RECTILINEAR CONGRUENCES

By

CHUAN-CHIH HSIUNG

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CONTENTS

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Introduction

In his prize memoir [11](1), Wilczynski has established the theory of a rectilinear congruence in ordinary three-dimensional projective space by using a system of linear partial differential equations. However, his method of deriving the system of the differential equations is not completely geometric. The author proposes to remedy this lack of geometric content in the present paper.

In §1 we introduce, by a purely geometric method, a completely integrable system of linear homogeneous partial differential equations which defines a rectilinear congruence in ordinary space except for a projective transformation. The integrability conditions of the system of the differential equations are also calculated.

In §2 we study the effect, on the differential equations of §1, of a group of transformations which leave invariant the focal nets N_y, N_z on the focal surfaces S_y, S_z of an integral rectilinear congruence $\mathcal{L}$ of these equations. Some invariants and covariants of these equations under

(1) Numbers in brackets refer to the bibliography at the end of the paper.

this group of transformations are also obtained and listed.

In §3 we calculate for the focal surfaces S_y, S_z local power series expansions, each set of which expresses a local nonhomogeneous projective coordinates of a point on one surface as a power series in the other two coordinates and represents the surface in the neighborhood of an ordinary point on it.

In §4 a canonical form of the system of the differential equations of §1 is obtained by a geometric determination. We then reduce the power series expansions of the focal surfaces S_y, S_z in §3 to canonical forms, and find the loci of some osculants associated with the plane sections of the focal surfaces S_y, S_z made by a variable plane through a generator yz of the congruence $\mathcal{L}$.

In §5, by means of the quadrics of Moutard for the tangents to the curves of the focal nets N_y, N_z of the focal surfaces S_y, S_z we study some special kinds of the congruence $\mathcal{L}$ and determine geometrically the unit point of the coordinate system for a general congruence $\mathcal{L}$.

In §6, we find the equations of the curves of Darboux and the pencils of quadrics of Darboux at the focal points y, z of the focal surfaces S_y, S_z and geometrically characterize the congruence $\mathcal{L}$ when one or both of its focal nets are Segre-Darboux nets. The conditions for the both surfaces S_y, S_z to be isothermally asymptotic at the same time are also deduced.

In §7, the Weingarten invariants and the tangential invariants of the focal nets N_y, N_z are derived and a simple geometrical characterization of a W congruence is given.

§8 contains the local power series expansions for the u-, v-curves of the focal net N_y on the focal surface S_y. These expansions express two local nonhomogeneous projective coordinates of a point on each curve as power series in the other coordinate, and represent the curve in the neighborhood of an ordinary point on it. Quadrics having contact of different orders with the u-, v-curves and the surface S_y at the point y are considered.

By <u>a line</u> ℓ_1(ℓ_1') we mean, as usual, any line through the point y(z) of the surface $S_y(S_z)$ and not lying in the tangent plane of the surface $S_y(S_z)$ at y(z); by <u>a line</u> ℓ_2 (ℓ_2') we mean, dually, any line in the tangent plane of the surface $S_y(S_z)$ at y(z) but not passing through the point y(z). In §9, we derive two correspondences between ℓ_1 and ℓ_2 and between ℓ_1' and ℓ_2', and present another new geometrical characterization of a W congruence.

§10 is concerned with the determination of the developables and focal surfaces of the axis congruences, and also of the ray congruences, of the focal nets N_y, N_z of the congruence $\mathcal{L}$. The condition for the focal net N_y or N_z to be harmonic is also obtained.

In §11, we study some covariant congruences associated with the congruence $\mathcal{L}$ by methods similar to those used in §10.

The last section is devoted to the derivation of the equations of the osculating linear complex along a generator of a W congruence and the associated linear complexes of the focal points y, z of the generator yz of the congruence $\mathcal{L}$.

1. Differential Equations and Integrability Conditions

First of all, we consider in ordinary projective space a congruence with two distinct proper focal surfaces S_y, S_z generated respectively by the two focal points y, z of a generator yz of the congruence. Let the parametric curves u, v on the surfaces S_y, S_z be taken as the curves of the conjugate nets N_y, N_z in which the developables of the congruence touch the surfaces S_y, S_z; and let the u-tangent at the point y of the surface S_y and the v-tangent at the point z of the surface S_z coincide in the generator yz. If we select two points η, ζ respectively on the v-tangent at the point y of the surface S_y and the u-tangent at the point z of the surface S_z, and if we suppose that the coordinates η, ζ of the points η, ζ are functions of u, v; then it can be shown that the coordinates y, z, η, ζ of corresponding points y, z, η, ζ satisfy a system of linear homogeneous partial differential equations of the form

$$
(1.1) \qquad \begin{cases} y_u = \alpha y + \beta z, \\ z_v = \gamma y + \delta z, \\ y_v = ay + c\eta, \\ z_u = b'z + c'\zeta, \\ y_{vv} = my + nz + p\eta + q\zeta, \\ z_{uu} = m'y + n'z + p'\eta + q'\zeta \qquad (cc'p'q\beta\gamma \neq 0), \end{cases}
$$

in which subscripts indicate partial differentiation and the coefficients are scalar functions of u, v.

The derivatives η_u, η_v and ζ_u, ζ_v may be written as linear combinations of y, z, η, ζ by using equations (1.1) and

$$
(y_u)_v = (y_v)_u, \qquad (z_u)_v = (z_v)_u,
$$
$$
y_{vv} = (y_v)_v, \qquad z_{uu} = (z_u)_u;
$$

the result is

$$
(1.2) \qquad \begin{cases} \eta_u = ey + fz + g\eta, \\ \eta_v = ry + nz/c + s\eta + q\zeta/c, \\ \zeta_u = m'y/c' + s'z + p'\eta/c' + r'\zeta, \\ \zeta_v = e'y + f'z + g'\zeta, \end{cases}
$$

in which the coefficients are defined by the following equations:

$$
(1.3)\quad \begin{cases}
ce = \alpha_v + \beta\gamma - a_u, & cf = \beta_v + \beta\delta - a\beta, \\
cr = m - a_v - a^2, & cs = p - c_v - ac, \\
g = \alpha - (\log c)_u; & \\
c'e' = \gamma_u + \alpha\gamma - b'\gamma, & c'f' = \delta_u + \beta\gamma - b'_v, \\
c'r' = q' - c'_u - b'c', & c's' = n' - b'_u - b'^2, \\
g' = \delta - (\log c')_v. &
\end{cases}
$$

The integrability conditions of equations (1.1) are found by the usual method from the equations

$$
(\eta_u)_v = (\eta_v)_u, \qquad (\zeta_u)_v = (\zeta_v)_u
$$

and the fact that the points y, z, η, ζ are linearly independent. These conditions are

$$
(1.4)\quad \begin{cases}
e_v + ae + f\gamma + gr = r_u + r\alpha + es + m'q/cc', \\
f_v + f\delta + gn/c = r\beta + (n/c)_u + b'n/c + fs + qs'/c, \\
ce + g_v = s_u + p'q/cc', \\
gq = c'n + c(q/c)_u + qr'; \\
f'_u + b'f' + e'\beta + g's' = s'_v + \delta s' + f'r' + p'n/cc', \\
e'_u + e'\alpha + g'm'/c' = s'\gamma + (m'/c')_v + am'/c' + e'r' + p'r/c', \\
c'f' + g'_u = r'_v + p'q/cc', \\
g'p' = cm' + c'(p'/c')_v + p's.
\end{cases}
$$

Making use of equations (1.3), the third and the seventh of equations (1.4) we obtain the equation

$$
(1.5)\qquad (a + g' + \delta + s)_u = (g + b' + \alpha + r')_v.
$$

It follows that there exists a function θ of u, v which is defined, except for an arbitrary additive constant, as a solution of the differential equations

(1.6) $$\theta_u = g + b' + \alpha + r', \qquad \theta_v = a + g' + \delta + s.$$

Accordingly, the following formula is valid:

(1.7) $$(y,\ z,\ \eta\ ,\ \zeta\ ) = e^{\theta},$$

where a determinant is indicated by writing only a typical row within parentheses.

2. Transformations, Invariants and Covariants

Let us consider the group of transformations on the the coordinates y, z, η , ζ and the parameters u, v:

(2.1) $$y = \lambda \bar{y}, \quad z = \mu \bar{z}, \quad \eta = \nu \bar{\eta}, \quad \zeta = \tau \bar{\zeta} \qquad (\lambda\mu\nu\tau \neq 0),$$

(2.2) $$\bar{u} = U(u), \quad \bar{v} = V(v) \qquad (U'V' \neq 0),$$

where λ , μ , ν , τ are scalar functions of u, v and the accent denotes differentiation with respect to the appropriate variable.

The effect of the transformation (2.1) on the system of equations (1.1) is to produce another system of equations of the same form whose coefficients, indicated by dashes, are given by the following formulas:

$$(2.3)\quad \begin{cases} \bar\alpha = \alpha - \lambda_u/\lambda, \quad \bar\beta = \beta\mu/\lambda, \quad \bar\gamma = \gamma\lambda/\mu, \quad \bar\delta = \delta - \mu_v/\mu, \\ \bar a = a - \lambda_v/\lambda, \quad \bar c = c\,\nu/\lambda, \quad \bar b' = b' - \mu_u/\mu, \quad \bar c' = c'\tau/\mu, \\ \bar m = \frac{1}{\lambda}(m\lambda - \lambda_{vv} - 2a\lambda_v + 2\lambda_v^2/\lambda), \quad \bar n = n\mu/\lambda, \\ \bar p = \frac{\nu}{\lambda}(p - 2c\lambda_v/\lambda), \quad \bar q = q\,\tau/\lambda, \\ \bar m' = m'\lambda/\mu, \quad \bar n' = \frac{1}{\mu}(n'\mu - \mu_{uu} - 2b'\mu_u + 2\mu_u^2/\mu), \\ \bar p' = p'\nu/\mu, \quad \bar q' = \frac{\tau}{\mu}(q' - 2c'\mu_u/\mu). \end{cases}$$

The effect of the transformation (2.2) on the system of equations (1.1) is to produce another system of equations of the same form whose coefficients, indicated by stars, are given by the following formulas:

$$(2.4)\quad \begin{cases} \alpha^* = \alpha/U', \quad \beta^* = \beta/U', \quad \gamma^* = \gamma/V', \quad \delta^* = \delta/V', \\ a^* = a/V', \quad c^* = c/V', \quad b'^* = b'/U', \quad c'^* = c'/U', \\ m^* = (m - aV''/V')/V'^2, \quad n^* = n/V'^2, \\ p^* = (p - cV''/V')/V'^2, \quad q^* = q/V'^2, \\ m'^* = m'/U'^2, \quad n'^* = (n' - b'U''/U')/U'^2, \\ p'^* = p'/U'^2, \quad q'^* = (q' - c'U''/U')/U'^2. \end{cases}$$

