A HYBRID SYSTEM APPROACH TO IMPEDANCE AND ADMITTANCE CONTROL

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ABSTRACT

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Impedance Control and Admittance Control are two different implementation methods that are used to achieve the same control objective of producing a desired relationship between applied force and displacement of a robotic system interacting with an environment. Due to the difference in implementation method, impedance control and admittance control have complementary performance and stability characteristics. Impedance control is stable for all contact environment stiffness and has good performance for stiff environments, but results in poor performance during interaction with soft environments. Admittance control results in good performance during interaction with soft environments but results in either poor performance and or unstable behaviour during interaction with stiff environments. We use a hybrid system framework to propose a family of controllers that attempt to interpolate the stability and performance characteristics of impedance control and admittance control. The advantages of this approach are first demonstrated through analysis and simulations of a single degree-of-freedom rigid linear model. The methodology is then extended to multi degree-of-freedom linear and non-linear models and single degree-of-freedom system with flexibility. Experimental results with a lightweight robotic arm are presented to demonstrate the usefulness of the approach.

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Chapter 1

Introduction

Most popular implementations for industrial robots are restricted to tasks which involve little to no exchange of mechanical energy between the end-effector and the environment. Instead they require precise position controls, such as in welding, painting, and pick and place operations.

The first control method is called "Hybrid Position and Force Control" and was developed by Raibert and Craig [23]. In hybrid position and force control the control input is decomposed into two orthogonal subsets, one subset for position control and one subset for force control. The force control subset is used to control the desired force of interaction of the end effector with the environment in certain directions, and the position control subset is used to control the position of the effector in the remaining directions. However, because the subsets of the control method are orthogonal, and dynamic coupling between the position of the effector and the force from the environment is not considered, it is not possible to accurately control either the end effector force or position. The compliance control method proposed by Mason is a variation of the hybrid position and force control method [14] and sufferes from the same drawbacks as "Hybrid Position and Force Control".

The second control method was proposed by Hogan and is known as "Impedance Control" [8]. In impedance control, the mechanical impedance of the manipulator is controlled to match that of a desired model. Therefore, impedance imposes a relationship between the position of the effector and the force produced by the environment, and provides a unifying methodology for all tasks ranging from motion in free-space to dynamic interaction with an environment.

Strategies that combine impedance control and hybrid position and force control have also been proposed. Anderson and Spong [3] proposed a method that uses feedback linearization as an inner loop. For the outer loop, they used a hybrid position and force type controller which uses impedance control instead of position control. Liu and Goldenberg [12] proposed a robust method for hybrid position and force control, it uses impedance control in the position control subspace, and an inertia normalization and damping in the force control subspace. A PI controller is used to achieve robustness, and the controller was implemented on a two degree of freedom direct drive robot.

The impedance control method may be implemented in one of two ways. Although Hogan [8] refers to both methods as impedance control, we make the distinction between the two. In the first method of implementation, which are referred to as "Impedance Control", the control force is derived from the desired mechanical impedance between the end efector position and force exerted between the end effector and the environment. The second implementation method is referred to as "Admittance Control" wherein the control input is a position controller that drives the end effector to the resulting mechanical admittance derived from the desired relationship between the force produced by the environment and the end-effector. Stability properties of impedance control and admittance control are discussed in [10].

Robotic systems controlled with impedance control typically have more stable dynamics when interacting with stiff environments, but have poor accuracy in free space when compared to admittance control. The accuracy problem of impedance control may be mitigated through hardware modifications such as low-friction joints and direct-drive actuators. Impedance control has been implemented ATR's Humanoid built by Sarcos [4] and in DLR's light-weight robot using inner loop torque sensing and control [18] as well in the Phantom haptic device using low-friction joints and low inertias.

Admittance control has complementary characteristics of impecance control and has high accuracy in free-space, but can result in instability when implemented with high contact stiffness. To improve stability characteristics, admittance control is implemented together with series elastic actuation but such hardware modification reduces the performance of the system in free-space. Also, admittance control usually requires high transmission drives, as compared to the direct drives that are used with impedance control, to achieve accurate position control and higher torque values for the inner loop position controller.

Impedance control and admittance control have complementary characteristics but neither provides good performance over the full range of control tasks from accurate position control in free-space to stability in contact with rigid environments. Although hardware modifications can be made to improve either controller, such modifications result in a predisposition to that controller as it degrades the performance of the other.

To overcome this obstacle and have complete flexibility over choosing the best controller for a given task, a hybrid switching control based on a hybrid systems framework [11] was proposed by Ott, Mukherjee, and Nakamura [19]. However, the proposed method was investigated with a single degree of freedom linear time invariant model. In this thesis, we extend this analysis to higher dimensions as well as the use of series elastic actuation and verify the results through experimentation.

Chapter 2

Background

2.1 Problem Statement

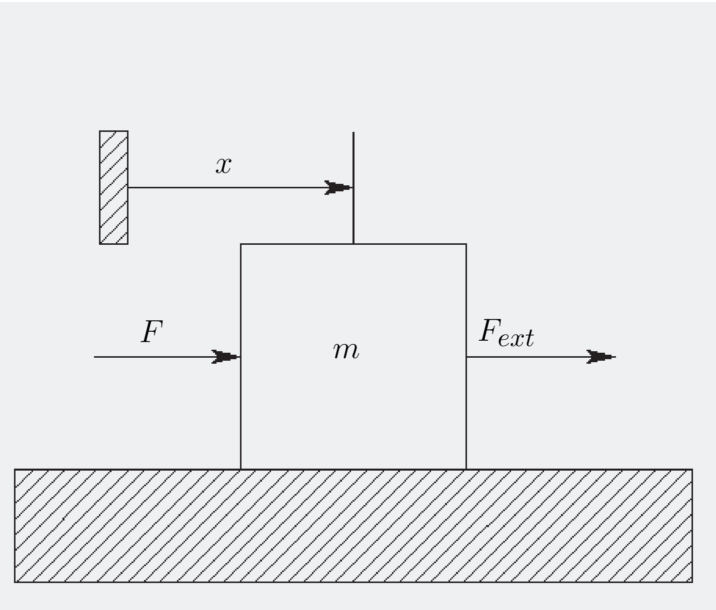

Figure 2.1: A single degree-of-freedom mass interacting with the environment

We begin with a review of the hybrid impedance admittance control problem as originally proposed by Ott, Mukherjee, and Nakamura for a single degree-of-freedom linear system [19]. Consider a single degree of freedom mass, m, that interacts with its environment as shown in figure 2.1. The displacement x with F and F_{ext} denoting the control force and force applied on the mass by the environment, respectively. Both F and F_{ext} are measured positive in the direction of positive displacement. The equation of motion is written as

$$m\ddot{x} = F + F_{ext} \tag{2.1}$$

The control objective of both impedance and the admittance control is to design the control force F that will result in the following dynamic relationship

$$M_\theta \ddot{e} + D_\theta \dot{e} + K_\theta e = F_{ext} \tag{2.2}$$

between

$$e = x - x_0 \tag{2.3}$$

which is the error in the position of the mass relative to a desired equilibrium trajectory x_0 and F_{ext}. Where M_θ, D_θ, and K_θ are positive constants that represent the desired inertia, damping and stiffness, respectively. For the single degree of freedom system, the differential map may be written as a transfer function between e and F_{ext} denoted by

$$G_d(s) = \frac{1}{M_\theta s^2 + D_\theta s + K_\theta} \tag{2.4}$$

2.2 Impedance Control

For impedance control, the control force is a mechanical impedance with the control plant and desired plant being a mechanical admittance. From equations (2.1) and (2.2) we have

$$\ddot{x} = m^{-1}(F + F_{ext}) \tag{2.5}$$

$$\ddot{x} = \ddot{x}_0 + M_\theta^{-1}(F_{ext} - K_\theta e - D_\theta \dot{e}) \tag{2.6}$$

Comparing equations (2.6) and (2.5) and solving for F we set

$$F = F_i = m\ddot{x}_0 + (mM_\theta^{-1} - 1)F_{ext} - mM_\theta^{-1}(D_\theta e + K_\theta e) \tag{2.7}$$

It can be verified that the substitution of equation (2.7) into (2.1) yields

$$M_\theta(\ddot{x} - \ddot{x}_0) + D_\theta(\dot{x} - \dot{x}_0) + K_\theta(x - x_0) = F_{ext} \tag{2.8}$$

2.3 Admittance Control

For admittance control, the control force is a position-controller designed to track the trajectory, $x = x_d$. Traditionally, tracking is implemented using a PD regulation controller of the form

$$F = F_a = K_p(x_d - x) - K_d\dot{x} \tag{2.9}$$

with positive gains k_p and k_d. The desired trajectory, x_d, is derived from the mechanical admittance in (2.2). Then, x_d may be written as the solution to the differential equation

$$M_\theta(\ddot{x}_d - \ddot{x}_0) + D_\theta(\dot{x}_d - \dot{x}_0) + K_\theta(x_d - x_0) = F_{ext} \tag{2.10}$$

Substituting (2.9) into (2.1) we find the complete system dynamics to be given by

$$m\ddot{x} + K_d\dot{x} + K_p(x - x_d) = F_{ext} \tag{2.11}$$

$$M_d(\ddot{x}_d - \ddot{x}_0) + D_\theta(\dot{x}_d - \dot{x}_0) + K_\theta(x_d - x_0) = F_{ext} \tag{2.12}$$

2.4 Motivation for Hybrid Control

The performance of impedance control in terms of position accuracy depends primarily on the back drive-ability of the system as well as unmodeled friction. For large unmodeled friction, the position accuracy in free space tends to be poor as it depends on the desired stiffness and damping which are generally chosen to be small to reduce input noise [26]. However, impedance control is robustly stable for model uncertainties and even when in contact with stiff environments, as analysed by [6], [2], and [15].

The performance of admittance control depends heavily on the position controller [22], which effectively compensates for unmodeled friction [26], [13]. Therefore, admittance control may be implemented in systems with poor back drive-ability and large unmodeled friction. However, we find that the position controller results in instability when the mass comes in contact with large stiffness environments even in the absence of uncertainties [10]. To compensate for the instability issue of admittance control researchers have proposed adaptive control methods for dealing with environments with unknown stiffness [25], [24].

The complementary characteristics of the two controllers is well known [27] and is illustrated with the help of fig. 2.2 bellow [19].

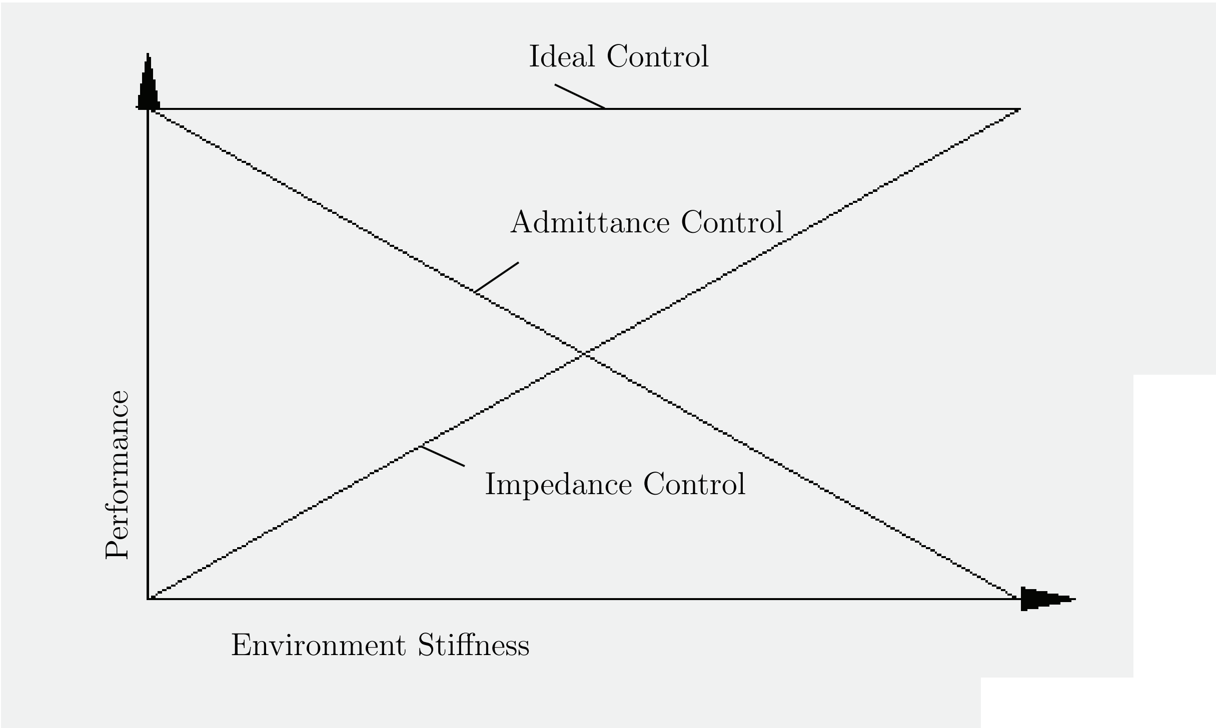

Figure 2.2: Qualitative illustaration of the performance of impedance control and admittance control for different environment stiffness

To overcome the limitations of both the impedance controller and the admittance controller we propose a hybrid controller in the next section that can potentially provide good performance for the entire spectrum of tasks ranging from motion in free space to dynamic interaction .

2.5 Hybrid Framework

For the single degree-of-freedom system described by equation (2.1), we propose to switch between the impedance controller and admittance controller as follows:

$$F = \begin{cases} F_i & : \quad t \in [t_0 + k\delta, t_0 + (k+1-n)\delta) \\ F_a & : \quad t \in [t_0 + (k+1-n)\delta, t_0 + (k+1)\delta) \end{cases} \tag{2.13}$$

where t_0 is the initial time, δ is the switching period, $n \in [0, 1]$ is the duty cycle, and k is a positive integer. F_i is given by equation (2.7), and F_a is given by equations (2.10) and (2.9).

If the environment is modelled as a linear spring, namely

$$F_{ext} = -k_e(x - x_0) \tag{2.14}$$

and the equilibrium trajectory, x_0 is assumed to be constant, *i.e.*,

$$\dot{x}_0 = \ddot{x}_0 = 0 \tag{2.15}$$

the hybrid system follows the descriptions

$$\begin{aligned} \dot{X}_i &= A_i X_i \; : \; t \in [t_0 + k\delta, t_0 + (k+1-n)\delta) \\ \dot{X}_a &= A_a X_a \; : \; t \in [t_0 + (k+1-n)\delta, t_0 + (k+1)\delta) \end{aligned} \tag{2.16}$$

where

$$X_i = (e \;\; \dot{e})^T \tag{2.17}$$

$$A_i = \begin{bmatrix} 0 & 1 \\ -(K_\theta + k_e)/M_\theta & -D_\theta/M_\theta \end{bmatrix} \tag{2.18}$$

$$X_a = (e \;\; \dot{e} \;\; e_d \;\; \dot{e}_d)^T \tag{2.19}$$

$$e_d = x_d - x_0 \tag{2.20}$$

and

$$A_a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(K_p + k_e)/m & -K_d/m & K_p/m & 0 \end{bmatrix} \tag{2.21}$$

When switching from the impedance controller to the admittance controller, two additional states are introduced. These states, e_d and $\dot{e}_d$, are chosen at the instant of switching to maintain continuity of the control force F and its derivative. Equation (2.9) gives the expression for the control force of the admittance controller

$$
\begin{aligned}
x_d &= x + \frac{1}{K_p}\left(F_i + K_d \dot{x}\right) \\
\Rightarrow e_d &= e + \frac{1}{K_p}\left(F_i + K_d \dot{e}\right) && (2.22) \\
\dot{x}_d &= \dot{x} + \frac{1}{K_p}\left(\dot{F}_i + K_d \ddot{x}\right) \\
\Rightarrow \dot{e}_d &= \dot{e} + \frac{1}{K_p}\left[\dot{F}_i + K_d(F_1 + F_{ext})\right] && (2.23)
\end{aligned}
$$

Replacing F_i in equations (2.22) and (2.23) with F_i from equation (2.7), it is possible to obtain an expression of the form

$$
X_a = S_{ai} X_i, \qquad S_{ai} = \begin{bmatrix} I \\ S \end{bmatrix} \tag{2.24}
$$

at the instant of switching, where I is the identity matrix, and s_{ij}, $i = 1, 2$, $j = 1, 2$ of the S matrix have the expression

$$
\begin{aligned}
s_{11} &= 1 - \frac{K_e}{K_p}\left(\frac{m}{M_\theta} - 1\right) - \frac{K_\theta}{K_p}\frac{m}{M_\theta} \\
s_{12} &= \frac{K_d}{K_p} - \frac{D_\theta}{K_p}\frac{m}{M_\theta} \\
s_{12} &= -\frac{m}{M_\theta}\frac{(K_\theta + k_e)}{K_p}\left(\frac{K_d}{m} - \frac{D_\theta}{M_{theta}}\right) \\
s_{22} &= 1 - \frac{k_e}{K_p}\left(\frac{m}{M_\theta} - 1\right) - \frac{D_\theta}{M_\theta}\left(\frac{K_d}{K_p} - \frac{D_\theta}{K_p}\frac{m}{M_\theta}\right) - \frac{K_\theta}{K_p}\frac{m}{M_\theta} && (2.25)
\end{aligned}
$$

When the system switched from the admittance controller to the impedance controller, the state mapping is given by

$$X_i = S_{ia} X_a, \qquad S_{ia} = [I \quad 0] \tag{2.26}$$

Where 0 is the 2×2 matrix of zeros.

2.6 Stability

Knowing the states at time $t = t_0 + k\delta$, the states at $t = t_0 + (k+1)\delta$, $k \in Z_+$ is a positive integer, can be obtained using equations (2.16), (2.24), and (2.26) as

$$\begin{aligned}
X_i(t_0 + (k+1-n)\delta) &= e^{A_i(1-n)\delta} X_i(t_0 + k\delta) \\
\Rightarrow X_a(t_0 + (k+1-n)\delta) &= S_{ai} e^{A_i(1-n)\delta} X_i(t_0 + k\delta) \\
\Rightarrow X_a(t_0 + (k+1)\delta) &= e^{A_a n\delta} X_a(t_0 + (k+1-n)\delta) \\
\Rightarrow X_i(t_0 + (k+1)\delta) &= S_{ia} e^{A_a n\delta} X_a(t_0 + (k+1-n)\delta) \\
\Rightarrow X_i(t_0 + (k+1)\delta) &= S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} X_i(t_0 + k\delta)
\end{aligned} \tag{2.27}$$

We now define the matrix A_{dis} to be

$$A_{dis} \triangleq S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)} \tag{2.28}$$

such that

$$X_i(t_0 + (k+1)\delta) = A_{dis} X_i(t_0 + k\delta) \tag{2.29}$$

We see that A_{dis} defines a discrete map of states from time $t = t_0 + k\delta$ to $t = t_0 + (k+1)\delta$, independent of k. We now define a Discrete Equivalent System based on the definition in Das and Mukherjee [5].

Definition 1. [Discrete Equivalent Subsystem (DES)] The time-invariant linear system

$$\dot{X} = A_{eq}X \tag{2.30}$$

is a DES of a switched linear system if the state variables of the DES assume identical values of a subset of the states of the switched system at regular intervals of time, starting from the same initial conditions

Based on definition 1, the system given by equation (2.30) is a DES of the switched system described by (2.16), (2.24), and (2.26) if

$$A_{eq} = \frac{1}{\delta}\ln[A_{dis}] \tag{2.31}$$

Where ln is defined as the principle matrix natural logarithm. Now, the following theorem on stability is presented [19].

Theorem 1. *The equilibrium $X_i = 0$ of the switched system described by equations (2.16), (2.24), and (2.26) is exponentially if A_{eq} of the DES system in equation (2.30) is Hurwitz.*

Proof. see [19]. □

Remark 1. The necessary and sufficient condition for A_{eq} in equation (2.31) to exist is that A_{dis} is non-singular [7]. Furthermore, we find that if λ_{dis} is an eigenvalue of A_{dis}, then the eigenvalue, λ_{eq} corresponding to A_{eq} is given by [7]

$$\lambda_{eq} = \frac{1}{\delta}\ln(|\lambda_{dis}|) + i\frac{1}{\delta}\arg(\lambda_{dis}) \tag{2.32}$$

where arg is taken to be the principle argument.

Remark 2. The stability analysis is based on switching from impedance control to admittance control and back to impedance control. This sequence leads to a DES defined by the principle matrix logarithm of the 2×2 matrix $S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)\delta}$ as shown in equation (2.27). However, if the hybrid system is switched from admittance control to impedance control and back to admittance control, the DES would be defined by the principle matrix logarithm of a 4×4 matrix $S_{ai}e^{A_i(1-n)\delta}S_{ia}e^{A_a n\delta}$ instead. This is problematic, since for any two matrices $A \in R^{m\times n}$ and $B \in R^{n\times n}$ with $m < n$, the eigenvalues of AB and BA are the same except for the $(n-m)$ additional eigenvalues of BA which are identically zero [9]. Therefore, the matrix $S_{ai}e^{A_i(1-n)\delta}S_{ia}e^{A_a n\delta}$ is singular. However, we note that the states corresponding to the zero eigenvalues are virtual states.

2.7 Stability of Nominal Plant

Theorem 1 provides a sufficient condition for exponential stability of the linear single degree of freedom switched system. However, the matrix A_{eq} is dependant on both the external stiffness, k_e, the switching period, δ, and the weight n. Therefore, before simulating a sample system we will analyse the stability properties of the system in the abscense of model uncertainties.

