CHARACTERIZATIONS OF THE SYMMETRIC DIFFERENCE AND THE STRUCTURE OF STONE ALGEBRAS

By

JACK GRESHAM ELLIOTT

AN ABSTRACT

Submitted to the School for Advanced Graduate Studies of Michigan State University of Agriculture and Applied Science in partial fulfillment of the requirements for the degree of

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JACK GRESHAM ELLIOTT ABSTRACT

It is well known that the symmetric difference in a Boolean algebra is a group operation. It is also an abstract metric operation, in the sense that it satisfies lattice relationships formally equivalent to the geometrical relationships defining a metric distance function. It is shown first that the symmetric difference in a Boolean algebra is the only binary relation which is simultaneously an abstract metric and a group operation. This characterization is then extended by successive relaxing of some of the group and metric postulates. Next the symmetric difference is characterized in several ways among the Boolean operations. Finally, the symmetric difference in a Boolean algebra is characterized as the only binary operation satisfying certain other side conditions.

Brouwerian algebras having a least element 0 and a greatest element I may be regarded as extensions of Boolean algebras in which the relationship $(a')' = a$ is replaced by the weaker relationship $\neg(\neg a) \leq a$, where a' and $\neg a$ denote respectively the Boolean and Brouwerian complements of a. While a Brouwerian algebra in which $a \cdot \neg a = 0$ for all a is necessarily a Boolean algebra, there exist Brouwerian algebras which are not Boolean algebras and in which $\neg a \cdot \neg(\neg a) = 0$ identically. M. H. Stone proposed the problem of characterizing those Brouwerian algebras (herein called Stone algebras) in which $\neg a \cdot \neg(\neg a) = 0$

for every a. It is first observed that a Brouwerian algebra B is a Stone algebra if, and only if, the subset R of elements x satisfying $\neg(\neg x) = x$ is a Boolean sub-algebra of B. Next it is shown that a Stone algebra B is the direct sum of the Boolean sub-algebra R and a Brouwerian sub-algebra T which is an ideal and which has in common with R only the element 0. A set-theoretic interpretation of this structure theorem is presented which furnished a technique for constructing Stone algebras.

In the concluding section it is shown that a wide class of Stone algebras, including all finite ones, are direct products of special Stone algebras each of whose greatest element I is join-irreducible. Finally, an example is presented which shows that not every Stone algebra may be characterized in this manner.

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To Verda

without whose patience and faith this thesis would

not have been written.

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Section 1. Introduction

Studies of Brouwerian algebras in which there is defined a binary operation analogous to the distance function of a metric space have been carried out by Nordhaus and Lapidus [11] and by Lapidus [8]. Their work generalized many of the earlier similar investigations of Ellis [5.6] and Blumenthal [4] in the field of Boolean algebras. Ellis, in particular, observed that in a Boolean algebra the symmetric difference operation satisfied lattice relationships formally equivalent to the postulates defining a metric distance function, and showed that many purely geometric concepts could be carried over into this new setting.

The first goal of this thesis is to show that in a Boolean algebra the symmetric difference is the only binary operation which satisfies the requirements of an abstract metric and is simultaneously a group operation.

One important difference between Boolean and Brouwerian algebras is the fact that the Boolean complement x' of an element x is disjoint from x, while the Brouwerian complement $\urcorner x$ of an element x is not necessarily disjoint from x. However, in many (but not all) Brouwerian algebras it is true that, given any element x, the elements $\urcorner x$ and $\urcorner\urcorner x$

are disjoint, where $\neg\neg x$ denotes the Brouwerian complement of $\neg x$. M. H. Stone has asked, "What is the most general Brouwerian algebra in which, for every x, the elements $\neg x$ and $\neg\neg x$ are disjoint?"[1]

The second goal of this thesis is to determine the basic structure of these Brouwerian algebras.

In Section 2 the symmetric difference operation in a Boolean algebra is characterized as the only binary operation which is at once an abstract metric and a group operation. By successive weakening or removal of some of the group and metric postulates generalizations of this result are obtained. Other characterizations of the symmetric difference among the class of Boolean operations are found, and Section 2 is concluded with further characterizations of the symmetric difference in a Boolean algebra as the only binary operation satisfying certain other side conditions.

In Section 3 there is determined the basic structure of those Brouwerian algebras in which, for every x, the elements $\neg x$ and $\neg\neg x$ are disjoint. An interesting characterization of a wide sub-class of these special Brouwerian algebras is presented in Section 4.

In the remainder of this section are presented fundamental definitions, concepts, and notation to be used throughout.

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[1] This question appears as Problem 70 of Birkhoff [3], where it is phrased in the dual setting of pseudo-complemented lattices.

<u>Definition</u>: A <u>partially ordered set</u> P is a set of elements a, b, c, ... together with a binary relation $a \geq b$ (read "a is over b", "a contains b", or "b is under a") subject to the following postulates:

P1: $a \geq a$

P2: If $a \geq b$ and $b \geq a$, then $a = b$

P3: If $a \geq b$ and $b \geq c$, then $a \geq c$.

<u>Definition</u>: An <u>upper bound</u> of a subset X of P is an element a such that $a \geq x$ holds for every x in X. An element b is the <u>least upper bound</u> of X if b is an upper bound of X and if $b \leq a$ holds for every upper bound a of X. A <u>lower bound</u> of X and the <u>greatest lower bound</u> of X are defined similarly.

<u>Definition</u>: A partially ordered set P is a <u>lattice</u> if for each pair of elements a, b the greatest lower bound of a and b and the least upper bound of a and b exist. The greatest lower bound of a and b is denoted by $a \cdot b$, or ab, and is referred to as the <u>product</u>, or <u>lattice product</u>, or <u>meet</u> of a and b; the least upper bound of a and b is written $a + b$ and is called the <u>sum</u>, or <u>lattice sum</u>, or <u>join</u> of a and b. It is shown in Birkhoff [3] that the meet and join operations satisfy the following laws:

L1 (Idempotent law): $a + a = a$ and $aa = a$.

L2 (Commutative law): $a + b = b + a$ and $ab = ba$.

L3 (Associative law): $a + (b + c) = (a + b) + c$
and $a(bc) = (ab)c$.

L4 (Absorption law): $a + ab = a$ and $a(a + b) = a$.

<u>Definition</u>: A <u>distributive lattice</u> is a lattice in which for every triple of elements a, b, c the following relationships hold:

L5: $a(b + c) = ab + ac.$

L6: $a + bc = (a + b)(a + c).$

Section 2. Characterizations of the Symmetric Difference Operation in a Boolean Algebra

Definition: A Boolean algebra is a distributive lattice with 0 and I in which for each element a there exists an element a' satisfying $a + a' = I$ and $aa' = 0$. The element a' is referred to as the complement (or Boolean complement) of a.

It can be shown that the complement a' of a is unique, and that complementation is ortho-complementation, i.e. that $(a')' = a$.

Definition: With each pair of elements a, b of an abstract set S let there be associated an element $f(a, b)$ of a lattice L with an 0. The binary function f is a metric function from S to L if the following three conditions hold:

M1: $f(a, b) = 0$ if, and only if, $a = b$,

M2: $f(a, b) = f(b, a)$,

M3: $f(a, b) + f(b, c) \geq f(a, c)$;

and we say that "S is lattice-metrized by f". A metric function f from a lattice L to itself is called a metric operation, and in this case L is called an auto-metrized lattice.

The properties M1, M2 and M3 are lattice-analogues of the familiar requirements of a distance function in a metric space. We carry the analogy further by referring to the lattice element $f(a,b)$ as the "distance between a and b", the elements $f(a,b)$, $f(b,c)$ and $f(a,c)$ as "sides of the triangle whose vertices are a, b and c", and in general using geometric terminology wherever such usage is convenient and suggestive. It is particularly convenient to refer to M3 as the "triangle inequality".

Definition: In a Boolean algebra the element $ab' + a'b$ is the symmetric difference of a and b.

Theorem 2.1 [Ellis, 5]: The symmetric difference in a Boolean algebra is a metric operation.

Proof: Let $d(a,b)$ denote the symmetric difference of a and b. First we observe that $d(a,b) = aa' + a'a = 0 + 0 = 0$. Next we show that $d(a,b) = 0$ implies $a = b$. $d(a,b) = ab' + a'b = 0$ can hold only if $ab' = a'b = 0$. To each side of the equation $ab' = 0$ we add ab, obtaining

$$ab' + ab = 0 + ab = ab. \tag{2.1}$$

Then $ab = ab' + ab = a(b' + b) = aI = a$, using the fact that a Boolean algebra is a distributive lattice. But $ab = a$ means that $a \leq b$. Similarly, from $a'b = 0$ we conclude that $b \leq a$. Therefore $a = b$, and M1 holds. Since the expression $ab' + a'b$ is symmetric in a and b, it follows that M2 holds. To prove M3, we will show that $[d(a,b) + d(b,c)]\cdot d(a,c) = d(a,c)$, which of course implies

$d(a,b) + a(b,c) \geq d(a,c)$. To this end, we write

$$(2.2) \qquad \left[d(a,b) + d(b,c)\right] \cdot d(a,c) = \left[(ab' + a'b) + (bc' + b'c)\right] \cdot (ac' + a'c)$$

$$= ab'ac' + a'bac' + bc'ac' + b'cac' + ab'a'c + a'ba'c + bc'a'c + b'ca'c$$

$$= ab'c' + abc' + a'bc + a'b'c$$

$$= ac'(b' + b) + a'c(b + b') = ac' + a'c = d(a,c)$$

and the proof is complete.

In the following theorem, we let $a*b$ denote a metric group operation in a Boolean algebra, and show that $a*b = ab' + a'b$ necessarily.

<u>Lemma 1</u>: If x, y and z are the sides of a triangle in a Boolean algebra, then $x + y = x + z = y + z$.

