LATTICE METRIZED SPACES

By

LEO LAPIDUS

AN ABSTRACT

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This thesis is a study in what has come to be known as the field of abstract distance spaces, in the spirit of the later work of Karl Menger, where the distances of the space are elements of some abstract algebraic structure. In particular, the distances of this study are elements of a lattice, more especially of a Brouwerian algebra, a generalization of a Boolean algebra.

First, a few lattice theoretic properties of Brouwerian algebras are developed in some detail, with considerable attention given the Brouwerian complement, which generalizes the familiar Boolean complement of set algebra.

Next, a Brouwerian algebra is metrized by symmetric difference, a generalization again of the well known symmetric difference of Boolean algebras. Many properties of the resulting Brouwerian spaces are then derived and numerous theorems are obtained which serve to characterize the Boolean algebras among the Brouwerian algebras. The congruence order of certain Brouwerian spaces relative to the class of lattice-metrized spaces is established.

In the final section, properties of lattice-metrized spaces in general are obtained and in particular many of the earlier results are extended. Finally, the notions of metric and lattice betweenness are analyzed. By studying the effects of their coincidence on the algebraic structure of the underlying lattice, the Boolean algebras are then characterized among the class of all lattices with I, in terms of the betweenness concepts.

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For

Don, David and Jeremy,

and

to my wife Hilda

without whose

self-sacrifice, infinite patience, and sympathetic understanding this entire project could not have been brought to a successful conclusion.

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Introduction

If, with each two elements of an abstract set is associated a number (real or complex), the resulting structure is known as an abstract distance space. (It is convenient and suggestive to refer to the elements of the set as "points" and to the number associated with a pair of points as the "distance" between them). This notion plays an important part in Frechet's 1906 thesis, although the concept was undoubtedly known to earlier workers in geometry.

The first systematic study of the geometric properties of these spaces was due to Karl Menger [9], who referred to these structures as semimetric spaces. In addition, however, to spaces in which distances were selected from among the real and complex numbers, Menger [10] and later Taussky [12] studied spaces whose distances were elements of a group. This has led more recently to the study of spaces whose distances were selected from even more general algebraic structures. In particular, Ellis [5], Blumenthal [3] and Elliott [4] have investigated spaces whose distances are elements of a lattice. This notion may be generalized in the following way:

If with each two elements (x,y) of an abstract set S, is associated an element $\underline{a}$ of a lattice L with least element 0, the association being denoted by $a = d(x,y)$, the resulting structure is called a lattice-metrized, (or more briefly an L-metrized) space, provided that (1) $d(x,y) = 0$ if and only if $x = y$, (2) $d(x,y) = d(y,x)$, and (3) for each three elements x,y,z of S, $d(x,y) + d(y,z) > d(x,z)$, where + is the addition of the lattice and > the order relation, read "over" (in the wide sense). That this association may be reasonably regarded as a "metrization" of S is suggested by the formal resemblance, at least, of the specified conditions to the usual postulates for distance in a metric space. If, in particular, $S \equiv L$, this association defines a binary operation on L, termed a <u>metric operation</u>, and the lattice is said to be autometrized.

The studies of Ellis, Blumenthal and Elliott referred to above were concerned with a particular autometrization of a Boolean algebra. Ellis observed that in such a lattice, the symmetric difference $ab' + a'b$ of two elements a and b where a' is the complement of a, is a metric operation in the sense described above. Since, however, a Boolean algebra may be metrized in other ways, (for example all distances between distinct pairs of points may be set equal to the same element $a \neq 0$), the term "autometrized Boolean algebra" will be used in this thesis, if the metrization is any one which satisfies the postulates (1),(2),(3), above, the designation "Boolean Geometry" being reserved for the special autometrization of symmetric difference.

It is well known that a Boolean algebra is a ring under the operation of symmetric difference as the addition of the ring, and, indeed, Elliott has shown that the only operation possible in a Boolean algebra which is simultaneously a metric operation and a group operation is the symmetric difference [4]. The group property of this operation has interesting consequences for Boolean geometries which will be discussed later.

This thesis is concerned with the extension of certain properties of Boolean geometries to a somewhat wider class of spaces called Brouwerian geometries. A further concern is with properties of L-metrized spaces in general. What is of special interest over and above the geometric properties per se of these spaces is the interplay between these geometric properties of a space and the algebraic structure of its underlying lattice.

Section 1 contains known results used throughout this thesis. In section 2 are developed in detail properties of Brouwerian algebras, many of which are stated without proof in [1], [2] and [8]. Brouwerian geometries are introduced and studied in section 3. In particular, numerous characterization theorems for Boolean algebras are obtained, and the congruence order of certain Brouwerian geometries is established. In section 4, L-metrized spaces in general are studied. Further, the notion of metric betweenness in these spaces is introduced, and consequences for the structure of the underlying lattice, of the coincidence of metric and lattice betweenness are derived.

Section 1. Preliminary Results

The following lattice theory results are used throughout this paper. Details may be found in [1].

A partially ordered set is a collection of elements together with a binary relation defined on the set, which is reflexive, asymmetric, and transitive. Denoting the relation by the symbol $<$, read "under", the three axioms satisfied by a partially ordered set are:

(1) For all a, $a < a$

(2) $a < b$ and $b < a$ imply $a = b$

(3) $a < b$ and $b < c$ imply $a < c$.

$a < b$ may also be written $b > a$ and read b is "over" a. If the order relation does not hold for two elements a and b, they are called non-comparable, otherwise, they are called comparable.

In representing a partially ordered set by a diagram, $a > b$ is indicated thus $\begin{matrix} \circ\ a \\ | \\ \circ\ b \end{matrix}$, whereas if a and b are not comparable, they appear thus, $a\circ \quad \circ b$.

By an upper bound to a subset X of a partially ordered set P is meant an element $a \in P$ such that $a > x$ for every $x \in X$. A least upper bound (or l.u.b.) is an upper bound which is under every other upper bound. Clearly a l.u.b. is unique. Lower bound and greatest lower bound (g.l.b.) are similarly defined. A <u>lattice</u> is a partially ordered set in which every pair of elements has a l.u.b. and a g.l.b. In this thesis, these are denoted respectively by $a + b$ and ab

and are called the sum and product respectively of a and b, (although "join", indicated by the symbol $\cup$, and "meet", indicated by the symbol $\cap$ are also found in the literature.) Each operation is called the "dual" of the other. They are readily shown to be idempotent, commutative and associative, and satisfy the absorption laws $a + ab = a$, and $a(a + b) = a$. Further, $a > b$ if and only if $a + b = a$ and $ab = b$. A lattice is said to be complete if any collection of elements has a g.l.b. and l.u.b. In particular, a complete lattice has a least element 0 and a greatest element I. A chain is a lattice in which each two elements are comparable, and is said to be linearly ordered.

A lattice in which $a < c$ implies $(a + b)c = a + bc$ is called <u>modular</u>, and this "weak" distributive law is called the modular law. A <u>distributive</u> lattice is one which satisfies the <u>distributive laws</u> $a + bc = (a + b)(a + c)$ and $a(b + c) = ab + ac$. Each of these implies the other.

Section 2. Brouwerian algebras

In this section, a Brouwerian algebra is defined and some of its most important properties are derived. Many of these will be used in later sections of this thesis.

Definition 2.1. If with each two elements a and b of a lattice L having a greatest element I, there is associated a least element x such that $b + x > a$, then x is denoted by $a - b$ and the lattice is called a Brouwerian algebra [8]. Thus, if $b + z > a$, then $z > x$. This association will be referred to as the subtraction operation.

It is of interest to note that such an algebra is equivalent to a lattice formulation due to G. Birkhoff [2] (and called by him a "Brouwerian logic") of Heyting's postulates for "intuitionist logic", (a logic consistent with the philosophy of the intuitionist school of mathematicians, whose leading exponent is L. J. Brouwer). It is also the dual of a relatively pseudo-complemented lattice [1].

It is convenient in practice to make use of the following

Theorem 2.1. A lattice L with an I is a Brouwerian algebra, if and only if, for each three elements x,y,z of L, $x - y < z \longleftrightarrow x < y + z$.

To show first that this characterization implies the original definition, since by the first postulate for a poset, $x - y < x - y$, it follows that $x - y < z \longrightarrow x < y + z$ implies $x < y + (x - y)$. Thus $(x - y)$ is an element of L which added to y yields an element over x.

And now $x < y + z \longrightarrow x - y < z$ implies that $x - y$ is the least such element.

To show that the original definition implies the above characterization, suppose $x - y$ is the least element of L such that $y + (x - y) > x$. Then $y + z > x$ implies $z > x - y$. On the other hand, $y + (x - y) > x$ implies $z + y + (x - y) > x$, and if $z > x - y$, then $z + (x - y) = z$, hence $z + y > x$. This completes the proof.

In what follows, if X is a subset of elements x_α of a lattice L, Πx_α designates the lattice product of the elements x_α of X.

Theorem 2.2. If L is a Brouwerian algebra, then

(i) L has a least element 0.

(ii) L is a distributive lattice, i.e. for all elements x, y, z, of L,

$$x + yz = (x + y)(x + z), \text{ and dually}$$

$$x(y + z) = xy + xz$$

(iii) The distributive law for finite addition with respect to infinite multiplication holds, i.e. if X is any subset of elements x_α of L such that Πx_α exists, then for every element a of L, $\Pi(a + x_\alpha)$ also exists and

(1) $a + \Pi x_\alpha = \Pi(a + x_\alpha)$

Proof of (i): This follows immediately from the definition of a Brouwerian algebra, since $I - I$ exists and clearly is the least element, denoted by 0.

Proof of (ii): In any lattice L, for all elements

x, y, z, of L,

$$(x + y) > (x + y)(x + z) \text{ and } (x + z) > (x + y)(x + z).$$

Thus $y, z \in U$, where U is the class of elements u_α such that $x + u_\alpha > (x + y)(x + z)$. By definition, $\Pi u_\alpha = (x + y)(x + z) - x$ exists, and since

$$y > (x + y)(x + z) - x, \text{ and } z > (x + y)(x + z) - x, \text{ then}$$

$$yz > (x + y)(x + z) - x, \text{ hence}$$

$yz + x > [(x + y)(x + z) - x] + x > (x + y)(x + z)$ by definition i.e. $yz + x > (x + y)(x + z)$.

But in any lattice, the one sided distributive law

$$yz + x < (x + y)(x + z) \text{ holds}$$

hence, $yz + x = (x + y)(x + z)$,

and the proof is complete.

<u>Proof of (iii)</u>. Clearly, $\Pi x_\alpha < x_\alpha$ implies $a + \Pi x_\alpha < a + x_\alpha$ for every α. Suppose, then, $p < a + x_\alpha$ for every α. Then $p < (a + \Pi x_\alpha) + x_\alpha$, hence by Theorem 2.1, $p - (a + \Pi x_\alpha) < x_\alpha$, and since this holds for every α, it follows that $p - (a + \Pi x_\alpha) < \Pi x_\alpha$, and Theorem 2.1 again implies $p < a + \Pi x_\alpha$.

Thus, $a + \Pi x_\alpha = \Pi(a + x_\alpha)$ by definition of the lattice product of a set of elements.

<u>Remark</u>. It should be noted that distributivity alone is insufficient to insure that a lattice be a Brouwerian algebra. The open subsets of the plane, for example, constitute a distributive lattice. But if a and b are two open circles with a non-null intersection, $a - b$ fails to exist, for the least set u, such that $b + u > a$ is the set

a - ab of elements of a not already in b. This set is not open, since its complement is not closed. Moreover, any <u>open</u> set containing those points of a not already in b must contain, in particular, a neighborhood of each point of a - ab which is an accumulation element of the complement of a - ab. Since each of these neighborhoods may be arbitrarily small, there is no least open set with the required property. However, one does have

<u>Theorem 2.3</u>. A <u>complete</u> lattice in which the distributive law for finite addition with respect to infinite multiplication holds is a Brouwerian algebra.

<u>Proof</u>. Let L be a complete lattice in which the distributive law for finite addition with respect to infinite multiplication holds, i.e. for every subset X of elements $x_\alpha \in L$, $a + \Pi x_\alpha = \Pi(a + x_\alpha)$. If $x, y \in L$, then since $y + x \geq x$, the class U of elements u_α such that $y + u_\alpha \geq x$ is not empty. Since L is complete, Πu_α and $\Pi(y + u_\alpha)$ exist. Moreover, $\Pi(y + u_\alpha) \geq x$, and since $\Pi(y + u_\alpha) = y + \Pi u_\alpha$ by hypothesis, it follows that $y + \Pi u_\alpha \geq x$. Hence $\Pi u_\alpha = x - y$, by definition of the subtraction operation, and L is a Brouwerian algebra.

<u>Corollary 2.3.1</u>

Every finite distributive lattice is a Brouwerian algebra.

