset P as the ordinal sum $T_k \oplus P$. Our main result in this section provides a way to compute the sorting generating function $f_{T_k \oplus P}(q)$ from $f_P(q)$.

Define a lower-triangular $n \times n$ matrix $X_n(k)$ whose (i, j) entry x_{ij} is given by

$$x_{ij} := \begin{cases} k! \binom{k+i-2}{k-1} & \text{if } i > j, \\ k! \binom{k+i-1}{k} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that given a labeling on a poset, the *standardization* of the restricted labeling on a subposet Q shifts the labels to those from 1 to $|Q|$; see definition 3.1.6.

Theorem 3.4.5. Let P be an n -element poset and $f_P(q) = \sum_{i=0}^{n-1} a_i q^i$ be the sorting generating function of P . Write the sorting generating function of $T_k \oplus P$ as $f_{T_k \oplus P}(q) = \sum_{i=0}^{n+k-1} a'_i q^i$. Let $v = (a_0, a_1, \dots, a_{n-1})^\top$ be the column vector of the coefficients of $f_P(q)$ and $v' = (a'_0, a'_1, \dots, a'_{n-1})^\top$ the column vector of the first n coefficients of $f_{T_k \oplus P}(q)$. Then

1. $X_n(k)v = v'$,
2. $a'_n = n!k! \binom{n+k-1}{k-1}$, and
3. $a'_i = 0$ for $i = n+1, n+2, \dots, n+k-1$.

Proof. Let $x_1, x_2, \dots, x_k$ be the elements of T_k . Since the roles of the x_i 's are symmetrical, it follows that permuting the labels of the x_i 's on any labeling $L \in \Lambda(T_k \oplus P)$ doesn't change $\text{or}(L)$. Therefore, we will compute the number of labelings that satisfy $L(x_1) < L(x_2) < \dots < L(x_k)$ and then multiply by $k!$.

Now, we will define a procedure that, given a labeling $L \in \Lambda(P)$ and a k -tuple of distinct numbers $I = (i_1, \dots, i_k) \in [n+k]^k$, produces a labeling $L^I \in \Lambda(T_k \oplus P)$ such that $L^I(x_s) = i_s$ for $1 \leq s \leq k$. Since we are counting labelings where the labels of the x_i 's are increasing, we will assume that $i_1 < i_2 < \dots < i_k$ for the rest of the proof.

To obtain L^I , first define labelings of $L^0, L^1, \dots, L^k$ of P , where $L^0 := L$ and for $s = 1, \dots, k$, recursively define L^s by

$$L^s(x) := \begin{cases} L^{s-1}(x) + 1 & \text{if } L^{s-1}(x) \geq i_s, \\ L^{s-1}(x) & \text{otherwise.} \end{cases}$$

Then define L^I on $T_k \oplus P$ by

$$L^I(x) := \begin{cases} i_s & \text{if } x = x_s, \\ L^k(x) & \text{if } x \in P. \end{cases} \quad (3.10)$$

In fig. 3.9, we give an example of defining L^I of $T_3 \oplus P$ on a 7-element poset P and with $I = (2, 4, 7)$. The labeling of P is given in the left figure, and the middle three figures illustrate the process mentioned above. The right figure is the resulting labeling L^I of $T_3 \oplus P$.

One can check that at each step $s = 1, \dots, k$ the standardization $\text{st}(L^s)$ is precisely L . Therefore, the standardization of $L^I|_P$ is $\text{st}(L^I|_P) = \text{st}(L^k) = L$. In other words, L^I is the unique labeling in

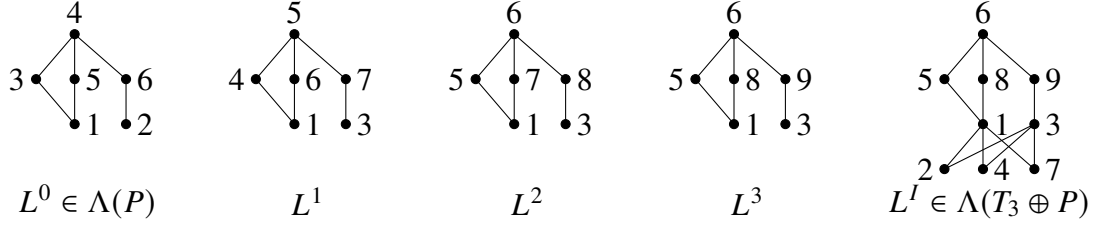


Figure 3.9 Defining L^I of $T_3 \oplus P$ with $I = (2, 4, 7)$

$\Lambda(T_k \oplus P)$ that assigns the label i_s to x_s for $s = 1, \dots, k$ and whose standardization when restricted to P is L . As a consequence, the set of labelings $\Lambda(T_k \oplus P)$ can be partitioned as

$$\Lambda(T_k \oplus P) = \bigsqcup_{L \in \Lambda(P)} \left\{ L^{[I]} : I \in \binom{[n]}{k} \right\}, \quad (3.11)$$

where $L^{[I]}$ contains the labeling L^I and the labelings obtained from L^I by permuting all the labels of the x_i 's.

Next, we proceed with the following two claims.

Claim 1. Given $L \in \Lambda(P)$ and $I = (i_1, \dots, i_k)$, the standardization of $L^I|_P$ is preserved under a sequence of promotions:

$$\text{st}((L_j^I)|_P) = L_j, \text{ for all } j \in \mathbb{Z}_{\geq 0}. \quad (3.12)$$

Proof of Claim 1. We will show Claim 1 by induction. When $j = 0$, the identity holds by the definition of L^I . Suppose it holds for some j and consider L_{j+1}^I . If $L_j^I(x_s) > 1$ for all $s = 1, 2, \dots, k$, then these minimal elements x_s 's are not in the $(j+1)$ -th promotion chain and the claim holds. On the other hand, if there exists an s such that $L_j^I(x_s) = 1$, then the $(j+1)$ -th promotion begins at x_s . Since $x_s \leq x$ for all $x \in P$, the next element in the promotion chain is $(L_j^I)^{-1}(y)$, where $y = \min\{L_j^I(z) : z \in P\}$. This element is exactly $(L_j^I)^{-1}(y) = (L_j)^{-1}(1)$. From this point on, the rest of the promotion chain is the same in L_j^I and L_j . Therefore, $\text{st}((L_j^I)|_P) = L_j$ for all $j \in \mathbb{Z}_{\geq 0}$. $\square$

Claim 2. Given $L \in \Lambda(P)$ and $I = (i_1, \dots, i_k)$, the order of L^I is given by

$$\text{or}(L^I) = \max(i_k - k, \text{or}(L)). \quad (3.13)$$

Proof of Claim 2. We observe that for some nonnegative integer j , L_j^I is a natural labeling if and only if two conditions are satisfied:

1. the set of labels $\{L_j^I(x_1), \dots, L_j^I(x_k)\}$ is $[k]$, and
2. $(L_j^I)|_P$ is a natural labeling.

By eq. (3.12), the second condition is satisfied if and only if $j \geq \text{or}(L)$. On the other hand, we show below that the first condition is satisfied if and only if $j \geq i_k - k$.

To see this, we notice that the first $i_1 - 1$ promotions only decrement the labels of $x_1, \dots, x_k$. Let $\mathcal{S}_j := \{L_j^I(x_1), \dots, L_j^I(x_k)\}$ and let s_j be the maximum value (possibly 0) such that $[s_j] \subseteq \mathcal{S}_j$. Then the minimum label in $L_j^I|_P$ is $s_j + 1$ and in the $(j + 1)$ -th promotion, $(L_j^I)^{-1}(s_j + 1)$ is part of the promotion chain, so $\mathcal{S}_{j+1} = [s_j] \cup \{y - 1 : y \in \mathcal{S}_j \setminus [s_j]\}$. Note that $s_{j+1} > s_j$ if and only if $s_j + 1 \in \{y - 1 : y \in \mathcal{S}_j \setminus [s_j]\}$. Thus, it follows by an inductive argument that $s_j \geq t$ if and only if $j \geq i_t - t$ which yields the desired result. Combining these two conditions implies that $\text{or}(L^I) = \max(i_k - k, \text{or}(L))$. $\square$

We are now ready to prove the first statement, in which we show that for $1 \leq s \leq n$, $k!$ times the number of labelings in $\Lambda(T_k \oplus P)$ with order $m - 1$ is equal to the m th row of $X_n(k)v$. By eq. (3.11), we can sum over all labelings $L \in \Lambda(P)$ and count the number of $I \in \binom{[n]}{k}$ such that $\text{or}(L^I) = m - 1$. We proceed by cases analysis of $\text{or}(L)$.

- Suppose $\text{or}(L) < m - 1$. Then in order for $\text{or}(L^I) = \max(i_k - k, \text{or}(L)) = m - 1$ to hold, it must be that $i_k - k = m - 1$. Fixing $i_k = k + m - 1$, there are $\binom{k+m-2}{k-1}$ ways to choose $i_1, \dots, i_{k-1}$ such that $\text{or}(L^I) = m - 1$.
- Suppose $\text{or}(L) = m - 1$. Then in order for $\text{or}(L^I) = \max(i_k - k, \text{or}(L)) = m - 1$ to hold, it must be that $i_k - k \leq m - 1$. Thus, $i_k \leq k + m - 1$ so there are $\binom{k+m-1}{k}$ ways to choose $i_1, \dots, i_k$ such that $\text{or}(L^I) = m - 1$.
- Suppose $\text{or}(L) > m - 1$. Then $\text{or}(L^I) = \max(i_k - k, \text{or}(L)) > m - 1$ so there are no choices of I that yield $\text{or}(L^I) = m - 1$.

After multiplying by $k!$ to account for the fact that permuting the labels of $x_1, \dots, x_k$ do not change the order of a labeling of $T_k \oplus P$, the first case yields the (m, j) entry of $X_n(k)$ when $j < m$, the middle case yields the (m, m) entry of $X_n(k)$, and the last case yields the (m, j) entry of $X_n(k)$ when $j > m$. This completes the proof of the first statement.

To prove the second statement, observe that since $\text{or}(L) \leq n - 1$ for any $L \in \Lambda(P)$, then $\text{or}(L^I) = \max(i_k - k, \text{or}(L)) = n$ if and only if $i_k - k = n$. Fixing $i_k = k + n$, there are $\binom{n+k-1}{k-1}$ choices for $i_1, \dots, i_{k-1}$, regardless of $\text{or}(L)$. Multiplying by $k!$ to account for permuting the labels of $x_1, \dots, x_k$ yields

$$a'_n = k! \binom{n+k-1}{k-1} (a_0 + a_1 + \dots + a_{n-1}) = n! k! \binom{n+k-1}{k-1}.$$

This completes the proof of the second statement.

Finally to prove the last statement, first observe that $i_k \leq k + n$ since there are only $k + n$ elements in $T_k \oplus P$. Thus, $i_k - k \leq n$. In addition, any labeling $L \in \Lambda(P)$ has $\text{or}(L) \leq n - 1$. It follows that $\text{or}(L^I) \leq n$ for any choice of $L \in \Lambda(P)$ and $I \in \binom{[n+k]}{k}$. Thus, there do not exist labelings $T_k \oplus P$ with order greater than n and hence $a'_i = 0$ for $i = n + 1, n + 2, \dots, n + k - 1$. This completes the proof of the last statement. $\square$

We would like to point out that if $k \geq 2$, then $T_k \oplus P$ has no tangled labelings.

Example 3.4.6. Let P be as in example 3.4.3. The sorting generating function of P is given by $f_P(q) = 2 + 4q$. Let v be the column vector $(2, 4, 0)^\top$. We show below how to obtain the sorting generating function of posets shown in fig. 3.10 from theorem 3.4.5.

For $T_1 \oplus P$,

$$X_3(1)v = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ 6 \end{pmatrix} \text{ and } a'_3 = 1(2 + 4 + 0) = 6.$$

Then $f_{T_1 \oplus P}(q) = 2 + 10q + 6q^2 + 6q^3$. For $T_2 \oplus P$,

$$X_3(2)v = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 6 & 0 \\ 6 & 6 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 32 \\ 36 \end{pmatrix} \text{ and } a'_3 = 8(2 + 4 + 0) = 48.$$

Then $f_{T_2 \oplus P}(q) = 4 + 32q + 36q^2 + 48q^3$. Finally, for $T_3 \oplus P$,

$$X_3(3)v = \begin{pmatrix} 6 & 0 & 0 \\ 18 & 24 & 0 \\ 36 & 36 & 60 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 132 \\ 216 \end{pmatrix} \text{ and } a'_3 = 60(2 + 4 + 0) = 360.$$

Then $f_{T_3 \oplus P}(q) = 12 + 132q + 216q^2 + 360q^3$.

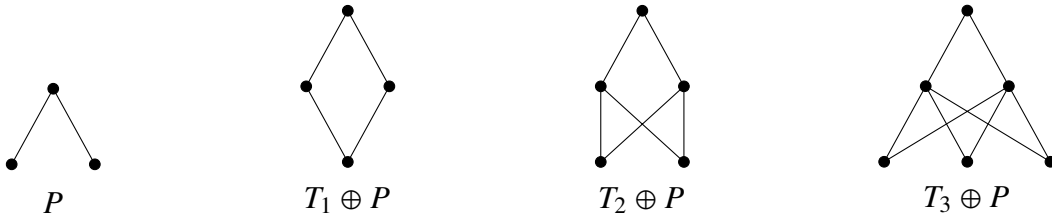


Figure 3.10 The posets obtained from P by attaching 1, 2 and 3 minimal elements

An analogous result for the cumulative generating function $g_{T_k \oplus P}(q)$ is stated below.

Theorem 3.4.7. *Let P be an n -element poset and $g_P(q) = \sum_{i=0}^{n-1} b_i q^i$ the cumulative generating function of P . Assume $g_{T_k \oplus P}(q) = \sum_{i=0}^{n+k-1} b'_i q^i$. Let $w = (b_0, b_1, \dots, b_{n-1})^\top$ be the column vector of the coefficients of $g_P(q)$ and $w' = (b'_0, b'_1, \dots, b'_{n-1})^\top$ be the column vector of the first n coefficients of $g_{T_k \oplus P}(q)$. Then*

1. $Y_n(k)w = w'$, where $Y_n(k)$ is the $n \times n$ diagonal matrix, the i th diagonal entry given by $\frac{(k+i-1)!}{(i-1)!}$.
2. $b'_i = (n+k)!$ for $i = n, n+1, \dots, n+k-1$.

Proof. Let R_n be the lower triangular matrix of size n whose lower triangular entries (including the diagonal entries) are 1. If $v = (a_0, a_1, \dots, a_{n-1})^\top$ is the column vector of the coefficients of $f_P(q)$, then it is easy to see that $R_n v = w$. One can also check that $Y_n(k)R_n = R_n X_n(k)$.

By theorem 3.4.5, the first part of the statement follows from the identities below.

$$Y_n(k)w = Y_n(k)R_n v = R_n X_n(k)v = R_n v' = w'.$$

Since $a'_i = 0$ for $i = n+1, n+2, \dots, n+k-1$, this implies that $b'_i = (n+k)!$ for $i = n, n+1, \dots, n+k-1$. □

We close this section with a special family of posets which are obtained from a given n -element poset P by attaching the chain with ℓ elements below P , that is, $\left(\bigoplus_{i=1}^{\ell} T_1\right) \oplus P$. For convenience, we denote it by $P^{(\ell)}$. Note that $P^{(\ell)}$ has $n + \ell$ elements.

We assume that the sorting and cumulative generating functions of $P^{(\ell)}$ are written as $f_{P^{(\ell)}}(q) = \sum_{i=0}^{n+\ell-1} a_i^{(\ell)} q^i$ and $g_{P^{(\ell)}}(q) = \sum_{i=0}^{n+\ell-1} b_i^{(\ell)} q^i$, respectively. Two propositions are stated below.

Proposition 3.4.8. *Let P be an n -element poset and $P^{(\ell)}$ the poset obtained from P by attaching the chain with ℓ elements below P . The last $\ell + 1$ coefficients of the cumulative generating function $g_{P^{(\ell)}}(q)$ are given by*

$$b_{n+\ell-(r+1)}^{(\ell)} = (n + \ell - r)^r (n + \ell - r)!, \quad (3.14)$$

for $0 \leq r \leq \ell$.

Moreover, $P^{(\ell)}$ satisfies conjecture 3.0.2 if and only if

$$b_{n-2}^{(\ell)} \geq (n-1)^{\ell+1} (n-1)!. \quad (3.15)$$

Proof. Applying theorem 3.4.7 with $k = 1$ repeatedly, we obtain

$$b_{n+\ell-(r+1)}^{(\ell)} = (n + \ell - r) b_{n+\ell-(r+1)}^{(\ell-1)} = \dots = (n + \ell - r)^t b_{n+\ell-(r+1)}^{(\ell-t)},$$

for $0 \leq r \leq \ell$ and for some non-negative integer t . When $(n + \ell - (r + 1)) - (\ell - t) = n - 1$, that is,

when $t = r$, $b_{n+\ell-(r+1)}^{(\ell-t)}$ is the leading coefficient of the $g_{P^{(\ell-t)}}(q)$. Hence, $b_{n+\ell-(r+1)}^{(\ell-t)} = (n + \ell - r)!$.

Therefore, $b_{n+\ell-(r+1)}^{(\ell)} = (n + \ell - r)^r (n + \ell - r)!$.

conjecture 3.0.2 implies that $a_{n-1} \leq (n-1)!$. Since $b_{n-1} = b_{n-2} + a_{n-1}$,

$$b_{n-2} = b_{n-1} - a_{n-1} \geq n! - (n-1)! = (n-1)(n-1)!.$$

We again apply theorem 3.4.7 with $k = 1$ repeatedly, then

$$b_{n-2}^{(\ell)} = (n-1)b_{n-2}^{(\ell-1)} = \cdots = (n-1)^\ell b_{n-2} \geq (n-1)^{\ell+1}(n-1)!.$$

The converse statement is argued in a similar way and will be omitted here. $\square$

We then state below the counterpart result of Proposition 3.4.8.

Proposition 3.4.9. *Let P be an n -element poset. For $0 \leq r \leq \ell - 1$, the number of r -tangled labelings of $P^{(\ell)}$ is given by*

$$a_{n+\ell-(r+1)}^{(\ell)} = \left((n+\ell-r)^{r+1} - (n+\ell-(r+1))^{r+1} \right) (n+\ell-(r+1))!. \quad (3.16)$$

Moreover, $P^{(\ell)}$ satisfies conjecture 3.0.2 if and only if

$$a_{n-1}^{(\ell)} \leq \left(n^{\ell+1} - (n-1)^{\ell+1} \right) (n-1)!. \quad (3.17)$$

Proof. Notice that $b_{n+\ell-(r+1)}^{(\ell)} - b_{n+\ell-(r+2)}^{(\ell)} = a_{n+\ell-(r+1)}^{(\ell)}$ for $0 \leq r \leq \ell - 1$. Then eq. (3.16) follows immediately from eq. (3.14).

By eq. (3.14) with $r = \ell$, $b_{n-1}^{(\ell)} = n^\ell n!$. Then eq. (3.17) is obtained from $a_{n-1}^{(\ell)} = b_{n-1}^{(\ell)} - b_{n-2}^{(\ell)}$ and eq. (3.15). The converse statement can be argued similarly and is omitted here. $\square$

We next show that our poset $P^{(\ell)}$ satisfies [32, Conjecture 23]. This conjecture states that for an n -element poset P , the number of labelings $L \in \Lambda(P)$ such that $L_{n-3} \notin \mathcal{L}(P)$ has an upper bound $3(n-1)!$.

Corollary 3.4.10. *Let P be an n -element poset and $\ell \geq 1$. The number of labelings $L \in \Lambda(P^{(\ell)})$ such that $L_{n+\ell-3} \notin \mathcal{L}(P^{(\ell)})$, that is, the total number of tangled and quasi-tangled labelings of $P^{(\ell)}$, equals*

$$3(n+\ell-1)! - (n+\ell-2)! \leq 3(n+\ell-1)!,$$

Proof. By proposition 3.4.9 with $r = 0$, the number of tangled labelings of $P^{(\ell)}$ is

$$a_{n+\ell-1}^{(\ell)} = (n + \ell - 1)!.$$

Take $r = 1$ in proposition 3.4.9, we obtain the number of quasi-tangled labelings of $P^{(\ell)}$, which is given by

$$\begin{aligned} a_{n+\ell-2}^{(\ell)} &= \left((n + \ell - 1)^2 - (n + \ell - 2)^2 \right) (n + \ell - 2)! \\ &= (2(n + \ell - 1) - 1) (n + \ell - 2)! \\ &= 2(n + \ell - 1)! - (n + \ell - 2)!. \end{aligned}$$

Summing these two numbers gives the desired result. $\square$

Remark 3.4.11. Let P be an n -element poset. We are able to give a simple and unified proof of some results given by Defant and Kravitz in [19] and by Hodges in [32].

- Take $\ell = 1$, the poset $P^{(1)}$ has one minimal element. By proposition 3.4.9 with $r = 0$, the number of tangled labelings of $P^{(1)}$ is given by

$$a_n^{(1)} = (n + 1 - n) n! = n!.$$

This gives an alternative proof of [19, Corollary 3.7] (for a connected poset).

- Take $\ell = 2$, the poset $P^{(2)}$ has one minimal element and this minimal element has exactly one parent. By proposition 3.4.9 with $r = 1$, the number of quasi-tangled labelings of $P^{(2)}$ is given by

$$a_n^{(2)} = \left((n + 1)^2 - n^2 \right) n! = (2n + 1)n! = 2(n + 1)! - n!.$$

This gives a simpler proof of [32, Corollary 10].

3.5 Ordinal Sum of Antichains

In this section, we consider a family of posets consisting of the *ordinal sum of antichains*. Let $C = (c_1, c_2, \dots, c_r)$ be an ordered sequence of r positive integers. Throughout this section, we

write $P_C = T_{c_r} \oplus T_{c_{r-1}} \oplus \cdots \oplus T_{c_1}$ for the ordinal sum of antichains of C . We completely determine the cumulative generating function of this family of posets. We also show various properties and a poset structure of its cumulative generating function.