From equations (1.3), (2.3), (2.4) we may obtain, after some calculation, the following functions which are absolute invariants under the transformation (2.1), and relative invariants under the transformation (2.2), of the system of equations (1.1):

$$
(2.5)\quad \left\{
\begin{array}{ll}
A = cm'/p', & A' = c'n/q, \\
B = m'\beta, & B' = n\gamma, \\
D = c'\beta/q, & D' = c\gamma/p', \\
G = m'n, & \\
H = \alpha_v + \beta\gamma - \delta_u - (\log\beta)_{uv}, & K' = \beta\gamma, \\
K = \beta\gamma, & H' = \delta_u + \beta\gamma - \alpha_v - (\log\gamma)_{uv}, \\
\mathfrak{K} = p'q/cc', & \mathfrak{H}' = p'q/cc', \\
\mathfrak{H} = c'f' + (c'n/q)_v, & \mathfrak{K}' = ce + (cm'/p')_u, \\
W_{(u)} = \mathfrak{K} - K, & W'_{(v)} = \mathfrak{H}' - H', \\
W_{(v)} = \mathfrak{H} - H, & W'_{(u)} = \mathfrak{K}' - K', \\
I = cf/\beta, & I' = c'e'/\gamma, \\
J = ce, & J' = c'f', \\
L = c's', & L' = cr, \\
M = 3a + g' - 2p/c + q_v/q, & M' = 3b' + g - 2q'/c' + p'_u/p', \\
N = \dfrac{n}{\beta}\,(2b' - q'/c' + \psi_u & N' = \dfrac{m'}{\gamma}(2a - p/c + \phi_v \\
\qquad - c'n/q + qs'/n), & \qquad - cm'/p' + p'r/m'), \\
P = 2a - p/c + \phi_v, & P' = 2b' - q'/c' + \psi_u.
\end{array}
\right.
$$

where

$$\phi = \log A, \qquad \psi = \log A'.$$

The effect of the transformation (2.2) on each of these invariants is given, with self-explanatory notation, by the following formulas:

$$(2.6)\quad \begin{cases}
A^* = (1/V')A, & A'^* = (1/U')A', \\
B^* = (1/U'^3)B, & B'^* = (1/V'^3)B', \\
D^* = (V'^2/U'^2)D, & D'^* = (U'^2/V'^2)D', \\
G^* = (1/U'^2V'^2)G, & \\
H^* = (1/U'V')H, & K'^* = (1/U'V')K', \\
K^* = (1/U'V')K, & H'^* = (1/U'V')H', \\
\mathfrak{H}^* = (1/U'V')\mathfrak{H}, & \mathfrak{K}'^* = (1/U'V')\mathfrak{K}', \\
\mathfrak{K}^* = (1/U'V')\mathfrak{K}, & \mathfrak{H}'^* = (1/U'V')\mathfrak{H}', \\
W^*_{(u)} = (1/U'V')W_{(u)}, & W'^*_{(v)} = (1/U'V')W'_{(v)}, \\
W^*_{(v)} = (1/U'V')W_{(v)}, & W'^*_{(u)} = (1/U'V')W'_{(u)}, \\
I^* = (1/V')I, & I'^* = (1/U')I', \\
J^* = (1/U'V')J, & J'^* = (1/U'V')J', \\
L^* = (1/U'^2)L, & L'^* = (1/V'^2)L', \\
M^* = (1/V')M, & M'^* = (1/U')M', \\
N^* = (1/V'^2)N, & N'^* = (1/U'^2)N', \\
P^* = (1/V')P, & P'^* = (1/U')P'.
\end{cases}$$

These invariants are obviously not all independent. Among them it is easy to obtain the following relations:

$$(2.7)\quad \begin{cases}
AA' = G/\mathfrak{H}, \quad BB' = GK, \quad DD' = K/\mathfrak{H}, \\
J = H + I_u = \mathfrak{H}' - A_u, \quad J' = K' + I'_v = \mathfrak{K} - A'_v, \\
N = [L + A'(P' - A')]/D, \quad N' = [L' + A(P - A)]/D'.
\end{cases}$$

In terms of the invariants, the integrability conditions (1.4) then can be written in the form

$$
(2.8) \quad \begin{cases}
J_v + J(P - \phi_v) + IK = L'_u + A\,\mathfrak{A}', \\
I_v + I(I + P - \phi_v) = L' + N, \\
K' - \mathfrak{A}' = 3(H' - J), \\
\alpha - r' - q_u/q = A'; \\
J'_u + J'(P' - \psi_u) + I'H' = L_v + A'\,\mathfrak{H}, \\
I'_u + I'(I' + P' - \psi_u) = L + N', \\
H - \mathfrak{H} = 3(K - J'), \\
\delta - s - p'_v/p' = A.
\end{cases}
$$

From equations (1.1), by differentiation and substitution we may obtain the Laplace equation of the focal net N_y:

$$
(2.9) \qquad y_{uv} = (\alpha_v + \beta\gamma - \alpha\delta - \alpha\beta_v/\beta)y + (\beta_v/\beta + \delta)y_u + \alpha\, y_v.
$$

It is easy to show that the point invariants of Laplace-Darboux of the net N_y are H, K defined by equations (2.5). Moreover, the point z is the ray-point y_{-1} of the v-curve corresponding to the point y or the minus-first Laplace transformed point of y with respect to the net N_y, and the ray-point y_1 of the u-curve corresponding to the point y or the first Laplace transformed point of y with respect to the net N_y is defined by the equation

$$
(2.10) \qquad y_1 = -Iy + c\eta.
$$

Similarly, the point invariants of Laplace-Darboux of the net N_z are H', K' defined by equations (2.5), the ray-point z_1 of the u-curve corresponding to the point z is the point y and the ray-point z_{-1} of the v-curve corresponding to the point z is

$$(2.11) \qquad z_{-1} = -I'z + c'\zeta .$$

3. Power Series Expansions for the Surfaces S_y, S_z

From system (1.1) and equations (1.2), (1.3) by differentiation and substitution, any derivative of y can be expressed as a linear combination of y, z, η, ζ. In particular, one obtains

$$(3.1) \quad \begin{cases} y_{uu} = (\alpha_u + \alpha^2)y + (\beta_u + \alpha\beta + b'\beta)z + c'\beta\zeta , \\ y_{uv} = (\alpha_v + a\alpha + \beta\gamma)y + (\beta_v + \beta\delta)z + c\alpha\eta , \\ y_{uuu} = (\alpha_{uu} + 3\alpha\alpha_u + \alpha^3 + m'\beta)y + (\beta_{uu} + 2\alpha_u\beta \\ \qquad + \alpha\beta_u + 2b'\beta_u + \alpha^2\beta + b'\alpha\beta + n'\beta)z + p'\beta\eta \\ \qquad + (2c'\beta_u + c'\alpha\beta + q'\beta)\zeta , \\ y_{uuv} = (\alpha_{uv} + a\alpha_u + 2\alpha\alpha_v + \beta_u\gamma + \beta\gamma_u + 2\alpha\beta\gamma \\ \qquad + a\alpha^2)y + (\beta_{uv} + \alpha_v\beta + \beta_u\delta + \alpha\beta_v + b'\beta_v \\ \qquad + \beta\delta_u + \alpha\beta\delta + b'\beta\delta + \beta^2\gamma)z + c(\alpha_u + \alpha^2)\eta \\ \qquad + c'(\beta_v + \beta\delta)\zeta , \\ y_{uvv} = (m_u + m\alpha + ep + m'q/c')y + (n_u + m\beta + b'n + fp \\ \qquad + qs')z + (p_u + gp + p'q/c')\eta + (q_u + c'n + qr')\zeta , \\ y_{vvv} = (m_v + am + n\gamma + e'q + pr)y + (n_v + n\delta + f'q + np/c)z \\ \qquad + (p_v + cm + ps)\eta + (q_v + g'q + pq/c)\zeta , \\ y_{uuuu} = (*)y + (*)z + (3p'\beta_u + p'_u\beta + 2p'\alpha\beta - c_u p'\beta/c \\ \qquad + p'q'\beta/c')\eta + (3c'\beta_{uu} + 3c'\alpha_u\beta + 2c'\alpha\beta_u \end{cases}$$

$$
\left|
\begin{aligned}
&+3q'\beta_u + q'_u\beta + c'\alpha^2\beta + c'n'\beta + q'\alpha\beta + q'r'\beta)\zeta ,\\
y_{vvvv} &= (*)y + (*)\eta + [n_{vv} + 2n_v\delta + n\,\delta_v + 2f'q_v + f'_v q\\
&\quad + np_v/c + (np/c)_v + mn + n\,\delta^2 + np\,\delta/c + f'q\,\delta\\
&\quad + f'g'q + f'pq/c + nps/c]z + (q_{vv} + 2g'q_v + g'_v q\\
&\quad + pq_v/c + 2p_v q/c - c_v pq/c^2 + g'pq/c + pqs/c\\
&\quad + mq + g'^2 q)\,\zeta ,
\end{aligned}
\right.
$$

where (*) denotes terms immaterial for our purpose.

The coordinates Y, where

$$Y = Y(u + \Delta u,\ v + \Delta v),$$

of any point Y near the point y on the focal surface S_y are given by the Taylor's series

$$(3.2)\qquad Y = y + y_u\,\Delta u + y_v\Delta v + \tfrac{1}{2}(y_{uu}\Delta u^2 + 2y_{uv}\Delta u\Delta v + y_{vv}\Delta v^2) + \ldots,$$

in which the increments Δu and Δv correspond to displacement on the surface S_y from the point y to the point Y.

If the points y, z, η, ζ are used as the vertices of the tetrahedron of reference, with unit point suitably chosen, then any point given by an expression of the form

$$(3.3)\qquad x_1 y + x_2 z + x_3\eta + x_4\zeta$$

has local coordinates proportional to $x_1, \ldots, x_4$. Substitution of the expressions (3.1) leads to the following power series expansions of the local coordinates of the point Y:

$$
(3.4)\quad
\begin{cases}
y_1 = 1 + \alpha\Delta u + a\Delta v + \frac{1}{2}(\alpha_u + \alpha^2)\Delta u^2 + (\alpha_v + a\alpha + \beta\gamma)\Delta u\Delta v \\
\qquad + \frac{1}{2}m\Delta v^2 + \ldots, \\
y_2 = \beta\Delta u + \frac{1}{2}(\beta_u + \alpha\beta + b'\beta)\Delta u^2 + (\beta_v + \beta\delta)\Delta u\Delta v + \frac{1}{2}n\Delta v^2 \\
\qquad + \frac{1}{6}(\beta_{uu} + 2\alpha_u\beta + \alpha\beta_u + 2b'\beta_u + \alpha^2\beta + b'\alpha\beta + n'\beta)\Delta u^3 \\
\qquad + \frac{1}{2}(\beta_{uv} + \alpha_v\beta + \beta_u\delta + \alpha\beta_v + b'\beta_v + \beta\delta_u + \alpha\beta\delta + \beta^2\gamma \\
\qquad + b'\beta\delta)\Delta u^2\Delta v + \frac{1}{2}(n_u + m\beta + b'n + fp + qs')\Delta u\Delta v^2 \\
\qquad + \frac{1}{6}(n_v + n\delta + f'q + np/c)\Delta v^3 + \ldots + \frac{1}{24}[n_{vv} + 2n_v\delta \\
\qquad + 2f'q_v + f'_v q + np_v/c + (np/c)_v + mn + n\delta^2 + np\delta/c \\
\qquad + f'q\delta + f'g'q + f'pq/c + nps/c]\Delta v^4 + \ldots, \\
y_3 = c\Delta v + c\alpha\Delta u\Delta v + \frac{1}{2}p\Delta v^2 + \frac{1}{6}(\beta_{uu} + 2\alpha_u\beta + \alpha\beta_u + 2b'\beta_u \\
\qquad + \alpha^2\beta + b'\alpha\beta + n'\beta)\Delta u^3 + \ldots + \frac{1}{6}(p_v + cm + ps)\Delta v^3 \\
\qquad + \ldots, \\
y_4 = \frac{1}{2}c'\beta\Delta u^2 + \frac{1}{2}q\Delta v^2 + \frac{1}{6}(2c'\beta_u + c'\alpha\beta + q'\beta)\Delta u^3 \\
\qquad + \frac{1}{2}c'(\beta_v + \beta\delta)\Delta u^2\Delta v + \frac{1}{2}(q_u + c'n + qr')\Delta u\Delta v^2 \\
\qquad + \frac{1}{6}(q_v + g'q + pq/c)\Delta v^3 + \frac{1}{24}(3c'\beta_{uu} + 3c'\alpha_u\beta \\
\qquad + 2c'\alpha\beta_u + 3q'\beta_u + q'_u\beta + c'\alpha^2\beta + c'n'\beta + q'\alpha\beta \\
\qquad + q'r'\beta)\Delta u^4 + \ldots + \frac{1}{24}(q_{vv} + 2g'q_v + g'_v q + pq_v/c \\
\qquad + 2p_v q/c - c_v pq/c^2 + g'pq/c + pqs/c + mq + g'^2 q)\Delta v^4 \\
\qquad + \ldots.
\end{cases}
$$