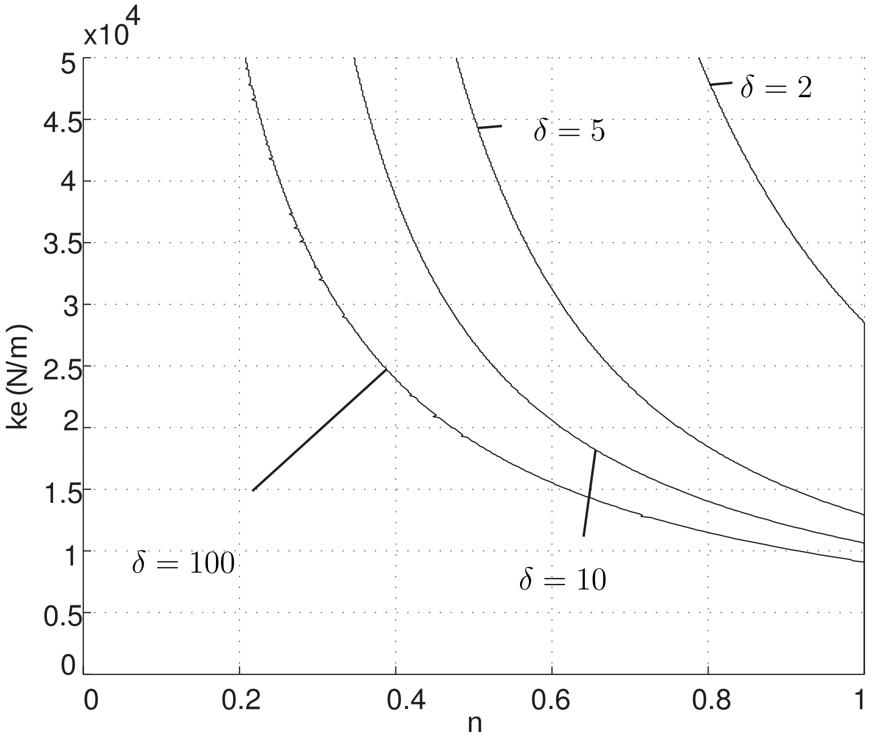

Figure 2.3: Stability boundaries for hybrid impedance and admittance control for different values of δ

Let the parameters of a single degree-of-freedom system be given by

$$
\begin{aligned}
m = 1\ kg \quad K_p = 10^6\ \tfrac{N}{m} \quad K_d = 1.4\sqrt{K_p m}\ \frac{Ns}{m} \\
M_\theta = 1\ kg \quad K_\theta = 100\ \tfrac{N}{m} \quad D_\theta = 1.4\sqrt{K_\theta M_\theta}\ \frac{Ns}{m}
\end{aligned} \tag{2.33}
$$

Figure 2.3 show stability boundaries of the system for different values of k_e, n, and δ. The solid lines in the figure represent the projection of the stability boundary for a constant δ in the (n, k_e) plane, with the lower left side of the solid lines denoting the stable region. We see from the figure that the region of stability increases as δ decreases. We notice also that the stability boundary does not necessarily meet the stability boundary of the admittance controller when $n = 1$. This is due to the resetting condition when switching from the impedance controller to the admittance controller, as found by the mapping S_{ai} as defined in (2.24).

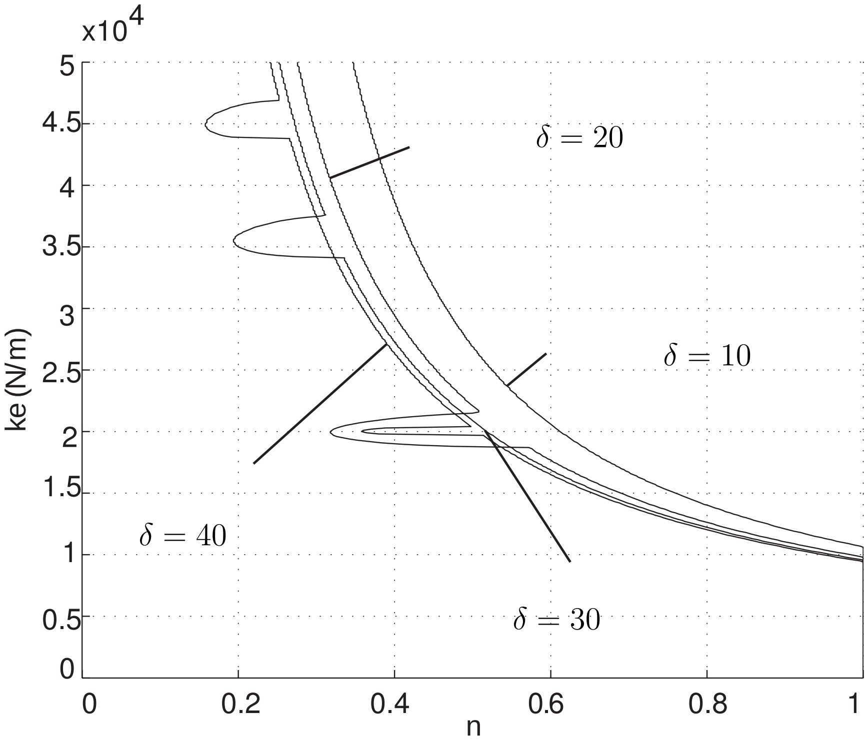

Figure 2.4: Stability boundaries for hybrid impedance and admittance control for δ values 10 ms, 20 ms, 30 ms, and 40 ms

We note that Figure 2.3 does not provide a complete picture of the change in the stability boundary for changing delta. We see in Figure 2.4 that at certain k_e values the largest n value in the stable region does not necessarily correspond to the smallest δ value.

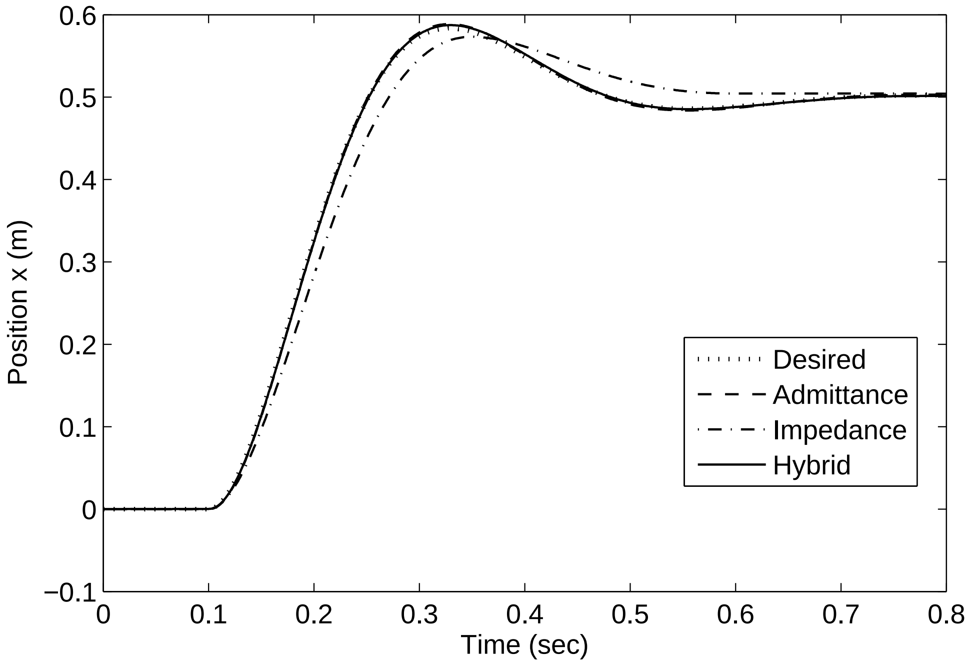

Figure 2.5: Soft Contact position response for $\delta = 50\ ms$ and $n = 0.9$

2.8 Simulations

We will now simulate a sample single degree-of-freedom system in order to study the performance when switching in the presence of uncertainties, like time delay on sensors, uncertain mass value, noise, and unmodeled damping. We will let the parameters of the system be given by

$$\begin{aligned} m = 1\ kg, \quad & K_p = 10^6\ \frac{N}{m}, \quad & K_d = 1.4\sqrt{K_p m}\ \frac{N\ s}{m}, \\ M_\theta = 0.8\ kg, \quad & K_\theta = 100\ \frac{N}{m}, \quad & D_\theta = 1.4\sqrt{K_\theta M_\theta}\ \frac{N\ s}{m} \end{aligned} \tag{2.34}$$

where the measured value of m is 80% the actual value. The simulations use a fixed time Diamond-Prince method with time step of $T = 0.001\ s$. We assume the measured value of x, $\dot{x}$, and F_{ext} have a delay of $T_d = 0.002\ s$, and the external input is corrupted by noise. We choose the function F_{ext} to be given by

$$F_{ext} = -k_e x \tag{2.35}$$

and x_0 is chosen to be a constant value of $1\ m$. We choose $\delta = 50\ ms$ for all simulations.

We begin by considering the soft environment stiffness to be given by $k_e = 100\ N/m$. We then wish to demonstrate how different values of n change the performance. Therefore, let us begin with $n = 0.9$

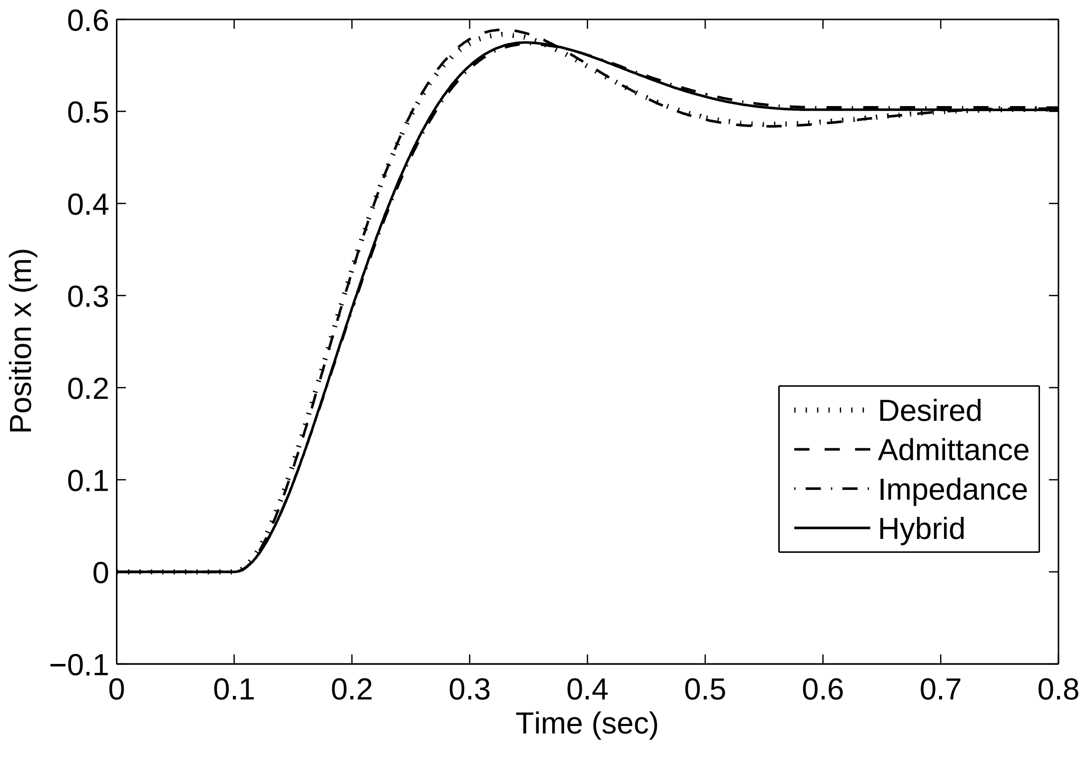

Figure 2.6: Soft contact position response for $\delta = 50\ ms$ and $n = 0.1$

Figure 2.5 shows the response of the impedance controller, admittance controller, ideal response in the abscence of uncertainties, and the hybrid controller when interacting with a soft environment and for $n = 0.9$. From the figure we see that the response of the admittance, hybrid, and ideal response are very close to each other while the response of the impedance controller lags behind the ideal response and results in a steady state error. This implies that for $\delta = 0.5$ the admittance controller has a very similar response to the hybrid controller as $n \to 1$. To further investigate, consider the same external stiffness, $k_e = 100\ N/m$, but let $n = 0.1$.

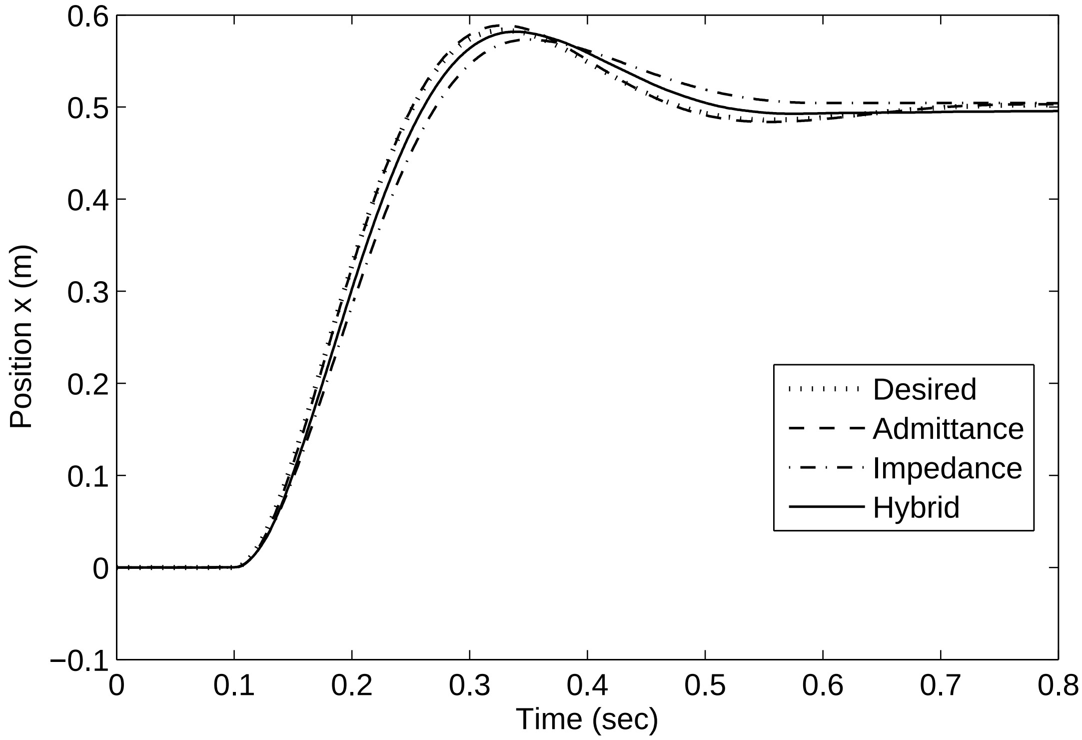

Figure 2.7: Soft contact position response for $\delta = 50\ ms$ and $n = 0.5$

Figure 2.6 again shows the response of the impedance controller, admittance controller, ideal response, and the hybrid controller when interacting with a soft environment and for $n = 0.1$. We now have the hybrid controller closely approximating the same performance as the impedance controller. This is expected since when $n = 0$ the hybrid controlled system is the impedance controlled system. Now, consider the system subject to the same external stiffness, $k_e = 100\ N/m$, but let $n = 0.5$.

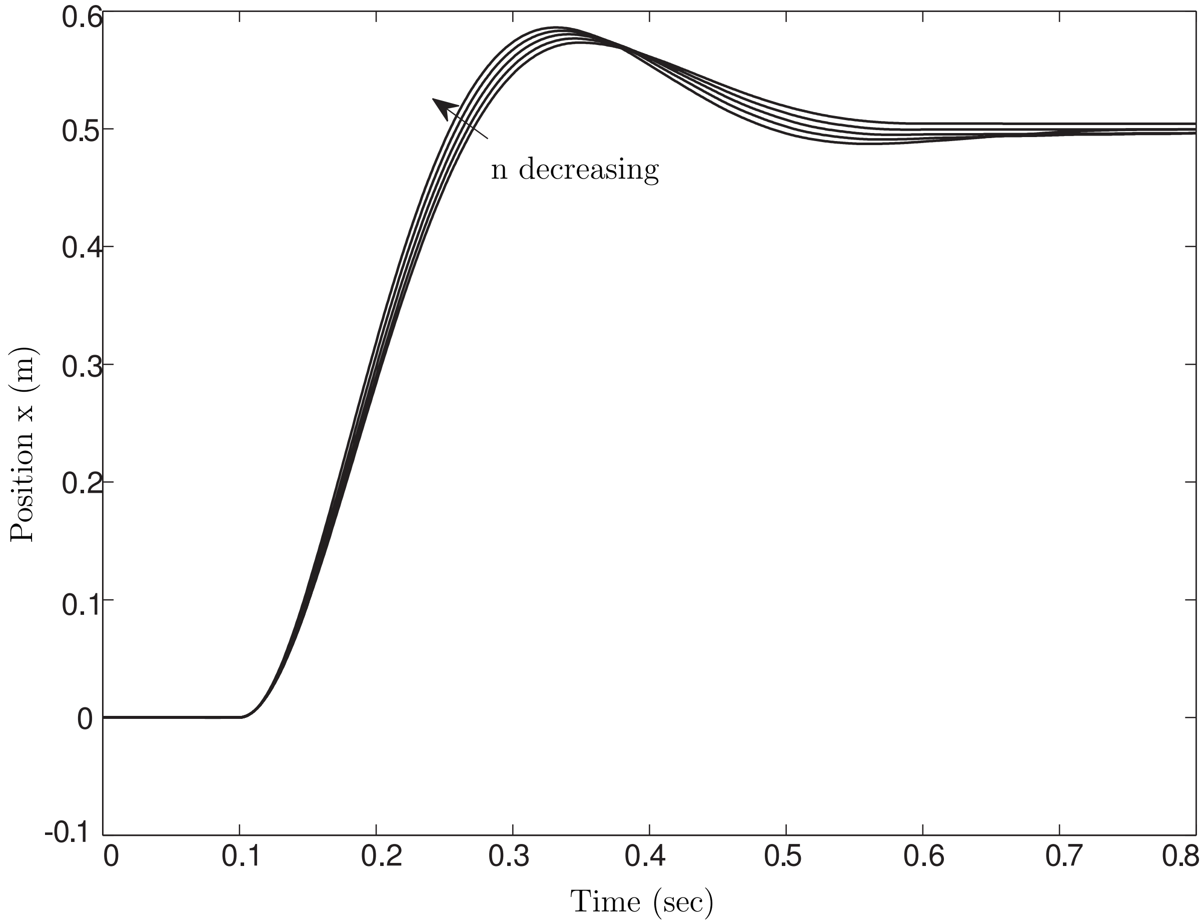

Figure 2.8: Soft Contact position response for $\delta = 50\ ms$ and a range of n values

Figure 2.7 again shows the response of the impedance controller, admittance controller, ideal response, and the hybrid controller when interacting with a soft environment and for $n = 0.5$. Now the hybrid controller approximates a mean performance between the impedance controller and the admittance controller. This implies that the switched system response resembles a weighted average between the impedance controller and the admittance controller with the weights being based on the value of n. This is shown in Figure 2.8 which plots the response of different n values in the hybrid controller for the same external stiffness, $k_e = 100\ N/m$.

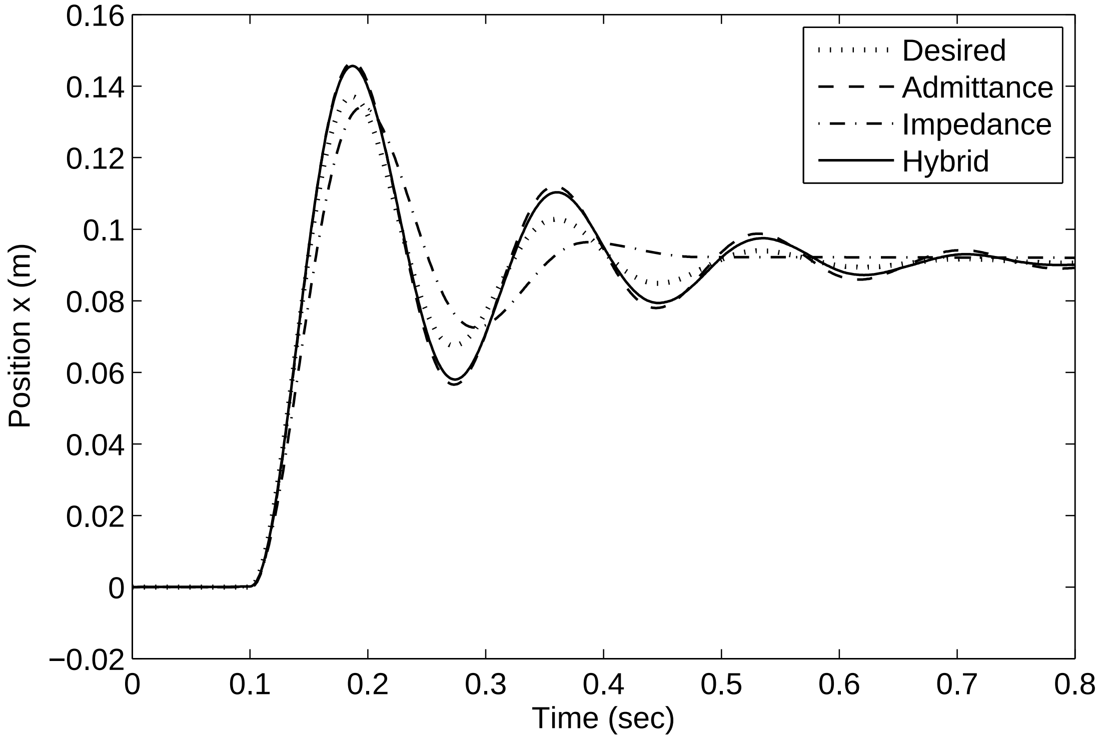

Figure 2.9: Stiff contact position response for $\delta = 50\ ms$ and $n = 0.9$

Figures 2.5-2.8 demonstrate the variation in the response of the hybrid system as a function of n, and how it interpolates between the response of the impedance controller and the admittance controller. However, the figures plot the results for a soft contact, for which the admittance controller has good performance. Therefore, we now investigate the scenarios where the system comes in contact with a hard surface, $k_e = 1000\ N/m$. We again begin with $n = 0.9$.

Figure 2.9 shows the response of the closed loops system under impedance control, admittance control, the hybrid control, and the ideal response when interacting with a stiff environment and $n = 0.9$. From the figure we see that the hybrid control and the admittance control closely approximate each other. This is consistent with the soft

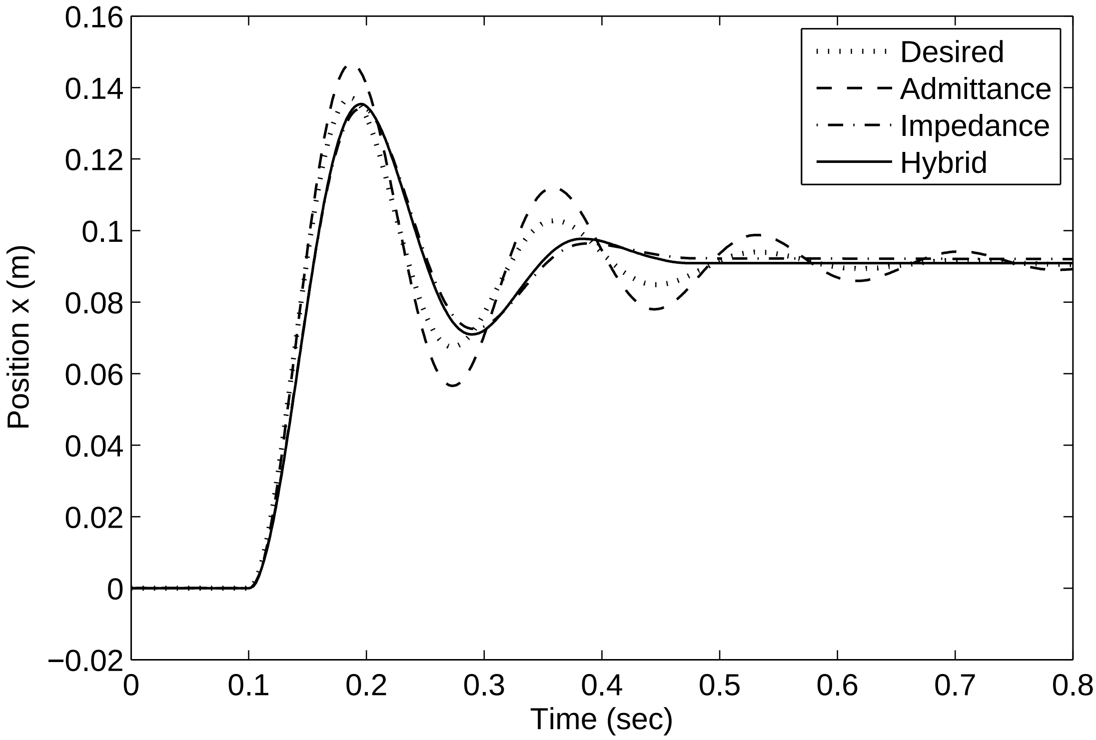

Figure 2.10: Stiff contact position response for $\delta = 50\ ms$ and $n = 0.1$

contact case. However, we see that the admittance controller does not approximate the desired performance. We then let $n = 0.5$ with the same external stiffness, $k_e = 1000\ N/m$.