<u>Proof</u>: Such a lattice is a complete lattice in which (1), of Theorem 2.2,(iii) holds.

Remark. A non-complete lattice which enjoys the distributive law in question may or may not be a Brouwerian algebra. The unit interval with an interior point deleted, and the unit square without the point (1,0) are lattices where $x_1, y_1 > x_2, y_2$ if and only if $x_1 \geq x_2$, and $y_1 \geq y_2$. Each enjoys the law in question whenever the products involved exist. Neither is complete. The former is a Brouwerian algebra; the latter is not, since $[I - (x,1)]$ fails to exist for any x, $(0 \leq x < 1)$.

Theorem 2.4. Every chain with I and O is a Brouwerian algebra.

Proof. For any two elements a,b of L with $a > b$, clearly $a - b = a$, and $b - a = 0$. Hence L is a Brouwerian algebra.

Definition 2.2 An element a of a subset X of a partially ordered set P is a minimal element of X, if for no element x of X is $a > x$.

Definition 2.3 A partially ordered set P is said to satisfy the descending chain condition, if and only if every non-void subset X of P contains a minimal element.

Theorem 2.5. Every distributive lattice L with a greatest element I, which satisfies the descending chain condition is a Brouwerian algebra.

Proof. For arbitrary elements a,b of L, since $b + a > a$, the set X of elements x_α such that $b + x_\alpha > a$ is non-void. Then X contains a minimal element x by hypothesis. Let y also be a minimal element of X. Now $xy \in X$, for xy exists and $b + x > a$, $b + y > a$ imply by the distributive law that

$(b + x)(b + y) = b + xy > a$. But then neither x nor y would be minimal in X since $xy < x$, and $xy < y$. Hence there can be at most one minimal element x in X.

Suppose x is not the least element in X. Then there exists an element z in X with $z \not> x$.

By our previous argument, $xz \in X$, i.e. $b + xz > a$. Since $xz < x$, again x is not minimal, a contradiction. Thus x is the least element of X, and $a - b$ exists.

Definition 2.3. A Boolean algebra is a complemented distributive lattice, i.e. corresponding to each element a of the lattice, there exists an element a' called the complement of a such that $a + a' = I$ and $aa' = 0$. It is readily shown that complements are unique and that complementation is orthocomplementation, i.e. $(a')' = a$.[1].

Theorem 2.6. Every Boolean algebra B is a Brouwerian algebra wherein $ab' = a - b$, for $a, b \in B$.

Proof. It will be shown that $b + ab' > a$, and

$$\text{if } b + x > a, \quad \text{then} \quad x > ab'.$$

Since $b + b' = I$ and $Ia = a$, $a > a$ implies $(b + b')a > a$, whence $ab + ab' > a$ by distributivity. Moreover, since $b > ab$ it follows that $b + ab' > ab + ab' > a$, i.e. $b + ab' > a$. Furthermore, if $b + x > a$, then $b'(b + x) > ab'$ and $bb' + b'x > ab'$, i.e. $b'x > ab'$ since $bb' = 0$. But $x > b'x > ab'$ so that $x > ab'$. Thus $ab' = a - b$, and B is a Brouwerian algebra.

Thus it is evident that Brouwerian algebras comprise a rather large class of lattices, including as they do, all chains with I and O, finite distributive lattices, distributive lattices satisfying the descending chain condition, complete lattices satisfying the distributive law for finite addition with respect to infinite multiplication, and the Boolean algebras. Attention is called finally, to the following

Theorem. The algebra of closed sets of a topological space, and every subalgebra of this algebra is a Brouwerian algebra. Conversely, every Brouwerian algebra is isomorphic to a subalgebra of the algebra of closed sets of a topological space. [8].

Theorem 2.7. In a Brouwerian algebra, the following relations hold:

(a) $a - b < a$ (b) $a < b$ if and only if $a - b = 0$

(c) $a - 0 = a$ (d) $a - a = 0$ (e) $a + (b - a) = a + b$

(f) $a < b$ implies $a - c < b - c$ (g) $(a - b) - b = a - b$

(h) $(a + b) - c = (a - c)+(b - c)$ (i) $a - bc = (a - b)+(a - c)$

(j) $a < b$ implies $c - b < c - a$ (k) $(a - b) + ab = a$

(l) $(a - b) + a = a$ (m) $a - b < a + b$

(n) $a - b = b - a$ if and only if $a = b$.

(o) $a - b < a - bc$

(p) $c + (a - b) = c + [(c + a) - (c + b)]$

Proof of Theorem 2.7.

(a) $a - b < a$

Proof. $a < a + b$ implies $a - b < a$ by Theorem 2.1

(b) $a < b$ if and only if $a - b = 0$

Proof. By Theorem 2.1, $a < b + 0$ implies $a - b < 0$ hence $a - b = 0$. Moreover, if $a - b = 0$, then $a - b < 0$ and again by Theorem 2.1 $a < b + 0$, i.e. $a < b$.

(c) $a - 0 = a$

Proof. By (a) $a - 0 < a$, and $a - 0 > a$ by definition

Hence $a - 0 = a$

(d) $a - a = 0$

Proof. This follows from the definition of subtraction.

(e) $a + (b - a) = a + b$

Proof. By (a) $b - a < b$. Hence $a + (b - a) < a + b$

but $a + (b - a) > b$ by definition of $(b - a)$. Hence

$a + (b - a) > b + a$, i.e. $a + (b - a) = a + b$

(f) $a < b$ implies $a - c < b - c$.

Proof. $b < c + (b - c)$ by definition. Moreover, $a < b$ implies $a < c + (b - c)$, hence $a - c < b - c$ by Theorem 2.1

(g) $(a - b) - b = a - b$.

Proof. $(a - b) - b < a - b$ by (a). To show that $(a - b) - b > a - b$, clearly, $(a - b) - b < (a - b) - b$,

$$
\begin{aligned}
\text{therefore } (a - b) &< [(a - b) - b] + b, \\
a &< [(a - b) - b] + b, \\
\text{and } \quad a - b &< [(a - b) - b], \text{ by Theorem 2.1}
\end{aligned}
$$

Hence $(a - b) - b = a - b$

(h) $(a + b) - c = (a - c) + (b - c)$

Proof. $a < c + (a - c)$ and $b < c + (b - c)$ by definition

Hence $a + b < c + (a - c) + (b - c)$, and by Theorem 2.1, it follows that $(a + b) - c < (a - c) + (b - c)$.

To show the reverse inequality,

by (a), $(a+b)>(a-c)+(b-c)=[(a-c)-c]+[(b-c)-c]$ by (g).

Hence $(a+b)-c> \left\{ [(a-c)-c]+[(b-c)-c] \right\} -c$

$> \left\{ [(a-c)-c]-c \right\} + \left\{ [(b-c)-c]-c \right\}$ by (f).

Therefore $(a + b) - c > (a - c) + (b - c)$ by (g).

Hence $(a + b) - c = (a - c) + (b - c)$

(i) $a - bc = (a - b) + (a - c)$

<u>Proof</u>. $a < b + (a - b)$ and $a < c + (a - c)$ by definition,

hence $a < [b + (a-b)][c + (a-c)]$

or $\quad a < bc + (a-b)c + b(a-c) + (a-b)(a-c)$

and $a - bc < (a-b)c + (a-b)(a-c) + b(a-c) < (a-b) + (a-c)$

i.e. $\quad a - bc < (a-b) + (a-c)$

To show that the reverse inequality holds,

since $a < a + b + bc$, then $a - b < a + bc$ by Theorem 2.1

But by (e), $a + bc = bc + (a-bc)$, hence

$$a - b < bc + (a-bc) \text{ and}$$

$$a < b + bc + (a-bc) \text{ by Theorem 2.1}$$

i.e. $\quad a < b + (a-bc)$, so again by Theorem 2.1

$$a - b < a - bc$$

In like manner we can show that

$$a - c < a - bc$$

Thus $(a-b) + (a-c) < a - bc$, and therefore

$$a - bc = (a-b) + (a-c).$$

(j) $a < b$ implies $c - b < c - a$

<u>Proof</u>. If $a < b$, then $ab = a$, hence

$$c - ab = c - a\text{, and by (i)}$$

$$c - a = (c-a) + (c-b) \text{ i.e.}$$

$$c - b < c - a$$

(k) $(a-b) + ab = a$

<u>Proof</u>. $a - b < a$ by (a), and $ab < a$, hence

$$(a-b) + ab < a$$

To show the reverse inequality, $(a-b) + b > a$ by Theorem 2.1 Hence $a(a-b) + ab > a$, and since $a - b < a$, $a(a-b) = a - b$, therefore $(a-b) + ab > a$, consequently

$$(a-b) + ab = a.$$

(l) $(a-b) + a = a$

<u>Proof</u>. Since $a - b < a$, it follows that $(a - b) + a = a$.

(m) $a - b < a + b$

<u>Proof</u>. $a - b < a < a + b$.

(n) $a - b = b - a$ if and only if $a = b$

<u>Proof</u>. The sufficiency is obvious. To show the necessity, $a - b < a$, $b - a < b$ by (a). But since $a - b = b - a$, $b - a < a$ and $a - b < b$. Hence by Theorem 2.1 $b < a$ and $a < b$, i.e. $a = b$.

(o) $a - b < a - bc$

<u>Proof</u>. Since $bc < b$, this follows from (j).

(p) $c + (a-b) = c + [(c+a) - (c+b)]$

<u>Proof</u>. Since $(c + a) - (c + b) < (c + a) - (c + b)$, it follows that $(c + a) < (c + b) + [(c + a) - (c + b)]$ and $(c + a) - b < c + [(c + a) - (c + b)]$ by Theorem 2.1.

Now $(a - b) < a < c + a$, hence $(a - b) - b < (c + a) - b$ by (a) and (f), and therefore, $a - b < (c + a) - b$ by (g). Thus $c + (a - b) < c + [(c + a) - (c + b)]$.

The reverse inequality is established thus: since $c + b + a > c$, it follows that $c + b > c - a$ by Theorem 2.1, hence $b + (c - b) > c - a$ by (e) and $b + (c - b) > (a + c) - a$ by Corollary 2.7 (h).1. so that

$$(b + a) + (c - b) > (a + c) \text{ by Theorem 2.1.}$$

Then $(a - b) + b + (c - b) > (a + c)$

and $(a - b) + (b + c) > a + c$ by (e) so that

$$(a - b) > (a + c) - (b + c) \text{ by Theorem 2.1 again,}$$

and $c + (a - b) > c + [(a + c) - (b + c)]$

Thus, finally, $c + (a - b) = c + [(a + c) - (b + c)]$

and the proof is complete.

Corollaries to Theorem 2.7

Corollary 2.7 (h).1. $(a + b) - b = a - b$.

Proof. Set $c = b$ in (h)

Corollary 2.7 (i).1. $a - ab = a - b$.

Proof. Set $c = a$ in (i)

Definition 2.5 The element $I - x$ of a Brouwerian algebra is called the Brouwerian complement of x, and is denoted by $\urcorner x$. Similarly, $\urcorner\urcorner x = I - \urcorner x$.

Remark. It is clear from Theorem 2.6 that the Brouwerian complement $\urcorner x$ coincides with the usual complement x' in the Boolean case. It should further be noted that in the Brouwerian algebra of the closed subsets of a topological space K, the Brouwerian complement of a (closed) subset A is merely the closure of the usual (Boolean) complement of the set A in the space of all subsets of K.

Theorem 2.8. In a Brouwerian algebra, the following relations hold:

(a) $a < b$ implies $\neg b < \neg a$ (b) $a + \neg a = I$

(c) $\neg 0 = I$, $\neg I = 0$ (d) $\neg\neg a < a$

(e) $\neg\neg\neg a = \neg a$ (f) $\neg(ab) = \neg a + \neg b$

(g) $\neg(a + b) < \neg a \cdot \neg b$ (h) $\neg(a \neg a) = I$

(i) $a = a \neg a + \neg\neg a$

(j) $\neg a = \neg b = I$ implies $\neg(a + b) = I$

(k) $\neg(a + b) = \neg\neg(\neg a \cdot \neg b)$

(l) $\neg(a - b) = \neg a + \neg\neg b$

(m) $(a + b) = I$ if and only if $a > \neg b$ and $b > \neg a$

(n) (i) $\neg a = \neg a - a$ (iii) $\neg\neg\neg a = \neg\neg\neg a - \neg\neg a$

(ii) $\neg\neg a = \neg\neg a - \neg a$ (iv) $\neg a = \neg a - \neg\neg a$

(o) $\neg x = 0$ implies $x = I$.

Proof of Theorem 2.8.

(a) $a < b$ implies $\neg b < \neg a$

Proof. This follows from the definition of $\neg x$ and Theorem 2.7 (j).

(b) $a + \neg a = I$

Proof. $a + \neg a > I$ by definition of $\neg a$. But $x > I$ implies $x = I$. Hence $a + \neg a = I$.