<u>Proof</u>: Since $x + y \geq z$ by M3, we add x to each side to get $x + y \geq x + z$. Similarly $x + z \geq y$ by M3, and adding x to each side yields $x + z \geq x + y$. This implies that $x + y = x + z$, and the proof for the other two cases is similar.

<u>Lemma 2</u>: If $a = b*c$, then $a*b = c$ and $a*c = b$.

<u>Proof</u>: $a = b*c$ implies $a*(b*c) = 0$ by M1. The associative law then gives $(a*b)*c = 0$, whence $a*b = c$ by M1. The proof for the other case is similar.

<u>Lemma 3</u>: $0*a = a$.

<u>Proof</u>: Let $0*a = x$. By Lemma 2, $a*x = 0$. Hence $a = x$ by M1.

<u>Lemma 4</u>: $a*I = a'$.

<u>Proof</u>: Let $a*a' = b$, and consider the triangle 0, a, a', the sides of which are $0*a = a$, $0*a' = a'$, $a*a' = b$. Lemma 1 gives us $a + b = a + a' = I$, and $a' + b = a + a' = I$.

Hence $I = (a + b)(a' + b) = b$. We conclude that $a*a' = I$, and Lemma 2 gives us $a*I = a'$ and $a'*I = a$.

Theorem 2.2: The only metric group operation in a Boolean algebra is the symmetric difference.

Proof: Let $x*y = p$. Consider the triangle I, x, y, the sides of which are $I*x = x'$, $I*y = y'$, $x*y = p$ by Lemma 4. From Lemma 1 we conclude that $x' + y' = x' + p$ and $x' + y' = y' + p$. Multiplying the first of these by x gives $xy' = xp$, and multiplying the second by y gives $x'y = yp$. Adding, we obtain $xy' + x'y = xp + yp = (x + y)p$. From the triangle 0, x, y, whose sides by Lemma 3 are $0*x = x$, $0*y = y$, and $x*y = p$, we obtain $x + y \geq p$ by the triangle inequality. Hence $(x + y)p = p$, and $xy' + x'y = p = x*y$, completing the proof.

We now extend Theorem 2.2 by relaxing some of the group requirements.

Definition: A semi-group is a system of elements together with an associative binary operation.

Theorem 2.3: The only metric semi-group operation in a Boolean algebra is the symmetric difference.

Proof: The only group property used in the proof of Theorem 2.2 was the associative law.

Definition: A binary operation $*$ is weakly associative if $a*(a*b) = (a*a)*b$.

Theorem 2.4: The only metric weakly associative operation in a Boolean algebra is the symmetric difference.

Proof: In the proof of Theorem 2.2, the associative law was used only to show that $a = b*c$ implies $b = a*c$ and $c = a*b$, i.e. in the proof of Lemma 2. We will show that these relations follow from the weak associative law and the fact that the symmetric difference is a metric operation. Then the proof of Theorem 2.1 suffices as a proof of this theorem. The fact that $0*a = a$ follows from M1 and the weak associative law, for $a*(a*0) = (a*a)*0 = 0*0 = 0$ implies $a = a*0$. Now let $a = b*c$, $x = a*b$, and $y = a*c$. Then

(2.3) $$x = b*a = b*(b*c) = (b*b)*c = 0*c = c$$

(2.4) $$y = c*a = c*(c*b) = (c*c)*b = 0*b = b.$$

Hence $a = b*c$ implies $b = a*c$ and $c = a*b$.

Theorems 2.3 and 2.4 were generalizations of Theorem 2.2 obtained through relaxations of the group postulates. In Theorem 2.5, which follows shortly, the associativity is abandoned.

Definition: A quasi-group is a system consisting of a set of elements, together with a binary operation which satisfies the law of unique solution, i.e. if $a = b*c$ and two of these are known, the third is uniquely determined. A loop is a quasi-group with a two-sided identity element.

Definition: The Ptolemaic inequality holds for a quadrilateral if the three products (meets) of opposite sides satisfy the triangle inequality (M3).

Theorem 2.5: The only metric loop operation in a Boolean algebra is the symmetric difference.

Before proceeding with the proof, some lemmas will be established.

<u>Lemma 1</u>: The loop identity is 0.

<u>Proof</u>: Let e denote the loop identity. Since $e*e = e$ by definition and $e*e = 0$ by M1, we have $e = 0$.

<u>Lemma 2</u>: $a*I = a'$ and $a*a' = I$.

<u>Proof</u>: By the law of unique solution there exists y such that $a*y = a'$. The sides of triangle 0, a, y are $a*y = a'$, $0*a = a$, and $0*y = y$. The triangle inequality implies

(2.5) $\quad a + y \geq a' \quad$ and $\quad a' + y \geq a.$

Thus

(2.6) $\quad aa' + a'y \geq a'$ and $\quad aa' + ay \geq a,$

or

(2.7) $\quad a'y \geq a' \quad$ and $ay \geq a.$

But these imply that $y \geq a'$ and $y \geq a$. Hence $y = I$, or $a*I = a'$. Consider the triangle 0, a, a' whose sides are $0*a = a$, $0*a' = a'$ and $a*a'$. Again the triangle inequality implies

(2.8) $\quad a + (a*a') \geq a' \quad$ and $a' + (a*a') \geq a.$

Multiplying the first of these by a' and the second by a gives

(2.9) $\quad a'(a*a') \geq a' \quad$ and $\quad a(a*a') \geq a.$

From these we conclude that $a*a' \geq a'$ and $a*a' \geq a$; hence $a*a' = I$.

<u>Lemma 3</u>: The Ptolemaic inequality holds in any quadrilateral 0, I, a, b.

<u>Proof</u>: In the quadrilateral 0, I, a, b, the side $0*a$ is $I*b$, the side $0*b$ is opposite $I*a$, and the side $0*I$ is

opposite $a*b$. We will show only that

$$(2.10) \qquad (0*a)(I*b) + (0*b)(I*a) \geq (0*I)(a*b)$$

or

$$(2.11) \qquad ab' + a'b \geq I \cdot (a*b) = a*b;$$

the proofs for the other two cases are similar. The triangle I, a, b has sides $I*a = a'$, $I*b = b'$ and $a*b$ by Lemma 2. The triangle inequality gives

$$(2.12) \qquad a' + b' \geq a*b.$$

By Lemma 1, the sides of the triangle 0, a, b are $0*a = a$, $0*b = b$ and $a*b$. The triangle inequality here yields

$$(2.13) \qquad a + b \geq a*b.$$

Hence

$$(2.14) \qquad (a + b)(a' + b') \geq a*b$$

or

$$(2.15) \qquad ab' + a'b \geq a*b,$$

which is what we set out to show.

Proof of Theorem 2.5: Let $a*b = x$. We know from Lemma 3 that $ab' + a'b \geq x$. We will complete the proof by showing $ab' + a'b \leq x$. Applying the Ptolemaic inequality to the quadrilateral 0, I, a, b', we have

$$(2.16) \qquad (0*a)(I*b') + (0*b')(I*a) \geq (0*I)(a*b')$$

or

$$(2.17) \qquad ab + a'b' \geq I \cdot (a*b') = a*b'.$$

Since $ab' + a'b \geq x$, we obtain

$$(2.18) \qquad (ab' + a'b)(ab + a'b') \geq x \cdot (a*b').$$

But $(ab' + a'b)(ab + a'b') = 0$, hence

$$(2.19) \qquad x \cdot (a*b') = 0.$$

The triangle a, b, b' has sides $a*b = x$, $a*b'$, and $b*b' = I$ by Lemma 2. Using the triangle inequality, we get

(2.20) $\quad x + (a*b') = I.$

Thus x is the complement of $a*b'$, i.e.

(2.21) $\quad x' = a*b'.$

A similar argument shows that

(2.22) $\quad x' = a'*b.$

Using the identity $u*v \leq u + v$, we have

(2.23) $\quad x' \leq a + b'$ and $x' \leq a' + b.$

Hence

(2.24) $\quad x' \leq (a + b')(a' + b) = ab + a'b'.$

By DeMorgan's laws, we get

(2.25) $\quad x \geq ab' + a'b.$

This, together with the earlier result $x \leq ab' + a'b$, shows that

(2.26) $\quad x = ab' + a'b$

and completes the proof of Theorem 2.5.

It might be conjectured that a metric quasi-group operation is a Boolean algebra is necessarily the symmetric difference. The following example shows that this is not the case. In the Boolean algebra of four elements 0, a, a' and I define "distances" as shown in Table 1.

Table 1

$\circledast$	0	a	a'	I
0	0	a'	a	I
a	a'	0	I	a
a'	a	I	0	a'
I	I	a	a'	0

Since each element appears once and only once in each row and column of Table 1, the law of unique solution holds. That M1 holds is shown by the fact that the elements on the main diagonal, and only those elements, are 0. The symmetry about the main diagonal implies that M2 holds. It is easily seen that the sides of any non-degenerate triangle are a, a' and I, hence M3 holds. This shows that $\circledast$ is indeed a metric quasi-group operation. However, $0 \circledast a = a'$, while $0 * a = 0a' + 0'a = a$. Hence $\circledast$ is not the symmetric difference.

Bernstein [1,2] characterized the possible group operations in a Boolean algebra among the class of Boolean operations. The author is indebted to Professor B. M. Stewart for pertinent observations which led to the following theorem. This theorem is similar to those in [1].

<u>Definition</u>: An operation $*$ is a Boolean operation in a Boolean algebra if

(2.27) $\quad x*y = Axy + Bxy' + Cx'y + Dx'y',$

where A, B, C and D are fixed elements of the Boolean algebra.

<u>Theorem 2.6</u>: Any Boolean group operation in a Boolean algebra is an abelian group operation, and is of the form

(2.28) $\quad x*y = e(xy + x'y') + e'(xy' + x'y)$

where e is the group identity.