The cumulative generating function of the k -element antichain T_k is $g_{T_k}(q) = k!(1 + q + q^2 + \cdots + q^{k-1})$. To find $g_{P_C}(q)$, we start from the antichain T_{c_1} and let $w = (c_1!, \dots, c_1!)^\top$ be the column vector consisting of the coefficients of $g_{T_{c_1}}(q)$. We next attach c_2 minimal elements to T_{c_1} ; the cumulative generating function $g_{T_{c_2} \oplus T_{c_1}}(q)$ is obtained by theorem 3.4.7. Recall that $Y_{c_1}(c_2)$ denotes the $c_1 \times c_1$ diagonal matrix whose i th diagonal entry is given by $\frac{(c_2+i-1)!}{(i-1)!}$. The matrix multiplication $Y_{c_1}(c_2)w$ gives the first c_1 coefficients of $g_{T_{c_1} \oplus T_{c_2}}(q)$ and the rest of coefficients are given by $(c_1 + c_2)!$. As a consequence, we can obtain g_{P_C} by applying theorem 3.4.7 repeatedly in this way. The explicit formula of $g_{P_C}(q)$ is summarized in the following proposition.

Proposition 3.5.1. *Let P_C be the ordinal sum of antichains of C , where $C = (c_1, c_2, \dots, c_r)$ is an ordered sequence of r positive integers. Write $g_{P_C}(q) = \sum_{s=0}^{c_1+\cdots+c_r-1} b_s q^s$ for the cumulative generating function of P_C . For each $0 \leq s < c_1 + \cdots + c_r$, let $j \in [r]$ be the unique integer such that*

$$\sum_{k=1}^{j-1} c_k \leq s < \sum_{k=1}^j c_k.$$

Then

$$b_s = (c_1 + c_2 + \cdots + c_j)! \prod_{m=j+1}^r \frac{(c_m + s)!}{s!}. \quad (3.18)$$

We now present the following symmetry property for the poset $B_{n,k} = T_n \oplus C_{k+1}$, where C_{k+1} is the chain of $k + 1$ elements and $n, k \in \mathbb{Z}_{\geq 0}$. This poset is sometimes called a *broom*.

Proposition 3.5.2. *Let $n, k \in \mathbb{Z}_{\geq 0}$. Write $f_{B_{n,k}}(q) = \sum_{s=0}^{n+k} a_s(n, k) q^s$ for the sorting generating function of $B_{n,k}$. Then*

$$a_s(n, k) = \begin{cases} (n+s)!(s+1)^{k+1-s} - (n+s-1)!s^{k+2-s}, & \text{for } s = 0, 1, \dots, k+1, \\ 0, & \text{for } s = k+2, k+3, \dots, n+k. \end{cases} \quad (3.19)$$

In particular, we have the symmetry property

$$a_k(n, k) = a_n(k, n), \text{ for } 0 \leq n \leq k \quad (3.20)$$

Proof. By proposition 3.5.1 with $c_1 = c_2 = \dots = c_{k+1} = 1$ and $c_{k+2} = n$, the cumulative generating function of $T_n \oplus C_{k+1}$ is given by $g_{T_n \oplus C_{k+1}}(q) = \sum_{s=0}^{n+k} b_s(n, k)q^s$, where

$$b_s(n, k) = (s+1)!(s+1)^{k-s} \frac{(n+s)!}{s!} = (n+s)!(s+1)^{k+1-s},$$

for $s = 0, 1, \dots, k$. We also have $b_s(n, k) = (n+k+1)!$ for $s = k+1, k+2, \dots, n+k$.

Then eq. (3.19) follows immediately from the fact that $a_s(n, k) = b_s(n, k) - b_{s-1}(n, k)$. The symmetry property (eq. (3.20)) can be verified directly using eq. (3.19). This completes the proof of proposition 3.5.2. $\square$

We next study problems originally proposed by Defant and Kravitz¹. Given an n -element poset P , are the coefficients of the sorting generating function $f_P(q)$ and the cumulative generating function $g_P(q)$ unimodal or log-concave? We prove that the coefficients of the cumulative generating function are log-concave for the ordinal sum of antichains and provide a counterexample to the conjecture that the coefficients of the sorting generating function of a general poset are unimodal.

Recall that a sequence of real numbers $(a_i)_{i=0}^n$ is called *unimodal* if there is an index j such that $a_0 \leq a_1 \leq a_2 \leq \dots \leq a_j \geq a_{j+1} \geq \dots \geq a_n$. We say this sequence is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for $i = 1, 2, \dots, n-1$. Note that a positive sequence is log-concave implies that this sequence is unimodal.

We show below that the coefficients of the cumulative generating function of P_C are log-concave.

Proposition 3.5.3. *Let P_C be the ordinal sum of antichains of C , where $C = (c_1, c_2, \dots, c_r)$ is a sequence of r positive integers. Let $g_{P_C}(q) = \sum_{s=0}^{c_1+\dots+c_r-1} b_s q^s$ be the cumulative generating function of P_C . Then the sequence $(b_s)_{s=0}^{c_1+\dots+c_r-1}$ is log-concave.*

¹The problems are stated as Conjecture 5.2 and Problem 5.3 in their 2020 preprint, but not in the published version [19].

Proof. We will show that $\frac{b_s^2}{b_{s-1}b_{s+1}} \geq 1$ for $s = 1, 2, \dots, c_1 + \dots + c_r - 2$ by direct computation using eq. (3.18). For $j = 1, 2, \dots, r$, let $\mathcal{I}_j = \{s : \sum_{k=1}^{j-1} c_k \leq s < \sum_{k=1}^j c_k\}$. The proof is based on the following four cases of the index s . We present the calculation for the first two cases below; the other two cases can be proved similarly and we leave them to the reader.

Case 1: $s - 1, s, s + 1 \in \mathcal{I}_j$ for some j . In this case

$$\begin{aligned} \frac{b_s^2}{b_{s-1}b_{s+1}} &= \frac{\left((c_1 + \dots + c_j)! \prod_{m=j+1}^r \frac{(c_m+s)!}{s!}\right)^2}{\left((c_1 + \dots + c_j)! \prod_{m=j+1}^r \frac{(c_m+s-1)!(c_m+s+1)!}{(s-1)!(s+1)!}\right)^2} \\ &= \prod_{m=j+1}^r \frac{(s+1)(c_m+s)}{s(c_m+s+1)} \\ &= \prod_{m=j+1}^r \frac{sc_m + s^2 + c_m + s}{sc_m + s^2 + s} \geq 1, \end{aligned}$$

since c_m and s are positive integers and thus the denominator is always smaller than the numerator.

Case 2: $s - 1, s \in \mathcal{I}_j$ and $s + 1 \in \mathcal{I}_{j+1}$ for some j . In this case, $s = \sum_{k=1}^j c_k - 1$, and

$$\begin{aligned} \frac{b_s^2}{b_{s-1}b_{s+1}} &= \frac{\left((c_1 + \dots + c_j)! \prod_{m=j+1}^r \frac{(c_m+s)!}{s!}\right)^2}{\left((c_1 + \dots + c_j)! \prod_{m=j+1}^r \frac{(c_m+s-1)!(c_m+s+1)!}{(s-1)!(s+1)!}\right) \left((c_1 + \dots + c_{j+1})! \prod_{m=j+2}^r \frac{(c_m+s+1)!}{(s+1)!}\right)} \\ &= \frac{(c_1 + \dots + c_j)!}{(c_1 + \dots + c_{j+1})!} \cdot \frac{(c_{j+1}+s)!(c_{j+1}+s)!(s-1)!}{(c_{j+1}+s-1)!s!} \cdot \prod_{m=j+2}^r \frac{(s+1)(c_m+s)}{s(c_m+s+1)} \\ &= \frac{(c_1 + \dots + c_j)!}{(c_1 + \dots + c_{j+1})!} \cdot \frac{(c_{j+1}+s)! \cdot (c_{j+1}+s)}{s! \cdot s} \cdot \prod_{m=j+2}^r \frac{(s+1)(c_m+s)}{s(c_m+s+1)} \\ &= \frac{(s+1)!}{(s+1+c_{j+1})!} \cdot \frac{(c_{j+1}+s)! \cdot (c_{j+1}+s)}{s! \cdot s} \cdot \prod_{m=j+2}^r \frac{(s+1)(c_m+s)}{s(c_m+s+1)} \\ &= \frac{(s+1)(c_{j+1}+s)}{s(c_{j+1}+s+1)} \cdot \prod_{m=j+2}^r \frac{(s+1)(c_m+s)}{s(c_m+s+1)} \geq 1 \end{aligned}$$

by similar reasoning as in Case 1.

We omit the calculation of showing $\frac{b_s^2}{b_{s-1}b_{s+1}} \geq 1$ for the last two cases, since they can be proved similarly.

Case 3: $s - 1 \in \mathcal{I}_j$ and $s, s + 1 \in \mathcal{I}_{j+1}$ for some j . In this case, $s = \sum_{k=1}^j c_k$.

Case 4: $s - 1 \in \mathcal{I}_j$, $s \in \mathcal{I}_{j+1}$ and $s + 1 \in \mathcal{I}_{j+2}$ for some j . In this case, $c_{j+1} = 1$ and $s = \sum_{k=1}^{j+1} c_k$. □

Remark 3.5.4. For the poset $P = T_2 \oplus T_2 \oplus T_2$ the sorting generating function is $f_P(q) = 8 + 64q + 216q^2 + 192q^3 + 240q^4$ and the cumulative generating function is $g_P(q) = 8 + 72q + 288q^2 + 480q^3 + 720q^4 + 720q^5$. One can see that the coefficients of $f_P(q)$ are not unimodal, which gives a counterexample to [32, Conjecture 29] (see also Conjecture 5.2 in the 2020 preprint of [19]). One can also check that the coefficients of $g_P(q)$ are log-concave.

We close this section with a new direction for studying the cumulative generating function of the ordinal sum of antichains P_C . One can ask: how do the cumulative generating functions $g_{P_C}(q)$ and $g_{P_{C'}}(q)$ compare when C' is a permutation of elements of C ? Given an ordered sequence of r distinct positive integers $C = (c_1, c_2, \dots, c_r)$ and a permutation π in the symmetric group on r elements $\mathfrak{S}_r$, define $\pi(C) = (c_{\pi(1)}, c_{\pi(2)}, \dots, c_{\pi(r)})$. The collection of the coefficients of the cumulative generating function of $P_{\pi(C)}$ for all $\pi \in \mathfrak{S}_r$ is defined to be

$$\mathbf{B}(C) = \left\{ \mathbf{b}_\pi = (b_0, b_1, \dots, b_{|C|-1}) : g_{P_{\pi(C)}}(q) = \sum_{i=0}^{|C|-1} b_i q^i \text{ for } \pi \in \mathfrak{S}_r \right\},$$

where $|C| = \sum_{i=1}^r c_i$. A natural partial order on $\mathbf{B}(C)$ is given by the following dominance relation.

Definition 3.5.5. For a pair of integer sequences $\mathbf{b} = (b_0, b_1, \dots, b_n)$ and $\mathbf{b}' = (b'_0, b'_1, \dots, b'_n)$, we say $\mathbf{b}'$ *dominates* $\mathbf{b}$, denoted by $\mathbf{b} \preceq \mathbf{b}'$, if $b_i \leq b'_i$ for $i = 0, 1, \dots, n$.

If $\mathbf{b}$ and $\mathbf{b}'$ denote the coefficients of the cumulative generating function of P and P' respectively, then the relation $\mathbf{b} \preceq \mathbf{b}'$ can be interpreted as saying that the labelings of P' require fewer promotions to be sorted compared to those of P . It is easy to check that $\preceq$ is a partial order on the set $\mathbf{B}(C)$.

Example 3.5.6. For $C = (1, 2, 3)$, the cumulative generating functions $P_{\pi(C)}$ for $\pi \in \mathfrak{S}_3$ are computed and their coefficients listed below:

$$\begin{aligned} \mathbf{b}_{123} &= (12, 144, 360, 720, 720, 720), & \mathbf{b}_{132} &= (12, 144, 288, 480, 720, 720), \\ \mathbf{b}_{213} &= (12, 96, 360, 720, 720, 720), & \mathbf{b}_{231} &= (12, 96, 360, 480, 600, 720), \\ \mathbf{b}_{312} &= (12, 72, 216, 480, 720, 720), & \mathbf{b}_{321} &= (12, 72, 216, 480, 600, 720). \end{aligned}$$

The Hasse diagram of $(\mathbf{B}(C), \preceq)$ is shown in the left of fig. 3.11. Observe that the subgraph consisting of all the black edges forms the Hasse diagram of the dual to the weak order on $\mathfrak{S}_3$ (see for instance [53, Exercises 3.183 and 3.185] for the definition of weak and strong order on $\mathfrak{S}_n$). The red edge $(\mathbf{b}_{312} \preceq \mathbf{b}_{213})$ shows a new cover relation which does not occur in the weak order on $\mathfrak{S}_3$.

Moreover, we draw the Hasse diagram of $(\mathbf{B}(C), \preceq)$ where $C = (1, 2, 3, 4)$ in the right picture of fig. 3.11. Similarly, the subgraph consisting of black edges forms the Hasse diagram of the dual to the weak order on $\mathfrak{S}_4$ while the red edges show new cover relations in our poset structure compared to the weak order of $\mathfrak{S}_4$. We formulate this observation more generally in the following theorem.

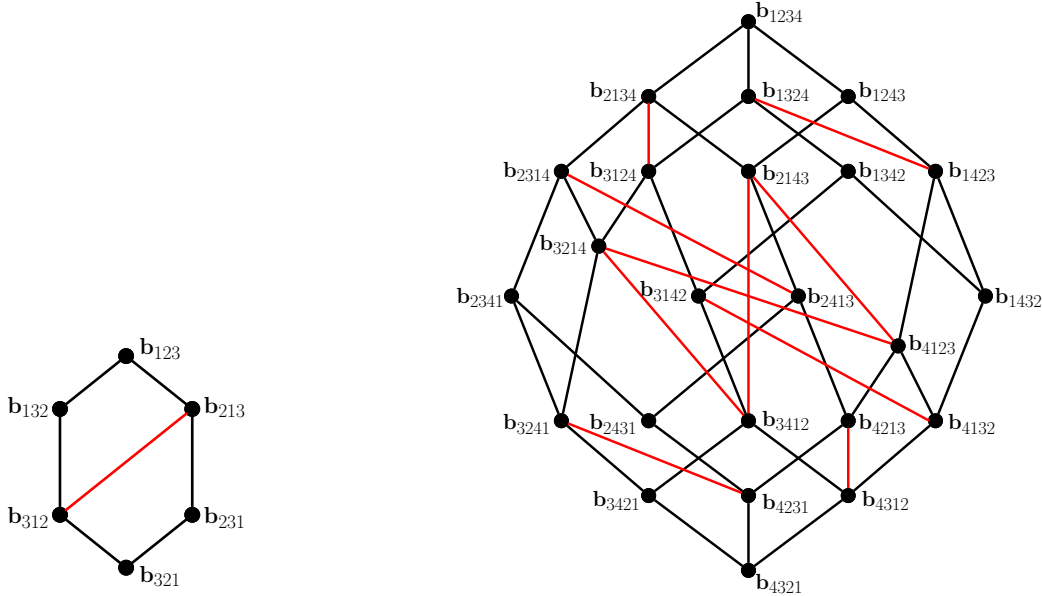


Figure 3.11 The Hasse diagram of $(\mathbf{B}(C), \preceq)$ where $C = (1, 2, 3)$ (left) and $C = (1, 2, 3, 4)$ (right), which contains a subposet (shown as a subgraph consisting of all the black edges) that is isomorphic to the weak order of $\mathfrak{S}_3$ (left) and $\mathfrak{S}_4$ (right). The new cover relations in our poset structure compared to the weak order of $\mathfrak{S}_3$ (left) and $\mathfrak{S}_4$ (right) are drawn in red

Theorem 3.5.7. *Given an ordered sequence of r distinct positive integers $C = (c_1, c_2, \dots, c_r)$. Let*

$$\mathbf{B}(C) = \left\{ \mathbf{b}_\pi = (b_0, b_1, \dots, b_{|C|-1}) : g_{P_{\pi(C)}}(q) = \sum_{i=0}^{|C|-1} b_i q^i \text{ for } \pi \in \mathfrak{S}_r \right\},$$

where $|C| = \sum_{i=1}^r c_i$. Then the poset $(\mathbf{B}(C), \preceq)$ is isomorphic to a refinement of the poset $(\mathfrak{S}_r, \leq)$, where $\leq$ is the weak order on $\mathfrak{S}_r$.

We first prove the following lemma which will be used to prove theorem 3.5.7.

Lemma 3.5.8. *Given an ordered sequence of r distinct positive integers $C = (c_1, c_2, \dots, c_r)$. Let $\pi = (i, i+1) \in \mathfrak{S}_r$ be a transposition. Let $\mathbf{b} = (b_0, \dots, b_{c_1+\dots+c_r-1})$ and $\mathbf{b}_\pi = (b'_0, \dots, b'_{c_1+\dots+c_r-1})$ be the coefficients of the cumulative generating functions $g_{P_C}(q)$ and $g_{P_{\pi(C)}}(q)$, respectively. If $c_i < c_{i+1}$, then $\mathbf{b}_\pi \preceq \mathbf{b}$ for $i = 1, 2, \dots, r-1$.*

Proof. For convenience, we write $\pi(C) = (d_1, d_2, \dots, d_r)$, where $d_i = c_{i+1}$, $d_{i+1} = c_i$ and $d_k = c_k$ for $k \neq i, i+1$. For $j = 1, 2, \dots, r$, let $\mathcal{I}_j = \{s : \sum_{k=1}^{j-1} c_k \leq s < \sum_{k=1}^j c_k\}$ and $\mathcal{I}'_j = \{s : \sum_{k=1}^{j-1} d_k \leq s < \sum_{k=1}^j d_k\}$. Since $c_k = d_k$ only differ for $k = i$ and $k = i+1$, proposition 3.5.1 implies that $b_s = b'_s$ for $s \in \mathcal{I}_j$ where $j \neq i, i+1$.

Notice that $\mathcal{I}_i \cup \mathcal{I}_{i+1} = \mathcal{I}'_i \cup \mathcal{I}'_{i+1}$ and $\mathcal{I}_i \subseteq \mathcal{I}'_i$, so it remains to check $b'_s/b_s \leq 1$ holds for the following three cases: (1) $s \in \mathcal{I}_i$, (2) $s \in \mathcal{I}'_i \setminus \mathcal{I}_i$, and (3) $s \in \mathcal{I}'_{i+1}$. This will imply that $\mathbf{b}_\pi \preceq \mathbf{b}$. For the last case, we obtain the equality $b_s = b'_s$ by proposition 3.5.1 immediately. The calculation for the first two cases is presented below.

Let $x^{\bar{n}} = \prod_{k=1}^n (x+k-1)$ denote the rising factorial of x .

Case 1: $s \in \mathcal{I}_i$. We may write $s = c_1 + \dots + c_{i-1} + t$, where $0 \leq t \leq c_i - 1$. Then for each such t ,

$$\begin{aligned} \frac{b'_s}{b_s} &= \frac{(d_1 + \dots + d_i)! \prod_{m=i+1}^r \frac{(d_m+s)!}{s!}}{(c_1 + \dots + c_i)! \prod_{m=i+1}^r \frac{(c_m+s)!}{s!}} \\ &= \frac{(c_1 + \dots + c_{i-1} + c_{i+1})! (c_i + s)!}{(c_1 + \dots + c_{i-1} + c_i)! (c_{i+1} + s)!} \\ &= \frac{(c_1 + \dots + c_{i-1} + c_{i+1})! (c_1 + \dots + c_{i-1} + c_i + t)!}{(c_1 + \dots + c_{i-1} + c_i)! (c_1 + \dots + c_{i-1} + c_{i+1} + t)!} \\ &= \frac{(c_1 + \dots + c_{i-1} + c_i + 1)^{\bar{t}}}{(c_1 + \dots + c_{i-1} + c_{i+1} + 1)^{\bar{t}}} \leq 1, \end{aligned}$$

because $c_i < c_{i+1}$.

Case 2: $s \in \mathcal{I}'_i \setminus \mathcal{I}_i$. We may write $s = c_1 + \dots + c_i + t$, where $0 \leq t \leq c_{i+1} - c_i - 1$. Then for each such t ,

$$\begin{aligned} \frac{b'_s}{b_s} &= \frac{(d_1 + \dots + d_i)! \prod_{m=i+1}^r \frac{(d_m+s)!}{s!}}{(c_1 + \dots + c_{i+1})! \prod_{m=i+2}^r \frac{(c_m+s)!}{s!}} \\ &= \frac{(c_1 + \dots + c_{i-1} + c_{i+1})! (c_i + s)!}{(c_1 + \dots + c_{i+1})! s!} \\ &= \frac{(c_1 + \dots + c_{i-1} + c_{i+1})! (c_1 + \dots + c_i + c_i + t)!}{(c_1 + \dots + c_i + c_{i+1})! (c_1 + \dots + c_i + t)!} \\ &= \frac{(c_1 + \dots + c_{i-1} + c_i + t + 1)^{c_{i+1} - c_i - t}}{(c_1 + \dots + c_i + c_i + t + 1)^{c_{i+1} - c_i - t}} \leq 1, \end{aligned}$$

by the same reasoning in Case 1. This completes the proof of lemma 3.5.8. $\square$

Proof of theorem 3.5.7. Without loss of generality, we assume the elements of C are written in the increasing order, $c_1 < c_2 < \dots < c_r$. The permutations $\pi \in \mathfrak{S}_r$ in this proof will be written in the one-line notation $\pi = p_1 p_2 \dots p_r$.