Figure 2.10 shows the responses of the closed loop system uner admittance control, impedance control, hybrid control, and the ideal response when interacting with a stiff environment and $n = 0.1$. From figure 2.10 we see that the hybrid controller response closely approximates the impedance controller. Again, this is not surprising as when $n = 0$ the hybrid controller is the impedance controller. Therefore, we consider the case where $n = 0.5$ for the same external stiffness, $k_e = 1000\ N/m$.

From figure 2.11 we see that the hybrid response again approximates a mean between the admittance controller and the impedance controller when interacting with a

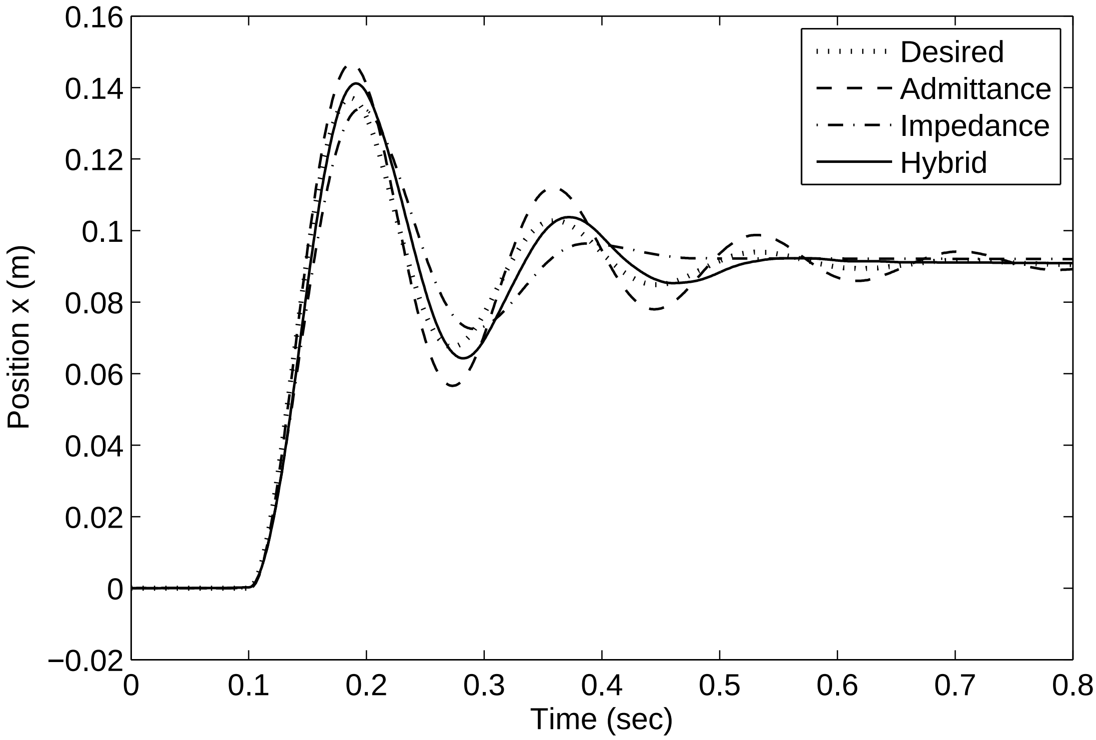

Figure 2.11: Stiff contact position response for $\delta = 50\ ms$ and $n = 0.5$

stiff environment. We again infer that changing the value of n from 1 to 0 changes the response of the hybrid controller from the admittance controller to the impedance controller in a similar manner to that of a weighted average based on the value of n. This is further demonstrated in Figure 2.12 which shows the response of the hybrid controller for different values of n for the same external stiffness, $k_e = 1000\ N/m$.

This simple example demonstrates the desired effect of switching between admittance and impedance in order to improve the overall performance for a range of external stiffness values.

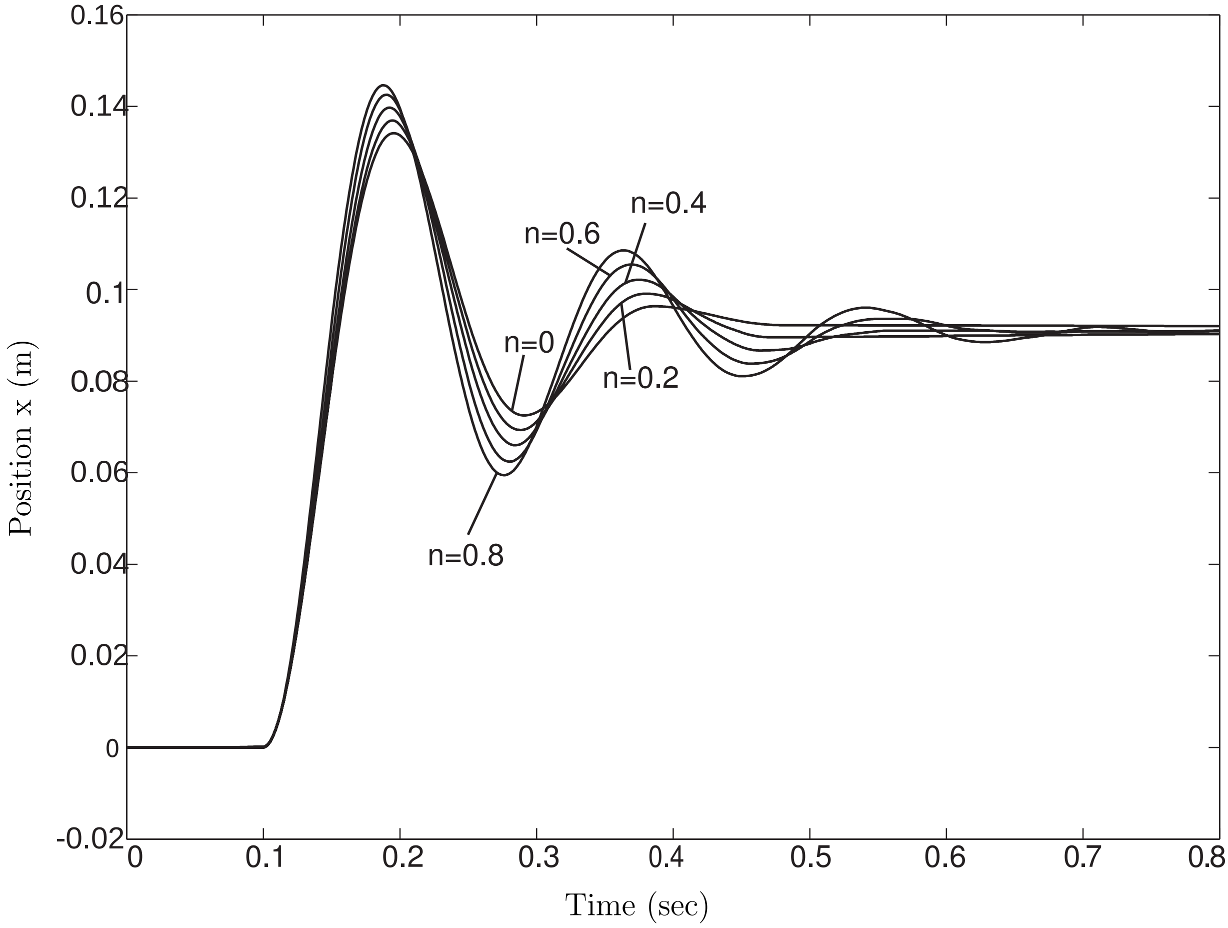

Figure 2.12: Stiff contact position response for $\delta = 50\ ms$ and a range of n values

Chapter 3

Preliminary Work

The previous chapter introduces the idea of switching between the impedance control and the admittance control in order to improve the the performance based on the stiffness of the environment. However, the proof of stability using a DES is inadequate since it is unable to handle analysis of switching for the sequence of admittance control to impedance control, and back to admittance control due to the singularity of the resulting matrix. Therefore, we begin this section by providing a new stability theorem that does not have the same issue. We then briefly investigate why we choose to switch controllers instead of producing a feedback control that is a weighted combination of the admittance control and impedance control.

The gains of the position controller used in the admittance controller are large and may be diificult to implement in experiments due to hardware restrictions. Therefore, we revisit the nominal plant stability and simulations of the previous section using smaller position gains. From this investigation we find that the position gains effect the performance of the switched system. Therefore, we propose a new method of defining the additional states of the admittance controller at the instant of switching to help compensate for the effect of smaller gains. Finally, we discuss a method for solving the new switching method using only measured values of the external force instead of using a known function for the external force.

3.1 Stability

Using the same notation used in chapter 2 for a single degree-of-freedom switched system we express the states at $t = t_0 + (k+1)\delta$ in terms of the states at time $t = t_0 + k\delta$ by, namely

$$\begin{aligned}
X_i(t_0 + (k+1-n)\delta) &= e^{A_i(1-n)\delta} X_i(t_0 + k\delta) \\
X_a(t_0 + (k+1-n)\delta) &= S_{ai} e^{A_i(1-n)\delta} X_i(t_0 + k\delta) \\
X_a(t_0 + (k+1)\delta) &= e^{A_a n\delta} X_a(t_0 + (k+1-n)\delta) \\
X_i(t_0 + (k+1)\delta) &= S_{ia} e^{A_a n\delta} X_a(t_0 + (k+1-n)\delta) \\
X_i(t_0 + (k+1)\delta) &= S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} X_i(t_0 + k\delta) \qquad (3.1)
\end{aligned}$$

We now define the matrix A_{dis}

$$A_{dis} = S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)} \qquad (3.2)$$

such that

$$X_i(t_0 + (k+1)\delta) = A_{dis} X_i(t_0 + k\delta) \tag{3.3}$$

the matrix A_{dis} defines a discrete map of states from time $t = t_0 + k\delta$ to $t = t_0 + (k+1)\delta$ independent of k. We now have the following theorem.

Theorem 2. *The origin of the switched system described by (2.16), (2.24), and (2.26) is asymptotically stable iff all eigenvalues of A_{dis} lie within the open unit circle about the origin.*

Proof. For convenience we define the following variables:

$$\begin{aligned} A_1(\tau) &= e^{A_i(1-n)\tau} &(3.4)\\ A_2(\tau) &= S_{ia} e^{A_a n\tau} S_{ai} e^{A_i(1-n)\delta} &(3.5)\\ M &= \max_{\tau\in[0,\delta]} \{\|A_1(\tau)\|^2, \|A_2(\tau)\|^2\} &(3.6)\end{aligned}$$

Sufficiency: From discrete control theory, the system described by equation (3.3) is asymptotically stable iff the eigenvalues of A_{dis} lie within the open unit circle about at the origin [16]. Then, from the discrete linear converse Lyapunov theorem there exists a $V(x)$ given by [16]

$$V(X_i(t_0 + k\delta)) = X_i^T(t_0 + k\delta) P X_i(t_0 + k\delta) \tag{3.7}$$

for a positive definite P, such that

$$V(X_i(t_0 + (k+1)\delta)) - V(X_i(t_0 + k\delta)) \leq X_i^T(t_0 + k\delta)QX_i(t_0 + k\delta) \tag{3.8}$$

where Q is negative definite. From equations (3.6), (3.7), and (3.1) we may find

$$V(X_i(t_0 + \epsilon)) \leq M[X_i^T(t_0)PX_i(t_0)] \quad \forall \epsilon \in [0, \delta] \tag{3.9}$$

$$\Rightarrow V(X_i(t_0 + k\delta + \epsilon)) \leq M[X_i^T(t_0 + k\delta)PX_i(t_0 + k\delta)] \quad \forall \epsilon \in [0, \delta] \tag{3.10}$$

From equation (3.7) and (3.10) that

$$V(X_i(t_0 + k\delta + \epsilon)) \leq MV(X_i(t_0 + k\delta)) \quad \forall \epsilon \in [0, \delta] \tag{3.11}$$

and since equation (3.8) the sequence of functions $V(X_i(t_0 + k\delta))$ is monotonically decreasing, we may write equation (3.11) as

$$V(X_i(t_0 + k\delta + \epsilon)) \leq MV(X_i(t_0)) \quad \forall \epsilon \in [0, \delta] \tag{3.12}$$

$$V(X_i(t)) \leq MV(X_i(t_0)) \quad \forall t \geq t_0 \tag{3.13}$$

For asymptotic stability we wish to show that $X_i(t) \to 0$ as $t \to \infty$. To do this we consider that, $V(X_i(t_0 + \delta))$, $V(X_i)(t_0 + 2\delta))$, ... is decreasing and lower bounded, therefore the limit of $V_(X_i)$ as $k \to \infty$ and may be given by $L \geq 0$ [21]. Then we have

$$
\begin{aligned}
&0 = L - L = \lim_{k\to\infty} V(X_i(t_0 + (k+1)\delta)) - \lim_{k\to\infty} V(X_i(t_0 + k\delta)) = \\
&\lim_{k\to\infty} \left[V(X_i(t_0 + (k+1)\delta)) - V(X_i(t_0 + k\delta))\right] \leq \\
&\lim_{k\to\infty} \left[X_i^T(t_0 + k\delta) Q X_i(t_0 + k\delta)\right] \leq 0
\end{aligned} \tag{3.14}
$$

therefore, $\|X_i(t_0+k\delta)\| \to 0$ as $k \to \infty$ since Q is negative definite. Therefore, $X_i(t) \to 0$ as $t \to \infty$.

Necessity for the proof of necessity we use contradiction. Assume that the switched system is asymptotically stable, but that A_{dis} has at least one eigenvalue not in the open unit circle. Then the discrete system is not asymptotically stable. Therefore, there exists a choice $X_i(t_0) \in B_\eta$, and an $N > 0$ such that

$$
\|X_i(t_0 + k\delta)\| \geq N \quad \forall k \geq 0 \tag{3.15}
$$

However, from the fact that the switched system is asymptotically stable we have

$$
\lim_{t\to\infty} \|X_i(t)\| = 0 \tag{3.16}
$$

implying that

$$
\lim_{k\to\infty} \|X_i(t_0 + k\delta)\| = 0 \tag{3.17}
$$

Therefore, there exists a $\lambda > 0$ such that if $k > \lambda$ then

$$\|X_i(t_0 + k\delta)\| < N \tag{3.18}$$

which is a contradiction of equation (3.15).

$\square$

Remark 3. The stability analysis is again based on switching from impedance control to admittance control, and then back to impedance control. This sequence leads to a 2×2 matrix A_{dis}, as shown in equation (3.2). If we instead switched from the admittance control to the impedance control, and back to the admittance control the matrix A_{dis} would be a 4×4 matrix given by

$$A_{dis} = S_{ai} e^{A_i(1-n)} S_{ia} e^{A_a n\delta} \tag{3.19}$$

However, this does not change the stability analysis, since for any two matrices $A \in R^{m \times n}$ and $B \in R^{n \times m}$ with $m < n$, the non-zero eigenvalues of AB and BA are the same except for the $(n - m)$ additional eigenvalues of BA which are identically zero [9], which is in the open unit circle.

3.2 Comparison of Additive Control

The control method proposed thus far is for a switching based control strategy. However, switching is only one possible method for hybrid control. Therefore, we will now briefly investigate a hybrid method that uses a convex combination of control inputs. Namely, let the input force be given by

$$F = nF_a + (1-n)F_i \tag{3.20}$$

where F_a is given by equations (2.10) and (2.9), F_i is given by equation (2.7), and $n \in [0,1]$. We again assume that x_0 is a constant point, and that F_{ext} is given by equation (2.14). Then by choosing n to be constant yields a time invariant linear differential equation of the form

$$\dot{X} = AX \tag{3.21}$$

where X is given by

$$X = [e\ , \quad \dot{e}\ , \quad e_d\ , \quad \dot{e}_d]^T \tag{3.22}$$

and A is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ A_1 & A_2 & n\frac{Kp}{m} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_e}{M_\theta} & 0 & -\frac{K_\theta}{M_\theta} & -\frac{D_\theta}{M_\theta} \end{bmatrix} \tag{3.23}$$

where A_1 and A_2 are given by

$$A_1 = -[n\frac{K_p + k_e}{m} + (1-n)\frac{K_\theta + k_e}{M_\theta}] \tag{3.24}$$

$$A_2 = -[n\frac{K_d}{m} + (1-n)\frac{D_\theta}{M_\theta}] \tag{3.25}$$

Then the system described by X is asymptotically stable if and only if A is Hurwitz.

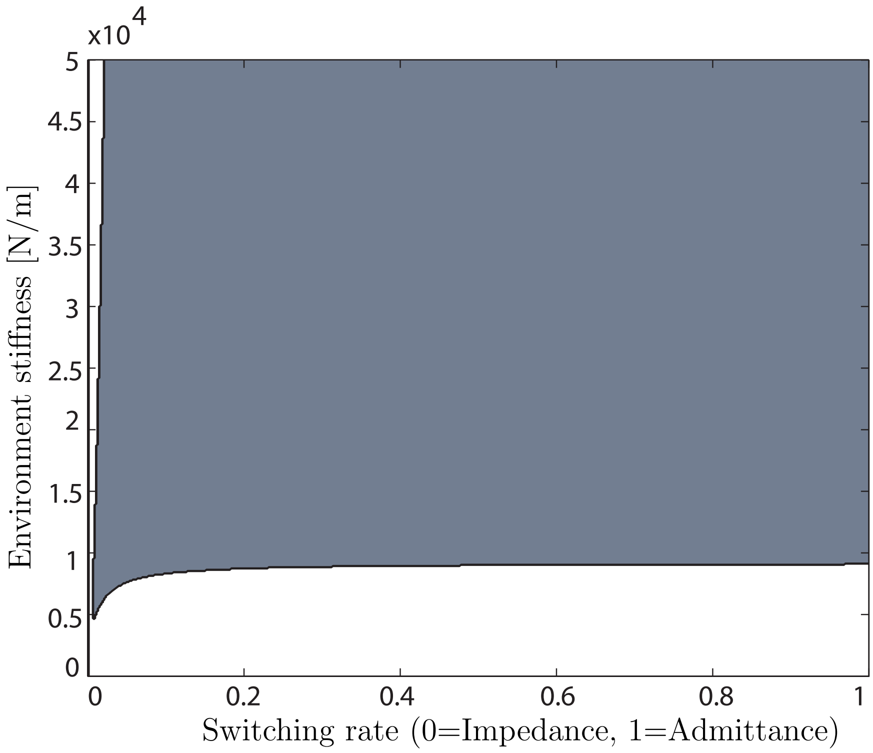

Figure 3.1: Stability boundaries for convex combination of impedance and admittance control

Therefore, we are able to compare the stability region produced by the convex combination of impedance and admittance control for different external stiffness compared to the switching method. To do this we consider the parameters of a linear single degree of freedom system given by (2.33). Figure 3.1 shows the stability properties of the combination of controllers for different values of n and k_e with the grey portion being unstable, and the white portion being stable. We see from the figure that the set of values (n, k_e) for the convex combination control method is very restrictive compared to the switching method. The resulting closed-loop system is unstable even for small values of n, which approximates impedance control.

3.3 Stability of Nominal Plant

We revisit stability of the nominal system using theorem 2. Additionally, we explore the effect of the gains used in the position controller. We note that the proportional value K_p used previously is very large and therefore the position control torque might be larger than the hardware permits. Therefore, let us again consider the single degree-of-freedom linear system with parameters given by (2.33) with the exception that

$$K_p = 10^4 \, \frac{N}{m} \qquad K_d = 1.4\sqrt{K_p \, m} \, \frac{N \, s}{m} \tag{3.26}$$

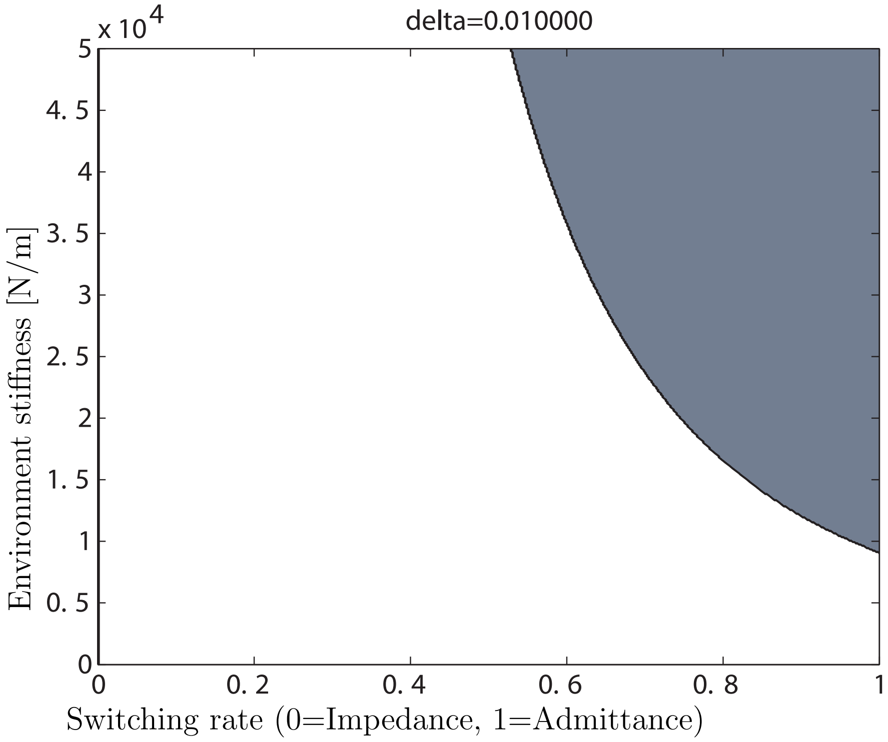

Figure 3.2: Stability boundaries for hybrid impedance and admittance control for $\delta = 10 \; ms$ and $K_p = 10000 \; N/m$

Figure 3.2 shows the region in which the controlled system is stable (in white), and the region in which the controlled system is unstable (in grey) for $\delta = 10 \; ms$ and for varying

k_e and n values. We notice that the separation between stable an unstable regions are smooth similar to the when K_p is large at the value $\delta = 10\ ms$ in Figure (2.3).