(c) $\neg 0 = I$, $\neg I = 0$

Proof. These follow directly from the definition.

(d) $\neg\neg a < a$.

Proof. This follows from (b) and Theorem 2.1

(e) $\neg\neg\neg a = \neg a$

Proof. By (d) and (a), $\neg\neg\neg a > \neg a$.

But again by (d), $\neg\neg(\neg a) < \neg a$. Hence $\neg\neg\neg a = \neg a$

(f) $\neg(ab) = \neg a + \neg b$

<u>Proof</u>. This follows directly from Theorem 2.7 (i)

(g) $\neg(a + b) < \neg a \cdot \neg b$.

<u>Proof</u>. By (d), $\neg\neg a < a, \neg\neg b < b$, hence $\neg\neg a + \neg\neg b < a + b$. By (a) $\neg(a + b) < \neg(\neg\neg a + \neg\neg b)$, hence $\neg(a + b) < \neg(\neg(\neg a \cdot \neg b)) < \neg a \cdot \neg b$ by (f) and (d).

(h) $\neg(a\neg a) = I$

<u>Proof</u>. By Theorem 2.7 (i),
$\neg(a\neg a) = I - a\neg a = (I - a) + (I - \neg a) = \neg a + \neg\neg a = I$ by (b).

(i) $a = a\neg a + \neg\neg a$

<u>Proof</u>. $a(\neg a + \neg\neg a) = a I = a$, and $a(\neg a + \neg\neg a) = a\neg a + \neg\neg a$ since $\neg\neg a < a$.

(j) $\neg x = \neg y = I$ imply $\neg(x + y) = I$

<u>Proof</u>. Let $x + y + t = I$ so that $y + t > \neg x$, i.e. $y + t = I$. Then, $t > \neg y$, i.e. $t = I$.

Thus the <u>only</u> element which, added to $(x + y)$ yields an element over I, is I itself. Hence I is the <u>least</u> element with this property, i.e. $\neg(x + y) = I$.

(k) $\neg(a + b) = \neg\neg(\neg a \cdot \neg b)$ [Dual of Theorem 12.21 (vi) p.42 [13].

<u>Proof</u>. $\neg(a\neg a) = \neg(b\neg b) = I$, by (h).

Hence by (j) with $x = a\neg a$, $y = b\neg b$, it follows that
$\neg(a\neg a + b\neg b) = \neg[(a+b)(a+\neg b)(\neg a+b)(\neg a+\neg b)] = I$.

Now $(a+b)\neg a\neg b < (a+b)(a+\neg b)(\neg a+b)(\neg a+\neg b)$ implies (by(a))
$\neg[(a+b)\neg a\neg b] > \neg[(a+b)(a+\neg b)(\neg a+b)(\neg a+\neg b)] = I$.

By (f) then, $\neg(a+b) + \neg(\neg a\neg b) = I$, hence $I - \neg(\neg a\neg b) < \neg(a+b)$

i.e. $\neg\neg(\neg a \neg b) < \neg(a+b)$. Further, $\neg\neg a < a$, $\neg\neg b < b$ imply $\neg\neg a + \neg\neg b < a+b$ so that by (a)

$\neg(a+b) < \neg(\neg\neg a + \neg\neg b) = \neg\neg(\neg a \cdot \neg b)$ (by f)

or $\neg(a+b) = \neg\neg(\neg a \cdot \neg b)$, which completes the proof.

(l) $\neg(a-b) = \neg(a\neg b) = \neg a + \neg\neg b$.

<u>Proof</u>. $\neg(a-b) = I - (a-b)$ is the least element which added to $(a-b)$ yields I. What must be shown is that (1) $(a-b) + \neg a + \neg\neg b = I$ and (2) if $(a-b) + x = I$, then $x > \neg a + \neg\neg b$.

To establish (1), let $a-b = y$, then $a < b + y$, $\neg a > \neg(b+y) = \neg\neg(\neg b \cdot \neg y)$, $\neg\neg a < \neg(\neg b \cdot \neg y) = \neg\neg b + \neg\neg y$, and $\neg\neg a - \neg\neg b < \neg\neg y$, i.e. $\neg\neg a - \neg\neg b < \neg\neg(a-b) < (a-b)$ by (a), (e), (k), (f), and (d).

Therefore $\neg\neg a < (a-b) + \neg\neg b$, i.e. $I - \neg a < (a-b) + \neg\neg b$, hence $I < (a-b) + \neg a + \neg\neg b$, i.e. $I = (a-b) + \neg a + \neg\neg b$.

To establish (2), since $(a-b) + \neg(a-b) = I$, it follows from the definition of $\neg(a-b)$ that $x > \neg(a-b)$. Now $a-b) < a$ and $a-b < \neg b$ imply $a-b < a\neg b$, hence $\neg(a\neg b) < \neg(a-b)$. Thus $x > \neg(a-b) > \neg(a\neg b) = \neg a + \neg\neg b$ by (f), which completes the proof.

(m) $a+b = I$ if and only if $a > \neg b$ and $b > \neg a$.

<u>Proof</u>. $a + b > I$ implies $a > \neg b$ and $b > \neg a$ by Theorem 2.1. Conversely, $a > I - b$ and $b > I - a$ imply $a + b > I$, i.e. $a + b = I$.

(n)(i) $\neg a = \neg a - a$

(ii) $\neg\neg a = \neg\neg a - \neg a$

(iii) $\neg\neg\neg a = \neg\neg\neg a - \neg\neg a$

(iv) $\neg a = \neg a - \neg\neg a$

Proof. (i), (ii) and (iii) follow from Theorem 2.7 (g), while (iv) follows from (n)(iii) and (e) of this theorem.

(o) $\neg x = 0$ implies $x = I$

Proof. $I - x = 0$ implies $I - x < 0$, hence $I < x$ by Theorem 2.1, hence $x = I$.

Corollaries to Theorem 2.8

Corollary 2.8 (k).1. $\neg\neg(a+b) = \neg\neg a + \neg\neg b$

Proof. $\neg\neg(a+b) = \neg\neg\neg(\neg a \cdot \neg b) = \neg(\neg a \cdot \neg b) = \neg\neg a + \neg\neg b$ by (e) and (f).

Corollary 2.8 (k).2. $\neg(x + y) = I$ implies $\neg x = \neg y = I$ (Converse of (j))

Proof. $I = \neg(x + y) = \neg\neg(\neg x \cdot \neg y)$, and $0 = \neg I = \neg\neg\neg(\neg x \cdot \neg y) = \neg(\neg x \cdot \neg y) = \neg\neg x + \neg\neg y$ by (e) and (f). But $\neg\neg x + \neg\neg y = 0$ implies $\neg\neg x = \neg\neg y = 0$, hence $\neg\neg\neg x = \neg\neg\neg y = \neg 0 = I$, and $\neg x = \neg y = I$ by (e).

Corollary 2.8 (k).3. $\neg\neg x = \neg\neg y = 0$ implies (1) $\neg(x + y) = I$ and (2) $\neg\neg(xy) = 0$.

Proof of (1). By Corollary 2.8 (k).1, $\neg\neg(x + y) = 0$, hence $\neg(x + y) = I$ by (o).

Proof of (2). $\neg\neg(xy) = \neg(\neg x + \neg y) = \neg\neg(\neg\neg x \cdot \neg\neg y) = \neg\neg 0 = 0$, by (f) and (d).

Remark. It should be noted that (2) holds independent of property (k), since $xy < x$ implies $\neg\neg(xy) < \neg\neg x = 0$

Corollary 2.8 (k).4. $\neg\neg(\neg a \cdot \neg\neg a) = 0$

Proof. Follows from (k), replacing b by $\neg a$.

Corollary 2.8.(1).1. $\neg(a - b) < \neg(\neg\neg a - \neg\neg b)$

Proof. Follows from complementation on $\neg\neg a - \neg\neg b < \neg\neg(a-b)$, and (e).

Corollary 2.8.(1).2. $\neg(a - \neg a) = \neg a$

Proof. Obtained by setting $b = \neg a$ in (1).

Corollary 2.8.(1).3. $\neg(\neg a - \neg\neg a) = \neg\neg a$

Proof. Obtained by setting $a = \neg a$ in Corollary 2.8.(1).2.

Corollary 2.8.(1).4. $\neg[(a - b) + (b - a)] = \neg(a + b) + \neg\neg(ab)$

Proof: $(a+b) - ab = [(a+b) - a] + [(a+b) - b] = (a-b) + (b-a)$ by Theorem 2.7 (i), (h), and (d).

Hence $\neg[(a-b) + (b-a)] = \neg[(a+b) - ab] = \neg(a+b) + \neg\neg(ab)$

Remark. Due to the prominence of the Boolean algebra of sets, for example, in many areas of mathematics, it is of some interest to see just how the Boolean algebras differ from the more general Brouwerian algebras. The following properties, which have been discussed in this section, show how these two classes of algebras compare

Boolean algebras	Brouwerian algebras
1.a) $x + x' = I$	1.b) $x + \neg x = I$
2.a) $x \cdot x' = 0$	2.b) $x\neg x \neq 0$ in general
3.a) $(x')' = x$	3.b) $\neg\neg x < x$
4.a) $(xy)' = x' + y'$	4.b) $\neg(xy) = \neg x + \neg y$
5.a) $(x + y)' = x' \cdot y'$	5.b) $\neg(x + y) < \neg x \cdot \neg y$

Clearly 1.) and 4.) hold equally in both cases. In any chain 5a holds, and a chain of more than two elements is

A Brouwerian algebra which is not Boolean, so that 5a) does not have sufficient strength to make a Brouwerian algebra Boolean. However, either of 2a) or 3a) does. Thus one has

Theorem 2.9. A Brouwerian algebra L is a Boolean algebra if and only if, for all elements x of L, $x \neg x = 0$

Proof. The necessity is part of the definition of a Boolean algebra. If, on the other hand, $x \neg x = 0$ for all x, since also $x + \neg x = I$, L is a complemented, distributive lattice, hence a Boolean algebra.

Theorem 2.10. A Brouwerian algebra L is a Boolean algebra if and only if, for all elements x of L, $\neg\neg x = x$.

Proof. The necessity is well known, following from the uniqueness of complements in the Boolean case [1]. The sufficiency follows readily from Theorem 2.8.(h), for, $\neg(x \neg x) = I$ implies $\neg\neg(x \neg x) = \neg I = 0$. But as follows from the hypothesis, $\neg\neg(x \neg x) = x \neg x$, hence $x \neg x = 0$ and L is a Boolean algebra by Theorem 2.9. A further criterion is given by

Theorem 2.11. A Brouwerian algebra L is a Boolean algebra, if and only if, each element z of L is the complement of some element x of L.

Proof. The necessity follows from ortho-complementation, since $z = \neg\neg z$. For the sufficiency, $z = \neg x$ implies $\neg\neg z = \neg\neg\neg x = \neg x = z$, i.e. $\neg\neg z = z$ for every element z of L, and L is a Boolean algebra by Theorem 2.10.

It may be observed finally, that a fundamental difference between Boolean algebras and Brouwerian algebras is that Boolean algebras are dual with respect to the sum and product operations while Brouwerian algebras may fail to be. [8].

Section 3. Brouwerian Geometries

In this section Brouwerian geometries are introduced and many of their properties are derived. These lead to numerous characterizations of Boolean algebras.

As indicated in the introduction, a lattice may be (auto)metrized in a variety of ways. In the case of Brouwerian algebras, the particular metrization which will be used here is that of symmetric difference, namely, $a * b = (a-b) + (b-a)$.

Theorem 3.1. The symmetric difference in a Brouwerian algebra is a metric operation.

Proof. Clearly (1) $a * b = (a-b) + (b-a) = b * a$
(2) $a * a = 0$; moreover, if $(a-b) + (b-a) = 0$, then by definition of the lattice sum, $0 > b - a$ and $0 > a - b$. Hence $a > b$, $b > a$ and $a = b$.
(3) The triangle inequality, $a * b < (b * c) + (a * c)$, is established as follows:

$$abc + (b * c)+(a * c) = abc + [(b-c)+(c-b)] + [(a-c)+(c-a)]$$
$$= abc + [(a-c)+(b-c)]+[(c-a)+(c-b)]$$
$$= abc + [(a+b)-c]+[c-ab] \text{ by Theorem 2.7 (h) and (i)}$$
$$= (ab)c + [c-ab] + [(a+b)-c]$$
$$= c + [(a+b)-c] \text{ by Theorem 2.7 (k)}$$
$$= c + (a+b) \text{ by Theorem (2.7)(e)}.$$

Hence

$$a+b+c < abc + (b * c) + (a * c) \text{ and}$$
$$(a+b+c) - abc < (b * c) + (a * c) \text{ by Theorem 2.1.}$$

Now $(a+b+c) - abc > a - abc > a - b$
and $(a+b+c) - abc > b - abc > b - a$ by Theorem 2.7 (f) and (j),
so that $(a-b) + (b-a) < (a + b + c) - abc$.
Therefore $(a * b) < (b * c) + (a * c)$

Theorem 3.2. In a Brouwerian algebra, $(a+b) - ab$ and $(a-ab) + (b-ab)$ are each equivalent to symmetric difference, $(a-b) + (b-a)$.