<u>Proof</u>: The proof consists of evaluating the "constants" A, B, C and D under the assumption that $*$ is a group operation. Repeatedly using (2.27), we write

(2.29) $\quad 0*D = A0D + B0D' + CID + DID' = CD,$

(2.30) $\quad 0*C' = A0C' + B0C + CIC' + DIC = DC.$

By the law of unique solution, this implies $D = C'$. Now

(2.31) $$D*0 = AD0 + BDI + CD'0 + DD'I = BD,$$

(2.32) $$B'*0 = AB'0 + BB'I + CB0 + DBI = DB.$$

Again by the law of unique solution, $D = B'$. Next

(2.33) $$\begin{aligned} A*D &= AAD + BAD' + CA'D + DA'D' \\ &= AD + AB \\ &= AD + AD' \\ &= A \end{aligned}$$

implies that $D = e$ by definition of the group identity. Hence (2.27) can be written

(2.34) $$x*y = Axy + e'xy' + e'x'y + ex'y'.$$

Now

(2.35) $$e = e*e = Aee + e'ee' + e'e'e + ee'e' = Ae.$$

Since $e = B'$, this gives $B' = AB'$. Next we observe that

(2.36) $$A'*B = AA'B + e'A'B' + e'AB + eAB' = AB + AB' = A,$$

(2.37) $$\begin{aligned} B*B &= ABB + e'BB' + e'B'B + eB'B' \\ &= AB + B' \\ &= AB + AB' \\ &= A. \end{aligned}$$

By the law of unique solution, we get $A' = B$. Collecting results, we can write

(2.38) $$e = D = B' = C' = A \quad \text{and} \quad e' = D' = B = C = A'.$$

Hence (2.27) becomes finally

(2.39) $$\begin{aligned} x*y &= exy + e'xy' + e'x'y + ex'y' \\ &= e(xy + x'y') + e'(xy' + x'y). \end{aligned}$$

The fact that $*$ is an abelian operation follows from the symmetry in x and y of the right side of (2.39).

<u>Corollary 1</u>: In a Boolean algebra, the only Boolean group operation with 0 as the group identity is the symmetric difference.

<u>Corollary 2</u>: In a Boolean algebra, the only Boolean group operation such that $0*0 = 0$ is the symmetric difference.

<u>Proof</u>: Using (2.28), we write

$$0 = 0*0 = e(0 \cdot 0 + II) + e'(0I + I0) = e, \tag{2.40}$$

and Corollary 2 follows from Corollary 1.

We notice that, in the proof of Theorem 2.6, no use was made of the associative law. Thus Theorem 2.6 may be generalized to get

<u>Theorem 2.7</u>: Any Boolean loop operation in a Boolean algebra is an abelian group operation, and is of the form

$$x*y = e(xy + x'y') + e'(xy' + x'y), \tag{2.41}$$

where e is the loop identity.

<u>Proof</u>: Exactly as in the proof of Theorem 2.6, it can be shown that $*$ is an abelian operation of the form cited in the theorem statement. We will now show that the associative law holds, in particular that

$$z*(x*y) = xyz + x'y'z + x'yz' + xy'z' \tag{2.42}$$

and

$$(z*x)*y = xyz + x'y'z + x'yz' + xy'z'. \tag{2.43}$$

In what follows, DeMorgan's laws are used repeatedly.

$$
\begin{aligned}
(2.44) \qquad z*(x*y) &= e\left[z(x*y) + z'(x*y)'\right] + e'\left[z(x*y)' + z'(x*y)\right] \\
&= (ez + e'z')(x*y) + (ez' + e'z)(x*y)' \\
&= (ez + e'z')\left[e(xy + x'y') + e'(xy' + x'y)\right] \\
&\quad + (ez' + e'z)\left[e(xy + x'y') + e'(xy' + x'y)\right]' \\
&= ez(xy + x'y') + e'z'(xy' + x'y) \\
&\quad + (ez' + e'z)\left[e(xy + x'y')\right]'\left[e'(xy' + x'y)\right]' \\
&= ez(xy + x'y') + e'z'(xy' + x'y) \\
&\quad + (ez' + e'z)\left[e' + (xy + x'y')'\right]\left[e + (xy' + x'y)'\right] \\
&= ez(xy + x'y') + e'z'(xy' + x'y) \\
&\quad + (ez' + e'z)\left[e'(xy' + x'y)' + e(xy + x'y')'\right. \\
&\quad \left. + (xy + x'y')'(xy' + x'y)'\right] \\
&= ez(xy + x'y') + e'z'(xy' + x'y) \\
&\quad + e'z(xy' + x'y)' + ez'(xy + x'y')' \\
&\quad + (ez' + e'z)(xy)'(x'y')'(xy')'(x'y)' \\
&= ez(xy + x'y') + e'z'(xy' + x'y) \\
&\quad + e'z(xy')'(x'y)' + ez'(xy)'(x'y')' \\
&\quad + (ez' + e'z)(x' + y')(x + y)(x' + y)(x + y') \\
&= ez(xy + x'y') + e'z'(xy' + x'y) \\
&\quad + e'z(x' + y)(x + y') + ez'(x' + y')(x + y) \\
&\quad + (ez' + e'z)(x'y + xy')(x'y' + xy) \\
&= ez(xy + x'y') + e'z'(xy' + x'y) \\
&\quad + e'z(xy + x'y') + ez'(xy' + x'y) \\
&= exyz + ex'y'z + e'xy'z' + e'x'yz' + e'xyz \\
&\quad + e'x'y'z + exy'z' + ex'yz'.
\end{aligned}
$$

Collecting terms, we obtain

$$
(2.45) \qquad z*(x*y) = xyz + x'y'z + x'yz' + xy'z'.
$$

To find $(z*x)*y$, we use the fact that $*$ is abelian to write

$$(z*x)*y = y*(z*x). \tag{2.46}$$

Replacing z by y, x by z, and y by x in (2.45), we get

$$(z*x)*y = zxy + z'x'y + z'xy' + zx'y'. \tag{2.47}$$

Hence

$$(z*x)*y = xyz + x'y'z + x'yz' + xy'z'. \tag{2.48}$$

The right sides of (2.45) and (2.48) are identical, which proves that the associative law holds. Since an associative loop is a group, the theorem follows.

Corollary 1: The only Boolean loop operation with 0 as the loop identity in a Boolean algebra is the symmetric difference.

Corollary 2: The only Boolean loop operation such that $0*0 = 0$ in a Boolean algebra is the symmetric difference.

Proof: Since

$$x*y = e(xy + x'y') + e'(xy' + x'y) \tag{2.49}$$

we can write that

$$0 = 0*0 = e(00 + II) + e'(0I + I0) = e. \tag{2.50}$$

Then Corollary 2 follows from Corollary 1.

It is interesting that the requirement that $*$ be a Boolean operation allowed us to remove the associative law from the assumptions needed to characterize the symmetric difference among the class of Boolean operations. It will be shown next that a similar phenomenon occurs with respect to the triangle inequality.

Definition: A binary operation is called semi-metric if it satisfies M1 and M2.

Theorem 2.8: The only Boolean semi-metric operation in a Boolean algebra is the symmetric difference.

Proof: According to Bernstein [1], a Boolean operation has the form

(2.51) $x{*}y = (I{*}I)xy + (I{*}0)\ xy' + (0{*}I)\ x'y + (0{*}0)x'y'.$

Thus

(2.52) $I{*}I = 0{*}0 = 0$

by M1, and

(2.53) $0{*}I = I{*}0$

by M2. Let $0{*}I = z$. Then (2.51) yields

(2.54) $I{*}z = zIz' + zI'z = 0 + 0 = 0,$

and $z = I$ by M1. Thus

(2.55) $x{*}y = xy' + x'y.$

Frink [7] has characterized the symmetric difference as the only Boolean group operation over which the meet distributes. In what follows, however, we will not restrict ourselves to Boolean operations.

Theorem 2.9: The only semi-metric group operation in a Boolean algebra over which the meet distributes is the symmetric difference.

Proof: The group identity is 0. Let a, b and c be the sides of the triangle l, m, n. Using the associative law, M1 and M2, it is seen that

(2.55)
$$\begin{aligned} a{*}b &= (l{*}m){*}(m{*}n) \\ &= l{*}[m{*}(m{*}n)] \\ &= l{*}[(m{*}m){*}n] \\ &= l{*}(0{*}n) \\ &= l{*}n \\ &= c \end{aligned}$$

Similarly it can be shown that

(2.56) $a{*}c = b$

and

(2.57) $b{*}c = a.$

Now, using (2.55) - (2.57) and the distributivity assumption,

(2.58) $$\left[(a{*}b) + (b{*}c)\right](a{*}c) = (c + a)(a{*}c) = \left[(a + c)a\right]{*}\left[(a + c)c\right] = a{*}c.$$

Recall that the lattice relation $(x + y)z = z$ implies $x + y \geq z$. Hence (2.58) yields

(2.59) $(a{*}b) + (b{*}c) \geq a{*}c.$

Similarly it can be shown that

(2.60) $(a{*}b) + (a{*}c) \geq b{*}c$

(2.61) $(b{*}c) + (a{*}c) \geq a{*}b.$

Thus M3 holds, and $*$ is a metric group operation. Then $*$ is the symmetric difference by Theorem 2.2.

It might be conjectured that the meet necessarily distributes over every semi-metric group operation in a Boolean algebra. That this is not the case is shown by the following example. In the Boolean algebra of eight elements, define an operation $\circledast$ by the following table:

Table 2

$\circledast$	0	a	b	c	a'	b'	c'	I
0	0	a	b	c	a'	b'	c'	I
a	a	0	b'	c'	I	b	c	a'
b	b	b'	0	a'	c	a	I	c'
c	c	c'	a'	0	b	I	a	b'
a'	a'	I	c	b	0	c'	b'	a
b'	b'	b	a	I	c'	0	a'	c
c'	c'	c	I	a	b'	a'	0	b
I	I	a'	c'	b'	a	c	b	0

Now 0 appears only on the main diagonal, and the table is symmetric about the main diagonal, so M1 and M2 hold. Clearly 0 is the group identity, and inverses are unique (each element is self-inverse). It has been verified that the associative law holds. Thus $\circledast$ is a semi-metric group operation. However

(2.62) $\quad a'(c \circledast a) = a'c' = b,$

while

(2.63) $\quad (a'c) * (a'a) = c \circledast 0 = c,$

which shows that the meet does not distribute over $\circledast$.