Division of the last three of the series (3.4) by the first gives the following expansions:

$$
\begin{cases}
y_2/y_1 = \beta\Delta u + \frac{1}{2}(\beta_u + b'\beta - \alpha\beta)\Delta u^2 + (\beta_v + \beta\delta - a\beta)\Delta u\Delta v \\
\qquad + n\Delta v^2/2 + \frac{1}{6}(\beta_{uu} - \alpha_u\beta - 2\alpha\beta_u + 2b'\beta_u + \alpha^2\beta \\
\qquad - 2b'\alpha\beta + n'\beta)\Delta u^3 + \ldots + \frac{1}{6}(n_v + n\delta + f'q + np/c \\
\qquad - 3an)\Delta v^3 + \ldots + \frac{1}{24}[n_{vv} + 2n_v\delta + n\delta_v + 2f'q_v
\end{cases}
$$

$$
(3.5)\quad \begin{cases}
\quad + f'_v q - 4an_v + np_v/c + (np/c)_v - 5mn + n\delta^2 \\
\quad - 4an\delta + np\delta/c - 4anp/c + f'q\delta + f'g'q \\
\quad - 4af'q + 12a^2 n + f'pq/c + nps/c\,]\,\Delta v^4 + \dots, \\
y_3/y_1 = c\Delta v + \tfrac{1}{2}(p - 2ac)\Delta v^2 + \tfrac{1}{6}p'\beta\Delta u^3 + \dots + \tfrac{1}{6}(p_v \\
\quad + ps - 3ap - 2cm + 6a^2 c)\Delta v^3 + \tfrac{1}{24}(3p'\beta_u + p'_u\beta \\
\quad - 2p'\alpha\beta - c_u p'\beta/c + p'q'\beta/c')\Delta u^4 + \dots, \\
y_4/y_1 = \tfrac{1}{2}c'\beta\Delta u^2 + \tfrac{1}{2}q\Delta v^2 + \tfrac{1}{6}(2c'\beta_u + q'\beta - 2c'\alpha\beta)\Delta u^3 \\
\quad + \tfrac{1}{2}(c'\beta_v + c'\beta\delta - ac'\beta)\Delta u^2\Delta v + \tfrac{1}{2}(q_u + c'n - q\alpha \\
\quad + qr')\Delta u\Delta v^2 + \tfrac{1}{6}(q_v + g'q - 3aq + pq/c)\Delta v^3 \\
\quad + \tfrac{1}{24}(3c'\beta_{uu} - 3c'\alpha_u\beta - 6c'\alpha\beta_u + 3q'\beta_u + q'_u\beta \\
\quad + c'n'\beta + 3c'\alpha^2\beta - 3q'\alpha\beta + q'r'\beta)\Delta u^4 + (*)\Delta u^3\Delta v \\
\quad + (*)\Delta u^2\Delta v^2 + (*)\Delta u\Delta v^3 + \tfrac{1}{24}(q_{vv} + g'_v q + 2g'q_v \\
\quad - 4aq_v + pq_v/c + 2p_v q/c - c_v pq/c^2 - 5mq + 12a^2 q \\
\quad - 4ag'q + g'^2 q - 4apq/c + g'pq/c + pqs/c)\Delta v^4 \\
\quad + \dots.
\end{cases}
$$

From the expansion (3.5) it is possible to calculate to as many terms as desired an expansion for y_4/y_1 as a power series in y_2/y_1, y_3/y_1. This calculation can be performed by setting y_4/y_1 equal to a power series in y_2/y_1, y_3/y_1 with undertermined coefficients and then demanding that the expansions (3.5) for y_2/y_1, y_3/y_1, y_4/y_1 shall satisfy this equation identically in Δu and Δv as far as the terms of a sufficiently high degree. The result to terms of the fourth degree is found, by the use of equations (2.5), to be

$$(3.6)\qquad \begin{aligned} y_4/y_1 = {} & \tfrac{1}{2}c'(y_2/y_1)^2/\beta + q(y_3/y_1)^2/2c^2 + B_1(y_2/y_1)^3 \\ & - c'I(y_2/y_1)^2(y_3/y_1)/2c\beta - \frac{qA'}{2c^2\beta}(y_2/y_1)(y_3/y_1)^2 \\ & + qM(y_3/y_1)^3/6c^3 + C_1(y_2/y_1)^4 + \cdots \\ & + q\left[M_v + M^2 + (P - \phi_v)M + 3L'\right](y_3/y_1)^4/24c^4 \\ & + \cdots, \end{aligned}$$

where

$$(3.7)\qquad \begin{cases} B_1 = (q'\beta + c'\alpha\beta - 3b'c'\beta - c'\beta_u)/6\beta^3, \\ C_1 = \dfrac{1}{24\beta^4}(-c'\beta_{uu} + c'\alpha_u\beta + 2c'\alpha\beta_u - 5b'c'\beta_u \\ \qquad + q'_u\beta - 3c'n'\beta - c'\alpha^2\beta + 5b'c'\alpha\beta - 3b'^2c'\beta \\ \qquad + q'r'\beta\,). \end{cases}$$

The local coordinates $z_1, \ldots, z_4$ of any point Z near the point z on the focal surface S_z and an analogous power series expansion for the surface S_z in the neighborhood of the point z can be obtained in a way similar to the foregoing, or else can be written immediately by making the substitutions

$$(3.8)\qquad \begin{pmatrix} y & \eta & u & 1 & 3 & \alpha & \beta & a & c & e & f & g & m & n & p & q & r & s \\ z & \zeta & v & 2 & 4 & \delta & \gamma & b' & c' & f' & e' & g' & n' & m' & q' & p' & s' & r' \end{pmatrix},$$

$$(3.9)\begin{pmatrix} A & B & D & G & H & K & \mathfrak{H} & \mathfrak{A} & W_{(u)} & W_{(v)} & I & J & L & M & N & P \\ A' & B' & D' & G & K' & H' & \mathfrak{A}' & \mathfrak{H}' & W'_{(v)} & W'_{(u)} & I' & J' & L' & M' & N' & P' \end{pmatrix}.$$

The result is

$$
(3.10)\quad \begin{cases}
z_1 = \gamma\Delta v + \tfrac{1}{2}m'\Delta u^2 + (\gamma_u + \alpha\gamma)\Delta u\Delta v + \tfrac{1}{2}(\gamma_v + \gamma\delta + a\gamma)\Delta v^2 \\
\quad + \tfrac{1}{6}(m'_u + m'\alpha + e'p' + m'q'/c')\Delta u^3 + \tfrac{1}{2}(m'_v + n'\gamma \\
\quad + am' + e'q' + p'r)\Delta u^2\Delta v + \tfrac{1}{2}(\gamma_{uv} + \alpha_v\gamma + a\gamma_u + \gamma_u\delta \\
\quad + \alpha\gamma_v + \gamma\delta_u + a\alpha\gamma + \alpha\gamma\delta + \beta\gamma^2)\Delta u\Delta v^2 + \tfrac{1}{6}(\gamma_{vv} \\
\quad + 2a\gamma_v + \gamma_v\delta + 2\gamma\delta_v + a\gamma\delta + \gamma\delta^2 + m\gamma)\Delta v^3 + \dots, \\
z_2 = 1 + b'\Delta u + \delta\Delta v + \tfrac{1}{2}n'\Delta u^2 + (\delta_u + \beta\gamma + b'\delta)\Delta u\Delta v \\
\quad + \tfrac{1}{2}(\delta_v + \delta^2)\Delta v^2 + \dots, \\
z_3 = \tfrac{1}{2}p'\Delta u^2 + \tfrac{1}{2}c\gamma\Delta v^2 + \tfrac{1}{6}(p'_u + gp' + p'q'/c')\Delta u^3 \\
\quad + \tfrac{1}{2}(p'_v + cm' + p's)\Delta u^2\Delta v + \tfrac{1}{2}c(\gamma_u + \alpha\gamma)\Delta u\Delta v^2 \\
\quad + \tfrac{1}{6}(2c\gamma_v + c\gamma\delta + p\gamma)\Delta v^3 + \dots, \\
z_4 = c'\Delta u + \tfrac{1}{2}q'\Delta u^2 + c'\delta\Delta u\Delta v + \dots;
\end{cases}
$$

$$
\begin{aligned}
(3.11)\quad z_3/z_2 = {} & \frac{c}{2\gamma}(z_1/z_2)^2 + p'(z_4/z_2)^2/2c'^2 + B'_1(z_1/z_2)^3 \\
& - cI'(z_1/z_2)^2(z_4/z_2)/2c'\gamma - \frac{p'A}{2c'^2\gamma}(z_1/z_2)(z_4/z_2)^2 \\
& + p'M'(z_4/z_2)^3/6c'^3 + C'_1(z_1/z_2)^4 + \dots \\
& + p'\left[M'^2 + M'_u + (P' - \psi_u)M' + 3L\right](z_4/z_2)^4/24c'^4 \\
& + \dots,
\end{aligned}
$$

where we have placed

$$
(3.12)\quad \begin{cases}
B'_1 = (p\gamma + c\gamma\delta - 3ac\gamma - c\gamma_v)/6\gamma^3, \\
C'_1 = \dfrac{1}{24\gamma^4}(-c\gamma_{vv} + 2c\gamma_v\delta + c\gamma\delta_v - 5ac\gamma_v + p_v\gamma \\
\quad - 3cm\gamma - c\gamma\delta^2 + 5ac\gamma\delta - 3a^2c\gamma + ps\gamma).
\end{cases}
$$

4. Canonical Form of the Differential Equations and Loci of Some Osculants

For the purpose of choosing for the points η, ζ two particular covariant points respectively on the v-tangent at the point y of the surface S_y and the u-tangent at the point z of the surface S_z, we recall the definitions [5] of two osculants associated with two ordinary points of the second kind of two plane curves. Suppose that O_1, O_2 are two ordinary points of the second kind of two plane curves C_1, C_2 respectively, so that O_1O_2 is the common tangent. Let K_1, K_2 be any four-point conics of the curves C_1, C_2 at the points O_1, O_2, and ℓ_1, ℓ_2 the polar lines of the points O_2, O_1 with respect to the conics K_1, K_2, respectively; then the intersection of the polar lines ℓ_1, ℓ_2 is called the principal point associated with the points O_1, O_2 of the curves C_1, C_2. Moreover, among the pencils of four-point conics of the curves C_1, C_2 at the points O_1, O_2, there are principal conics which pass through the principal point.