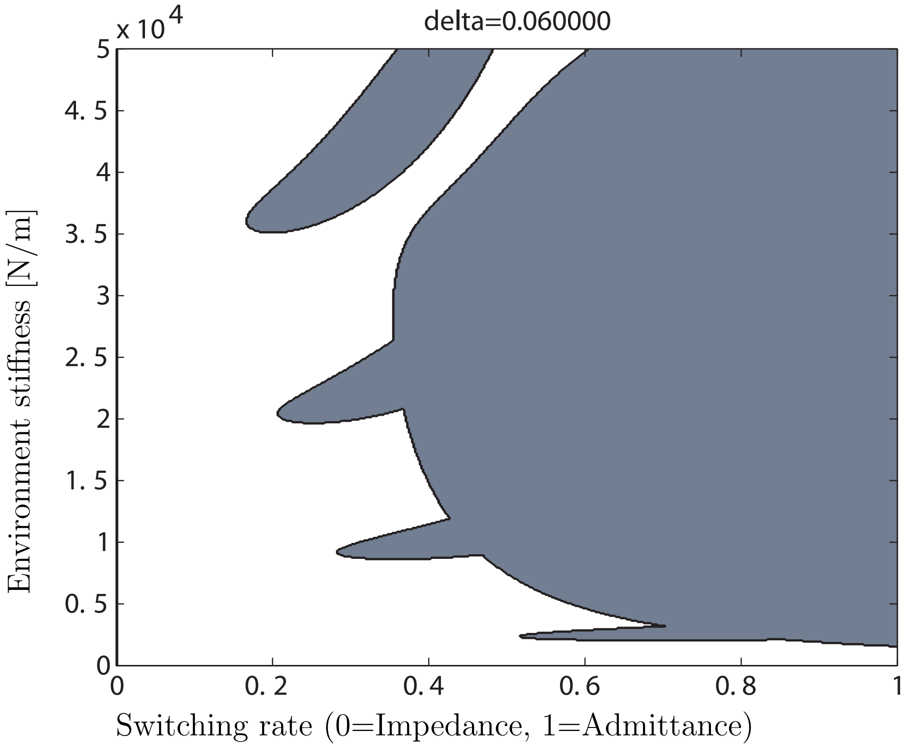

Figure 3.3: Stability boundaries for hybrid impedance and admittance control for $\delta = 60\ ms$ and $K_p = 10000\ N/m$

While figure 3.2 is informative, it does not provide a complete picture of the region of stability to compare to the case with larger value K_p. A plot of the stability region for $\delta = 50ms$ in Figure 3.3 shows similar features to figure 2.4. However, we notice that when $k_e > 35000\ N/m$ the regions where the system is unstable are not connected. This indicates that reducing the value of K_p causes the the boundary between stable and unstable pairs (n, k_e) to deform irregularly as δ increases.

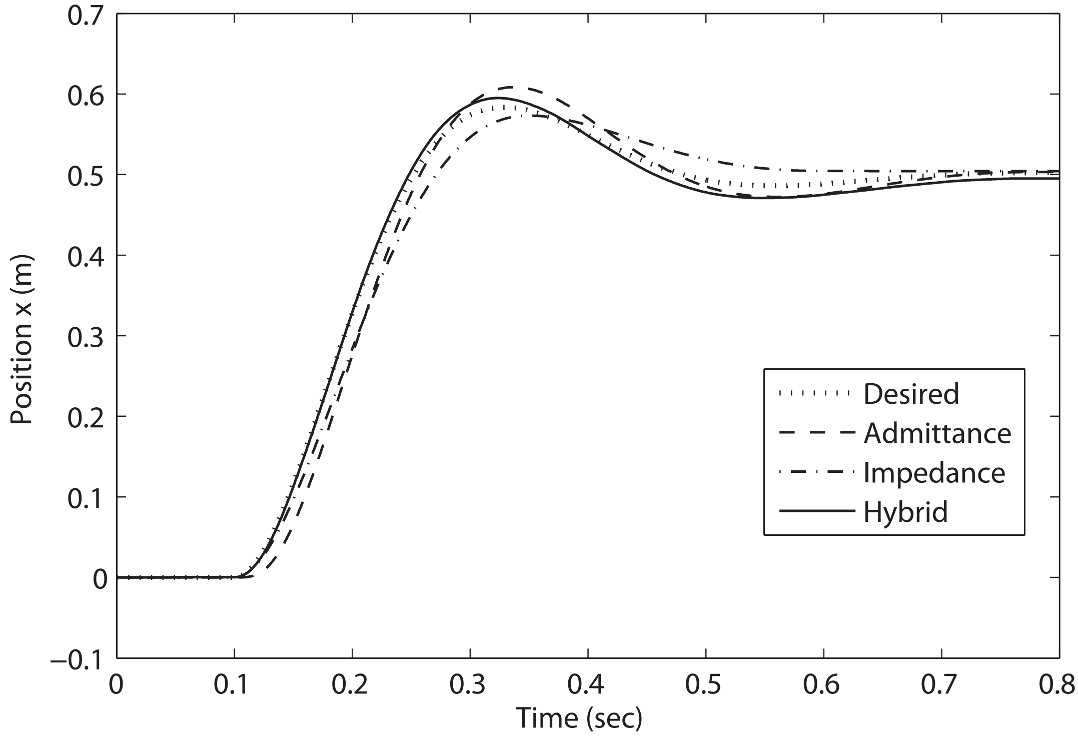

Figure 3.4: Soft contact position response for $K_p = 10000\ N/m$, $\delta = 50\ ms$, and $n = 0.9$

3.4 Simulations

We see from the previous section that the position gains can change the stability characteristics of the hybrid system. We now wish to see how a similar change in gains effect performance. Let us again consider the single degree of freedom system whose model is given by equation (2.1). Let us also choose the parameters of the system to be given by (2.34), except we now change K_p to be

$$K_p = 10^4\ \frac{N}{m} \tag{3.27}$$

which also changes the value of K_d since it is based on the value of K_p. The external stiffness is chosen to be $k_e = 100\ N/m$. We again begin with $n = 0.9$. Figure 3.4 shows

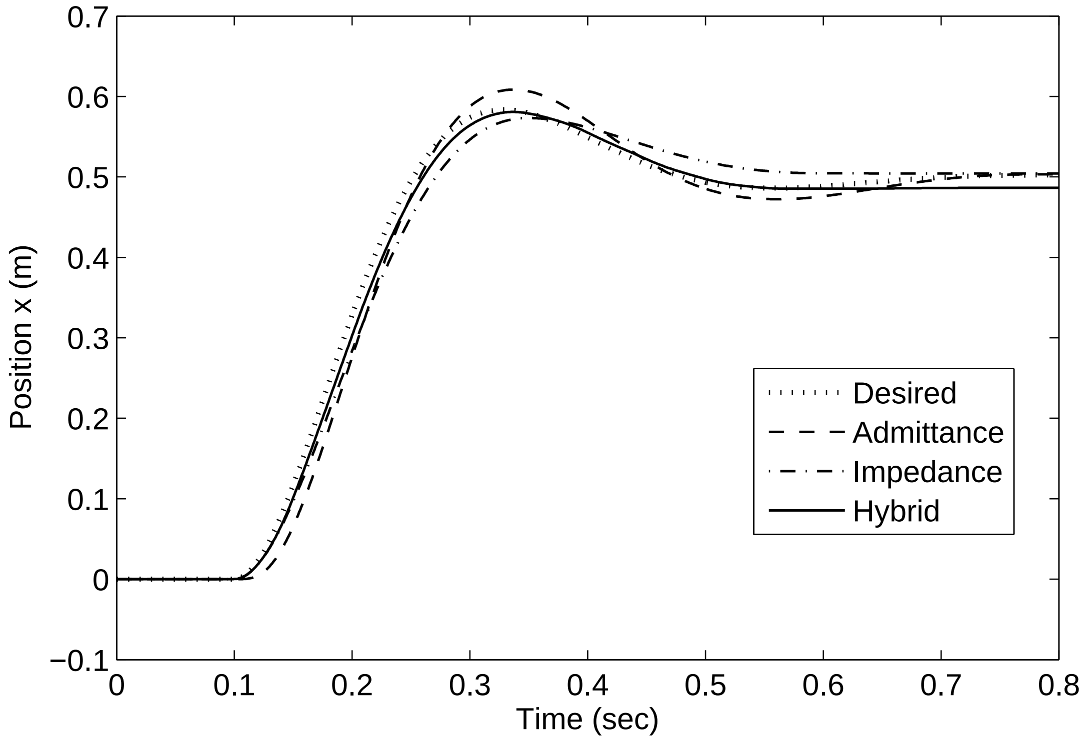

Figure 3.5: Soft contact position response for $K_p = 10000\ N/m$, $\delta = 50\ ms$, and $n = 0.5$

the response of the admittance controller, impedance controller, hybrid controller, and the ideal response. We notice immediately that the response of the hybrid controller is not a good approximation of the admittance controller as it was before, instead we see that the hybrid controller results in a steady state error as well as the fact that the admittance controller no longer closely approximates the ideal response. To further investigate we consider Figure 3.5 which shows the response of the admittance, impedance, and hybrid controllers as well as the desired response for $n = 0.5$. We notice that contrary to the case where K_p is very large, for K_p smaller the hybrid controller no longer approximates an average between the impedance control response and the hybrid response when $n = 1$. Instead the hybrid controller produces a larger steady state error when $n = 0.5$ than the impedance controller. The deviation in behaviour is because the position controller used in the admittance controller does not track a trajectory, but instead assumes that x_d is a sequence of fixed points with a disturbance F_{ext}. Thus, the gains of position controller greatly effect the performance of the admittance controller.

3.5 Change of Switching Conditions

A large factor in the performance of the switched system is the position control gain used in the admittance algorithm. This is partly because the control force in (2.9) is based on the assumption that x_d is a constant. Therefore, choosing the states x_d and $\dot{x}_d$ (when switching from the impedance controller to the admittance controller) such that the force and force derivative are continuous, is not necessarily optimal. We change the algorithm as follows for the single degree-of-freedom case.

We have the impedance control torque to be given by (2.7). We then change the admittance torque to be given by a PD controller instead of just a position controller giving

$$F = F_a = -K_p(x - x_d) - K_d(\dot{x} - \dot{x}_d) \tag{3.28}$$

with x_d, $\dot{x}_d$ to again be given by solutions to (3.28). We again let the external force, F_{ext}

$$F_{ext} = -k_e(x - x_0) \tag{3.29}$$

with x_0 again being a constant.

We then consider the hybrid switching as proposed in (2.13) giving

$$
\begin{aligned}
\dot{X}_i &= A_i X_i \ : \ t \in [t_0 + k\delta, t_0 + (k+1-n)\delta) \\
\dot{X}_a &= A_a X_a \ : \ t \in [t_0 + (k+1-n)\delta, t_0 + (k+1)\delta)
\end{aligned} \tag{3.30}
$$

where

$$X_i = (e \quad \dot{e})^T \tag{3.31}$$

$$e = q - q_0 \tag{3.32}$$

$$A_i = \begin{bmatrix} 0 & 1 \\ -M_\theta^{-1}(K_\theta + k_e) & -M_\theta^{-1} D_\theta \end{bmatrix} \tag{3.33}$$

$$X_a = (e \quad \dot{e} \quad e_d \quad \dot{e}_d)^T \tag{3.34}$$

$$e_d = q_d - q_0 \tag{3.35}$$

and

$$A_a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -M^{-1}(K_p + k_e) & -M^{-1}K_d & M^{-1}K_p & M^{-1}K_d \\ 0 & 0 & 0 & 1 \\ -M_\theta^{-1}k_e & 0 & -M_\theta^{-1}K_\theta & M_\theta^{-1}D_\theta \end{bmatrix} \tag{3.36}$$

When switching from the admittance to the impedance controller, we have

$$X_i = S_{ia}X_a, \quad S_{ia} = [I \;\; 0] \tag{3.37}$$

at the instant of switching.

When switching form impedance controller to the admittance controller we have two additional states, x_d and $\dot{x}_d$, which must be defined. We wish to find x_d such that the control force is continuous. Using (3.28) as the expression for the control force of the admittance control we have

$$x_d((t_k + (1-n)\delta)^+) = \lim_{t \to (t_k + (1-n)\delta)^-} \left[x + K_p^{-1}(F_i + K_d(\dot{x} - \dot{x}_d)) \right] \tag{3.38}$$

$$\Rightarrow e_d((t_k + (1-n)\delta)^+) = \lim_{t \to (t_k + (1-n)\delta)^-} \left[e + K_p^{-1}(F_i + K_d(\dot{e} - \dot{e}_d)) \right] \tag{3.39}$$

Instead of defining $\dot{x}_d$ such that the derivative of the force is continuous at the instant of switching, we consider it to be chosen later. We may then obtain the expression

$$X_a = S_{ai}X_i + B_{ai}\dot{e}_d, \quad S_{ai} = \begin{bmatrix} I \\ S \end{bmatrix} \tag{3.40}$$

at the instant of switching. Where S is given by

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & 0 \end{bmatrix} \tag{3.41}$$

$$S_{11} = 1 - K_p^{-1} k_e (M M_d^{-1} - 1) - K_p^{-1} K_\theta M M_\theta^{-1} \tag{3.42}$$

$$S_{12} = K_p^{-1} K_d - K_p^{-1} D_\theta M M_\theta^{-1} \tag{3.43}$$

and B_{ai} is given by

$$B_{ai} = \begin{bmatrix} 0 \\ 0 \\ -K_p^{-1} K_d \\ 1 \end{bmatrix} \tag{3.44}$$

For brevity, let us define t_k such that $t_k = t_0 + k\delta$, $k \in Z_+$. Then, knowing the states at time $t = t_k$, the states at $t = t_k + \delta$ can be obtained using equations (3.30), (3.37), and (3.40) as

$$
\begin{aligned}
X_i(t_k + (1-n)\delta) &= e^{A_i(1-n)\delta} X_i(t_k) \\
X_a(t_k + (1-n)\delta) &= S_{ai} e^{A_i(1-n)\delta} X_i(t_k) + B_{ai}\dot{e}_d \\
X_a(t_k + \delta) &= e^{A_a n\delta} X_a(t_k + (1-n)\delta) \\
X_i(t_k + \delta) &= S_{ia} e^{A_a n\delta} X_a(t_k + (1-n)\delta) \\
X_i(t_k + \delta) &= S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} X_i(t_k) \\
&\quad + S_{ia} e^{A_a n\delta} B_{ai}\dot{e}_d
\end{aligned} \tag{3.45}
$$

We define A_{dis} as follows

$$A_{dis} = S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} \tag{3.46}$$

and B_{dis} as follows

$$B_{dis} = S_{ia} e^{A_a n\delta} B_{ai} \tag{3.47}$$

This gives

$$X_i(t_k + \delta) = A_{dis} X_i(t_k) + B_{dis}\dot{e}_d \tag{3.48}$$

Let us now define the following as the desired behaviour of the closed loop system.

$$\dot{X}_{des} = A_{des} X_{des} \tag{3.49}$$

with X_{des}, and A_{des} given by

$$X_{des} = [e, \quad \dot{e}]^T \tag{3.50}$$

$$A_{des} = \begin{bmatrix} 0 & 1 \\ -M_\theta^{-1}(K_\theta + k_e) & -M_\theta^{-1} D_\theta \end{bmatrix} \tag{3.51}$$

Solving (3.49) from time t_k to time $t_k + \delta$ gives

$$X_{des}(t_k + \delta) = e^{A_{des}\delta} X_{des}(t_k) \tag{3.52}$$

Let A_d be given by

$$A_d = e^{A_{des}\delta} \tag{3.53}$$

such that

$$X_{des}(t_k + \delta) = A_d X_{des}(t_k) \tag{3.54}$$

In the next section, we define $\dot{e}_d$ at the instant when we switch from the impedance controller to the admittance controller. We note that the matrix A_{dis} and the vector B_{dis} depend on the value of n, and that B_{dis} is zero when $n = 0$ since B_{ai} is in the null space of S_{ia}. However, we also find that $A_{dis} = A_d$ when $n = 0$ making the case trivial.

3.5.1 Matching Eigenvalues

The first method of choosing $\dot{e}_d$ is have it be a feedback of the form

$$\dot{e}_d = K_{ei}\ X_i(t_k) \tag{3.55}$$

where K_{ei} is a matrix chosen such that the eigenvalues of A_d are the same as the eigenvalues of $A_{dis}+B_{dis}K_{ei}$. This method produces similar response times from the switched system and the desired system. While the requirement that the pair (A_{dis}, B_{dis}) match all eigenvalues is not very restrictive, the resulting gains do not necessarily produce a smooth space with respect to variations in (n, k_e). Also, matching the eigenvalues does not take the eigenvectors into consideration producing different responses as the order of the system increases.

3.5.2 Minimizing Discrete Difference Between Desired and Actual Behaviour

The goal is then to minimize $|X_{des}(t_k+\delta) - X_i(t_k+\delta)|$. To solve this we may write

$$[A_d - A_{dis}]\, X_i(t_k) - B_{dis}\dot{e}_d = 0 \tag{3.56}$$

Using the Moore-Penrose inverse we find

$$\dot{e}_d = \left[B_{dis}^T B_{dis}\right]^{-1} B_{dis}^T [A_d - A_{dis}] X_i(t_k) \tag{3.57}$$

We define K_h to be

$$K_h = \left[B_{dis}^T B_{dis}\right]^{-1} B_{dis}^T [A_d - A_{dis}] \tag{3.58}$$

such that

$$\dot{e}_d = K_h X_i(t_k) \tag{3.59}$$

Then, we have the discrete mapping of the switched system to be given by

$$X_i(t_k + \delta) = (A_{dis} + B_{dis} K_h) X_i(t_k) \tag{3.60}$$

3.6 Linear Separation of External Force in Switching Condition

The change of switching conditions mentioned in the previous section is based on prior knowledge of the external force F_{ext} . However, the form of the equation of the external force is not generally known, but can be measured using a force sensor. Therefore, we now rederive the method in the last section accounting for only measurements of the external force and not the actual equation in (3.29).

3.6.1 Derivation

Let us consider the external force as an unknown input into the closed loop differential equation. Then the system dynamics when the impedance controller is applied in equation (2.7) may be written as

$$\dot{X}_i = A_i X_i + B_i F_{ext} \tag{3.61}$$

where X_i is given by

$$X_i = \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \tag{3.62}$$

A_i is given by

$$A_i = \begin{bmatrix} 0 & 1 \\ -M_\theta^{-1} K_\theta & -M_\theta^{-1} K_\theta \end{bmatrix} \tag{3.63}$$

and B_i is given by

$$B_i = \begin{bmatrix} 0 \\ M_\theta^{-1} \end{bmatrix} \tag{3.64}$$

The system dynamics when the admittance control is applied according to equations (3.30), (3.34), and (3.36) may be written as

$$\dot{X}_a = A_a X_a + B_a F_{ext} \tag{3.65}$$

where X_a is given by

$$X_a = \begin{bmatrix} e \\ \dot{e} \\ e_d \\ \dot{e}_d \end{bmatrix} \quad (3.66)$$

A_a in equation (3.65) is given by

$$A_a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -M^{-1}K_p & -M^{-1}K_d & M^{-1}K_p & M^{-1}K_d \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -M_\theta^{-1}K_\theta & M_\theta^{-1}D_\theta \end{bmatrix} \quad (3.67)$$

And B_a is given by

$$B_a = \begin{bmatrix} 0 \\ M^{-1} \\ 0 \\ M_\theta^{-1} \end{bmatrix} \quad (3.68)$$

Solving equations (3.61) and (3.65) give the general solutions

$$X_i(t) = e^{A_i(t-t_0)}X_i(t_0) + \int_{t_0}^{t} e^{A_i t-\tau} B_a F_{ext}(\tau) d\tau \quad (3.69)$$

and

$$X_a(t) = e^{A_a(t-t_0)}X_a(t_0) + \int_{t_0}^{t} e^{A_a t-\tau} B_i F_{ext}(\tau) d\tau \quad (3.70)$$

respectively. Then, for switched system we have

$$\dot{X}_i = A_i X_i + B_i F_{ext} \quad \forall t \in [t_k, t_k + (1-n)\delta) \tag{3.71}$$

$$\dot{X}_a = A_a X_a + B_a F_{ext} \quad \forall t \in [t_k + (1-n)\delta, t_k + \delta) \tag{3.72}$$

for a positive integer, k, and for $0 \leq n \leq 1$.