<u>Theorem 2.10</u>: The only semi-metric semi-group operation in a Boolean algebra over which the meet distributes is the symmetric difference.

<u>Proof</u>: M1 guarantees that $a*a = 0$. Thus if 0 is an identity element, then each element of the Boolean algebra is its own inverse. But the associative law and M1 give us

(2.66) $\quad (0*a)*a = 0*(a*a) = 0*0 = 0$

whence $0*a = a$, again by M1, and 0 is an identity element. If e is any element such that $e*a = a$ holds for all a, then $e*e = e$. But $e*e = 0$ by M1, so 0 is a unique identity. Thus $*$ is a group operation and Theorem 2.10 now follows from Theorem 2.9.

<u>Theorem 2.11</u>: In a Boolean algebra, the only semi-metric weakly associative operation over which the meet distributes is the symmetric difference.

<u>Proof</u>: Using M1 and weak associativity,

(2.64) $\quad (0*a)*a = 0*(a*a) = 0*0 = 0$

implies

$$(2.65) \qquad 0 * a = a.$$

By the distributivity assumption and M2

$$(2.66) \qquad ab'(a*b) = (ab'a)*(ab'b) = (ab')*0 = ab'$$

yields

$$(2.67) \qquad ab' \leq a*b.$$

Similarly

$$(2.68) \qquad a'b \leq a*b,$$

hence

$$(2.69) \qquad ab' + a'b \leq a*b.$$

Now

$$(2.70) \qquad ab(a*b) = (aba)*(abb) = (ab)*(ab) = 0,$$

and therefore

$$(2.71) \qquad \left[ab(a*b)\right]' = I.$$

By DeMorgan's laws

$$(2.72) \qquad (ab)' + (a*b)' = I.$$

Then

$$(2.73) \qquad (ab)\left[(ab)' + (a*b)'\right] = (ab)$$

gives

$$(2.74) \qquad (ab)(a*b)' = (ab)$$

which implies

$$(2.75) \qquad ab \leq (a*b)'.$$

Next we observe that

$$(2.76) \qquad (a + b)(a*b) = \left[(a + b)a\right] * \left[(a + b)b\right] = a*b,$$

or

$$(2.77) \qquad a + b \geq a*b.$$

Hence

$$(2.78) \qquad (a + b)' \leq (a*b)'$$

or

$$(2.79) \qquad a'b' \leq (a*b)'.$$

Thus (2.75) and (2.79) yield

$$(2.80) \qquad ab + a'b' \leq (a*b)'$$

or

$$(2.81) \qquad (ab + a'b')' \geq a*b.$$

Again applying DeMorgan's laws, we get

$$(2.82) \qquad ab' + a'b \geq a*b.$$

But (2.69) and (2.82) together imply

$$(2.83) \qquad ab' + a'b = a*b$$

and the theorem is proved.

<u>Corollary</u>: In a Boolean algebra, the only semi-metric operation $*$ such that $0*a = a$ for every a and such that the meet distributes over $*$ is the symmetric difference.

<u>Proof</u>: In the proof of Theorem 2.11 the weak associativity property was used only to show that $0*a = a$ for every a in the Boolean algebra.

<u>Definition</u>: As usual, let $*$ denote the symmetric difference. A binary operation o is called <u>quasi-analytical</u> (Marczewski [10]) when

$$(2.84) \qquad (aob)*(cod) \leq (a*c) + (b*d)$$

for all quadruples a, b, c, d of a Boolean algebra.

<u>Theorem 2.12</u> (Marczewski): The only quasi-analytical group operation in a Boolean algebra with 0 as the group identity is the symmetric difference.

Proof: Marczewski showed in his proof that the operation o is Boolean. It then follows from Corollary 1 to Theorem 2.6 or from Bernstein's results [1] that o is a metric operation, whereupon the theorem follows from Theorem 2.2. Following is an independent proof of Marczewski's theorem. First we note that $a = a^{-1}$, for

(2.85) $$a = a*0 = (ao0)*(aoa^{-1}) \leq (a*a) + (0*a^{-1}) = a^{-1}$$

and

(2.86) $$a^{-1} = a^{-1}*0 = (a^{-1}o0)*(a^{-1}oa) \leq (a^{-1}*a^{-1}) + (0*a) = a$$

give us respectively $a \leq a^{-1}$ and $a^{-1} \leq a$.

Since $a = a^{-1}$, we have $aoa = 0$. Let $aob = 0$. But $aoa = 0$, hence $a = b$ by the law of unique solution and M1 holds. To prove M2, we write

(2.87) $$\begin{aligned}(aob)o(boa) &= ao\left[bo(boa)\right] \\ &= ao\left[(bob)oa\right] \\ &= ao(0oa) \\ &= aoa \\ &= 0.\end{aligned}$$

Thus $aob = boa$ by M1.

Let a, b and c be sides of a triangle l, m, n, with $a = lom$, $b = mon$ and $c = lon$. Then

(2.88) $$aob = (lom)o(mon) = lon = c$$

(2.89) $$aoc = (lom)o(lon) = (mol)o(lon) = mon = b$$

(2.90) $$boc = (mon)o(lon) = (mon)o(nol) = mol = a$$

Now

(2.91) $c = a \circ b = (0 \circ 0) * (a \circ b) \leq (0 * a) + (0 * b) = a + b$

(2.92) $b = a \circ c = (0 \circ 0) * (a \circ c) \leq (0 * a) + (0 * c) = a + c$

(2.93) $a = b \circ c = (0 \circ 0) * (b \circ c) \leq (0 * b) + (0 * c) = b + c$

proves M3. Hence o is the symmetric difference by Theorem 2.2.

Section 3. Structure of Stone Algebras

Definition: A Brouwerian algebra is a lattice L in which for every pair of elements a, b there exists an element x such that

(3.1) $b + x \geq a$

and

(3.2) $b + y \geq a$ implies $y \geq x$.

In other words x is the "smallest" element such that $b + x \geq a$. The element x is the difference of a and b, and is denoted by $a - b$. It may be verified (see McKinsey and Tarski, [9]) that

(3.3) $a - b \leq c$ if and only if $a \leq b + c$.

Examples of Brouwerian algebras are numerous; among the Brouwerian algebras are all Boolean algebras, all chains with 0, all finite distributive lattices, all distributive lattices in which descending chains are finite, and all complete and completely distributive lattices.

Theorem 3.1: A Brouwerian algebra is a distributive lattice.

Proof: We will show that

(3.4) $a + y_1y_2 = (a + y_1)(a + y_2)$.

Let

(3.5) $\quad b = (a + y_1)(a + y_2).$

Then

(3.6) $\quad a + y_1 \geq b$ and $a + y_2 \geq b$

implies

(3.7) $\quad y_1 \geq b - a$ and $y_2 \geq b - a$

by (3.3). This gives

(3.8) $\quad y_1 y_2 \geq b - a.$

We can now write

(3.9) $\quad a + y_1 y_2 \geq a + (b - a) \geq b,$

where the last inequality follows from (3.1).

Having

(3.10) $\quad a + y_1 y_2 \geq (a + y_1)(a + y_2),$

it remains to show that the reverse inequality also holds. But in any lattice

(3.11) $\quad a \leq a + y_1$, and $y_1 y_2 \leq a + y_1$ implies

(3.12) $\quad a + y_1 y_2 \leq a + y_1.$

Similarly $a + y_1 y_2 \leq a + y_2$. Hence

(3.13) $\quad a + y_1 y_2 \leq (a + y_1)(a + y_2).$

This shows that (3.4) holds. Also valid is the dual of (3.4), i.e. the expression

(3.14) $\quad a(y_1 + y_2) = ay_1 + ay_2$

obtained from (3.4) by interchanging "+" and ".".

Definition: If the Brouwerian algebra has a greatest element I, the element I - a is the Brouwerian complement of a, and is denoted by $\neg a$. Similarly $I - \neg a = \neg\neg a$, $I - \neg\neg a = \neg\neg\neg a$, and so on.

In what follows, we restrict ourselves to Brouwerian algebras having an 0 and an I.

It is shown in [9] and [13] that

$$
\begin{aligned}
&\text{(a)} \quad a \le b \text{ implies } \neg a \ge \neg b \\
&\text{(b)} \quad \neg\neg a \le a \\
(3.15) \qquad &\text{(c)} \quad \neg\neg\neg a = \neg a \\
&\text{(d)} \quad \neg(ab) = \neg a + \neg b \\
&\text{(e)} \quad \neg(a + b) = \neg\neg(\neg a \neg b).
\end{aligned}
$$

M. H. Stone has asked the question: "What is the most general Brouwerian algebra B in which $\neg a \neg\neg a = 0$ holds for every element a in B?". This problem appears in its dual form as "Problem 70" of Birkhoff [3]. A simple example of a Brouwerian algebra in which this property does not hold is the lattice whose five elements are 0, ab, a, b, a + b = I, for in this lattice $\neg a = b$, $\neg b = \neg\neg a = a$, but $\neg a \neg\neg a = ba \ne 0$. On the other hand, this property holds in every Boolean algebra, and in every chain with an 0 and an I.

Definition: A Stone algebra is a Brouwerian algebra in which $\neg a \neg\neg a = 0$ identically.