Proof. Immediate from Theorem 2.7 (h), (i), and (d).

Definition 3.1. A Brouwerian algebra (auto)metrized by the symmetric difference is called a Brouwerian geometry.

It is often convenient to employ geometrical language and regard a triple of elements a,b,c as the vertices of a triangle with sides $a * b$, $a * c$, and $b * c$. Theorem 3.1 asserts that the sides of any triangle in a Brouwerian geometry satisfy the triangle inequality. The notation $\Delta(a,b,c)$ will be used to designate the triangle with vertices a,b,c.

One reason why a Boolean geometry has so many novel properties is that the symmetric difference in that instance is a group operation. This is not true for Brouwerian geometries, since Brouwerian symmetric difference is not in general associative, (although it does have the remaining group properties). In a Brouwerian chain, for example, (a chain (auto)metrized by symmetric difference), for $a > b$, $(a * a) * b = b$, whereas $a * (a * b) = 0$. Thus, because of the group property, a Boolean geometry can have no isosceles triangles, since the equation $a * x = b$ has one and only one solution. Brouwerian geometries, on the other hand, may abound in isosceles triangles, as happens, for example, in a Brouwerian chain where one has

Theorem 3.3. Every triangle of a Brouwerian chain C is isosceles.

Proof. For any two elements $a > b$ of C, $a * b = a - b = a$, since $b - a = 0$ by Theorem 2.7 (b). Hence for any three elements $a > b > c$ of C, $a * b = a * c = a$. However,

Theorem 3.4. A Brouwerian geometry is a Boolean geometry, if and only if it is free of isosceles triangles.

Proof. The necessity, as already indicated, is clear. To establish the sufficiency, let x be an arbitrary element of the Brouwerian geometry L, and consider the $\Delta(0, I, x\urcorner x)$. Now $I * x = (I-x)+(x-I) = \urcorner x$ for all x, hence $I * 0 = \urcorner 0 = I$, while $I * x\urcorner x = \urcorner(x\urcorner x) = I$ by Theorem 2.8 (c) and (h). Since there are no isosceles triangles, it follows that $0 * x\urcorner x = x\urcorner x = 0$ and L is a Boolean algebra by Theorem 2.9. Then $a - b = ab'$, $b - a = a'b$ and $a * b = ab' + a'b$, so that L is a Boolean geometry.

Corollary 3.4.1. A Brouwerian algebra is a Boolean algebra if and only if symmetric difference is a group operation.

Proof. It has already been indicated, as is well known, that symmetric difference in a Boolean algebra is a group operation. If it is a group operation in a Brouwerian algebra L, then the associated geometry contains no isosceles triangle, and L is a Boolean algebra.

Remark. The above corollary holds if the word "group" is replaced by the word "associative".

Definition 3.2. Three elements a,b,c of a lattice, L, are said to satisfy the triangle inequality if each is under the sum of the other two, written (a,b,c)T.

Remark. Although three elements a,b,c of an (auto)metrized lattice may be in this relation, there may not be any triangle $\Delta(x,y,z)$ in L which has sides a,b,c.(When such a triangle exists, it will be designated as T(a,b,c) rather than $\Delta(x,y,z)$ if the sides, rather than the vertices are to be emphasized).

Theorem 3.5. In a Brouwerian geometry, the relation (a,b,c)T is equivalent to each of the relations

(1) $a + b = a + c = b + c = a + b + c$

(2) $a - b < c$, $b - c < a$, $c - a < b$

(3) $a * b < c < a + b$

(4) $b - a = c - a$, $a - b = c - b$, $a - c = b - c$.

Proof.

(1) $a + b = a + c = b + c = a + b + c$

If (a,b,c)T, then $a < b + c$ implies $a + b + c < b + c$, but since $a + b + c > b + c$, $a + b + c = b + c$. Similarly for the remaining equalities in (1). In words, the sum of two sides of a triangle equals the sum of any other two sides.

(2) $a - b < c$, $b - c < a$, $c - a < b$.

These follow from (a,b,c)T and Theorem 2.1. In words, the difference of two sides of a triangle is under the third side.

(3) $a * b < c < a + b$.

This follows from (2) by adding inequalities.

(4) $b - a = c - a$, $a - b = c - b$, $a - c = b - c$.

By Corollary 2.7 (h), 1, for any two elements a and b, $(a + b) - a = b - a$, and $(a + c) - a = c - a$ likewise.

By (1) of this theorem $a + b = a + c$, hence $b - a = c - a$.

Similarly for the remaining equalities.

To prove the converse,

(1) $a + b + c = a + b$ implies $(a + b) > c$

(2) $a - b < c$ implies $a < b + c$ by Theorem 2.1

(3) $(a - b) < (a - b) + (b - a) < c$ implies $a < b + c$ by Theorem 2.1 again. Moreover $(b - a) < (a - b) + (b - a) < c$ hence $b < a + c$.

(4) $a - c < b - c$ implies $a < (b - c) + c$ by Theorem 2.1.

But $b - c < b$ by Theorem 2.7 (a). Hence,

$$a < (b - c) + c < b + c, \text{ i.e. } a < b + c.$$

The remaining relationships are similarly obtained.

<u>Corollary 3.5.1</u>. If a and b are two elements of a Brouwerian geometry, then $(a,b,a * b)T$ and $ab + (a * b) = a + b$.

<u>Proof</u>. In $\Delta(0,a,b)$, $a * 0 = a$, $b * 0 = b$, hence $(a,b,a * b)T$.

Further, by Theorem 3.5(1), $a + (a * b) = a + b$

Interchanging a and b and multiplying the two equalities gives, using the distributive law, $[a + (a * b)][b + (a * b)]$ $= (a + b)(a + b)$, or $ab + (a * b) = a + b$.

<u>Theorem 3.6.</u> In a Brouwerian geometry the base of an isosceles triangle is "under" the vertex.

Proof.

In the isosceles $\Delta(a,b,c)$, if $b * c = x$, $a * b = a * c = y$, then by Corollary 3.5.1,

$(a,b,y)T$, $(a,c,y)T$ and $(b,c,x)T$,

Hence, by Theorem 3.5(4),

$$b-c = x-c,\ c-b = x-b,\ y-c = a-c,\ y-b = a-b$$

Now in $T(x,y,y)$ clearly $x < y$, hence

$$x-c < y-c \quad \text{and} \quad x-b < y-b \quad \text{by Theorem 2.7(f).}$$

Hence $x = (b-c) + (c-b) = (x-c) + (x-b) < (y-c) + (y-b)$, therefore, $x < (a-b) + (a-c) < a$, the latter by Theorem 2.7(a), which completes the proof.

It has been observed that whereas a Boolean geometry has no isosceles triangles, these may abound in a Brouwerian geometry. However,

Theorem 3.7. A Brouwerian geometry contains no equilateral triangle.

Proof. In $\Delta(a,b,c)$, if $a * b = b * c = a * c = x$, then by Corollary 3.5.1, $(a,x,b)T$, hence $a + x = b + x = a + b$. But by Theorem 3.6, $x < a$, $x < b$, hence $a + x = a$, $b + x = b$, so that $a = b$, which contradicts our assumption that a,b,c are pairwise distinct vertices of a triangle. More generally, a Brouwerian geometry contains no equilateral "n-gon" for n odd. [7]

Theorem 3.8. A Brouwerian geometry is a chain if and only if every triangle is isosceles.

Proof: The necessity was proved in Theorem 3.3.

The sufficiency is shown as follows:

If every triangle of a Brouwerian geometry L is isosceles, consider $\Delta(0,a,b)$ where a and b are arbitrary distinct elements of L. Then $a * b = a$ or $a * b = b$, since $a * 0 = a$ and $b * 0 = b$. If $a * b = a$, then $b < a$ and L is a chain.

It has been indicated earlier that Brouwerian symmetric difference lacks the associative property which makes the Boolean symmetric difference a group operation and leads in that case to the "uniqueness of solution" property. It is instructive to see just how much of the associativity of the Boolean symmetric difference remains in the symmetric difference operation in the Brouwerian case.

Ellis [5] has shown that in a Boolean geometry, for any two pairwise distinct elements a and b, if $a * b = c$, then $a * c = b$ and $b * c = a$. In words, if a side of a triangle equals the vertex opposite it, the same is true of the other two sides. This is conveniently described by

<u>Definition 3.3</u>. If each side of a triangle equals the opposite vertex, the triangle is said to have the property of (triangular) fixity.

It is readily extended by means of

<u>Definition 3.4</u>. If x,y,z are the <u>sides</u> of a triangle in an (auto)metrized space L, the triangle with <u>vertices</u> x,y,z is called the <u>first distance triangle</u> of the original one.

Similarly, the triangle whose vertices are the sides of the first distance triangle is the <u>second distance triangle</u>.

Theorem 3.9. In a Boolean geometry, every first distance triangle has fixity.

Proof: In $\Delta(a,b,c)$ let $a * b = x$, $b * c = y$, $a * c = z$. Then in the first distance triangle

$$\Delta(x,y,z),\ x * y = (a * b)*(b * c)$$
$$= (a * c)*(b * b) \text{ associativity)}$$

i.e. $x * y = a * c = z$ since $b * b = 0$.

Similarly for the remaining distances.

Corollary 3.9.1. (Ellis) In a Boolean geometry, $a * b = c$ implies $a * c = b$, $b * c = a$.

Remark. It is clear from the proof of Theorem 3.9 that the associativity alone of the metric operation implies the result. However, triangular fixity and the associativity of the metric are not quite equivalent. Nevertheless, the following corollary and Theorem 3.10 show how close this comes to being the case.

Corollary 3.9.2. If the metric operation of an autometrized lattice is associative, every first distance triangle with pairwise distinct vertices has fixity.

Theorem 3.10. If every first distance triangle of an autometrized lattice L has fixity the metric operation on pairwise distinct elements is associative.

Proof. Let x,y,z be pairwise distinct elements of an autometrized lattice L, but otherwise arbitrary. Then, $\Delta(x,y,z)$ is the first distance triangle of $T(x, y, z)$, and since fixity implies $x * y = z$, $y * z = x$, and $x * z = y$, it follows that $(x * y) * z = z * z = 0 = x * x = x * (y * z)$.

In Brouwerian geometries, although fixity no longer holds in general, what does hold is

<u>Theorem 3.11</u>. In a Brouwerian geometry, each side of a first distance triangle is <u>under</u> the opposite vertex.

<u>Proof</u>: In $\Delta(a,b,c)$ let $b * c = x$, $a * c = y$, $a * b = z$, then $(x,y,z)T$ and $x * y < z$ by Theorem 3.5(3). Since $\Delta(x,y,z)$ is the first distance triangle of $\Delta(a,b,c)$, the theorem is proved. Furthermore,

<u>Theorem 3.12</u>. In a Brouwerian geometry, every <u>second</u> distance triangle has fixity.

<u>Proof</u>. Define x,y,z as in Theorem 3.11. Then $\Delta(x,y,z)$ with $u = y * z$, $v = x * z$, and $w = x * y$ is the first distance triangle of $\Delta(a,b,c)$, while $\Delta(u,v,w)$ with $v * w = p$, $u * w = q$, and $u * v = r$ is the second distance triangle. It is sufficient to prove that $p = u$. By Theorem 3.11, $p < u$. It remains to show $u < p$. Since $(x,y,z)T$, $(x,w,y)T$, and $(x,v,z)T$, then (4) of Theorem 3.5, implies
$y - z = x - z = v - z$, and $z - y = x - y = w - y$. The inequality $w < z$ by Theorem 3.11 implies by (j) of Theorem 2.7 that $v - z < v - w$. Likewise, $v < y$ implies $w - y < w - v$. Moreover, since $y - z = v - z$, it follows that $y - z < v - w$, and $z - y = w - y$ implies $z - y < w - v$.
Hence $u = (y - z)+(z - y) < (v - w)+(w - v)$ i.e. $u < p$.

<u>Corollary 3.12.1.</u> In a Brouwerian geometry, every nth distance triangle has fixity for $n > 1$.

<u>Theorem 3.13</u>. A Brouwerian geometry is a Boolean geometry if and only if every first distance triangle has fixity.

<u>Proof</u>: The necessity was proved in Theorem 3.9. Conversely, A Brouwerian geometry which is not a Boolean geometry would contain an isosceles triangle, say $T(x,x,y)$. In the first distance triangle, $\Delta(x,x,y)$, fixity would imply $y = x * x = 0$, a contradiction. Hence the theorem.