Define the map $\varphi : (\mathfrak{S}_r, \leq) \rightarrow (\mathbf{B}(C), \preceq)$ by sending a permutation $\pi = p_1 p_2 \dots p_r$ to $\mathbf{b}_{\text{rev}(\pi)}$, where $\text{rev}(\pi) = p_r p_{r-1} \dots p_1$ is the reverse of π , and $\mathbf{b}_{\text{rev}(\pi)}$ is the sequence of the coefficients of $g_{P_{\text{rev}(\pi)}(C)}(q)$. Let σ be the adjacent transposition that swapped the elements at positions i and $i+1$. Let $\pi \in \mathfrak{S}_r$ be a permutation such that $\pi \leq \sigma\pi$ in the weak order. One may write $\pi = p_1 p_2 \dots p_r$ with $p_i < p_{i+1}$, and $\sigma\pi = p_1 \dots p_{i-1} p_{i+1} p_i p_{i+2} \dots p_r$.

We show that if $\pi \leq \sigma\pi$ in $(\mathfrak{S}_r, \leq)$, then $\varphi(\pi) \preceq \varphi(\sigma\pi)$ in $(\mathbf{B}(C), \preceq)$. Intuitively, $\text{rev}(\pi)(C) = \{c_{p_r}, \dots, c_{p_1}\}$ and $\text{rev}(\sigma\pi)(C) = \{c_{p_r}, \dots, c_{p_{i+2}}, c_{p_i}, c_{p_{i+1}}, c_{p_{i-1}}, \dots, c_{p_1}\}$. Since $p_i < p_{i+1}$ and $c_{p_i} < c_{p_{i+1}}$ (by the assumption that c_i 's are increasing as i increases), by lemma 3.5.8, we obtain $\mathbf{b}_{\text{rev}(\pi)} \preceq \mathbf{b}_{\text{rev}(\sigma\pi)}$.

Therefore, $\varphi(\pi) = \mathbf{b}_{\text{rev}(\pi)} \preceq \mathbf{b}_{\text{rev}(\sigma\pi)} = \varphi(\sigma\pi)$. The poset $(\mathbf{B}(C), \preceq)$ is thus isomorphic to a refinement of $(\mathfrak{S}_r, \leq)$. $\square$

We would like to point out that $(\mathbf{B}(C), \preceq)$ is not a subposet of the strong order of $\mathfrak{S}_n$ in general. Take $C = (1, 2, 3, 4)$ as an example (see the right picture of fig. 3.11 again); the cover relation

$\mathbf{b}_{4123} \preceq \mathbf{b}_{3214}$, under the inverse of the map φ defined in the proof of theorem 3.5.7, does not relate in the strong order of $\mathfrak{S}_4$. One can also check that $(\mathbf{B}(C), \preceq)$ is not graded in general.

3.6 Future Work

We present some future directions from this work. In this chapter, we propose the $(n - 2)!$ conjecture (conjecture 3.0.5), stating that the number of tangled x -labelings (the label of x fixed as $n - 1$) of an n -element poset P is bounded by $(n - 2)!$. In section 3.2 and section 3.3, we prove that inflated rooted forest posets and shoelace posets satisfy the $(n - 2)!$ conjecture. We also obtain the exact enumeration of tangled labelings of the W -poset (a special case of the shoelace poset) in theorem 3.3.7. One can define inflated shoelace posets in analogy with inflated rooted forest posets. An interesting question would be to investigate whether inflated shoelace posets satisfy the $(n - 2)!$ conjecture. Other general classes of posets that would be of interest to study include posets related to Young tableaux.

In section 3.4, we explicitly determine the number of k -sorted labelings of the poset $T_s \oplus P$ from P (attach s minimal elements to P) via the matrix multiplication stated in theorem 3.4.5. However, obtaining the number of k -sorted labelings of the poset $P \oplus T_s$ from P (attach s maximal elements to P) does not seem to have such a nice pattern. There may exist some other ways to express them. We leave this direction to be pursued by the interested reader.

In section 3.5, we introduce the new poset structure $(\mathbf{B}(C), \preceq)$ and show that it contains a subposet which is isomorphic to the weak order of the symmetric group (theorem 3.5.7). It would be an interesting follow-up to fully characterize our poset $(\mathbf{B}(C), \preceq)$ as a poset on permutations.

CHAPTER 4

TWINNING AND THE CHROMATIC SYMMETRIC FUNCTION

The *chromatic symmetric function* of a graph $G = (V, E)$ is defined by Stanley [47] to be

$$X_G(\mathbf{x}) = \sum_{\kappa} \prod_{v \in V} x_{\kappa(v)},$$

where the sum is over all proper colorings $\kappa : V \rightarrow \mathbb{Z}_{>0}$ of G by positive integers. The goal of this chapter is to study the effect that a small change to the graph G has on $X_G(\mathbf{x})$. Specifically, we look at the change in $X_G(\mathbf{x})$ when one *twins* (or *clones*) a vertex v of a graph G , that is, when one adds a vertex v' incident to v and all its neighbors, to produce a new graph G_v . Precise definitions of this operation and related terms appear in Section 4.1.

Question 4.0.1. For a given vertex v of a graph G , how are the polynomials $X_{G_v}(\mathbf{x})$ and $X_G(\mathbf{x})$ related?

In the seminal paper [47], Stanley proved that the chromatic symmetric functions of paths and cycles are *e*-positive, that is, their expansion in the basis of elementary symmetric functions has nonnegative coefficients. As observed in [47], the result for paths is originally due to Carlitz, Scoville, and Vaughan in a different context [10, p.242]. More generally, spurred by the following conjecture of Stanley and Stembridge, much of the research on the chromatic symmetric function has centered around the incomparability graph $\text{Inc}(P)$ of a $(3 + 1)$ -free poset P , defined as a poset containing no induced subposet isomorphic to the disjoint union of a 3-chain and a 1-chain. We note that Hikita did very recently prove the Conjecture in [31].

Conjecture 4.0.2 ([47, 49]). *If P is a $(3 + 1)$ -free poset, then $X_{\text{Inc}(P)}(\mathbf{x})$ is *e*-positive.*

To *twin* a poset P at a vertex v , producing P_v , is to add an element v' such that v' is incomparable to w if and only if either $w = v$ or w is incomparable to v . Note that if $G = \text{Inc}(P)$, then $\text{Inc}(P_v) = G_v$. This next simple lemma is the main motivation for considering the twinning operation. Its proof is immediate from the fact that if $u < v$, then $u < v'$, and if $v < w$ then $v' < w$.

Lemma 4.0.3. *The twin of a $(3 + 1)$ -free poset is $(3 + 1)$ -free.*

One can therefore make a weakened version of the Stanley–Stembridge conjecture, first appearing in the work of Foley, Hoàng, and Merkel [24].

Conjecture 4.0.4 ([24]). *If P is $(3+1)$ -free and $X_{\text{Inc}(P)}(\mathbf{x})$ is e -positive, then $X_{\text{Inc}(P_v)}(\mathbf{x})$ is e -positive for any $v \in P$.*

Remark 4.0.5. Li, Li, Wang, and Yang [33, Theorem 3.6] prove that the twinning operation on graphs does not always preserve e -positivity. They give an example of a graph G [33, Theorem 4.1] that is not an incomparability graph of a $(3 + 1)$ -free poset, but whose chromatic symmetric function $X_G(\mathbf{x})$ is e -positive, and show that for a certain vertex v of G , the chromatic symmetric function for the twinned graph G_v does not expand positively even in the Schur basis, and so it cannot be e -positive. This suggests that there is something special about the twinning operation on $(3 + 1)$ -free posets.

In 2001, Gebhard and Sagan [27] introduced the stronger notion of (e) -positivity of chromatic symmetric functions in noncommuting variables. This implies e -positivity for chains of complete graphs [27, Corollary 7.7], and includes twins of paths as a special case. Later, Dahlberg and van Willigenburg [14] gave a direct proof of e -positivity for *lollipop* graphs, which are a special case of [27, Corollary 7.7], and which again include paths twinned at a leaf.

Throughout the chapter, we refer to a property of the *chromatic symmetric function of the graph* G as being a property of *the graph* G . For instance, we use interchangeably the phrases “the generating function of the chromatic symmetric function of a graph” and “the generating function of a graph”. Similarly, we use “the chromatic symmetric function of the graph G is e -positive” and “the graph G is e -positive” interchangeably.

This chapter studies the effect of twinning on the e -expansions of the chromatic symmetric function of certain graphs. We specifically look at twins of path and cycle graphs, a few of which are shown in Figure 4.2. A summary of our progress on Question 4.0.1 follows.

Our first main contribution is a series of explicit e -positive formulas for the generating function of the following families of twinned graphs:

1. The path twinned at one leaf (Proposition 4.2.9)
2. The path twinned at both leaves (Theorem 4.2.14)
3. The path twinned at an interior vertex (Theorem 4.2.24)
4. The cycle twinned at a vertex (Theorem 4.2.29)

The fourth family, examined in detail in Section 4.2.3, and culminating in Theorem 4.2.29, was not known to be e -positive until now. The first three families appear in [27] and were shown to have the stronger property of (e) -positivity of their chromatic symmetric functions with noncommuting variables [27, Theorems 7.6 and 7.8]. The e -positivity of the first graph was later re-established directly in [14]. The explicitly e -positive expressions for the generating functions that we give in Proposition 4.2.9, Theorem 4.2.14 and Theorem 4.2.24 are special cases of K-chains and slightly melting K-chains considered by Foster Tom in [58]. In Corollary 4.2.4 we provide a new e -positive expression for the generating function of the path that isolates the terms containing e_1 . Our derivations make crucial use of the triple deletion formula of Orellana and Scott [37].

For all but the third family, the expression we obtain for the generating function has the form

$$\sum_{n \geq 0} X_{G_n} z^n = \frac{f_G}{1 - \sum_{i \geq 2} (i-1) e_i z^i} + h_G$$

where f_G and h_G are some e -positive functions depending on the family and h_G has finite degree. The third family has h_G with infinite degree. Note that the denominator in the rational expression above coincides with the denominator in the generating function for both the path and the cycle (see Theorem 4.2.1). This allows us to easily obtain explicit formulas for the coefficients of the elementary symmetric functions (see Corollary 4.2.10, Corollary 4.2.15, Corollary 4.2.30). We state our formulas for the coefficients using a new statistic $\varepsilon(\lambda)$ associated with a partition λ (see Section 4.1.3). This statistic appears naturally when computing the e -coefficients of the path and cycle, and appears to be of independent interest.

The identities presented in Section 4.2.1, particularly in Lemma 4.2.3 and its proof, are the starting point for our e -positivity results. They also seem interesting in their own right.

Our second main contribution is an e -positive recurrence relation for each of the families listed above, as well as a graph appearing in the computation for the twinned cycle that we call the moose graph, which has been shown to be e -positive as a special case of hat graphs [63, Theorem 3.9].

1. The path twinned at one leaf (Proposition 4.3.2)
2. The path twinned at both leaves (Proposition 4.3.3)
3. The path twinned at an interior vertex (Theorem 4.3.4)
4. The cycle twinned at a vertex (Theorem 4.3.6)
5. The moose graph (Proposition 4.3.7)

4.1 Preliminaries

In this section, we define the basic notions used throughout the chapter, as well as discuss previous results. We assume a familiarity with symmetric functions as in [48, Chapter 7] or [35].

A *graph* G is a pair of sets (V, E) where V is the set of *vertices* and E is a set of 2-element subsets of vertices, called *edges*. We denote edges by $\{u, v\}$ or simply by uv . We assume that V and E are both finite, and that the graph is simple (i.e., there are no loops and no multiple edges). A *leaf* of a graph is a vertex contained within exactly one edge. An *internal vertex* is a vertex contained within at least two edges. Two graphs that are important for this chapter are the *path* P_n , which has vertex set $V = [n] = \{1, \dots, n\}$ and edge set $E = \{\{i, i+1\} \mid i \in [n-1]\}$, and the *cycle* C_n , which also has vertex set $V = [n]$ and edge set $E = \{\{i, i+1\} \mid i \in [n-1]\} \cup \{1, n\}$. We illustrate them in Figure 4.1.

A *proper coloring* of a graph $G = (V, E)$ is a function $\kappa : V \rightarrow \mathbb{Z}_{>0}$ such that if $uv \in E$, then $\kappa(u) \neq \kappa(v)$. Let $\mathbf{x} = (x_1, x_2, \dots)$ be an infinite set of commuting variables. The *chromatic*

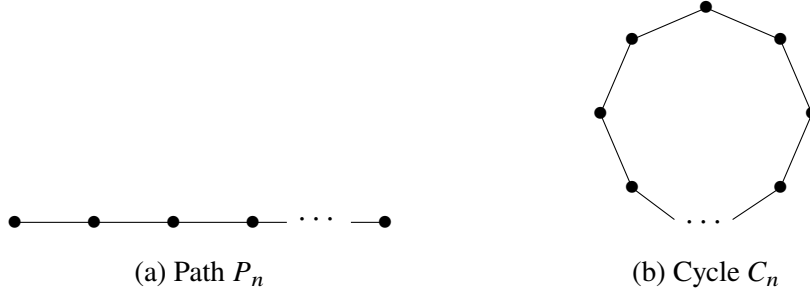


Figure 4.1 The path and cycle graphs

symmetric function of a graph $G = (V, E)$ is defined to be

$$X_G := X_G(\mathbf{x}) = \sum_{\kappa} \prod_{v \in V} x_{\kappa(v)},$$

where the sum is over all proper colorings κ of G . This symmetric function was first introduced by Stanley in [47] and has been studied by numerous authors since then. One central goal has been to characterize graphs G for which X_G is e -positive, i.e., X_G has nonnegative coefficients in the elementary symmetric function basis. We give an overview of the previous e -positivity results in Section 4.1.2.

4.1.1 Graph Operations

We begin by defining the two operations on graphs that appear in this chapter.

Given an edge ϵ of a graph G , the *deletion of ϵ in G* is the graph, denoted by $G - \epsilon$, obtained by removing the edge ϵ from G . The following formulas of Orellana and Scott are used extensively in our arguments and we refer to them as the *triple deletion arguments*.

Proposition 4.1.1 (Triple Deletion Formula [37, Theorem 3.1]). *Let G be any graph. Suppose edges $\epsilon_1, \epsilon_2, \epsilon_3$ form a triangle in G . Then,*

$$X_G = X_{G-\epsilon_1} + X_{G-\epsilon_2} - X_{G-\{\epsilon_1, \epsilon_2\}}.$$

Notice that Proposition 4.1.1 requires the graph to contain a triangle. However, one can use this formula to derive other relations for graphs that do not necessarily contain a triangle. An example of such a relation is the following.

Corollary 4.1.2 ([37, Corollary 3.2]). *Let $\epsilon_1 = vv_1 \in E$, $\epsilon_2 = vv_2 \in E$ and suppose $\epsilon_3 = v_1v_2 \notin E$. Then*

$$X_G = X_{(G-\epsilon_1) \cup \epsilon_3} + X_{G-\epsilon_2} - X_{(G-\{\epsilon_1, \epsilon_2\}) \cup \epsilon_3}.$$

We now introduce the main operation studied in this chapter.

Definition 4.1.3. Given a graph G and a vertex v , the *twin of G at v* is the graph, denoted by G_v , obtained by adding a new vertex v' and connecting v' to v and to all of its neighbors. We refer to this operation as the *twinning* of a graph and to the resulting graph G_v as the *twinned graph*. By extension, $G_{v,w}$ denotes the graph G twinned at the vertices v and w in succession.

A simple example is the complete graph $G = K_n$ on n vertices. For any vertex v , $(K_n)_v$ is the complete graph K_{n+1} . We illustrate in Figure 4.2 the twinned path at a leaf and at an interior vertex, and the twinned cycle.

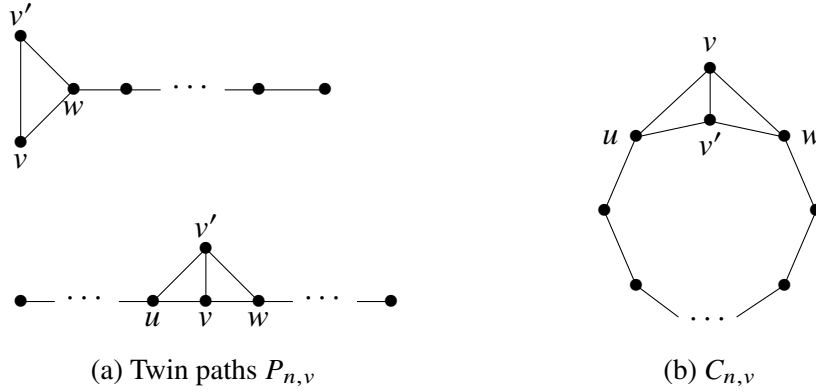


Figure 4.2 Twinning of the path and cycle graphs

Twinning a non-isolated vertex always produces a triangle, so the triple deletion argument is a natural method to reduce the twinned graph back to the original one, as the following result shows.

Corollary 4.1.4. *Let H be a graph on n vertices and let u be a vertex of H . Let H' be the graph obtained by adding a new vertex v and the edge uv to H and let H'' be the graph obtained by adding a new vertex w and the edge vw to H' . Finally let H'_v be the graph H' twinned at vertex v , with v' denoting the new vertex. Then*

$$X_{H'_v} = 2(X_{H''} - e_2 X_H).$$

Proof. This is clear by the triple deletion argument using the edges uv, uv' of the triangle $\{u, v, v'\}$ as shown in Figure 4.3. Note also that $X_{P_2} = 2e_2$. $\square$

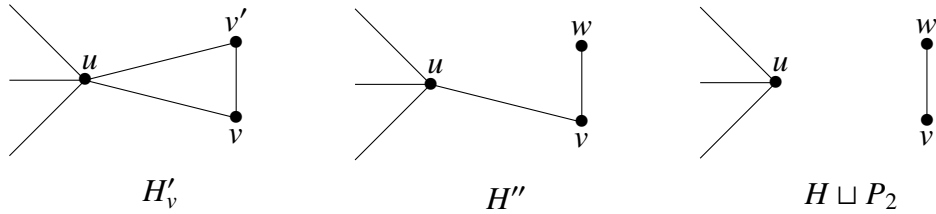


Figure 4.3 The triple deletion argument used in Corollary 4.1.4

4.1.2 Known e -positivity Results

Stanley defined the chromatic symmetric function X_G of a graph G in 1995. Since then, many families of graphs have been examined. We provide an extensive, but by no means exhaustive, list of known e -positivity results as of May 2024 in Table 4.1. We do not define these classes of graphs, but instead provide references containing their definitions as well as proofs of their e -positivity classification. By convention, a family of graphs listed as “not e -positive” means that there is at least one graph in that class that is not e -positive. The table is roughly sorted chronologically by reference, and it is condensed so that some subclasses of other results are omitted.

Graph	Positivity	Reference
Paths	e -positive	[47, Proposition 5.3]
Cycles	e -positive	[47, Proposition 5.4]
Complete graphs	e -positive	[49, Equation 3.1]
Co-triangle-free graphs	e -positive	[49, Theorem 4.3]
K_α -chains	e -positive	[27, Corollary 7.7]
Diamond and path chains	e -positive	[27, Theorem 7.8]
(claw, P_4)-free graphs	e -positive	[59, Theorem 1.4]
(claw, diamond)-free graphs	not e -positive	[30, Lemma 7]
(claw, co-claw)-free graphs	not e -positive	[30, Lemma 7]
(claw, K_4)-free graphs	not e -positive	[30, Lemma 7]
(claw, $4K_1$)-free graphs	not e -positive	[30, Lemma 7]
(claw, $2K_2$)-free graphs	not e -positive	[30, Lemma 7]
(claw, C_4)-free graphs	not e -positive	[30, Lemma 7]
(claw, paw)-free graphs	e -positive	[30, Theorem 3]
(claw, co-paw)-free graphs	e -positive	[30, Theorem 4]
Generalized bull graphs	e -positive	[12, Theorem 3.7]
Lollipops and lariats	e -positive	[14, Theorem 9]
P_3 -free graphs	e -positive	[24, Theorem 5]
(claw, K_3)-free graphs	e -positive	[24, Theorem 5]
(claw, co- P_3)-free graphs	e -positive	[24, Theorem 5]
(co-claw)-free unit interval graphs	e -positive	[24, Theorem 18]
Generalized pyramid graphs	e -positive	[34, Theorem 7]
$2K_2$ -free unit interval graphs	e -positive	[34, Theorem 13]
Triangular ladders	e -positive	[13, Theorem 22]
Star graphs	not e -positive	[16, Example 11]
Saltire and augmented saltire graphs	not e -positive	[15, Lemmas 4.4, 4.9]
Triangular tower graphs	not e -positive	[15, Lemma 5.4]
Tadpole graphs	e -positive	[33, Theorem 3.1]
Line graphs of tadpole graphs	e -positive	[62, Corollary 3.3]
Cycle-chord graphs	e -positive	[61], [62, Corollary 4.6]
Kayak paddle graphs	e -positive	[2, Proposition 6.7]
Generalized nets	not e -positive	[25, Theorem 1]
Melting K_α -chains	e -positive	[58, Corollary 4.18]

Table 4.1 Known e -positivity results and their references

4.1.3 A New Statistic on Partitions

In this subsection, we introduce a new statistic on the set of partitions that will allow us to describe e -coefficients more compactly.

Recall that a *partition* λ of n , written $\lambda \vdash n$, is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_\ell)$ that sum to n , that is, $\sum_i \lambda_i = n$. We write $\ell = \ell(\lambda)$ for the *length* of the partition, that is, the number of entries in the sequence. A partition λ of n can also be written as $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots, n^{m_n} \rangle$, where $m_i = m_i(\lambda) \geq 0$ denotes the multiplicity of the part i in λ . The *support* of λ , denoted $\text{supp}(\lambda)$, is the set of distinct parts appearing in λ , that is, $\text{supp}(\lambda) := \{i \in \mathbb{Z}_{>0} : m_i(\lambda) \geq 1\}$.