Let us consider a general plane π through the generator yz:

$$x_4 - \lambda x_3 = 0 \qquad (\lambda \neq 0). \tag{4.1}$$

It is obvious that the plane π intersects the surfaces S_y, S_z in two curves C_y, C_z having y, z as two ordinary points of the second kind. By virtue of equations (3.6), (4.1) we may obtain the power series expansion for the projection of the section C_y in the plane $x_4 = 0$, namely,

$$(4.2) \qquad x_3/x_1 = \frac{1}{\lambda}\left[\frac{c'}{2\beta}(x_2/x_1)^2 + B_1(x_2/x_1)^3 + \cdots\right].$$

The equations of any four-point conic of the section C_y at the point y are found, from equation (4.2), to be (4.1) and

$$(4.3) \qquad x_1x_3 - c'x_2^2/2\beta\lambda - 2\beta B_1x_2x_3/c' + kx_3^2 = 0,$$

where k is a parameter. The polar line of the point z with respect to this conic is given by equations (4.1) and

$$(4.4) \qquad c'^2x_2 + 2\beta^2B_1\lambda x_3 = 0.$$

Elimination of λ between equations (4.1), (4.4) shows that as the plane π revolves about the line yz, the locus of this polar line is a plane through the line $z\zeta$:

$$(4.5) \qquad c'^2x_2 + 2\beta^2B_1x_4 = 0.$$

Similarly, as the plane π revolves about the line yz, the locus of the polar line of the point y with respect to any four-point conic of the section C_z at the point z is a plane through the line $y\eta$:

$$(4.6) \qquad c^2x_1 + 2\gamma^2B_1'x_3 = 0.$$

Thus we reach the following conclusion:

<u>As the plane</u> π <u>revolves about the line</u> yz, <u>the locus</u>

<u>of the principal point associated with the points</u> y, z <u>of the plane sections</u> C_y, C_z <u>describes a line</u> ℓ, <u>whose equations are</u> (4.5), (4.6)

This line ℓ intersects the lines $x_1 = x_3 = 0$ and $x_2 = x_4 = 0$ in two points. If we choose these two points respectively for the points η and ζ, then

(4.7) $$B_1 = 0, \quad B_1' = 0;$$

that is,

(4.8) $$p = c(\gamma_v/\gamma + 3a - \delta), \qquad q' = c'(\beta_u/\beta + 3b' - \alpha).$$

Hereafter it will be supposed that the differential equations (1.1) are in the canonical form for which the conditions (4.7) or (4.8) are satisfied. Accordingly, by means of equations (1.3), (2.5), (4.8) the power series expansions (3.6), (3.11) for the surfaces S_y, S_z in the neighborhood of the points y, z become, respectively,

(4.9) $$\begin{aligned} y_4/y_1 = {} & c'(y_2/y_1)^2/2\beta + q(y_3/y_1)^2/2c^2 \\ & - c'I(y_2/y_1)^2(y_3/y_1)/2c\beta - qA'(y_2/y_1)(y_3/y_1)^2/2c^2\beta \\ & + qM(y_3/y_1)^3/6c^3 - c'L(y_2/y_1)^4/8\beta^3 \quad \ldots \\ & + q\left[M_v + M^2 + (P - \phi_v)M + 3L'\right](y_3/y_1)^4/24c^4 + \ldots, \end{aligned}$$

(4.10) $$\begin{aligned} z_3/z_2 = {} & c(z_1/z_2)^2/2\gamma + p'(z_4/z_2)^2/2c'^2 \\ & - cI'(z_1/z_2)^2(z_4/z_2)/2c'\gamma - p'A(z_1/z_2)(z_4/z_2)^2/2c'^2\gamma \\ & + p'M'(z_4/z_2)^3/6c'^3 - cL'(z_1/z_2)^4/8\gamma^3 + \ldots \\ & + p'\left[M_u' + M'^2 + (P' - \psi_u)M' + 3L\right](z_4/z_2)^4/24c'^4 \\ & + \ldots. \end{aligned}$$

From equations (1.1), it follows immediately that <u>the</u>

point ζ is in the osculating plane of the v-curve of the net N_y at the point y if, and only if, $n = 0$. A similar argument holds for the u-curve of the net N_z at the point z. Moreover, from equation (2.10), the point y_1 coincides with the point η if, and only if, $I = 0$; and similarly, the point z_{-1} coincides with the point ζ in case $I' = 0$.

Buzano [3] and Bompiani [2] have shown the existence of a projective invariant, together with metric and projective characterizations, determined by the neighborhood of the second order of two surfaces σ, σ^* at two ordinary points O, O* in ordinary space under the conditions that the tangent planes of the surfaces σ, σ^* at the points O, O* be distinct and have OO* for the common line. For the focal surfaces S_y, S_z at the points y, z, this invariant is easily found, from Bompiani's note [2] and expansions (4.9), (4.10), to be $\mathfrak{H}/(16K)$.

As the plane π (4.1) revolves about the line yz the locus of the principal conic having four-point contact with the section C_y at the point y is a quadric cone with vertex at the point η, whose equation is found, by means of equations (4.1), (4.3) and the conditions (4.7), to be

$$(4.11) \qquad 2\beta x_1 x_4 - c' x_2^2 = 0.$$

Similarly, the locus of the principal conic having four-point contact with the section C_z at the point z is a quadric cone with vertex at the point ζ:

(4.12) $$2\gamma x_2x_3 - cx_1^2 = 0.$$

From equations (2.5), (2.10), (2.11), we find the equations of the line y_1z_{-1} to be

(4.13) $$cx_1 + Ix_3 = 0, \qquad c'x_2 + I'x_4 = 0,$$

which intersects the plane (4.1) in a point with the coordinates

(4.14) $$(-I/c,\ -I'\lambda/c',\ 1,\ \lambda).$$

If a four-point conic (4.3) at the point y of the section C_y of the surface S_y with the plane (4.1) passes through the point (4.14), then we can determine, by observing the condition $B_1 = 0$, the parameter k in equation (4.3):

(4.15) $$k = I/c + I'^2\lambda/(2c'\beta),$$

and the equation, other than (4.1), of the conic becomes

(4.16) $$x_1x_3 - (c'/2\beta\lambda)x_2^2 + (I/c + I'^2\lambda/2c'\beta)x_3^2 = 0.$$

As the plane (4.1) revolves about the line yz, the locus of this conic is a quadric cone with vertex at the point y_1, whose equation is found, by eliminating λ from equations (4.1), (4.16), to be

(4.17) $$x_1x_4 - (c'/2\beta)x_2^2 + (I/c)x_3x_4 + (I'^2/2c'\beta)\ x_4^2 = 0.$$

Similarly, we may obtain the quadric cone with vertex at the point z_{-1},

$$(4.18) \qquad x_2x_3 - (c/2\gamma)x_1^2 + (I^2/2c\gamma)x_3^2 + (I'/c')x_3x_4 = 0.$$

The line $\eta\zeta$ intersects the quadric cones (4.17), (4.18) in the points η, ζ and two other points with the coordinates

$$(4.19) \qquad (0,\ 0,\ cI'^2,\ -2c'\beta I),$$

$$(4.20) \qquad (0,\ 0,\ 2c\gamma I',\ -c'I^2).$$

The two points (4.19), (4.20) coincide in neither η nor ζ if, and only if,

$$(4.21) \qquad II' = 4K,$$

and they are separated harmonically by the points η, ζ in case

$$(4.22) \qquad II' + 4K = 0.$$

5. Moutard Quadrics of the Surfaces S_y, S_z

It is known that for a nonasymptotic tangent at an ordinary point of a surface there is a quadric of Moutard, which is the locus of the osculating conic at the point of the plane section of the surface made by a variable plane through the tangent. In this section we shall find the equations of the quadrics of Moutard for the tangents yz, $y\eta$ of the surface S_y and also for the tangents zy, $z\zeta$ of the surface S_z.

By means of equation (4.1) and the series (4.9), we may obtain the expansion for the projection in the plane $x_4 = 0$ of the section C_y of the surface S_y made by a general plane (4.1):

$$(5.1) \quad x_3/x_1 = \frac{c'}{2\beta\lambda}\left(\frac{x_2}{x_1}\right)^2 + \frac{c'}{4\beta^2\lambda^3}\left(\frac{c'q}{2c^2} - \frac{c'I\lambda}{c} - \frac{L\lambda^2}{2\beta}\right)\left(\frac{x_2}{x_1}\right)^4 + \cdots.$$

The osculating conic of the section C_y at the point y is given by equations (4.1) and

$$(5.2) \quad x_1x_3 - \frac{c'}{2\beta\lambda}x_2^2 - \frac{1}{c'\lambda}\left(\frac{c'q}{2c^2} - \frac{c'I\lambda}{c} - \frac{L\lambda^2}{2\beta}\right)x_3^2 = 0.$$

As the plane (4.1) revolves about the line yz, the locus of this osculating conic is the quadric $Q_y^{(u)}$ of Moutard for the tangent yz of the surface S_y, <u>whose equation is found, by eliminating</u> λ <u>from equations</u> (4.1), (5.2), to be

$$(5.3) \quad x_1x_4 - \frac{c'}{2\beta}x_2^2 - \frac{q}{2c^2}x_3^2 + \frac{I}{c}x_3x_4 + \frac{L}{2c'\beta}x_4^2 = 0.$$

Similarly, <u>the equation of the quadric</u> $Q_z^{(v)}$ <u>of Moutard for the tangent yz of the surface</u> S_z <u>is</u>

$$(5.4) \quad x_2x_3 - \frac{c}{2\gamma}x_1^2 + \frac{L'}{2c\gamma}x_3^2 + \frac{I'}{c'}x_3x_4 - \frac{p'}{2c'^2}x_4^2 = 0.$$

In order to find the equation of the quadric $Q_y^{(v)}$ of Moutard for the v-tangent of the surface S_y at the point y, let us consider a general plane through the line $y\eta$:

$$(5.5) \qquad x_4 - \mu x_2 = 0 \qquad (\mu \neq 0).$$

By means of equation (5.5) and the series (4.9), we may obtain the expansion for the projection in the plane $x_4 = 0$ of the section Γ_y of the surface S_y made by the plane (5.5):

$$(5.6) \qquad \frac{x_2}{x_1} = \frac{q}{2c^2\mu}\left(\frac{x_3}{x_1}\right)^2 + \frac{qM}{6c^3\mu}\left(\frac{x_3}{x_1}\right)^3 + \frac{q}{4c^4\mu^3}\left\{\frac{c'q}{2\beta} - \frac{qA'\mu}{\beta}\right.$$
$$\left. + \frac{\mu^2}{6}\left[M_v + M^2 + (P - \phi_v)M + 3L'\right]\right\}\left(\frac{x_3}{x_1}\right)^4 + \cdots.$$

The osculating conic of the section Γ_y at the point y is given by equations (5.5) and

$$(5.7) \qquad x_1x_2 - \frac{1}{q\mu}\left\{\frac{c'q}{2\beta} - \frac{qA'\mu}{\beta} + \frac{\mu^2}{6}\left[M_v - M^2/3\right.\right.$$
$$\left.\left. + (P - \phi_v)M + 3L'\right]\right\} x_2^2 - \frac{M}{3c}x_2x_3 - \frac{q}{2c^2\mu}x_3^2 = 0.$$

Elimination of μ from equations (5.5), (5.7) gives <u>the equation of the quadric $Q_y^{(v)}$ of Moutard for the v-tangent of the surface S_y at the point</u> y:

$$(5.8) \qquad x_1x_4 - c'x_2^2/2\beta - qx_3^2/2c^2 + A'x_2x_4/\beta - Mx_3x_4/3c$$
$$+ \frac{1}{18q}\left[M^2 - 3M_v - 3(P - \phi_v)M - 9L'\right] x_4^2 = 0.$$

Similarly, <u>the equation of the quadric $Q_z^{(u)}$ of Moutard</u>

<u>for the u-tangent of the surface</u> S_z <u>at the point z is</u>

$$(5.9) \qquad x_2x_3 - \frac{c}{2\gamma}x_1^2 + \frac{A}{\gamma}x_1x_3 + \frac{1}{18p'}\left[M'^2 - 3M'_u - 3(P' - \psi_u)M' - 9L\right]x_3^2 - \frac{M'}{3c'}x_3x_4 - \frac{p'}{2c'^2}x_4^2 = 0.$$

It is obvious that if the quadrics $Q_y^{(u)}$, $Q_z^{(v)}$ pass through the points ζ, η then the focal nets N_y, N_z are restricted, respectively, by the conditions $L = 0$, $L' = 0$. The polar planes of the points η, ζ with respect to the quadric $Q_y^{(u)}$ and with respect to the quadric $Q_z^{(v)}$ are, respectively,

$$(5.10) \qquad qx_3 - cIx_4 = 0,$$

$$(5.11) \qquad x_1 + Ix_3/c + Lx_4/c'\beta = 0,$$

$$(5.12) \qquad x_2 + L'x_3/c\gamma + I'x_4/c' = 0,$$

$$(5.13) \qquad c'I'x_3 - p'x_4 = 0.$$

If the planes (5.10), (5.13) pass through the points ζ, η then $I = 0$, $I' = 0$, respectively. Furthermore, these two planes coincide in case the focal nets N_y, N_z are restricted by the condition

$$(5.14) \qquad II' = \wp .$$

The line $\eta\zeta$ intersects the two planes (5.11), (5.12) in two points with coordinates

(5.15) $(0,\ 0,\ cL,\ -c'\beta I), \quad (0,\ 0,\ c\gamma I',\ -c'L')$.

