RATIONALITY OF BRAUER-SEVERI SURFACE BUNDLES

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ABSTRACT

Rationality problems of complex algebraic geometry has a long history. Recent developments, usually referred as specialization method in the literature, have given fruitful new examples of non-rational varieties.

We give a sufficient condition for a Brauer-Severi surface bundle over a rational 3-fold to not be stably rational. Additionally, we present an example that satisfies this condition and demonstrate the existence of families of Brauer-Severi surface bundles whose general members are smooth and not stably rational.

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CHAPTER 1

INTRODUCTION

Rationality questions aim to determine the parametrization of multivariable polynomials. To avoid unnecessary complexity, we focus only on parametrizations involving polynomials or rational functions (quotients of polynomials). Such a parametrization is commonly referred to as a **birational parametrization**. A first example would be the parametrization of the unit circle $x^2 + y^2 - 1 = 0$.

Example 1.0.1. $x = cos(\theta); y = sin(\theta)$ is not a birational parametrization of the unit circle. Because these expressions involve power series instead of polynomials. However, $x = \frac{-2t}{1+t^2}; y = \frac{1-t^2}{1+t^2}$ is a birational parametrization of unit circle.

Given a complex projective integral variety X. We say X is rational if it admits such a birational parametrization. A formal definition is usually given from the geometric point of view:

Definition 1.0.2. Let X be a projective integral variety over $\mathbb{C}$. We say that X is **rational** if X is birational to a projective space $\mathbb{P}^n_k$ for some natural number n. X is **stably rational** if there exists a natural number m such that $X \times_k \mathbb{P}^m_k$ is rational.

It is clear from the definition that rational varieties are stably rational. However the converse is false, the first example is given in [BCTSSD85]. Their example admits a conic bundle structure over a rational surface. In [HPT18], an example of a quadratic surface bundle over $\mathbb{P}^2$ that is not stably rational and has a nontrivial unramified Brauer group is constructed. This example is realized as a divisor in $\mathbb{P}^2 \times \mathbb{P}^3$ of bidegree (2,2), with the quadratic surface bundle structure induced by the first projection. In [ABvBP20], this elegant example was examined from a different perspective: it naturally becomes a conic bundle over $\mathbb{P}^3$ via the second projection. This structure allowed the authors to establish a sufficient condition [ABvBP20, Thm. 2.6] for a conic bundle over $\mathbb{P}^3$ to not be stably rational.

All of the above examples illustrate a slogan: It is easier to determine rationality of varieties admitting a fibration structure. Why is this true? Because there is a very powerful tool to show that a variety is not stably rational, the unramified Brauer group.

Let X be a projective integral variety over C. By definition, the unramified Brauer group of X is a specific subgroup of the Brauer group of the function field of X, which is a stably birational invariant. When X admits a fibration structure over a base, we can analyze which Brauer classes originate from the base. This approach can lead to the construction of a nontrivial unramified Brauer class for X, which is sufficient to show that X is not stably rational.

In practice, we usually allow X to be singular. The specialization method, introduced by Voisin in [Voi15] and further developed by Colliot-Thélène and Pirutka in [CTP16], as well as by Schreieder in [Sch19a, Proposition 26], gives us a way to get smooth non-stably rational varieties from a singular one. For example, the authors in [ABvBP20] also introduced a new example of a flat family of conic bundles over $\mathbb{P}^3$, where a very general member is not stably rational, using a theorem of Voisin [Voi15, Thm. 2.1], in the form of [CTP16, Thm. 2.3].

On the other hand, to apply this technic, we require the singular locus of X is mild. To be more precise, we would like X admits a desingularization which is universally CH_0-trivial:

Definition 1.0.3. A **universally** CH_0**-trivial** desingularization of X is a projective morphism $f : \tilde{X} \to X$, such that $\tilde{X}$ is smooth and for any field extension $F/\mathbb{C}$, the map induced by f:

$$\mathrm{CH}_0(\tilde{X_F}) \to \mathrm{CH}_0(X)$$

is an isomorphism. Here $\mathrm{CH}_0(X)$ denotes the 0-th Chow group, that is the group of 0-cycles module rational equivalence [Ful98, Chapter 1].

In a summary, to apply this specialization method, there are two requirements:

- Constructing a specific example with meaningful, nontrivial stably birational invariants that can be specialized, such as the unramified Brauer group.
- Showing that this example admits a universally CH_0-trivial desingularization and can specialize to varieties of interest.

However, it is usually hard to construct an explicit desingularization of a given variety. Furthermore, it is also hard to check the universally CH_0-trivial property using Definition 1.0.3. A good refinement to check this property is given in [CTP16, Proposition 1.8], but one still need to construct an explicit smooth model of the given variety. Recently, Schreieder gave an alternate approach in a series of papers: [Sch19a, Proposition 26] and [Sch19b]. Instead of constructing such a desingularization, Schreieder's result allows a purely cohomological criteria.

In a recent paper [Pir23], Pirutka introduced the notion of the relative unramified cohomology group, which combines the approach of constructing nontrivial unramified Brauer class via fibrations ([AM72],[CTO89]) and the method given in [Sch19a] and [Sch19b] to avoid geometric construction of universally CH_0-trivial desingularization.

The present thesis is going to study stable rationality of Brauer-Severi surface bundles over rational 3-folds using the specialization method mentioned above. We start with the definitions:

Definition 1.0.4 ([GS06, Def. 5.1.1])**.** A **Brauer-Severi variety** of dimension n over a field k is a projective algebraic variety X over k such that the base extension $X_{\bar{k}} := X \times_k \bar{k}$ becomes isomorphic to $\mathbb{P}^n_{\bar{k}}$, where $\bar{k}$ is an algebraic closure of k.

Remark 1.0.5. 1. Projective spaces are considered as trivial Brauer-Severi varieties.

2. 1-dimensional Brauer-Severi varieties are precisely smooth projective plane conics ([GS06, Chpater 5]).

3. Considering 2, a conic bundle, which by definition is a flat projective surjective morphism of varieties with geometric fibers isomorphic to projective plane conics and general fiber smooth, is a synonym for 1-dimensional Brauer-Severi bundle or Brauer-Severi curve bundle.

In this thesis, we mainly focus on the structure of 2-dimensional Brauer-Severi bundle:

Definition 1.0.6 ([KT19, Def. 4.1])**.** Let B be a locally Noetherian scheme, in which 3 is invertible in the local rings. A **Brauer-Severi surface bundle** over B is a flat projective morphism $\pi : Y \to B$ such that the fiber over every geometric point of B is isomorphic to one of the following:

- $\mathbb{P}^2$

- The union of three standard Hirzebruch surfaces $\mathbb{F}_1$, meeting transversally, such that any pair of them meets along a fiber of one and the (-1)-curve of the other.

- An irreducible scheme whose underlying reduced subscheme is isomorphic to the cone over a twisted cubic curve.

Remark 1.0.7. In the case of conic bundles, the degenerate fibers are simply two distinct lines or a double line. The 2 dimensional case is more complex.

Kresch and Tschinkel introduced good models of Brauer-Severi surface bundles using the concept of a root stack in [KT19]. With this definition, they constructed a flat family of Brauer-Severi surface bundles over $\mathbb{P}^2$ [KT20, Thm. 1], in which the general member is smooth and not stably rational. A natural next step following these advancements is to investigate the stable rationality of Brauer-Severi surface bundles over $\mathbb{P}^3$.

This paper begins by generalizing Theorem 2.6 of [ABvBP20] to Brauer-Severi surface bundles over 3-folds. After constructing a new (singular) example with a nontrivial unramified Brauer group, we obtain the following result:

Theorem 1.0.8. *There exists a flat projective family of Brauer-Severi surface bundles over $\mathbb{P}^3_{\mathbb{C}}$, where a general fiber in this family is smooth and not stably rational.*

The structure of this thesis is as follows: In Chapter 2, we review basic facts about Brauer groups (Section 2.1, Section 2.2) and introduce the unramified Brauer group (Section 2.3), a stably birational invariant. By definition, this is the subgroup of the Brauer group of the function field, whose elements arise from the Brauer classes of a smooth model. Recently, significant progress has been made using the unramified Brauer group and the specialization method to show that a very general member of certain classes of varieties is not stably rational ([Pir18, Section 2.1]).

In Chapter 3, after recalling a criterion for stable rationality of conic bundles over 3-folds ([ABvBP20, Thm. 2.6]) in Section 3.1. We give the restrictions of discriminant locus of Brauer-Severi surface bundles in Section 3.2. Then we provide the key technical theorem in Section 3.3.

This provides a tool to construct an explicit (singular) Brauer-Severi surface bundle over $\mathbb{P}^3_{\mathbb{C}}$ with a nontrivial unramified Brauer group, which we explore in Chapter 4. Section 4.1 consists of explicit construction of our example, and in Section 4.2, we verify that this example is indeed a Brauer-Severi surface bundle.

Chapter 5 it ought to verify the example we constructed can be used as a reference variety. We recall the main developments of specialization methods dated back to 70s until now in Section 5.1. In Section 5.2, we show that our example 4.1.1 satisfies the hypotheses required by the specialization method introduced by Voisin in 2014 [Voi15], and further developed by Colliot-Thélène and Pirutka in 2016 [CTP16]. Following Schreieder's approach [Sch19a, Proposition 26], we verify this using a purely cohomological criterion. In Section 5.3, we prove Theorem 1.0.8 by constructing a flat family of Brauer-Severi surface bundles over $\mathbb{P}^3_{\mathbb{C}}$, where the general member is smooth and includes Example 4.1.1 as a member. Detailed calculations are provided separately in Appendix.

CHAPTER 2

BACKGROUND

In this chapter, we recall the necessary background that will be used later.

2.1 Brauer groups

For a more detailed treatment of Brauer groups of fields and schemes, see [GS06] and [CTS21]. let k be a field, A be an associative k-algebra. A is called **central** if its center is k. A is called **simple** if it has no non-trivial two-sided ideals. A **central simple k-algebra(CSA)** is a finite dimensional k-algebra that is both central and simple.

Example 2.1.1. Here are first examples of central simple algebras:

- Hamilton quaternions over $\mathbb{R}$. Generally, any central division algebra is a CSA.
- The $n \times n$ matrix algebra $M_n(k)$, more generally, if D is a central division algebra, then $M_n(D)$ is a CSA.

Wedderburn's theorem provides a converse to the second type of examples above:

Theorem 2.1.2 (Wedderburn,[GS06, Thm. 2.1.3])**.** *For any CSA A, there is a central division algebra D and an integer $n \geq 1$, such that*

$$A \cong D \otimes_k M_n(k) = M_n(D)$$

And the division algebra D is unique up to isomorphism.

If A is a CSA over k and K/k is a field extension such that $A \otimes_k K \cong M_n(K)$ for suitable n. Then K is called a **splitting field** of A. Splitting field of a given CSA always exists according to the definition. In fact, we can do better:

Theorem 2.1.3 ([GS06, Thm. 2.2.1])**.** *Let k be a field and let A be a finite-dimensional k-algebra. Then A is a CSA if and only if there is a positive intger n and a finite field extension K/k such that*

$$A \otimes_k K \cong M_n(K)$$

As a direct corollary of Theorem 2.1.3, we know the dimension of a given CSA as a k-algebra is a square, We call the integer $\sqrt{dim_k A}$ **degree** of A. By Wedderburn's Theorem 2.1.2, assume $A \cong M_n(D)$, we define **index** of A to be the degree of D.

Definition 2.1.4. Two CSA A and B are called **Brauer equivalent** if there are $n, m > 0$ such that

$$A \otimes_k M_n(k) \cong B \otimes_k M_m(k)$$

This is an equivalent relation on the set of all central simple algebras over k. Given a CSA A, use $[A]$ denote its Brauer equivalence class.

Definition 2.1.5. The **Brauer group** of a field k, denote by $\mathrm{Br}(k)$, is the set of Brauer equivalent classes of central simple algebras over k, with a structure of torsion abelian group with the binary operation given by:

$$[A] + [B] = [A \otimes_k B]$$

The **index** of a Brauer class $[A]$ is defined to be the index of A, and the order $[A]$ in $\mathrm{Br}(k)$ is called the **period** of $[A]$.

Example 2.1.6.

- $\mathrm{Br}(k) = 0$, if k is separably closed or is finite.
- (Frobenius) $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$.
- (Tsen) let k be a field of transcendence degree 1 over an algebraically closed field, then $\mathrm{Br}(k) = 0$.

We can also describe the Brauer group using Galois cohomology:

Theorem 2.1.7 ([GS06, Thm. 4.4.7])**.** *Let m be a positive integer that is invertible in k, then*

$$\mathrm{Br}(k) \cong H^2(k, \mathbb{G}_m), \mathrm{Br}(k)[m] \cong H^2(k, \mu_m)$$

Situations are become complex when we passes to schemes, there are naturally two ways to define a Brauer group of a scheme: namely following the idea of Definition 2.1.5 or using cohomological description similar to Theorem 2.1.7. However, the two constructions are not equivalent. For details in this direction, see [CTS21, Chapter 3,4]. In this thesis, when talking about Brauer group of schemes, we adopt the following definition:

Definition 2.1.8. Let X be a quasi-compact k-scheme, the **(cohomological) Brauer group**, denote by $\mathrm{Br}(X)$, is defined to be the torsion subgroup of $H^2_{ét}(X, \mathbb{G}_m)$. In particular, assume m is a positive integer that is invertible in k, the m-tosion part Brauer group of X is $\mathrm{Br}(X)[m] = H^2_{ét}(X, \mu_m)$.

2.2 Cyclic algebras

Through out this section, we assume the base field k contains all primitive roots of unity. p is a prime number.

Definition 2.2.1. For any $a, b \in k^*$, and a primitive p-th root of unity $\omega \in k^*$, we define the **cyclic algebra**, denote by $(a, b)_\omega$, to be the k-algebra by the following presentation:

$$(a, b)_\omega = \langle x, y | x^p = a, y^p = b, xy = \omega yx \rangle ,$$

Remark 2.2.2.
- when $p = 2$, $\omega = -1$, we get back the definition of quaternion algebras.
- One directly check $(a, b)_\omega$ is a CSA over k. Futhermore, its either a division algebra of degree (index) p, or the matrix algebra $M_p(k)$. The later case happens if and only if ,by [GS06, Corollary 4.7.7], b is a norm from the field extension $k(\sqrt[m]{a})/k$.