Now we are ready to introduce the new statistic.

Definition 4.1.5. For a partition λ , define $\varepsilon(\lambda)$ to be the quantity

$$\varepsilon(\lambda) := \ell(\lambda)! \prod_{j \in \text{supp}(\lambda)} \frac{(j-1)^{m_j(\lambda)}}{m_j(\lambda)!} \quad \text{with} \quad \varepsilon(\emptyset) = 1. \quad (4.1)$$

Moreover, for a partition λ of n and a part a such $m_a(\lambda) \geq 1$, let $\lambda - a$ denote the partition of $n - a$ obtained by deleting one part equal to a from λ . By convention, if a is not a part of λ , we set $\varepsilon(\lambda - a) = 0$.

For example, $\varepsilon((n)) = n - 1$ and $\varepsilon((2^n)) = 1$ for any positive integer n . Additionally, $\varepsilon(\lambda) = 0$ if λ contains a 1. In Table 4.2, we include several examples of partitions λ together with their statistic $\varepsilon(\lambda)$.

λ	(2)	(3)	(4)	(2,2)	(5)	(3,2)	(6)	(4,2)	(3,3)	(2,2,2)
$\varepsilon(\lambda)$	1	2	3	1	4	4	5	6	4	1

λ	(7)	(5,2)	(4,3)	(3,2,2)	(8)	(6,2)	(5,3)	(4,4)	(4,2,2)	(3,3,2)	(2,2,2,2)
$\varepsilon(\lambda)$	6	8	12	6	7	10	16	9	9	12	1

Table 4.2 Examples of $\varepsilon(\lambda)$ for some partitions λ

Remark 4.1.6. Note that in (4.1),

$$\frac{\ell(\lambda)!}{m_j(\lambda)!} = \binom{\ell(\lambda)}{m_1(\lambda), \dots, m_n(\lambda)},$$

which is always a nonnegative integer. Thus, $\varepsilon(\lambda)$ is also a nonnegative integer. In fact, $\varepsilon(\lambda) = 0$ if and only if 1 is a part in λ , i.e., $m_1(\lambda) \geq 1$.

Next, we present other properties of $\varepsilon(\lambda)$.

Lemma 4.1.7. *Let λ and μ be partitions of n and m , respectively. Then, we have the following:*

1. For $j \in \text{supp}(\lambda)$, $(j-1)\varepsilon(\lambda-j) = m_j(\lambda) \frac{\varepsilon(\lambda)}{\ell(\lambda)}$;
2. $\varepsilon(\lambda) = \sum_{j \in \text{supp}(\lambda)} (j-1)\varepsilon(\lambda-j)$; and
3. $\varepsilon(\lambda)\varepsilon(\mu) = \varepsilon(\lambda \cup \mu) \binom{\ell(\lambda \cup \mu)}{\ell(\lambda)}^{-1} \prod_{j \in \text{supp}(\lambda \cup \mu)} \binom{m_j(\lambda \cup \mu)}{m_j(\lambda)}$ where $\lambda \cup \mu$ is the partition of $n+m$ formed by listing the parts of λ and μ together in decreasing order.

Proof. 1. Note first that both sides are identically zero if $1 \in \text{supp}(\lambda)$. For $j \in \text{supp}(\lambda)$ with $j \neq 1$, this identity follows from the definition by noticing that

$$\varepsilon(\lambda) = (j-1) \frac{\ell(\lambda)}{m_j(\lambda)} \left((j-1)^{m_j(\lambda)-1} \prod_{l \in \text{supp}(\lambda), l \neq j} (l-1)^{m_l(\lambda)} \frac{(\ell(\lambda)-1)!}{(m_j(\lambda)-1) \prod_{l \neq j} m_l(\lambda)!} \right).$$

2. This identity follows from the definition of $\varepsilon(\lambda)$, using $\sum_{j \in \text{supp}(\lambda)} m_j(\lambda) = \ell(\lambda)$.

3. This identity follows by expanding $\varepsilon(\lambda \cup \mu)$, using $m_j(\lambda \cup \mu) = m_j(\lambda) + m_j(\mu)$ and $\ell(\lambda \cup \mu) = \ell(\lambda) + \ell(\mu)$. □

Remark 4.1.8. Intuitively, the formula for $\varepsilon(\lambda)$ can be interpreted as the number of pairs (w, f) of words w on the set $\{1, \dots, \ell(\lambda)\}$ of type λ , i.e. with λ_i occurrences of the letter i , together with a function $f : \{1, \dots, \ell(\lambda)\} \rightarrow \mathbb{Z}$ satisfying $1 \leq f(j) \leq \lambda_j - 1$ for each $j \in \{1, \dots, \ell(\lambda)\}$. These are exactly the *codes* of Stembridge [54] with no fixed points and can be used to prove Lemma 4.1.7 combinatorially. For example, the right-hand side of part (b) can be interpreted as the number of ways of making a code of type λ from a code whose type has length $\ell(\lambda) - 1$.

4.2 e -positivity via Generating Functions

For given family of graphs $G = \{G_n\}_{n \geq 0}$, one can show e -positivity of X_{G_n} by showing that its generating function

$$\mathcal{X}_G(z) = \sum_{n \geq 0} X_{G_n} z^n$$

can be written in the form

$$\mathcal{X}_G(z) = \frac{P(z)}{1 - Q(z)}, \quad (4.2)$$

where $P(z)$ and $Q(z)$ are e -positive formal power series in z . For the path P_n and the cycle C_n , it is known from Stanley's original paper [47] that this can be done. (See also [10, p.242] for paths.)

Theorem 4.2.1 ([47, Propositions 5.3 and 5.4]).

$$\begin{aligned} \mathcal{X}_P(z) &:= \sum_{n \geq 0} X_{P_n} z^n = \frac{\sum_{i \geq 0} e_i z^i}{1 - \sum_{i \geq 1} (i-1) e_i z^i}, \\ \mathcal{X}_C(z) &:= \sum_{n \geq 2} X_{C_n} z^n = \frac{\sum_{i \geq 2} i(i-1) e_i z^i}{1 - \sum_{i \geq 1} (i-1) e_i z^i}. \end{aligned}$$

Note in particular that $X_{P_0} = 1$.

In this section, we establish identities of the form (4.2) for several families of twinned graphs by applying generating function techniques to the relations obtained from the triple deletion argument.

It is useful to convert the preceding result to a recurrence relation for the chromatic symmetric function as follows. We will use this formulation several times in this chapter, notably in the proofs of Lemma 4.2.23 and Proposition 4.2.25, as well as in Section 4.3 .

Proposition 4.2.2. *We have the following recurrence relations:*

1. For $n \geq 3$,

$$\begin{aligned} X_{P_n} &= n e_n + \sum_{j=2}^{n-1} (j-1) e_j X_{P_{n-j}} \\ &= n e_n + \sum_{i=1}^{n-2} (n-i-1) e_{n-i} X_{P_i}, \end{aligned}$$

with initial conditions $X_{P_0} = 1$, $X_{P_1} = e_1$ and $X_{P_2} = 2e_2$.

2. For $n \geq 4$,

$$X_{C_n} = n(n-1)e_n + \sum_{j=2}^{n-2} (j-1)e_j X_{C_{n-j}},$$

with initial conditions $X_{C_1} = 0$, $X_{C_2} = 2e_2$, and $X_{C_3} = 6e_3$.

4.2.1 Symmetric Function Identities and Technical Lemmas

In this section, we examine more closely the relationship between the generating function $E(z)$ for the elementary symmetric functions, and the generating functions $\mathcal{X}_P(z)$ and $\mathcal{X}_C(z)$ for the chromatic symmetric functions of the path and the cycle. We also present some formulas for several families of coefficients appearing in the e -expansion of $\mathcal{X}_P(z)$ and $\mathcal{X}_C(z)$. We start by introducing some definitions and notation to facilitate our study.

Let $E(z) := \sum_{i \geq 0} e_i z^i$ be the generating function for the elementary symmetric functions and define

$$D(z) := E(z) - zE'(z) = 1 - \sum_{i \geq 2} (i-1)e_i z^i.$$

Theorem 4.2.1 can then be rewritten as:

$$\mathcal{X}_P(z) = \frac{E(z)}{D(z)} \quad \text{and} \quad \mathcal{X}_C(z) = \frac{z^2 E''(z)}{D(z)}. \quad (4.3)$$

It will be useful for our study to collect here the definitions of several e -positive series and their truncations and tails. Considering $k \geq 2$ whenever it appears, we define

$$\begin{aligned} E_{\geq k}(z) &= \sum_{i \geq k} e_i z^i, \\ K(z) &= \sum_{i \geq 2} i e_i z^i, & K_{\geq k}(z) &= \sum_{i \geq k} i e_i z^i, \\ G(z) &= 1 - D(z) = \sum_{i \geq 2} (i-1) e_i z^i, & G_{\geq k}(z) &= \sum_{i \geq k} (i-1) e_i z^i, \\ \frac{1}{D(z)} &= \sum_{i \geq 0} G(z)^i, & G_{\leq k}(z) &= \sum_{2 \leq i \leq k} (i-1) e_i z^i = G(z) - G_{\geq k+1}(z). \end{aligned} \quad (4.4)$$

The next lemma collects some e -positivity results concerning the generating functions introduced above.

Lemma 4.2.3.

1. The following expressions are e -positive:

- a) $z^2E''(z) - zE'(z) + e_1z$;
- b) $2z^2E''(z) - 3zE'(z) + 3e_1z + 2e_2z^2$; and
- c) $z^2E''(z) - 3zE'(z) + 3E(z) + e_2z^2$.

2. The following expressions can be written as rational functions with e -positive numerators:

- a) $\mathcal{X}_P(z) - (1 + e_1z)$; and
- b) $(1 + e_1z)\mathcal{X}_C(z) - \mathcal{X}_P(z) + 1 + e_1z$.

Proof.

1. The e -positivity results follow, respectively, from the identities:

- a) $z^2E''(z) - zE'(z) = -e_1z + \sum_{i \geq 3} i(i-2)e_iz^i$;
- b) $2z^2E''(z) - 3zE'(z) = -3e_1z - 2e_2z^2 + \sum_{i \geq 3} (2i^2 - 5i)e_iz^i$; and
- c) $z^2E''(z) - 3zE'(z) + 3E(z) = 3 - e_2z^2 + \sum_{i \geq 4} (i-1)(i-3)e_iz^i$.

2. For the results concerning e -positive numerators, we have that:

a) By (4.3),

$$\begin{aligned} \mathcal{X}_P(z) - (1 + e_1z) &= \frac{zE'(z) + e_1z[zE'(z) - E(z)]}{D(z)} \\ &= \frac{\sum_{i \geq 2} ie_iz^i + e_1z \sum_{i \geq 2} (i-1)e_iz^i}{D(z)}. \end{aligned}$$

b) By the previous item,

$$\begin{aligned} (1 + e_1z)\mathcal{X}_C(z) - \mathcal{X}_P(z) + 1 + e_1z &= \frac{(1 + e_1z)[z^2E''(z) - zE'(z)] + e_1zE(z)}{D(z)} \\ &= \frac{(1 + e_1z)F_1(z) + e_1z(E(z) - 1 - e_1z)}{D(z)}, \end{aligned}$$

where $F_1(z) = \sum_{i \geq 3} i(i-2)e_i z^i$ and $E(z) - 1 - e_1 z = \sum_{i \geq 2} e_i z^i$ are e -positive. $\square$

Using Lemma 4.2.3, we get an e -positive expression for the generating function for paths that isolates those terms containing e_1 and that is different from the one in Theorem 4.2.1.

Corollary 4.2.4. *We have*

$$\mathcal{X}_P(z) = \frac{K(z)}{D(z)} + e_1 z \frac{G(z)}{D(z)} + (1 + e_1 z).$$

We now analyze these generating functions to extract closed formulas for the coefficients in the e -expansions. Recall the statistic on partitions $\varepsilon(\lambda)$ introduced in Section 4.1.3.

We start with a result that shows the relation between the coefficients of $G(z)^k$ and $\frac{1}{D(z)}$ in their e -expansion and $\varepsilon(\lambda)$.

Lemma 4.2.5. *The coefficient of $e_\lambda z^{|\lambda|}$ in $G(z)^k$ is $\varepsilon(\lambda)$ and hence*

$$\frac{1}{D(z)} = \sum_{\lambda} \varepsilon(\lambda) e_\lambda z^{|\lambda|},$$

where the sum is over all partitions λ .

Proof. This follows by manipulating the formal series directly:

$$G(z)^k = \left(\sum_{i \geq 2} (i-1) e_i z^i \right)^k = \sum_{\substack{\lambda \\ \ell(\lambda)=k}} e_\lambda z^{|\lambda|} \prod_{i \geq 2} (i-1)^{m_i(\lambda)} = \sum_{\substack{\lambda \\ \ell(\lambda)=k}} \varepsilon(\lambda) e_\lambda z^{|\lambda|}. \quad \square$$

We end this subsection by showing that several families of coefficients in the e -expansion of $\mathcal{X}_P(z)$ and $\mathcal{X}_C(z)$ can be expressed compactly in terms of $\varepsilon(\lambda)$. (See also [64].)

Proposition 4.2.6. *Given a graph G , let c_λ be the coefficient of $z^{|\lambda|} e_\lambda$ in $\mathcal{X}_G$, that is, $\mathcal{X}_G = \sum c_\lambda z^{|\lambda|} e_\lambda$.*

Then, we have the following:

$$1. \text{ For } G = P_n, c_\lambda = \varepsilon(\lambda) + \sum_{a \in \text{supp}(\lambda)} \varepsilon(\lambda - a) = \sum_{a \in \text{supp}(\lambda)} a \varepsilon(\lambda - a).$$

$$\text{In particular, if } \lambda = 1 \cup \mu \text{ for some partition } \mu, \text{ then } c_\lambda = \sum_{\substack{a \in \text{supp}(\mu) \\ a \geq 2}} (a-1) \varepsilon(\mu - a).$$

Moreover, we can also extract particular coefficients like

$$c_{(n)} = n, \quad c_{(n-1,1)} = n - 2, \quad c_{(2^k)} = 2, \quad \text{and} \quad c_{(2^k,1)} = 1.$$

2. For $G = C_n$, we have that

$$c_\lambda = \sum_{a \in \text{supp}(\lambda)} a(a-1) \varepsilon(\lambda - a).$$

Proof. We use the generating functions in (4.3).

1. The first expression comes directly from the path generating function $\mathcal{X}_P(z)$ and the second expression also follows from Lemma 4.2.3. The equality of the two expressions and the case when $\lambda = 1 \cup \mu$ follow using Lemma 4.1.7.
2. The formula for this coefficient comes directly from the cycle generating function $\mathcal{X}_C(z)$. $\square$

4.2.2 Generating Functions for Twinned Paths

In this section, we focus on studying the various ways to twin a path. The following is a key result.

Lemma 4.2.7. *For $k \geq 2$, the rational function $\frac{1 - G_{\leq k}(z)}{D(z)}$ and the function $\mathcal{X}_P(z)(1 - G_{\leq k}(z))$ are e -positive.*

Proof. For the rational function, we have that

$$\frac{1 - G_{\leq k}(z)}{D(z)} = \frac{1 - G(z) + G_{\geq k+1}(z)}{1 - G(z)} = 1 + \frac{G_{\geq k+1}(z)}{D(z)}. \quad (4.5)$$

This is e -positive since $G_{\geq k+1}(z) = \sum_{i \geq k+1} (i-1)e_i z^i$ and $\frac{1}{D(z)}$ expands e -positively in powers of $G(z)$. The e -positivity of the second function follows from (4.3) and the identity

$$\mathcal{X}_P(z)(1 - G_{\leq k}(z)) = E(z) + \mathcal{X}_P(z) G_{\geq k+1}(z). \quad \square$$

4.2.2.1 Paths Twinned at a Leaf

The recurrence for the chromatic symmetric function of twinned paths at a leaf (i.e., a vertex of degree 1) appears in Dahlberg and van Willigenburg [14, Equation 5], where the graph $P_{n,v}$ with v a leaf is called the *lariat graph* L_{n+3} . Its chromatic symmetric function had been considered earlier by Wolfe in [64], and e -positivity was first established by Gebhard and Sagan in [27, Corollary 7.7].

Proposition 4.2.8. *Let v be a leaf of the path P_n . The generating function for the chromatic symmetric function of the twin $P_{n,v}$ of a path on n vertices satisfies the following identity:*

$$2 + 2e_1z + \sum_{n \geq 1} X_{P_{n,v}} z^{n+1} = 2(1 - e_2z^2)X_P(z).$$

Proof. By [14, Equation 5], the chromatic symmetric function of $P_{n,v}$, with $n \geq 1$, is given by

$$X_{P_{n,v}} = 2X_{P_{n+1}} - X_{P_2}X_{P_{n-1}}. \quad (4.6)$$

The proof now follows by using the generating function $X_P(z)$. □

Now we are ready to derive a generating function for paths twinned at a leaf. Although the e -positivity was established in [27, Corollary 7.7] and again in [14], as mentioned earlier, our contribution here is to give the manifestly e -positive generating function below for $X_{P_{n,v}}$, using only symmetric functions, which enables a more efficient coefficient extraction.

Proposition 4.2.9. *Let $X_{P_v}(z)$ be the generating function for the twinned path at a leaf, that is,*

$X_{P_v} := \sum_{n \geq 1} X_{P_{n,v}} z^{n+1}$. *Then*

$$\frac{1}{2}X_{P_v}(z) = K(z) \frac{G_{\geq 3}(z)}{D(z)} + e_1zG(z) \frac{G_{\geq 3}(z)}{D(z)} + e_2z^2 + \sum_{i \geq 3} ie_i z^i + e_1zG_{\geq 3}(z).$$

In particular $X_{P_{n,v}}$ is e -positive, and the initial values are

$$X_{P_{1,v}} = 2e_2, \quad X_{P_{2,v}} = 2(3e_3), \quad X_{P_{3,v}} = 2(4e_4 + 2e_1e_3), \quad X_{P_{4,v}} = 2(4e_2e_3 + 3e_1e_4 + 5e_5).$$

An e -positive expression without denominators in terms of the path generating function X_P is

$$\frac{1}{2}X_{P_v}(z) = X_P(z)G_{\geq 3}(z) + \sum_{i \geq 2} e_i z^i.$$

Proof. By Lemma 4.2.7 and Corollary 4.2.4, we have that

$$\begin{aligned}
\frac{1}{2}\mathcal{X}_{P_v}(z) &= (1 - e_2 z^2)\mathcal{X}_P - (1 + e_1 z) = K(z) \frac{1 - e_2 z^2}{D(z)} + e_1 z G(z) \frac{1 - e_2 z^2}{D(z)} - e_2 z^2 (1 + e_1 z) \\
&= (K(z) + e_1 z G(z)) \frac{G_{\geq 3}(z)}{D(z)} + (K(z) + e_1 z G(z)) - e_2 z^2 (1 + e_1 z) \\
&= (K(z) + e_1 z G(z)) \frac{G_{\geq 3}(z)}{D(z)} + (K(z) - e_2 z^2) + e_1 z (G(z) - e_2 z^2).
\end{aligned}$$

Since $K(z) - e_2 z^2 = e_2 z^2 + \sum_{i \geq 3} i e_i z^i$ and $G(z) - e_2 z^2 = \sum_{i \geq 3} (i - 1) e_i z^i$, the result follows.

The second expression is obtained from the first by rewriting the formula in Corollary 4.2.4 as follows:

$$\mathcal{X}_P(z) = \frac{K(z) + e_1 z G(z)}{D(z)} + (1 + e_1 z). \quad \square$$

Corollary 4.2.10. *Let c_λ be the coefficient of $e_\lambda z^{|\lambda|}$ in $\mathcal{X}_{P_v}$, that is $\mathcal{X}_{P_v} = \sum c_\lambda e_\lambda z^{|\lambda|}$, where v is a leaf of the path P_n . The following is a list of closed formulas for all the coefficients c_λ involved in the general expression of $\mathcal{X}_{P_v}(z)$:*

1. $c_{(k)} = 2k$, $k \geq 3$, and $c_{(2)} = 2$;
2. $c_{(k-1,1)} = 2(k-2)$, $k \geq 4$;
3. $c_{(k-2,2)} = 4(k-3)$, $k \geq 5$;
4. $c_{(i,j)} = 2i(j-1) + 2j(i-1) = 2(2ij - i - j)$, $i > j \geq 3$;
5. $c_{(i,i)} = 2i(i-1)$, $i \geq 3$.
6. $c_{(3,2^k)} = 8$ and $c_{(3,2^k,1)} = 4$, $k \geq 2$.
7. If $c_{1 \cup \mu} \neq 0$ and $\ell(\mu) \geq 2$, then $1 \notin \text{supp}(\mu)$ and there exists $a \geq 3$ such that $a \in \text{supp}(\mu)$. In particular $c_{(2^k,1)} = 0$. The coefficient of $e_{1 \cup \mu}$ is equal to twice the coefficient of e_μ in $G(z)G_{\geq 3}(z)G(z)^{\ell(\mu)-2}$, and it equals

$$2 \sum_{\substack{(a,b) \\ a,b \in \text{supp}(\mu) \\ a \geq 2, b \geq 3}} (a-1)(b-1)\varepsilon((\mu-a)-b).$$

8. Assume $1 \notin \text{supp}(\lambda)$ and $\ell(\lambda) \geq 2$. If $c_\lambda \neq 0$, then λ contains at least one part of size at least 3. In particular $c_{(2^k)} = 0$. The coefficient c_λ is equal to twice the coefficient of e_λ in $K(z)G_{\geq 3}(z)G(z)^{\ell(\lambda)-2}$, and it equals

$$2 \sum_{\substack{(a,b) \\ a,b \in \text{supp}(\lambda) \\ a \geq 2, b \geq 3}} a(b-1)\varepsilon((\lambda-a)-b).$$

Note that cases (c)-(f) are particular cases of (g) and (h).