The points (5.15) coincide in neither η nor ζ if, and only if,

(5.16) $LL' = II'K$,

and they are separated harmonically by the points η, ζ in case

(5.17) $LL' + II'K = 0$.

On the other hand, the polar planes of the points z, η, ζ with respect to the quadric $Q_y^{(v)}$ have the equations, respectively,

(5.18) $c'x_2 - A'x_4 = 0$,

(5.19) $3qx_3 + cMx_4 = 0$,

(5.20) $x_1 + \frac{A'}{\beta} x_2 - \frac{M}{3c} x_3 + \frac{1}{9q}\left[M^2 - 3M_v - 3(P - \phi_v)M - 9L'\right] x_4 = 0$.

If the planes (5.18), (5.19) pass through the point ζ, then $n = 0$, $M = 0$, respectively; and a similar argument can made with regard to the quadric $Q_z^{(u)}$. Moreover, the plane (5.19) coincides with the polar plane of the point ζ with respect to the quadric $Q_z^{(u)}$ in case

(5.21) $MM' = 9\wp$.

The line $y\eta$ intersects the planes (5.11), (5.20) in two points with coordinates

$$(5.22) \qquad (I,\ 0,\ -c,\ 0), \qquad (M,\ 0,\ 3c,\ 0).$$

These two points coincide in neither y nor η if, and only if,

$$(5.23) \qquad 3I + M = 0.$$

Further, if the points (5.22) are separated harmonically by the points y, η then

$$(5.24) \qquad 3I = M.$$

Likewise, we can discuss the conditions similar to (5.23), (5.24).

Finally, if the planes (5.10), (5.13) coincide, and if the points (4.19), (4.20) are coincident or separated harmonically by the points η, ζ, then the focal nets N_y, N_z are restricted by the conditions (5.14) and

$$(5.25) \qquad \mathfrak{H} = \pm 4K.$$

Now we are in a position to determine geometrically the unit point of the coordinate system. To this end, we observe that the locus of a moving point, whose polar planes with respect to the quadrics $Q_y^{(v)}$, $Q_z^{(u)}$ intersect the generator yz at the same point, is a quadric, its equation can easily be found to be

(5.26) $$cc'x_1x_2 - cA'x_1x_4 - c'Ax_2x_3 + (AA' - \beta\gamma)x_3x_4 = 0.$$

This quadric cuts the cubic curve of intersection, besides the line $\eta\zeta$, of the two cones (4.11), (4.12) in two points. It is easily seen that _if m' and n not vanish at the same time for the congruence yz, then we may take one of these two points as the unit point of the coordinate system, and therefore we have_

(5.27) $$c = 2\gamma, \quad c' = 2\beta, \quad 3p'q + 4m'n = 4(m'q + np').$$

6. Segre-Darboux Nets on the Surfaces S_y, S_z

It is known that associated with an ordinary point O of a surface S there are three-parameter family of quadrics each of which has second order contact with the surface S at the point O. A general quadric of this family intersects the surface S in a curve with a triple point at O. In particular, if the three triple-point tangents of the curve of intersection coincide, then the quadric is called _a quadric of Darboux_, and the corresponding coincident triple-point tangents and the curve of intersection are respectively called _a tangent_ and _a curve of Darboux_ at the point O of the surface S.

In order to find the equations of the configurations mentioned above for the surface S_y at the point y, we first consider the quadrics having second order contact with the surface S_y at the point y. The equation of a general one of these qudrics is obtained by writing the equation of the most general nonsingular quadric and demanding that the series (3.4) satisfy this equation identically in Δu, Δv as far as the terms of the second degree. The result can be written in the form

$$(6.1) \qquad x_1x_4 - \frac{c'}{2\beta} x_2^2 - \frac{q}{2c^2} x_3^2 + (k_2x_2 + k_3x_3 + k_4x_4)x_4 = 0,$$

where k_2, k_3, k_4 are parameters. Each quadric (6.1) cuts the surface S_y in a curve with a triple point at y, whose tangents are in the directions satisfying the equation

$$(6.2) \qquad 3c'\beta^2k_2du^3 + 3c'\beta(ck_3 - I)du^2dv + 3q(\beta k_2 - A')dudv^2 + q(3ck_3 + M)dv^3 = 0.$$

It is not difficult to verify that if the binary cubic form that appears in equations (6.2) is a perfect cube of a linear form, then

$$(6.3) \qquad k_2 = A'/4\beta, \qquad k_3 = (I - M)/4c,$$

and therefore the equations of the curves and the pencil of quadrics of Darboux at the point y of the surface S_y are, respectively,

$$(6.4) \qquad 3c'\beta A'du^3 - 3c'\beta(3I + M)du^2dv - 9qA'dudv^2 + q(3I+M)dv^3 = 0,$$

$$(6.5) \qquad 4c^2\beta x_1x_4 - 2c^2c'x_2^2 - 2q\beta x_3^2 + c^2A'x_2x_4 + c\beta(I-M)x_3x_4 + k_4x_4^2 = 0.$$

Similarly, the curves and the pencil of quadrics of Darboux at the point z of the surface S_z are respectively given by the equations

$$(6.6)\qquad p'(3I'+M')du^3-9p'Adu^2dv-3c\gamma(3I'+M')dudv^2+3c\gamma Adv^3=0,$$

$$(6.7)\qquad 2c'^2cx_1{}^2-c'^2Ax_1x_3-4c'^2\gamma x_2x_3-c'\gamma(I'-M')x_3x_4+3p'\gamma x_4{}^2 + k_3x_3{}^2=0,$$

where k_3 is a parameter.

The equations in local coordinates of the tangents of Darboux at the point y, z of the surfaces S_y, S_z may easily be obtained from equations (6.4), (6.6), (7.4), (7.5); the result we shall omit here.

The polar line of the line $y\zeta$ with respect to the pencil (6.5) of quadrics of Darboux at the point y of the surface S_y is

$$(6.8)\qquad x_4=4c\beta x_1+cA'x_2+\beta(I-M)x_3=0,$$

which intersects the line yz in a point

$$(6.9)\qquad (-A',\ 4\beta,\ 0,\ 0),$$

and passes through the point z or η in case $n=0$ or $I=M$. Similarly, the polar line of the line $z\eta$ with respect to the pencil (6.7) of quadrics of Darboux at the point z of the surface S_z is

$$(6.10)\qquad x_3=c'Ax_1+4c'\gamma x_2+\gamma(I'-M')x_4=0,$$

which intersects the line yz in a point

(6.11) $(4\gamma, -A, 0, 0)$,

and passes through the point y or ζ in case $m' = 0$ or $I' = M'$. Moreover, if the two points (6.9), (6.11) are coincident or separated harmonically by the points y, z, then the focal nets N_y, N_z are restricted by the conditions

(6.12) $AA' = \pm 16K$.

From equations (6.4), (6.6) it is easily seen that the curves of Darboux correspond on the two surfaces S_y, S_z if, and only if,

(6.13) $W_{(u)} = 0, \quad (3I + M)(3I' + M') = 9AA'$.

It is known [4, p. 283] that if the curves of Darboux correspond on the two focal surfaces of a congruence, the congruence is a W congruence and both surfaces have the property of being isothermally asymptotic. Thus <u>the conditions for the both surfaces S_y, S_z to be isothermally asymptotic at the same time can be reduced to the form</u> (6.13).

On the other hand, a necessary and sufficient condition for the u-curves or the v-curves of the surface S_y to be curves of Darboux is $n = 0$ or (5.23). It is known that if the curves of one family of a conjugate net are curves of Darboux, then the curves of the other family must be the corresponding curves of Segre. Such a net is called <u>a Segre-Darboux net</u>. Combining the above results and the ones in §§4, 5 we obtain the following theorem:

The focal net N_y is a Segre-Darboux net with the u-curves as curves of Darboux if, and only if, the point ζ is in the osculating plane of the v-curve of the net N_y at the point y. The focal net N_y is a Segre-Darboux net with the v-curves as curves of Darboux if, and only if, the line $y\eta$ intersects, in the same point, the polar planes of the point ζ with respect to the quadrics $Q_y^{(u)}$, $Q_y^{(v)}$ of Moutard at the point y of the surface S_y.

We shall call a congruence a Segre-Darboux congruence when the two focal nets of a congruence both are Segre-Darboux nets. Noticing the theroem concerning the other focal net N_z and similar to the above one, we may obtain a geometric interpretation for a Segre-Darboux congruence yz.

7. W Congruences

The differential equation of the asymptotic curves on the surface S_y is

$$\mathbf{L}du^2 + 2\mathbf{M}dudv + \mathbf{N}dv^2 = 0, \tag{7.1}$$

where the coefficients **L**, **M**, **N** are the determinants of the fourth order defined by

$$\mathbf{L} = (y_{uu}, y, y_u, y_v), \quad \mathbf{M} = (y_{uv}, y, y_u, y_v), \quad \mathbf{N} = (y_{vv}, y, y_u, y_v). \tag{7.2}$$

By means of equations (1.1), (3.1), it is easy to write equation (7.1) in the form

$$(7.3) \qquad c'\beta\, du^2 + q dv^2 = 0.$$

The equations, in local coordinates, of the asymptotic tangents to the surface S_y at the point y may easily be obtained from the expansion (4.9) of the surface S_y, or else from equation (7.3) and the fact that the tangent at the point y to a curve C_λ belonging to the family defined on the surface S_y by the differential equation

$$(7.4) \qquad dv - \lambda\, du = 0,$$

where λ is a function of u, v, has the equations in local coordinates

$$(7.5) \qquad x_4 = \beta x_3 - c\lambda x_2 = 0.$$

Similarly, the differential equation of the asymptotic curves on the surface S_z is

$$(7.6) \qquad p' du^2 + c\gamma\, dv^2 = 0.$$

The asymptotic curves on the focal surfaces S_y, S_z correspond in case equations (7.3), (7.6) are equivalent. Then the u-tangents of the net N_y or the v-tangents of the net N_z form a congruence of the special type called <u>a W congruence</u>. Thus we reach the result that <u>the u-tangents</u>

of the net N_y or the v-tangents of the net N_z form a W congruence in case the Weingarten invariant $W_{(u)}$ or $W'_{(v)}$ defined by equations (2.5) vanishes.