The importance of cyclic algebras is that they served as nice representatives of Brauer classes, as a result of Merkurjev-Suslin theorem:

Theorem 2.2.3 ((Merkurjev-Suslin),[GS06, Thm. 8.6.5])**.** *With the above assumptions on k and p, let $\alpha \in \mathrm{Br}(k)[p]$, then there exist a finite number of cyclic algebras of degree p $\{(a_i, b_i)\}_{i=1}^t$, such that*

$$\alpha = [(a_1, b_1)_\omega \otimes_k \cdots \otimes_k (a_t, b_t)_\omega]$$

In the rest of the thesis, we will focus on cyclic algebras of degree 3. Note that when k contains a primitive cubic roots of unity, we have a non-canonical isomorphism of group schemes :$\mu_3 \cong \mathbb{Z}/3\mathbb{Z}$, hence the 3-torsion of the Brauer group of a field k would be

$$\mathrm{Br}(k)[3] \cong H^2(k, \mathbb{Z}/3\mathbb{Z})$$

2.3 Unramified Brauer group and purity

Let L be the function field of an integral scheme Z over the field of complex numbers, $\mathbb{C}$. Let ν be a discrete valuation of L with residue field $k(\nu)$, we have the following residue maps [GS06, Section 6.8]:

$$\partial_\nu^1 : H^1(L, \mathbb{Z}/3) \to H^0(k(\nu), \mathbb{Z}/3)$$

$$\partial_\nu^2 : H^2(L, \mathbb{Z}/3) \to H^1(k(\nu), \mathbb{Z}/3)$$

By Kummer theory [GS06, Prop. 4.3.6], we can identify these residue maps as:

$$\partial_\nu^1 : L^\times / L^{\times 3} \to \mathbb{Z}/3$$

$$\partial_\nu^2 : \mathrm{Br}(L)[3] \to k(\nu)^\times / k(\nu)^{\times 3}$$

Lemma 2.3.1. *With notations as above, the two residue maps are defined by:*

$$\partial_\nu^1([a]) = \nu(a)(mod\, 3)$$

$$\partial_\nu^2([(a,b)_\omega]) = (-1)^{\nu(a)\nu(b)} \frac{a^{\nu(b)}}{b^{\nu(a)}} \mod k(\nu)^{\times 3}$$

Proof. See [IOOV17, Thm. 2.18] □

Definition 2.3.2. The 3-torsion of **unramified Brauer group** of a field L over another field k, denoted by $H^2_{nr}(L/k, \mathbb{Z}/3\mathbb{Z})$ or $\mathrm{Br}_{nr}(L/k)[3]$, is the intersection of kernels of all residue maps ∂_ν^2, where ν take values in all divisorial valuations of L which are trivial on k.

From the discussions in [CT95] and [CTS21, Corollary 6.2.10], the unramified Brauer group of L is a stably birational invariant for any model X of L, whether the model is nonsingular or singular. Here, a model X of L refers to an integral projective variety with function field L. If the model is nonsingular, we have the following:

Lemma 2.3.3. *Let k be a field with characteristic not 3. Let X be a regular, proper, integral variety over k with function field L, then we have*

$$\mathrm{Br}(X)[3] \cong \mathrm{Br}_{nr}(L/k)[3]$$

Proof. This is a direct result of [CTS21, Prop. 3.7.8] □

Given a 3-torsion Brauer class in the Brauer group of the function field L, it is not easy to determine whether it belongs to the unramified subgroup using the definition alone, as there are usually too many divisorial valuations to consider. In [CT95], several theorems are established that reduce the number of valuations needed to check whether a Brauer class is unramified. Specifically, it suffices to check valuations corresponding to prime divisors in a smooth model. This result follows from the **purity** property of unramified Brauer groups. For more details on these theorems, see [CT95] and [CTS21, Section 3.7]. A similar discussion can be found in [ABvBP20, Section 2].

CHAPTER 3

BRAUER-SEVERI SURFACE BUNDLES

3.1 Review of conic bundles

In this section, we make a brief review of a formula for the unramified Brauer group of centain conic bundle fourfolds, all of the contents in this section can be found in [ABvBP20]. We start with the definition:

Definition 3.1.1. Let B be a Noetherian scheme over $\mathbb{C}$, a **conic bundle** over B is a flat projective morphism $\pi : Y \to B$ such that the fiber over every geometric point of B is isomorphic to one of the following:

- $\mathbb{P}^1$
- Two distinct lines intersect at one point
- Double lines

In [ABvBP20], The authors focused on conic bundles over $\mathbb{P}^3_{\mathbb{C}}$. Let$\pi : Y \to \mathbb{P}^3_{\mathbb{C}}$ be such a conic bundle, and let $S \subset \mathbb{P}^3_{\mathbb{C}}$ be its **discriminant locus**, which by definition is the subset of points in $\mathbb{P}^3_{\mathbb{C}}$ whose fibers are not smooth plane conics. One can show that S is indeed a divisor of pure dimension with its natural determinantal scheme structure, and since $\mathbb{P}^3_{\mathbb{C}}$ is Noetherian, S has finitely many irreducible components, denoted by $S_1, \cdots S_n$.

Definition 3.1.2 ([ABvBP20, Def. 2.4])**.** The discriminant locus is **good** if S is reduced and for each irreducible component S_i, the fiber Y_s for general $s \in S_i$ consists of two distinct lines, and the natural double cover of S_i induced by π is irreducible.

With the above requirements of discriminant locus of conic bundles, one can prove the following formula for the unramified Brauer group:

Theorem 3.1.1 ([ABvBP20, Thm. 2.6])**.** Let k be an algebraically closed field of characteristic not 2 and let $\pi : Y \to B$ be a conic bundle over a smooth projective threefold B over k. Assume

$\mathrm{Br}(B)[2] = 0, H^3_{\text{ét}}(B, \mathbb{Z}/2) = 0$, (e.g. $B = \mathbb{P}^3$). Let $\alpha \in \mathrm{Br}(K)[2]$ be the Brauer class in $K = k(B)$ corresponding to the generic fiber of π. Assume the discriminant locus is good (Definition 3.1.2) with components $S_1, \cdots, S_n$. And also assume that

1)Through any irreducible curve in B, there pass at most two surfaces from the set $S_1, \cdots, S_n$.

2) Through any point of B, there pass at most three surfaces from the set $S_1, \cdots, S_n$.

3)For all $i \neq j$, S_i and S_j are factorial at every point of $S_i \cap S_j$.

Let

$$\gamma_i = \partial^2_{S_i}(\alpha) \in H^1(k(S_i), \mathbb{Z}/2)$$

Define a subgroup Γ of the group $\bigoplus_{i=1}^n H^1(k(S_i), \mathbb{Z}/2)$ by

$$\Gamma = \bigoplus_{i=1}^{n} < \gamma_i > \cong (\mathbb{Z}/2)^n$$

We will write elements of Γ as $(x_1, x_2 \cdots, x_n)$ with $x_i \in \{0, 1\}$.

Let $H \subset \Gamma$ consists of those elements $(x_1, \cdots, x_n) \in (\mathbb{Z}/2)^n$ such that $x_i = x_j$ whenever there exists an irreducible components C of $S_i \cap S_j$, such that

$$(\partial^2_{S_i}(\alpha), \partial^2_{S_j}(\alpha)) = (0,0)\text{or}(1,1)$$

Let $H' \subset H$ consists of those elements $(x_1, \cdots, x_n) \in (\mathbb{Z}/2)^n$ such that $x_i = x_j$ whenever there exists an irreducible components C of $S_i \cap S_j$, such that

$$(\partial^2_{S_i}(\alpha), \partial^2_{S_j}(\alpha)) = (0,0) \text{ and } \gamma_i|_C \text{ and } \gamma_j|_C \text{ are not both zero in } H^1(k(C), \mathbb{Z}/2)$$

Then $H^2_{nr}(k(Y)/k, \mathbb{Z}/2)$ contains the subquotient $H'/< 1, \cdots, 1 >$.

These theorem applies to a famous example given in [HPT18]: Let Y_{HTP} denote the divisor of bi-degree $(2,2)$ in $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^3_{\mathbb{C}}$ given by :

$$YZS^2 + XZT^2 + XYU^2 + (X^2 + Y^2 + Z^2 - 2(XY + XZ + YZ))V^2 = 0$$

where we assume the coordinates in $\mathbb{P}^3_{\mathbb{C}}$ are $[S : T : U : V]$ and the coordinates in $\mathbb{P}^2_{\mathbb{C}}$ are $[X : Y : Z]$. By considering the projection from Y_{HTP} to $\mathbb{P}^2_{\mathbb{C}}$, the authors in [HPT18] proved this is not stably rational as a quadratic surface bundle. In [ABvBP20], the authors considering the projection from Y_{HTP} to $\mathbb{P}^3_{\mathbb{C}}$ instead, and Theorem 3.1.1 double checked that Y_{HTP} is not stably rational using its conic bundle structure over $\mathbb{P}^2_{\mathbb{C}}$.

3.2 Discriminant locus

Now we generlize the results from previous sections to the case of Brauer-Severi surface bundles. The first step is to find suitable discriminant locus.

Let $\pi : Y \to B$ be a Brauer-Severi surface bundle (Definition 1.0.6) over a smooth projective rational threefold B over a field k whose generic fiber is smooth, and let $S = \{b \in B | \pi^{-1}(b) \text{ is singular}\}$ denote its discriminant locus. We consider S with its reduced closed subscheme structure in B, and since B is Noetherian, S consists of finitely many irreducible components, say $S_1, \cdots, S_n$.

In the case of conic bundles considered in [ABvBP20, Def. 2.4], the authors focused on a special kind of conic bundles with a *good* discriminant locus. We will generalize the definition of a *good* discriminant locus to Brauer-Severi surface bundles in this context:

Definition 3.2.1. We say the discriminant locus S is **good** if the following conditions are satisfied:

1. Each irreducible component of S is reduced. (This is assumed above.)
2. The fiber Y_s over a general $s \in S_i$ for each irreducible component S_i is geometrically the union of three standard Hirzebruch surfaces $\mathbb{F}_1$ described in Definition 1.0.6.
3. The natural triple cover of S_i induced by $\pi : Y \to B$ is irreducible.
4. By (3), the fiber $Y_{F_{S_i}}$ over the generic point of S_i is irreducible. Thus, there is a natural map τ from the cubic classes of function field of S_i to the cubic classes of function field of $Y_{F_{S_i}}$. We assume the cubic extension over the function field of S_i induced by $\pi : Y \to B$ generates the kernel of τ.

Remark 3.2.2. The last requirement is automatically satisfied in the case of conic bundles with the suitable definition given in [ABvBP20, Thm. 2.6]. However, in our case, this is not generally true. Note that the example we provided meets all these requirements (See Example 4.1.1).

Remark 3.2.3. By Lemma 2.3.1, those S_i are precisely those irreducible surfaces such that the Brauer class of the generic fiber have a nontrivial residue along S_i. In particular, the discriminant locus is a divisor of the base with pure-dimensional irreducible components.

We end this section with a lemma that generalizes [ABvBP20, Lemma 2.3] to the 3-torsion case:

Lemma 3.2.4. *Let S be a smooth nonsplit Brauer-Severi surface over an arbitrary field K (in particular, $S_{\bar{K}} \cong \mathbb{P}^2_{\bar{K}}$). Then the pullback map* $\mathrm{Br}(K) \to \mathrm{Br}(S)$ *induces an exact sequence:*

$$0 \to \mathbb{Z}/3 \to \mathrm{Br}(K) \to \mathrm{Br}(S) \to 0,$$

where the kernel is generated by the Brauer class $\alpha \in \mathrm{Br}(K)[3]$ determined by S. Furthermore, if characteristic of $K \neq 3$ and K contains a primitive 9-th root of unit, then the above exact sequence restricts to :

$$0 \to \mathbb{Z}/3 \to \mathrm{Br}(K)[3] \to \mathrm{Br}(S)[3] \to \mathbb{Z}/3 \to 0$$

Proof. We first prove the exactness of the first sequence. Recall that the kernel $\ker(\mathrm{Br}(K) \to \mathrm{Br}(S))$ is given by Amitsur's theorem [GS06, Thm. 5.4.1]. To show surjectivity of $\mathrm{Br}(K) \to \mathrm{Br}(S)$, consider the separable closure K^s of K, and let $\Gamma = \mathrm{Gal}(K^s/K)$. Then we have the Hochschild-Serre spectral sequence

$$H^p(\Gamma, H^q(S_{K^s}, \mathbb{G}_m)) \Rightarrow H^{p+q}(S, \mathbb{G}_m).$$

The low degree exact sequence reads

$$0 \to \mathrm{Pic}(S) \to \mathrm{Pic}(S_{K^s})^\Gamma \to \mathrm{Br}(K) \to \ker\left(\mathrm{Br}(S) \to \mathrm{Br}(S_{K^s})^\Gamma\right) \to H^1(\Gamma, \mathbb{Z})$$

By the definition of a Brauer-Severi Variety, we know $S_{K^s} \cong \mathbb{P}^2_{K^s}$. Hence we have $\mathrm{Br}(S_{K^s}) \cong \mathrm{Br}(K^s) = 0$. Since we clearly have $H^1(\Gamma, \mathbb{Z}) = 0$, it follows that $\mathrm{Br}(K) \to \mathrm{Br}(S)$ is surjective.

For the second part, consider any $a \in \mathrm{Br}(S)[3]$. There exists a lift $a' \in \mathrm{Br}(K)$ of a such that $3a' \mapsto 0 \in \mathrm{Br}(S)$. Therefore, $3a' \in \ker(\mathrm{Br}(K) \to \mathrm{Br}(S)) \cong \mathbb{Z}/3$. In other words, $9a' = 0 \in$

$\mathrm{Br}(K)$. Hence, $\mathrm{Br}(K)[9]$ surjects onto $\mathrm{Br}(S)[3]$. Notice that by the description of the kernel in Amitsur's theorem, we clearly have

$$\ker\left(\mathrm{Br}(K)[3] \to \mathrm{Br}(S)[3]\right) \cong \mathbb{Z}/3.$$

To compute the cokernel, consider the short exact sequence of trivial Γ-modules:

$$1 \to \mu_3 \to \mu_9 \to \mu_3 \to 1,$$

and its associated long exact sequence:

$$\cdots \to H^1(K, \mu_3) \xrightarrow{\partial^1} H^2(K, \mu_3) \to H^2(K, \mu_9) \to H^2(K, \mu_3) \xrightarrow{\partial^2} H^3(K, \mu_3) \to \cdots$$

We claim the boundary maps ∂^1, ∂^2 in the above exact sequence are zero. Indeed, let ω be a primitive third root of unit. By the proof of [TOP17, Lemma 4.1], ∂^1 is given by the cup product with $[\omega] \in H^1(K, \mathbb{Z}/3)$. By our assumption, ω is a cube in K, hence $[\omega] = 0 \in H^1(K, \mathbb{Z}/3) \cong K^*/(K^*)^3$. This shows $\partial^1 = 0$.