<u>Theorem 3.14</u>. A Brouwerian algebra is a Boolean algebra if and only if it admits a metric group operation. Furthermore, the operation must be symmetric difference.

<u>Proof</u>. If a Brouwerian algebra is a Boolean algebra it admits a metric group operation, namely symmetric difference. As Elliott has shown [4], this is the only metric group operation possible in a Boolean algebra. Conversely, if $\emptyset$ is a metric group operation in a Brouwerian algebra L, since $0 \emptyset 0 = 0$, the zero element of L is the group identity and $a \emptyset 0 = a$. The $\Delta(0,a,b)$ implies $a - b < a \emptyset b$ and $b - a < a \emptyset b$ by Theorem 2.1 so that $a * b < a \emptyset b$. Hence, in particular, $I \emptyset a\urcorner a > I * a\urcorner a = I$, i.e. $I \emptyset a\urcorner a = I$. But since $I \emptyset 0 = I$, it follows that $a\urcorner a = 0$ since $I \emptyset x = I$ must have a unique solution, $\emptyset$ being a group operation. Therefore by Theorem 2.9, L is a Boolean algebra, and $\emptyset$ is consequently the symmetric difference.

<u>Remark</u>. Since any finite distributive lattice for example is a Brouwerian algebra, the above result implies that any <u>finite</u> distributive lattice which admits a metric group operation is a Boolean algebra. It is natural to inquire as to whether

this is true for arbitrary distributive lattices. The theorems which follow have a direct bearing on this question.

Definition 3.5. If, to each member α of an index set A, there corresponds a lattice L_α, by the direct product, $\{L_\alpha\}$, of the lattices, L_α, is meant the set of all functions x on A such that $x(\alpha) \in L_\alpha$ for each α of A. Thus, an element $\{x_\alpha\}$ of $\{L_\alpha\}$ is a collection of elements $\{x_\alpha\}$ selected one from each lattice L_α. Furthermore, $\{x_\alpha\} > \{y_\alpha\}$ means x_α, $y_\alpha \in L_\alpha$ and $x_\alpha > y_\alpha$ for every α of A. The lattice L_α is the α th coordinate lattice, and the element x_α is the α th coordinate of the element $\{x_\alpha\}$ of the direct product $\{L_\alpha\}$.

Theorem 3.15. The direct product $\{L_\alpha\}$ of a collection of lattices L_α is a lattice, with addition and multiplication in $\{L_\alpha\}$ componentwise.

Proof. If x_α, $y_\alpha \in L_\alpha$, then $x_\alpha + y_\alpha > x_\alpha$, and $x_\alpha + y_\alpha > y_\alpha$ for every α of A. Hence $\{x_\alpha + y_\alpha\} > \{x_\alpha\}$, and $\{x_\alpha + y_\alpha\} > \{y_\alpha\}$ by definition. Furthermore, if $\{z_\alpha\} > \{x_\alpha\}$ and $\{z_\alpha\} > \{y_\alpha\}$, then, in L_α, $z_\alpha > x_\alpha$, $z_\alpha > y_\alpha$ and $z_\alpha > x_\alpha + y_\alpha$. Hence $\{z_\alpha\} > \{x_\alpha + y_\alpha\}$ for every α of A, and $\{x_\alpha + y_\alpha\} = \{x_\alpha\} + \{y_\alpha\}$. Dually, $\{x_\alpha\}\{y_\alpha\} = \{x_\alpha y_\alpha\}$, and $\{L_\alpha\}$ is a lattice.

Definition 3.6. Let S be a sublattice of a direct product $\{L_\alpha\}$ of lattices L_α. Then S is a sublattice of the direct product $\{S_\alpha\}$ of the sublattices S_α of elements of L_α appearing as α-components (or α-coordinates) of elements of S. S is called a subdirect product of the

S_α and denoted by $\{S_\alpha\}_S$. It should be noted that <u>every</u> element of S_α is an α - component of <u>some</u> element of S.

To clarify the concepts involved in the above definition, let $\{L_\alpha\}$ be the direct product $\{XY\}$ of the chains X and Y consisting of the points 1,2, and 3 of the x-axis and the points 1 and 2 of the y-axis respectively. $\{XY\}$ consists, then, of the six points (1,1), (1,2), (2,1), (2,2), (3,1), (3,2) of the xy-plane.

The subset S of points (1,1) and (2,2) constitutes a sub-lattice of $\{XY\}$ and at the same time a sublattice of the direct product of the sublattices X_s of X (consisting of the points 1 and 2 of the x-axis) and Y_s of Y (consisting of the points 1 and 2 of the y-axis, i.e. $Y_s \equiv Y$ in this case.) Then S is a subdirect product of the sublattices X_s and Y_s, but <u>not</u> a subdirect product of X and Y, since the point 3 of X for example, does not appear as a coordinate of any point of the sublattice S. The sublattice S^* of $\{XY\}$, consisting of the points (1,1), (2,2), and (3,2), on the other hand, <u>is</u> a subdirect product of X and Y, since each point of X and each point of Y appears as a coordinate of some point of S^*.

Attention is called to the following representation theorem for distributive lattices [1]p.140:

<u>Theorem.</u> Any distributive lattice (of more than one element) is isomorphic with a subdirect product of chains of length two (i.e. chains consisting of just two elements each.)

One shows readily that a direct product of distributive lattices is a distributive lattice, and in particular, a direct product of Boolean algebras is a Boolean algebra.

Theorem 3.16. Any direct product of Brouwerian algebras is a Brouwerian algebra and subtraction is component-wise.

Proof. Let $\{L_\alpha\}$ be a direct product of Brouwerian algebras L_α.

Then, for every α,

and for $x_\alpha, y_\alpha \in L_\alpha$,

by definition, $y_\alpha + (x_\alpha - y_\alpha) > x_\alpha$,

and if z_α is any element such that

$$y_\alpha + z_\alpha > x_\alpha,$$

then $z_\alpha > x_\alpha - y_\alpha$.

Now, since $y_\alpha + (x_\alpha - y_\alpha) > x_\alpha$, it follows

that $\{y_\alpha\} + \{x_\alpha - y_\alpha\} > \{x_\alpha\}$ by definition

of a direct product of lattices.

Moreover, since $y_\alpha + z_\alpha > x_\alpha$

implies $z_\alpha > x_\alpha - y_\alpha$,

then $\{y_\alpha\} + \{z_\alpha\} > \{x_\alpha\}$

implies $\{z_\alpha\} > \{x_\alpha - y_\alpha\}$.

Thus $\{x_\alpha\} - \{y_\alpha\}$ exists and equals $\{x_\alpha - y_\alpha\}$.

Hence L is a Brouwerian algebra and subtraction is component-wise.

<u>Corollary 3.16.1</u>. Any direct product of Boolean algebras is a Boolean algebra.

<u>Theorem 3.17</u>. If a distributive lattice representable as a direct product of chains (each having a greatest element I) admits a metric group operation, $\emptyset$, it must be a Boolean algebra.

<u>Proof</u>. It will be shown that in these circumstances, the chains must each be of length two. Let $\{C_\alpha\}$ be the direct product of a collection of chains C_α. Let O_α and I_α be the least and greatest elements of C_α respectively. Then $\{O_\alpha\}$ and $\{I_\alpha\}$ are the least and greatest elements of $\{C_\alpha\}$. Let $\{x_\alpha, O_\beta\}$ be an element of $\{C_\alpha\}$ having one component $x_\alpha \neq O_\alpha$ and all other components O_β, $\beta \neq \alpha$. Let $x_\alpha \in C_\alpha$ with $O_\alpha < x_\alpha < I_\alpha$. Since $\emptyset$ is a metric group operation, $\{O_\alpha\} \emptyset \{O_\alpha\} = \{O_\alpha\}$ implies $\{O_\alpha\}$ is the group identity. Hence $\{O_\alpha\} \emptyset \{x_\alpha, O_\beta\} = \{x_\alpha, O_\beta\}$ and $\{O_\alpha\} \emptyset \{I_\alpha\} = \{I_\alpha\}$. Thus $\Delta(\{O_\alpha\}, I, \{x_\alpha, O_\beta\})$ implies $(\{x_\alpha, O_\beta\}, \{I_\alpha\}, \{\{x_\alpha, O_\beta\} \emptyset \{I_\alpha\}\})T$.

Letting $\{z_\alpha\} = \{\{x_\alpha, O_\beta\} \emptyset \{I_\alpha\}\}$, the triangle inequality implies (1) $\{x_\alpha, O_\beta\} + \{z_\alpha\} = \{I_\alpha\}$.

Now, since addition in $\{C_\alpha\}$ is coordinatewise, for a given element $\{a_\alpha\} \in \{C_\alpha\}$, the relation

(2) $\{a_\alpha\} + \{x_\alpha\} = \{I_\alpha\}$

implies (3) $a_\alpha + x_\alpha = I_\alpha$ for every α.

Moreover, since C_α is a chain, then for $a_\alpha \neq I_\alpha$, (3) holds if and only if $x_\alpha = I_\alpha$. (For $a_\alpha = I_\alpha$, x_α

may be arbitrary). Thus the solution of (2) is $\{x_\alpha\} = \{I_\alpha\}$, if and only if $a_\alpha = I_\alpha$ for no α. Consequently, in (1), $\{z_\alpha\} = \{I_\alpha\}$.

Thus $\{0_\alpha\} \emptyset \{I_\alpha\} = \{x_\alpha , 0_\beta\} \emptyset \{I_\alpha\} = \{I_\alpha\}$.

But this contradicts the unique solution property of the group operation $\emptyset$. Hence an element x_α with $0_\alpha < x_\alpha < I_\alpha$ cannot exist for any α, and each chain contains precisely two elements. Since such a chain is a Boolean algebra, $\{C_\alpha\}$ is a Boolean algebra by Corollary 3.16.1

Definition 3.7. A (distributive) lattice representable as a subdirect product $\{C_\alpha\}_s$ of γ chains (γ any cardinal), will be called γ-dimensional.

Theorem 3.18. A finite dimensional lattice $\{C_\alpha\}_s$ which admits a metric group operation is a Boolean algebra.

Proof. It will be shown that each chain C_α of $\{C_\alpha\}_s$ has only two elements.

Assume as in Theorem 3.17 the existence of an element $x_\alpha \in C_\alpha$ such that $0_\alpha < x_\alpha < I_\alpha$. If $\{x_\alpha , 0_\beta\} \in \{C_\alpha\}_S$, Theorem 3.17 applies. If this is not the case, then $\{C_\alpha\}_s$ contains an element $\{y_\alpha\} \neq \{x_\alpha , 0_\beta\}$, but with $y_\alpha = x_\alpha$ for some α. This means that at least one coordinate other than the α th coorinate does not equal 0_β for any β.

Case 1. $y_\alpha \neq I_\alpha$ for any α. A consideration of $\Delta(\{0_\alpha\}, \{y_\alpha\}, \{I_\alpha\})$ leads by the argument of Theorem 3.17 to the same contradiction.

Case 2. At least one coordinate of $\{y_\alpha\}$ other than x_α is equal to I_α for some α of the index set A. Consider again

$\Delta(\{0_\alpha\}, \{y_\alpha\}, \{I_\alpha\})$. To satisfy the triangle inequality in this case, viz.

$\{y_\alpha\} + \{\{y_\alpha\} \varnothing \{I_\alpha\}\} = \{I_\alpha\}$, does not require that $\{y_\alpha\} \varnothing \{I_\alpha\} = \{I_\alpha\}$, as was observed in Theorem 3.17. Now if $\{z_\alpha\} = \{y_\alpha\} \varnothing \{I_\alpha\}$, then $\{w_\alpha\} = \{y_\alpha\}\{z_\alpha\} \in \{C_\alpha\}_s$, since $\{y_\alpha\} \in \{C_\alpha\}_s$, $\{z_\alpha\} \in \{C_\alpha\}_s$ and $\{C_\alpha\}_s$ is a sublattice of $\{C_\alpha\}$. Moreover, for every α, $w_\alpha = y_\alpha$ or $w_\alpha = z_\alpha$ according as $y_\alpha < z_\alpha$ or $z_\alpha < w_\alpha$ since C_α is a chain. Hence, if for some α, $y_\alpha = I_\alpha$ but $z_\alpha \neq I_\alpha$, then $w_\alpha = z_\alpha \neq I_\alpha$, and if $y_\alpha \neq I_\alpha$, then $z_\alpha = I_\alpha$ and again, $w_\alpha \neq I_\alpha$. In any case, therefore, $\{w_\alpha\} = \{y_\alpha\}\{z_\alpha\}$ has at least one fewer coordinates I_α than does $\{y_\alpha\}$. If now $w_\alpha = I_\alpha$ for no α, Case 1 again applies. If this is not the case, iterate the above process with $\{w_\alpha\}$ in the role of $\{y_\alpha\}$. Since the lattice is finite dimensional, in a finite number of steps there will emerge an element $\{u_\alpha\}$ such that $u_\alpha = I_\alpha$ for no α. The argument of Theorem 3.17 will then apply. Thus each chain must contain precisely two elements. Hence the lattice is finite and is therefore a Boolean algebra in accordance with the remark following Theorem 3.14.