4.2.2.2 Paths Twinned at Both Leaves

In this section, we consider the twinned path $P_{n,w,v}$ at both leaves, which we label with w and v . The e -positivity of its chromatic symmetric function is a consequence of [27, Corollary 7.7], whose proof relies on the theory of symmetric functions in noncommuting variables. Here we derive an e -positive generating function using only symmetric function identities.

Unlike the other families of graphs, here one needs to pay special attention to the smaller values of n . We consider the special case of the path on two vertices first. Twinning both vertices produces the twin of the cycle graph C_3 at one vertex, which is also the complete graph K_4 , as shown in Figure 4.4, and therefore we have the following.

Lemma 4.2.11. *For the path P_2 twinned at both vertices, $X_{P_{2,v,w}} = X_{C_{3,v}} = 24e_4$.*

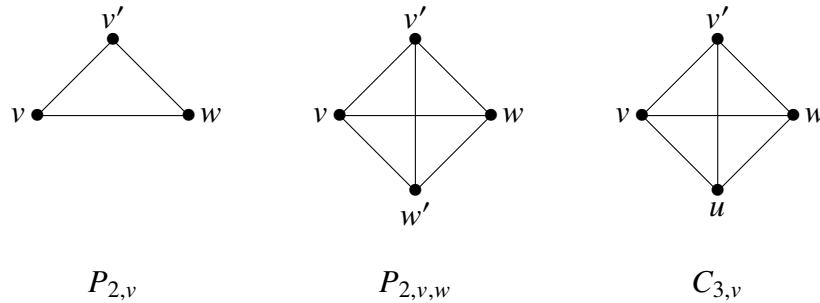


Figure 4.4 Demonstration that $P_{2,v,w} = C_{3,v}$

For the general case, we start with a consequence of the triple deletion argument.

Corollary 4.2.12. *Let v, w be the two leaves of the path P_n , and let $P_{n,v,w}$ be the path twinned at both leaves. Then, for $n \geq 3$,*

$$X_{P_{n,v,w}} = 2X_{P_{n+1,v}} - 2e_2X_{P_{n-1,v}} = 4(X_{P_{n+2}} - 2e_2X_{P_n} + e_2^2X_{P_{n-2}}). \quad (4.7)$$

This relation allows us to give the following generating function identity.

Proposition 4.2.13. *For the graph $P_{n,v,w}$, we have*

$$\frac{1}{4} \sum_{n \geq 3} X_{P_{n,v,w}} z^{n+2} = (1 - e_2 z^2) \frac{1}{2} X_{P_v} + \frac{1}{2} \alpha, \quad (4.8)$$

where $\alpha = 2e_2^2 z^4 - (8e_4 z^4 + 4e_3 e_1 z^4 + 6e_3 z^3 + 2e_2 z^2)$.

Proof. Multiply both sides of the first equality in Corollary 4.2.12 by z^{n+2} and sum for $n \geq 3$:

$$\begin{aligned} \sum_{n \geq 3} X_{P_{n,v,w}} z^{n+2} &= 2 \sum_{n \geq 3} X_{P_{n+1,v}} z^{n+2} - 2e_2 \sum_{n \geq 3} X_{P_{n-1,v}} z^{n+2} \\ &= 2 \sum_{n \geq 4} X_{P_{n,v}} z^{n+1} - 2e_2 z^2 \sum_{n \geq 2} X_{P_{n,v}} z^{n+1} \\ &= 2(1 - e_2 z^2) X_{P_v} - 2(X_{P_{3,v}} z^4 + X_{P_{2,v}} z^3 + X_{P_{1,v}} z^2 - 2e_2 z^2 X_{P_{1,v}} z^2) \\ &= 2(1 - e_2 z^2) X_{P_v} - 2[(8e_4 + 4e_3 e_1) z^4 + 6e_3 z^3 + 2e_2 z^2 - 2e_2^2 z^4] \end{aligned}$$

where the computations for $X_{P_{n,v}}$ follow from Proposition 4.2.9. □

The next theorem follows from Corollary 4.2.12 and manipulation of the formal series.

Theorem 4.2.14. *The generating function $\frac{1}{4} \sum_{n \geq 3} X_{P_{n,v,w}} z^{n+2}$ has the following e -positive expansion:*

$$\begin{aligned} \frac{1}{4} \sum_{n \geq 3} X_{P_{n,v,w}} z^{n+2} &= (K(z) + e_1 z G(z)) \frac{G_{\geq 3}(z)^2}{D(z)} + e_1 z G_{\geq 3}(z)^2 \\ &\quad + \left(G_{\geq 3}(z) \sum_{i \geq 3} i e_i z^i + e_1 z \sum_{i \geq 4} (i-1) e_i z^i + e_2 z^2 \sum_{i \geq 3} (i-2) e_i z^i \right) + \sum_{i \geq 5} i e_i z^i. \end{aligned}$$

An e -positive expression without denominators in terms of the path generating function X_P is

$$X_P(z) G_{\geq 3}^2(z) + K_{\geq 5}(z) + G_{\geq 3}(z) \sum_{i \geq 3} e_i z^i + e_1 z G_{\geq 4}(z) + e_2 z^2 \sum_{i \geq 3} (i-2) e_i z^i.$$

Proof. We use the generating function in Proposition 4.2.9 to expand $\frac{1}{2}(1 - e_2 z^2) \mathcal{X}_{P_{n,v}}$ as

$$\begin{aligned} \frac{1}{2}(1 - e_2 z^2) \mathcal{X}_{P_v} = & (1 - e_2 z^2) (K(z) + z e_1 G(z)) \frac{G_{\geq 3}(z)}{D(z)} \\ & + (1 - e_2 z^2) \left(e_2 z^2 + \sum_{i \geq 3} i e_i z^i + z e_1 G_{\geq 3}(z) \right). \end{aligned}$$

By (4.5), $\frac{1 - e_2 z^2}{D(z)} = 1 + \frac{G_{\geq 3}(z)}{D(z)}$, and we can rewrite the above expression as

$$\begin{aligned} \frac{1}{2}(1 - e_2 z^2) \mathcal{X}_{P_v} = & (K(z) + z e_1 G(z)) G_{\geq 3}(z) \left(1 + \frac{G_{\geq 3}(z)}{D(z)} \right) \\ & + (1 - e_2 z^2) \left(e_2 z^2 + \sum_{i \geq 3} i e_i z^i + z e_1 G_{\geq 3}(z) \right). \end{aligned}$$

Next, we arrange the expression above so that the term $-\frac{1}{2}\alpha$ appears:

$$\begin{aligned} \frac{1}{2}(1 - e_2 z^2) \mathcal{X}_{P_v} = & (K(z) + z e_1 G(z)) \frac{G_{\geq 3}(z)^2}{D(z)} - \frac{1}{2}\alpha + \sum_{i \geq 5} i e_i z^i + e_1 z \sum_{i \geq 4} (i - 1) e_i z^i \\ & + (K(z) + z e_1 G(z)) G_{\geq 3}(z) - e_2 z^2 \sum_{i \geq 3} i e_i z^i - e_2 z^2 (z e_1) G_{\geq 3}(z). \end{aligned}$$

Thus, we have that

$$\begin{aligned} \frac{1}{4} \sum_{n \geq 3} \mathcal{X}_{P_{n,v,w}} z^{n+2} = & \frac{1}{2}(1 - e_2 z^2) \mathcal{X}_{P_v} + \frac{1}{2}\alpha \\ = & (K(z) + z e_1 G(z)) \frac{G_{\geq 3}(z)^2}{D(z)} + \sum_{i \geq 5} i e_i z^i + e_1 z \sum_{i \geq 4} (i - 1) e_i z^i \end{aligned} \quad (4.9)$$

$$+ (K(z) + z e_1 G(z)) G_{\geq 3}(z) - e_2 z^2 \sum_{i \geq 3} i e_i z^i - e_2 z^2 (z e_1) G_{\geq 3}(z), \quad (4.10)$$

where the terms in line (4.9) are e -positive. Thus, we only need to show that the terms in line (4.10) are also e -positive. Note that

$$K(z) G_{\geq 3}(z) = \left(2e_2 z^2 + \sum_{i \geq 3} i e_i z^i \right) G_{\geq 3}(z) = 2e_2 z^2 G_{\geq 3}(z) + G_{\geq 3}(z) \sum_{i \geq 3} i e_i z^i.$$

Together with the fact that $G(z) - e_2 z^2 = G_{\geq 3}(z)$, line (4.10) can be written as

$$\begin{aligned}
& (K(z) + z e_1 G(z)) G_{\geq 3}(z) - e_2 z^2 \sum_{i \geq 3} i e_i z^i - e_2 z^2 (z e_1) G_{\geq 3}(z) \\
&= G_{\geq 3}(z) \sum_{i \geq 3} i e_i z^i + e_2 z^2 \left(2 \sum_{i \geq 3} (i-1) e_i z^i - \sum_{i \geq 3} i e_i z^i \right) + e_1 z (G(z) - e_2 z^2) G_{\geq 3}(z) \\
&= G_{\geq 3}(z) \sum_{i \geq 3} i e_i z^i + e_2 z^2 \sum_{i \geq 3} (i-2) e_i z^i + e_1 z G_{\geq 3}(z)^2. \tag{4.11}
\end{aligned}$$

Since the expression in (4.11) is also e -positive, the result follows.

The second expression involving $\mathcal{X}_P$ follows as in the proof of Proposition 4.2.9. $\square$

In particular, we can extract the following formulas for the coefficients.

Corollary 4.2.15. *Let c_λ be the coefficient of $e_\lambda z^{|\lambda|}$ in $\mathcal{X}_{P_{v,w}}$, that is, $\mathcal{X}_{P_{v,w}} = \sum c_\lambda e^\lambda z^{|\lambda|}$. We have the following list of closed formulas:*

1. For $k \geq 3$, $c_{(k+2)} = 4(k+2)$, $c_{(k,2)} = 4(k-2)$, and $c_{(k+1,1)} = 4k$.
2. For $i \geq 3$, $c_{(i,i)} = 4(i-1)i$, and for $i, j \geq 3, i \neq j$, $c_{(i,j)} = 4(j-1)i + 4(i-1)j$.
3. For $i, j \geq 3, i \neq j$, $c_{(i,i,1)} = 4(i-1)^2$, $c_{(i,j,1)} = 8(i-1)(j-1)$, $i, j \geq 3, i \neq j$, and zero otherwise.
4. If $c_{1 \cup \mu} \neq 0$, then $1 \notin \text{supp}(\mu)$.
5. For all $k \geq 0$, $c_{(3^2, 2^{k+1})} = 32$ and $c_{(3^2, 2^{k+1}, 1)} = 16$.

4.2.2.3 Paths Twinned at an Interior Vertex

In this section, we establish an e -positive generating function for the path $P_{n,\ell}$ twinned at an interior vertex ℓ , where we label the vertices of P_n by $1, 2, \dots, n$ from left to right. As stated in the introduction, the e -positivity can also be deduced from [27, Theorem 7.8].

As in the preceding section, we first derive a triple deletion formula for the chromatic symmetric function of $P_{n,\ell}$, (Proposition 4.2.19), and then deduce an e -positive generating function for its chromatic symmetric function (Theorem 4.2.24). We begin with some definitions.

Definition 4.2.16. For $n \geq 2$ and $1 \leq \ell \leq n-1$, let $\tilde{T}_{n,\ell}$ (T for *triangle*) denote the graph obtained from the path graph P_n by adding a vertex adjacent to both ℓ and $\ell+1$. For $n \geq 1$ and $1 \leq \ell \leq n$, let $F_{n,\ell}$ (F for *flagpole*) denote the graph obtained from P_n by adding a vertex adjacent to ℓ .

By the triple deletion argument illustrated in Figure 4.5, we have the following result.

Lemma 4.2.17. For $n \geq 3$ and $2 \leq \ell \leq n-1$, we have

$$X_{P_{n,\ell}} = 2X_{\tilde{T}_{n,\ell-1}} - X_{\tilde{T}_{\ell,\ell-1}}X_{P_{n-\ell}}.$$

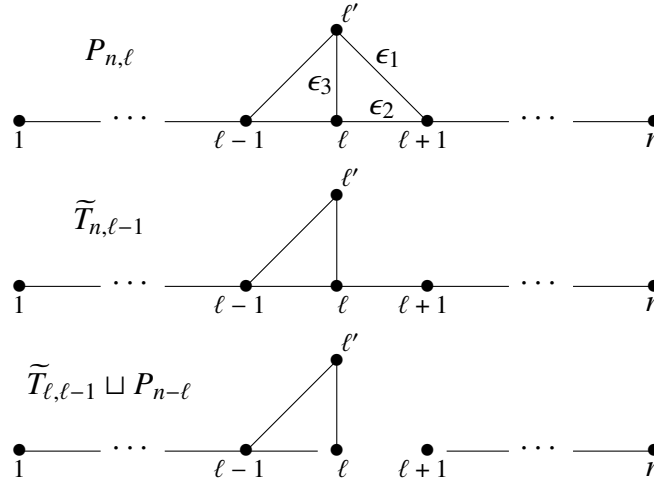


Figure 4.5 The triple deletion argument applied as in Lemma 4.2.17

By carefully applying the triple deletion argument to various $P_{n,\ell}$, we can deal with the triangles $\tilde{T}_{n,\ell}$ by “shifting” them around. Note that $\tilde{T}_{n,\ell}$ has a triangle, and so the triple deletion argument applies to two different sets of edges, to which we refer as *left* and *right shifts*. We illustrate them on the left-hand side and right-hand side of Figure 4.6, respectively.

Lemma 4.2.18 (Left and Right Shift Lemma). For $n \geq 3$ and $2 \leq \ell \leq n-1$, we have

$$\begin{array}{c|c} X_{\tilde{T}_{n,\ell}} = X_{F_{n,\ell}} + X_{P_{n+1}} - X_{P_{\ell+1}}X_{P_{n-\ell}} & X_{\tilde{T}_{n,\ell}} = X_{F_{n,\ell+1}} + X_{P_{n+1}} - X_{P_{\ell}}X_{P_{n-\ell+1}} \\ \text{Left shift} & \text{Right shift} \end{array}$$

Our next step is to use Lemma 4.2.18 to obtain a formula equivalent to that in Lemma 4.2.17 which does not involve twinning paths.

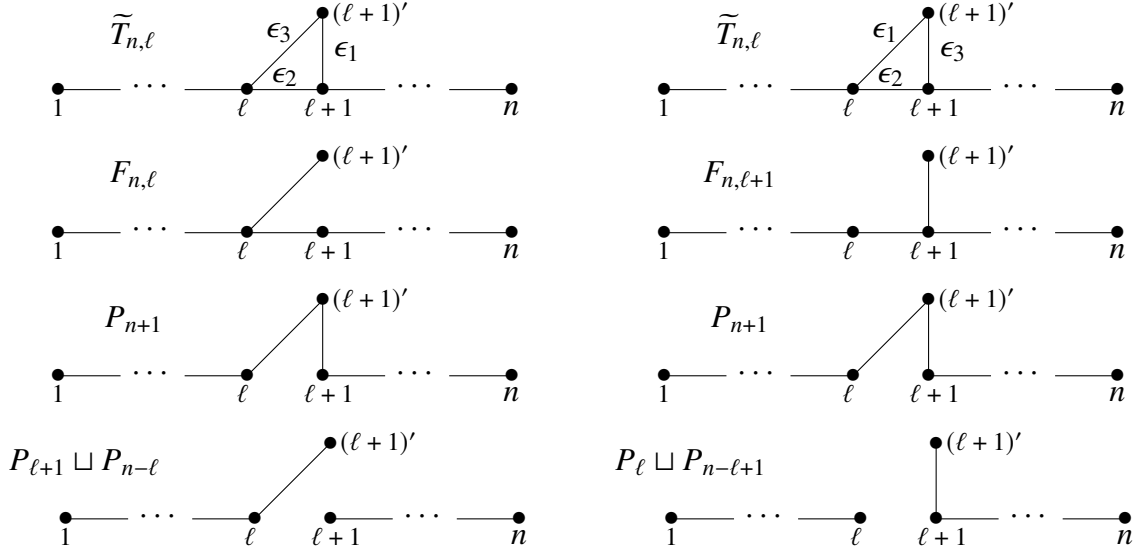


Figure 4.6 Illustration of Lemma 4.2.18

Proposition 4.2.19. For $n \geq 3$ and $2 \leq \ell \leq n-1$, we have

$$X_{P_{n,\ell}} = -2X_{P_{\ell-1}}X_{P_{n-\ell+2}} + 2e_1X_{P_n} + 4X_{P_{n+1}} - 2X_{P_\ell}X_{P_{n-\ell+1}} + 2e_2X_{P_{\ell-1}}X_{P_{n-\ell}} - 2X_{P_{\ell+1}}X_{P_{n-\ell}}.$$

Proof. Applying the Left and Right Shift Lemmas at $\ell - k - 1$ implies that

$$X_{F_{n,\ell-k}} = X_{F_{n,\ell-k-1}} + X_{P_{\ell-k-1}}X_{P_{n-\ell+k+2}} - X_{P_{\ell-k}}X_{P_{n-\ell+k+1}}. \quad (4.12)$$

By applying (4.12) repeatedly, we get that

$$X_{F_{n,\ell}} = X_{F_{n,1}} + \sum_{k=0}^{\ell-2} (X_{P_{\ell-k-1}}X_{P_{n-\ell+k+2}} - X_{P_{\ell-k}}X_{P_{n-\ell+k+1}}). \quad (4.13)$$

Since $F_{n,1}$ is precisely P_{n+1} , we can telescope the sum in (4.13) to obtain

$$X_{F_{n,\ell}} = X_{P_{n+1}} + X_{P_1}X_{P_n} - X_{P_\ell}X_{P_{n-\ell+1}}. \quad (4.14)$$

Recall the formula in Lemma 4.2.17:

$$X_{P_{n,\ell}} = 2X_{\tilde{T}_{n,\ell-1}} - X_{\tilde{T}_{\ell,\ell-1}}X_{P_{n-\ell}}.$$

We apply the Left Shift Lemma to $X_{\tilde{T}_{n,\ell-1}}$ and the Right Shift Lemma to $X_{\tilde{T}_{\ell,\ell-1}}$, and obtain

$$X_{P_{n,\ell}} = 2(X_{F_{n,\ell-1}} + X_{P_{n+1}} - X_{P_\ell}X_{P_{n-\ell+1}}) - (X_{F_{\ell,\ell}} + X_{P_{\ell+1}} - X_{P_{\ell-1}}X_{P_2})X_{P_{n-\ell}}.$$

Since $F_{\ell,\ell}$ is $P_{\ell+1}$, we can rewrite the last equation as:

$$\begin{aligned} X_{P_{n,\ell}} &= 2(X_{F_{n,\ell-1}} + X_{P_{n+1}} - X_{P_\ell} X_{P_{n-\ell+1}}) - (2X_{P_{\ell+1}} - X_{P_{\ell-1}} X_{P_2}) X_{P_{n-\ell}} \\ &= 2X_{F_{n,\ell-1}} + [2X_{P_{n+1}} - 2X_{P_\ell} X_{P_{n-\ell+1}} + X_{P_{\ell-1}} X_{P_2} X_{P_{n-\ell}} - 2X_{P_{\ell+1}} X_{P_{n-\ell}}]. \end{aligned}$$

Substituting in (4.14) for $X_{F_{n,\ell-1}}$, we have:

$$\begin{aligned} X_{P_{n,\ell}} &= 2(X_{P_{n+1}} + X_{P_1} X_{P_n} - X_{P_{\ell-1}} X_{P_{n-\ell}}) \\ &\quad + [2X_{P_{n+1}} - 2X_{P_\ell} X_{P_{n-\ell+1}} + X_{P_{\ell-1}} X_{P_2} X_{P_{n-\ell}} - 2X_{P_{\ell+1}} X_{P_{n-\ell}}]. \end{aligned}$$

Finally, the formula in the statement follows by collecting all the terms and evaluating $X_{P_1} = e_1$ and $X_{P_2} = 2e_2$. $\square$

Now we investigate the generating function of $X_{P_{n,\ell}}$. For this, we introduce two families of polynomials in the variable z with coefficients in the ring of symmetric functions.

Definition 4.2.20. 1. For $\ell \geq 2$, we define the following polynomial of degree $\ell + 1$ in z :

$$f_\ell(z) := 2 + e_1 z - X_{P_{\ell-1}} z^{\ell-1} (1 - e_2 z^2) - X_{P_\ell} z^\ell - X_{P_{\ell+1}} z^{\ell+1}.$$

2. For $\ell \geq 2$, we define the following polynomial of degree $\ell + 1$:

$$g_\ell(z) := - \sum_{j=0}^{\ell} X_{P_j} z^j - (1 + e_1 z) \sum_{j=0}^{\ell-2} X_{P_j} z^j - (X_{P_{\ell+1}} - e_2 X_{P_{\ell-1}}) z^{\ell+1}.$$

The following result gives an identity for the generating function for the chromatic symmetric function of the twinned path in terms of the generating function for the chromatic symmetric function of the path and the new families of polynomials introduced.