To show $\partial^2 = 0$, consider the following commutative diagram given in [GS06, Lemma 7.5.10]:

$$\begin{array}{ccc} \mu_3 \otimes K_2^M(K) & \xrightarrow{\{,\}} & K_3^M(K)/3K_3^M(K) \\ \Big\downarrow{\scriptstyle \omega\cup h^2_{K,3}} & & \Big\downarrow{\scriptstyle h^3_{K,3}} \\ H^2(K, \mu_3^{\otimes 3}) & \xrightarrow{\tilde{\partial^2}} & H^3(K, \mu_3^{\otimes 3}) \end{array}$$

Notations in the above diagram are explained in [GS06, Lemma 7.5.10]. Notice since μ_3 are trivial Γ-modules, we have the following isomorphism ([TOP17, Lemma 4.1]):

$$\phi^i_j : H^i(K, \mathbb{Z}/3) \cong H^i(K, \mu_3^{\otimes j})$$

$$\alpha \mapsto \alpha \cup \underbrace{(\omega \cup \cdots \cup \omega)}_{j \text{ copies}}$$

Since the upper horizontal map in the commutative diagram is given by the symbol product with $0 = [\omega] \in H^1(K, \mathbb{Z}/3)$, it follows that this map is 0. By the Merkurjev-Suslin theorem ([GS06, Theorem 8.6.5]), the left vertical map is surjective, hence $\tilde{\partial^2} = 0$. Given that cup products "commute" with boundary homomorphisms ([NSW08, Proposition 1.4.3]), we have the commutative diagram involving ∂^2:

$$\begin{array}{ccc} H^2(K,\mu_3^{\otimes 3}) & \xrightarrow{\tilde{\partial}^2} & H^3(K,\mu_3^{\otimes 3}) \\ {\scriptstyle (\phi_3^2)^{-1}}\downarrow & & {\scriptstyle (\phi_3^3)^{-1}}\downarrow \\ H^2(K,\mathbb{Z}/3) & \xrightarrow{\partial^2} & H^3(K,\mathbb{Z}/3) \end{array}$$

From this, it is clear that $\partial^2 = 0$.

Next notice that K has characteristic $\neq 3$, so we have $\mathrm{Br}(K)[n] \cong H^2(K,\mu_n)$ when n is a power of 3. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}(K)[3] & \longrightarrow & \mathrm{Br}(K)[9] & \longrightarrow & \mathrm{Br}(K)[3] & \longrightarrow & 0 \\ & & {\scriptstyle \sigma_3}\downarrow & & {\scriptstyle \sigma_9}\downarrow & & {\scriptstyle \sigma_3}\downarrow & & \\ 0 & \longrightarrow & \mathrm{Br}(S)[3] & \longrightarrow & \mathrm{Br}(S)[9] & \longrightarrow & \mathrm{Br}(S)[3] & & \end{array}$$

Then the snake lemma gives us

$$\ker(\sigma_9) \xrightarrow{\phi_1} \ker(\sigma_3) \to \mathrm{coker}(\sigma_3) \xrightarrow{\phi_2} \mathrm{coker}(\sigma_9).$$

Since ϕ_1 is multiplication by 3 and $\ker(\sigma_9) \cong \mathbb{Z}/3$, $\phi_1 = 0$. The map ϕ_2 is also zero as $\mathrm{Br}(K)[9]$ maps onto $\mathrm{Br}(S)[3]$. Hence we have $\mathrm{coker}(\sigma_3) \cong \ker(\sigma_3) \cong \mathbb{Z}/3$. □

3.3 Brauer groups of Brauer-Severi surface bundles

As mentioned in the introduction, we interest in Brauer-Severi surface bundles over $\mathbb{P}^3_{\mathbb{C}}$. In this section, we present sufficient conditions under which a Brauer-Severi surface bundle 5-fold is not stably rational. These conditions are derived by generalizing [ABvBP20, Thm. 2.6] to the 3-torsion case. However, the details in these two cases are quite different.

Theorem 3.3.1. Let k be an algebraically closed field of characteristic $\neq 3$ and let $\pi : Y \to B$ be a Brauer-Severi surface bundle over a smooth projective threefold B over k with a smooth generic fiber. Assume $\mathrm{Br}(B)[3] = 0$ and $H^3_{\text{ét}}(B,\mathbb{Z}/3) = 0$. (For example, take $B = \mathbb{P}^3$.) Let $\alpha \in \mathrm{Br}(K)[3]$ be the Brauer class over $K = k(B)$ corresponding to the generic fiber of π, and it can be represented by a cyclic algebra of index 3. Assume the discriminant locus is good [Definition 3.2.1] with irreducible components $S_1, \cdots, S_n$. Further suppose the following conditions also hold:

1. Any irreducible curve in B is contained in at most two surfaces from the set $\{S_1, \cdots, S_n\}$.

2. Through any point of B, there pass at most three surfaces from the set $\{S_1, \cdots, S_n\}$.

3. For all $i \neq j$, S_i and S_j are factorial at every point of $S_i \cap S_j$.

Let $\gamma_i = \partial^2_{S_i}(\alpha) \in H^1(k(S_i), \mathbb{Z}/3)$. Let Γ be the subgroup of $\bigoplus_{i=1}^n H^1(k(S_i), \mathbb{Z}/3)$ given by $\Gamma = \bigoplus_{i=1}^n \langle\gamma_i\rangle \cong (\mathbb{Z}/3)^n$. We write elements of Γ as $(x_1, x_2 \cdots, x_n)$ with $x_i \in \{0, 1, 2\}$.

Let $H \subset \Gamma$ consist of those elements $(x_1, \cdots, x_n) \in (\mathbb{Z}/3)^n$ such that $x_i = x_j$ whenever there exists an irreducible component C of $S_i \cap S_j$, such that

$$(\partial^1_C(\gamma_i), \partial^1_C(\gamma_j)) = (1, 2) \text{ or } (2, 1).$$

Let $H' \subset H$ be the subgroup consisting of elements $(x_1, \cdots, x_n) \in (\mathbb{Z}/3)^n$ such that $x_i = x_j$ whenever there exists an irreducible components C of $S_i \cap S_j$, such that

$$(\partial^1_C(\gamma_i), \partial^1_C(\gamma_j)) = (0, 0), \text{ and } \gamma_i|_C \text{ and } \gamma_j|_C \text{ are not both trivial in } H^1(k(C), \mathbb{Z}/3).$$

Then $H^2_{nr}(k(Y)/k, \mathbb{Z}/3)$ contains the subquotient $H'/\langle 1, \cdots, 1\rangle$.

Proof. First, note that under these assumptions, Y is necessarily integral (see Corollary 4.2.6), hence we can talk about its function field $k(Y)$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \mathbb{Z}/3 & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & H^2_{nr}(k(Y)/Y, \mathbb{Z}/3) & \longrightarrow & H^2_{nr}(k(Y)/K, \mathbb{Z}/3) & \xrightarrow{\oplus\partial^2_T} & \bigoplus_{T \in Y_B^{(1)}} H^1(k(T), \mathbb{Z}/3) & & \\
 & & & & \uparrow{\scriptstyle\sigma} & & \uparrow{\scriptstyle\tau} & & \\
 & & \mathrm{Br}_{nr}(K)[3] = 0 & \longrightarrow & H^2(K, \mathbb{Z}/3) & \xrightarrow{\oplus\partial^2_S} & \bigoplus_{S \in B^{(1)}} H^1(k(S), \mathbb{Z}/3) & \xrightarrow{\oplus\partial^1_C} & \bigoplus_{C \in B^{(2)}} \mathbb{Z}/3 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \langle\alpha\rangle & & \Gamma & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We make some observations related to this diagram:

1. By definition, $H^2_{nr}(k(Y)/Y, \mathbb{Z}/3)$ denotes all those classes in $H^2(k(Y), \mathbb{Z}/3)$ which are unramified with respect to divisorial valuations corresponding to prime divisors on Y, Since the singular locus of Y has codimension ≥ 2, we can also characterize $H^2_{nr}(k(Y)/Y, \mathbb{Z}/3)$ as all those classes in $H^2(k(Y), \mathbb{Z}/3)$ that are unramified with respect to divisorial valuations which have centers on Y which are not contained in Y_{sing} [ABvBP20, Cor. 2.2].

2. $H^2_{nr}(k(Y)/K, \mathbb{Z}/3)$ denotes those classes in $H^2(k(Y), \mathbb{Z}/3)$ which are killed by residue maps associated to divisorial valuations that are trivial on K, hence correspond to prime divisors of Y dominating the base B. We use $Y^{(1)}_B$ to denote all prime divisors in Y that do not dominate the base B. Then the upper row is exact by definition.

3. The second row is obtained from Bloch-Ogus complex [BO74], which is exact under the assumptions
$$\mathrm{Br}(B)[3] = 0, \text{and } H^3_{\text{ét}}(B, \mathbb{Z}/3) = 0.$$

4. The left vertical row is exact by Lemma 3.2.4, because we have
$$H^2_{nr}(k(Y)/K, \mathbb{Z}/3) \cong H^2_{nr}(K(S_0)/K, \mathbb{Z}/3) \cong Br(S_0)[3],$$
where S_0 is the Brauer-Severi surface (over K) corresponding to the generic fiber α. Hence S_0 is smooth and we have the last isomorphism in the above statement.

5. In the right vertical row, the map τ is induced by the field extensions $k(S) \subset k(T)$, coincides with the induced map
$$k(S)^\times/k(S)^{\times 3} \to k(T)^\times/k(T)^{\times 3}.$$
If S is not contained in the discriminant locus, the generic fiber of $T \to S$ is geometrically integral. Then $k(S)$ is algebraically closed in $k(T)$, and thus the induced map above is injective. If $S = S_i$ is a component of the discriminant locus, then after taking the base change to the cubic extension $F/k(S_i)$ defined by the residue class $\gamma_i \in H^1(k(S_i), \mathbb{Z}/3)$, the

generic fiber of $T_i \to S_i$ is a union of three Hirzebruch surfaces $\mathbb{F}_1$, meeting transversally so that any pair of them meet along a fiber of one and a (-1)-curve of the other. (This is correct because T_i is the unique irreducible componenet dominate S_i in the preimage of S_i under π, with the third assumption in Definition 3.2.1). In this case, the low degree long exact sequence from the Hochschild-Serre spectral sequence

$$H^p(\mathrm{Gal}(F/k(S_i)), H^q(\mathrm{Spec}(F), \mathbb{Z}/3)) \Rightarrow H^{p+q}(\mathrm{Spec}(k(S_i)), \mathbb{Z}/3)$$

implies the kernel of the natural map $H^1(k(S_i), \mathbb{Z}/3) \to H^1(F, \mathbb{Z}/3)$ is generated by γ_i. By the last assumption in Definition 3.2.1, we know $\ker(\tau) = \Gamma$.

Then we can prove that $H^2_{nr}(k(Y)/Y, \mathbb{Z}/3)$ lies in the image of σ. In fact, let $\xi \in H^2_{nr}(k(Y)/Y, \mathbb{Z}/3)$ and denote by ξ again its the image in $H^2_{nr}(k(Y)/Y, \mathbb{Z}/3)$. Then ξ is killed by $\oplus\partial^2_T$. If ξ is not in the image of σ, it lifts to a class $\xi' \in H^2(K, \mathbb{Z}/9)$ by Lemma 3.2.4. We have the following exact sequence which is similar to the second row in above diagram with coefficients $\mathbb{Z}/9$:

$$0 \to H^2(K, \mathbb{Z}/9) \xrightarrow{\oplus\partial^2_S} \bigoplus_{S \in B^{(1)}} H^1(k(S), \mathbb{Z}/9) \xrightarrow{\oplus\partial^1_C} \bigoplus_{C \in B^{(2)}} \mathbb{Z}/9$$

Hence at least one residue $\partial^2_S(\xi')$ must have order 9 (since $\oplus\partial^2_S$ is injective both for 3-torsion and 9-torsion cases). On the other hand,

$$\ker\Big(\bigoplus_{S \in B^{(1)}} H^1(k(S), \mathbb{Z}/9) \to \bigoplus_{T \in Y_B^{(1)}} H^1(k(T), \mathbb{Z}/9)\Big) \cong \Gamma$$

This is correct because (again we use F to denote a separable closure of $k(S_i)$) in the long exact sequence associate to

$$H^p(\mathrm{Gal}(F/k(S_i)), H^q(\mathrm{Spec}(F), \mathbb{Z}/9)) \Rightarrow H^{p+q}(\mathrm{Spec}(k(S_i)), \mathbb{Z}/9)$$

We have

$$0 \to H^1(Gal(F/k(S_i)), \mathbb{Z}/9) \to H^1(\mathrm{Spec}(k(S_i)), \mathbb{Z}/9) \to H^1(\mathrm{Spec}(F), \mathbb{Z}/9)$$

As we can calculate étale cohomology of spectrum of a field using Galois cohomology, we have the kernel:

$$H^1(Gal(F/k(S_i)), \mathbb{Z}/9) \cong Hom_{cont}(\mathbb{Z}/3, \mathbb{Z}/9) \cong \mathbb{Z}/3$$

While the kernel for those S doesn't belongs to the discriminant locus is clearly zero by the same argument in 3-torsion case.

Now we notice that Γ has no elements of order 9, this means $\partial^2_S(\xi^{'})$ can't be mapped to 0 in $\bigoplus_{T\in Y^{(1)}_B} H^1(k(T), \mathbb{Z}/3)$, hence a contradiction.

The above diagram chasing in fact gives us

$$H^2_{nr}(k(Y)/Y, \mathbb{Z}/3) \cong \Gamma \cap \ker(\oplus\partial^1_C)/\langle\alpha\rangle \cong H/\langle\alpha\rangle\,.$$

Next we determine classes in H that are in $H^2_{nr}(k(Y)/k, \mathbb{Z}/3)$. In particular, we show that the subgroup H' defined earlier is contained in $H^2_{nr}(k(Y)/k, \mathbb{Z}/3)$. We do this by checking whether the classes in H' are unramified with respect to all divisorial valuations μ of $k(Y)$ (and not just those that come from prime divisors on Y). Consider a class $\beta \in H$, viewed as an element in $H^2(K, \mathbb{Z}/3)$. Denote by β' the image of β in $H^2(k(Y), \mathbb{Z}/3)$. We aim to show that β' is unramified on Y if β is in H'. Using the definition of H, it is sufficient to check this for valuations whose centers on B has codimension at least 1. In the following, we use $\mathcal{O}$ to denote the local ring of μ in B.

Case 1: The center of μ on B is not contained in the discriminant locus: In this case, for any surface S passing through the center of μ, we have

$$\partial^2_S(\beta) = 0.$$

Then [ABvBP20, Proposition 2.1] tells us β is in the image of $H^2_{\text{ét}}(\mathcal{O}, \mathbb{Z}/3)$. Hence $\beta' = \sigma(\beta)$ is also unramified with respect to μ in this case.

Case 2: The center of μ on B is contained in the discriminant locus, but not in the intersection of two or more components: Now the center is contained in S_i for a unique i. Recall that the i^{th} component x_i of $\oplus\partial^2_S(\beta)$ is 0, 1 or 2. If $x_i = 0$, by an argument same as Case 1, β is in the image of $H^2_{\text{ét}}(\mathcal{O},\mathbb{Z}/3)$. Similarly, if $x_i = 1$, $\beta - \alpha$ is in the image of $H^2_{\text{ét}}(\mathcal{O},\mathbb{Z}/3)$. Finally, if $x_i = 2$, $\beta - 2\alpha$ is in the image of $H^2_{\text{ét}}(\mathcal{O},\mathbb{Z}/3)$. Notice that

$$\beta' = \sigma(\beta) = \sigma(\beta - \alpha) = \sigma(\beta - 2\alpha).$$

So in all three conditions, we have β' is unramified with respect to μ in this case.