Definition 3.8. A space S is said to be congruent to a space S' if there exists a one-to-one distance preserving map of S onto S'.

A space S is said to have congruence order k relative to a class of spaces M containing S, provided that any space of M is congruent (isometric) to a subset of S whenever each k of its points are, and k is the smallest number with this property. This concept is due to Menger [9], (p.116), who proved, for example, that the congruence order of n-dimensional Euclidean space, E_n, relative to the class of metric spaces is $n + 3$. In [5] it was shown that the congruence order of a Boolean geometry relative to the class of L-metrized spaces is three. It is natural to seek the congruence order of Brouwerian geometries. The theorems which follow bear directly on this question.

Theorem 3.19. If the distance function of an autometrized space is a group operation, the congruence order of the space relative to the class of L-metrized spaces is three.

Proof: Let L denote the autometrized space whose distance function $*$ is a group operation, and suppose S is any L-metrized space with the property that every three of its points can be congruently embedded in L. Consider any fixed element a of S, x an arbitrary element, and let $d(a,x) = u$, where $u \in L$. If $\bar{a}$ is any point of L, then there exists uniquely a point $\bar{x} \in L$ such that $\bar{a} * \bar{x} = u$, since the distance function is a group operation. This implies the mapping $x \longrightarrow \bar{x}$ of S onto a subset of L is clearly single valued. Moreover if $y \in S$, $y \neq x$, and $d(a,y) = u$, then the isosceles triangle with vertices (a,x,y) is congruent, by hypothesis to an isosceles triangle in L, a contradiction. Thus the inverse mapping is single valued and the described mapping is one-to-one. In particular, if $x = a$, then $u = 0$ and $a \longrightarrow \bar{a}$.

We prove next that this mapping is distance preserving. Let $y \in S$ and $d(a,y) = v$. Then $\overline{a} * \overline{y} = v$. Suppose $d(x,y) = w$, $w \in L$. Then a triangle exists in L with sides u,v,w. However, if two sides of a triangle in L are respectively equal to two sides of another triangle in L, then the third sides are equal. This follows from the associative law of the group operation, since if $\Delta(a,b,c)$ in L has $a * b = u$, $b * c = v$, then $(a * b)*(b * c) = a * (b * b) * c = a * c$, i.e. the third side of the triangle is uniquely determined by the other two sides. Thus $\overline{x} * \overline{y} = w$. A three point space with all distances equal to the same non-zero element of L shows that the congruence order is not two, since L contains no equilateral triangle.

<u>Corollary 3.19.1</u>. (Ellis). The congruence order of a Boolean geometry relative to the class of L-metrized spaces is three.

<u>Theorem 3.20</u>. A Brouwerian geometry is a Boolean geometry if and only if it has congruence order three relative to the class of L-metrized spaces.

<u>Proof</u>. In view of Corollary 3.19.1, it is necessary only to show that a Brouwerian geometry with congruence order three is a Boolean geometry. Suppose $x \urcorner x \neq 0$, and consider the L-metrized space consisting of the four distinct elements a,b,c,d with $a * c = b * d = x \urcorner x$ and the remaining distances equal to I. Each three points of S are congruently embeddable on the points 0, I, $x \urcorner x$, of the Brouwerian geometry, and by hypothesis, the entire space is so embeddable. This configuration, however, is impossible in an arbitrary Brouwerian

geometry, for if a,b,c,d map respectively into a_1,b_1,c_1,d_1, of L, then $x\urcorner x < a_1$, $x\urcorner x < c_1$ since the vertex of an isosceles triangle is "over" the base. Then $x\urcorner x < a_1c_1$, and $a_1c_1 + x\urcorner x = a_1c_1$. But by Corollary 3.5.1, $a_1c_1 + a_1 * c_1 = a_1 + c_1$, so that $a_1c_1 + x\urcorner x = a_1 + c_1$. Hence $a_1c_1 = a_1 + c_1$, therefore $a_1 = c_1$, i.e. $a_1 * c_1 = x\urcorner x = 0$, a contradiction, and L is a Boolean geometry.

Theorem 3.21. A Brouwerian chain has congruence order four relative to the class of L-metrized spaces.

Proof. Since any chain with I, O is a Brouwerian algebra, then metrized by symmetric difference, (wherein $a - b = a$ for $a > b$) it is a Brouwerian geometry. Let C be such a chain, i.e. Brouwerian. Then (1) every triangle is isosceles, (2) if $a > b > c > d$, then $a * b = a * c = a * d = a$, $b * c = b * d = b$, $c * d = c$, (3) Opposite sides of a quadruple cannot be equal, since a Brouwerian geometry has no equilateral triangle.

Let S be any L-metrized space with the property that every four points of S are congruently embeddable in C. If S contains a point O' which is not the vertex of an isosceles triangle, let O' be mapped into the O of C and every point x' of S into its distance x from O'. This establishes a one-to-one distance preserving map $x' \longrightarrow x$ of S onto a subset of C. The one-to-oneness is obvious. That the mapping is a congruence is seen as follows: If x' and y' are distinct elements of S whose distances from O' are x and y respectively, then d(x,y) equals x or y, according as

$x > y$ or $y > x$, since by hypothesis, the three points O', x', y' are congruently embeddable in C.

If S contains no point O' as described above, then every point is the vertex of an isosceles triangle. However, two isosceles triangles with the same vertex must have their legs equal, otherwise a quadruple including the vertex is determined, which maps into a quadruple in C not satisfying (3) above. Let each point x' of S therefore be mapped into x, the leg of an isosceles triangle with vertex x'. This is a one-to-one mapping of S onto a subset of C. The single valuedness of the mapping is obvious. That the inverse mapping is single valued is seen as follows: If x' and y' are distinct points of S, each the vertex of an isosceles triangle with leg x, say, then a quadruple including x' and y' is determined having opposite sides equal to x, a contradiction, since this implies a similar configuration in C, violating (3).

To see that the mapping is a congruence, consider two distinct points x', y' of S whose images under the mapping are x and y. (Suppose $x > y$, so $x * y = x$.) Then x' is the vertex of an isosceles triangle with leg x. Let $x = x' * u'$, and let $y' * u' = z$. Then $z \neq x$ by (3) so that $d(x', y') = x$ or z. If $d(x', y') = z$, then $z = y$ by definition of the mapping and $x < y$, a contradiction. Thus $d(x', y') = x = x * y$. Thus the congruence order is, at most, four. That it is not less than four is shown by a four point L-metrized space S with two opposite distances equal to a, and the remaining distances equal to b where $a < b$ and a, b are distinct elements of C. Each three points are embeddable in C, but the entire space is not, since (3) is violated. This completes the proof. Examples show that a Brouwerian geometry may have congruence order four without being a chain.

Remark: Theorem 3.21 is the analogue of a classical metric theorem, namely: The congruence order of the Euclidean line (E_1) relative to the class of semi-metric spaces is four. Moreover its congruence order relative to the class of semi-metric spaces containing more than four points is three. One says in this case, that the quasi-congruence order of E_1 is three. This is not so for a Brouwerian chain, however, as shown by the following example. Let S be an L-metrized space of five points a,b,c,d,e with $a * e = b * e = c * e = d * e = x$, $a * b = b * c = c * d = d * a = y$, and $a * c = b * d = z$. Let the Brouwerian chain C contain the elements $x > y > z > 0$. Then each three points of S are congruently embeddable in C. but S is not, for the quadruple (a,b,c,d) is not congruently embeddable in C, since it violates (3) of Theorem 3.21.

Definition 3.9. The direct product $\{A_\alpha\}$ of a collection of autometrized lattices A_α is defined as the lattice direct product with the metric operation coordinate-wise.

Theorem 3.22. If each autometrized lattice A_α of a direct product $\{A_\alpha\}$ has congruence order k relative to the class of L-metrized spaces, then the congruence order of $\{A_\alpha\}$ relative to that class is also k.

Proof. Let S be an L-metrized space, each k of whose points are congruently embeddable in $\{A_\alpha\}$. Let p_i, p_j be arbitrary distinct points of S with $d(p_i,p_j) = \{z_\alpha\} \in \{A_\alpha\}$.

For each α, let $A^{(\alpha)}$ be a space whose points correspond one-to-one to the points of S, with $p_i^{(\alpha)}, p_j^{(\alpha)}$ the respective correspondents of p_i, p_j, with $d(p_i^{(\alpha)}, p_j^{(\alpha)}) = z_\alpha \in A_\alpha$. Thus the distance between any two points of $A^{(\alpha)}$ is the α th coordinate of the distance between the corresponding two points of S.

For each triangle $\Delta(p_i,p_j,p_k)$ of S with sides $\{z_\alpha\}$, $\{u_\alpha\}$, $\{v_\alpha\}$, $(\{z_\alpha\}, \{u_\alpha\}, \{v_\alpha\})T$ implies (by definition of the lattice direct product) $(z_\alpha, u_\alpha, v_\alpha)T$ in A_α, for every α. In particular, then, if, say, $z_\alpha = 0_\alpha$ for some α, it follows that $(z_\alpha, u_\alpha, v_\alpha)T$ implies $u_\alpha > v_\alpha$, $v_\alpha > u_\alpha$, hence $v_\alpha = u_\alpha$. Thus, if in $A^{(\alpha)}$ for some α, $d(p_i^{(\alpha)}, p_j^{(\alpha)}) = 0_\alpha$, we shall identify $p_i^{(\alpha)}$ with $p_j^{(\alpha)}$, and only in these circumstances. Then, clearly, for every α, $A^{(\alpha)}$ is an L-metrized space.

Consider now an arbitrary k-tuple (P), (a set of k points, $p_1, p_2, \ldots p_k) \in S$, containing p_i, p_j, with $d(p_i,p_j) = \{z_\alpha\}$. Since (P) is congruently embeddable in $\{A_\alpha\}$ by hypothesis, let (Q) be a k-tuple of $\{A_\alpha\}$ to which (P) is congruent. Let $q_i : \{x_\alpha\}$, $q_j : \{y_\alpha\} \in (Q)$ be respective correspondents of p_i, p_j under this congruence, so that $d(q_i,q_j) = \{z_\alpha\} = \{x_\alpha * y_\alpha\}$, (since distance in $\{A_\alpha\}$ is coordinate-wise). Then, by definition of the autometrized direct product, there exists a k-tuple (Q_α) in A_α for every α, with $q_{i\alpha} : x_\alpha$, $q_{j\alpha} : y_\alpha \in A_\alpha$ corresponding respectively to q_i, q_j of $\{A_\alpha\}$ wherein $d(q_{i_\alpha}, q_{j_\alpha}) = x_\alpha * y_\alpha$. Clearly, the k-tuple $Q^{(\alpha)}$ in $A^{(\alpha)}$ corresponding to the k-tuple (P) in S is congruent to the k-tuple Q_α in A_α, for every α. Thus each k points of $A^{(\alpha)}$ are congruently embeddable in A_α, hence $A^{(\alpha)}$ is congruently embeddable in A_α for every α, under the hypotheses of the theorem.

Now, let p_i, p_j be arbitrary distinct points of S with $d(p_i, p_j) = \{z_\alpha\} \in \{A_\alpha\}$, let $p_i^{(\alpha)}, p_j^{(\alpha)} \in A^{(\alpha)}$ be as previously constructed, and let $q_{i_\alpha} : x_\alpha$, $q_{j_\alpha} : y_\alpha \in A_\alpha$ be respective correspondents of $p_i^{(\alpha)}, q_i^{(\alpha)}$ under the congruence of $A^{(\alpha)}$ into A_α already established. Then $d(q_{i_\alpha}, q_{j_\alpha}) = x_\alpha * y_\alpha = z_\alpha$, and there exist points $q_i : \{x_\alpha\}$, $q_j : \{y_\alpha\}$ in $\{A_\alpha\}$ with $q_i * q_j = \{x_\alpha * y_\alpha\} = \{z_\alpha\}$. Then the mapping $p_i \longrightarrow q_i$, $p_j \longrightarrow q_j$ of S into $\{A_\alpha\}$ is a congruence, established as follows:

According to the construction of the spaces $A^{(\alpha)}$, the mapping $p_i \longrightarrow p_i^{(\alpha)}$, $p_j \longrightarrow p_j^{(\alpha)}$ of S onto $A^{(\alpha)}$ for every α is one-to-one. Moreover, under the established congruence of $A^{(\alpha)}$ into A_α, for every α, the mapping $p_i^{(\alpha)} \longrightarrow q_{i_\alpha}$, $p_j^{(\alpha)} \longrightarrow q_{j_\alpha}$ is also one-to-one. Finally the mapping $q_{i_\alpha} \longrightarrow q_i, q_{j_\alpha} \longrightarrow q_j$ of A_α into $\{A_\alpha\}$ for every α is one-to-one by the definition of the direct product. Therefore the mapping $p_i \longrightarrow q_i, p_j \longrightarrow q_j$ of S into $\{A_\alpha\}$ is one-to-one as well. Since $d(p_i, p_j) = \{z_\alpha\} = q_i * q_j$, the mapping is a congruence, as claimed.