Let B denote a Brouwerian algebra with 0 and I, and let X denote the set of elements of B satisfying $\neg x \neg\neg x = 0$. If x and y are in X, then

$$
\begin{aligned}
(3.16) \qquad \neg(x + y)\neg\neg(x + y) &= \neg\neg(\neg x \neg y)(\neg\neg x + \neg\neg y) \\
&\le \neg x \neg y(\neg\neg x + \neg\neg y) \\
&= \neg x \neg y \neg\neg x + \neg x \neg y \neg\neg y \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

and

$$(3.17)\qquad \neg(xy)\neg\neg(xy) = (\neg x + \neg y)\neg\neg(xy)$$
$$= (\neg x + \neg y)\neg\neg(\neg\neg x\neg\neg y)$$
$$\leq (\neg x + \neg y)\neg\neg x\neg\neg y$$
$$= \neg x\neg\neg x\neg\neg y + \neg y\neg\neg x\neg\neg y$$
$$= 0 + 0 = 0$$

show that X is a sub-lattice of B, since X is partially ordered by the partially ordering of B. Further, using the relationship $\neg(a - b) = \neg a + \neg\neg b$[1], we see that

$$(3.18)\qquad \neg(x - y)\neg\neg(x - y) = (\neg x + \neg\neg y)\left[\neg\neg(\neg\neg x\neg y)\right]$$
$$\leq (\neg x + \neg\neg y)\neg\neg x\neg y$$
$$= \neg x\neg\neg x\neg y + \neg\neg y\neg\neg x\neg y$$
$$= 0 + 0 = 0$$

That is, if x and y are in X, then so are x + y, xy and x - y. This proves the

<u>Theorem 3.2</u>: In a Brouwerian algebra B with 0 and I, the collection of elements x satisfying $\neg x\neg\neg x = 0$ is a Stone sub-algebra.

Birkhoff [3] has shown that in any Brouwerian algebra B the subset R of elements satisfying $\neg\neg r = \neg r$ is a Boolean algebra under the operations a + b and $a \odot b = \neg\neg(ab)$. In a Stone algebra, however, the subset R is a Boolean sub-algebra of B, i.e. R is a Boolean algebra under the operations a + b and ab which hold in B. This is, in fact, a characterization of Stone algebras, as is shown by the following theorem.

---

[1] This result is shown in [8].

Theorem 3.3: A Brouwerian algebra B is a Stone algebra if and only if R is a Boolean sub-algebra of B.

Before proceeding with the proof of this theorem, some lemmas will be established which not only facilitate the proof but also add some insight into the structure of Stone algebras. Let Q denote the set of all elements of B satisfying $a\urcorner a = 0$.

Lemma 1: Q is a subset of R.

Proof: If a is in Q, then $a\urcorner a = 0$ implies

(3.19) $$\urcorner\urcorner a = \urcorner\urcorner a + a\urcorner a = (\urcorner\urcorner a + a)(\urcorner\urcorner a + \urcorner a) = a\,I = a,$$

and a is in R.

Lemma 2: Q is a sub-lattice of B.

Proof: Let a and b be in Q. Then

(3.20) $$\begin{aligned}(a + b)\urcorner(a + b) &\leq (a + b)(\urcorner a\urcorner b)\\ &= a\urcorner a\urcorner b + b\urcorner a\urcorner b\\ &= 0 + 0\\ &= 0\end{aligned}$$

and

(3.21) $$(ab)\urcorner(ab) = ab(\urcorner a + \urcorner b) = ab\urcorner a + ab\urcorner b = 0 + 0 = 0$$

show that a + b and ab are in Q.

Lemma 3: B is a Stone algebra if and only if Q = R.

Proof: Let Q = R. Recalling that $\urcorner a = \urcorner\urcorner\urcorner a$, it follows that $\urcorner a$ is an element of R, for every a in B. Since Q = R, we have that $\urcorner a\urcorner\urcorner a = 0$, and B is a Stone algebra. Conversely, let $\urcorner x\urcorner\urcorner x = 0$ hold for every x in B. If x is in R, then $\urcorner\urcorner x = x$ implies $0 = x\urcorner x$. Hence R is a subset of Q. Using Lemma 1 we conclude that R = Q.

<u>Proof of Theorem 3.3</u>: Let B be a Stone algebra. Then $R = Q$ by Lemma 3, hence if x is in R then $x\neg x = 0$. Since $x + \neg x = I$ identically, it is seen that $\neg x$ is a Boolean complement of x, and is unique since B is a distributive lattice. By Lemma 2, $R = Q$ is a sub-lattice of B. Hence R is itself a complemented distributive lattice under the operations of B, i.e. R is a Boolean sub-algebra of B. Conversely, assume that R is a Boolean sub-algebra of B. Then if x is in R, there exists an element x' in R satisfying $x + x' = I$ and $xx' = 0$. Since $x + \neg x = I$, we have $(x + \neg x)(x + x') = x + x'\neg x = I$. This, together with $x(x'\neg x) = 0$, implies that $x' = x'\neg x$ since B is a distributive lattice. Thus $x' \leq \neg x$. But x' satisfies $x + x' = I$, hence $\neg x \leq x'$ by definition of the operation $\neg$. This shows that $x' = \neg x$, and hence $xx' = x\neg x = 0$. From this it follows that R is contained in Q. Applying Lemma 1, we have that $R = Q$. Then B is a Stone algebra by Lemma 3, and the proof is complete.

This theorem suggests that Stone algebras may, in a sense, be built up from Boolean algebras. This is indeed the case, and in the remainder of this section we present a characterization theorem which gives some insight into the general structure of Stone algebras.

<u>Definition</u>: An ideal J in a lattice K is a subset of K having the properties

(3.22) x and y in J implies $x + y$ is in J.

(3.33) x in J and $y \leq x$ implies y is in J.

Let L be a distributive lattice with 0 and I, R be a Boolean sub-algebra of L containing 0 and I, and T be an ideal in L having the properties

(3.24) (a) The only element in L common to both R and T is 0.

(b) T is a Brouwerian sub-algebra of L.

<u>Remark</u>: $t_1 + t_2 = I$ holds for no pair of elements t_1, t_2 of T.

<u>Proof</u>: If $t_1 + t_2 = I$ for some pair of elements t_1, t_2 of T, then the fact that T is an ideal would imply that I is in T. This is impossible by (3.24a).

<u>Remark</u>: The relationship $t \geq r \neq 0$ holds for no elements t in T and r in R.

<u>Proof</u>: Assume $t \geq r$. Since T is an ideal, it follows that r is in T. Then, by (3.24), $r = 0$.

Let B denote the direct sum $R \oplus T$ of R and T, i.e. the set of elements of L of the form $r + t$, where r is in R and t is in T.

<u>Theorem 3.4</u>: B is a lattice.

<u>Proof</u>: Let $r_1 + t_1$ and $r_2 + t_2$ be elements of B. Then

$$(3.25) \qquad (r_1 + t_1) + (r_2 + t_2) = (r_1 + r_2) + (t_1 + t_2) = r_3 + t_3,$$

where $r_3 = r_1 + r_2$ is in R since R is a Boolean sub-algebra of L and $t_3 = t_1 + t_2$ is in T since T is an ideal of L.

Since L is a distributive lattice, we observe that

$$(3.26) \qquad (r_1 + t_1)(r_2 + t_2) = r_1r_2 + (r_1t_2 + r_2t_1 + t_1t_2)$$
$$= r_3 + t_3,$$

where $r_3 = r_1r_2$ is in R since R is a Boolean sub-algebra of L and $t_3 = r_1t_2 + r_2t_1 + t_1t_2$ is in T since T is an ideal of L.

<u>Lemma 1</u>: $r - (r_1 + t_1) = rr_1'$.

<u>Proof</u>: Using the fact that L is a distributive lattice, we write

$$(3.27) \qquad (r_1 + t_1) + rr_1' = r_1 + rr_1' + t_1$$
$$= (r_1 + r)(r_1 + r_1') + t_1$$
$$= (r_1 + r)\ I + t_1$$
$$= r_1 + r + t_1$$
$$\geq r.$$

Thus rr_1' satisfies the first part (3.1) of the definition of the difference of r and $(r_1 + t_1)$. We show next that if $(r_1 + t_1) + x \geq r$ then $x \geq r_1r_1'$. Let x be any element of B, say $x = r_2 + t_2$, and assume

$$(3.28) \qquad (r_1 + t_1) + (r_2 + t_2) \geq r.$$

Then

$$(3.29) \qquad (r_1 + r_2) + (t_1 + t_2) \geq r.$$

Since R is a Boolean sub-algebra of L, there exists an element $(r_1 + r_2)'$ in R such that $(r_1 + r_2)(r_1 + r_2)' = 0$.

Hence

$$(3.30) \qquad (r_1 + r_2)(r_1 + r_2)' + (t_1 + t_2)(r_1 + r_2)' \geq r(r_1 + r_2)'$$

or

$$(3.31) \qquad (t_1 + t_2)(r_1 + r_2)' \geq r(r_1 + r_2)'.$$

The left side of (3.25) is in T, since T is an ideal of L, and the right side is in R since R is a sub-algebra of B. But in an earlier remark we showed that $t \geq r \neq 0$ is impossible. Hence

(3.32) $\quad r(r_1 + r_2)' = 0.$

Since R is a Boolean algebra, DeMorgan's laws hold. Hence

(3.33) $\quad r' + r_1 + r_2 = I.$

Multiplying both sides by rr_1', we get $(rr_1')r_2 = (rr_1')$ which in turn implies that

(3.34) $\quad r_2 \geq rr_1'.$

Hence

(3.35) $\quad r_2 + t_2 \geq r_2 \geq rr_1'$

and the proof of Lemma 1 is complete.

<u>Lemma 2</u>: $t - (r_1 + t_1) = t - \left[t(r_1 + t_1)\right].$

<u>Proof</u>: The right side exists since T is itself a Brouwerian algebra. Let $y = t - \left[t(r_1 + t_1)\right]$. Then

(3.36)
$$\begin{aligned}(r_1 + t_1) + y &= (r_1 + t_1) + \left[t - t(r_1 + t_1)\right] \\ &\geq t(r_1 + t_1) + \left[t - t(r_1 + t_1)\right] \\ &\geq t\end{aligned}$$

by definition of the difference operation. If x in B satisfies

(3.37) $\quad (r_1 + t_1) + x \geq t$

then

(3.38) $\quad t(r_1 + t_1) + tx \geq t.$

Appealing to the second part (3.2) of the definition of the difference operation, we see that

(3.39) $\quad tx \geq y,$

i.e. $y = t - t(r_1 + t_1)$ is by definition the least element satisfying $t(r_1 + t_1) + y \geq t$. Hence

(3.40) $\quad x \geq tx \geq y,$

which completes the proof of Lemma 2.