Case 3: The center μ on B is a curve C that is an irreducible component of $S_i \cap S_j$: In this case, we again check the possible values of x_i and x_j in $\oplus\partial^2_S(\beta)$. We have the following cases:

Case 3(a): If $x_i = x_j$, then the argument in Case 2 above gives us that at least one of $\beta, \beta - \alpha$ or $\beta - 2\alpha$ lies in the image of $H^2_{\text{ét}}(\mathcal{O},\mathbb{Z}/3)$. So we are done in this situation.

Case 3(b): $(x_i, x_j) = (0,1)$ or $(1,0)$: By symmetry, we can assume $(x_i, x_j) = (1,0)$. Notice that

$$3 \,\Big|\, \left(\partial^1_C(\gamma_i) + \partial^1_C(\gamma_j)\right)$$

by the exactness of the second row in the diagram. Then we must have

$$\partial^1_C(\gamma_i) = \partial^1_C(\gamma_j) = 0$$

This means that a rational function representing the class

$$\gamma_i \in H^1(k(S_i),\mathbb{Z}/3) = k(S_i)^\times / k(S_i)^{\times 3}$$

has a zero or a pole of order divisible by 3 along C. Without loss of generality, we may assume that the function associated with γ_i is contained in the local ring $\mathcal{O}_{S_i,C}$ of C in S_i.We call this function f_{γ_i}. Let t be a local parameter for C in $\mathcal{O}_{S_i,C}$. Such a local parameter exists as C is a Cartier divisor on S_i, which in turn follows since S_i is assumed to be factorial along C. Then $\frac{f_{\gamma_i}}{t^{\mu_C(f_{\gamma_i})}}$ is a unit, and hence any preimage in $\mathcal{O}$ is also a unit (See Remark 3.3.1 below). Call such a preimage u_{γ_i}, which may be viewed as a rational function in K. Assume π_{S_i} is a local parameter of S_i in $\mathcal{O}$. Consider the symbol algebra $(u_{\gamma_i}, \pi_{S_i}) \in H^2(K,\mathbb{Z}/3)$. Let S be a surface containing C. By Lemma

2.3.1, we have

$$\partial_S^2(u_{\gamma_i}, \pi_{S_i}) = (-1)^{\mu_S(u_{\gamma_i})\mu_S(\pi_{S_i})} \frac{\bar{u}_{\gamma_i}^{\mu_S(\pi_{S_i})}}{\pi_{S_i}^{\mu_S(u_{\gamma_i})}} = \begin{cases} \bar{u}_{\gamma_i} & \text{if } S = S_i \\ 0 & \text{if } S \neq S_i \end{cases}$$

On the other hand, $\bar{u}_{\gamma_i} = \gamma_i$ by construction, so we have

$$\partial_{S_i}^2(u_{\gamma_i}, \pi_{S_i}) = \gamma_i = \partial_{S_i}^2(\beta)$$

$$\partial_{S_j}^2(u_{\gamma_i}, \pi_{S_i}) = 0 = \partial_{S_j}^2(\beta)$$

Also $\partial_S^2(\beta) = 0$ if S is a surface passing through C other than S_i and S_j.(In fact, by our assumption, such an S is not in the discriminant locus and so this agrees with this conclusion.) Hence [ABvBP20, Proposition 2.1] tells us $\beta - (u_{\gamma_i}, \pi_{S_i})$ is in the image of $H^2_{\text{ét}}(\mathscr{O}, \mathbb{Z}/3)$. Hence

$$\partial_\mu^2(\sigma(\beta - (u_{\gamma_i}, \pi_{S_i}))) = 0$$

It then suffices to show that

$$\partial_\mu^2(\sigma((u_{\gamma_i}, \pi_{S_i}))) = \pm\bar{u}_{\gamma_i}^{\mu(\pi_{S_i})} = 0 \in H^1(k(\mu), \mathbb{Z}/3)$$

By assumption, $\bar{u}_{\gamma_i}|_C$ is trivial, hence so is $\bar{u}_{\gamma_i}^{\mu(\pi_{S_i})}$ as the center of μ is C.

Case 3(c): $(x_i, x_j) = (0, 2)$ or $(2, 0)$: By symmetry, we can assume $(x_i, x_j) = (2, 0)$. Now the proof is essentially same as Case 3(b), which shows that

$$\partial_\mu^2(\sigma(\beta - 2(u_{\gamma_i}, \pi_{S_i}))) = 0.$$

It follows that $\partial_\mu^2(\sigma(\beta)) = 0$. and so β' is unramified along μ.

Case 3(d): $(x_i, x_j) = (1, 2)$ or $(2, 1)$: Assume $(x_i, x_j) = (2, 1)$. Then applying Case $3(b)$ to the class $\beta - \alpha$, we see that this case is also proved.

Case 4: The center of μ on B is a point $P \in C$, here C is as in case 3, and S_i, S_j are the only surfaces among the $S_1, \cdots, S_n$ that pass through P. As we have seen in the discussion of Case 3, we can reduce to the case when $(x_i, x_j) = (1, 0)$. Hence we again have $\partial_C^1(\gamma_i) = \partial_C^1(\gamma_j) = 0$.

In fact, this is true for any curve C' that contains P and is contained in $S_i \cup S_j$. Choose a function $f_{\gamma_i} \in k(S_i)$ representing the class γ_i. Then let $C_1, \cdots, C_N$ be all irreducible curves through P that are either a zero or a pole for the function f_{γ_i}. Pick local equations t_ℓ of C_ℓ in $\mathcal{O}_{S_i,P}$, and consider the following rational function on S_i:

$$\frac{f_{\gamma_i}}{\left(t_1^{\mu_{C_1}(f_{\gamma_i})} \cdots t_N^{\mu_{C_N}(f_{\gamma_i})}\right)}.$$

Since S_i is assumed to be factorial, in particular, normal at P, the above rational function is a unit locally around P. Hence it can be lifted to a unit in $\mathcal{O}$. Then we can repeat the rest of the proof as in Case 3(b). (Notice that every element in $k(P)$ is a cube since k is algebraically closed, so the last step of Case 3(b) is automatically true.)

Case 5: The center of μ on B is a point P that lies on exactly three distinct surfaces S_i, S_j, S_l: We consider the possible values of (x_i, x_j, x_l). If $x_i = x_j = x_l$, then one of $\beta, \beta - \alpha$ or $\beta - 2\alpha$ is unramified. By symmetry and up to subtraction by α or 2α, the only remaining cases are $(1,0,0)$, $(1,1,0)$, and $(2,1,0)$. Notice that $(2,1,0) = (1,1,0) + (1,0,0)$, and that the case $(1,1,0)$ is equivalent to the case $(2,0,0)$. Hence, we only need to consider the case $(1,0,0)$, which is same as Case 4. Now the rest of the proof is same as in Case 4.

□

Remark 3.3.1. In Case 3(b) in Theorem 3.3.1, we claimed that if $\bar{x} \in \mathcal{O}_{S_i,C}$ is a unit, then any preimage x in $\mathcal{O} = \mathcal{O}_{B,C}$ is also a unit. In fact, we have

$$\mathcal{O}_{S_i,C} \cong \mathcal{O}/(\pi_{S_i}).$$

As $\bar{x}$ is a unit in $\mathcal{O}_{S_i,C}$, there exist a $\bar{y} \in \mathcal{O}_{S_i,C}$ such that $\bar{x}\bar{y} = 1 \in \mathcal{O}_{S_i,C}$. Hence there exist $t \in \mathcal{O}$, such that

$$xy = 1 + \pi_{S_i} t \in \mathcal{O}$$

Notice that $\pi_{S_i} t$ is contained in the maximal ideal of $\mathcal{O}$, so $1 + \pi_{S_i} t$ is a unit in $\mathcal{O}$. Hence any preimage x is also a unit in $\mathcal{O}$.

We prove an immediate corollary in which we weaken the hypothesis about factoriality when $n = 2$. In this case, the discriminant locus has exactly two irreducible components. We prove that it is sufficient to have only one of them factorial at their intersection to make the unramified Brauer group nontrivial:

Corollary 3.3.2. *Assume $n = 2$. We continue with the same hypothesis as in the theorem except the following change: we replace the requirement* (3) *by the following:*
(3') S_1 is factorial at every point of $S_1 \cap S_2$.
Then $H^2_{nr}(k(Y)/k, \mathbb{Z}/3)$ is nontrivial and hence Y is not stably rational.

Proof. In this case, the Brauer class β in H' whose representative is $(1, 0)$ can be lifted to a nontrivial unramified Brauer class in $H^2(k(Y)/k, \mathbb{Z}/3)$ □

CHAPTER 4

FLATNESS AND EXAMPLES

4.1 A distinguished example

In this section, we will construct a Brauer-Severi surface bundle over $\mathbb{P}^3$ that is stably non-rational. We use Corollary 3.3.2 for this purpose.

Example 4.1.1. Consider the following two surfaces in $\mathbb{P}^3_{\mathbb{C}} = \textbf{Proj}\,\mathbb{C}[x_0, x_1, x_2, x_3]$:

$$S_1 : \{x_0^9 + (x_1^3 - x_2^3)(x_2^3 - x_3^3)(x_3^3 - x_1^3) = 0\}$$

$$S_2 : \{\left(x_0^9 + (x_1^3 - x_2^3)(x_2^3 - x_3^3)(x_3^3 - x_1^3)\right)\left(x_0^9 - x_1^3x_2^3x_3^3\right) + x_1^6x_2^6x_3^6 = 0\}$$

In the following, we use F_{S_1}, F_{S_2} to denote the equation defines S_1, S_2 separately. We start by checking that both S_1 and S_2 are irreducible and reduced:

- S_1 is irreducible and reduced. This follows directly from the fact that the singular locus of S_1 has dimension 0. In fact, S_1 only singular at 12 isolated points:

  $$[0:1:0:0]\,,\ [0:0:1:0]\,,\ [0:0:0:1]$$

  $$[0:\omega:1:1]\,,\ [0:1:\omega:1]\,,\ [0:1:1:\omega]$$

  $$[0:\omega^2:1:1], [0:1:\omega^2:1],\ [0:1:1:\omega^2]$$

  $$[0:\omega^2:\omega:1]\,, [0:\omega:\omega^2:1]\,, [0:1:1:1]$$

  Here ω is a primitive 3^{rd} roots of unity. If S_1 is not reduced, then the singular locus would have dimension 2. If S_1 is not irreducible, the singular locus would have dimension at least 1 by Bèzout theorem.

- S_2 is irreducible and reduced. We may rewrite the equation defining S_2 as:

  $$x_0^{18} + P(x_1, x_2, x_3)x_0^9 - x_1^3x_2^3x_3^3P(x_1, x_2, x_3)$$

  where $P(x_1, x_2, x_3) = (x_1^3 - x_2^3)(x_2^3 - x_3^3)(x_3^3 - x_1^3) - x_1^3x_2^3x_3^3$. We may consider the above polynomial as an element in $\mathbb{C}[x_0, x_1, x_2, x_3] = \mathbb{C}[x_1, x_2, x_3][x_0]$, which is a UFD. Hence to check it is irreducible, it is sufficient to use Eisenstein's criterion: We need to find a prime ideal $\mathfrak{p}$ in $\mathbb{C}[x_1, x_2, x_3][x_0]$, such that

$$P(x_1, x_2, x_3) \in \mathfrak{p},$$

$$x_1^3 x_2^3 x_3^3 P(x_1, x_2, x_3) \in \mathfrak{p} \text{ and}$$

$$x_1^3 x_2^3 x_3^3 P(x_1, x_2, x_3) \notin \mathfrak{p}^2.$$

It is evident that an appropriate prime ideal exists if $P(x_1, x_2, x_3)$ has an irreducible factor with multiplicity 1 and is coprime to $x_1 x_2 x_3$. In fact, any irreducible factor of $P(x_1, x_2, x_3)$ is inherently coprime to $x_1 x_2 x_3$. Therefore it suffices to provide a single regular point of $P(x_1, x_2, x_3)$ to show the existence of such an irreducible factor. Finally, we directly check that $(1, 1, 0)$ is a regular point of $P(x_1, x_2, x_3)$. Hence S_2 is irreducible.

Now as we have already shown S_2 is irreducible, it is sufficient to find a smooth point in S_2 to show it is reduced. Indeed, one can easily check that $[(-56)^{\frac{1}{9}} : 1 : 2 : 0]$ is indeed a smooth point of S_2.

We choose rational triple covers of S_1 and S_2 defined by:

$$\gamma_1 = \overline{\frac{x_2^3 - x_3^3}{x_0^3}} \in H^1(\mathbb{C}(S_1), \mathbb{Z}/3) \cong \mathbb{C}(S_1)^\times / \mathbb{C}(S_1)^{\times 3}$$

$$\gamma_2 = \overline{\frac{x_0^9 - x_1^3 x_2^3 x_3^3}{x_0^9}} \in H^1(\mathbb{C}(S_2), \mathbb{Z}/3) \cong \mathbb{C}(S_2)^\times / \mathbb{C}(S_2)^{\times 3}$$

We claim the triple covers γ_1, γ_2 are not trivial: In fact, by Lemma 2.3.1, the residue of γ_1 of a valuation centered at the point $[0 : \omega : 1 : 1]$ is $1 \in \mathbb{Z}/3$. Hence γ_1 is not trivial. To show γ_2 is not trivial is equivalent to show F_{S_1} is not a cubic in the function field of S_2. And it's true because the residue of $\overline{\frac{F_{S_1}}{x_0^9}}$ of a valuation centered at the point $[0 : 0 : 1 : 1]$ is $1 \in \mathbb{Z}/3$. Hence γ_1, γ_2 are not trivial.

Consider the corresponding Bloch-Ogus exact sequence:

$$0 \longrightarrow Br(\mathbb{C}(\mathbb{P}^3_{\mathbb{C}}))[3] \overset{\oplus \partial^2_S}{\longrightarrow} \bigoplus_{S \in (\mathbb{P}^3_{\mathbb{C}})^{(1)}} H^1(k(S), \mathbb{Z}/3) \overset{\oplus \partial^1_C}{\longrightarrow} \bigoplus_{C \in (\mathbb{P}^3_{\mathbb{C}})^{(2)}} H^0(k(C), \mathbb{Z}/3)$$

We have $\oplus\partial^1_C(\gamma_1) = \oplus\partial^1_C(\gamma_2) = 0$. In fact, it is easy to check that for any curve C such that γ_1(or γ_2) has a zero or pole along C, the order is divided by 3. Hence

$$(1, \cdots, 1, \gamma_1, 1, \cdots, 1, \gamma_2, 1, \cdots) \in \bigoplus_{S \in (\mathbb{P}^3_{\mathbb{C}})^{(1)}} H^1(k(S), \mathbb{Z}/3)$$

can be lifted to a Brauer class $[\mathscr{A}] = [(\frac{F_{S_2}(x_2^3 - x_3^3)}{x_0^{21}}, \frac{F_{S_1}}{x_0^9})_\omega] \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^3_{\mathbb{C}}))[3]$. This can be directly checked by Lemma 2.3.1.