Corollary 3.22.1. Any autometrized lattice L which is a direct product $\{C_\alpha\}$ of Brouwerian chains C_α has congruence order four.

Proof. The congruence order of each Brouwerian chain C_α relative to the class of L-metrized spaces is four. (Theorem 3.21).

Corollary 3.22.2. If each autometrized lattice A_α of a direct product $\{A_\alpha\}$ has a finite congruence order relative to the class of L-metrized spaces, the congruence order of $\{A_\alpha\}$ is the maximum of the congruence orders of the A_α, if it exists.

Proof. Let the maximum congruence order be the k of the theorem. If the congruence order of A_α for some α is $n \leq k$, then, since each k points of $A^{(\alpha)}$ are congruently embeddable in A_α, each n points are certainly likewise embeddable, hence so is $A^{(\alpha)}$. All other details of the proof are identical with those of the theorem.

Remark. The congruence order of an arbitrary Brouwerian geometry relative to the class of L-metrized spaces is still an open question.

(This rectifies an earlier statement of L.M.Kelly and the writer [Bulletin of the American Mathematical Society, Vol. 62 Number 2, March 1956, pp.172-3] that the congruence order in the general case was shown to be four. Further investigation to date of subdirect products of Brouwerian algebras continues to suggest strongly that this is indeed the case.)

Section 4. General Theorems; Betweenness

In this section properties of L-metrized spaces in general are established. Further, metric and lattice betweenness are introduced, and the consequences of their coincidence is studied.

<u>Theorem 4.1.</u> In any L-metrized space, the sum of the distances of the elements of a subset S from any element of the subset is constant and equal to the sum of all the distances of the subset, provided the sums exist.

<u>Proof</u>. Let p be any element of S, x and y arbitrary distinct elements of S, with $d(x,y) = d_{xy}$, $d(p,x) = p_x$ and $d(p,y) = p_y$. Then the triangle inequality asserts that $p_x + p_y > d_{xy}$,

hence $\sum\limits_{x,y \in S} p_x + p_y = \sum\limits_{x \in S} p_x > d_{xy}$ and $\sum\limits_{x \in S} p_x > \sum\limits_{x,y \in S} d_{xy}$.

But $\sum\limits_{x,y \in S} d_{xy} > \sum\limits_{x \in S} p_x$, hence $\sum\limits_{x \in S} p_x = \sum\limits_{x,y \in S} d_{xy}$.

<u>Corollary 4.1.1</u>. In any L-metrized space, the sum of any two sides of a triangle equals the perimeter.

<u>Corollary 4.1.2</u>. In any L-metrized space, the sum of two sides of a triangle equals the sum of any other two sides.

<u>Definition 4.1.</u> Three points u,v,w, of an L-metrized space are called linear if $d(u,v) + d(v,w) = d(u,w)$.

<u>Definition 4.2</u>. In any L-metrized space, a triangle with sides a,b,c will be designated T(a,b,c). (Note: this implies (a,b,c)T).

Corollary 4.1.3. In any L-metrized space, the vertices of an isosceles triangle are linear.

Proof. In $T(a,a,b)$, $a + b = a + a = a$

Theorem 4.2. In any L-metrized space, a triangle $T(a,b,c)$ is isosceles if and only if a,b,c form a chain (or a,b,c are pair-wise comparable).

Proof. If $a = b$, say, then $a + a = a > c$.

Conversely, since $a + b = b + c = a + c$, $a > b > c$ implies $a = b$.

Theorem 4.3. In any L-metrized space, if, in $T(a,b,c)$, $a > b$, then $a > c$ and $b + c = a$ (i.e. the vertices are linear). Moreover, if $T(a,b,c)$ is not isosceles, then b and c are non-comparable.

Proof. Since $a + b = b + c = a + c$, $a > b$ implies $a = a + c = b + c$ i.e. $a > c$. If b and c were comparable, the elements would form a chain and the triangle would be isosceles.

Corollary 4.3.1. In any L-metrized space, a non-isosceles triangle has either precisely one pair of non-comparable sides, or all three sides are pair-wise non-comparable.

Proof. Exactly two pairs of non-comparable sides leads immediately to a contradiction of the theorem.

Corollary 4.3.2. In any L-metrized space, if precisely one pair of sides b,c of a triangle are non-comparable, the third side a is uniquely determined as their sum, $a = b + c$, and again the vertices are linear.

<u>Corollary 4.3.3.</u> In any L-metrized space, a triangle with precisely one pair of non-comparable sides is uniquely determined by them.

<u>Corollary 4.3.4.</u> In any L-metrized space, the vertices of a triangle are linear, if and only if the triangle has a pair of comparable sides.

<u>Proof.</u> The sufficiency of the condition is a conclusion of the theorem. On the other hand if $a = b + c$, $a > b$ and $a > c$ by definition of the lattice sum.

<u>Corollary 4.3.5.</u> In any L-metrized space, the only triangles whose vertices are non-linear are those whose sides are pair-wise non-comparable.

<u>Proof.</u> Any triangle without this property satisfies the conditions of the theorem.

<u>Definition 4.3.</u> An L-metrized space whose distance lattice is a chain C, is called a C-metrized space.

<u>Theorem 4.4.</u> Every triangle of a C-metrized space is isosceles.

<u>Proof.</u> This theorem follows from Theorem 4.2.

<u>Corollary 4.4.1.</u> Every triangle of an autometrized chain is isosceles.

<u>Remark.</u> Every triangle of an L-metrized space may be isosceles, even though its distance lattice is not a chain. An example is the autometrized four element Boolean algebra a, b, I, O with $d(a,b) = a$, $d(O,I) = b$ and all other distances equal to I.

Definition 4.4. An autometrized lattice L as well as its metric operation is called regular if $a * 0 = a$ for every element a of L. (Thus Boolean and Brouwerian geometries are readily seen to be regular.)

It should be noted that every lattice admits the regular metric operation $a * b = a + b$ for $a \neq b$, and $a * a = 0$, for $a * 0 = a + 0 = a$, and since for any $\Delta(a,b,c)$, $(a * b) + (b * c) = (a + b) + (b + c) = a + b + c$, $a + b + c \geq a + c = a * c$, i.e. the triangle inequality holds.

Theorem 4.5. In a regular autometrized lattice, $(a,b,a * b)T$.

Proof. Evident from a consideration of $\Delta(a,0,b)$.

Although, as indicated above, an autometrized lattice in which every triangle is isosceles need not be a chain, one does have

Theorem 4.6. A regular autometrized lattice L in which every triangle is isosceles is a chain.

Proof. Let a and b be arbitrary distinct elements of L and consider $\Delta(a,b,0)$. Since $a * 0 = a$ and $b * 0 = b$, it follows that $a * b = a$ or $a * b = b$ so that either $a > b$ or $b > a$.

Theorem 4.7. In a regular autometrized chain, for $a > b$, $a * b = a$.

Proof. In $\Delta(a,b,0)$, $a * 0 = a$, $b * 0 = b$. Since every triangle is isosceles, $a * b = a$ for $a > b$.

Corollary 4.7.1. Every regular autometrized chain is a Brouwerian geometry.

Proof. A chain is a Brouwerian algebra and for $a > b$, $a * b = a = (a-b) + (b-a)$.

Corollary 4.7.2. A regular autometrized chain has no equilateral triangles.

Proof. A Brouwerian geometry has no equilateral triangles.

Theorem 4.8. An autometrized chain C of more than three elements is regular, if and only if, every quadruple of elements $a > b > c > d$ has the distance pattern,

$a * b = a * c = a * d = a,\ b * c = b * d = b,\ c * d = c.$

Proof. The regularity and the fact that every triangle is isosceles yields the above pattern.

Conversely, if x is an arbitrary element of C, then in any quadruple $z > y > x > 0$, the prescribed pattern yields $x * 0 = x$.

Remark. A non-regular autometrized chain may have a quadruple whose distances have the indicated pattern but the vertices of the quadruple need not be those indicated.

Theorem 4.9. An autometrized lattice L is regular if and only if $a + (a * b) = a + b$ for all elements a,b of L.

Proof. If L is regular, then $(a,b,a * b)T$ by Theorem 4.5 and the condition follows from Corollary 4.1.2. On the other hand the condition implies, in particular, for the elements 0,x, that $0 + (0 * x) = 0 + x = x$, i.e. $0 * x = x$.

Corollary 4.9.1. In a regular autometrized lattice L, $a * b < a + b$ for all elements a,b, of L.

Proof. Implied by $(a,b,a * b)T$ in $\Delta(a,b,0)$.

Remark. The condition $a * b < a + b$ is not sufficient to ensure regularity as shown by the chain $I > a > b > 0$ with $I * a = I * b = I * 0 = I,\ a * b = a * 0 = b * 0 = b.$

Definition 4.5. An L-metrized space is called distributive if its distance lattice is distributive.

Corollary 4.9.2. In a regular distributive autometrized lattice, $ab + (a * b) = a + b$.

Proof. Identical with that of Corollary 3.5.1.

Remark. The above condition even with regularity is not sufficient to yield distributivity. The condition holds for every pair of elements of the non-modular five element lattice [1](page 6 figure 1d.) with elements I, O, $a > b$, and c, if the autometrization is regular, $a * b = a$, and all other distances are equal to I.

Definition 4.6. An autometrized lattice is called symmetric if the distance between every two of its elements is equal to the distance between their sum and product. Thus Brouwerian, and Boolean geometries are readily seen to be symmetric.

Theorem 4.10. An autometrized lattice which is symmetric and contains no isosceles triangle is distributive.

Proof. A non-distributive lattice must contain one of the two special five element lattices shown below. [1](page 134). The symmetric property implies in each case the existence of an isosceles triangle, contrary to assumption.

In figure 1, if $x * y = u * v = y * z = x * z$, then x,y,z are vertices of an isosceles triangle.

In figure 2, if $y * z = x * z = u * v$, then x,y,z are vertices of an isosceles triangle.

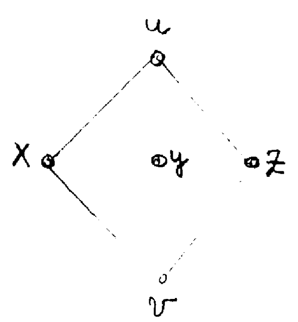

fig.1

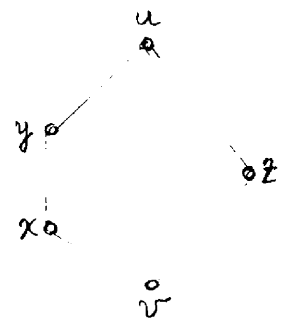

fig.2

Definition 4.7. In an L-metrized space, the element b is metrically between a and c, if d(a,b) + d(b,c) = d(a,c). The points (a,b,c) are said to be linear as already indicated, and the relation is written (a,b,c)M.

It should be noted at the outset that this is not a betweenness relation in the usual sense, since it fails in many instances to have the special inner point property. i.e. (a,b,c)M and (a,c,b)M may both persist even though b and c do not coincide. None the less, it is convenient to use the terminology which has been indicated. However, the relation does have the other basic betweenness property, viz., symmetry in the outer points, i.e. (a,b,c)M if and only if (c,b,a)M, since the metric operation is commutative.

Theorem 4.11. Three linear points of an L-metrized space fail to have the special inner point property if and only if they are the vertices of an isosceles triangle.

Proof. If in Δ(a,b,c), d(a,b) = d(a,c), then (a,b,c)M and (a,c,b)M by Corollary 4.1.2. Conversely, if these relations hold, then d(a,b) + d(b,c) = d(a,c) and d(a,c) + d(b,c) = d(a,b). Hence d(a,c) ≥ d(a,b) and d(a,b) ≥ d(a,c), i.e. d(a,b) = d(a,c).

<u>Definition 4.8</u>. If a betweenness relation R has the property that $(a,b,c)R$ and $(a,x,b)R$ imply $(x,b,c)R$, the relation is said to have transitivity t_1 . If the same two relations imply $(a,x,c)R$, the relation is said to have transitivity t_2. [11].

<u>Theorem 4.12</u>. Metric betweenness has transitivity t_2.

<u>Proof</u>. If $*$ denotes the metric operation, then $(a,b,c)M$ and $(a,x,b)M$ imply $(a * b)+(b * c)=(a * c)$ and $(a * x)+(x * b)=(a * b)$. Hence, $(a * x)+(x * b)+(b * c) =(a * c)$. Now, since $x * c < (x * b)+(b * c)$ by the triangle inequality, it follows that $(a * x)+(x * c) < a * c$, but the triangle inequality implies that $(a * x)+(x * c) > a * c$, hence , $(a * x)+(x * c) = (a * c)$, i.e. $(a,x,c)M$ as claimed.