<u>Theorem 3.5</u>: B is a Brouwerian algebra.

<u>Proof</u>: Let $(r + t)$ and $(r_1 + t_1)$ be any two elements of B. We will show that $(r + t) - (r_1 + t_1)$ exists in B, in particular that

(3.41) $\quad (r + t) - (r_1 + t_1) = \left[r - (r_1 + t_1)\right] + \left[t - (r_1 + t_1)\right]$

Let

(3.42) $\quad x = r - (r_1 + t_1)$ and $y = t - (r_1 + t_1)$.

The existence of x and y is guaranteed by Lemmas 1 and 2. Further,

(3.43) $\quad x + (r_1 + t_1) = \left[r - (r_1 + t_1)\right] + (r_1 + t_1) \geq r$

and

(3.44) $\quad y + (r_1 + t_1) = \left[t - (r_1 + t_1)\right] + (r_1 + t_1) \geq t,$

by the definitions of x and y. Combining (3.43) and (3.44), we get

(3.45) $\quad x + y + (r_1 + t_1) \geq r + t.$

We will complete the proof by showing that if an element z of B satisfies $z + (r_1 + t_1) \geq r + t$ then $z \geq x + y$. Now

(3.46) $\quad z + (r_1 + t_1) \geq r + t \geq r \geq r - (r_1 + t_1) = x$

gives $z \geq x$ by definition of the difference operation, and similarly

(3.47) $\quad z + (r_1 + t_1) \geq r + t \geq t \geq t - (r_1 + t_1) = y$

yields $z \geq y$. Hence $z \geq x + y$ and the proof is complete.

<u>Theorem 3.6</u>: B is a Stone algebra.

<u>Proof</u>: Let $r_1 + t_1$ denote an arbitrary element of B. We will use the relationship

(3.48) $\quad r - (r_1 + t_1) = rr_1'$

of Lemma 1 to obtain $\urcorner(r_1 + t_1)$ and $\urcorner\urcorner(r_1 + t_1)$. By definition of the operation $\urcorner$, we have that

(3.49) $\quad \urcorner(r_1 + t_1) = I - (r_1 + t_1) = Ir_1' = r_1'$

and

(3.50) $\quad \urcorner\urcorner(r_1 + t_1) = I - \urcorner(r_1 + t_1) = I - r_1' = (r_1')' = r_1.$

Since R is itself a Boolean algebra, we have

(3.51) $\quad \urcorner(r_1 + t_1)\urcorner\urcorner(r_1 + t_1) = r_1'r_1 = 0.$

Thus B is a Stone algebra.

<u>Structure Theorem</u>: If B is the direct sum of R and T, where R is a Boolean sub-algebra (with least element 0 and greatest element I) of a distributive lattice L with 0 and I and T is an ideal of L such that

(a) the only element of L common to both R and T is 0

(b) T is a Brouwerian sub-algebra of L,

then B is a Stone algebra. Further, every Stone algebra may be so described.

The first part of the Structure Theorem has already been proved. The remainder of this section, except for some remarks at the end, will be used to prove the last part of the theorem.

<u>Definition</u>: Let T denote the set of elements of B satisfying $\urcorner x = I$ and, as before, let R denote the collection of

elements of B satisfying $\neg\neg x = x$.

Theorem 3.7: R is a Boolean sub-algebra of B.

Proof: This has already been proved in Theorem 3.2.

Theorem 3.8: T is an ideal of B.

Proof: Let a and b be elements of T. Then $\neg a = \neg b = I$, and

$$(3.52) \qquad \neg(a + b) = \neg\neg(\neg a \neg b) = \neg\neg I = I,$$

by (3.15e). Hence a + b is in T. If a is in T, and $c \leq a$, then $\neg c \geq \neg a = I$ by (3.15a). Thus $\neg c = I$, c is in T, and T is an ideal of B.

Theorem 3.9: The only element of B common to both R and T is 0.

Proof: Assume that an element a is in both R and T. Then $\neg a = I$, and $0 = a\neg a = aI = a$.

Theorem 3.10: T is a Brouwerian sub-algebra of B.

Proof: In the proof of Theorem 3.8 we showed that a + b is in T whenever a and b are in T. Now

$$(3.53) \qquad \neg(ab) = \neg a + \neg b = I + I = I$$

by (3.15d). Hence ab is in T, and T is a sub-lattice of B. It remains to prove that a - b is in T if a and b are in T. But $a - b \leq a$ by (3.3). Hence a - b is in T since T is an ideal of B.

Theorem 3.11: Every element b of B can be written in the form $b = r + t$, where r is in R and t is in T.

Proof: Since B is a distributive lattice, we may write

$$(3.54) \qquad \neg\neg b + b\neg b = (\neg\neg b + b)(\neg\neg b + \neg b).$$

But

(3.55) $\neg\neg b + b = b$

by (3.15b), and

(3.56) $\neg\neg b + \neg b = I$

by definition of the difference operation. Hence

(3.57) $\neg\neg b + b\neg b = (\neg\neg b + b)(\neg\neg b + \neg b) = bI = b$

holds for every element b in B. Now $\neg\neg b$ is in R, since

(3.58) $\neg\neg(\neg\neg b) = \neg(\neg\neg\neg b) = \neg(\neg b) = \neg\neg b$

by (3.15c). Further, $b\neg b$ is in T, for

(3.59) $\neg(b\neg b) = \neg b + \neg\neg b = I$

by (3.15d). Thus we may set $\neg\neg b = r$ and $b\neg b = t$, and the desired representation is obtained.

Theorems 3.7 through 3.11 complete the proof of the Structure Theorem.

More insight into the make-up of Stone algebras may be obtained by interpreting the preceding work in terms of set theory.

Definition: A ring of sets is a collection $\mathcal{C}$ of sets A, B, C,... such that if A and B belong to $\mathcal{C}$ so does the set sum $A \cup B$ and the set product $A \cap B$. A Boolean ring of sets is a ring of sets which contains with any member A the set complement A' of A.

Definition: Given two members A and B of a ring of sets $\mathcal{C}$, $A \underset{\mathcal{C}}{-} B$ denotes the smallest set of all sets X in $\mathcal{C}$ satisfying $B \cup X \supset A$ whenever this smallest set exists. $\mathcal{C}$ is a Brouwerian ring of sets if, for every pair of members A, B, $A \underset{\mathcal{C}}{-} B$ exists in $\mathcal{C}$.

An example of a Brouwerian ring of sets which is not a Boolean ring of sets is the collection $\mathcal{K}$ of all closed subsets of the plane. In $\mathcal{K}$, $A \underset{\mathcal{K}}{-} B$ is the intersection of A and the closure of the complement of B. The collection $\mathcal{O}$ of all open subsets of the plane is a ring of sets which is not a Brouwerian ring of sets. For, let A and B be open sets, neither containing the other, such that $A \cap B$ is not empty. The smallest set satisfying $B \cup X \supset A$ is $A \cap B'$, which is not in $\mathcal{O}$. It is easily seen that there is no smallest open set containing $A \cap B'$, hence $A \underset{\mathcal{O}}{-} B$ does not in general exist in $\mathcal{O}$.

Let $\mathcal{C}$ be a ring of sets containing the null set $\emptyset$ and a greatest set I, and let R be a Boolean sub-ring of $\mathcal{C}$ which also contains $\emptyset$ and I. Let $\mathcal{T}$ be a Brouwerian sub-ring of $\mathcal{C}$ which is an ideal and which has in common with $\mathcal{R}$ only the null set $\emptyset$. Finally, let $\mathcal{B} = \mathcal{R} \oplus \mathcal{T}$ denote the collection of all sets of the form $R \cup T$, where R is in $\mathcal{R}$ and T is in $\mathcal{T}$.

<u>Set-Theoretic Structure Theorem</u>: Every ring of sets $\mathcal{B} = \mathcal{R} \oplus \mathcal{T}$, where $\mathcal{R}$ and $\mathcal{T}$ satisfy the conditions laid down in the preceding paragraph, is a Stone algebra, and every Stone algebra can be so described.

<u>Proof</u>: The first part of the theorem follows from Theorem 3.5 and 3.6. Let B denote an arbitrary Stone algebra. Then $B = R \oplus T$ where R is the set of elements of B satisfying $\neg\neg r = r$ and T is the set of elements of B satisfying $\neg t = I$. Since any distributive lattice is

isomorphic with a ring of sets [cf. Birkhoff, p. 140] we know that B is isomorphic with a ring $\mathcal{B}$ of sets. The Boolean sub-ring $\mathcal{R}$ and the Brouwerian sub-ring $\mathcal{T}$ are the respective images, under the isomorphism, of R and T. A direct application of Theorems 3.7 through 3.11 can now be made to complete the proof of this theorem.

This set-theoretic representation furnishes a method of constructing Stone algebras. Let $\mathcal{U}$ denote an algebra of sets, $\mathcal{V}$ an arbitrary collection of elements of $\mathcal{U}$ together with their complements, and $\mathcal{R}$ the collection of elements of $\mathcal{V}$ together with their pairwise sums and products. If R_1 and R_2 are in $\mathcal{R}$, it is clear that $R_1 \cup R_2$ and $R_1 \cap R_2$ are also in $\mathcal{R}$. Further, if R is in $\mathcal{R}$, then R' is in $\mathcal{R}$. For, if R is in $\mathcal{V}$ then R' is in $\mathcal{V}$ which is contained in $\mathcal{R}$. If R is not in $\mathcal{V}$, then either $R = V_1 \cup V_2$ or $R = V_1 \cap V_2$, where V_1 and V_2 are elements of $\mathcal{V}$. In the first case $R' = V_1' \cap V_2'$ and in the second case $R' = V_1' \cup V_2'$. Since V_1' and V_2' are elements of $\mathcal{V}$, it follows that in either case R' is in $\mathcal{R}$. Hence $\mathcal{R}$ is a Boolean ring of sets.