From equation (2.10), (1.1), (1.2), (1.3) by differentiation and substitution, any derivative of y_1 can be expressed as a linear combination of y, z, η , ζ . In particular, one obtains

$$(7.7)\quad \begin{cases} y_{1u} = (H - cf\alpha/\beta)y + c\alpha\eta, \\ y_{1v} = [cE - acf/\beta - (cf/\beta)_v]y + nz + (cs + c_v - c^2f/\beta)\eta + q\zeta, \\ y_{1uu} = (*)y + \beta Hz + (*)\eta, \\ y_{1uv} = (*)y + n\alpha z + (*)\eta + q\alpha\zeta, \\ y_{1vv} = (*)y + (n_v - n\beta_v/\beta + np/c + f'q)z + (*)\eta \\ \qquad\qquad + q(p/c - c'_v/c' - \beta_v/\beta + q_v/q)\zeta . \end{cases}$$

From equations (7.7) and the ones similar to (7.1), (7.2), the differential equation of the asymptotic curves on the surface S_{y_1} sustaining the first Laplace transformed net N_{y_1} of the net N_y is found to be

$$(7.8)\qquad c'\beta Hdu^2 + q\mathfrak{R}dv^2 = 0.$$

Thus the v-tangents of the net N_y form a W congruence in case the Weingarten invariant $W_{(v)}$ defined by equations (2.5) vanishes.

Similarly, *the u-tangents of the net* N_z *form a W congruence if, and only if, the Weingarten invariant* $W'_{(u)}$ *vanishes*.

From the relation between the Weingarten and the Laplace-Darboux invariants of a conjugate net, we know immediately that $\mathfrak{H}$, $\mathfrak{K}$ and $\mathfrak{H}'$, $\mathfrak{K}'$ are respectively *the tangential invariants of the focal nets* N_y *and* N_z.

Now we proceed to give a simple geometric interpretation for the condition for the u-tangents or the v-tangents of the net N_y to form a W congruence.

The equation of any quadric having the lines yz, $y\eta$, $z\zeta$ as generators can be written in the form

$$x_1x_4 + k_1x_2x_3 + k_2x_3x_4 = 0, \tag{7.9}$$

where k_1, k_2 are parameters. A general quadric (7.9) intersects the surface S_y in a curve with a double point at y, whose tangents are found, from the series (4.9), to be

$$x_4 = (c'/\beta)x_2^2 + 2k_1x_2x_3 + qx_3^2/c^2 = 0. \tag{7.10}$$

If these two tangents coincide, then

$$k_1^2 = c'q/c^2\beta. \tag{7.11}$$

Similarly, the quadric (7.9) cuts the surface S_z in a curve with a cusp at z if, and only if,

$$k_1^2 = c'^2\gamma/cp'. \tag{7.12}$$

The conditions (7.11), (7.12) hold simultaneously in case $W_{(u)} = 0$. Thus we arrive at the following theorem:

A necessary and sufficient condition for the congruence yz to be a W congruence is that there exists a quadric (and therefore one-parameter family of such quadrics), which has yz, $y\eta$, $z\zeta$ as generators and whose curves of intersection with the focal surfaces S_y, S_z have cusps at the points y, z respectively.

Similar statements can be obtained for both the congruences $y\eta$ and $z\zeta$.

A conjugate net whose Weingarten invariants both vanish is called [9, p. 1077] an R net, each family of curves of an R net has tangents that form a W congruence. From the above theorem, we may also interpret geometrically a congruence when either of its focal nets or both are R nets.

8. Curves of the Focal Nets N_y, N_z

By putting Δv to zero, expansions (3.5) become the nonhomogeneous local coordinates of a point Y near the point y and on the u-curve through y of the focal net N_y, and therefrom we may easily obtain the power series expansions for the u-curve of the focal net N_y in the neighborhood of the point y, namely,

$$(8.1)\quad \begin{cases} \dfrac{y_3}{y_1} = \dfrac{p'}{6\beta^2}\left(\dfrac{y_2}{y_1}\right)^3 + \dfrac{p'M'}{24\beta^3}\left(\dfrac{y_2}{y_1}\right)^4 + \cdots, \\[2ex] \dfrac{y_4}{y_1} = \dfrac{c'}{2\beta}\left(\dfrac{y_2}{y_1}\right)^2 - \dfrac{c'L}{8\beta^3}\left(\dfrac{y_2}{y_1}\right)^4 + \cdots. \end{cases}$$

Similarly, by putting Δu to zero in expansions (3.5), we reach <u>the following power series expansions for the v-curve of the focal net N_y in the neighborhood of the point</u> y:

$$(8.2)\quad \begin{cases} \dfrac{y_2}{y_1} = \dfrac{n}{2c^2}\left(\dfrac{y_3}{y_1}\right)^2 + \dfrac{n\,(M + \mathfrak{A}/A')}{6c^3}\left(\dfrac{y_3}{y_1}\right)^3 + \dfrac{n}{24c^4}\left[M_v + M^2 \right. \\[1ex] \qquad \left. + 3L' + (\mathfrak{A}_v + 2M\mathfrak{A})/A' + (P - \phi_v)(M + \mathfrak{A}/A')\right]\left(\dfrac{y_3}{y_1}\right)^4 \\[1ex] \qquad + \cdots, \\[2ex] \dfrac{y_4}{y_1} = \dfrac{q}{2c^2}\left(\dfrac{y_3}{y_1}\right)^2 + \dfrac{qM}{6c^3}\left(\dfrac{y_3}{y_1}\right)^3 + \dfrac{q}{24c^4}\left[M_v + M^2 + (P - \phi_v)M \right. \\[1ex] \qquad \left. + 3L'\right]\left(\dfrac{y_3}{y_1}\right)^4 + \cdots. \end{cases}$$

Analogous expansions for the u-, v-curves at the point z of the focal net N_z can be written by making the substitutions (3.8), (3.9) on the symbols.

From expansions (8.1), (8.2), the quadrics (6.1) having second order contact with the surface S_y at the point y are the quadrics having third order contact with both the u-, v-curves of the surface S_y at the point y in case

$$(8.3) \qquad k_2 = 0, \qquad k_3 = -M/3c.$$

Thus we find that <u>any quadric of the pencil</u>

$$(8.4) \qquad x_1x_4 - (c'/2\beta)x_2^2 - qx_3^2/2c^2 - Mx_3x_4/3c + k_4x_4^2 = 0,$$

<u>where k_4 is arbitrary, has third order contact with the parametric curves of the surface S_y at the point</u> y. If a unique quadric of this pencil is desired, we may choose the one that passes through the covariant point ζ. The polar plane of the point η with respect to any quadric of the pencil (8.4) is the plane (5.19).

Among the quadrics (6.1) there is a pencil having fourth order contact with the u-curve (8.1) of the surface S_y at the point y. For this pencil we find

$$(8.5) \qquad k_2 = 0, \qquad k_4 = L/2c'\beta \;,$$

with k_3 arbitrary. The quadric $Q_y^{(u)}$ (5.3) of Moutard for the u-tangent of the surface S_y is a unique quadric of this pencil [8, p. 698], and for this quadric we have $k_3 = I/c$.

Among the quadrics (6.1) there is also a pencil having fourth order contact with the v-curve (8.2) of the surface S_y at the point y. For this pencil we find

$$(8.6)\quad \begin{cases} k_3 = -M/3c, \\ k_4 = \frac{1}{18q}\left[M^2 - 3M_v - 3(P - \phi_v)M - 9L' + 9nA'/\beta - 18nk_2\right]. \end{cases}$$

9. Correspondences Associated with the Focal Nets N_y, N_z

Let us consider a curve C_λ passing through the point y and belonging to the family (7.4) defined on the focal surface S_y; and let the parametric representation of the curve C_λ be

$$(9.1)\qquad u = u(w), \quad v = v(w).$$

The parametric u-, v-tangents at points of the curve C_λ generate two non-developable ruled surfaces $R_y^{(u)}$, $R_y^{(v)}$ respectively. The points

$$(9.2)\qquad T = ty - z, \qquad \bar{T} = \bar{t}y - \eta,$$

where t, $\bar{t}$ are functions of u, v, lie on the parametric u-, v-tangents through the point y respectively, and the line ℓ_2 determined by them lies in the tangent plane $x_4 = 0$ of the focal surface S_y at the point y.

From system (1.1) and equations (1.2), one easily obtains

$$(9.3)\quad \begin{cases} T_u = (t_u + t\alpha)y + (t\beta - b')z - c'\zeta, \\ T_v = (t_v + at - \gamma)y + ct\eta - \delta z, \end{cases}$$

$$\begin{cases} \bar{T}_u = (\bar{t}_u + \bar{t}\alpha - e)y + (\bar{t}\beta - f)z - g\eta, \\ \bar{T}_v = (\bar{t}_v + a\bar{t} - r)y - nz/c + (c\bar{t} - s)\eta - q\zeta/c. \end{cases}$$

The tangent planes to the ruled surface $R_y^{(u)}$ at the point T and to the ruled surface $R_y^{(v)}$ at the point $\bar{T}$ intersect in a line ℓ_1 which passes through the point y. There must be a point on the line ℓ_1 given by an expression of the form

$$X = \zeta + k_1 z + k_2 \eta .$$

The plane of the points y, X, T is to be tangent to the ruled surface $R_y^{(u)}$ at T, and therefore must contain the tangent to the curve traced by T as the point y moves along the curve C_λ. It is clear from the first two of equations (9.3) that a point on this tangent is given by

$$T_w = (*)y + (*)z + ctv_w \eta - c'u_w \zeta ;$$

that is to lie in the plane yXT, which can occur if and only if $k_2 = -ct\lambda/c'$. Similarly, the plane $yX\bar{T}$ is tangent to the ruled surface $R_y^{(v)}$ at $\bar{T}$ if and only if $k_1 = -c\,(\bar{t}\beta - f - n\lambda/c)/q\lambda$. Thus we obtain a correspondence(2)

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(2) This correspondence has been introduced by the author [6, pp. 381-382]. The line ℓ_1 (9.5) is the analogue for a conjugate parametric net of the R'_λ-associate of the reciprocal of ℓ_2 (9.4) defined by Bell [1, pp. 390-391].

between the line ℓ_2 in the plane $x_4 = 0$:

(9.4) $$x_4 = x_1 + tx_2 + \bar{t}x_3 = 0,$$

and the line ℓ_1 through the point y:

(9.5) $$\begin{cases} x_3 + ct\lambda x_4/c' = 0, \\ x_2 + c(\bar{t}\beta - f - n\lambda/c)x_4/q\lambda = 0. \end{cases}$$

Conversely, if equations (9.5) are written in the form

(9.6) $$x_2/\mathfrak{A} = x_3/\mathfrak{B} = x_4/\mathfrak{C} ,$$

then

(9.7) $$\begin{cases} \rho\, t = - c'\mathfrak{B}/c\lambda, \\ \rho\, \bar{t} = - q\lambda\mathfrak{A}/c\beta + (f + n\lambda/c)\mathfrak{C}/\beta , \\ \rho = \mathfrak{C} \quad . \end{cases}$$

When the line (9.4) passes through a fixed point $(x_1^*, x_2^*, x_3^*, 0)$, the corresponding line (9.5) describes a plane, its equation is

(9.8) $$\frac{q\lambda}{c\beta} x_3^* x_2 + \frac{c'}{c\lambda} x_2^* x_3 + \left[x_1^* + (f + n\lambda/c)x_3^*/\beta\right] x_4 = 0.$$

Consequently, a point $(x_1^*, x_2^*, x_3^*, 0)$ in the plane $x_4 = 0$ and the plane $(0, u_2, u_3, u_4)$ through the point y are in the following correspondence:

$$(9.9)\qquad \begin{cases} \sigma u_2 = -\, q\lambda/(c\beta)\; x_3^*, \\ \sigma u_3 = -\, c'/(c\lambda)\; x_2^*, \\ \sigma u_4 = x_1^* + \frac{1}{\beta}\,(f + n\lambda/c)\; x_3^*, \end{cases} \qquad (\sigma \neq 0).$$

This correspondence becomes a polarity with respect to a quadric (and therefore ∞^1 quadrics) if and only if λ satisfies

$$(9.10)\qquad \lambda^2 = c'\beta/q,$$

that is, if and only if the tangent of the curve C_λ at the point y is an associate conjugate tangent of the parametric conjugate net N_y. The equation of any one of these quadrics is then of the form

$$(9.11)\qquad c\sqrt{D}\; x_1x_4 - c'\; x_2x_3 + (A' + I\sqrt{D})\; x_3x_4 + k_4 x_4{}^2 = 0,$$

where k_4 is a parameter.