By Theorem 4.2.1, the cyclic algebra $\mathscr{A}$ gives out a Brauer-Severi surface bundle $Y \to \mathbb{P}^3_{\mathbb{C}}$. This Brauer-Severi surface bundle has a good discriminant locus. We prove this by checking the conditions in Definition 3.2.1. Here is the list of corresponding arguments:

1. We already proved that S_1 and S_2 are reduced.

2. The behavior of a general fiber over S_1 and S_2 is given by [Mae97, Thm. 2.1].

3. The induced triple cover over S_1 and S_2 are irreducible because γ_1, γ_2 are not trivial.

4. To show the last requirement in Definition 3.2.1 is true, we have the following commutative diagram:

$$\begin{array}{ccc} F^\times/F^{\times 3} & \xrightarrow{a} & F(u,v)^\times/F(u,v)^{\times 3} \\ b\uparrow & & d\uparrow \\ k(S_i)^\times/k(S_i)^{\times 3} & \xrightarrow{\tau_i} & k(T_i)^\times/k(T_i)^{\times 3} \end{array}$$

   Where T_i is defined right after the large diagram in Theorem 3.3.1. For S_1, b is induced by the cubic extension defined by γ_1, d is induced by the cubic extension defined by $\frac{F_{S_2}(x_2^3 - x_3^3)}{x_0^{21}}$, which is equal to the cubic class defined by γ_1 ([Art82a, Thm. 2.1].) Note that a is injective, and $\ker(b) =< \gamma_1 >$. On the other hand, a diagram chasing as in part (5) in proof of Theorem 3.3.1 shows that $\ker(\tau_1)$ contains $< \gamma_1 >$. This forces d to be injective and $\ker(\tau_1) =< \gamma_1 >$. Same argument works for S_2.

On the other hand, we list all irreducible components of $S_1 \cap S_2$:

$$D = 6D_1 + 6D_2 + 6D_3$$

Where

$$D_1 = \{x_1 = 0, x_0^9 - x_2^6 x_3^3 + x_2^3 x_3^6 = 0\}$$

$$D_2 = \{x_2 = 0, x_0^9 - x_3^6 x_1^3 + x_3^3 x_1^6 = 0\}$$

$$D_3 = \{x_3 = 0, x_0^9 - x_1^6 x_2^3 + x_1^3 x_2^6 = 0\}$$

One can easily check they are indeed irreducible using Eisenstein's criterion by viewing those polynomials as elements in $\mathbb{C}[x_1, x_2, x_3][x_0]$. Notice that D_1 passes through only two singular points of S_1: $[0:0:1:0]$ and $[0:0:0:1]$. It is straightforward to check D_1 is indeed a Cartier divisor of S_1, even along these two singular points:

Lemma 4.1.2. *D_i are Cartier divisors of S_1.*

Proof. By symmetry, it is sufficient to check the behavior of D_1 at singular points of S_1. Notice that D_1 only passes through two singular points of S_1: $[0:0:1:0]$ and $[0:0:0:1]$. Let $P = [0:0:0:1]$, then we have the local ring:

$$\mathcal{O}_{S_1,P} = (\mathbb{C}[x_0, x_1, x_2]/(x_0^9 + (x_1^3 - x_2^3)(x_2^3 - 1)(1 - x_1^3)))_{(x_0,x_1,x_2)}$$

By expanding the equation defining S_1, we have

$$x_0^9 - x_2^6 + x_2^3 = x_1^3(x_1^3 x_2^3 - x_2^6 - x_1^3 + 1) \in \mathcal{O}_{S_1,P}$$

Notice $x_1^3 x_2^3 - x_2^6 - x_1^3 + 1$ is a unit in $\mathcal{O}_{S_1,P}$, hence the ideal defining D_1, which is $(x_1, x_0^9 - x_2^6 + x_2^3)$, is generated by one element x_1. Similarly one can do the calculation for the point $[0:0:1:0]$. As a result, D_1 is a Cartier divisor of S_1.

□

Finally, we need to check both $\gamma_1|_{D_i}$ and $\gamma_2|_{D_i}$ are trivial for $i \in \{1, 2, 3\}$. These are directly following from the choices of γ_1 and γ_2. Hence in this example, using notations in Theorem 3.3.1, we have $H' = H = \Gamma = \mathbb{Z}/3 \times \mathbb{Z}/3$. By Corollary 3.3.2, the unramified Brauer group of Y contains a subgroup $\mathbb{Z}/3$, hence Y is not stably rational.

4.2 Flatness

In this section, we check the cyclic algebra

$$\mathscr{A} = \left(\frac{F_{S_2}(x_2^3 - x_3^3)}{x_0^{21}}, \frac{F_{S_1}}{x_0^9}\right)_\omega$$

indeed gives us a Brauer-Severi surface bundle over $\mathbb{P}^3_{\mathbb{C}}$ as in Definition 1.0.6. We keep the notation in Example 4.1.1 through out this section. The definition of a general Brauer-Severi scheme is given by Van den Bergh in [VdB88]. In [See99], Seelinger gave an alternating description of Brauer-Severi scheme which is easier to use in our case. See also Section 1 in [Mae97] for the discussion of the following definitions:

Definition 4.2.1. Let Λ be a sheaf of $\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}}}$ algebra that is torsion free and coherent as an $\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}}}$ module. We say Λ is an $\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}}}$-order in $\mathscr{A}$ if Λ contains $\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}}}$ and

$$\Lambda \otimes_{\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}}}} \mathbb{C}(\mathbb{P}^3_{\mathbb{C}}) \cong \mathscr{A}$$

Definition 4.2.2. For each point $p \in \mathbb{P}^3_{\mathbb{C}}$, let $\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}},p}$ denote the regular local ring of $\mathbb{P}^3_{\mathbb{C}}$ at p. We say a finitely generated $\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}},p}$ algebra Λ_p is an $\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}},p}$-order in $\mathscr{A}$, if Λ_p is torsion free and

$$\Lambda_p \otimes_{\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}},p}} \mathbb{C}(\mathbb{P}^3_{\mathbb{C}}) \cong \mathscr{A}$$

Remark 4.2.3. In this paper, we always assume an order is locally free.

Recall that (3.4) of [Mae97] describes an $\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}}}$-order which we again denote by Λ in the following, we denote its localization at a point p by Λ_p.

Definition 4.2.4. Let V_Λ (respectively, V_{Λ_p}) be the functor from the category of $\mathbb{P}^3_{\mathbb{C}}$-schemes (respectively, Spec$(\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}},p})$-schemes) to the category of sets:

$$V_\Lambda(S) = \{[z] \in G_n[(\Lambda \otimes_{\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}}}} S)^\vee] \mid z \cdot u = N_S(u)z\ ,\ \forall u \in (\Lambda \otimes_{\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}}}} S)^*\}$$

$$V_{\Lambda_p}(S) = \{[z] \in G_n[(\Lambda_p \otimes_{\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}},p}} S)^\vee] \mid z \cdot u = N_S(u)z\ ,\ \forall u \in (\Lambda \otimes_{\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}},p}} S)^*\}$$

where $\vee$ denotes the dual sheaf, $*$ denotes the unit group , N_S is the reduced norm and G_n denotes the functor of Grassmannian of n-quotients([VdB88, Def.1]). These functors are represented by schemes as these are closed subschemes of the Grassmannian, which we call the Brauer-Severi scheme (associated to Λ, Λ_p) and again denote them by V_Λ, V_{Λ_p}.

Theorem 4.2.1. $Y = V_\Lambda$ is a Brauer-Severi surface bundle over $\mathbb{P}^3_{\mathbb{C}}$.

Proof. According to Definition 4.2.4, for every closed point p in $\mathbb{P}^3_{\mathbb{C}}$, we have the following commutative diagram of schemes:

$$\begin{array}{ccc} V_\Lambda \times_{\mathbb{P}^3_{\mathbb{C}}} \operatorname{Spec}(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p}) \cong V_{\Lambda_p} & \to & V_\Lambda \\ \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec}(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p}) & \longrightarrow & \mathbb{P}^3_{\mathbb{C}} \end{array}$$

We first show π is a flat morphism. In order to do so, it suffices to show π_p is flat for all closed points $p \in \mathbb{P}^3_{\mathbb{C}}$. Indeed, if this is done, the flat locus of π would be an open subset of $\mathbb{P}^3_{\mathbb{C}}$ containing all closed points, hence is equal to $\mathbb{P}^3_{\mathbb{C}}$. Furthermore, by the "Miracle flatness" theorem [sta23] and the fact that $\operatorname{Spec}(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p})$ is regular, it suffices to show each V_{Λ_p} is Cohen-Macaulay and each fiber of π_p has the same dimension. We do this by a case-by-case argument for all closed points in $\mathbb{P}^3_{\mathbb{C}}$:

Case 1: $p \notin S_1 \cup S_2$. It is well know that Λ is an Azumaya algebra outside of discriminant locus [Art82a]. All fibers of π_p are smooth Brauer-Severi surfaces and furthermore V_{Λ_p} is regular, hence Cohen-Macaulay. By the "Miracle flatness" theorem, π_p is flat in this case.

Case 2: $p \in S_1 \cup S_2$ and $p \notin S_1 \cap S_2$ and $p \notin S_1 \cap \{x_2^3 - x_3^3 = 0\}$. Following ideas from Artin [Art82a], we may write Λ_p as the symbol algebra $(f_p, g_p)_\omega$. That is, Λ_p over $\operatorname{Spec}(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p})$ is generated by x, y subject to the relations

$$x^3 = f_p, y^3 = g_p, xy = \omega yx.$$

Since $p \notin S_1 \cap \{x_2^3 - x_3^3 = 0\}$, it follows that f_p is a unit in $\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p}$. Let $R_p = \mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p}[T]/(T^3 - f_p)$, then

$$\operatorname{Spec}(R_p) \xrightarrow{\tau} \operatorname{Spec}(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p})$$

is an étale neighborhood of $\mathrm{Spec}(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p})$ with τ faithfully flat as it surjects on the underlying topological space. By faithfully flat descent, it suffices to show $V_{\Lambda_p} \otimes R_p$ is flat over $\mathrm{Spec}(R_p)$. In [Art82a], Artin noticed $V_{\Lambda_p} \otimes R_p$ can be viewed as a subalgebra of the 3 by 3 matrices algebra over R_p by setting

$$x = \begin{bmatrix} T & 0 & 0 \\ 0 & T\omega & 0 \\ 0 & 0 & T\omega^2 \end{bmatrix}, y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ g_p & 0 & 0 \end{bmatrix}$$

And $V_{\Lambda_p} \otimes R_p$ can be embedded into $\mathbb{P}^2_{R_p} \times \mathbb{P}^2_{R_p} \times \mathbb{P}^2_{R_p}$ by the following 9 equations with a cyclic permutations in indices:

$$g_p\xi_{11}\xi_{22} = \xi_{12}\xi_{21}$$

$$g_p\xi_{11}\xi_{23} = \xi_{13}\xi_{21}$$

$$g_p\xi_{11}\xi_{32} = g_p\xi_{12}\xi_{31}$$

$$g_p\xi_{11}\xi_{33} = \xi_{13}\xi_{31}$$

$$\xi_{12}\xi_{23} = \xi_{13}\xi_{22}$$

$$g_p\xi_{12}\xi_{33} = \xi_{13}\xi_{32}$$

$$g_p\xi_{21}\xi_{32} = g_p^2\xi_{22}\xi_{31}$$

$$g_p\xi_{21}\xi_{33} = g_p^2\xi_{23}\xi_{31}$$

$$g_p\xi_{22}\xi_{33} = \xi_{23}\xi_{32}$$

Here we use $[\xi_{11} : \xi_{12} : \xi_{13}]$, $[\xi_{21} : \xi_{22} : \xi_{23}]$, $[\xi_{31} : \xi_{32} : \xi_{33}]$ to denote the coordinates in $\mathbb{P}^2_{R_p} \times \mathbb{P}^2_{R_p} \times \mathbb{P}^2_{R_p}$. Note that even though Artin's original calculation assume the local ring is a DVR, [Art82a, Prop 3.6] does work for any regular local rings [Mae97, Thm 2.1]. If g_p is part of a regular system of parameters of R_p, then sections 4 of [Art82a] tells us $V_{\Lambda_p} \otimes R_p$ is indeed regular. If p is a singular point of S_1 or S_2 which doesn't lie in $S_1 \cap S_2$), $V_{\Lambda_p} \otimes R_p$ is not regular. However, from the above equations, a direct calculations show that on each standard affine chart (e.g. $\{\xi_{11} = \xi_{21} = \xi_{31} = 1\}$), $V_{\Lambda_p} \otimes R_p$ can be defined by 4 equations. Hence $V_{\Lambda_p} \otimes R_p$ has an open cover with each a complete intersection in $\mathbb{A}^6_{R_p}$, furthermore the coordinate ring of each affine chart is again a complete intersection as a $\mathbb{C}$-algebra by counting dimensions. Hence $V_{\Lambda_p} \otimes R_p$ is Cohen-Macaulay.

Consider the points (not necessarily closed) $q \in \mathrm{Spec}(R_p)$. If $g_p \in m_q$, the fiber over q is the union of three standard Hirzebruch surfaces $\mathbb{F}_1$, meeting transversally, such that any pair of them meet along a fiber of one and the (-1)-curve of the other ([Art82a, Prop. 3.10]). If $g_p \notin m_q$, the fiber over q is completely determined by $[\xi_{11} : \xi_{12} : \xi_{13}]$, hence is isomorphic to $\mathbb{P}^2_{R_p}$. So, in particular, the closed fiber is the union of three $\mathbb{F}_1$ as desired and all fibers have same relative dimension. Again by the "Miracle flatness" theorem, $V_{\Lambda_p} \otimes R_p$ is flat over $\mathrm{Spec}(R_p)$. As τ is faithfully flat, π_p is flat in this case.

Case 3: $p \in S_1 \cap S_2$ or $p \in S_1 \cap \{x_2^3 - x_3^3 = 0\}$. We again write $\Lambda_p = (f_p, g_p)_\omega$. A same calculation as in [Mae97, Prop. (2.2),Lemma (2.3)] shows that each fiber over $\mathrm{Spec}(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}, p)$ has the same relative dimension and the closed fiber is a cone over a twisted cubic as described in Definition 1.0.6. Furthermore, in [Mae97, Lemma 2.4], Maeda shows the following facts:

a) V_{Λ_p} has an open affine cover

$$V_{\Lambda_p} = U_1 \cup U_2 \cup U_3$$

where U_1 and U_2 are hypersurfaces in $\mathbb{A}^3_{\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p}}$.

b) U_3 is a $(3,3)-$ complete intersection in $\mathbb{A}^4_{\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p}}$.

Notice that here U_1, U_2, U_3 do not have to be regular as f_p, g_p are not part of a local parameters in the maximal ideal of the local ring at p, for some p. For example, when $p = [0 : 0 : 1 : 0]$. However, we can still conclude that U_1, U_2, U_3 are Cohen-Macaulay since they are complete intersection, hence so is V_{Λ_p}. So we have π_p is flat by the "Miracle flatness" theorem.