<u>Remark</u>. In general, metric betweenness fails to have t_1, notably in the case of an isosceles triangle, for if $\Delta(a,b,c)$ has $d(a,b) = d(a,c)$, then $(a,b,c)M$ and $(a,c,b)M$ hold, but $c,b,c)M$ is not valid. Indeed, this may happen in an L-metrized space without isosceles triangles, as the following example shows:

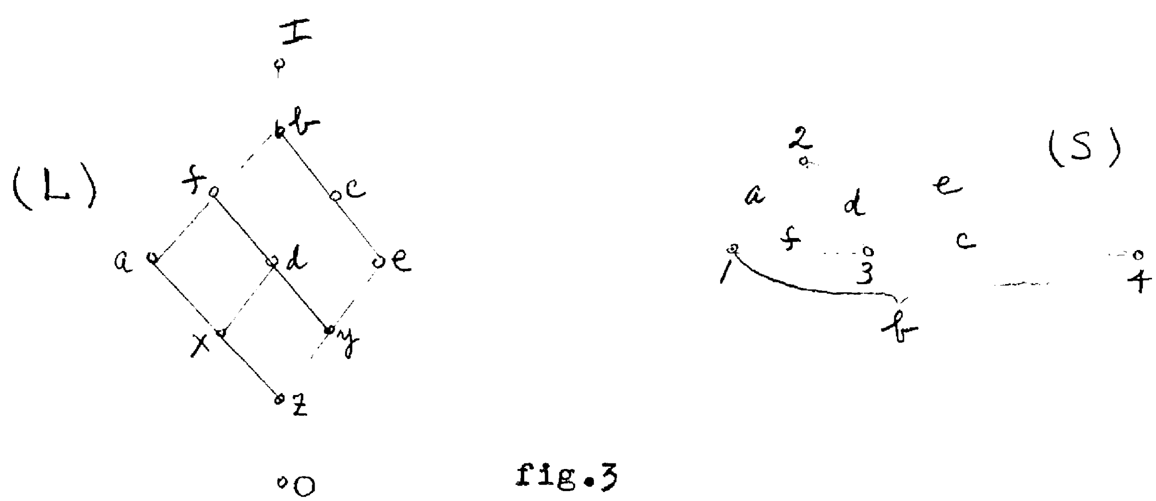

fig.3

In S, (figure 3), (1,3,4)M and (1,2,3)M but (2,3,4)M does not hold.

<u>Definition 4.9</u>. An element b of a lattice is "<u>lattice between</u>" a and c, written (a,b,c)L, if and only if $ab + bc = b = (a + c)(b + c)$, [11] p.105. This relation has transitivity t_1[11], but not in general t_2, since t_2 implies modularity (and conversely) [11]. It is a betweenness relation in the usual sense. Further, in any lattice, (a,b,c)L implies $ac < b < a + c$, but not, in general conversely. However, in a <u>distributive</u> lattice, (a,b,c)L if and only if $ac < b < a + c$ [11]. Metric and lattice betweenness will be said to coincide in an autometrized lattice provided (a,b,c)M if and only if (a,b,c)L.

<u>Theorem 4.13</u>. In an autometrized lattice L, metric and lattice betweenness coincide, if and only if (1) metric betweenness has t_1, (2) L is symmetric, (3) $a < b < c$ implies (a,b,c)M.

<u>Proof</u>. Use is made here of a theorem due to Pitcher and Smiley [11, Theorem 10.1]. In verifying the hypotheses of that theorem, one observes that (1) implies that L is free of isosceles triangles and so by Theorem 4.11 that metric betweenness has the special inner point property. (2) and (3) imply $(a, a + b, b)M$ and $(a, ab, b)M$, for (3) implies $(a + b, a, ab)M$ and $(a + b, b, ab)M$, hence, letting $x = (a + b) * a$, $u = a * ab$, $y = (a + b) * b$, and $v = b * ab$, we have $x + u = y + v = (a + b) * ab$. Moreover, by (2) $(a + b) * ab = a * b$, so that $x + y + u + v = a * b$. Hence

$x + y < a * b$, and $u + v < a * b$. But $x + y > a * b$ and $u + v > a * b$ by the triangle inequality, hence, $x + y = u + v = a * b$, i.e. $(a, a + b, b)M$ and $(a, ab, b)M$.

Conversely, if metric and lattice betweenness coincide, (1) and (3) are immediate. To establish the symmetric property, consider the quadruple $(a, b, ab, a + b)$. $(a + b, a, ab)L$ and $(a + b, b, ab)L$ imply, by the assumed coincidence,

(i) $[(a+b) * a] + (a * ab) = [(a+b) * b] + [b * ab] = (a+b) * ab$.

Similarly, $(a, a + b, b)L$ and $(a, ab, b)L$ imply

(ii) $[(a+b) * a] + [(a+b) * b] = (a * ab) + (b * ab) = a * b$.

Adding first and second members of (i) gives the same result as adding first and second members of (ii). Idempotency yields

$$(a + b) * ab = a * b.$$

Thus the theorem is established.

<u>Corollary 4.13.1.</u> An autometrized lattice in which metric and lattice betweenness coincide is distributive.

<u>Proof</u>. By Theorem 4.10

<u>Theorem 4.14</u>. (Ellis) In a Boolean geometry, metric and lattice betweenness coincide.

<u>Proof</u>. A somewhat briefer proof than Ellis has given is as follows: $(a, b, c)L$ implies $ac < b < a + c$, in any lattice. Taking complements of $ac < b$ yields $b' < a' + c'$. Moreover,

$(a * b) + (b * c) = ab' + a'b + bc' + b'c$

$= (a + c)b' + b(a' + c')$. But $b < a + c$ implies

$b(a' + c') < (a + c)(a' + c')$ and $b' < a' + c'$ implies

$b'(a+c)<(a+c)(a'+c')$ so that $b(a'+c')+b'(a+c)<(a+c)(a'+c')$, i.e. $(a * b) + (b * c) < a * c$. But in any case, $(a * b)+(b * c)> a * c$, hence $(a * b)+(b * c) = a * c$. i.e. $(a,b,c)M$.

Conversely $(a,b,c)M$, i.e. $(ab'+a'b)+(bc'+b'c)=ac'+a'c$ implies $(a + c)b' + b(a' + c') = ac' + a'c$. Multiplying the latter first by ac, then by $a'c'$, we obtain $acb' = 0$ and $a'c'b = 0$. But $xy' = 0$ implies $xy + xy' = xy$ or $x = xy$ so that $x < y$. Hence $ac < b$. Similarly $b(a'c') = 0$ implies $b(a+c)' = 0$, hence $b < a + c$. i.e. $ac < b < a + c$ or since a Boolean algebra is distributive, $(a,b,c)L$. This completes the proof.

Remark. Since t_1 fails for an isosceles triangle, it fails in any Brouwerian geometry which is non-Boolean, hence metric and lattice betweenness do not in general coincide in Brouwerian geometries. Indeed this coincidence to-gether with regularity is sufficient to characterize Boolean geometries among the Brouwerian geometries as the following theorems show.

Theorem 4.15. An autometrized lattice with an I is a Boolean geometry if and only if it is regular, and metric and lattice betweenness coincide.

Proof. Theorem 4.14 establishes the necessity, since symmetric difference is a regular metric operation. To prove the sufficiency, let $a * I = x$, $I * x = y$ and consider the points $(0,I,a,x,ax)$. Regularity implies $(a,I,x)T$ and $(x,I,y)T$ or $a + x = I$ and $x + y = I$. Moreover $(a,I,x)L$

implies $x + y = a * x$ since MB $\equiv$ LB. Hence, $a * x = I$. Symmetry (by Theorem 4.13) implies further that $a * x = (a + x) * ax = I * ax = I$. But since $I * 0 = I$, $\Delta(0,I,ax)$ is isosceles. This is impossible since t_1 must hold under our assumptions. Therefore $ax * 0 = ax = 0$. Thus the lattice is complemented, and, being distributive by Corollary 4.13.1 is a Boolean algebra.

To show that $a * b$ is symmetric difference, consider the points $(0,a,b,I)$, where $a * I = a'$, $b * I = b'$, since, in a Boolean algebra, complements are unique, and $(a,I,a * I)T$ (due to regularity) implies, in particular, that $a + (a * I) = I$. Clearly $a * b < a + b$, and $a * b < a' + b'$(since $\Delta(a,b,I)$ implies $(a',b',a * b)T$), whence $a * b < (a+b)(a' + b')$ $= ab' + a'b$. But, $(a * b)+ ab = a + b$ (Corollary 4.9.2), implies that $(a+b) - ab < a * b$, **since a** Boolean algebra **is**, a fortiori, a Brouwerian algebra. Thus $a * b = (a-b)+(b-a)$, $a * b$ is therefore Boolean symmetric difference, and the lattice is a Boolean geometry.

<u>Corollary 4.15.1.</u> A lattice with an I is a Boolean algebra if and only if it admits a metric group operation under which metric and lattice betweenness coincide.

<u>Proof:</u> The necessity is clear. The sufficiency is assured since a metric group operation is regular.

<u>Definition 4.10.</u> An L-metrized space S is said to be of constant width, if (1) There exists in S, a maximal distance, m, i.e. such that $m > x$ for every distance x in S, and (2) Corresponding to each element $a \in S$ there exists

an element $b \in S$ such that $d(a,b) = m$.

Theorem 4.16. A lattice with an I is a Boolean algebra if and only if it admits a metrization such that the space is of constant width, and metric and lattice betweenness coincide.

Proof: The necessity is clear, since (1) Boolean symmetric difference is a metrization under which metric and lattice betweenness coincide and (2) being a group operation, the equation $a * x = I$ always has a solution, namely $y = a'$. (Clearly I is the maximal distance in the space and it actually occurs, since $I * O = I$)

To establish the sufficiency, let a and b be arbitrary distinct elements of the lattice. Then, since metric and lattice betweenness coincide, $(I,a,O)M$ and $(I,b,O)M$, i.e. $(a * O) + (a * I) = (b * O) + (b * I) = I * O$. Since, in $\Delta(I,a,b)$, $(a * b) < (a * I) + (b * I)$, then $a * b < I * O$ since $(a * I) + (b * I) < (I * O)$, and $I * O$ is the maximal distance which occurs. Now, since the space is of constant width, corresponding to an element x, there exists an element y, such that $x * y = I * O$. Moreover, y is unique, since there can be no isosceles triangles. Furthermore, since the lattice must be symmetric by Theorem 4.13, $(x+y) * xy = I * O$, and $[(x+y),(xy),O]M$ implies $[(x+y)*(xy)]+[(xy) * O]=(x+y) * O$, i.e. $(I * O)+((xy) * O)=(x+y)*O$ But $I * O > (xy) * O$, hence $(x+y) * O = I * O$. Since there can be no isosceles triangles, $x + y = I$. Hence $I * xy = I * O$, and again, the absence of isosceles triangles implies $xy = 0$. Thus the lattice is a Boolean algebra.

Corollary 4.16.1. A finite autometrized lattice in which metric and lattice betweenness coincide is a Boolean algebra.

Proof. By Theorem 4.13 the lattice can have no isosceles triangles. If the lattice has n elements, then each element is a "vertex" of $(n-1)$ distances. Since none of these can be repeated, each element of the lattice (except 0) must occur at each vertex as a distance, hence being finite, the lattice has an I which occurs at each vertex. I being clearly maximal, the lattice is of constant width, and is therefore a Boolean algebra.

Remark. In the infinite case even though no distance may be repeated at a vertex, there is no assurance that every distance must occur. Thus whether in an autometrized lattice with an I, the coincidence of metric and lattice betweenness in and of itself is sufficient to characterize a Boolean algebra is still an open question.

Ellis has observed [5] that the group of motions of a Boolean geometry is simply transitive, i.e. for any two points a,b of the geometry, there is a motion (a one-to-one distance preserving map of the space onto itself) which carries b into a. An examination of the Brouwerian chain of three elements shows that the group of motions of a Brouwerian geometry is not simply transitive, and, indeed

Theorem 4.17. A Brouwerian geometry is a Boolean geometry if and only if its group of motions is simply transitive.

Proof. The necessity as indicated has been observed by Ellis. Suppose then that a Brouwerian geometry L has a simply transitive group of motions, and consider that motion which carries I into O. Then if $x \longrightarrow y$, $I * x = O * y = \neg x = y$ i.e. $x \longrightarrow \neg x$. Hence $O \longrightarrow I$, $x \neg x \longrightarrow \neg(x \neg x) = I$, so that $O = x \neg x$ and L is a Boolean geometry.

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