Proposition 4.2.21. *Let $2 \leq \ell \leq n - 1$. The generating function for the chromatic symmetric function of the twinned path $P_{n,\ell}$, twinned at vertex ℓ , can be written in terms of the path generating function X_P as follows:*

$$\sum_{n \geq \ell+1} X_{P_{n,\ell}} z^{n+1} = 2X_P(z) f_\ell(z) + 2g_\ell(z). \quad (4.15)$$

Proof. We reorder the terms appearing in the recurrence in Proposition 4.2.19 to make the source of the factor $f_\ell(z)$ accompanying X_P more transparent:

$$X_{P_{n,\ell}} = 4X_{P_{n+1}} + 2e_1X_{P_n} - 2X_{P_{\ell-1}}(X_{P_{n-\ell+2}} - e_2X_{P_{n-\ell}}) - 2X_{P_\ell}X_{P_{n-\ell+1}} - 2X_{P_{\ell+1}}X_{P_{n-\ell}}. \quad (4.16)$$

Multiplying by z^{n+1} and summing over $n \geq \ell + 1$ gives

$$\begin{aligned} & \sum_{n \geq \ell+1} X_{P_{n,\ell}} z^{n+1} \\ &= 2[2 + e_1z - X_{P_{\ell-1}}z^{\ell-1}(1 - e_2z^2) - X_{P_\ell}z^\ell - z^{\ell+1}X_{P_{\ell+1}}]X_P \\ & - 4 \sum_{j=0}^{\ell} X_{P_j}z^j - 2ze_1 \sum_{j=0}^{\ell-1} X_{P_j}z^j + 2z^{\ell-1}(1 + ze_1)X_{P_{\ell-1}} + 2X_{P_\ell}z^\ell - 2(X_{P_{\ell+1}} - e_2X_{P_{\ell-1}})z^{\ell+1}. \end{aligned}$$

Here we have made the substitutions $\sum_{j=0}^2 X_{P_j}z^j = 1 + e_1z + 2e_2z^2$, $\sum_{j=0}^1 X_{P_j}z^j = 1 + e_1z$ and $X_{P_0} = 1$.

The expression for $f_\ell(z)$ follows immediately from the first line above.

Now rewrite the second line as

$$\begin{aligned} & -4X_{P_\ell}z^\ell - 4X_{P_{\ell-1}}z^{\ell-1} - 4 \sum_{j=0}^{\ell-2} X_{P_j}z^j - 2z^\ell e_1X_{P_{\ell-1}} - 2ze_1 \sum_{j=0}^{\ell-2} X_{P_j}z^j \\ & + 2X_{P_{\ell-1}}z^{\ell-1} + 2e_1X_{P_{\ell-1}}z^\ell + 2X_{P_\ell}z^\ell - 2(X_{P_{\ell+1}} - e_2X_{P_{\ell-1}})z^{\ell+1}, \end{aligned}$$

which in turn yields the expression for $g_\ell(z)$ in Definition 4.2.20. $\square$

Although $f_\ell(z)$ is not e -positive, we can conclude the following.

Corollary 4.2.22. *The e -positivity of $X_{P_{n,\ell}}$ is equivalent to the e -positivity of $X_P(z)f_\ell(z)$.*

Proof. The degree of $g_\ell(z)$ as a polynomial in z is $\ell + 1$, while the left-hand side of (4.15) has lowest degree $\ell + 2$ in z . We conclude that all terms in $g_\ell(z)$ are necessarily canceled out by identical terms in $X_P(z)f_\ell(z)$. $\square$

Our next result rewrites $f_\ell(z)$ as a positive expansion of other functions that were introduced in (4.4).

Lemma 4.2.23. For $\ell \geq 2$, we have

$$f_\ell(z) = \sum_{i=3}^{\ell+1} (i-2)e_i z^i + 2(D + G_{\geq \ell+2}) + \sum_{i=1}^{\ell-2} (D + G_{\geq \ell+2-i})X_{P_i} z^i. \quad (4.17)$$

Proof. By Definition 4.2.20,

$$f_\ell(z) = 2 + e_1 z + X_{P_{\ell-1}} e_2 z^{\ell+1} - X_{P_{\ell-1}} z^{\ell-1} - X_{P_\ell} z^\ell - X_{P_{\ell+1}} z^{\ell+1}.$$

For $\ell = 2$, recall that $X_{P_3} = e_2 e_1 + 3e_3$, and $X_{P_1} = e_1$ and $X_{P_2} = 2e_2$. Then we have

$$\begin{aligned} f_2(z) &= 2 - 2e_2 z^2 - 3e_3 z^3 = 2(1 - e_2 z^2 - 2e_3 z^3) + e_3 z^3 \\ &= 2(1 - G(z) + G_{\geq 4}(z)) + e_3 z^3 = 2(D(z) + G_{\geq 4}(z)) + e_3 z^3. \end{aligned}$$

Let φ_ℓ denote the right-hand side of (4.17). We show that φ_ℓ and f_ℓ satisfy the same recurrence relation. It is straightforward to see that for $\ell \geq 2$,

$$f_{\ell+1} - f_\ell = X_{P_\ell} e_2 z^{\ell+2} - X_{P_{\ell+2}} z^{\ell+2} - X_{P_{\ell-1}} e_2 z^{\ell+1} + X_{P_{\ell-1}} z^{\ell-1}. \quad (4.18)$$

Next we look at $\varphi_{\ell+1} - \varphi_\ell$. Observe from (4.4) that $G_{\geq m+1} - G_{\geq m} = -(m-1)e_m z^m$. We therefore have

$$\begin{aligned} \varphi_{\ell+1} &= \sum_{i=3}^{\ell+2} (i-2)e_i z^i + 2(D + G_{\geq \ell+3}) + \sum_{i=1}^{\ell-1} (D + G_{\ell+3-i})X_{P_i} z^i \\ \varphi_\ell &= \sum_{i=3}^{\ell+1} (i-2)e_i z^i + 2(D + G_{\geq \ell+2}) + \sum_{i=1}^{\ell-2} (D + G_{\ell+2-i})X_{P_i} z^i \end{aligned}$$

and hence we obtain, for $\ell \geq 2$,

$$\varphi_{\ell+1} - \varphi_\ell = \ell e_{\ell+2} z^{\ell+2} - 2(\ell+1)e_{\ell+2} z^{\ell+2} - \sum_{i=1}^{\ell-2} (\ell+1-i)e_{\ell+2-i} z^{\ell+2-i} X_{P_i} z^i + (D + G_{\geq 4})X_{P_{\ell-1}} z^{\ell-1}.$$

The path recurrence relation in Proposition 4.2.21 tells us that

$$X_{P_{\ell+2}} = (\ell+2)e_{\ell+2} + \sum_{j=1}^{\ell-2} (\ell+1-j)e_{\ell+2-j} X_{P_j} + 2e_3 X_{P_{\ell-1}} + e_2 X_{P_\ell}.$$

Together with $D + G_{\geq 4} = 1 - G_{\leq 3} = 1 - e_2 z^2 - 2e_3 z^3$, this gives

$$\varphi_{\ell+1} - \varphi_\ell = (-X_{P_{\ell+2}} + 2e_3 X_{P_{\ell-1}} + e_2 X_{P_\ell}) z^{\ell+2} + (1 - e_2 z^2 - 2e_3 z^3) X_{P_{\ell-1}} z^{\ell-1}.$$

The terms containing $e_3 X_{P_{\ell-1}}$ cancel, and the remaining expression coincides with the one for $f_{\ell+1} - f_\ell$ in (4.18). Hence φ_ℓ and f_ℓ satisfy the same recurrence relation. Since their initial values also coincide, the claim follows. $\square$

The preceding efforts culminate in the following e -positivity result, as announced at the start of this section.

Theorem 4.2.24. *For $\ell \geq 2$, we have the e -positive expansion*

$$X_P f_\ell = X_P \sum_{i=3}^{\ell+1} (i-2) e_i z^i + 2(E + X_P G_{\geq \ell+2}) + \sum_{i=1}^{\ell-2} (E + X_P G_{\geq \ell+2-i}) X_{P_i} z^i. \quad (4.19)$$

Hence the generating function $\sum_{n \geq \ell+1} X_{P_{n,\ell}} z^{n+1}$ is e -positive.

Proof. The expression for f_ℓ in Lemma 4.2.23 immediately allows us to conclude (4.19), using the fact that $X_P(z)D(z) = E(z)$. It then suffices to observe that the generating function $\sum_{n \geq \ell+1} X_{P_{n,\ell}} z^{n+1}$ is comprised precisely of all the terms of degree $\geq \ell+2$ in the e -positive rational expression

$$\begin{aligned} 2X_P f_\ell &= \frac{2E f_\ell}{D} \\ &= \frac{2E \left(\sum_{i=3}^{\ell+1} (i-2) e_i z^i + G_{\geq \ell+2} + \sum_{i=0}^{\ell-2} G_{\geq \ell+2-i} X_{P_i} z^i \right)}{1 - \sum_{i \geq 2} (i-1) e_i z^i} + 2(1+E) \sum_{i=0}^{\ell-2} X_{P_i} z^i. \quad \square \end{aligned}$$

From Theorem 4.2.24 and a tedious computation of $X_P f_\ell + g_\ell$, we obtain the following cancellation-free e -positive expression for $\frac{1}{2} \sum_{n \geq \ell+1} X_{P_{n,\ell}} z^{n+1}$. (Note that the sum is zero if the range of summation is empty.)

Proposition 4.2.25. *For integers $n \geq 3$ and $2 \leq \ell \leq n-1$, the twin $P_{n,\ell}$ of the path P_n at the degree 2 vertex ℓ is e -positive. In particular, we have*

$$\begin{aligned} \frac{1}{2} \sum_{n \geq \ell+1} X_{P_{n,\ell}} z^{n+1} &= \ell e_{\ell+1} z^{\ell+1} \left(\sum_{i=1}^{\ell-2} X_{P_i} z^i \right) + \sum_{i=3}^{\ell} (i-1) e_i z^i \left(\sum_{j=0}^{i-4} X_{P_{\ell-2-j}} z^{\ell-2-j} \right) + E_{\geq \ell+2} \\ &\quad + E_{\geq \ell+2} \sum_{i=0}^{\ell-2} X_{P_i} z^i + \left(\sum_{i \geq \ell-1} X_{P_i} z^i \right) \left(\sum_{i=2}^{\ell+1} (i-2) e_i z^i \right) \\ &\quad + 2X_P G_{\geq \ell+2} + X_P \sum_{i=1}^{\ell-2} G_{\geq \ell+2-i} X_{P_i} z^i. \end{aligned}$$

4.2.3 Generating Function for Twinned Cycles

In this section, we establish a new result, the e -positivity of the chromatic symmetric function of the twinned cycle. Again, our goal of obtaining an e -positive generating function for the twinned cycle will begin with a formula for the chromatic symmetric function of the twinned cycle, which is derived using the triple deletion formula.

We start by introducing two more families of graphs. Consider the twinned cycle graph $C_{n,v}$ where v' is the twinned vertex of v and u and w are the adjacent vertices to v and v' . Let D_{n+1} be the graph obtained from $C_{n,v}$ by removing the edge uv and let Tad_{n+1} be the graph obtained from $C_{n,v}$ by removing the edges uv and vv' . We illustrate these two definitions in Figure 4.7.

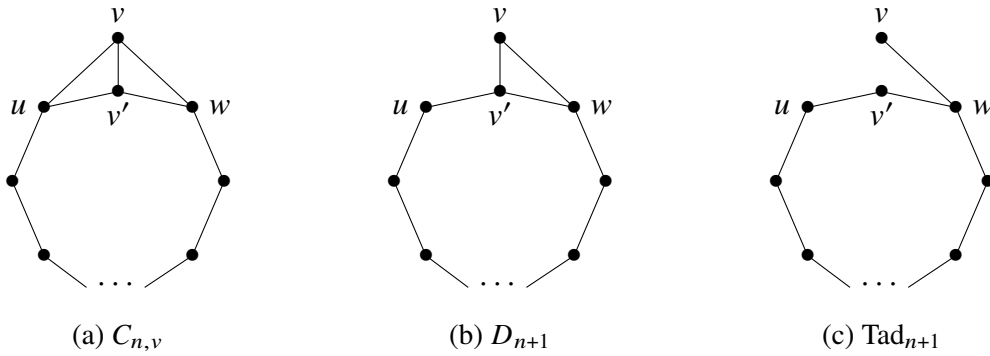


Figure 4.7 $C_{n,v}$, D_{n+1} , and Tad_{n+1}

Lemma 4.2.26. For $n \geq 3$:

$$X_{C_{n,v}} = 4X_{C_{n+1}} + 2e_1X_{C_n} - 6X_{P_{n+1}} + 2e_2X_{P_{n-1}}.$$

Proof. Consider $n \geq 3$. By the triple deletion argument applied to $\epsilon_1 = uv$ and $\epsilon_2 = uv'$, we get that

$$X_{C_{n,v}} = 2X_{D_{n+1}} - X_{P_{n,v}} = 2X_{D_{n+1}} - 2X_{P_{n+1}} + X_{P_2}X_{P_{n-1}}. \quad (4.20)$$

In D_{n+1} , applying the triple deletion argument to $\epsilon_1 = vw$ and $\epsilon_2 = vv'$ gives

$$X_{D_{n+1}} = 2X_{\text{Tad}_{n+1}} - e_1X_{C_n}, \quad (4.21)$$

while applying the triple deletion argument to $\epsilon_1 = vw$ and $\epsilon_2 = v'w$ gives

$$X_{D_{n+1}} = X_{\text{Tad}_{n+1}} + X_{C_{n+1}} - X_{P_{n+1}}.$$

Subtracting both expressions for $X_{D_{n+1}}$ we obtain that

$$X_{\text{Ta}d_{n+1}} = X_{C_{n+1}} + e_1 X_{C_n} - X_{P_{n+1}},$$

and therefore

$$X_{D_{n+1}} = 2X_{C_{n+1}} + e_1 X_{C_n} - 2X_{P_{n+1}}. \quad (4.22)$$

Finally, putting together (4.20) and (4.22), we have

$$X_{C_{n,v}} = 4X_{C_{n+1}} + 2e_1 X_{C_n} - 6X_{P_{n+1}} + 2e_2 X_{P_{n-1}},$$

as claimed. $\square$

Let $\mathcal{X}_{C_v}$ be the generating function for the twinned cycle, that is, $\mathcal{X}_{C_v}(z) := \sum_{n \geq 3} X_{C_{n,v}} z^{n+1}$. By Lemma 4.2.26 we have the following expression for $\mathcal{X}_{C_v}$.

Corollary 4.2.27. *The generating function of the twinned cycle can be written as*

$$\mathcal{X}_{C_v}(z) = 2(2 + e_1 z) \mathcal{X}_C - 2(3 - e_2 z^2) \mathcal{X}_P + 6(1 + e_1 z) + 2e_2 z^2 - 6e_3 z^3.$$

Proof. The generating function follows by multiplying the formula in Lemma 4.2.26 by z^{n+1} and summing over all $n \geq 3$. In particular, taking into account the initial terms that do not appear and using the initial values $X_{C_1} = 0$, $X_{P_1} = e_1$, $X_{C_2} = 2e_2 = X_{P_2}$, $X_{C_3} = 6e_3$, and $X_{P_3} = e_2 e_1 + 3e_3$, we get the following expressions in terms of the generating functions for the cycle and the path:

1. $4 \sum_{n \geq 3} X_{C_{n+1}} z^{n+1} = 4(\mathcal{X}_C - z^2 \mathcal{X}_{C_2} - z^3 \mathcal{X}_{C_3}),$
2. $2(e_1 z) \sum_{n \geq 3} X_{C_n} z^n = 2e_1 z(\mathcal{X}_C - z^2 \mathcal{X}_{C_2}),$
3. $6 \sum_{n \geq 3} X_{P_{n+1}} z^{n+1} = 6(\mathcal{X}_P - 1 - z \mathcal{X}_{P_1} - z^2 \mathcal{X}_{P_2} - z^3 \mathcal{X}_{P_3}),$ and
4. $2e_2 z^2 \sum_{n \geq 3} X_{P_{n-1}} z^{n-1} = 2e_2 z^2(\mathcal{X}_P - 1 - z \mathcal{X}_{P_1}).$

Putting all this together gives the generating function as stated. $\square$

Consider the following e -positive generating functions that appear in the proof of Lemma 4.2.3:

$$F_2 = \sum_{i \geq 3} (2i^2 - 5i)e_i z^i \quad \text{and} \quad F_3 = \sum_{i \geq 4} [(i-1)(i-3)]e_i z^i.$$

Our goal is to show that the expression given in Corollary 4.2.27 is indeed e -positive.

Lemma 4.2.28. *The twinned cycle generating function, scaled by $\frac{1}{2}$, can be written as*

$$\frac{1}{2}\mathcal{X}_{C_v} = \frac{1}{D(z)} [F_2 + e_1 z F_3 + e_2 z^2 (E - 1 - e_1 z)] - \frac{e_2 z^2}{D(z)} + e_2 z^2 - 3e_3 z^3.$$

Proof. By Corollary 4.2.27,

$$\frac{1}{2}\mathcal{X}_{C_v} = (2 + e_1 z)\mathcal{X}_C - 3(\mathcal{X}_P - 1 - e_1 z) + e_2 z^2(\mathcal{X}_P + 1) - 3e_3 z^3. \quad (4.23)$$

From the proof of Lemma 4.2.3, we have the following

$$\mathcal{X}_P - (1 + e_1 z) = \frac{z(1 + e_1 z)E'(z) - e_1 z E(z)}{D(z)} \quad \text{and} \quad \mathcal{X}_C = \frac{z^2 E''(z)}{D(z)},$$

and by definition $\mathcal{X}_P(z) = \frac{E(z)}{D(z)}$ and $\mathcal{X}_C(z) = \frac{z^2 E''(z)}{D(z)}$. Substituting these into (4.23), we get

$$\begin{aligned} \frac{1}{2}\mathcal{X}_{C_v} &= \frac{1}{D(z)} [(2 + e_1 z)z^2 E''(z) - 3(zE'(z) + z^2 e_1 E'(z) - e_1 z E(z)) + e_2 z^2 E(z)] + e_2 z^2 - 3e_3 z^3 \\ &= \frac{1}{D(z)} [(2z^2 E'' - 3zE') + e_1 z(z^2 E'' - 3e_1 z E' + 3E) + e_2 z^2 E(z)] + e_2 z^2 - 3e_3 z^3 \\ &= \frac{1}{D(z)} [F_2 - 3e_1 z - 2e_2 z^2 + e_1 z(F_3 - e_2 z^2 + 3) + e_2 z^2 E(z)] + e_2 z^2 - 3e_3 z^3, \end{aligned}$$

where the final equality comes from Lemma 4.2.3. The statement then follows after further algebraic manipulations. $\square$

Theorem 4.2.29. *The generating function for $\frac{1}{2}X_{C_{n,v}}$ has the following e -positive rational expression:*

$$\frac{1}{2} \sum_{n \geq 3} X_{C_{n,v}} z^{n+1} = \sum_{i \geq 4} (2i^2 - 5i)e_i z^i + \frac{e_1 z F_3 + F_2 \sum_{i \geq 3} (i-1)e_i z^i + e_2 z^2 \sum_{i \geq 3} (2i^2 - 6i + 2)e_i z^i}{1 - \sum_{i \geq 2} (i-1)e_i z^i}.$$

Proof. Let $E_{\geq 2} := E - 1 - e_1 z = \sum_{i \geq 2} e_i z^i$. Then by Lemma 4.2.28, we have

$$\begin{aligned} \frac{1}{2}\mathcal{X}_{C_v} &= \frac{1}{D(z)} \left(F_2 + e_1 z F_3 + e_2 z^2 E_{\geq 2} \right) - \frac{e_2 z^2}{D(z)} + e_2 z^2 - 3e_3 z^3 \\ &= F_2 - 3e_3 z^3 + \frac{e_1 z F_3}{D(z)} + \left[F_2 \left(\frac{1}{D(z)} - 1 \right) + \frac{e_2 z^2 E_{\geq 2}}{D(z)} - \frac{e_2 z^2}{D(z)} + e_2 z^2 \right]. \end{aligned} \quad (4.24)$$

The first two terms in (4.24) can be written as:

$$F_2 - 3e_3z^3 = \sum_{i \geq 4} (2i^2 - 5i)e_iz^i. \quad (4.25)$$

For the generating function in the brackets of (4.24), we have

$$\begin{aligned} & F_2 \left(\frac{1}{D(z)} - 1 \right) + \frac{e_2z^2E_{\geq 2}}{D(z)} - \frac{e_2z^2}{D(z)} + e_2z^2 \\ &= F_2 \sum_{k \geq 1} G^k + e_2z^2E_{\geq 2} \sum_{k \geq 0} G^k - e_2z^2 \sum_{k \geq 1} G^k \\ &= F_2 \sum_{k \geq 1} G^k + e_2z^2E_{\geq 2} \sum_{k \geq 1} G^{k-1} - e_2z^2 \sum_{k \geq 1} G^k \\ &= \sum_{k \geq 1} G^{k-1} \left(GF_2 - e_2z^2(G - E_{\geq 2}) \right) \\ &= \sum_{k \geq 1} G^{k-1} \left(GF_2 - e_2z^2F_2 + e_2z^2F_2 - e_2z^2 \sum_{i \geq 3} (i-2)e_iz^i \right) \\ &= \sum_{k \geq 1} G^{k-1} \left(F_2G_{\geq 3} + e_2z^2 \sum_{i \geq 3} (2i^2 - 6i + 2)e_iz^i \right) \\ &= \frac{F_2G_{\geq 3} + e_2z^2 \sum_{i \geq 3} (2i^2 - 6i + 2)e_iz^i}{D(z)}, \end{aligned} \quad (4.26)$$

where the penultimate equality follows from the definitions of $G_{\geq 3}$ and F_2 . Combining (4.24), (4.25), and (4.26), we obtain the desired expression. $\square$

From the generating function in Theorem 4.2.29, we can readily extract the e -coefficients of $X_{C_{n,v}}$.

Corollary 4.2.30. *Let λ be a partition of $k \geq 3$, $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots, k^{m_k} \rangle$, and let c_λ be the coefficient of $e_\lambda z^{|\lambda|}$ in $\frac{1}{2}X_{C_v}$. We have the following list of expressions for the coefficients:*

1. $c_{(k)} = k(2k - 5)$.
2. If $m_1 > 1$, then $c_\lambda = 0$.
3. If $m_1 = 1$ and $\lambda = \mu \cup 1$ (so that $m_1(\mu) = 0$), then

$$c_\lambda = \sum_{\substack{i \geq 4 \\ i \in \text{supp}(\mu)}} (i-1)(i-3)\varepsilon(\mu-i).$$

Note that this is 0 unless λ has a part of size at least 4.