At time t_k the system is switched from the admittance controlled system to the impedance controlled system. We then have the change of states given by the mapping

$$X_i = S_{ia} X_a \qquad S_{ia} = [I \quad 0] \tag{3.73}$$

At the time $t_k + (1-n)\delta$, the system is switched from the impedance controlled system to the admittance controlled system. This results in additional states e_d and $\dot{e}_d$ where e_d is chosen such that the control force is continuous and $\dot{e}_d$ is chosen later as in the previous section. From (3.38) we have

$$e_d((t_k + (1-n)\delta)^+) = \lim_{t \to (t_k + (1-n)\delta)^-} \left[e + K_p^{-1}(F_1 + K_d(\dot{e} - \dot{e}_d)) \right] \tag{3.74}$$

Then, the change in states at time $t_k + (1-n)\delta$ may be given by

$$X_a = S_{ai} X_i + K_{ai} F_{ext} + B_{ai} \dot{e}_d \qquad S_{ia} = \begin{bmatrix} I \\ S \end{bmatrix} \tag{3.75}$$

where S is given by

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & 0 \end{bmatrix} \tag{3.76}$$

$$S_{11} = 1 - K_p^{-1} K_\theta M M_\theta \tag{3.77}$$

$$S_{12} = K_p^{-1} K_d - K_p^{-1} D_\theta M M_\theta \tag{3.78}$$

B_{ai} is given by

$$B_{ai} = \begin{bmatrix} 0 \\ 0 \\ -K_p^{-1} K_d \\ 1 \end{bmatrix} \tag{3.79}$$

and K_{ai} is given by

$$K_{ai} = \begin{bmatrix} 0 \\ 0 \\ K_p^{-1}(M M_\theta^{-1} - 1) \\ 0 \end{bmatrix} \tag{3.80}$$

Solving for the general solution of the switched system from time t_k to time $(t_k + \delta)$ we find

$$
\begin{aligned}
X_i(t_k+\delta) =& S_{ia}e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} X_i(t_k) \\
&+ S_{ia}e^{A_a(t_k+\delta)} \int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau} B_a F_{ext}(\tau) d\tau \\
&+ S_{ia}e^{A_a n\delta} S_{ai} e^{A_i(t_k+(1-n)\delta)} \int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau} B_i F_{ext}(\tau) d\tau \\
&+ S_{ia}e^{A_a n\delta} K_{ai} F_{ext}(t_k+(1-n)\delta) + S_{ia}e^{A_a n\delta} B_{ai}\dot{e}_d
\end{aligned}
\tag{3.81}
$$

We now write the desired behaviour in equation (3.49) as

$$
X_{des} = A_{des}X_{des} + B_{des}F_{ext} \tag{3.82}
$$

Where A_{des} is given by

$$
A_{des} = \begin{bmatrix} 0 & 1 \\ -M_\theta^{-1}K_\theta & -M_\theta^{-1}D_\theta \end{bmatrix} \tag{3.83}
$$

and B_{des} given by

$$
B_{des} = \begin{bmatrix} 0 \\ M_\theta^{-1} \end{bmatrix} \tag{3.84}
$$

Then, the general solution of X_{des} from t_k to $(t_k+\delta)$ is given by

$$
X_{des}(t) = e^{A_{des}(t-t_0)} X_{des}(t_k) + \int_{t_k}^{t_k+\delta} e^{A_{des}(t-\tau)} B_{des1} F_{ext}(\tau) d\tau \tag{3.85}
$$

As in section 3.5.2 we have the goal is to choose $\dot{x}_d$ at $t_k + (1-n)\delta$ such that

$$\|X_{des}(t_k + \delta) - X_i(t_k + \delta)\| = 0 \tag{3.86}$$

Substituting equations (3.81) and (3.85) into (3.86) we get

$$\begin{aligned}
&e^{A_{des}\delta} X_{des}(t_k) + e^{A_{des}(t_k+\delta)} \int_{t_k}^{t_k+\delta} e^{-A_{des}\tau} B_{des} F_{ext}(\tau) d\tau - \\
&[S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} X_i(t_k) + S_{ia} e^{A_a(t_k+\delta)} \int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau} B_a F_{ext}(\tau) d\tau \\
&+ S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(t_k+(1-n)\delta)} \int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau} B_i F_{ext}(\tau) d\tau \\
&+ S_{ia} e^{A_a n\delta} K_{ai} F_{ext}(t_k + (1-n)\delta + S_{ia} e^{A_a n\delta} B_{ai} \dot{e}_d] = 0
\end{aligned} \tag{3.87}$$

To solve equation (3.87) we will consider $\dot{x}_d$ to be of the form

$$\dot{e}_d = u_h + u_p \tag{3.88}$$

where u_h minimizes the homogeneous portion of (3.87), when $F_{ext} \equiv 0$. And u_p minimizes the additional terms when $F_{ext} \neq 0$. Then, to find u_h we find equation (3.87) may be written as

$$[e^{A_{des}\delta} X_{des}(t_k) - S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} X_i(t_k)] - e^{A_a n\delta} B_{ai} u_h = 0 \tag{3.89}$$

Letting $X_{des}(t_k) = X(t_k)$, (3.89) simplifies to

$$[e^{A_{des}\delta} - S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)\delta}]X_i(t_k) - e^{A_a n\delta}B_{ai}u_h = 0 \tag{3.90}$$

using the Moore-Penrose inverse we find u_h to be given by

$$u_h = \left[(S_{ia}e^{A_a n\delta}B_{ai})^T S_{ia}e^{A_a n\delta}B_{ai}\right]^{-1}(S_{ia}e^{A_a n\delta}B_{ai})^T$$
$$[e^{A_{des}\delta} - S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)\delta}]X_i(t_k) \tag{3.91}$$

We will then let K_h be given by

$$K_h = \left[(S_{ia}e^{A_a n\delta}B_{ai})^T S_{ia}e^{A_a n\delta}B_{ai}\right]^{-1}(S_{ia}e^{A_a n\delta}B_{ai})^T$$
$$[e^{A_{des}\delta} - S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)\delta}] \tag{3.92}$$

such that

$$u_h = K_h X_i(t_k) \tag{3.93}$$

Since switching occurs at $t_k + (1-n)\delta$ it is desirable to write u_h in the form

$$u_h = K_h(e^{A_i(1-n)\delta})^{-1}X_i(t_k + (1-n)\delta) \tag{3.94}$$

By substituting (3.94) and (3.88) into (3.87) we find

$$
\begin{aligned}
&e^{A_{des}\delta}X_{des}(t_k) + e^{A_{des}(t_k+\delta)}\int_{t_k}^{t_k+\delta} e^{-A_{des}\tau}B_{des}F_{ext}(\tau)d\tau - \\
&[S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)\delta}X_i(t_k) + S_{ia}e^{A_a(t_k+\delta)}\int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau}B_aF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(t_k+(1-n)\delta)}\int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau}B_iF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_a n\delta}B_{in}K_h(e^{A_i(1-n)\delta})^{-1}X_i(t_k+(1-n)\delta) \\
&+ S_{ia}e^{A_a n\delta}K_{ai}F_{ext}(t_k+(1-n)\delta) + S_{ia}e^{A_a n\delta}B_{ai}u_p] = 0
\end{aligned} \tag{3.95}
$$

Using equation (3.69) we write (3.95) as

$$
\begin{aligned}
&\{e^{A_{des}\delta}X_{des}(t_k) - S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)\delta}X(t_k) - S_{ia}e^{A_a n\delta}B_{ai}K_hX_i(t_k)\}+ \\
&e^{A_{des}(t_k+\delta)}\int_{t_k}^{t_k+\delta} e^{-A_{des}\tau}B_{des}F_{ext}(\tau)d\tau - \\
&[S_{ia}e^{A_a(t_k+\delta)}\int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau}B_aF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(t_k+(1-n)\delta)}\int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau}B_iF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_a n\delta}B_{ai}K_he^{A_it_k}\int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau}B_iF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_a n\delta}K_{ai}F_{ext}(t_k+(1-n)\delta) + S_{ia}e^{A_a n\delta}B_{ai}u_p] = 0
\end{aligned} \tag{3.96}
$$

From equations (3.89), (3.93), and from linearity we have the choice of u_p can only be chosen to minimize portions of (3.96) containing the external force. Thus, (3.96) simplifies to.

$$
\begin{aligned}
&e^{A_{des}(t_k+\delta)}\int_{t_k}^{t_k+\delta} e^{-A_{des}\tau}B_{des}F_{ext}(\tau)d\tau - \\
&[S_{ia}e^{A_a(t_k+\delta)}\int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau}B_aF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_an\delta}S_{ai}e^{A_i(t_k+(1-n)\delta)}\int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau}B_iF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_an\delta}B_{ai}K_he^{A_it_k}\int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau}B_iF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_an\delta}K_{ai}F_{ext}(t_k+(1-n)\delta) + S_{ia}e^{A_an\delta}B_{ai}u_p] = 0
\end{aligned}
\tag{3.97}
$$

Solving for u_p gives

$$
\begin{aligned}
u_p = &\left[(S_{ia}e^{A_an\delta}B_{ai})^TS_{ia}e^{A_an\delta}B_{ai}\right]^{-1}(S_{ia}e^{A_an\delta}B_{ai})^T \\
&\{e^{A_{des}(t_k+\delta)}\int_{t_k}^{t_k+\delta} e^{-A_{des}\tau}B_{des}F_{ext}(\tau)d\tau - \\
&[S_{ia}e^{A_a(t_k+\delta)}\int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau}B_aF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_an\delta}S_{ai}e^{A_i(t_k+(1-n)\delta)}\int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau}B_iF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_an\delta}B_{in}K_he^{A_it_k}\int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau}B_iF_{ext}(\tau)d\tau] \\
&- S_{ia}e^{A_an\delta}K_{ai}F_{ext}(t_k+(1-n)\delta)\}
\end{aligned}
\tag{3.98}
$$

If we define the functions F_p and H_p as

$$F_p = \left[(S_{ia}e^{A_a n\delta}B_{ai})^T S_{ia}e^{A_a n\delta}B_{ai}\right]^{-1}(S_{ia}e^{A_a n\delta}B_{ai})^T$$
$$\{e^{A_{des}(t_k+\delta)}e^{-A_{des}\tau}B_{des} - S_{ia}e^{A_a(t_k+\delta)}e^{-A_a\tau}B_a\} \quad (3.99)$$

$$H_p = -\left[(S_{ia}e^{A_a n\delta}B_{ai})^T S_{ia}e^{A_a n\delta}B_{ai}\right]^{-1}(S_{ia}e^{A_a n\delta}B_{ai})^T S_{ia}e^{A_a n\delta}K_{ai} \quad (3.100)$$

Then u_p may be expressed in the compact form:

$$u_p = \int_{t_k}^{t_k+\delta} F_p(t-\tau)F_{ext}(\tau)d\tau + H_p F_{ext}(t_k + (1-n)\delta) \quad (3.101)$$

We then find that the choice $\dot{e}_d = u_h + u_p$ is the same as the choice of $\dot{e}_d$ in section 3.5.2 due to uniqueness of the convolution mapping. However, the value of u_p must be solved numerically based on measured values of the external force on-line.

Chapter 4

N-DOF Rigid Joint Models

The theory in the previous chapters are applicable to a single degree-of-freedom linear model. In this chapter we now wish to extend the theory to multi-dimensional systems. We proceed by considering a multi-degree-of-freedom linear rigid joint system and deriving switching conditions based on continuity of control force and its derivative. We then generalize the control method and switching conditions for multi-degree-of-freedom non-linear systems.

4.1 Linear N-DOF

4.1.1 Equations of Motion

Assuming rigid joints, the system dynamics of a general linear N-DOF is given by

$$M\ddot{q} + C\dot{q} + Gq = \tau + J^T F_{ext} \tag{4.1}$$

where M is the symmetric positive definite mass matrix, C is a symmetric matrix acting as a form of damping, and G acts as a stiffness matrix. F_{ext} is the external force applied to the system, J is the Jacobian matrix transforming rotational coordinates to Euclidean coordinates, and τ is the input torque. The control objective is to the design τ such that $(q, \dot{q})$ satisfies the system of differential equations

$$M_\theta(\ddot{q} - \ddot{q}_0) + D_\theta(\dot{q} - \dot{q}_0) + K_\theta(q - q_0) = J^T F_{ext} \tag{4.2}$$

where M_θ, D_θ, and K_θ are positive definite matrices.

4.1.2 Impedance Control

In impedance control, we let the controller be a mechanical impedance. Therefore, by comparing equations (4.1) and (4.2), and solving for τ we find the impedance torque to be given by

$$\begin{aligned} \tau = \tau_i = M\ddot{q}_0 + C\dot{q} + Gq + MM_D^{-1}\left[-K_\theta(q - q_0) - D_\theta\dot{q}\right] \\ +(MM_d^{-1} - I)J^T F_{ext} \end{aligned} \tag{4.3}$$

Indeed, substituting (4.3) into (4.1) results in

$$M_\theta(\ddot{q} - \ddot{q}_0) + D_\theta(\dot{q} - \dot{q}_0) + K_\theta(q - q_0) = J^T F_{ext} \tag{4.4}$$

4.1.3 Admittance Control

For the admittance control, the control torque is a position controller to a trajectory q_d which is generated by the mechanical admittance. Therefore we consider the control torque to be given by

$$\tau = \tau_a = -K_p(q - q_d) - K_d(\dot{q}) + C\dot{q} + Gq \tag{4.5}$$

where K_p and K_d are positive definite matrices, and q_d is given by the solution to the desired behavior

$$M_\theta(\ddot{q}_d - \ddot{q}_0) + D_\theta(\dot{q}_d - \dot{q}_0) + K_\theta(q_d - q_0) = J^T F_{ext} \tag{4.6}$$

Substituting (4.5) into (4.1) results in the dynamic equations

$$M\ddot{q} + K_d\dot{q} + K_p(q - q_d) = J^T F_{ext} \tag{4.7}$$

$$M_d(\ddot{q}_d - \ddot{q}_0) + D_\theta(\dot{q}_d - \dot{q}_0) + K_\theta(q_d - x_0) = J^T F_{ext} \tag{4.8}$$

If q converges to q_d, by the virtue of gains chosen in (4.7), then (4.8) gives (4.2).

4.1.4 Hybrid Framework

For the linear N-DOF system described by equation (4.1), we propose the following switching torque

$$\tau = \begin{cases} \tau_i & : \quad t \in [t_0 + k\delta, t_0 + (k+1-n)\delta) \\ \tau_a & : \quad t \in [t_0 + (k+1-n)\delta, t_0 + (k+1)\delta) \end{cases} \tag{4.9}$$

where t_0 is the initial time, δ is the switching period, $n \in [0, 1]$ is the duty cycle, k is a positive integer. τ_i is given by equation (4.3), and τ_a is given by equations (4.5) and (4.6).

If the environment is modeled as a linear spring

$$J^T F_{ext} = -k_e(q - q_0) \tag{4.10}$$

for a positive definite definite matrix k_e, and the virtual trajectory q_0 is assumed to be constant, *i.e.*

$$\dot{q}_0 = \ddot{q}_0 = 0 \tag{4.11}$$

the hybrid system follows the descriptions

$$\dot{X} = \gamma(t) A_i X + (1 - \gamma(t)) A_a X \qquad \forall t = t_j \tag{4.12}$$

$$\Delta X = \Phi_j X \qquad \forall t = t_j \tag{4.13}$$

where t_j defines the instants of switching between the impedance and admittance control torques and may be written as

$$t_j \in \{t | t = t_0 + k\delta,\ k \in Z_+\} \cup \{t | t = t_0 + (k+1-n)\delta,\ k \in Z_+,\ n \in [0,1]\} \quad (4.14)$$

and $\gamma(t)$ indicates which controller, impedance or admittance, is being applied at any given time, and may be written as

$$\gamma(t) = \begin{cases} 1 & : \ t \in (t_0 + k\delta, t_0 + (k+1-n)\delta) \\ 0 & : \ t \in (t_0 + (k+1-n)\delta, t_0 + (k+1)\delta) \end{cases} \quad (4.15)$$

The state X and the matrices A_i and A_a are given by the relations:

$$X = (e \ \ \dot{e} \ \ e_d \ \ \dot{e}_d)^T \quad (4.16)$$

$$e = q - q_0 \quad (4.17)$$

$$e_d = q_d - q_0 \quad (4.18)$$

$$A_i = \begin{bmatrix} 0 & I & 0 & 0 \\ -M_\theta^{-1}(K_\theta + k_e) & -M_\theta^{-1}D_\theta & 0 & 0 \\ 0 & 0 & 0 & I \\ -M_\theta^{-1}k_e & 0 & -M_\theta^{-1}K_\theta & M_\theta^{-1}D_\theta \end{bmatrix} \quad (4.19)$$

and

$$A_a = \begin{bmatrix} 0 & I & 0 & 0 \\ -M^{-1}(K_p + k_e) & -M^{-1}K_d & M^{-1}K_p & 0 \\ 0 & 0 & 0 & I \\ -M_\theta^{-1}k_e & 0 & -M_\theta^{-1}K_\theta & -M_\theta^{-1}D_\theta \end{bmatrix} \tag{4.20}$$

When switching from the impedance controller to the admittance controller, additional states are introduced into the control torque. The states, e_d and $\dot{e}_d$, are chosen at the instant of switching to maintain continuity of the control torque τ and its derivative. By setting equations (4.3) and (4.5) equal to each other we find

$$e_d((t_k + (1-n)\delta)^+) = \lim_{t \to (t_k + (1-n)\delta)^-} \left[e + K_p^{-1}(\tau_i - C\dot{q} - Gq + K_d\dot{e}) \right] \tag{4.21}$$

By differentiating (4.21) we have

$$\dot{e}_d((t_k + (1-n)\delta)^+) = \lim_{t \to (t_k + (1-n)\delta)^-} \left[\dot{e} + K_p^{-1}(\dot{\tau}_i - C\ddot{q} - G\dot{q} + K_d\ddot{e}) \right] \tag{4.22}$$

Substituting equation (4.3) for τ_i in equations (4.21) and (4.22), it is possible to obtain an expression of the form

$$X^+(t_0 + (k+1-n)\delta) = S_{ai}X^-(t_0 + (k+1-n)\delta), \quad S_{ai} = \begin{bmatrix} I & 0 \\ S & 0 \end{bmatrix} \tag{4.23}$$

at the instant of switching. Where $X^+(t)$ denotes the right limit, $X^-(t)$ denotes the left limit, I is the identity matrix, and S is of the form

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{4.24}$$

where

$$\begin{aligned}
S_{11} &= I - K_p^{-1}(MM_\theta^{-1} - 1)k_e - K_p^{-1}MM_\theta^{-1}K_\theta && (4.25)\\
S_{12} &= K_p^{-1}K_d - K_p^{-1}D_\theta MM_\theta^{-1} && (4.26)\\
S_{21} &= -K_p^{-1}K_dM_\theta^{-1}(K_\theta + k_e) - K_p^{-1}MM_\theta^{-1}D_\theta M_\theta^{-1}K_\theta && (4.27)\\
S_{22} &= I - K_p^{-1}(MM_\theta^{-1} - I)k_e - K_p^{-1}MM_\theta^{-1}K_\theta \\
&\quad -K_p^{-1}K_dM_\theta^{-1}D_\theta + K_p^{-1}MM_\theta^{-1}D_\theta M_\theta^{-1}D_\theta && (4.28)
\end{aligned}$$

Then, we find that Φ_j to be given by

$$\Phi_j(t_0 + (k+1-n)\delta) = S_{ai} - I = \begin{bmatrix} 0 & 0 \\ S_{ai} & -I \end{bmatrix} \tag{4.29}$$

When the system is switched from the admittance controller to the impedance controller, the state mapping is given by

$$X^+(t_0 + k\delta) = S_{ia}X^-(t_0 + k\delta), \qquad S_{ia} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{4.30}$$

Where 0 is the $n \times n$ matrix of zeros. We then find Φ_j at this instant to be given by

$$\Phi_j(t_0 + k\delta) = S_{ia} - I = \emptyset \tag{4.31}$$

4.1.5 Stability

Knowing the states at time $t = t_0 + k\delta$, the states at $t = t_0 + (k+1)\delta$, $k \in Z_+$ is a positive integer, can be obtained using equations (4.12), (4.23), and (4.30) as

$$\begin{aligned}
X^-(t_0 + (k+1-n)\delta) &= e^{A_i(1-n)\delta} X^+(t_0 + k\delta) \\
X^+(t_0 + (k+1-n)\delta) &= S_{ai} e^{A_i(1-n)\delta} X^+(t_0 + k\delta) \\
X^-(t_0 + (k+1)\delta) &= e^{A_a n\delta} X^+(t_0 + (k+1-n)\delta) \\
X^+(t_0 + (k+1)\delta) &= S_{ia} e^{A_a n\delta} X^+(t_0 + (k+1-n)\delta) \\
X^+(t_0 + (k+1)\delta) &= S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} X^+(t_0 + k\delta) \qquad (4.32)
\end{aligned}$$

We now define a matrix A_{dis} to be

$$A_{dis} = S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)} \tag{4.33}$$

such that

$$X^+(t_0 + (k+1)\delta) = A_{dis} X^+(t_0 + k\delta) \tag{4.34}$$

With A_{dis} being a $n \times n$ matrix now instead of a 2×2 matrix. However, theorem 2 holds independently of the size of A_{dis}, and therefore holds here.

4.2 Non-linear N-DOF Model

For the linear case we found that the single degree-of-freedom system can be generalized to n dimensions. We now wish to generalize the concept to a non-linear system. However, we note that the position controller used in the single degree-of-freedom and the linear multiple degree-of-freedom cases rely on $F_{ext} = 0$ when $x = x_0$ and that $\dot{x}_0 = 0$. Therefore, we additionally consider a position control that is better suited for tracking a trajectory, and is independent of when the external force is zero.

4.2.1 Equations of Motion

Consider a general non-linear dynamic system given by

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau + J^T F_{ext} \tag{4.35}$$

where $M(q)$ is the mass inertia matrix, $C(q,\dot{q})$ is the matrix representing the Coriolis effects, and $G(q)$ is the gravitational matrix. Our goal is to design τ such that $(q,\dot{q})$ form the solution to the differential equation

$$M_\theta(\ddot{q} - \ddot{q}_0) + D_\theta(\dot{q} - \dot{q}_0) + K_\theta(q - q_0) = J^T F_{ext} \tag{4.36}$$

Where the virtual trajectory, q_0 is then assumed to be a smooth bounded function in time.

4.2.2 Impedance Controller

For the impedance controller we consider a force feedback controller that attempts to cancel the non-linear dynamics and then feedback the desired control objective. This is achieved by a control torque of the form:

$$\begin{aligned}\tau = \tau_i &= C(q,\dot{q})\dot{q} + G(q) + M(q)\ddot{q}_0 \\ &- M(q)M_\theta^{-1}[K_\theta(q-q_0) + D_\theta(\dot{q}-\dot{q}_0)] + (M(q)M_\theta^{-1} - I)J^T F_{ext}(q,\dot{q})\end{aligned} \tag{4.37}$$

By substituting (4.37) into (4.35) gives

$$M_\theta(\ddot{q} - \ddot{q}_0) + D_\theta(\dot{q} - \dot{q}_0) + K_\theta(q - q_0) = J^T F_{ext}(q,\dot{q}) \tag{4.38}$$

4.2.3 Admittance Controller

For the tracking controller we cancel the non-linear dynamics and produces a controller that allows q to track a desired trajectory q_d. Thus, we consider a control torque given by

$$\begin{aligned}\tau = \tau_a = C(q,\dot{q})\dot{q} + G(q) + M(q)\ddot{q}_d - J^T F_{ext} + \\ M(q)M_d^{-1}[-K_p(q-q_d) - K_d(\dot{q}-\dot{q}_d)]\end{aligned} \tag{4.39}$$

Substituting equation (4.39) into (4.35) we have

$$M_d(\ddot{q} - \ddot{q}_d) + K_d(\dot{q} - \dot{q}_d) + K_p(q - q_d) = 0 \tag{4.40}$$

For positive definite values M_d, K_d, and K_p, q converges to q_d for any initial value $q(t_0)$.

The desired trajectory q_d is obtained by solving the differential equation

$$M_\theta(\ddot{q}_d - \ddot{q}_0) + D_\theta(\dot{q}_d - \dot{q}_0) + K_\theta(q_d - q_0) = J^T F_{ext} \tag{4.41}$$

The admittance controlled system is represented by the dynamics

$$M_d(\ddot{q} - \ddot{q}_d) + K_d(\dot{q} - \dot{q}_d) + K_p(q - q_d) = 0 \tag{4.42}$$

$$M_\theta(\ddot{q}_d - \ddot{q}_0) + D_\theta(\dot{q}_d - \dot{q}_0) + K_\theta(q_d - q_0) = J^T F_{ext} \tag{4.43}$$

4.2.4 Hybrid Framework

We define a switching controller based on the hybrid system framework. Let us begin by defining the force caused by interaction with the environment to be modeled as a linear spring

$$J^T F_{ext}(q) = -k_e(q - q_{ext}) \tag{4.44}$$

for a positive definite definite matrix k_e, and q_{ext} is a constant offset.