In a similar manner, with regard to the focal net N_z and the focal surface S_z we may obtain the polarity with respect to the following pencil of quadrics

$$(9.12)\qquad c'\sqrt{D'}\; x_2x_3 - c\; x_1x_4 + (A + I'\sqrt{D'})\; x_3x_4 + k_3 x_3{}^2 = 0,$$

where k_3 is a parameter.

The polar planes of any point $(x_1^*, x_2^*, 0, 0)$ except the points y, z on the line yz with respect to any quadrics (9.11), (9.12) are respectively

$$(9.13)\quad \begin{cases} c' x_2^* x_3 - c\sqrt{D}\; x_1^* x_4 = 0, \\ c'\sqrt{D'}\; x_2^* x_3 - c x_1^* x_4 = 0, \end{cases}$$

which coincide if and only if $K = \mathfrak{H}$. Thus we obtain another geometric interpretation of a W congruence:

A necessary and sufficient condition for the congruence yz to be a W congruence is that the polar planes of any point, except the points y, z, on the generator yz with respect to the two pencils (9.11), (9.12) of quadrics be coincident.

10. Axis Congruences and Ray Congruences

The axis at the point y of the net N_y is defined to be the line of intersection of the osculating planes of the u-, v-curves of the net N_y at the point y. From equations (1.1), (3.1) it is easy to obtain the equations of this axis, namely,

$$(10.1)\qquad x_3 = qx_2 - nx_4 = 0.$$

The point Y defined by

$$(10.2)\qquad Y = ky + nz + q\zeta \qquad\qquad (k \text{ scalar})$$

is on the axis. When the point y varies along a curve C_λ of the family (7.4) on the surface S_y, the point Y generates a curve whose tangent at Y is determined by Y and Y', which can be expressed as a linear combination of y, z, η, ζ by using equations (1.1), (1.2), (1.3), namely,

(10.3) $$Y' = [k' + \alpha k + m'q/c' + (ak + n\gamma + e'q)\lambda]\, y$$
$$+[\beta k + b'n + n_u + qs' + (n_v + n\delta + f'q)\lambda]\, z$$
$$+(ck\lambda + p'q/c')\eta + [c'n + q_u + qr' + (q_v + q\delta$$
$$-c_v'/c')\lambda]\,\zeta .$$

In the expression (10.3) equating the coefficient of η to zero and the ratio of the coefficients of z and ζ to n/q and making use of equations (2.5), we obtain two conditions on the functions k and λ which are necessary and sufficient that the axis may generate a developable surface and have Y for focal point when the point Y varies along the curve C_λ ; these two conditions are

(10.4) $$\mathfrak{P} + k\lambda = 0, \qquad c'\beta N + c'\beta k + q\mathfrak{A}\lambda = 0.$$

Elimination of λ from these two equations gives

(10.5) $$c'\beta k^2 + c'\beta Nk - q\mathfrak{P}\mathfrak{A} = 0.$$

If k_1 and k_2 are the two roots of this equation regarded as a quadratic equation in k, the two points Y_1 and Y_2 defined by the formula (10.2) with k_1 and k_2, respectively, in place of k are the focal points of the axis, and the loci of these two focal points when u and v vary are the focal surfaces of the axis congruence of the net N_y. On the other hand, on eliminating k from equations (10.4) and replacing λ by dv/du, we obtain <u>the differential equation of the axis curves of the net</u> N_y, in which the developables of the axis congruence intersect the surface S_y, namely,

$$(10.6) \qquad c'\beta \mathfrak{H}\, du^2 - c'\beta N du dv - q \mathfrak{A} dv^2 = 0.$$

The tangents of these axis curves are called the axis tangents of the net N_y.

Since at each point of a harmonic conjugate net the axis tangents separate the tangents of the net harmonically [13, p. 215], it follows immediately that <u>the net</u> N_y <u>is harmonic in case</u> $N = 0$. Similarly, <u>the net</u> N_z <u>is harmonic if, and only if,</u> $N' = 0$. From equations (7.3), (10.6) we may also verify the result [12, p. 319] that the axis curves of the nets N_y, N_z form conjugate nets in case $\mathfrak{H} = \mathfrak{A}$, $\mathfrak{H}' = \mathfrak{A}'$, respectively.

We shall next determine the developables and focal surfaces of the ray congruence of the net N_y. The point X defined by

$$(10.7) \qquad X = y_1 + kz \qquad (k \text{ scalar})$$

is on the ray of the net N_y corresponding to the point y. When the point y varies along a curve of the family (7.4) on the surface S_y, the point X generates a curve whose tangent at X is determined by X and the point X', which can be expressed as a linear combination of y, z, η, ζ by means of equations (1.1), (1.2), (1.3), (2.5), namely,

$$(10.8) \qquad X' = \{H - cf\alpha/\beta + [a_v - \delta_v + a^2 - a\delta - (\log\beta)_{vv}$$

$$- a(\log\beta)_v + cr + \beta\gamma k]\lambda\} y + [c_u + cg + c(a - \delta + s + c_v/c - \beta_v/\beta)\lambda] z + c[\alpha + (c_v/c - cf/\beta + s)\lambda]\eta + (q\lambda + c'k)\zeta \qquad (X' = dX/du, \ldots).$$

In the expression (10.8) equating the coefficient of ζ to zero and the ratio of the coefficients of y and η to $-f/\beta$ and making use of equations (1.3), we obtain two conditions on the functions k and λ which are necessary and sufficient that the ray may generate a developable surface and have X for focal point when the point y varies along the curve C_λ ; these two conditions are

$$(10.9) \qquad \begin{cases} c'k + q\lambda = 0, \\ \beta H + [\beta\gamma k - c(f_v + f\delta - r\beta - fs)]\lambda = 0. \end{cases}$$

From the second and the fourth of the integrability conditions (1.4) and equations (2.5), we may further reduce the two conditions (10.9) to

$$(10.10) \qquad c'k + q\lambda = 0, \quad H + (\gamma k - N)\lambda = 0.$$

Elimination of λ from these two equations gives

$$(10.11) \qquad c'\gamma k^2 - c'Nk - qH = 0.$$

If k_1, k_2 are the two roots of this equation, then the corresponding points X_1, X_2 are the focal points of the ray, and the loci of these two focal points when u and v vary are the focal surfaces of the ray congruence of the net N_y. On the other hand, on eliminating k from equations (10.10) and replacing λ by dv/du, we obtain <u>the differential equation of the ray curves of the net</u> Ny, in which the developables of the ray congruence intersect the surface S_y, namely,

$$(10.12) \qquad c'Hdu^2 - c'Ndudv - q\gamma\, dv^2 = 0.$$

The tangents of these ray curves, called the ray tangents of the net N_y, separate the tangents of the net N_y harmonically in case $N = 0$, that is, in case the net is harmonic [13]. The ray curves of the net N_y form a conjugate net [12] in case $H = K$.

In a similar way, we can discuss the ray congruence of the net N_z.

11. <u>The Congruences</u> $z\eta$, $y\zeta$ <u>and the Principal Congruence</u> $\eta\zeta$

As the point y varies on the surface S_y, the line $z\eta$ describes a congruence. The point P defined by

$$(11.1) \qquad P = \eta + kz \qquad\qquad (k \text{ scalar})$$

is on the line $z\eta$. When the point y varies along a curve of the family (7.4) on the surface S_y, the point P generates a curve whose tangent at P is determined by P and the point P', which can be expressed as a linear combination of y, z, η, ζ by means of equations (1.1), (1.2), namely,

$$(11.2)\qquad P' = (e + r\lambda + \gamma k\lambda)y + (f + n\lambda/c + b'k + \delta k\lambda + k')z + (g + s\lambda)\eta + (q\lambda + cc'k)\zeta/c \qquad (P' = dP/du,\ \ldots).$$

In the expression (11.2) equating the coefficients of y, ζ to zero, we obtain two conditions on the functions k and λ which are necessary and sufficient that the line $z\eta$ may generate a developable surface and have P for focal point when the point y varies along the curve C_λ ; these two conditions are

$$(11.3)\qquad cc'k + q\lambda = 0, \qquad J + (c\gamma k + L')\lambda = 0.$$

Elimination of λ from these two equations gives

$$(11.4)\qquad c^2c'\gamma k^2 + cc'L'k - qJ = 0.$$

If k_1, k_2 are the two roots of this equation, then the corresponding points P_1, P_2 are the focal points of the line $z\eta$, and the loci of these two focal points when u and v vary are the focal surfaces of the congruence $z\eta$ of the net N_y. On the other hand, on eliminating k from equations (11.3) and replacing λ by dv/du, we obtain the differential equation of the curves in which the developables of the congruence $z\eta$ of the net N_y intersect the surface S_y, namely,

$$(11.5)\qquad c'Jdu^2 + c'L'dudv - q\gamma dv^2 = 0.$$

The tangents of the curves (11.5) separate the tangents of the net N_y harmonically in case $L' = 0$. Moreover, the curves (11.5) form a conjugate net if, and only if, $J = K$. Finally, from equations (2.7), (2.10) it follows that if the point y_1 coincides with the point η and if the curves (11.5) form a conjugate net, then the conjugate net N_y has equal Laplace-Darboux invariants.

Similarly, we can discuss the congruence $y\zeta$.

Finally, we shall call the congruence $\eta\zeta$ the principal congruence of the congruence yz. The point Q defined by

(11.6) $$Q = \eta + k\zeta \qquad \text{(k scalar)}$$

is on the line $\eta\zeta$. When the point y varies along a curve of the family (7.4) on the surface S_y, the point Q generates a curve whose tangent at Q is determined by Q and the point Q' given by

(11.7) $$Q' = (e + r\lambda + m'k/c' + e'k\lambda)y + (f + n\lambda/c + s'k + f'k\lambda)z + (g + s\lambda + p'k/c')\eta + (g\lambda/c + r'k + g'k\lambda + k')\zeta .$$

In this expression equating the coefficients of y, z to zero, we obtain two conditions on the functions k and λ which are necessary and sufficient that the line $\eta\zeta$ may generate a developable surface and have Q for focal point when the point y varies along the curve C_λ ; these two conditions are

$$(11.8)\qquad e + r\lambda + m'k/c' + e'k\lambda = 0, \qquad f + n\lambda/c + s'k + f'k\lambda = 0.$$

Elimination of λ from these two equations gives

$$(11.9)\qquad c^2m'(B'I'L - GJ')k^2 + cc'G(II'K + LL' - JJ' - G)k + c'^2n(BIL - GJ) = 0.$$

If k_1, k_2 are the two roots of this equation, then the corresponding points Q_1, Q_2 are the focal points of the line $\eta\zeta$, and the loci of these two focal points when u and v vary are the focal surfaces of the principal congruence $\eta\zeta$. On the other hand, elimination of k from equations (11.9) and replacement of λ by dv/du yield the differential equation

$$(11.10)\qquad (BI - JL)du^2 + (II'K - LL' - JJ' + G)du\,dv + (B'I' - J'L')dv^2 = 0.$$

The developables of the principal congruence $\eta\zeta$ intersect the surfaces S_y, S_z in the curves respectively represented on S_y, S_z by this differential equation. We shall call these curves *the principal curves* of the nets N_y, N_z. The tangents of these principal curves will be called *the principal tangents* of the nets N_y, N_z.