4. If $m_1 = m_2 = 0$, then

$$c_\lambda = \sum_{a \in \text{supp}(\lambda)} (2a^2 - 5a) \varepsilon(\lambda - a).$$

5. If $m_1 = 0$ and $m_2 = \ell(\lambda)$, then $c_\lambda = 0$.

6. If $m_1 = 0$ and $1 \leq m_2 < \ell(\lambda)$, then

$$c_\lambda = \sum_{\substack{a, b \geq 3 \\ (a, b) \in \text{supp}(\lambda)}} \varepsilon(\lambda - a - b)(2a^2 - 5a)(b - 1) + \sum_{\substack{c \geq 3 \\ c \in \text{supp}(\lambda)}} \varepsilon(\lambda - c - 2)(2c^2 - 6c + 2)$$

where $(a, b) \in \text{supp}(\lambda)$ means both a and b are in $\text{supp}(\lambda)$ if $a \neq b$, and $m_a \geq 2$ if $a = b$.

4.3 e -positivity via Recurrences

In this section, we reprove several e -positivity results for certain classes of graphs by exhibiting an e -positive recurrence relation. The recurrence relations for paths and cycles from Proposition 4.2.2 serve as the model for those of this section, and in fact will play a key role in our derivations. We will also need explicit expressions for some coefficients, which are readily extracted from Proposition 4.2.2. These are recorded in the next result.

Corollary 4.3.1. *Given a graph G , let $[e_\lambda]X_G$ denote the coefficient of e_λ in the chromatic symmetric function of G , X_G .*

- For $n \geq 2$, $[e_n]X_{P_n} = n$, $[e_{n-1}e_1]X_{P_n} = n - 2$, and $[e_n]X_{C_n} = n(n - 1)$.
- For $n \geq 5$, $[e_{n-2}e_2]X_{P_n} = 3n - 8$ and $[e_{n-2}e_2]X_{C_n} = n(n - 3)$.
- For $k \geq 2$ and $r \geq 1$, $[(e_k)^r]X_{P_{kr}} = k(k - 1)^{r-1}$ and $[e_{(kr)}]X_{C_{kr}} = k(k - 1)^r$.
- $[e_2^2]X_{C_4} = 2$.

The rest of this section follows the structure of Section 4.2.

4.3.1 Recurrences for Twinned Paths

In this section we derive recurrence formulas for the chromatic symmetric function for a path twinned at one or both leaves or at an internal vertex.

4.3.1.1 Paths Twinned at a Leaf

In the next proposition, we give formulas for the chromatic symmetric function $X_{P_{n,v}}$ of the path twinned at a leaf, one in terms of the path chromatic symmetric function, and the other a recurrence in the spirit of Proposition 4.2.2. The recurrence given below makes the e -positivity transparent.

Proposition 4.3.2. *Let v be a leaf of the path P_n . Then, for $n \geq 4$,*

$$X_{P_{n,v}} = 2(n+1)e_{n+1} + 2 \sum_{j=3}^n (j-1)e_j X_{P_{n+1-j}}.$$

Thus $X_{P_{n,v}}$ is e -positive. Moreover, for $n \geq 4$, $X_{P_{n,v}}$ satisfies the e -positive recurrence

$$X_{P_{n,v}} = \sum_{j=2}^{n-2} (j-1)e_j X_{P_{n-j,v}} + 2(n+1)e_{n+1} + 2(n-1)e_n e_1 + 2(n-3)e_{n-1} e_2,$$

with initial values $X_{P_{1,v}} = 2e_2$, $X_{P_{2,v}} = 6e_3$, and $X_{P_{3,v}} = 8e_4 + 4e_3 e_1$.

Proof. The first expression follows from Proposition 4.2.21 and Proposition 4.2.8, noting that

$$\begin{aligned} X_{P_{n,v}} &= 2X_{P_{n+1}} - X_{P_2} X_{P_{n-1}} = 2(n+1)e_{n+1} + 2 \sum_{j=2}^n (j-1)e_j X_{P_{n+1-j}} - 2e_2 X_{P_{n-1}} \\ &= 2(n+1)e_{n+1} + 2 \sum_{j=3}^n (j-1)e_j X_{P_{n+1-j}} + 2e_2 X_{P_{n-1}} - 2e_2 X_{P_{n-1}}. \end{aligned}$$

For the second recurrence, we apply the triple deletion argument to $P_{n,v}$ followed by Proposi-

tion 4.2.21 to both terms. Thus,

$$\begin{aligned}
X_{P_{n,v}} &= 2X_{P_{n+1}} - X_{P_2}X_{P_{n-1}} = 2X_{P_{n+1}} - 2e_2X_{P_{n-1}} \\
&= 2(n+1)e_{n+1} + 2 \sum_{j=2}^n (j-1)e_j X_{P_{n+1-j}} - 2e_2 \left[(n-1)e_{n-1} + \sum_{j=2}^{n-2} (j-1)e_j X_{P_{n-1-j}} \right] \\
&= 2(n+1)e_{n+1} - 2(n-1)e_{n-1}e_2 + 2 \sum_{j=2}^{n-2} (j-1)e_j [2X_{P_{n+1-j}} - 2e_2X_{P_{n-1-j}}] \\
&\quad + 2(n-2)e_{n-1}X_{P_2} + 2(n-1)e_nX_{P_1} \\
&= 2(n+1)e_{n+1} + 2 \sum_{j=2}^{n-2} (j-1)e_j X_{P_{n-j,v}} + 2(n-1)e_ne_1 + 2(n-3)e_{n-1}e_2,
\end{aligned}$$

where the final step follows by the triple deletion argument applied to $P_{n-j,v}$. As this is an e -positive recursion with e -positive initial conditions, by induction it follows that $X_{P_{n,v}}$ is e -positive for all n . $\square$

4.3.1.2 Paths Twinned at Both Leaves

Our new contribution is the e -positive recurrence below.

Proposition 4.3.3. *For $n \geq 6$, the chromatic symmetric function $X_{P_{n,v,w}}$ for the path P_n twinned at both leaves v, w satisfies the recurrence*

$$\begin{aligned}
\frac{1}{4}X_{P_{n,v,w}} &= \frac{1}{4} \sum_{j=3}^{n-3} (j-1)e_j X_{P_{n-j,v,w}} \\
&\quad + (n+2)e_{n+2} + ne_{n+1}e_1 + 3(n-2)e_{n-1}e_3 + 2(n-3)e_{n-2}e_3e_1 + 4(n-3)e_{n-2}e_4 \\
&\quad + e_2 \left[\frac{1}{4}X_{P_{n-2,v,w}} - 2e_n - (n-4)e_{n-2}e_2 - (n-2)e_{n-1}e_1 \right],
\end{aligned}$$

with the initial conditions

$$\begin{aligned}
X_{P_{2,v,w}} &= 24e_4, & X_{P_{4,v,w}} &= 24e_3^2 + 8e_4e_2 + 16e_5e_1 + 24e_6, \\
X_{P_{3,v,w}} &= 4e_3e_2 + 12e_4e_1 + 20e_5, & X_{P_{5,v,w}} &= 16e_3e_3e_1 + 68e_4e_3 + 12e_5e_2 + 20e_6e_1 + 28e_7.
\end{aligned}$$

Moreover, despite the negative terms, the expression is e -positive.

Proof. By the triple deletion argument, we have that

$$X_{P_{n,v,w}} = 2X_{P_{n+1,v}} - 2e_2X_{P_{n-1,v}}.$$

Applying the triple deletion argument again to both twinned terms, we have that $X_{P_{n+1,v}} = 2X_{P_{n+2}} - 2e_2X_{P_n}$ and $X_{P_{n-1,v}} = 2X_{P_n} - 2e_2X_{P_{n-2}}$. Thus, for $n \geq 3$,

$$\frac{1}{4}X_{P_{n,v,w}} = X_{P_{n+2}} + e_2^2X_{P_{n-2}} - 2e_2X_{P_n}. \quad (4.27)$$

Now we prove the recurrence relation by strong induction on n . The initial conditions are checked directly. For $n \geq 6$, using repeatedly (4.27) and Proposition 4.2.21, we write

$$\begin{aligned} \frac{1}{4}X_{P_{n,v,w}} &= X_{P_{n+2}} - 2e_2X_{P_n} + e_2^2X_{P_{n-2}} \\ &= (n+2)e_{n+2} + \sum_{j=2}^{n+1} (j-1)e_jX_{P_{n+2-j}} + (n-2)e_{n-2}e_2^2 + e_2^2 \sum_{j=2}^{n-3} (j-1)e_jX_{P_{n-2-j}} \\ &\quad - 2ne_ne_2 - 2e_2 \sum_{j=2}^{n-1} (j-1)e_jX_{P_{n-j}} \\ &= \frac{1}{4} \sum_{j=2}^{n-3} (j-1)e_jX_{P_{n-j,v,w}} \\ &\quad + (n+2)e_{n+2} + ne_{n+1}e_1 + 3(n-2)e_{n-1}e_3 + 2(n-3)e_{n-2}e_3e_1 + 4(n-3)e_{n-2}e_4 \\ &\quad - 2e_ne_2 - (n-4)e_{n-2}e_2^2 - (n-2)e_{n-1}e_2e_1. \end{aligned}$$

To rearrange this into the announced form, we peel off the $j = 2$ term from the sum and group it with the negative terms:

$$\begin{aligned} \frac{1}{4}X_{P_{n,v,w}} &= \frac{1}{4} \sum_{j=3}^{n-3} (j-1)e_jX_{P_{n-j,v,w}} \\ &\quad + (n+2)e_{n+2} + ne_{n+1}e_1 + 3(n-2)e_{n-1}e_3 + 2(n-3)e_{n-2}e_3e_1 + 4(n-3)e_{n-2}e_4 \\ &\quad + e_2 \left[\frac{1}{4}X_{P_{n-2,v,w}} - 2e_n - (n-4)e_{n-2}e_2 - (n-2)e_{n-1}e_1 \right]. \end{aligned}$$

Next we prove that the term within brackets

$$\left[\frac{1}{4}X_{P_{n-2,v,w}} - 2e_n - (n-4)e_{n-2}e_2 - (n-2)e_{n-1}e_1 \right] \quad (4.28)$$

is e -positive. This follows by comparing the coefficients of the e -functions involved. Consider (4.27) applied to $\frac{1}{4}X_{P_{n-2,v,w}}$, and use the coefficients described in Corollary 4.3.1. We obtain the following formulas for the coefficients:

$$[e_n]\frac{1}{4}X_{P_{n-2,v,w}} = n, \quad [e_{n-1}e_1]\frac{1}{4}X_{P_{n-2,v,w}} = n-2, \quad \text{and} \quad [e_{n-2}e_2]\frac{1}{4}X_{P_{n-2,v,w}} = n-4.$$

In particular, notice that the coefficients of e_n and $e_{n-1}e_1$ are zero and that the coefficient of e_ne_2 is $n-2$, which is positive for $n \geq 6$. Thus, the negative terms appearing in (4.28) are absorbed by terms in $\frac{1}{4}X_{P_{n-2,v,w}}$, and (4.28) is e -positive.

Finally, as this is an e -positive recurrence with e -positive initial conditions, by induction it follows that $X_{P_{n,v,w}}$ is e -positive for all n . $\square$

4.3.1.3 Paths Twinned at an Interior Vertex

Next we provide an e -positive recurrence relation for path graphs twinned at an interior vertex.

Theorem 4.3.4. *The chromatic symmetric function $X_{P_{n,\ell}}$ for the path P_n twinned at the interior vertex ℓ satisfies the e -positive recurrence for $\ell \geq 2$, $n \geq \ell + 1$, and $n \geq 4$,*

$$\begin{aligned} X_{P_{n,\ell}} = & \sum_{j=2}^{n-\ell-1} (j-1)e_j X_{P_{n-j,\ell}} + 4(n+1)e_{n+1} + 2ne_1e_n + 2e_1 \sum_{j=n-\ell+2}^{n-1} (j-1)e_j X_{P_{n-j}} \\ & + 4 \sum_{j=n-\ell+3}^n (j-1)e_j X_{P_{n+1-j}} + 2 \sum_{j=n-\ell+1}^{n-\ell+2} (j-2)e_j X_{P_{n+1-j}} + (n-\ell-2)e_{n-\ell} X_{P_{\ell,\ell}}. \end{aligned}$$

Thus, for $n \geq 3$ and $2 \leq \ell \leq n-1$, $X_{P_{n,\ell}}$ is e -positive.

Proof. Fix $\ell \geq 2$, and consider $n \geq \ell + 1$ with $n \geq 4$. We start with the recurrence relation in the statement of Proposition 4.2.19. This can be rewritten as

$$\begin{aligned} X_{P_{n,\ell}} = & 4X_{P_{n+1}} + 2e_1X_{P_n} + 2e_2X_{P_{\ell-1}}X_{P_{n-\ell}} \\ & - 2X_{P_{\ell+1}}X_{P_{n-\ell}} - 2X_{P_{\ell}}X_{P_{n-\ell+1}} - 2X_{P_{\ell-1}}X_{P_{n-\ell+2}}. \end{aligned} \tag{4.29}$$

From (4.29) and using Proposition 4.2.21, we have that

$$\begin{aligned}
X_{P_{n,\ell}} = & 4(n+1)e_{n+1} + 4 \sum_{j=2}^n (j-1)e_j X_{P_{n+1-j}} + 2e_1 \left[n e_n + \sum_{j=2}^{n-1} (j-1)e_j X_{P_{n-j}} \right] \\
& + 2e_2 X_{P_{\ell-1}} \left[(n-\ell)e_{n-\ell} + \sum_{j=2}^{n-\ell-1} (j-1)e_j X_{P_{n-\ell-j}} \right] \\
& - 2X_{P_{\ell+1}} \left[(n-\ell)e_{n-\ell} + \sum_{j=2}^{n-\ell-1} (j-1)e_j X_{P_{n-\ell-j}} \right] \\
& - 2X_{P_\ell} \left[(n+1-\ell)e_{n+1-\ell} + \sum_{j=2}^{n-\ell} (j-1)e_j X_{P_{n+1-\ell-j}} \right] \\
& - 2X_{P_{\ell-1}} \left[(n+2-\ell)e_{n+2-\ell} + \sum_{j=2}^{n-\ell+1} (j-1)e_j X_{P_{n+2-\ell-j}} \right].
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
& - 2X_{P_{\ell-1}} \left[(n+2-\ell)e_{n+2-\ell} + \sum_{j=2}^{n-\ell+1} (j-1)e_j X_{P_{n+2-\ell-j}} \right].
\end{aligned} \tag{4.31}$$

Notice that, in the six summands above, for each fixed j , the terms attached to the factor $(j-1)e_j$, when collected together, match the six terms in the right-hand side of the recurrence (4.29) applied to $X_{P_{n-j,\ell}}$. Grouping the remaining terms into an expression Y if they have a positive sign, or Z if they have a negative sign, we obtain

$$X_{P_{n,\ell}} = \sum_{j=2}^{n-\ell-1} (j-1)e_j X_{P_{n-j,\ell}} + Y - Z.$$

The positive terms Y are given by

$$\begin{aligned}
Y = & 4(n+1)e_{n+1} + 2ne_n e_1 + 2e_2 X_{P_{\ell-1}}(n-\ell)e_{n-\ell} \\
& + 4 \sum_{j=n-\ell}^n (j-1)e_j X_{P_{n+1-j}} + 2e_1 \sum_{j=n-\ell}^{n-1} (j-1)e_j X_{P_{n-j}} \\
= & 4(n+1)e_{n+1} + 2ne_n e_1 + 4 \sum_{j=n-\ell+3}^n (j-1)e_j X_{P_{n+1-j}} + 2e_1 \sum_{j=n-\ell+2}^{n-1} (j-1)e_j X_{P_{n-j}} \\
& + 4 \underbrace{\sum_{j=n-\ell}^{n-\ell+2} (j-1)e_j X_{P_{n+1-j}} + 2e_1 \sum_{j=n-\ell}^{n-\ell+1} (j-1)e_j X_{P_{n-j}} + 2e_2 X_{P_{\ell-1}}(n-\ell)e_{n-\ell}}_{Y_1},
\end{aligned}$$

where the last line is obtained by splitting the summations. The negative terms Z are given by

$$\begin{aligned} Z = & 2X_{P_{\ell+1}}(n-\ell)e_{n-\ell} + 2X_{P_{\ell}}(n+1-\ell)e_{n+1-\ell} + 2X_{P_{\ell-1}}(n+2-\ell)e_{n+2-\ell} \\ & + 2X_{P_{\ell}}(n-\ell-1)e_{n-\ell}X_{P_1} + 2X_{P_{\ell-1}} \sum_{j=n-\ell}^{n-\ell+1} (j-1)e_jX_{P_{n+2-\ell-j}}, \end{aligned}$$

where the last two terms come from the sums in (4.30) and (4.31). We rewrite Z so that $Y - Z$ is easier to analyze.

$$\begin{aligned} Z = & 2 \sum_{j=n-\ell}^{n-\ell+2} j e_j X_{P_{n+1-j}} \\ & + 2X_{P_{\ell}}(n-\ell-1)e_{n-\ell}X_{P_1} + 2X_{P_{\ell-1}}(n-\ell)e_{n-\ell+1}X_{P_1} + 2X_{P_{\ell-1}}(n-\ell-1)e_{n-\ell}X_{P_2} \\ = & 2 \sum_{j=n-\ell}^{n-\ell+2} j e_j X_{P_{n+1-j}} + 2e_1 \sum_{j=n-\ell}^{n-\ell+1} (j-1)e_jX_{P_{n-j}} + 2X_{P_2}X_{P_{\ell-1}}(n-\ell-1)e_{n-\ell} \end{aligned}$$

Using $X_{P_2} = 2e_2$, we have

$$Y_1 - Z = 2 \sum_{j=n-\ell}^{n-\ell+2} (j-2)e_jX_{P_{n+1-j}} - 2X_{P_{\ell-1}}(n-\ell-2)e_{n-\ell}e_2,$$

because the terms with the factor $e_1 = X_{P_1}$ can be seen to vanish identically. By splitting the sum,

$Y_1 - Z$ can be rewritten as

$$Y_1 - Z = 2 \sum_{j=n-\ell+1}^{n-\ell+2} (j-2)e_jX_{P_{n+1-j}} + 2(n-\ell-2)e_{n-\ell}(X_{P_{\ell+1}} - e_2X_{P_{\ell-1}}).$$

In the last term on the right-hand side, the factor of $2(X_{P_{\ell+1}} - e_2X_{P_{\ell-1}}) = 2X_{P_{\ell+1}} - X_{P_2}X_{P_{\ell-1}} = X_{P_{\ell,\ell}}$ is precisely the chromatic symmetric function of the path P_{ℓ} twinned at a leaf. Thus, putting all of this together, we obtain the recurrence relation from the statement.

Finally, we can deduce the e -positivity. For the initial values $n = \ell + 1, \ell + 2, \ell + 3$, $X_{P_{n,\ell}}$ is e -positive by Proposition 4.2.25. We proceed by strong induction on n to show the claimed e -positivity for $X_{P_{n,\ell}}$ for $n \geq \ell + 4$. Our induction hypothesis is that $X_{P_{m,\ell}}$ is e -positive for all $m < n, m \geq \ell + 1$. We only need to look at $Y_1 - Z$ since that is the part containing negative terms. By Proposition 4.3.2 we know that $Y_1 - Z$ is e -positive. Hence, by induction the proof is complete. $\square$

As an application of the preceding recurrence, we have the following corollary.

Corollary 4.3.5 (See also [27, Theorem 7.8]). *Consider the path P_n on n vertices, with $n \geq 4$. Let $2 \leq \ell \leq n - 2$ and let v be the leaf n . Then the chromatic symmetric function of $P_{n,\ell,v}$ is e -positive.*

Proof. The triple deletion argument implies that

$$X_{P_{n,\ell,v}} = 2X_{P_{n+1,\ell}} - X_{P_2}X_{P_{n-1,\ell}} = 2(X_{P_{n+1,\ell}} - e_2X_{P_{n-1,\ell}}).$$

Examining the recurrence for $X_{P_{n,\ell}}$ in Theorem 4.3.4 one sees that, when $n - \ell - 1 \geq 2$, the initial term in the first sum in the expression for $X_{P_{n,\ell}}$ is $e_2X_{P_{n-2,\ell}}$, making $X_{P_{n,\ell}} - e_2X_{P_{n-2,\ell}}$ e -positive. Replacing n by $n + 1$ now gives e -positivity of $X_{P_{n+1,\ell}} - e_2X_{P_{n-1,\ell}}$ for $n - \ell \geq 2$. $\square$

4.3.2 Recurrence for Twinned Cycles

In this section we derive an e -positive recursive formula for the twinned cycle, analogous to those for the twinned path from the last section. We give a similar formula for another family of graphs that we call moose graphs.

4.3.2.1 Twinned Cycles

We start with the cycle graph.

Theorem 4.3.6. *The chromatic symmetric function $X_{C_{n,v}}$ for the cycle C_n twinned at a vertex v is e -positive. For $n \geq 5$, it satisfies the e -positive recurrence*

$$\begin{aligned} X_{C_{n,v}} = & \sum_{k=3}^{n-2} (k-1)e_k X_{C_{n-k,v}} + 2(n+1)(2n-3)e_{n+1} + 2(n-1)(n-3)e_n e_1 \\ & + e_2 [X_{C_{n-2,v}} - 2(n-3)e_{n-1}], \end{aligned}$$

with initial conditions

$$X_{C_{1,v}} = 2e_2, \quad X_{C_{2,v}} = 6e_3, \quad X_{C_{3,v}} = 24e_4, \quad \text{and} \quad X_{C_{4,v}} = 50e_5 + 6e_4e_1 + 4e_3e_2.$$

Proof. We proceed by induction on n . The initial cases $n \leq 4$ are verified by direct computation using Theorem 4.2.1 and Lemma 4.2.26. Note that these initial terms are all e -positive.