From among the members of $\mathcal{U}$ not already in $\mathcal{R}$ choose a sub-collection $\mathcal{T}$ in such a way that:

(a) If T is in $\mathcal{T}$, the set complement T' is not in $\mathcal{T}$.

(b) If T_1 and T_2 are in $\mathcal{T}$, so is $T_1 \cup T_2$.

(c) If T is in $\mathcal{T}$, so are all sets of $\mathcal{U}$ contained in T.

(d) The collections $\mathcal{T}$ and $\mathcal{R}$ have in common only the null set.

It is seen from (b) and (c) that $\mathcal{T}$ is a ring of sets, and (a) implies that $\mathcal{T}$ is not a Boolean ring of sets. If T_1 and T_2 are in $\mathcal{T}$, the set $T_1 \underset{\mathcal{U}}{-} T_2$ exists in $\mathcal{U}$ since $\mathcal{U}$ is an algebra of sets. But it is clear that $T_1 \underset{\mathcal{U}}{-} T_2 \subset T_1$, so that $T_1 \underset{\mathcal{U}}{-} T_2 = T_1 \underset{\mathcal{T}}{-} T_2$ exists in $\mathcal{T}$ and $\mathcal{T}$ is a Brouwerian ring of sets. Intuitively, the Brouwerian ring $\mathcal{T}$ of sets serves to "fill out" the Boolean "skeleton" $\mathcal{R}$. The desired Stone algebra $\mathcal{B}$ is now obtained by forming the direct sum $\mathcal{B} = \mathcal{R} \oplus \mathcal{T}$.

Section 4. Characterization of Certain Stone Algebras

In this section we characterize a wide sub-class of Stone algebras. These Stone algebras are shown to be factorable into a direct product of Brouwerian algebras of a rather special kind called T-algebras.

Definition: An element a of a lattice L is join-irreducible if $x + y = a$ implies $x = a$ or $y = a$.

Definition: A Brouwerian algebra with I is a T-algebra if I is join-irreducible.

T-algebras may be constructed in the following manner. To any Brouwerian algebra L adjoin a new element J in such a way that J is properly over every element of L. Let $\tilde{L}$ denote the resulting lattice. It is seen that the adjoining of J to L leaves unchanged all the original differences $a - b$ of elements of L. If x is an element of L, then $J - x = J$ since for no $y \neq J$ can the relationship $x + y = J$ hold. (Recall that J is properly over every element of L, and that $x + y$ is an element of L). This shows that there exists in $\tilde{L}$ the difference of any two elements, i.e. that $\tilde{L}$ is a Brouwerian algebra.

One of the results proved in this section is that the direct product of T-algebras is a Stone algebra. Thus a large collection of Stone algebras can be constructed by taking an arbitrary collection of arbitrary Brouwerian algebras, converting each Brouwerian algebra into a T-algebra by adjoining an element J, and forming the direct product of the resulting T-algebras.

Important concepts used throughout the rest of this section are presented in the following definitions.

Definition: Let the set C be the indexing set for a collection of join-irreducible elements $a_\gamma, \gamma \in C$. The collection $\{a_\gamma\}$ is a representation of I if

(4.1) $$I = \bigvee_{\gamma \in C} a_\gamma.$$

The representation is irredundant if $\gamma \neq \gamma'$ implies $a_\gamma a_{\gamma'} = 0$.

Definition: A lattice L is complete if every subset of L has a greatest lower bound and a least upper bound.

Definition: A lattice L is completely distributive if arbitrary sums distribute over arbitrary products, and dually.

Remark: Let D be the indexing set for an arbitrary subset of L, and let a_γ', $\delta \in D$, denote the Boolean complement of a_δ. For our purposes the full power of the complete distributive law is not needed; instead, it suffices that

(4.2) $$\bigwedge(a_\delta + a_\delta') = \bigvee_{(i)} \left[\bigwedge_{\delta \in D} a_\delta{}^{(i)}\right]$$

where $\bigwedge_{\delta \in D} a_\delta{}^{(i)}$ denotes a product formed by choosing, for each $\delta \in D$, either a_δ or a_δ', and $\bigvee_{(i)} \left[\bigwedge_{\delta \in D} a_\delta{}^{(i)}\right]$ denotes the union of all such products. The following example illustrates the notation; the complete distributive law we require is the generalization of the following law:

$$(4.3) \qquad (a + a')(b + b')(c + c') = abc + abc' + ab'c + ab'c' + a'bc + a'bc' + a'b'c + a'b'c'.$$

Definition: Let A and B denote two algebraic systems having the same operations. The direct product $A \times B$ of A and B is the set whose elements are pairs (a,b), $a \in A$ and $b \in B$, and whose operations are performed component-wise:

$$(4.4) \qquad f\left[(a_1,b_1),(a_2,b_2)\right] = \left[f(a_1,a_2),\ f(b_1,b_2)\right].$$

The direct product of an arbitrary number of algebraic systems, all having the same operations, is defined similarly.

Lemma 1: The direct product of an arbitrary collection of Stone algebras is itself a Stone algebra.

Proof: Let A be the indexing set for a collection of Stone algebras S_α, $\alpha \in A$. Let

$$(4.5) \qquad S = \prod_{\alpha \in A} S_\alpha$$

denote the direct product of the Stone algebras S_α. An element x of S has components x_α, where $x_\alpha \in S_\alpha$. Then the element $\neg x = I - x$ of S has components $\neg x_\alpha = I_\alpha - x_\alpha$, and $\neg\neg x \in S$ has components $\neg\neg x_\alpha = I_\alpha - \neg x_\alpha$, since the difference operation is performed componentwise. Since the product operation is also performed componentwise, the element $\neg x \neg\neg x \in S$ has components $\neg x_\alpha \neg\neg x_\alpha$. But each S_α is a Stone algebra, hence $\neg x_\alpha \neg\neg x_\alpha = 0_\alpha$. Thus the components of $\neg x \neg\neg x$ are all 0, and S is a Stone algebra.

Lemma 2: Every T-algebra is a Stone algebra.

<u>Proof</u>: Let $x \neq I$. Then $\urcorner x + x = I$ implies that $\urcorner x = I$, since I is join-irreducible. Hence $\urcorner\urcorner x = 0$, and $\urcorner x \urcorner\urcorner x = I0 = 0$ holds for every $x \neq I$. The proof is completed by noting that $\urcorner I \urcorner\urcorner I = 0I = 0$.

The principal result of this section is presented in the next two theorems.

<u>Theorem 4.1</u>: If B is a complete Stone algebra, and if I has a representation as an irredundant join of join-irreducible elements, then B is isomorphic with a direct product of T-algebras.

<u>Theorem 4.2</u>: If B is a complete and completely distributive Stone algebra, then I can be represented as an irredundant join of join-irreducible elements.

<u>Proof of Theorem 4.1</u>: Let C be the indexing set for the set of join-irreducible elements $a_\gamma, \gamma \in C$, making up the representation of I, so that

(4.6) $$I = \bigvee_{\gamma \in C} a_\gamma .$$

Let A_γ denote the set of elements $x \in B$ satisfying $x \leq a_\gamma$; and let D denote the direct product of the sets A_γ. The proof consists of three parts. In the first part it is shown that A_γ is a T-algebra. A one-to-one correspondence is established between B and D in the second part of the proof, and in the third part this correspondence is shown to be an isomorphism.

A_γ is clearly a sub-lattice of B. If u and v are in A_γ, then the fact that $u - v \leq u$ means that $u - v$ is also in A_γ, so that A_γ is itself a Brouwerian algebra.

The element a_γ (which plays the role of I in A_γ) is join-irreducible, hence A_γ is a T-algebra, by definition.

Let d be an element of D having components d_γ, where $d_\gamma \in A_\gamma$. The correspondent x in B of d in D is defined as

(4.7) $$x = \bigvee_{\gamma \in C} d_\gamma .$$

The sum exists since each d_γ is in B, and B is complete. Let $\bar{d}$ denote another element of D, having components $\bar{d}_\gamma$, and assume that

(4.8) $$\bigvee_{\gamma \in C} d_\gamma = \bigvee_{\gamma \in C} \bar{d}_\gamma ,$$

i.e. assume that d and $\bar{d}$ map into the same element x of B. If $a_\beta, \beta \in C$, is one of the elements making up the representation if I, then from (4.8) we may write that

(4.9) $$a_\beta \bigvee_{\gamma \in C} d_\gamma = a_\beta \bigvee_{\gamma \in C} \bar{d}_\gamma .$$

Using the infinite distributive law, which holds in B since B is complete, the above expression becomes

(4.10) $$\bigvee_{\gamma \in C} (a_\beta d_\gamma) = \bigvee_{\gamma \in C} (a_\beta \bar{d}_\gamma) .$$

The fact that the representation is irredundant implies that the elements a_γ are pairwise disjoint. Since $d_\gamma \leq a_\gamma$ implies $a_\beta d_\gamma \leq a_\beta a_\gamma = 0$ for $\beta \neq \gamma$, expression (4.10) reduces to

(4.11) $$a_\beta d_\beta = a_\beta \bar{d}_\beta .$$

But $d_\beta \leq a_\beta$, and $\bar{d}_\beta \leq a_\beta$, hence $d_\beta = \bar{d}_\beta$. Since β was an arbitrary member of the indexing set C, this shows that $d = \bar{d}$, i.e. that the correspondence defined in (4.7) is a one-to-one mapping of D into B. We will complete the second part of the proof of Theorem 4.1 by showing that every element in B is the image of an element of D. Let y be an element of B. Then ya_γ is in A_γ, and the element d_y, whose components are ya_γ, is in D. The image of d_y is

(4.12) $$\bigvee_{\gamma \in C}(ya_\gamma) = y\bigvee_{\gamma \in C} a_\gamma = yI = y,$$

again using the infinite distributive law.