As the base field is $\mathbb{C}$, the argument above shows that the fiber over each geometric point is indeed one of the three cases in Definition 1.0.6. This shows that V_Λ is a Brauer-Severi surface bundle over $\mathbb{P}^3_{\mathbb{C}}$. We denote it by Y in the following sections of this paper as before. $\square$

Now we explain which surfaces in $\mathbb{P}^3_{\mathbb{C}}$ admit an associated Brauer-Severi surface bundle:

Definition 4.2.5. Let S be a reduced surface in $\mathbb{P}^3_{\mathbb{C}}$ with irreducible components $S = S_1 \cup S_2 \cup \cdots \cup S_m$. Then we say S admits a nontrivial triple cover étale in codimension 1 if there is nontrivial element in

$$\bigoplus_{i=1}^{n} H^1(\mathbb{C}(S_i), \mathbb{Z}/3) \cap \ker(\bigoplus_{C}(\partial^1_C))$$

Where C runs over all irreducible curves in $\mathbb{P}^3$, ∂^1_C is the residue map as in Definition 2.3.1.

It is clear that any surface admits a nontrivial triple cover étale in codimension 1 will give us a 3-torsion Brauer class in $\mathbb{C}(\mathbb{P}^3_{\mathbb{C}})$ by Bloch-Ogus sequence as discussed in Example 4.1.1.

So with the proof of Theorem 4.2.1, we have:

Corollary 4.2.6. *Let $S \subset \mathbb{P}^3_{\mathbb{C}}$ be a reduced surface which admits a nontrivial triple cover étale in codimension 1 (Definition 4.2.5). Assume the 3-torsion Brauer class given by the Bloch-Ogus sequence is represented by a cyclic algebra $\mathscr{A}$ of degree 3. Then there exists a Brauer-Severi surface bundle $Y_S \to \mathbb{P}^3_{\mathbb{C}}$ with discriminant locus S associated to $\mathscr{A}$. Furthermore, Y_S is integral.*

Proof. The first part of this corollary directly follows from a similar discussion of local structures as in Theorem 4.2.1. Next, we show that Y_S is reduced. Indeed, the map $Y_S \to \mathbb{P}^3_{\mathbb{C}}$ is projective, hence closed. Then for any point $y \in Y_S$, there is a point y' lying in a closed fiber such that y specializes to y'. Since any localization of a reduced ring is again reduced, it suffices to check the local ring $\mathcal{O}_{Y_S, y'}$ is reduced. Further more it suffices to assume y' is a closed point. This can be directly checked using the explicit equations given in the proof of Theorem 4.2.1. (Details are discussed in Lemma .0.6.)

On the other hand, Y_S is irreducible because $\mathbb{P}^3_{\mathbb{C}}$ is irreducible, π is flat hence open and the fact that there exists a dense collection of points in $\mathbb{P}^3_{\mathbb{C}}$ whose fiber is irreducible ([sta24a]). Hence Y_S is integral.

□

CHAPTER 5

THE SPECIALIZATION METHOD AND DESINGULARIZATION

5.1 The specialization method

In this section, we review the developments of the specialization methods. More details can be found in [Pir18], [Tsc20] and [CT19].

This idea was firstly introduced by Clemens in [Cle75], a modern version of his theorem is summarized as follows:

Theorem 5.1.1 ([Cle75],[Bea77],[Tsc20, Thm. 1])**.** Let $\phi : \mathcal{X} \to B$ be a flat family of projective threefolds with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber $X := \mathcal{X}_b$ satisfies the following conditions:

- (S) Singularities: X has at most rational double points.
- (O) Obstruction: the intermediate Jacobian of a desingularization $\tilde{X}$ of X is not a product of Jacobians of curves.

Then there exists a Zariski open subset $B^\circ \subset B$ such that for all $b' \in B^\circ$, the fiber $\mathcal{X}_{b'}$ is not rational.

For the definition and first properties of the intermediate Jacobian, we refer to [CG72]. This idea was further developed and modified within last 15 years. A novel idea of obstruction to stable rationality appears firstly to Voisin in [Voi15]:

Definition 5.1.1. Let X be a projective variety of dimension n over a field k, assume $X(k) \neq 0$. Let $[\Delta_X] \in \mathrm{CH}_n(X \times_k X)$ be the diagonal Chow class, namely $\Delta_X = \{x, x\} \subset X \times_k X$. We say X has **integral Chow decomposition of the diagonal**, if

$$[\Delta_X] = [X \times x] + [Z] \in \mathrm{CH}_n(X \times_k X).$$

Here x is a k-point and Z is a n-cycle on $X \times_k X$ whose support has form $D \times_k X$ with D a closed subvariety of X has codimension at least 1.

If X does not admit an integral Chow decomposition of the diagonal, then X is not stably rational ([Tsc20]). Hence, the result in [Voi15] can be summarized as:

Theorem 5.1.2 ([Voi15, Thm .2.1],[Tsc20, Thm. 2])**.** Let $\phi : \mathcal{X} \to B$ be a flat family of projective varieties with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber $X := \mathcal{X}_b$ satisfies the following conditions:

- (S) Singularities: X has at most rational double points.
- (O) Obstruction: there exist a desingularization $\tilde{X}$ of X which does not admit an integral Chow decomposition of the diagonal.

Then a very general (which means the complement of a countable union of Zariski closed subsets of B) fiber of ϕ does not admit an integral decomposition of the diagonal, and in particular, is not stably rational.

In the literature, the special fiber as in Theorem 5.1.1 and Theorem 5.1.2 is called a **reference variety**. Further developments of specialization method are trying to allow other type of singularities and find practically useful obstructions of reference variety. In the present thesis we focus on the refinement given in [CTP16] and [HPT18]:

Theorem 5.1.3 ([HPT18, Thm. 4])**.** Let $\phi : \mathcal{X} \to B$ be a flat family of projective complex varieties with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber $X := \mathcal{X}_b$ is integral and satisfies the following conditions:

- (S) Singularities: X admits an universally CH_0-trivial desingularization (Definition 1.0.3).
- (O) Obstruction: The unramified Brauer group (Definition 2.3.2) of X is nontrivial.

Then a very general fiber of ϕ is not stably rational.

Theorem 5.1.3 is widely used in practice. However, one still need to construct a resolution of the reference variety, which can be very hard for high dimension varieties. In [Sch19a, Proposition 26] and [Sch19b], Schreieder provide a new refinement which only involve a purely cohomological criteria:

Theorem 5.1.4 ([Sch19a, Proposition 26])**.** Let $\phi : \mathscr{X} \to B$ be a flat family of projective complex varieties with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber $X := \mathscr{X}_b$ is integral satisfies the following conditions:

- (O) Obstruction: The unramified Brauer group of X is nontrivial.
- (S) Singularities: Suppose there exists a resolution of singularities $\tau : \tilde{X} \to X$, let U be the smooth locus of X, and α be a nontrivial unramified Brauer class. We require the restriction of α to the function field of each irreducible component of $\tilde{X} - \tau^{-1}(U)$ is trivial (See Lemma 5.2.1).

Then a very general fiber of ϕ is not stably rational.

Note that Schreieder's result can be further generalized by replacing resolutions of singularities by alterations ([Sch19b]). In a very recent paper, Pirutka modified Schreieder's idea in [Pir23]. By introducing the notation of relative unramified Brauer group of a fibration, one can check the singularity requirements by passing to the completion of local rings.

5.2 Desingularization

In this section, we show that Example 4.1.1 is a reference variety by applying Theorem 5.1.4.

Lemma 5.2.1 ([Sch21a, Proposition 4.8(a)])**.** *Let Y be a projective variety over a field k. Let $E \subset Y$ be an irreducible subvariety such that the local ring of Y at the generic point η_E of E, denoted by $\mathscr{O}_{Y,\eta_E}$, is a regular local ring. Then there exists a restriction map:*

$$\mathrm{Res}^Y_E : H^2_{nr}(k(Y)/k, \mathbb{Z}/3) \to H^2(k(E), \mathbb{Z}/3)$$

Proof. Let $\alpha \in H^2_{nr}(k(Y)/k, \mathbb{Z}/3)$ be an unramified Brauer class (Definition 2.3.2). Notice that by assumption, $\mathscr{O}_{Y,\eta_E}$ is a regular local ring with residue field $k(E)$ and fraction field $k(Y)$. We have the following diagram:

$$
\begin{array}{ccc}
0 & & \\
\downarrow & & \\
H^2(\mathcal{O}_{Y,\eta_E}, \mathbb{Z}/3) & \longrightarrow & H^2(k(E), \mathbb{Z}/3) \\
\downarrow & \nearrow & \\
H^2(k(Y), \mathbb{Z}/3) & & \\
\downarrow{\scriptstyle \oplus\partial_\nu^2} & & \\
\bigoplus H^1(k(\nu), \mathbb{Z}/3) & &
\end{array}
$$

here the left column is given by [CT95, Theorem 3.8.3]. The horizontal map is given by the functoriality in étale cohomology. Now that α is an unramified Brauer class, it is killed by $\oplus\partial_\nu^2$. Hence α comes from a class in $H^2(\mathcal{O}_{Y,\eta_E}, \mathbb{Z}/3)$, which can be further mapped to $H^2(k(E), \mathbb{Z}/3)$ by the horizontal map. □

We use notation in Example 4.1.1 and Lemma 5.2.1. Let $U \subset Y$ be the smooth locus of Y. Let $\alpha_1 \in Br(\mathbb{C}(\mathbb{P}^3_{\mathbb{C}}))[3]$ be the Brauer class which has nontrivial residue γ_1 along S_1, and trivial residues everywhere else. By Lemma 2.3.1, α_1 can be represents by the cyclic algebra

$$
(\frac{x_2^3 - x_3^3}{x_0^3}, \frac{F_{S_1}}{x_0^9})_\omega.
$$

By arguments in Example 4.1.1, α_1 can be lifted to a nontrivial unramified Brauer class

$$
\tilde{\alpha}_1 \in H^2_{nr}(\mathbb{C}(Y)/\mathbb{C}, \mathbb{Z}/3)
$$

Now we prove that the second hypothesis in [Sch19a, Proposition 26] is true in our case:

Lemma 5.2.2. *Let $\pi : Y \to \mathbb{P}^3_{\mathbb{C}}$ be the Brauer-Severi surface bundle in Example 4.1.1. Let U be the smooth locus of Y. Then there exists a resolution of singularities $f : \tilde{Y} \to Y$, such that for each irreducible component E of $\tilde{Y} - f^{-1}(U)$, $\mathrm{Res}_E^{\tilde{Y}}(\tilde{\alpha}_1)$ is trivial.*

Proof. The existence of resolution of singularities of Y is guaranteed by Hironaka's theorem [Hir64]. We can further assume without loss of generality that each E is a prime divisor of $\tilde{Y}$.

Recall that Y has singular locus of codimension at least 2. So for every irreducible component E of $\tilde{Y} - f^{-1}(U)$, $f(E)$ has dimension at most 3. On the other hand, since that the generic fiber of π

is smooth, and the generic fiber over each irreducible component of the discriminant locus (namely $S_1 \cup S_2$) is the union of three standard Hirzebruch surfaces $\mathbb{F}_1$ described in Definition 1.0.6. We know $\pi(f(E))$ has dimension at most 1 (This follows from that local model of π over S_1 and S_2 is smooth, see the discussion in Theorem 4.2.1). In other words, each E in $\tilde{Y}$ would dominate a curve or a point in $S_1 \cup S_2$.

In the following of this proof, let $K = \mathbb{C}(\mathbb{P}^3_{\mathbb{C}})$ be the function field of $\mathbb{P}^3_{\mathbb{C}}$. Denote by p_E the generic point of $\pi(f(E))$, by K_{P_E} the field of fractions of the regular complete local ring $\hat{\mathcal{O}}_{\mathbb{P}^3_{\mathbb{C}},p_E}$. We give a case by case argument according to the generic point $p_E \in \mathbb{P}^3_{\mathbb{C}}$ of $\pi(f(E))$:

Case 1: $p_E \notin S_1$. Then $\mathrm{Res}^{\tilde{Y}}_E(\tilde{\alpha}_1) = 0$ simply follows from the fact that p_E does not belongs to the discriminant locus of α_1 (see e.g. the proof of [Sch19b, Proposition 5.1(2)]).

Case 2: $p_E \in S_1 - S_2$. Consider the following commutative diagram coming from functoriality:

$$\begin{array}{ccc} H^2(K(Y_K), \mathbb{Z}/3\mathbb{Z}) & \rightarrow & H^2(K_{p_E}(Y_K), \mathbb{Z}/3\mathbb{Z}) \\ \uparrow & & \uparrow \\ H^2(K, \mathbb{Z}/3\mathbb{Z}) & \longrightarrow & H^2(K_{p_E}, \mathbb{Z}/3\mathbb{Z}) \end{array}$$

Because $\tilde{\alpha}_1$ is unramified, by [Pir23, Proposition 2.5], it suffices to check $\alpha_1 = 0$ in $H^2(K_{p_E}(Y_K), \mathbb{Z}/3\mathbb{Z})$. Indeed, as Y_K is the Brauer-Severi surface associate to the cyclic algebra $\mathscr{A} = (\frac{F_{S_2}(x_2^3-x_3^3)}{x_0^{21}}, \frac{F_{S_1}}{x_0^9})_\omega$ and F_{S_2} is a nonzero cube when $F_{S_1} = 0$ in the residue field of $\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p_E}$. By Cohen's structure theorem [Coh46, Theorem 15], the residue field embeds into K_{P_E}. Hence, after taking base changes to K_{p_E},

$$(\frac{F_{S_2}(x_2^3 - x_3^3)}{x_0^{21}}, \frac{F_{S_1}}{x_0^9})_\omega \cong (\frac{x_2^3 - x_3^3}{x_0^3}, \frac{F_{S_1}}{x_0^9})_\omega.$$

This implies that Y_K is also the Brauer-Severi surface associate to α_1, we conclude that $\alpha_1 = 0$ in $H^2(K_{p_E}(Y_K), \mathbb{Z}/3\mathbb{Z})$ by Amitsur's theorem [GS06, Theorem 5.4.1].

Case 3: $p_E \in S_1 \cap S_2$ is a closed point, and $p_E \notin \{x_2^3 - x_3^3 = 0\}$. Then notice α_1 can be represented by the cyclic algebra

$$(\frac{x_2^3 - x_3^3}{x_0^3}, \frac{F_{S_1}}{x_0^9})_\omega \cong (\frac{(x_2 - x_3)^6}{(x_2^3 - x_3^3)^2}, \frac{F_{S_1}}{(x_2^3 - x_3^3)^3})_\omega.$$

Then $\frac{(x_2-x_3)^6}{(x_2^3-x_3^3)^2}$ is nonzero in the residue field of $\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}},p_E}$, which is $\mathbb{C}$. Hence $\frac{(x_2-x_3)^6}{(x_2^3-x_3^3)^2}$ is a cube in the residue field as $\mathbb{C}$ is algebraically closed. By Cohen's structure theorem, the residue field $\mathbb{C}$ embeds into K_{P_E}, hence $\frac{(x_2-x_3)^6}{(x_2^3-x_3^3)^2}$ is also a cube in K_{P_E}. This shows that α_1 is trivial in $H^2(K_{P_E}, \mathbb{Z}/3\mathbb{Z})$. By the same commutative diagram as in case 2, it is clear that $\alpha_1 = 0$ in $H^2(K_{p_E}(Y_K), \mathbb{Z}/3\mathbb{Z})$. We then have $\mathrm{Res}_E^{\tilde{Y}}(\tilde{\alpha}_1) = 0$ by [Pir23, Proposition 2.5].