Now, let us define a function r which forms a solution to (4.36) generated by an arbitrary set of initial conditions, namely

$$M_\theta(\ddot{r} - \ddot{q}_0) + D_\theta(\dot{r} - \dot{q}_0) + K_\theta(r - q_0) = -k_e(r - q_{ext}) \tag{4.45}$$

Where $(r(t_0), \dot{r}(t_0)) \in R^{2n}$. We then have that the control objective is achieved if there exists a pair $(r(t_0), \dot{r}(t_0))$, and a t_j such that

$$|q - r| = 0 \quad \forall\, t \geq t_j \tag{4.46}$$

We then define the variable $e = q - r$ and $e_d = q_d - r$. Then, the system under impedance control may be given by

$$\dot{X}_i = A_i X_i \tag{4.47}$$

where

$$X_i = (e\ ,\ \dot{e}\ ,\ \ e_d\ ,\ \ \dot{e}_d)^T \tag{4.48}$$

$$A_i = \begin{bmatrix} 0 & I & 0 & 0 \\ -M_\theta^{-1}(K_\theta + k_e) & -M_\theta^{-1} D_\theta & 0 & 0 \\ 0 & 0 & 0 & I \\ -M_\theta^{-1} k_e & 0 & -M_\theta^{-1} K_\theta & -M_\theta^{-1} D_\theta \end{bmatrix} \tag{4.49}$$

Likewise, the system under the admittance control may be given by

$$\dot{X}_a = A_a X_a \tag{4.50}$$

where

$$X_a = (e\ ,\ \dot{e}\ ,\ \ e_d\ ,\ \dot{e}_d)^T \tag{4.51}$$

$$A_a = \begin{bmatrix} 0 & I & 0 & 0 \\ A_{a1} & A_{a2} & A_{a3} & A_{a4} \\ 0 & 0 & 0 & I \\ -M_\theta^{-1}k_e & 0 & -M_\theta^{-1}K_\theta & -M_\theta^{-1}D_\theta \end{bmatrix} \tag{4.52}$$

and

$$A_{a1} = -(M_\theta^{-1}k_e + M_d^{-1}K_p) \tag{4.53}$$

$$A_{a2} = -M_d^{-1}K_d \tag{4.54}$$

$$A_{a3} = M_d^{-1}K_p - M_\theta^{-1}K_\theta \tag{4.55}$$

$$A_{a4} = M_d^{-1}K_d - M_\theta^{-1}D_\theta \tag{4.56}$$

Both X_i and X_a have the same number of states, and both are invariant when $X_i = X_a = 0$ which corresponds to $\|q - r\| = 0$.

Now, we define the set of times σ to be

$$\sigma = \{t|t = t_0 + k\delta,\ k \in Z_+\} \cup \{t|t = t_0 + (k+1-n)\delta,\ k \in Z_+,\ n \in [0,1]\} \tag{4.57}$$

Then we may write the general hybrid control law as

$$\dot{X} = \gamma(t)A_iX + (1-\gamma(t))A_aX \qquad \forall t = t_j \tag{4.58}$$

$$\Delta X = \Phi_j X \qquad \forall t = t_j \tag{4.59}$$

where $t_j \in \sigma$, and $\gamma(t) \in [0, 1]$ for all t. Since we are only concerned with switching we have

$$\gamma(t) = \begin{cases} 1 & : \quad t \in (t_0 + k\delta, t_0 + (k+1-n)\delta) \\ 0 & : \quad t \in (t_0 + (k+1-n)\delta, t_0 + (k+1)\delta) \end{cases} \tag{4.60}$$

namely, we either apply all impedance control or all admittance control, but not both at the same time. We must then define Φ_j which causes an impulse in the system at times $t = t_j$. Since the times $t_j \in \sigma$ and σ is the union of two sets, we will investigate each individually.

First, we will investigate is $\{t | t = t_0 + (k+1-n)\delta,\ k \in Z_+,\ n \in (0, 1)\}$. In this case we see that we switch from the impedance controller to the admittance controller. When we switch we wish to ensure that the control torque is smooth at that instant. Setting equations (4.37) and (4.39) equal we find

$$M(q)\ddot{q}_d - J^T F_{ext} + M(q)M_d^{-1}[-K_p(q-q_d) - K_d(\dot{q}-\dot{q}_d)]$$

$$= M(q)\ddot{q}_0 - M(q)M_\theta^{-1}[K_\theta(q-q_0) + D_\theta(\dot{q}-\dot{q}_0)]$$

$$+(M(q)M_\theta^{-1} - I)J^T F_{ext} \tag{4.61}$$

$$\Rightarrow \ddot{q}_d + M_d^{-1}[-K_p(q-q_d) - K_d(\dot{q}-\dot{q}_d)] =$$

$$\ddot{q}_0 - M_\theta^{-1}[K_\theta(q-q_0) + D_\theta(\dot{q}-\dot{q}_0)] + M_\theta^{-1}J^T F_{ext} \tag{4.62}$$

$$\Rightarrow M_\theta M_d^{-1}[-K_p(q-q_d) - K_d(\dot{q}-\dot{q}_d)] =$$

$$M_\theta(\ddot{q}_0 - \ddot{q}_d) - K_\theta(q-q_0) + D_\theta(\dot{q}-\dot{q}_0) + J^T F_{ext} \tag{4.63}$$

We see from (4.63) that the choice $q = q_d$ and $\dot{q} = \dot{q}_d$ satisfies the equation and all its derivatives. Therefore, we have

$$X^+(t_0 + (k+1-n)\delta) = S_{ai}X^+(t_0 + (k+1-n)\delta), \qquad S_{ai} = \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix} \tag{4.64}$$

at the instant of switching from impedance control to admittance control. We then have that Φ_j to be given by

$$\Phi_j = S_{ai} - I = \begin{bmatrix} 0 & 0 \\ I & -I \end{bmatrix} \tag{4.65}$$

For the set $\{t | t = t_0 + k\delta,\ k \in Z_+\}$ the system is switched from the admittance controller to the impedance controller. During this switching instant we choose to keep the states continuously differentiable. Thus the state mapping is given by

$$X^{+}(t_0 + k\delta) = S_{ia}X^{-}(t_0 + k\delta), \qquad S_{ia} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{4.66}$$

and does not result in an impulse, meaning Φ_j is given by

$$\Phi_j = S_{ia} - I = \emptyset \tag{4.67}$$

4.2.5 Stability

Knowing the states at time $t = t_0 + k\delta$, the states at $t = t_0 + (k+1)\delta$, $k \in Z_+$ is a positive integer, can be obtained using equations (4.58), (4.59), (4.64), and (4.66) as

$$\begin{aligned}
X^{-}(t_0 + (k+1-n)\delta) &= e^{A_i(1-n)\delta}X_i(t_0 + k\delta) \\
X^{+}(t_0 + (k+1-n)\delta) &= S_{ai}e^{A_i(1-n)\delta}X_i(t_0 + k\delta) \\
X^{-}(t_0 + (k+1)\delta) &= e^{A_a n\delta}X_a(t_0 + (k+1-n)\delta) \\
X^{+}(t_0 + (k+1)\delta) &= S_{ia}e^{A_a n\delta}X_a(t_0 + (k+1-n)\delta) \\
X^{+}(t_0 + (k+1)\delta) &= S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)\delta}X_i(t_0 + k\delta) \qquad (4.68)
\end{aligned}$$

We now define a matrix A_{dis} to be

$$A_{dis} = S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)} \tag{4.69}$$

such that

$$X^{+}(t_0 + (k+1)\delta) = A_{dis} X^{+}(t_0 + k\delta) \tag{4.70}$$

With A_{dis} being a $n \times n$ matrix now instead of a 2×2 matrix. We again note that theorem 2 holds independently of the size of A_{dis}, and therefore holds here.

Chapter 5

Flexible Joint Model

All development thus far is for a rigid joint model which is characteristic of a directly driven system. However, in order to achieve large torque values gearing of the actuator is required. This leads to some flexibility between the actuator and the joint. Furthermore, many modern robotic control drives use series elastic actuation which improves the stability of position control methods and enforces smoothness of torques on the link.

For this chapter we rederive the impedance control, admittance control, and hybrid methods for a single degree-of-freedom flexible joint system using continuity of control force and control force derivative. We then rederive the alternate switching conditions for the system with joint flexibility. Finally we examine a simple nonlinear system with flexibility, and examine the changes the impedance, admittance, and hybrid controller, as well as the proof of stability of the hybrid controller.

5.1 Linear 1-DOF Model

We begin with the simplest case as in chapter 2, a single degree-of-freedom linear system. However, to add flexibility to the system we consider the actuator to be coupled to the system through a spring.

5.1.1 Equations of Motion

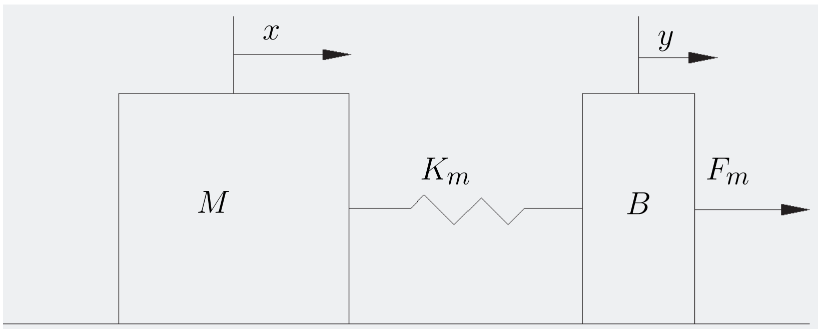

Figure 5.1: Single degree-of-freedom flexible joint model

For the single degree of freedom linear flexible joint model as shown in figure 5.1.1 the system dynamics are given by

$$M\ddot{x} = F + F_{ext} \tag{5.1}$$

$$B\ddot{y} + F = F_m \tag{5.2}$$

$$F = K_m(y - x) \tag{5.3}$$

Where x represents link side dynamics and y represents actuator side dynamics, M is a positive constant corresponding to the mass of the system, K_m is the stiffness between the actuator and the joint, F_{ext} is the external force applied to the system, and F_m is the input force.

We choose the desired dynamics to be given by

$$M_\theta(\ddot{x} - \ddot{x}_0) + D_\theta(\dot{x} - \dot{x}_0) + K_\theta(x - x_0) = F_{ext} \tag{5.4}$$

We note that the values that we can measure are y, $\dot{y}$, F, and $\dot{F}$. We cannot explicitly measure x or $\dot{x}$.

5.1.2 Passive Impedance Controller

For the passive impedance controller, we assume a controller of the form [17]:

$$\begin{aligned} F_m = F_i = BB_d^{-1}\{MM_\theta^{-1}[-K_c(y - y_0) - D_c(\dot{y} - \dot{y}_0)] \\ +(MM_\theta^{-1} - 1)F_{ext} + M\ddot{x}_0\} + (I - BB_d^{-1})\tau \end{aligned} \tag{5.5}$$

Where B_d is a normalization of the acutator mass B. By substituting equation (5.5) into equation (5.2) we find the dynamics to be given by

$$M\ddot{x} = F + F_{ext} \tag{5.6}$$

$$\begin{aligned} B_d\ddot{y} + MM_\theta^{-1}[D_c(\dot{y} - \dot{y}_0) + K_c(y - y_0)] \\ -(MM_\theta^{-1} - 1)F_{ext} + M\ddot{x}_0 = -F \end{aligned} \tag{5.7}$$

By combining equations (5.6) and (5.7) we get

$$M_\theta(\ddot{x} - \ddot{x}_0) + D_c(\dot{y} - \dot{y}_0) + K_c(y - y_0) = F_{ext} - M_\theta M^{-1} B_d \ddot{y} \tag{5.8}$$

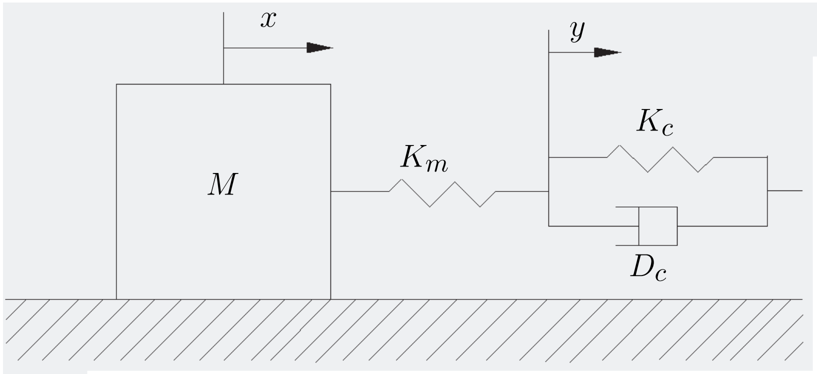

Figure 5.2: Single degree-of-freedom passive impedance for flexible joint model

We choose B_d to be as small as possible such that the effect of $\ddot{y}$ is negligible. The resulting model is shown in figure 5.1.2. We then wish to find the values of D_c, K_c, and y_0 such that (5.8) produces the same result as (5.4).

To find y_0 we consider the static solutions of (5.6), (5.7), and (5.4) when $F_{ext} = 0$. This yields the equations

$$x_{ss} = y_{xx} \tag{5.9}$$

$$x_{ss} = x_0 \tag{5.10}$$

$$y_{ss} = y_0 \tag{5.11}$$

where x_{ss} represents the static value of x, and y_{ss} represents the static value of y. Solving (5.9)-(5.11) we have the equation

$$y_0 = x_0 \tag{5.12}$$

To solve for K_c we consider the static solutions of (5.6), (5.7), and (5.4) when $F_{ext} \neq 0$ this yields

$$K_m(x_{ss} - y_{ss}) = F_{ext} \tag{5.13}$$

$$K_\theta(x_{ss} - x_0) = F_{ext} \tag{5.14}$$

$$K_c(y_{ss} - y_0) = F_{ext} \tag{5.15}$$

Solving equations (5.13)-(5.15) we find

$$K_c = \left[K_\theta^{-1} - K_m^{-1}\right]^{-1} \tag{5.16}$$

Finally, to find D_c we consider that all internal forces in figure 5.1.2 must be equal. This gives the equation

$$D_c(\dot{y} - \dot{y}_0) + K_c(y - y_0) = D_\theta(\dot{x} - \dot{x}_0) + K_\theta(x - x_0) \tag{5.17}$$

From equations (5.13)-(5.15) we assume

$$K_c(y - y_0) = K_\theta(x - x_0) \tag{5.18}$$

Giving

$$D_c(\dot{y} - \dot{y}_0) = D_\theta(\dot{x} - \dot{x}_0) \tag{5.19}$$

Then we have

$$D_c = D_\theta \frac{\dot{x} - \dot{x}_0}{\dot{y} - \dot{y}_0} \tag{5.20}$$

Since, we always will initialize the system from rest we then assume $\dot{x} - \dot{x}_0 \approx \dot{y} - \dot{y}_0$ for K_m large. This gives

$$D_c = D_\theta \tag{5.21}$$

Therefore, We, for small B_d, have (5.8) is approximated by

$$M_\theta(\ddot{x} - \ddot{x}_0) + D_\theta(\dot{y} - \dot{y}_0) + K_c(y - y_0) = F_{ext} \tag{5.22}$$

which is equivalent to

$$M_\theta(\ddot{x} - \ddot{x}_0) + D_\theta(\dot{x} - \dot{x}_0) + K_\theta(x - x_0) = F_{ext} \tag{5.23}$$

5.1.3 Admittance Controller

For the admittance controller, let the position controller be given by [1]

$$F_m = F_a = BB_d^{-1}[-K_p(y - y_d) - K_d\dot{y} - K_T K_m^{-1}(F - F_d)$$
$$-K_S K_m^{-1}\dot{F}] + (I - BB_d^{-1})F \qquad (5.24)$$

This produces the dynamic equations

$$M\ddot{x} = \tau + F_{ext} \qquad (5.25)$$
$$B_d\ddot{y} + K_d\dot{y} + K_p(y - y_d) +$$
$$K_S K_m^{-1}\dot{F} + K_T K_m^{-1}(F - F_d) = -F \qquad (5.26)$$

Combining equations (5.25) and (5.26) we find

$$M\ddot{x} + K_d\dot{y} + K_p(y - y_d)$$
$$+ K_S K_m^{-1}\dot{F} + K_T K_m^{-1}(F - F_d) = F_{ext} - B_d\ddot{y} \qquad (5.27)$$

We again choose B_d small such that the effects of $\ddot{y}$ is negligible. We then choose x_d, $\dot{x}_d$ to be given by the mechanical admittance in equation (5.4). Namely, let x_d, and $\dot{x}_d$ be the solution to

$$M(\ddot{x}_d - \ddot{x}_0) + D_\theta(\dot{x}_d - \dot{x}_0) + K_\theta(x_d - x_0) = F_{ext} \qquad (5.28)$$

We then assume that the stiffness K_m is large enough such that $y \approx x$ and $\dot{y} \approx \dot{x}$. Thus, we choose $y_d = x_d$ and $\dot{y}_d = \dot{x}_d$ making $F_d = 0$. We then find that the full dynamics of the system to be given by

$$M\ddot{x} = F + F_{ext} \tag{5.29}$$

$$B_d\ddot{y} + K_d\dot{y} + K_p(y - y_d) + K_S K_m^{-1}\dot{F} + K_T K_m^{-1} F = -F \tag{5.30}$$

$$M(\ddot{y}_d - \ddot{x}_0) + D_\theta(\dot{y}_d - \dot{x}_0) + K_\theta(y_d - x_0) = F_{ext} \tag{5.31}$$

5.1.4 Hybrid Framework

For the single degree-of-freedom system described by equations (5.1)-(5.3), we propose to switch the controller between impedance and admittance as follows

$$F_m = \begin{cases} F_1 & : \ t \in [t_0 + k\delta, t_0 + (k+1-n)\delta) \\ F_2 & : \ t \in [t_0 + (k+1-n)\delta, t_0 + (k+1)\delta) \end{cases} \tag{5.32}$$

where t_0 is the initial time, δ is the switching period, $n \in [0, 1]$ is the duty cycle, k is a positive integer. F_1 is given by eq. (5.5), and F_2 is given by equations (5.24) and (5.28).

Let the environment is modeled as a linear spring

$$F_{ext} = -k_e(x - x_0) \tag{5.33}$$

and the virtual trajectory, x_0 is assumed to be constant,

$$\dot{x}_0 = \ddot{x}_0 = 0 \tag{5.34}$$

The hybrid system then follows the descriptions

$$\begin{aligned} \dot{X}_i &= A_i X_i \ : \ t \in [t_0 + k\delta, t_0 + (k+1-n)\delta) \\ \dot{X}_a &= A_a X_a \ : \ t \in [t_0 + (k+1-n)\delta, t_0 + (k+1)\delta) \end{aligned} \tag{5.35}$$

where

$$X_i = (e \ , \ \ \dot{e} \ , \ \ e_\theta \ , \ \ \dot{e}_\theta)^T \tag{5.36}$$

$$e = x - x_0 \tag{5.37}$$

$$e_\theta = y - y_0 \tag{5.38}$$

$$A_i = \begin{bmatrix} 0 \quad 1 \quad 0 \quad 0 \\ -M^{-1}(K_m + k_e) \ \ 0 \ \ M^{-1}K_m \ \ 0 \\ 0 \quad 0 \quad 0 \quad 1 \\ A_{i1} \ \ 0 \ \ - B_d^{-1} M M_\theta^{-1}(K_c + K_m) \ \ - B_d^{-1} M M_\theta^{-1} K_d \end{bmatrix} \tag{5.39}$$

$$A_{i1} = B_d^{-1}(M M_\theta^{-1} K_m + (M M_\theta^{-1} - 1)k_e) \tag{5.40}$$

$$X_a = (e \ , \ \ \dot{e} \ \ e_\theta \ , \ \ \dot{e}_\theta \ , \ \ e_d \ , \ \ \dot{e}_d)^T \tag{5.41}$$

$$e_d = y_d - x_0 \tag{5.42}$$

and

$$A_a = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -M^{-1}(K_m - k_e) & 0 & M^{-1}K_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ A_{a1} & A_{a2} & A_{a3} & A_{a4} & A_{a5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -M_\theta^{-1}k_e & 0 & 0 & 0 & -M_\theta^{-1}K_\theta & -M_\theta^{-1}D_\theta \end{bmatrix} \tag{5.43}$$

$$A_{a1} = B_d^{-1}(K_m + K_T K_m^{-1}) \tag{5.44}$$

$$A_{a2} = B_d^{-1} K_S K_m^{-1} \tag{5.45}$$

$$A_{a3} = -B_d^{-1}(K_m + K_p + K_T K_m^{-1}) \tag{5.46}$$

$$A_{a4} = -B_d^{-1}(K_S K_m^{-1} + K_d) \tag{5.47}$$

$$A_{a5} = B_d^{-1} K_p \tag{5.48}$$

When switching from the impedance controller to the admittance controller, two additional states are introduced. These states, e_d and $\dot{e}_d$, are chosen at the instant of switching to maintain continuity of the control force F and its derivative. Equation (5.24) gives the expression for the control force of the admittance controller and may be written as

$$\begin{aligned} y_d = &\quad y + \frac{1}{K_p}\left(\frac{B_d}{B}F_i + K_d\dot{y} + \frac{K_T F}{K_m} + \frac{K_S}{K_m}\dot{F} - +(1-\frac{B_d}{B})F\right) \\ \Rightarrow e_d = &\quad e_\theta + \frac{1}{K_p}\left(\frac{B_d}{B}F_i + K_d\dot{e}_\theta + \frac{K_T F}{K_m} + \frac{K_S}{K_m}\dot{F} - +(1-\frac{B_d}{B})F\right) \end{aligned} \tag{5.49}$$

and differentiating equation (5.49) gives

$$\begin{aligned} \dot{y}_d = &\quad \dot{y} + \frac{1}{K_p}\left(\frac{B_d}{B}\dot{F}_i + K_d\ddot{y} + \frac{K_T \dot{F}}{K_m} + \frac{K_S}{K_m}\ddot{F} - +(1-\frac{B_d}{B})\dot{F}\right) \\ \Rightarrow \dot{e}_d = &\quad \dot{e}_\theta + \frac{1}{K_p}\left(\frac{B_d}{B}\dot{F}_i + K_d\ddot{e}_\theta + \frac{K_T \dot{F}}{K_m} + \frac{K_S}{K_m}\ddot{F} - +(1-\frac{B_d}{B})\dot{F}\right) \end{aligned} \tag{5.50}$$

Substituting equation (5.5) for F_1 in equations (5.49) and (5.50), it is possible to obtain an expression of the form

$$X_a = S_{ai}X_i, \qquad S_{ai} = \begin{bmatrix} I \\ S \end{bmatrix} \tag{5.51}$$

at the instant of switching. Where I is the identity matrix, and S is given by

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \end{bmatrix} \tag{5.52}$$

Where

$$
\begin{aligned}
s_{11} &= -K_p^{-1}K_T \\
s_{12} &= -K_p^{-1}K_S \\
s_{13} &= K_P^{-1}[K_P + K_T - MM_\theta^{-1}K_c - (MM_\theta^{-1} - 1)k_e] \\
s_{14} &= K_p^{-1}(K_d + K_S - MM_\theta^{-1}D_\theta) \\
s_{21} &= -K_p^{-1}K_T \\
s_{22} &= K_p^{-1}[K_S\frac{(K_m + k_e)}{M} + (K_d + K_S - \frac{M}{M_\theta}D_\theta)\frac{K_m}{B_d}] \\
s_{23} &= -K_p^{-1}[\frac{K_S K_m}{M} + (K_d + K_S - \frac{MD_\theta}{M_\theta})\frac{(K_m + MM_\theta^{-1}K_c)}{B_d}] \\
s_{24} &= K_p^{-1}\{[K_p + K_T - \frac{MK_c}{M_\theta} - (\frac{M}{M_\theta} - 1)k_e] - \\
&\quad (K_d + K_S - \frac{MD_\theta}{M_\theta})\frac{MD_\theta}{B_d M_\theta}\}
\end{aligned} \tag{5.53}
$$

When the system switched from the admittance controller to the impedance controller, the state mapping is given by

$$X_i = S_{ia}X_a, \qquad S_{ia} = [I \quad 0] \tag{5.54}$$

Where 0 is the 2×2 matrix of zeros.