Assume we have shown the claim to be true for $X_{C_{m,v}}$ for all $m < n$. Now we rewrite the expression in Lemma 4.2.26 using Proposition 4.2.2 to obtain:

$$\begin{aligned}
X_{C_{n,v}} = & 4(n+1)ne_{n+1} + 4 \sum_{k=2}^{n-1} (k-1)e_k X_{C_{n+1-k}} \\
& + 2e_1 \left[n(n-1)e_n + \sum_{k=2}^{n-2} (k-1)e_k X_{C_{n-k}} \right] \\
& - 6 \left[(n+1)e_{n+1} + \sum_{k=2}^n (k-1)e_k X_{P_{n+1-k}} \right] \\
& + 2e_2 \left[(n-1)e_{n-1} + \sum_{k=2}^{n-2} (k-1)e_k X_{P_{n-1-k}} \right]. \tag{4.32}
\end{aligned}$$

Applying Lemma 4.2.26 again, we can collect the four summations above into one sum and three additional terms as follows:

$$\begin{aligned}
& \sum_{k=2}^{n-2} (k-1)e_k \left[4X_{C_{n+1-k}} + 2e_1 X_{C_{n-k}} - 6X_{P_{n+1-k}} + 2e_2 X_{P_{n-1-k}} \right] \\
& + 4(n-2)e_{n-1}X_{C_2} - 6(n-2)e_{n-1}X_{P_2} - 6(n-1)e_nX_{P_1} \\
& = \sum_{k=2}^{n-2} (k-1)e_k X_{C_{n-k,v}} + 4(n-2)e_{n-1}X_{C_2} - 6(n-2)e_{n-1}X_{P_2} - 6(n-1)e_nX_{P_1}.
\end{aligned}$$

Combining this expression with the remaining terms from (4.32), we obtain

$$\begin{aligned}
X_{C_{n,v}} = & \sum_{k=2}^{n-2} (k-1)e_k X_{C_{n-k,v}} + 2(n+1)(2n-3)e_{n+1} + 2(n-1)(n-3)e_n e_1 \\
& - 2(n-3)e_{n-1}e_2. \tag{4.33}
\end{aligned}$$

We isolate the term $k = 2$ from the summation and regroup it with the last term in (4.33), so that it becomes

$$\begin{aligned}
X_{C_{n,v}} = & \sum_{k=3}^{n-2} (k-1)e_k X_{C_{n-k,v}} + 2(n+1)(2n-3)e_{n+1} + 2(n-1)(n-3)e_n e_1 \\
& + e_2 \left[X_{C_{n-2,v}} - 2(n-3)e_{n-1} \right],
\end{aligned}$$

as stated in the theorem.

Now we want to show e -positivity. By the induction hypothesis, only the last term requires scrutiny. Using Corollary 4.3.1 and Lemma 4.2.26, the coefficient of e_{n-1} in $X_{C_{n-2},v}$ is $2(n-1)(2n-7)$. Therefore, the coefficient of e_{n-1} in $X_{C_{n-2},v} - 2(n-3)e_{n-1}$ is $2(n-1)(2n-7) - 2(n-3) = 4(n^2 - 5n + 5)$, which is nonnegative for $n \geq 4$. Thus, by the induction hypothesis, the formula in the statement for $X_{C_{n,v}}$ is indeed an e -positive recurrence for the twinned cycles for $n \geq 5$. $\square$

4.3.2.2 The Moose Graph

We define the *moose graph* A_{n+2} to be the graph on $n+2$ vertices and $n+1$ edges, obtained from the cycle graph C_n by attaching a leaf to each of the vertices v, w of an edge vw in C_n ,

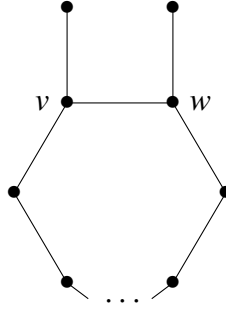


Figure 4.8 The moose graph A_{n+2}

We provide an e -positive recurrence relation for the chromatic symmetric function of the moose graph. This graph was shown to be e -positive as a special case in [63, Theorem 3.9]. We omit the proof.

Proposition 4.3.7. *For $n \geq 2$, the chromatic symmetric function of the moose graph A_{n+2} is e -positive. For $n \geq 4$, it satisfies the e -positive recurrence*

$$X_{A_{n+2}} = \left(\sum_{j=2}^{n-2} (j-1)e_j X_{A_{n+2-j}} \right) + (n+2)(n-1)e_{n+2} + 2e_1 e_{n+1}(n^2 - n - 1) + (n-1)(n-2)e_1^2 e_n + 2e_2 e_n,$$

with initial values

$$X_{A_4} = X_{P_4} = 2e_2^2 + 2e_3e_1 + 4e_4,$$

$$X_{A_5} = 2e_3e_1^2 + 2e_3e_2 + 10e_4e_1 + 10e_5,$$

$$X_{A_6} = 2e_2^3 + 2e_3e_2e_1 + 6e_4e_1^2 + 6e_4e_2 + 22e_5e_1 + 18e_6.$$

4.4 Future Directions

This work and the work of Tom [58] provide explicit e -positive generating functions for the chromatic symmetric function of twinned paths and cycles, and suggests that it may be worthwhile to undertake a similar study for twins of other graph families.

More generally, an examination of Table 4.1 shows that much of the recent literature focuses on establishing Gebhard and Sagan's (e) -positivity of the chromatic symmetric function in *noncommuting* variables. Although (e) -positivity implies e -positivity as a symmetric function in ordinary commuting variables, in such cases an explicit e -positive generating function or recurrence would be desirable. We propose the following future investigations in this direction:

1. For the twinned cycle graph, is the chromatic symmetric function in noncommuting variables (e) -positive?
2. Are there pleasing e -positive symmetric function expansions for those families whose e -positivity is known only via the stronger (e) -positivity property? Specific examples that may admit nice generating functions are the triangular ladder [13, 48], the kayak paddle graphs [2] and the tadpole graph [27, 33].

It would also be interesting to examine twinning for the *chromatic quasisymmetric function* of Shareshian and Wachs [45, 46] since the main class of posets of study for these, whose incomparability graphs are unit interval graphs, is also closed under the appropriately defined twinning operation for labeled graphs.

BIBLIOGRAPHY

- [1] A. Schrijver A. Brouwer. On the period of an operator defined on antichains. *Math Centrum Report ZW 24/74*, 1974.
- [2] F. Aliniaiefard, V. Wang, and S. van Willigenburg. The chromatic symmetric function of a graph centred at a vertex. *arXiv:2108.04850*, page preprint, 2021.
- [3] Kenneth Appel and Wolfgang Haken. The solution of the four-color-map problem. *Scientific American*, 237:108, 1977.
- [4] DREW ARMSTRONG, CHRISTIAN STUMP, and HUGH THOMAS. A uniform bijection between nonnesting and noncrossing partitions. *Transactions of the American Mathematical Society*, 365(8):4121–4151, 2013. ISSN 00029947. URL <http://www.jstor.org/stable/23513509>.
- [5] Esther Banaian, Kyle Celano, Megan Chang-Lee, Laura Colmenarejo, Owen Goff, Jamie Kimble, Lauren Kimpel, John Lentfer, Jinting Liang, and Sheila Sundaram. The e -positivity of the chromatic symmetric function for twinned paths and cycles, 2024. URL <https://arxiv.org/abs/2405.17649>.
- [6] Margaret Bayer, Herman Chau, Mark Denker, Owen Goff, Jamie Kimble, Yi-Lin Lee, and Jinting Liang. Promotion, tangled labelings, and sorting generating functions, 2024. URL <https://arxiv.org/abs/2411.12034>.
- [7] David Bessis and Victor Reiner. Cyclic sieving of noncrossing partitions for complex reflection groups. *Annals of Combinatorics*, 15(2):197 – 222, 2011. ISSN 02180006. URL <https://search.ebscohost.com/login.aspx?direct=true&db=a9h&AN=61844247&site=eds-live&authtype=sso&custid=s8364774>.
- [8] George D. Birkhoff. A determinant formula for the number of ways of coloring a map. *Annals of Mathematics*, 14(1/4):42–46, 1912. ISSN 0003486X, 19398980. URL <http://www.jstor.org/stable/1967597>.
- [9] P. J. Cameron and D. G. Fon-Der-Flaass. Orbits of antichains revisited. *European J. Combin.*, 16(6):545–554, 1995. ISSN 0195-6698. doi: 10.1016/0195-6698(95)90036-5. URL [https://doi-org.proxy1.cl.msu.edu/10.1016/0195-6698\(95\)90036-5](https://doi-org.proxy1.cl.msu.edu/10.1016/0195-6698(95)90036-5).
- [10] L. Carlitz, R. Scoville, and T. Vaughan. Enumeration of pairs of sequences by rises, falls and levels. *Manuscripta Math.*, 19(3):211–243, 1976. ISSN 0025-2611,1432-1785. doi: 10.1007/BF01170773. URL <https://doi.org/10.1007/BF01170773>.
- [11] A. Cayley. Xxviii. on the theory of the analytical forms called trees. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 13(85):172–176, 1857. doi:

10.1080/14786445708642275. URL <https://doi.org/10.1080/14786445708642275>.

- [12] S. Cho and J. Huh. On e -positivity and e -unimodality of chromatic quasisymmetric functions. *SIAM Journal on Discrete Mathematics*, 33:2286–2315, 2019.
- [13] S. Dahlberg. A new formula for Stanley’s chromatic symmetric function for unit interval graphs and e -positivity for triangular ladder graphs. *Sém. Lothar. Combin.*, 82B:Art. 59, 12, 2020.
- [14] S. Dahlberg and S. van Willigenburg. Lollipop and lariat symmetric functions. *SIAM Journal on Discrete Mathematics*, 32(2):1029 – 1039, 2018. doi: 10.1137/17M1144805.
- [15] S. Dahlberg, A. Foley, and S. van Willigenburg. Resolving Stanley’s e -positivity of claw-contractible-free graphs. *Journal of the European Mathematical Society*, 22(8):2673–2696, 2020.
- [16] S. Dahlberg, A. She, and S. van Willigenburg. Schur and e -positivity of trees and cut vertices. *Electron. J. Combin.*, 27(1):Paper No. 1.2, 22, 2020. ISSN 1077-8926. doi: 10.37236/8930. URL <https://doi.org/10.37236/8930>.
- [17] Pranjal Dangwal, Jamie Kimble, Jinting Liang, Jianzhi Lou, Bruce E. Sagan, and Zach Stewart. Rowmotion on rooted trees, 2022. URL <https://arxiv.org/abs/2208.12155>.
- [18] Pranjal Dangwal, Jamie Kimble, Jinting Liang, Jianzhi Lou, Bruce E. Sagan, and Zach Stewart. Rowmotion on rooted trees. *Seminaire Lotharingien de Combinatoire*, 2023.
- [19] Colin Defant and Noah Kravitz. Promotion Sorting. *Order*, 40(1):199–216, 2023. ISSN 0167-8094,1572-9273. doi: 10.1007/s11083-022-09603-9. URL <https://doi.org/10.1007/s11083-022-09603-9>.
- [20] Narsingh Deo. *Graph Theory with Applications to Engineering and Computer Science*. Prentice-Hall, 1974.
- [21] Michel Marie Deza and Komei Fukuda. Loops of clutters. In *Coding Theory and Design Theory*, pages 72–92, New York, NY, 1990. Springer New York. ISBN 978-1-4613-8994-1.
- [22] David Einstein and James Propp. Piecewise-linear and birational toggling. In *26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014)*, Discrete Math. Theor. Comput. Sci. Proc., AT, pages 513–524. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2014.
- [23] Sergi Elizalde, Matthew Plante, Tom Roby, and Bruce E. Sagan. Rowmotion on fences. 2021.
- [24] A. M. Foley, C. T. Hoàng, and O. D. Merkel. Classes of graphs with e -positive chromatic

- symmetric function. *Electron. J. Combin.*, 26(3):Paper No. 3.51, 19, 2019. ISSN 1077-8926. doi: 10.37236/8211. URL <https://doi.org/10.37236/8211>.
- [25] A. M. Foley, J. Kazdan, L. Kröll, S. Martínez Alberga, O. Melnyk, and A. Tenenbaum. Spiders and their kin: an investigation of Stanley’s chromatic symmetric function for spiders and related graphs. *Graphs Combin.*, 37(1):87–110, 2021. ISSN 0911-0119,1435-5914. doi: 10.1007/s00373-020-02230-4. URL <https://doi.org/10.1007/s00373-020-02230-4>.
 - [26] D.G. Fon-Der-Flaass. Orbits of antichains in ranked posets. *European Journal of Combinatorics*, 14(1):17–22, 1993. ISSN 0195-6698. doi: <https://doi.org/10.1006/eujc.1993.1003>. URL <https://www.sciencedirect.com/science/article/pii/S0195669883710036>.
 - [27] D. D. Gebhard and B. E. Sagan. A chromatic symmetric function in noncommuting variables. *Journal of Algebraic Combinatorics*, 13(3):227–255, 2001. doi: 10.1023/a:1011258714032.
 - [28] Darij Grinberg and Tom Roby. Iterative properties of birational rowmotion II: rectangles and triangles. *Electron. J. Combin.*, 22(3):Paper 3.40, 49, 2015.
 - [29] Darij Grinberg and Tom Roby. Iterative properties of birational rowmotion I: generalities and skeletal posets. *Electron. J. Combin.*, 23(1):Paper 1.33, 40, 2016.
 - [30] A. M. Hamel, C. T. Hoàng, and J. E. Tuero. Chromatic symmetric functions and H -free graphs. *Graphs Combin.*, 35(4):815–825, 2019. ISSN 0911-0119,1435-5914. doi: 10.1007/s00373-019-02034-1. URL <https://doi.org/10.1007/s00373-019-02034-1>.
 - [31] Tatsuyuki Hikita. A proof of the stanley-stembridge conjecture, 2024. URL <https://arxiv.org/abs/2410.12758>.
 - [32] Eliot Hodges. On promotion and quasi-tangled labelings of posets. *Annals of Combinatorics*, 28(2):529–554, 2024. ISSN 0219-3094. doi: 10.1007/s00026-023-00646-2. URL <https://doi.org/10.1007/s00026-023-00646-2>.
 - [33] E. Y. H. Li, G. M. X. Li, D. G. L. Wang, and A. L. B. Yang. The twinning operation on graphs does not always preserve e -positivity. *Taiwanese Journal of Mathematics*, 25(6):1089 – 1111, 2021. doi: 10.11650/tjm/210703. URL <https://doi.org/10.11650/tjm/210703>.
 - [34] G. M. X. Li and A. L. B. Yang. On the e -positivity of $(\text{claw}, 2K_2)$ -free graphs. *Electron. J. Combin.*, 28(2):Paper No. 2.40, 14, 2021. ISSN 1077-8926. doi: 10.37236/9910. URL <https://doi.org/10.37236/9910>.
 - [35] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford University Press, 1998.
 - [36] Gregg Musiker and Tom Roby. Paths to understanding birational rowmotion on products of two chains. *Algebraic Combinatorics*, 2:275–304, 2019. doi: 10.5802/alco.43.<https://doi.org/10.5802/alco.43>.

[//alco.centre-mersenne.org/articles/10.5802/alco.43/](http://alco.centre-mersenne.org/articles/10.5802/alco.43/).

- [37] R. Orellana and G. Scott. Graphs with equal chromatic symmetric functions. *Discrete Mathematics*, 320:1–14, 2014. ISSN 0012-365X. doi: <https://doi.org/10.1016/j.disc.2013.12.006>. URL <https://www.sciencedirect.com/science/article/pii/S0012365X13004974>.
- [38] Dmitri I. Panyushev. On orbits of antichains of positive roots. *European Journal of Combinatorics*, 30(2):586–594, 2009. ISSN 0195-6698. doi: <https://doi.org/10.1016/j.ejc.2008.03.009>. URL <https://www.sciencedirect.com/science/article/pii/S0195669808001455>.
- [39] J. Propp and T. Roby. Homomesy in products of two chains. *Electronic Journal of Combinatorics*, 22, 2015.
- [40] Brendon Rhoades. Cyclic sieving, promotion, and representation theory. *J. Combin. Theory Ser. A*, 117(1):38–76, 2010. ISSN 0097-3165,1096-0899. doi: 10.1016/j.jcta.2009.03.017. URL <https://doi.org/10.1016/j.jcta.2009.03.017>.
- [41] Tom Roby. Dynamical algebraic combinatorics and the homomesy phenomenon. In *Recent trends in combinatorics*, volume 159 of *IMA Vol. Math. Appl.*, pages 619–652. Springer, [Cham], 2016. doi: 10.1007/978-3-319-24298-9_25. URL https://doi-org.proxy1.cl.msu.edu/10.1007/978-3-319-24298-9_25.
- [42] David B. Rush and XiaoLin Shi. On orbits of order ideals of minuscule posets. *Journal of Algebraic Combinatorics*, 37, 2013. doi: 10.1007/s10801-012-0380-2.
- [43] M. P. Schützenberger. Quelques remarques sur une construction de Schensted. *Math. Scand.*, 12:117–128, 1963. ISSN 0025-5521,1903-1807. doi: 10.7146/math.scand.a-10676. URL <https://doi.org/10.7146/math.scand.a-10676>.
- [44] M. P. Schützenberger. Promotion des morphismes d’ensembles ordonnés. *Discrete Math.*, 2:73–94, 1972. ISSN 0012-365X,1872-681X. doi: 10.1016/0012-365X(72)90062-3. URL [https://doi.org/10.1016/0012-365X\(72\)90062-3](https://doi.org/10.1016/0012-365X(72)90062-3).
- [45] J. Shareshian and M. L. Wachs. Chromatic quasisymmetric functions and Hessenberg varieties. In *Configuration spaces*, volume 14 of *CRM Series*, pages 433–460. Ed. Norm., Pisa, 2012.
- [46] J. Shareshian and M. L. Wachs. Chromatic quasisymmetric functions. *Advances in Mathematics*, 295:497–551, 2016. doi: 10.1016/j.aim.2015.12.018.
- [47] R. P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111(1):166–194, 1995. ISSN 0001-8708. doi: 10.1006/aima.1995.1020.
- [48] R. P. Stanley. *Enumerative Combinatorics. volume 2*. Cambridge University Press, 1999.

- [49] R. P. Stanley and J. R. Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted position. *Journal of Combinatorial Theory, Series A*, 62(2):261–279, 1993.
- [50] Richard P. Stanley. Acyclic orientations of graphs. *Discrete Mathematics*, 5(2):171–178, 1973. ISSN 0012-365X. doi: [https://doi.org/10.1016/0012-365X\(73\)90108-8](https://doi.org/10.1016/0012-365X(73)90108-8). URL <https://www.sciencedirect.com/science/article/pii/0012365X73901088>.
- [51] Richard P. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1(1):9–23, 1986. ISSN 0179-5376. doi: 10.1007/BF02187680. URL <https://doi-org.proxy1.cl.msu.edu/10.1007/BF02187680>.
- [52] Richard P. Stanley. Promotion and evacuation. *Electron. J. Combin.*, 16(2):Research Paper 9, 24, 2009. ISSN 1077-8926. doi: 10.37236/75. URL <https://doi.org/10.37236/75>.
- [53] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012. ISBN 978-1-107-60262-5.
- [54] J. R. Stembridge. Eulerian numbers, tableaux, and the Betti numbers of a toric variety. *Discrete Mathematics*, 99(1-3):307–320, 1992. doi: 10.1016/0012-365x(92)90378-s.
- [55] Jessica Striker. Rowmotion and generalized toggle groups. *Discrete Math. Theor. Comput. Sci.*, 20(1):Paper No. 17, 26, 2018.
- [56] Jessica Striker and Nathan Williams. Promotion and rowmotion. *European J. Combin.*, 33(8):1919–1942, 2012. ISSN 0195-6698,1095-9971. doi: 10.1016/j.ejc.2012.05.003.
- [57] H. Thomas and N. Williams. Rowmotion in slow motion. *Proceedings of the London Mathematical Society*, 119(5):1149–1178, 2019. doi: <https://doi.org/10.1112/plms.12251>. URL <https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/plms.12251>.
- [58] F. Tom. A signed e -expansion of the chromatic symmetric function and some new e -positive graphs. *arXiv:2311.08020*, page preprint, 2024.
- [59] S. Tsujie. The chromatic symmetric functions of trivially perfect graphs and cographs. *Graphs and Combinatorics*, 34(5):1037–1048, jul 2018. doi: 10.1007/s00373-018-1928-2. URL <https://doi.org/10.1007%2Fs00373-018-1928-2>.
- [60] Michelle L. Wachs. On the (co)homology of the partition lattice and the free Lie algebra. volume 193, pages 287–319. 1998. doi: 10.1016/S0012-365X(98)00147-2. URL [https://doi-org.proxy1.cl.msu.edu/10.1016/S0012-365X\(98\)00147-2](https://doi-org.proxy1.cl.msu.edu/10.1016/S0012-365X(98)00147-2). Selected papers in honor of Adriano Garsia (Taormina, 1994).
- [61] D. G. L. Wang. All cycle-chords are e -positive. *arXiv:2405.01155*, page preprint, 2024.

- [62] D. G. L. Wang and M. M. Y. Wang. The e -positivity of two classes of cycle-chord graphs. *J. Algebraic Combin.*, 57(2):495–514, 2023. ISSN 0925-9899,1572-9192. doi: 10.1007/s10801-022-01175-6. URL <https://doi.org/10.1007/s10801-022-01175-6>.
- [63] D. G. L. Wang and J. Z. F. Zhou. Composition method for chromatic symmetric functions: Neat noncommutative analogs. *arXiv:2401.01027*, page preprint, 2024.
- [64] M. Wolfe. Symmetric chromatic functions. *Pi Mu Epsilon Journal*, 10(8):643–657, 1998. ISSN 0031952X. URL <http://www.jstor.org/stable/24337882>.