Case 4: p_E is one of $[0:0:1:1]$,$[0:0:1:\omega]$,$[0:0:1:\omega^2]$. In these cases, by the discussions in Case 4 of the proof of Theorem 3.3.1, we can choose another appropriate representing algebras of α_1:

$$(\frac{x_2^6}{(x_1^3-x_2^3)(x_3^3-x_1^3)}, \frac{F_{S_1}}{x_0^9})_\omega.$$

It is straight forward to check this is a representing algebra of α_1 by applying Lemma 2.3.1. And $\frac{x_2^6}{(x_1^3-x_2^3)(x_3^3-x_1^3)}$ is a nontrivial unit in the residue field of $\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}},p_E}$, which is $\mathbb{C}$. Hence the remaining proof can be done exactly same as in Case 3.

Case 5: $p_E = [0:1:0:0]$. In this case, the proof are the same as in Case 4, by using the following representing algebra of α_1:

$$(\frac{x_1^6}{(x_1^3-x_2^3)(x_3^3-x_1^3)}, \frac{F_{S_1}}{x_0^9})_\omega.$$

Case 6: p_E is one of the generic point of D_1, D_2 and D_3(Example 4.1.1). Note that by the defining equations of D_1, D_2 and D_3, $\frac{x_2^3-x_3^3}{x_0^3}$ is always a nontrivial cube in the residue field of $\mathscr{O}_{\mathbb{P}^3_{\mathbb{C}},p_E}$. Again as the residue field embeds into K_{p_E}, we get $\alpha_1 = 0$ in $H^2(K_{p_E}(Y_K), \mathbb{Z}/3\mathbb{Z})$.

This completes the proof. □

5.3 Main result

With Example 4.1.1, we prove Theorem 1.0.8:

Theorem 1.0.8. *There exists a flat projective family of Brauer-Severi surface bundles over $\mathbb{P}^3_{\mathbb{C}}$, where a general fiber in this family is smooth and not stably rational.*

Proof. By [Sch19a, Proposition 26] and Lemma 5.2.2,we know the Brauer-Severi surface bundle constructed in Example 4.1.1 can be used as a reference variety.

To finish the proof, we need to construct a flat family of Brauer-Severi surface bundles over $\mathbb{P}^3_{\mathbb{C}}$ with Example 4.1.1 as one closed fiber with smooth general fiber. Start with the cyclic algebra from Example 4.1.1:

$$\mathscr{A} = (\frac{F_{S_2}(x_2^3 - x_3^3)}{x_0^{21}}, \frac{F_{S_1}}{x_0^9})_\omega$$

We consider two regular surfaces in $\mathbb{P}^3_{\mathbb{C}}$:

$$G_1 = \{x_0^9 - x_1^9 + x_2^8 x_3 + x_3^8 x_2 = 0\}$$

$$G_2 = \{x_0^{21} + x_1^{21} + x_2^{21} - x_3^{21} = 0\}$$

By Lemma .0.3, both G_1 and G_2 are regular surfaces in $\mathbb{P}^3_{\mathbb{C}}$, and they intersect transversally. Consider the following pencil of cyclic algebras:

$$\mathscr{A}_{[t_0:t_1]} = (\frac{t_0 F_{S_2}(x_2^3 - x_3^3) + t_1(G_2 - F_{S_2}(x_2^3 - x_3^3))}{x_0^{21}}, \frac{t_0 F_{S_1} + t_1(G_1 - F_{S_1})}{x_0^9})_\omega$$

We denote

$$t_0 F_{S_2}(x_2^3 - x_3^3) + t_1(G_2 - F_{S_2}(x_2^3 - x_3^3))$$

and

$$t_0 F_{S_1} + t_1(G_1 - F_{S_1})$$

by $F_{S_2}^{[t_0:t_1]}$ and $F_{S_1}^{[t_0:t_1]}$ respectively. By Lemma .0.4 and Lemma .0.5, when $[t_0 : t_1] \neq [1 : 0]$, both $F_{S_2}^{[t_0:t_1]} = 0$ and $F_{S_1}^{[t_0:t_1]} = 0$ are irreducible surfaces in $\mathbb{P}^3$. Using Lemma 2.3.1, the induced triple covers are given by

$$\gamma_1^{[t_0:t_1]} = \frac{F_{S_2}^{[t_0:t_1]}}{x_0^{21}}, \gamma_2^{[t_0:t_1]} = \frac{x_0^9}{F_{S_1}^{[t_0:t_1]}}$$

Similar to the discussion in Example 4.1.1, note that the residue of $\gamma_1^{[t_0:t_1]}$ of the valuation centered at the point $[0 : \xi : 1 : 1]$ is $1 \in \mathbb{Z}/3$, where ξ satisfies $\xi^9 - 2 = 0$. And the residue of $\gamma_2^{[t_0:t_1]}$ of the valuation centered at the point $[0 : \psi : 1 : -1]$ is $2 \in \mathbb{Z}/3$, where $\psi^{21} + 2 = 0$. Hence both

$\gamma_1^{[t_0:t_1]}$ and $\gamma_2^{[t_0:t_1]}$ are irreducible. By Corollary 4.2, for any $[t_0 : t_1] \in \mathbb{P}^1_{\mathbb{C}}$, there exists an integral Brauer-Severi surface bundle $\mathscr{Y}_{[t_0:t_1]} \to \mathbb{P}^3_{\mathbb{C}}$.

By viewing $\mathscr{A}_{[t_0:t_1]}$ as a simple algebra over $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^3_{\mathbb{C}}$ and applying the construction of Theorem 4.2.1 again, we have a Brauer-Severi surface bundle over $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^3_{\mathbb{C}}$ which can be viewed as a 1 dimensional family of of Brauer-Severi surface bundles over $\mathbb{P}^3_{\mathbb{C}}$. Let $\mathscr{Y} \to \mathbb{P}^1_{\mathbb{C}}$ denote this family of Brauer-Severi surface bundles. We claim this is indeed a flat family. By [Har77, Proposition 9.7], It is sufficient to check $\mathscr{Y}$ is integral. $\mathscr{Y}$ is irreducible because each closed fiber is an irreducible variety and the morphism $\mathscr{Y} \to \mathbb{P}^1_{\mathbb{C}}$ is projective, hence closed. Now let $\hat{\mathscr{Y}}$ be the closed sub-scheme of $\mathscr{Y}$ with the same underlying topological space equipped with the reduced scheme structure, we have the following Cartesian diagram:

$$\begin{array}{ccc} \hat{\mathscr{Y}}_{[t_0:t_1]} & \longrightarrow & \hat{\mathscr{Y}} \\ \downarrow i_{[t_0:t_1]} & & \downarrow i \\ \mathscr{Y}_{[t_0:t_1]} & \longrightarrow & \mathscr{Y} \\ \downarrow & & \downarrow \\ \{[t_0 : t_1]\} & \to & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

On one hand, $i_{[t_0:t_1]}$ is a homeomorphism on topological spaces as the pullback of schemes by monomorphism coincide with topological pullback according to the explicit construction of fiber product of ringed spaces. On the other hand, $i_{[t_0:t_1]}$ is a closed immersion as a base change of the closed immersion i. Since $\mathscr{Y}_{[t_0:t_1]}$ is reduced as discussed above, we know $i_{[t_0:t_1]}$ is an isomorphism.

Finally, by replacing $\mathscr{Y}$ by $\hat{\mathscr{Y}}$ if necessary, we get a 1 dimensional flat family of Brauer-Severi surface bundles over $\mathbb{P}^3_{\mathbb{C}}$, with a special fiber $\mathscr{Y}_{[1:0]}$ (Example 4.1.1 and Lemma 5.2.2) and a regular fiber $\mathscr{Y}_{[1:1]}$ ([Mae97, Theorem 2.1]), hence we are done. □

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APPENDIX

Lemma .0.1. *Let* $P_1 = P_1(x_1, x_2, x_3) = (x_1^3 - x_2^3)(x_2^3 - x_3^3)(x_3^3 - x_1^3)$, *let* $P_2 = P_2(x_1, x_2, x_3) = x_1^3 x_2^3 x_3^3$. *Then for any* $[t_1 : t_2] \neq [0 : 1]$ *or* $[1 : 0]$ *or* $[1 : \sqrt{-27}]$ *or* $[1 : -\sqrt{-27}]$, *singular locus of the curve* $t_1 P_1 + t_2 P_2 = 0$ *in* $\mathbb{P}^2_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x_1, x_2, x_3]$ *are precisely the three points:*

$$[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$$

Proof. Taking partial derivatives:

$$t_1 \frac{\partial P_1}{\partial x_1} + t_2 \frac{\partial P_2}{\partial x_1} = 0$$
$$t_1 \frac{\partial P_1}{\partial x_2} + t_2 \frac{\partial P_2}{\partial x_2} = 0$$
$$t_1 \frac{\partial P_1}{\partial x_3} + t_2 \frac{\partial P_2}{\partial x_3} = 0$$

Consider a point with $x_1 = 0$ lying in the curve. Then $P_2 = 0$, and this forces $P_1 = x_2^3 x_3^3 (x_3^3 - x_2^3) = 0$ as $t_1 \neq 0$. If one of x_2 or x_3 is 0, then we get the singularities listed in the statement of this lemma. If not, we have $x_2^3 - x_3^3 = 0$, and the above partial derivatives can be simplified to obtain $x_2 = 0$, and hence $x_3 = 0$, which is a contradiction. Similar calculations work when $x_2 = 0$ or $x_3 = 0$. So all singularities when some x_i equal to 0 are precisely the three points listed above.

In the following we assume all of x_1, x_2 and x_3 are nonzero. It is clear that for $i \neq j$, we have $x_i \frac{\partial P_2}{\partial x_i} = x_j \frac{\partial P_2}{\partial x_j}$. The above partial derivatives also tell us

$$x_1 \frac{\partial P_1}{\partial x_1} = x_2 \frac{\partial P_1}{\partial x_2} = x_3 \frac{\partial P_1}{\partial x_3}.$$

By Euler's theorem on homogeneous polynomial, we know

$$9P_1 = x_1 \frac{\partial P_1}{\partial x_1} + x_2 \frac{\partial P_1}{\partial x_2} + x_3 \frac{\partial P_1}{\partial x_3}.$$

Hence we have

$$x_1 \frac{\partial P_1}{\partial x_1} = 3P_1,$$

which simplifies to

$$(x_2^3 - x_3^3)(x_2^3 x_3^3 - x_1^6) = 0$$

Similarly,

$$(x_3^3 - x_1^3)(x_1^3 x_3^3 - x_2^6) = 0$$

$$(x_1^3 - x_2^3)(x_1^3 x_2^3 - x_3^6) = 0$$

By our assumption here, $P_2 \neq 0$, so $P_1 \neq 0$. Hence $x_i^3 \neq x_j^3$, so we have

$$x_2^3 x_3^3 - x_1^6 = 0$$

$$x_1^3 x_3^3 - x_2^6 = 0$$

$$x_1^3 x_2^3 - x_3^6 = 0$$

Hence the original partial derivatives simplify to

$$3t_1(x_3^3 - x_2^3) + t_2 x_1^3 = 0$$

$$3t_1(x_2^3 - x_1^3) + t_2 x_3^3 = 0$$

$$3t_1(x_1^3 - x_3^3) + t_2 x_2^3 = 0$$

Viewing this as three linear equations of x_i^3, we can calculate the determinant of the coefficient matrix as $-t_2(27t_1^2 + t_2^2)$. Hence when $[t_1 : t_2]$ is not one of the four cases listed in the statement, the determinant is non-zero, we know $x_1^3 = x_2^3 = x_0^3 = 0$ is the only solution, which is impossible. □

Lemma .0.2. *Using notations in Example 4.1.1, singular locus of S_2 consists of three lines:*

$$L_1 = \{x_0 = 0, x_1 = 0\}$$

$$L_2 = \{x_0 = 0, x_2 = 0\}$$

$$L_3 = \{x_0 = 0, x_3 = 0\}$$

Each D_i exactly passes through 5 singular points of S_2, they are as follows:

$$D_1 : [0:0:0:1], [0:0:1:0], [0:0:1:1], [0:0:1:\omega], [0:0:1:\omega^2]$$

$$D_2 : [0:0:0:1], [0:1:0:0], [0:1:0:1], [0:1:0:\omega], [0:1:0:\omega^2]$$

$$D_3 : [0:0:1:0], [0:1:0:0], [0:1:1:0], [0:1:\omega:0], [0:1:\omega^2:0]$$

Proof. Let $P_1 = P_1(x_1, x_2, x_3) = (x_1^3 - x_2^3)(x_2^3 - x_3^3)(x_3^3 - x_1^3)$, and let $P_2 = P_2(x_1, x_2, x_3) = x_1^3 x_2^3 x_3^3$. Then any singular point of S_2 satisfies the one of the two systems of equations:

$$\begin{cases} x_0 = 0 \\ P_2 \frac{\partial P_1}{\partial x_1} + (P_1 - 2P_2)\frac{\partial P_2}{\partial x_1} = 0 \\ P_2 \frac{\partial P_1}{\partial x_2} + (P_1 - 2P_2)\frac{\partial P_2}{\partial x_2} = 0 \\ P_2 \frac{\partial P_1}{\partial x_3} + (P_1 - 2P_2)\frac{\partial P_2}{\partial x_3} = 0 \end{cases} . \tag{.0.1}$$

or

$$\begin{cases} x_0^9 = -\frac{1}{2}(P_1 - P_2) \\ (P_1 + P_2)\frac{\partial P_1}{\partial x_1} + (P_1 - 3P_2)\frac{\partial P_2}{\partial x_1} = 0 \\ (P_1 + P_2)\frac{\partial P_1}{\partial x_2} + (P_1 - 3P_2)\frac{\partial P_2}{\partial x_2} = 0 \\ (P_1 + P_2)\frac{\partial P_1}{\partial x_3} + (P_1 - 3P_2)\frac{\partial P_2}{\partial x_3} = 0 \end{cases} . \tag{.0.2}$$

In case (A.1), clearly points in $L_1 \cup L_2 \cup L_3$ are solutions of this system of equations. Assume $x_i \neq 0, i = 1, 2, 3$. Then multiply the second equation of (A.1) by x_1, multiply the third equation of (A.1) by x_2, multiply the last equation of (A.1) by x_3 and add the resulting equations. By Euler's theorem on homogeneous polynomial, we have

$$P_1 - P_2 = 0$$

and the partial derivatives can be simplified as

$$x_0 = 0$$

$$\frac{\partial P_1}{\partial x_1} - \frac{\partial P_2}{\partial x_1} = 0$$

$$\frac{\partial P_1}{\partial x_2} - \frac{\partial P_2}{\partial x_2} = 0$$

$$\frac{\partial P_1}{\partial x_3} - \frac{\partial P_2}{\partial x_3} = 0$$

Hence by Lemma .0.1, the set of all singularities in this case is identified to $L_1 \cup L_2 \cup L_3$.