5.1.5 Stability

Knowing the states at time $t = t_0 + k\delta$, the states at $t = t_0 + (k+1)\delta$, $k \in Z_+$ is a positive integer, can be obtained using equations (5.35), (5.51), and (5.54) as

$$
\begin{aligned}
X_i(t_0 + (k+1-n)\delta) &= e^{A_i(1-n)\delta} X_i(t_0 + k\delta) \\
X_a(t_0 + (k+1-n)\delta) &= S_{ai} e^{A_i(1-n)\delta} X_i(t_0 + k\delta) \\
X_a(t_0 + (k+1)\delta) &= e^{A_a n\delta} X_a(t_0 + (k+1-n)\delta) \\
X_i(t_0 + (k+1)\delta) &= S_{ia} e^{A_a n\delta} X_a(t_0 + (k+1-n)\delta) \\
X_i(t_0 + (k+1)\delta) &= S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} X_i(t_0 + k\delta) \qquad (5.55)
\end{aligned}
$$

We now define a matrix A_{dis} to be

$$
A_{dis} = S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)} \qquad (5.56)
$$

such that

$$
X_i(t_0 + (k+1)\delta) = A_{dis} X_i(t_0 + k\delta) \qquad (5.57)
$$

With A_{dis} being a $n \times n$ matrix now instead of a 2×2 matrix. However, theorem 2 holds independently of the size of A_{dis}, and therefore holds here.

5.1.6 Simulations and Performance

Let us now consider a linear single degree-of-freedom system described by equations (5.1)-(5.3) and with parameters given by

$$m = 3.6\ kg \quad B = 1.5308\ kg \quad K_m = 22000\ \frac{N}{m}$$
$$K_p = 113580\ \frac{N}{m} \quad K_d = 361.2\ \frac{N\ s}{m} \quad K_T = 119470\ \frac{N}{m}$$
$$K_S = 3076.4\ \frac{N\ s}{m} \quad M_\theta = 3.6344\ kg \quad B_\theta = 0.34017\ kg$$
$$K_\theta = 100\ \frac{N}{m} \quad D_\theta = 23.872\ \frac{N\ s}{m} \tag{5.58}$$

We then choose the external force to be given by

$$F_{ext} = -k_e x \tag{5.59}$$

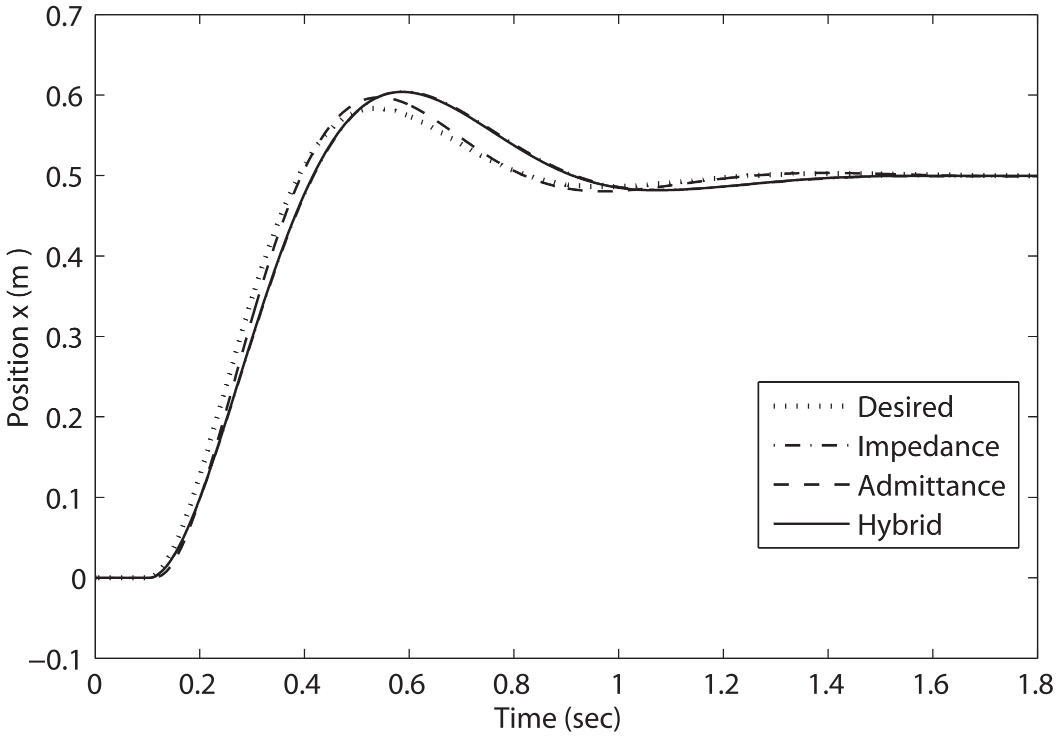

Figure 5.3: Soft contact position response for $\delta = 50\ ms$ and $n = 0.1$

and x_0 is chosen as a constant input of 1 m with a system initial condition given as $x(t_0) = y(t_0) = 0$. The switching time, δ, is chosen to be 50 ms. We then add a mass error to the system of 80% as well as a feedback time delay, input noise, and unmodeled friction. To simulate a soft stiffness, k_e is initially chosen to have a value of 100 N/m.

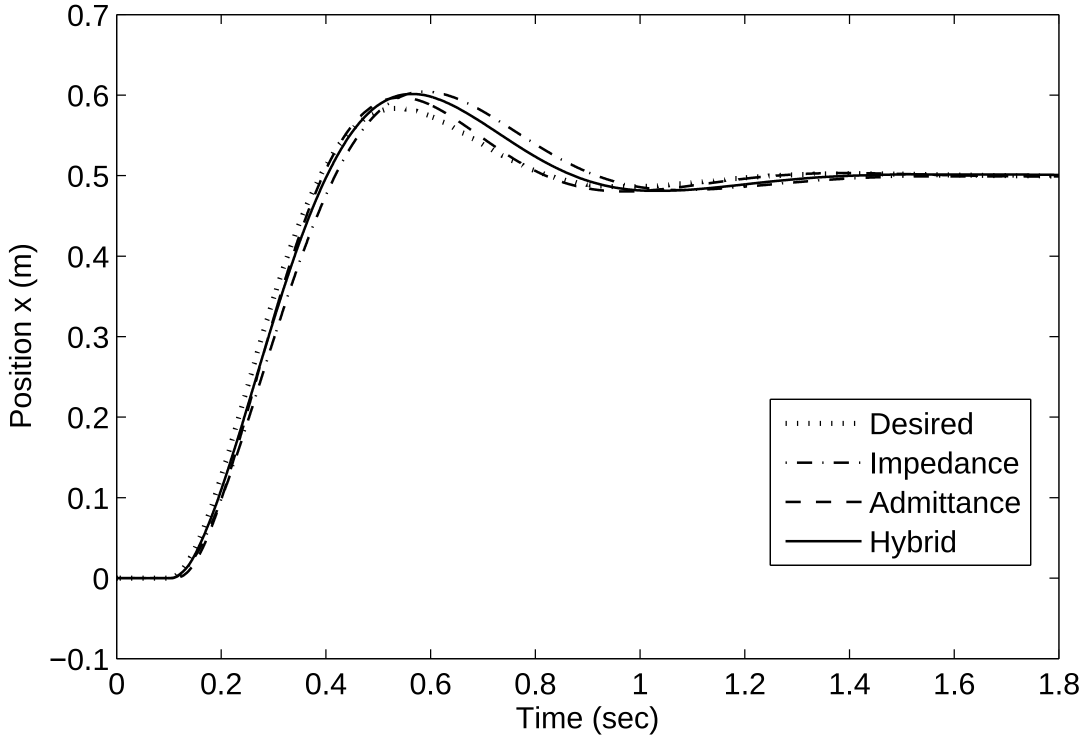

Figure 5.4: Soft contact position response for $\delta = 50$ ms and $n = 0.5$

Figures 5.3, 5.4, and 5.5 show the response of the passive impedance controlled system, the admittance controlled system, the hybrid controlled system, and the desired response for $n = 0.1$, $n = 0.5$, and $n = 0.9$ respectively. We see from figure 5.3 that the hybrid controlled system response is almost identical to the passive impedance controlled system. This is expected since the hybrid controlled system is the same as the passive impedance controlled system when $n = 0$. Then from figure 5.4 we see that increasing n from 0.1 to 0.5 improves the performance of the hybrid system and causes the resulting response to approach that of the admittance controlled system.

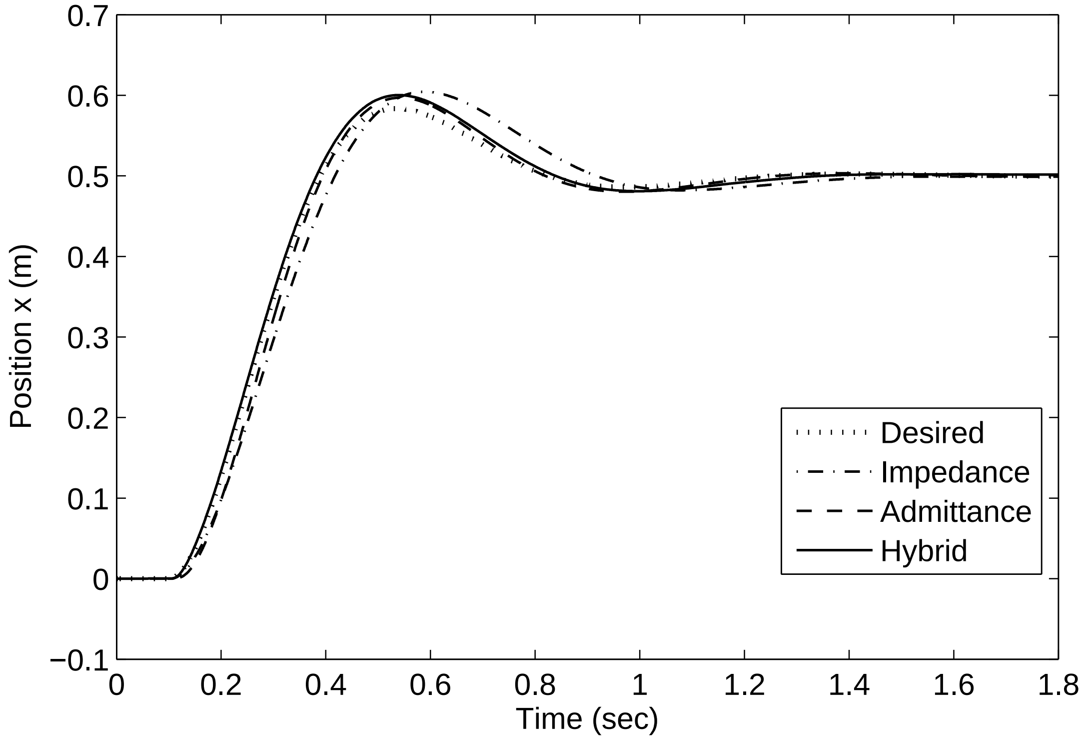

Figure 5.5: Soft contact position response for $\delta = 50\ ms$, and $n = 0.9$

We again see from figure 5.5 that again increasing n from 0.5 to 0.9 improves the performance of the hybrid system and causes the response to approach the response curve of of the admittance controlled system. However, we notice that as n goes to 1, while the hybrid system does approach the admittance controlled system it does not converge to it. This is due to the flexibility in the position controller used as the inner loop of the admittance controller.

We have shown that the hybrid control method can improve the performance of the impedance control for a soft environment, but tuning of the position controller is required to reach the performance of the admittance controller. Now we wish to see the performance of the hybrid control method when the system is in contact with a stiff

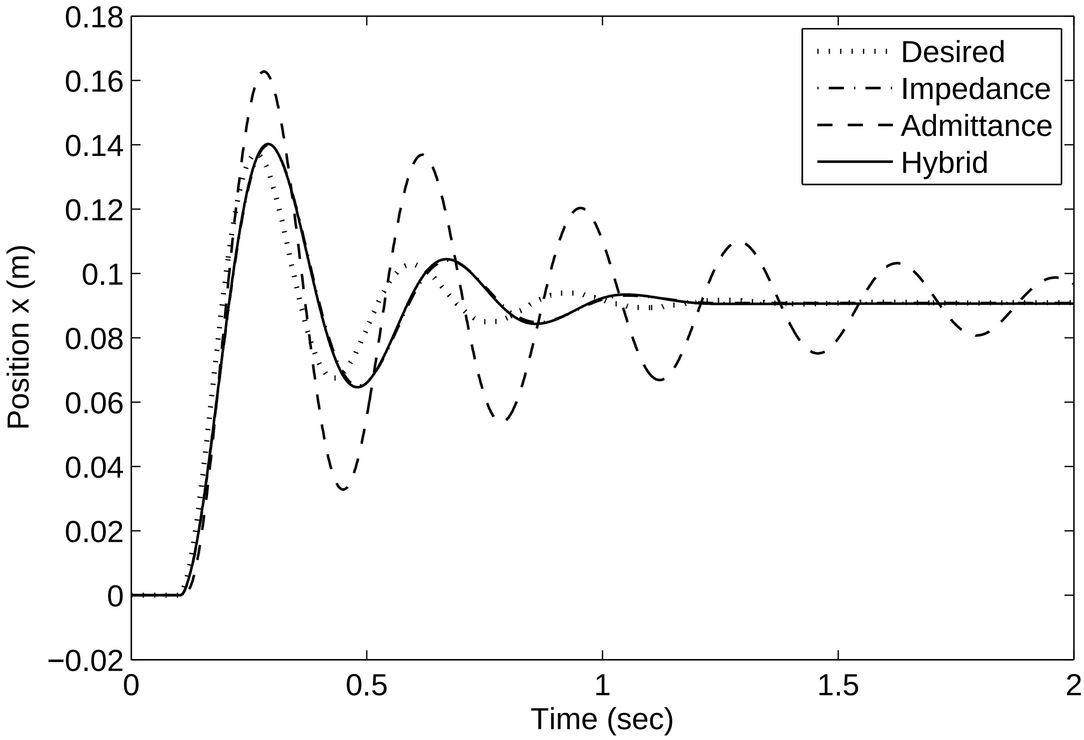

Figure 5.6: Stiff contact position response for $\delta = 50\ ms$ and $n = 0.1$

environment. Therefore, to simulate a stiff environment we choose $k_e = 1000\ N/m$. Then, Figures 5.6, 5.7, and 5.8 show the response of the passive impedance controlled system, the admittance controlled system, the hybrid controlled system, and the desired response for $n = 0.1$, $n = 0.5$, and $n = 0.9$ respectively for the stiff environment. We see from figure 5.6 that when $n = 0.1$ the response of the hybrid system is again almost identical to the passive impedance controlled system. This is again is not surprising since when $n = 0$ the hybrid controlled system is identical to the passive impedance controlled system.

From figure 5.7 we see that increasing n from 0.1 to 0.5 shifts the response of the hybrid system toward that of the admittance controlled system and away from the passive impedance controlled system. Then, from figure 5.7 we again see that increasing n from

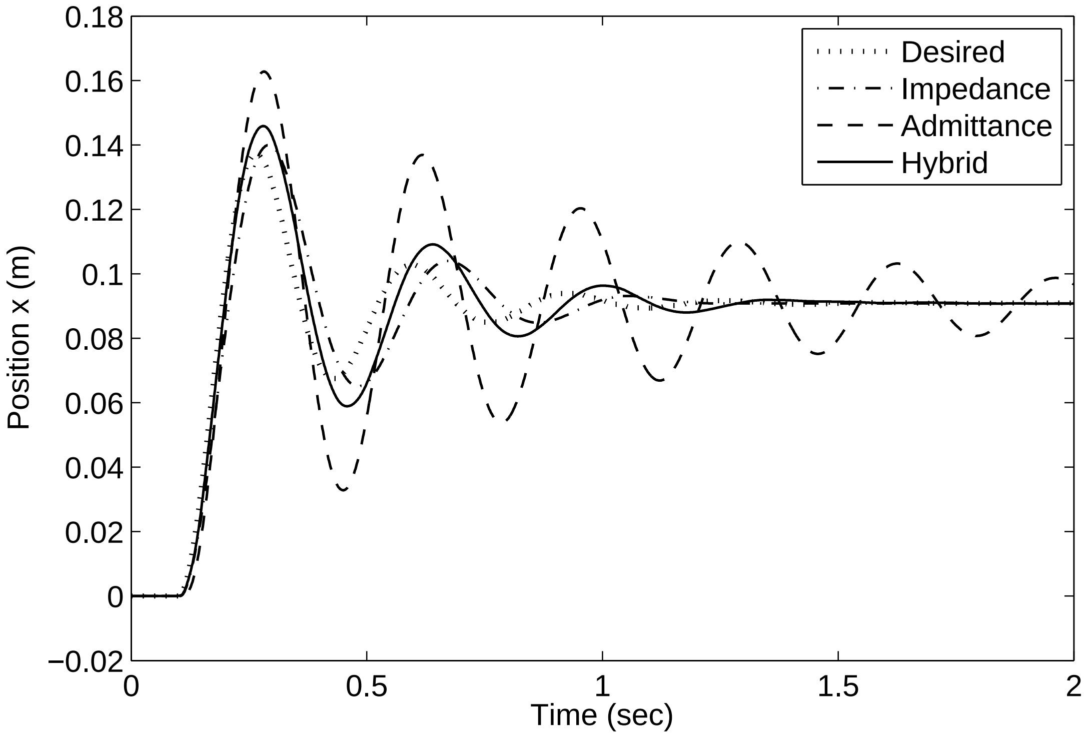

Figure 5.7: Stiff contact position response for $\delta = 50\ ms$ and $n = 0.5$

0.5 to 0.9 shifts the response of the hybrid system closer to the response of the admittance controlled system. However, the deviation between the response of the hybrid controlled system and the admittance controlled system remains as n goes to 1. This again is due in part to tuning of the inner loop position controller used in the admittance control algorithm.

Therefore, we find that through the switching method it is possible to vary the response of the hybrid method between the the admittance controller response and the passive impedance controller response possibly producing better performance than either. However, the variation in response of the hybrid controlled system is highly dependant on the tuning of the inner loop position controller used in the admittance algorithm.

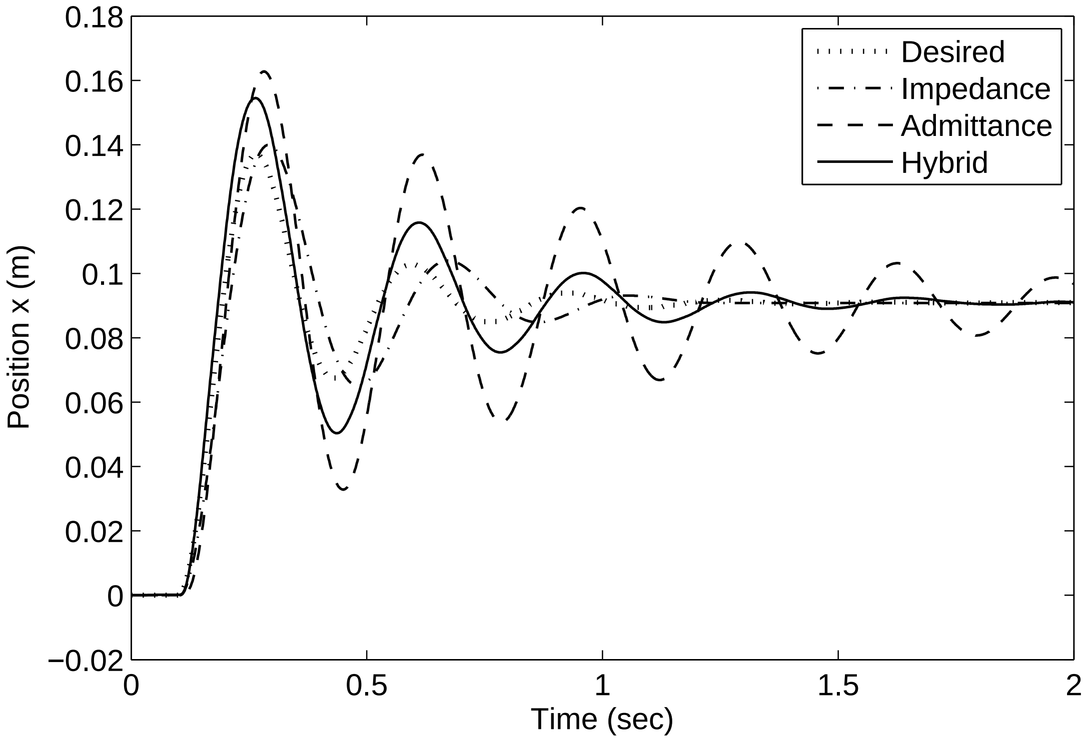

Figure 5.8: Stiff contact position response for $\delta = 50\ ms$ and $n = 0.9$

5.2 Change of Switching Conditions

A large factor in the performance of the switches system is the position control used as the control force in the admittance algorithm. This is partly because the control force in (5.24) is based on the assumption that x_d is a constant. Then, choosing the states x_d and $\dot{x}_d$ when switching from the impedance to admittance controller such that the control force and its derivative are continuous is not necessarily optimal. Therefore, we change the algorithm as follows.