The principal tangents of one net separate the tangents of the same net harmonically in case

(11.11) $$G + II'K = JJ' + LL'.$$

Moreover, necessary and sufficient conditions for the principal curves to form conjugate nets on the surfaces S_y, S_z are, respectively,

(11.12) $$\mathfrak{A}(BI - JL) + DH(B'I' - J'L') = 0,$$

(11.13) $$D'K'(BI - JL) + \mathfrak{H}'(B'I' - J'L') = 0.$$

Finally, the principal curves form conjugate nets on the both surfaces S_y, S_z at the same time if, and only if,

(11.14) $$HKK' = \mathfrak{H}\,\mathfrak{A}\,\mathfrak{H}'.$$

12. Osculating Linear Complexes and Associated Linear Complexes

In this section we shall find the equations of the osculating linear complex along a generator of a W congruence and the associated linear complexes of the focal points of a generator of a congruence(3).

First of all, we introduce line coordinates. In the customary way, the plückerian homogeneous coordinates of a line joining two points with coordinates $(y_1, \ldots, y_4)$, $(z_1, \ldots, z_4)$ are defined to be the six numbers ω_{12}, ω_{13},

(3) Cf. [10, p. 167]; [7, pp. 300-303].

$\omega_{14}, \omega_{23}, \omega_{42}, \omega_{34}$ given by

$$\omega_{ik} = y_i z_k - y_k z_i \qquad (i, k = 1, \ldots, 4). \tag{12.1}$$

From the expansions (3.4), (3.10), the local coordinates ω_{ik} of the generator YZ near the generator yz of the congruence are found to be represented by the series

$$\left\{\begin{aligned}
\omega_{12} &= 1 + (b' + \alpha)\Delta u + (a + \delta)\Delta v + \tfrac{1}{2}(\alpha_u + \alpha^2 + 2b\gamma + n')\Delta u^2 \\
&\quad + (\alpha_v + \delta_u + ab' + a\alpha + \alpha\delta + \beta\gamma + b'\delta)\Delta u \Delta v \\
&\quad + \tfrac{1}{2}(\delta_v + m + 2a\delta + \delta^2)\,\Delta v^2 + \ldots, \\
\omega_{13} &= \tfrac{1}{2}p'\,\Delta u^2 - \tfrac{1}{2}c\gamma\,\Delta v^2 + \ldots, \\
\omega_{14} &= c'\,\Delta u + \tfrac{1}{2}(q' + 2c'\alpha)\,\Delta u^2 + c'(a + \delta)\Delta u \Delta v + \ldots, \\
\omega_{23} &= -\,c\Delta v - c(b' + \alpha)\Delta u \Delta v - \tfrac{1}{2}(p + 2c\delta)\,\Delta v^2 + \ldots, \\
\omega_{42} &= -\,\tfrac{1}{2}c'\beta\,\Delta u^2 + \tfrac{1}{2}q\,\Delta v^2 + \ldots, \\
\omega_{34} &= cc'\,\Delta u\,\Delta v + \ldots .
\end{aligned}\right. \tag{12.2}$$

We observe that the complex $\omega_{13} = 0$ is special and consists of all lines intersecting the line $z\zeta$. Similarly, the complex $\omega_{42} = 0$ consists of all lines intersecting the line $y\eta$; and the complex $\omega_{34} = 0$ consists of all lines intersecting the generator yz. The lines common to all three of these complexes form two flat pencils, one with its center at the point y and lying in the tangent plane of the surface S_z at the point z, and the other with its center at z and lying in the tangent plane of S_y at y. The lines of these two pencils are called the central rays of the generator yz.

The central rays can also be characterized in another way. If we write the equation

$$(12.3) \qquad a_{34}\omega_{12} + a_{42}\omega_{13} + a_{23}\omega_{14} + a_{14}\omega_{23} + a_{13}\omega_{42} + a_{12}\omega_{34} = 0$$

of a general inear complex and demand that it be satisfied by the series (12.2) for ω_{ik} identically in Δu, Δv as far as the terms of the first degree, we obtain the conditions $a_{34} = a_{23} = a_{14} = 0$. Thus the equation of the most general linear complex having first order contact with the congruence yz along a generator yz is found to be

$$(12.4) \qquad a_{42}\omega_{13} + a_{13}\omega_{42} + a_{12}\omega_{34} = 0.$$

Such a complex contains the generator yz and a consecutive generator of every ruled surface in the congruence and containing yz. There are obviously ∞^2 such linear complexes; all of them have in common the central rays of the generator yz and no other lines.

If we go on and seek to determine a linear complex having second order contact with the congruence yz along a generator yz, we find that such a complex exists if, and only if, the congruence is a W congruence. In this case its equation is

$$(12.5) \qquad q\,\omega_{13} + c\,\gamma\,\omega_{42} = 0,$$

and it is called <u>the osculating linear complex</u> along the generator yz of the W congruence yz.

There is a unique complex associated with one focal point y of a generator yz of a congruence yz in the following manner. The complex contains yz and all its central rays through y. Moreover, for every ruled surface of the congruence through yz, the complex contains also the generator YZ consecutive to yz and all its central rays through its focal point Y consecutive to y. This complex is called <u>the associated linear complex</u> of the focal point y of the generator yz of the congruence. There is, of course, a complex similarly associated with the other focal point z. In order to find the equation of the associated linear complex of the point y we proceed as follows. A linear complex containing the line yz (1, 0, 0, 0, 0, 0) must have $a_{34}=0$. A central ray through the point y(1, 0, 0, 0) and an arbitrary point (0, h, 0, k) on the line $z\zeta$ has coordinates (h, 0, k, 0, 0, 0), and belongs to the complex in case also $a_{23}=0$. The series (12.2) show that the complex contains an arbitrary generator YZ consecutive to yz in case also $a_{14}=0$. The equation of the complex now has the form (12.4). We proceed to calculate the coordinates of an arbitrary central ray of a neighboring generator YZ through the point Y. The local coordinates of the points Y, Z are respectively given by the series (3.4), (3.10). The local coordinates of the point Z_u on the u-tangent of the surface S_z at the point z are found similarly, by expanding Z_u in powers of Δu, Δv, to be given by certain series which turn out to be precisely the partial derivatives of the series (3.10) with respect to Δu:

$$(12.6)\qquad \begin{cases} x_1 = m' \Delta u + (\gamma_u + \alpha\gamma)\Delta v + \ldots, \\ x_2 = b' + n' \Delta u + (\delta_u + \beta\gamma + b'\delta)\Delta v + \ldots, \\ x_3 = p' \Delta u + \ldots, \\ x_4 = c' + q' \Delta u + c'\delta \Delta v + \ldots . \end{cases}$$

The local coordinates of a point $hZ + kZ_u$ can easily be written, and then the needed line coordinates of the central ray joining the point Y to the point $hZ + kZ_u$ are found to be

$$(*,\ kp'\Delta u + \ldots,\ *,\ *,\ -kc'\beta\Delta u + \ldots,\ kcc'\Delta v + \ldots).$$

These coordinates satisfy equation (12.4) identically in h, k and to terms of the first degree in Δu, Δv in case

$$a_{12} = 0, \qquad a_{42}p' - a_{13}c'\beta = 0.$$

Thus the local equation of <u>the associated complex of the point</u> y of the generator yz is found to be

$$(12.7)\qquad c'\beta\,\omega_{13} + p'\,\omega_{42} = 0.$$

Similarly, the equation of <u>the associated complex of the</u> point z is

(12.8) $$q\,\omega_{13} + c\,\gamma\,\omega_{42} = 0.$$

These complexes are the same and are the osculating linear complex if, and only if, the congruence is a W congruence.

If the congruence is not a W congruence the associated complexes (12.7), (12.8) are distinct and determine a pencil of linear complexes, whose equation can be written in the form

$$h(c'\beta\,\omega_{13} + p'\,\omega_{42}) + k(q\,\omega_{13} + c\,\gamma\,\omega_{42}) = 0.$$

The special complexes of this pencil are given by

$$(hc'\beta + kq)(hp' + kc\gamma) = 0,$$

and consequently their equations are $\omega_{42} = 0$ and $\omega_{13} = 0$ respectively. The first of these, as we have already observed, consists of all lines intersecting the line $y\eta$, and the second bears the same relation to the line $z\zeta$. Therefore <u>the two lines</u> $y\eta$, $z\zeta$ <u>are the directrices of the congruence of intersection of the associated complexes</u> (12.7), (12.8). These two lines are sometimes called <u>the principal focal rays</u> of the generator yz, all of the focal rays being the tangents of the surface S_y at the point y and the tangents of S_z at the point z.

BIBLIOGRAPHY

1. P. O. Bell, A study of curved surfaces by means of certain associated ruled surfaces, Transactions of the American Mathematical Society, vol. 46 (1939), pp. 389-409.

2. E. Bompiani, Invarianti proiettivi di una particolare coppia di elementi superficiali del 2° ordine, Bollettino della Unione Matematica Italiana, vol. 14(1935), pp. 237-243.

3. P. Buzano, Invariante proiettivo di una particolare coppia di elementi di superficie, Bollettino della Unione Matematica Italiana, vol. 14(1935), pp. 93-98.

4. G. Fubini and E. Čech, Geometria Proiettiva Differenziale, Bologna, vol. 1, 1926.

5. C. C. Hsiung, Projective differential geometry of a pair of plane curves, Duke Mathematical Journal, vol. 10(1943), pp. 539-546.

6. _______, A study on the theory of conjugate nets, American Journal of Mathematics, vol. 69(1947), pp. 379-390.

7. E. P. Lane, Projective Differential Geometry of Curves and Surfaces, The University of Chicago Press, 1932.

8. E. P. Lane and M. L. MacQueen, The curves of a conjugate net, Duke Mathematical Journal, vol. 5(1939), pp. 692-704.

9. G. Tzitzéica, Sur certains réseaux conjugés, Comptes Rendus de l'Académie des Sciences, Paris, vol. 152(1911), pp. 1077-1079.

10. Waelsch, Zur Infinitesimalgeometrie der Strahlen-congruenzen und Flächen, Sitzungsberichte der Wiener Akademie der Wissenschaften, C, Abt. IIa (1891).

11. E. J. Wilczynski, Sur la théorie générale des congruences, Mémoires publiés par la classe des sciences de l'Académie Royale de Belgique, ser. 2, vol. 3 (1911).

12. _______, The general theory of congruences, Transactions of the American Mathematical Sciety, vol. 16(1915), pp. 311-327.

13. _______, Geometrical significance of isothermal conjugacy, American Journal of Mathematics, vol. 42(1920), pp. 211-221.