In case (A.2), it is easy to check if some $x_i = 0, i = 1, 2, 3$, then either we are reduced to case (A.1) or we obtain that all of them are 0. So we again assume $x_i \neq 0, i = 1, 2, 3$, and use the same trick as the previous case. By Euler's theorem on homogeneous polynomial, we have

$$0 = P_1^2 + 2P_1P_2 - 3P_2^2 = (P_1 - P_2)(P_1 + 3P_2)$$

If $P_1 - P_2 = 0$, then $x_0 = 0$, we are reduced to the case (A.1). So the only new possibility is $P_1 + 3P_2 = 0$. Then the partial derivatives are exactly the partial derivatives for the curve $P_1 + 3P_2 = 0$. Again by Lemma .0.1, the set of singularities of S_2 are $L_1 \cup L_2 \cup L_3$. □

Lemma .0.3. *Let*

$$G_1 = \{x_0^9 - x_1^9 + x_2^8 x_3 + x_3^8 x_2 = 0\}$$

$$G_2 = \{x_0^{21} + x_1^{21} + x_2^{21} - x_3^{21} = 0\}$$

Then G_1 and G_2 are regular surface in $\mathbb{P}^3_{\mathbb{C}}$. And furthermore G_1 and G_2 intersect transversally.

Proof. G_2 is clearly a regular surface in $\mathbb{P}^3_{\mathbb{C}}$. Taking partial derivatives of defining equation of G_1, the singular points are defined by the equations:

$$9x_0^8 = 0$$

$$-9x_1^8 = 0$$

$$8x_2^7 x_3 + x_3^8 = 0$$

$$8x_3^7 x_2 + x_2^8 = 0$$

This gives $x_0 = x_1 = 0$. Note that if one of x_2 or x_3 is 0, so is the other. Assume $x_2 \neq 0$ and $x_3 \neq 0$, we get $8x_2^7 + x_3^7 = 0, 8x_3^7 + x_2^7 = 0$. Then again $x_2 = x_3 = 0$, a contradiction. Hence G_1 is also a regular surface in $\mathbb{P}^3_{\mathbb{C}}$.

To check G_1 intersects G_2 transversally, we prove by contradiction: Assume there is a point $P = [x_0 : x_1 : x_2 : x_3] \in G_1 \cap G_2$, such that there exists a nonzero complex number k, with

$$21x_0^{20} = k(9x_0^8)$$

$$-21x_1^{20} = k(9x_1^8)$$

$$21x_2^{20} = k(8x_2^7x_3 + x_3^8)$$

$$-21x_3^{20} = k(8x_3^7x_2 + x_2^8)$$

Where the left side of each equations is the partial derivatives of defining equations of G_1, and the right hand side is k times the partial derivatives of defining equations of G_2. We split into several cases:

Case 1: $x_2 = 0$ or $x_3 = 0$. In this case, we clearly have $x_2 = x_3 = 0$, hence $x_0 \neq 0$ and $x_1 \neq 0$. So the partial derivatives with respect x_0 and x_1 tells us:

$$x_0^{12} = \frac{9}{21}k = -x_1^{12}$$

As $P \in G_1 \cap G_2$, we also have

$$x_0^9 - x_1^9 = 0$$

$$x_0^{21} + x_1^{21} = 0$$

This forces $x_0 = x_1 = 0$, which makes this case impossible.

Case 2: $x_2 \neq 0$ and $x_3 \neq 0$. Then the partial derivatives with respect to x_2 and x_3 shows that

$$\frac{x_2^{20}}{8x_2^7x_3 + x_3^8} = \frac{k}{21} = -\frac{x_3^{20}}{8x_3^7x_2 + x_2^8}$$

(note that the denominator is nonzero, otherwise by the partial derivatives with respect to x_2 and x_3, one of x_2 or x_3 is 0. This contradicts to the assumption.) This can be simplified to

$$\frac{x_2^{21}}{8x_2^7 + x_3^7} = -\frac{x_3^{21}}{8x_3^7 + x_2^7}$$

Since $x_3 \neq 0$, we assume $x_3 = 1$ without lose of generality. Let $t = x_2^7$, we see the above relation shows that t is a root of the following polynomial:

$$t^4 + 8t^3 + 8t + 1 = 0$$

Now we have three subcases:

Sub-case 1: $x_0 = 0$ and $x_1 = 0$. Then the defining polynomial of G_2 tells us

$$t^3 - 1 = 0$$

This contradicts to the relation: $t^4 + 8t^3 + 8t + 1 = 0$.

Sub-case 2: $x_0 \neq 0$ and $x_1 \neq 0$. In this case, a similar argument as in Case 1 shows that

$$x_0^{12} + x_1^{12} = 0$$

So we may assume $x_0 = \tau x_1$, where τ satisfies $\tau^{12} + 1 = 0$. Plug these information into defining equations of G_1 and G_2, we get:

$$G_1 : (\tau^9 - 1)x_1^9 + x_2x_3(t+1) = 0$$

$$G_2 : -(\tau^9 - 1)x_1^{21} + t^3 - 1 = 0$$

Note that $\tau^9 - 1 \neq 0$, and hence the above two equations shows that $t + 1 \neq 0$ and $t^3 - 1 \neq 0$. Taking ratio of the above two equations, we have:

$$-x_1^{12} = \frac{t^3 - 1}{x_2x_3(t+1)}$$

Compare the above relation with G_1, we have:

$$x_1^3 = \frac{(\tau^9 - 1)(t^3 - 1)}{x_2^2(t+1)^2}$$

Hence we get:

$$\frac{t^3 - 1}{\tau^9 - 1} = x_1^{21} = \frac{(\tau^9 - 1)^7(t^3 - 1)^7}{t^2(t+1)^{14}}$$

This is simplified to

$$(\tau^9 - 1)^8(t^3 - 1)^6 - t^2(t+1)^{14} = 0$$

That is

$$(\tau^9 - 1)^8 = \frac{t^2(t+1)^{14}}{(t^3 - 1)^6}$$

Since t satisfies $t^4 + 8t^3 + 8t + 1 = 0$, we list all roots of this polynomial:

$$t_1 = -2 - \frac{3}{\sqrt{2}} - \sqrt{\frac{15}{2} + 6\sqrt{2}}$$

$$t_2 = -2 - \frac{3}{\sqrt{2}} + \sqrt{\frac{15}{2} + 6\sqrt{2}}$$
$$t_3 = -2 + \frac{3}{\sqrt{2}} - \sqrt{-1}\sqrt{-\frac{15}{2} + 6\sqrt{2}}$$
$$t_4 = -2 + \frac{3}{\sqrt{2}} + \sqrt{-1}\sqrt{-\frac{15}{2} + 6\sqrt{2}}$$

By taking norm of both sides of $(\tau^9 - 1)^8 = \frac{t^2(t+1)^{14}}{(t^3-1)^6}$ for each value of t above, we see all four possible values of t are impossible. (Indeed, one can check the norm of the left hand side has two possible estimated values: 0.118 or 135.882, while the norm of the right hand side has two possible estimated values: 0.00238 or 14.2778.)

Sub-case 3: One of x_0 or x_1 is 0, and the other is nonzero. A similar discussion as in Sub-case 2 gives us

$$1 = \frac{t^2(t+1)^{14}}{(t^3-1)^6}$$

Again by taking norms of both sides, we see this is also impossible.

This completes the calculation. □

Lemma .0.4. *Use notations of Example 4.1.1 and Lemma .0.2, let*

$$G_1 = \{x_0^9 - x_1^9 + x_2^8 x_3 + x_3^8 x_2 = 0\}$$

Then for any $[t_0 : t_1] \neq [1 : 0] \in \mathbb{P}^1_{\mathbb{C}}$,

$$F_{S_1}^{[t_0:t_1]} = t_0 F_{S_1} + t_1(G_1 - F_{S_1}) = 0$$

defines irreducible surfaces in $\mathbb{P}^3_{\mathbb{C}}$.

Proof. We can view $F_{S_1}^{[t_0:t_1]}$ as a polynomial in x_0:

$$F_{S_1}^{[t_0:t_1]} = t_0 x_0^9 + t_1(-x_1^9 + x_2^8 x_3 + x_3^8 x_2) + (t_0 - t_1)P_1 = 0$$

Hence by the Eisenstein's criterion, it suffices to show the curve in $\mathbb{P}^2_{\mathbb{C}}$ defined by the constant term

$$t_1(-x_1^9 + x_2^8 x_3 + x_3^8 x_2) + (t_0 - t_1)P_1 = 0$$

has a regular point. It is clear that $[0 : 1 : 0]$ is such a point. □

Lemma .0.5. *Use notations of Example 4.1.1 and Lemma .0.2, let*

$$G_2 = \{x_0^{21} + x_1^{21} + x_2^{21} - x_3^{21} = 0\}$$

Then for any $[t_0 : t_1] \neq [1 : 0] \in \mathbb{P}^1_{\mathbb{C}}$,

$$F_{S_2}^{[t_0:t_1]} = t_0 F_{S_2}(x_2^3 - x_3^3) + t_1(G_2 - F_{S_2}(x_2^3 - x_3^3)) = 0$$

define irreducible surfaces in $\mathbb{P}^3_{\mathbb{C}}$.

Proof. View $F_{S_2}^{[t_0:t_1]}$ as a polynomial of x_0. As any factor of a homogeneous polynomial is also homogeneous, it suffices to show the constant term with respect to x_0 is itself irreducible. That is we need to show that

$$t_1 x_1^{21} - (t_0 - t_1)(x_2^3 - x_3^3) P_2(P_1 - P_2) + t_1(x_2^{21} - x_3^{21}) = 0$$

is irreducible. View the above polynomial as a polynomial of x_1, using Eisenstein criterion with the prime factor $(x_2 - x_3)$, we get the conclusion. □

Lemma .0.6. *With notations as in the proof of Theorem 4.2.1 and Corollary 4.2.6, we have that* Y_S *is reduced.*

Proof. As stated in the proof of Corollary 4.2.6, it suffices to check that over any closed point $p \in \mathbb{P}^3_{\mathbb{C}}$, and any point y lying in the fiber over p, the local ring $\mathcal{O}_{V_{\Lambda_p},y} \cong \mathcal{O}_{Y_S,y}$ is reduced.

In the proof of Theorem 4.2.1, we provide open affine covers for each local model V_{Λ_p}. Hence it suffice to show the coordinate ring of each open affine set appearing in these open affine covers is reduced. We discuss the cases as in the proof of Theorem 4.2.1 separately. First, Case 1 is trivial as the local model is regular.

In Case 2, we consider the the affine chart $\{\xi_{11} = \xi_{21} = \xi_{31} = 1\}$. Then its coordinate ring is

$$R_p[\xi_{12}, \xi_{13}, \xi_{22}, \xi_{23}, \xi_{32}, \xi_{33}]/(g_p\xi_{22} - \xi_{12}, g_p\xi_{23} - \xi_{13}, g_p\xi_{32} - g_p\xi_{12}, g_p\xi_{33} - \xi_{13})$$

It is easy to check that this defining ideal is radical. All the other affine charts can be checked similarly.

In Case 3, by [Mae97, Lemma 2.4], the local model has an open affine cover consisting of three open affine charts. The first two affine charts are both hypersurfaces in $\mathbb{A}^3_{R_P}$ and since each is defined by an irreducible polynomial, each affine chart is reduced. For the third chart, we need to be careful since it is not a hypersurface. Its coordinate ring is given by

$$R_p[x, y, z, w]/(F_1, F_2),$$

$$F_1 = y^3 - \omega x^2 w + (1 - \omega)xyz - f_p,$$

$$F_2 = z^3 - \omega^2 x w^2 - (1 - \omega)xyz - g_p.$$

for some $f_p.g_p \in R_p$. Here ω is a primitive third root of unity. Recall that $R_p = \mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p} = \mathbb{C}[\bar{x}_1, \bar{x}_2, \bar{x}_3]_{S_0}$, where $\bar{x}_i$ is a coordinate for some standard affine chart in $\mathbb{P}^3_{\mathbb{C}}$, and S_0 is the multiplicative set $\mathbb{C}[\bar{x}_1, \bar{x}_2, \bar{x}_3] - m_p$ with m_p the maximal ideal associates to p. Hence:

$$R_p[x, y, z, w]/(F_1, F_2) \cong (\mathbb{C}[\bar{x}_1, \bar{x}_2, \bar{x}_3, x, y, z, w]/(F_1, F_2))_{S_0}.$$

We check that this algebra is reduced using Serre's criterion. Namely, we verify whether our ring satisfies (R_0) and (S_1) [sta24b]. Note that F_1, F_2 do not have common factors in the polynomial ring $\mathbb{C}[\bar{x}_1, \bar{x}_2, \bar{x}_3, x, y, z, w]$. Hence F_1, F_2 form a regular sequence, and so We have that $\mathbb{C}[\bar{x}_1, \bar{x}_2, \bar{x}_3, x, y, z, w]/(F_1, F_2)$ is a complete intersection. In particular, this affine chart is Cohen-Macaulay. This shows $\mathbb{C}[\bar{x}_1, \bar{x}_2, \bar{x}_3, x, y, z, w]/(F_1, F_2))_{S_0}$ is also Cohen-Macaulay and hence satisfies Serre's condition (S_1). On the other hand, one easily checks that F_1 and F_2 intersect transversally by showing that the rows of the jacobian matrix are never proportional along their intersection whenever both of them are nonzero. Note that this follows immediately since the part of the jacobian matrix corresponding to the variables x, y, z, and w already satisfies this property. Hence the 2×7 Jacobian matrix of F_1, F_2 is not of full rank if and only if at least one of the two rows is zero. By the Jacobian criterion, these are precisely the singular points. We easily see that these points correspond to prime ideals in $\mathbb{C}[\bar{x}_1, \bar{x}_2, \bar{x}_3, x, y, z, w]$ containing one of the following ideals: (x, y, f_p),(z, x, y, g_p),(z, x, w, g_p) or (z, y, w, g_p). Hence singular set of $\mathbb{C}[\bar{x}_1, \bar{x}_2, \bar{x}_3, x, y, z, w]/(F_1, F_2)$ has codimension at least 3, which remains true by passing to the localization with respect to S_0 as f_p, g_p lives in the maximal ideal of $\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}},p}$. Thus

$R_p[x,y,z,w]/(F_1,F_2)$ is regular in codimension 0, namely (R_0). Hence this affine chart is also reduced. This completes the proof. $\square$