That this one-to-one correspondence is operation-preserving follows from the fact that if d and $\bar{d}$ are elements of D satisfying $d \leq \bar{d}$, then the components d_γ of d and $\bar{d}_\gamma$ of $\bar{d}$ indivually satisfy $d_\gamma \leq \bar{d}_\gamma$. Hence

(4.13) $$\bigvee_{\gamma \in C} d_\gamma \leq \bigvee_{\gamma \in C} \bar{d}_\gamma,$$

and the correspondence is order-preserving. But all the operations in B are defined in terms of the order relation; hence the correspondence is an isomorphism and the proof of Theorem 4.1 is complete.

<u>Proof of Theorem 4.2</u>: If B is a T-algebra the theorem is trivial. If not, the set R of elements of B satisfying $\neg\neg r = r$ contains elements other than 0 and I. For, if B is not a T-algebra, then there exists elements x and y, both different from I, such that $x + y = I$. This implies that $\neg(x + y) = 0$ and $\neg\neg(x + y) = I$. If $\neg x = I$, then

(4.14) $$I = \neg\neg(x + y) = \neg\left[\neg\neg(\neg x \neg y)\right] = \neg\left[\neg\neg(\neg y)\right] = \neg\neg y \leq y$$

implies $y = I$ which is a contradiction. Hence $\neg x \neq I$. Finally, $x \neq I$ yields $\neg x \neq 0$. Thus the element $\neg x$, which is in R since $\neg\neg(\neg x) = \neg x$, is different from 0 and I.

Let A be the indexing set for R. If $r_\alpha, \alpha \in A$, is an element of R, so is its Boolean complement r_α', since R is a Boolean sub-algebra of B by Theorem 3.2. We form the product

$$(4.15) \qquad I = \bigwedge_{\alpha \in A}(r_\alpha + r_\alpha').$$

The product is I, as shown, since each term of the product is I. Using the complete distributive law (4.2), (4.15) becomes

$$(4.16) \qquad I = \bigvee_{(i)} \left[\bigwedge_{\alpha \in A} r_\alpha{}^{(i)}\right].$$

Let $x = \bigwedge_{\alpha \in A} r_\alpha{}^{(i)}$. We will show that x is in R, i.e. that every term of (4.16) is in R. From $x \leq r_\alpha{}^{(i)}$ it follows that $\neg x \geq \neg r_\alpha{}^{(i)} = r_\alpha{}^{(i)\prime}$.

Hence

$$(4.17) \qquad \neg x \geq \bigvee_{\alpha \in A} r_\alpha{}^{(i)\prime}.$$

Let $y = \bigvee_{\alpha \in A} r_\alpha{}^{(i)\prime}$. Since $y + r_\alpha{}^{(i)} = I$ holds for every α in A, we have

$$(4.18) \qquad I = \bigwedge_{\alpha \in A}(y + r_\alpha{}^{(i)}) = y + \bigwedge_{\alpha \in A} r_\alpha{}^{(i)} = y + x.$$

By definition of $\neg x$, this means $y \geq \neg x$, i.e.

$$(4.19) \qquad \neg x \leq \bigvee_{\alpha \in A} r_\alpha{}^{(i)\prime}.$$

From (4.17) and (4.19), we have

$$(4.20) \qquad \neg x = \bigvee_{\alpha \in A} r_\alpha{}^{(i)\prime}.$$

It is clear that, for every α in A, $x r_\alpha{}^{(i)\prime} = 0$. Hence

$$(4.21) \qquad 0 = \bigvee_{\alpha\in A} xr_\alpha(i)' = x\bigvee_{\alpha\in A} r_\alpha(i)' = x\rceil x.$$

This shows that x is in Q and hence in R by Lemma 1 to Theorem 3.3, page 29.

Not every term of (4.16) is 0, since the sum of the terms is I. After discarding from (4.16) those terms which are 0, the remaining terms may be relabelled so that (4.16) becomes

$$(4.22) \qquad I = \bigvee_{\delta\in D} a_\delta.$$

It will be shown next that D is the indexing set for the atoms of R, i.e. those elements a_δ of R such that $0 \lneqq r \lneqq a_\delta$ holds for no element r in R. After that we will show that the representation (4.22) is irredundant, and the proof will be completed by showing each element a_δ is join-irreducible.

Suppose that an element r of R satisfied

$$(4.23) \qquad 0 \leq r \leq a_\delta, \quad 0 \neq r, \quad a_\delta \neq r$$

for some δ in D. By the manner in which a_δ was obtained, we observe that the expression for a_δ contains the letter r, with or without a prime. If r appears as one of the members of the expression for a_δ, then $r \geq a_\delta$, which violates (4.23). On the other hand, if r' appears as one of the members of the expression for a_δ, then $ra_\delta = 0$, which also contradicts (4.23). We conclude that no element r of R can satisfy (4.23), i.e. that a_δ is an atom of R. We remark parenthetically that a_δ may not be an atom of B.

Again using the fact that each element r_α of **R**, with or without a prime, appears as a member of the expression for a_δ, we see that $\delta_1 \neq \delta_2$ implies a_{δ_1} and a_{δ_2} are different, i.e. at least one of the elements is primed in one term and not in the other. It follows that

(4.24) $\quad a_{\delta_1} a_{\delta_2} = 0 \quad \text{for} \quad \delta_1 \neq \delta_2,$

which shows that the representation (4.22) is irredundant.

Assume that there exists elements x, y of B, each different from 0 and from a_δ, which satisfy

(4.25) $\quad 0 \leq x \leq a_\delta, \quad 0 \leq y \leq a_\delta, \quad x + y = a_\delta$

for some δ in D. Recalling that $\neg\neg x \leq x$, that $\neg\neg x$ is in R, and that a_δ is an atom of R, we have $\neg\neg x = 0$. Hence $\neg x = I$, and, similarly, $\neg y = I$. Hence

(4.26) $\quad \neg a_\delta = \neg(x + y) = \neg\neg(\neg x \neg y) = \neg\neg(II) = \neg\neg I = I.$

But if $\neg a_\delta = I$, then $0 = \neg\neg a_\delta = a_\delta$ since a_δ is in R. This is a contradiction, for in the construction of (4.22) only the terms of (4.16) different from 0 were retained. Thus (4.25) is impossible, and a_δ is join-irreducible. This completes the proof of Theorem 4.2.

<u>Corollary 1</u>: Every complete and completely distributive Stone algebra is isomorphic with a direct product of T-algebras.

<u>Proof</u>: This is a direct consequence of Theorems 4.1 and 4.2.

<u>Corollary 2</u>: Any finite Stone algebra is isomorphic with

a direct product of T-algebras.

Proof: Any finite Stone algebra is complete and completely distributive.

Noticing that the essence of the proof of Theorem 4.2 was the discovery of the atoms of R, we are led to

Theorem 4.3: A Stone algebra B is isomorphic with a direct product of T-algebras if every descending chain in R is finite.

Proof: Since R is a Boolean algebra in which descending chains are finite, we know that R itself is finite (cf. Birkhoff [3], p. 159). Hence the atoms of R can be determined; it can be shown as in Theorem 4.2 that I is an irredundant join of the atoms of R and that the atoms of R are join-irreducible elements of B. The proof is completed by applying Theorem 4.2.

One further extension of Theorem 4.2 is obtained by noticing that the use of the infinite distributive law in the proof was confined to elements of R.

Theorem 4.4: If B is a Stone algebra in which the Boolean sub-algebra R is complete and completely distributive, then B is isomorphic with a direct product of T-algebras.

Proof: Exactly as in Theorems 4.1 and 4.2.

An example of a Stone algebra which is r], p. 184).le is the "measure algebra" $\overline{M}$ (see Birkhoff [3], p. 184). This algebra may be constructed as follows. Let M denote the set of Lebesgue measurable subsets of the unit interval. Divide M into equivalence classes by placing in the same class any two subsets whose symmetric difference is a set of measure zero. The equivalence classes are ordered by set inclusion. It is known that the resulting algebra $\overline{M}$ is a complete Boolean algebra without atoms. Since $\overline{M}$ is a Boolean algebra, it follows that $\overline{M}$ is a Stone algebra. However, $\overline{M}$ cannot be factored into a direct product of T-algebras since it has no join-irreducible elements.

It might be conjectured that all Stone algebras are direct products, the factors being either T-algebras or Boolean algebras without atoms. That this is not the case is shown by the following example, due to L. M. Kelly. First consider a non-atomic Boolean algebra, and consider its representation as a Boolean ring $\mathcal{R}$ of sets. $\mathcal{R}$ may be regarded as embedded in an algebra of sets which of course contains points. Let T be one of these points, and let the set $\mathcal{T}$ consist of T together with the null set. It is easily verified that the conditions of the Set-Theoretic Structure Theorem (p. 38) are satisfied, hence $\mathcal{B} = \mathcal{R} \oplus \mathcal{T}$ is a Stone algebra. Since $\mathcal{B}$ contains only one join-irreducible element, namely T, the only possible factorization of $\mathcal{B}$ of the conjectured type is $\mathcal{B} = \mathcal{R} \times \mathcal{T}$. In the direct product $\mathcal{R} \times \mathcal{T}$, the four elements (R,T), (R,D), (R',T) and (R',0) are distinct, where R denotes some member of $\mathcal{R}$ different from 0 and I. But the point T lies in either R or R', hence in the direct sum $\mathcal{R} \oplus \mathcal{T}$ the four elements R + T, R + 0, R' + T and R' + 0 are not distinct. Thus no one-to-one correspondence can be set up between $\mathcal{R} \oplus \mathcal{T}$ and $\mathcal{R} \times \mathcal{T}$, i.e. the Stone algebra $\mathcal{R} \oplus \mathcal{T}$ cannot be factored in the conjectured manner.

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