We have the impedance control torque to be given by (5.5). We then change the admittance torque to be given by a PD controller instead of just a position controller giving

$$F_m = F_a = BB_D^{-1}[-K_p(y - y_d) - K_d(\dot{y} - \dot{y}_d) - K_T K_m^{-1}(F - F_d)$$
$$-K_S K_m^{-1}(\dot{F} - \dot{F}_d)] + (I - BB_d^{-1})F \quad (5.60)$$

with x_d, $\dot{x}_d$ to again be given by (5.28). We again let the external force, F_{ext}

$$F_{ext} = -k_e(x - x_0) \quad (5.61)$$

with x_0 again being a constant.

We then consider the hybrid switching as proposed in (5.32). Then the switched system may be written as

$$\begin{aligned} \dot{X}_i &= A_i X_i \ : \ t \in [t_0 + k\delta, t_0 + (k + 1 - n)\delta) \\ \dot{X}_a &= A_a X_a \ : \ t \in [t_0 + (k + 1 - n)\delta, t_0 + (k + 1)\delta) \end{aligned} \quad (5.62)$$

where

$$X_i = (e \ , \ \ \dot{e} \ , \ \ e_\theta \ , \ \ \dot{e}_\theta)^T \quad (5.63)$$

$$\begin{aligned} e &= x - x_0 \\ e_\theta &= y - y_0 \end{aligned} \quad (5.64)$$

$$A_i = \begin{bmatrix} 0 \quad 1 \quad 0 \quad 0 \\ -M^{-1}(K_m + k_e) \;\; 0 \;\; M^{-1}K_m \;\; 0 \\ 0 \quad 0 \quad 0 \quad 1 \\ A_{i1} \;\; 0 \;\; -B_d^{-1}MM_\theta^{-1}(K_c + K_m) \;\; -B_d^{-1}MM_\theta^{-1}K_d \end{bmatrix} \tag{5.65}$$

$$A_{i1} = B_d^{-1}(MM_\theta^{-1}K_m + (MM_\theta^{-1} - 1)k_e) \tag{5.66}$$

$$X_a = (e \;,\;\; \dot{e} \;, e_\theta \;,\;\; \dot{e}_\theta \;,\;\; e_d \;,\;\; \dot{e}_d)^T \tag{5.67}$$

$$e_d = x_d - x_0 \tag{5.68}$$

and

$$A_a = \begin{bmatrix} 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \\ -M^{-1}(K_m - k_e) \;\; 0 \;\; M^{-1}K_m \;\; 0 \;\; 0 \;\; 0 \\ 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \\ A_{a1} \;\; A_{a2} \;\; A_{a3} \;\; A_{a4} \;\; A_{a5} \;\; A_{a6} \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \\ -M_\theta^{-1}k_e \;\; 0 \;\; 0 \;\; 0 \;\; -M_\theta^{-1}K_\theta \;\; -M_\theta^{-1}D_\theta \end{bmatrix} \tag{5.69}$$

$$A_{a1} = B_d^{-1}(K_m + K_T K_m^{-1}) \tag{5.70}$$

$$A_{a2} = B_d^{-1}K_S K_m^{-1} \tag{5.71}$$

$$A_{a3} = -B_d^{-1}(K_m + K_p + K_T K_m^{-1}) \tag{5.72}$$

$$A_{a4} = -B_d^{-1}(K_S K_m^{-1} + K_d) \tag{5.73}$$

$$A_{a5} = B_d^{-1} K_p \tag{5.74}$$

$$A_{a6} = B_d^{-1} K_d \tag{5.75}$$

We remember that when switching from the admittance to the impedance controller, we have

$$X_i = S_{ia} X_a, \qquad S_{ia} = [I \quad 0] \tag{5.76}$$

at the instant of switching.

When switching form impedance controller to the admittance controller we again have two additional sets of states x_d and $\dot{x}_d$ which must be defined. We again wish to find x_d such that the control force is continuous. Using (5.60) as the expression for the control force of the admittance control we have

$$\begin{aligned} y_d = &\quad y + \frac{1}{K_p}\left(\frac{B_\theta}{B}F_i + K_d\dot{y} + \frac{K_T F}{K_m} + \frac{K_S}{K_m}\dot{F} - +(1 - \frac{B_\theta}{B})F\right) \\ \Rightarrow e_d = &\quad e_\theta + \frac{1}{K_p}\left(\frac{B_\theta}{B}F_i + K_d\dot{e}_\theta + \frac{K_T F}{K_m} + \frac{K_S}{K_m}\dot{F} - +(1 - \frac{B_\theta}{B})F\right) \end{aligned} \tag{5.77}$$

Instead of defining $\dot{y}_d$ such that the derivative of the force is continuous at the instant of switching, we consider it to be chosen later. We may then obtain the expression

$$X_a = S_{ai} X_i + B_{ai} \dot{e}_d, \qquad S_{ai} = \begin{bmatrix} I \\ S \end{bmatrix} \tag{5.78}$$

at the instant of switching. Where S is given by

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.79}$$

$$\begin{aligned} s_{11} &= -K_p^{-1} K_T \\ s_{12} &= -K_p^{-1} K_S \\ s_{13} &= K_P^{-1} [K_P + K_T - M M_\theta^{-1} K_c - (M M_\theta^{-1} - 1) k_e] \\ s_{14} &= K_p^{-1} (K_d + K_S - M M_\theta^{-1} D_\theta) \end{aligned} \tag{5.80}$$

and B_{ai} is given by

$$B_{ai} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -K_p^{-1} K_d \\ 1 \end{bmatrix} \tag{5.81}$$

For brevity, let us define t_k such that $t_k = t_0 + k\delta$, $k \in Z_+$. Then, knowing the states at time $t = t_k$, the states at $t = t_k + \delta$ can be obtained using equations (5.62), (5.76), and (5.78) as

$$
\begin{aligned}
X_i(t_k + (1-n)\delta) &= e^{A_i(1-n)\delta} X_i(t_k) \\
X_a(t_k + (1-n)\delta) &= S_{ai} e^{A_i(1-n)\delta} X_i(t_k) + B_{ai}\dot{e}_d \\
X_a(t_k + \delta) &= e^{A_a n\delta} X_a(t_k + (1-n)\delta) \\
X_i(t_k + \delta) &= S_{ia} e^{A_a n\delta} X_a(t_k + (1-n)\delta) \\
X_i(t_k + \delta) &= S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} X_i(t_k) \\
&\quad + S_{ia} e^{A_a n\delta} B_{ia}\dot{e}_d \qquad (5.82)
\end{aligned}
$$

We then let A_{dis} be given by

$$A_{dis} = S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} \qquad (5.83)$$

and B_{dis} be given by

$$B_{dis} = S_{ia} e^{A_a n\delta} B_{ia} \qquad (5.84)$$

giving

$$X_i(t_k + \delta) = A_{dis} X_i(t_k) + B_{dis}\dot{e}_d \qquad (5.85)$$

We now wish to define the desired closed loop behavior by

$$\dot{X}_{des} = A_{des}\dot{X}_{des} \tag{5.86}$$

with

$$X_{des} = [e\ , \quad \dot{e}\ , \quad e_\theta\ , \quad \dot{e}_\theta]^T \tag{5.87}$$

However, since the the desired system given by equation (5.4) does not contain e_θ or $\dot{e}_\theta$ we cannot define A_{des} the same way we did for the rigid joint model. We will instead leave the discussion of the choice of A_{des} for later.

5.2.1 Matching Eigenvalues

The first method of choosing $\dot{e}_d$ is have it be a feedback of the form

$$\dot{e}_d = K_{ei}X_i(t_k) \tag{5.88}$$

where K_{ei} is a matrix chosen such that the eigenvalues of $e^{A_{des}\delta}$ are the same as the eigenvalues of $A_{dis} + B_{dis}K_{ei}$. This method produces similar response times from the switched system and the desired system. While the requirement that the pair (A_{dis}, B_{dis}) to match all eigenvalues is not very restrictive, the resulting gains do not necessarily produce a smooth space with respect to variations in (n, k_e). Also, matching the eigenvalues does not take the eigenvectors into consideration producing different responded as the order of the system increases. Therefore, we will not consider this method further for the flexible joint model.

5.2.2 Minimizing Discrete Difference Between Desired and Actual Behavior

For the flexible joint model we have additional states, $(e_\theta, \dot{e}_\theta)$, which were not present in the rigid body case. However, we note that these additional states do not appear in the equation for the desired behavior. Therefore, the desired equation of motion of the states $(e_\theta, \dot{e}_\theta)$ must be determined before proceeding to the

5.2.2.1 Choosing Desired Behavior based on Passive Impedance

The first method investigated is to choose A_{des} as an ideal system being controlled by the passive impedance controller. A_{des} is then given by

$$A_{des} = \begin{bmatrix} \begin{matrix} 0 & 1 & 0 & 0 \end{matrix} \\ \begin{matrix} -M^{-1}(K_m + k_e) & 0 & M^{-1}K_m & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 1 \end{matrix} \\ \begin{matrix} A_{des1} & 0 & -B_d^{-1}MM_\theta^{-1}(K_c + K_m) & -B_d^{-1}MM_\theta^{-1}K_d \end{matrix} \end{bmatrix} \tag{5.89}$$

$$A_{des1} = B_d^{-1}(MM_\theta^{-1}K_m + (MM_\theta^{-1} - 1)k_e) \tag{5.90}$$

Solving (5.86) from time t_k to $t_k + \delta$ gives

$$X_{des}(t_k + \delta) = e^{A_{des}\delta} X_{des}(t_k) \tag{5.91}$$

Let A_d be given by

$$A_d = e^{A_{des}\delta} \tag{5.92}$$

such that

$$X_{des}(t_k + \delta) = A_d X_{des}(t_k) \tag{5.93}$$

The goal is then to minimize $|X_{des}(t_k + \delta) - X_i(t_k + \delta)|$. To solve this we may write

$$[A_d - A_{dis}]\, X_i(t_k) - B_{dis}\dot{e}_d = 0 \tag{5.94}$$

using the Moore-Penrose inverse we find

$$\dot{e}_d = \left[B_{dis}^T B_{dis}\right]^{-1} B_{dis}^T [A_d - A_{dis}] X_i(t_k) \tag{5.95}$$

We will then let K_h be given by

$$K_h = \left[B_{dis}^T B_{dis}\right]^{-1} B_{dis}^T [A_d - A_{dis}] \tag{5.96}$$

such that

$$\dot{e}_d = K_h X_i(t_k) \tag{5.97}$$

Then, we have the discrete mapping of the switched system to be given by

$$X_i(t_k+\delta) = (A_{dis} + B_{dis}K_h)X_i(t_k) \tag{5.98}$$

5.2.2.2 Choosing Desired Behavior based on Rigid Impedance

For the second method we choose A_{des} to be the desired rigid joint impedance decoupled from the additional states $(e_\theta, \dot{e}_\theta)$. We thus let A_{des} to be given by

$$A_{des} = \begin{bmatrix} A_{des1} & 0 \\ 0 & 0 \end{bmatrix} \tag{5.99}$$

with 0 being a 2×2 matrix of zeros, and A_{des1} is given by

$$A_{des1} = \begin{bmatrix} 0 & 1 \\ -\frac{K_\theta + k_e}{M_\theta} & -\frac{D_\theta}{M_\theta} \end{bmatrix} \tag{5.100}$$

Solving for (5.86) from time t_k to $t_k + \delta$ we have

$$X_{des}(t_k+\delta) = \begin{bmatrix} e^{A_{des1}\delta} & 0 \\ 0 & 0 \end{bmatrix} X_{des}(t_k) \tag{5.101}$$

Let A_d be given by

$$A_d = \begin{bmatrix} A_{d1} & 0 \\ 0 & 0 \end{bmatrix} \tag{5.102}$$

where A_{d1} is given by

$$A_{d1} = e^{A_{des1}\delta} \tag{5.103}$$

such that

$$X_{des}(t_k + \delta) = A_d X_{des}(t_k) \tag{5.104}$$

Our goal is to minimize $|CX_{des}(t_k + \delta) - CX_i(t_k + \delta)|$, where C is given by

$$C = [I \;\; 0] \tag{5.105}$$

We then have

$$[CA_d - CA_{dis}]\, X_i(t_k) - CB_{dis}\dot{e}_d = 0 \tag{5.106}$$

Using the Moore-Penrose inverse we set

$$\dot{e}_d = \left[(CB_{dis})^T (CB_{dis})\right]^{-1} (CB_{dis})^T [A_{d1} - CA_{dis}] X_i(t_k) \tag{5.107}$$

$$\dot{e}_d = K_h X_i(t_k) \tag{5.108}$$

Where K_h is defined as

$$K_h = \left[(CB_{dis})^T (CB_{dis})\right]^{-1} (CB_{dis})^T [A_{d1} - CA_{dis}] \tag{5.109}$$

The discrete mapping of the switched system is then given by

$$X_i(t_k + \delta) = (A_{dis} + B_{dis} K_h) X_i(t_k) \tag{5.110}$$

We see that as $n \to 0$ the switched system is dominated by the passive impedance controller. However, the passive impedance does not exactly reproduce the rigid impedance causing the resulting gains of K_h grow unbounded.

5.2.2.3 Choosing Desired Behavior based on Combination of Passive and Rigid Impedance

Since the passive impedance controller does not exactly produce the desired rigid joint behavior. Therefore, the first method is not the most appropriate method. However, as noted the second method which does use the desired rigid joint behavior is not ideal since it produces unbounded control gains as $n \to 0$. We therefore propose to use a gain K_h that is a convex combination of the two desired behaviors.

$$K_h = \gamma K_{h1} + (1 - \gamma) K_{h2}, \quad \gamma \in [0, 1] \tag{5.111}$$

where K_{h1} is given by (5.96) and K_{h2} is given by (5.109). The variable γ is chosen such that $\gamma = 1$ when $n = 0$ and $\gamma = 0$ when $n = 1$. Since there are infinitely many functions, $\gamma(n)$ which satisfy the boundary conditions, trial and error is used to find an appropriate combination at each n value.

5.3 Linear Separation of External Force in Switching Condition

In the change of switching conditions section we determined a set of values of e_d and $\dot{e}_d$ at the instant the system is switched from the impedance controller to the admittance controller. However, we notice that the choice of $\dot{e}_d$ is dependant on the value of k_e, which may not be known. Therefore, we wish to re derive the conditions used in the previous section with the external force not modeled as a spring force but as a measured variable. We will again use the assumption that the system is a single degree of freedom linear model given by equations (5.1)-(5.3)

5.3.1 Derivation

Let us consider the external force as an unknown input into the closed loop differential equation. Then the hybrid switched system dynamics may be written as

$$\begin{aligned} \dot{X}_i &= A_i X_i + B_i F_{ext} \quad \forall t \in [t_k, t_k + (1-n)\delta) \\ \dot{X}_a &= A_a X_a + B_a F_{ext} \quad \forall t \in [t_k + (1-n)\delta, t_k + \delta) \end{aligned} \tag{5.112}$$

for a positive integer, k, and for $0 \leq n \leq 1$. Where X_i is given by

$$X_i = [e \;,\;\; \dot{e} \;,\;\; e_\theta \;,\;\; \dot{e}_\theta]^T \tag{5.113}$$

A_i is given by

$$A_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -M^{-1}K_m & 0 & M^{-1}K_m & 0 \\ 0 & 0 & 0 & 1 \\ B_d^{-1}K_m & 0 & -B_d^{-1}(K_c + K_m) & -B_d^{-1}K_d \end{bmatrix} \tag{5.114}$$

B_i is given by

$$B_i = \begin{bmatrix} 0 \\ M_\theta^{-1} \\ 0 \\ (MM_\theta^{-1} - 1) \end{bmatrix} \tag{5.115}$$

X_a is given by

$$X_a = [e\ ,\ \ \dot{e}\ ,\ \ e_\theta\ ,\ \ \dot{e}_\theta\ ,\ \ e_d\ ,\ \ \dot{e}_d]^T \tag{5.116}$$

A_a is given by

$$A_a = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -M^{-1}K_m & 0 & M^{-1}K_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ A_{a1} & A_{a2} & A_{a3} & A_{a4} & A_{a5} & A_{a6} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -M^{-1}K_\theta & -M^{-1}D_\theta \end{bmatrix} \tag{5.117}$$

$$A_{a1} = B_d^{-1}(K_m + K_T K_m^{-1}) \quad (5.118)$$

$$A_{a2} = B_d^{-1} K_S K_m^{-1} \quad (5.119)$$

$$A_{a3} = -B_d^{-1}(K_m + K_p + K_T K_m^{-1}) \quad (5.120)$$

$$A_{a4} = -B_d^{-1}(K_S K_m^{-1} + K_d) \quad (5.121)$$

$$A_{a5} = B_d^{-1} K_p \quad (5.122)$$

$$A_{a6} = B_d^{-1} K_d \quad (5.123)$$

and B_a is given by

$$B_a = \begin{bmatrix} 0 \\ M^{-1} \\ 0 \\ 0 \\ 0 \\ M_\theta^{-1} \end{bmatrix} \quad (5.124)$$

Solving the differential equations in (5.112) give the general solutions

$$X_i(t) = e^{A_i(t-t_0)} X_i(t_0) + \int_{t_0}^{t} e^{A_i t-\tau} B_a F_{ext}(\tau) d\tau \quad (5.125)$$

and

$$X_a(t) = e^{A_a(t-t_0)} X_a(t_0) + \int_{t_0}^{t} e^{A_a t-\tau} B_i F_{ext}(\tau) d\tau \quad (5.126)$$

for a positive integer, k, and for $0 \leq n \leq 1$.

At time t_k the system is switched from the admittance controlled system to the impedance controlled system. We then have the change of states given by the mapping

$$X_i = S_{ia} X_a \qquad S_{ia} = [I \;\; 0] \tag{5.127}$$

At time $t_k + (1-n)\delta$, the system is switched from the impedance controlled system to the admittance controlled system. This results in additional states e_d and $\dot{e}_d$ where e_d is chosen such that the control force is continuous and $\dot{e}_d$ is chosen to satisfy equation (5.139). From (5.77) we have

$$\begin{aligned} y_d = & \quad y + \frac{1}{K_p}\left(\frac{B_\theta}{B}F_i + K_d\dot{y} + \frac{K_T F}{K_m} + \frac{K_S}{K_m}\dot{F} - (1 - \frac{B_\theta}{B})F\right) \\ \Rightarrow e_d = & \quad e_y + \frac{1}{K_p}\left(\frac{B_\theta}{B}F_i + K_d\dot{e}_y + \frac{K_T F}{K_m} + \frac{K_S}{K_m}\dot{F} - (1 - \frac{B_\theta}{B})F\right) \end{aligned} \tag{5.128}$$

The states at time $t_k + (1-n)\delta$ are given by the realtion

$$X_a = S_{ai} X_i + K_{ai} F_{ext} + B_{ai} \dot{e}_d \qquad S_{ia} = \begin{bmatrix} I \\ S \end{bmatrix} \tag{5.129}$$

where S is given by

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.130}$$

$$
\begin{aligned}
s_{11} &= -K_p^{-1}K_T \\
s_{12} &= -K_p^{-1}K_S \\
s_{13} &= K_P^{-1}[K_P + K_T - MM_\theta^{-1}K_c] \\
s_{14} &= K_p^{-1}(K_d + K_S - MM_\theta^{-1}D_\theta)
\end{aligned}
\tag{5.131}
$$

B_{ai} is given by

$$
B_{ai} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -K_p^{-1}K_d \\ 1 \end{bmatrix}
\tag{5.132}
$$

and K_{ai} is given by

$$
K_{ai} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ K_p^{-1}(MM_\theta^{-1} - 1) \\ 0 \end{bmatrix}
\tag{5.133}
$$

For the switched system, the states from time t_k to time $t_k + \delta$ can be expressed by the mapping:

$$\begin{aligned}X(t_k+\delta) =& S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)\delta}X_i(t_k)\\&+ S_{ia}e^{A_a(t_k+\delta)}\int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau}B_aF_{ext}(\tau)d\tau\\&+ S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(t_k+(1-n)\delta)}\int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau}B_iF_{ext}(\tau)d\tau\\&+ S_{ia}e^{A_a n\delta}K_{ai}F_{ext}(t_k+(1-n)\delta) + S_{ia}e^{A_a n\delta}B_{ai}\dot{e}_d\end{aligned} \tag{5.134}$$

We consider the desired behavior of the closed-loop system to be that of the system under passive impedance control. The desired dynamics can be written as

$$X_{des} = A_{des}X_{des} + B_{des}F_{ext} \tag{5.135}$$

A_{des} is given by

$$A_{des} = \begin{bmatrix} 0 & 1 & 0 & 0\\ -M^{-1}K_m & 0 & M^{-1}K_m & 0\\ 0 & 0 & 0 & 1\\ B_d^{-1}K_m & 0 & -B_d^{-1}(K_c+K_m) & -B_d^{-1}K_d \end{bmatrix} \tag{5.136}$$

and B_{des} given by

$$B_{des} = \begin{bmatrix} 0 \\ M_\theta^{-1} \\ 0 \\ (MM_\theta^{-1} - 1) \end{bmatrix} \tag{5.137}$$

The general solution of X_{des} from t_k to $t_k + \delta$ is given by

$$X_{des}(t) = e^{A_{des}(t-t_0)} X_{des}(t_k) + \int_{t_k}^{t_k+\delta} e^{A_{des}(t-\tau)} B_{des1} F_{ext}(\tau) d\tau \tag{5.138}$$

As in section (5.2.2.1) we have the goal is to choose $\dot{e}_d$ at $t_k + (1-n)\delta$ such that

$$\|X_{des}(t_k + \delta) - X(t_k + \delta)\| = 0 \tag{5.139}$$

Substituting equations (5.134) and (5.138) into (5.139) we get

$$\begin{aligned}
&e^{A_{des}\delta} X_{des}(t_k) + e^{A_{des}(t_k+\delta)} \int_{t_k}^{t_k+\delta} e^{-A_{des}\tau} B_{des} F_{ext}(\tau) d\tau - \\
&[S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} X_i(t_k) + S_{ia} e^{A_a(t_k+\delta)} \int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau} B_a F_{ext}(\tau) d\tau \\
&+ S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(t_k+(1-n)\delta)} \int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau} B_i F_{ext}(\tau) d\tau \\
&+ S_{ia} e^{A_a n\delta} K_{ai} F_{ext}(t_k + (1-n)\delta + S_{ia} e^{A_a n\delta} B_{ai} \dot{e}_d] = 0
\end{aligned} \tag{5.140}$$

To solve equation (5.140) we will consider $\dot{e}_d$ to be of the form

$$\dot{e}_d = u_h + u_p \tag{5.141}$$

where u_h minimizes the homogeneous portion of (5.140), when $F_{ext} \equiv 0$, and u_p minimizes the additional terms when $F_{ext} \neq 0$. To find u_h we let $F_{ext} \equiv 0$ and $u_p = 0$ and equation (5.140) can then be written as

$$[e^{A_{des}\delta} X_{des}(t_k) - S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta} X(t_k)] - e^{A_a n\delta} B_{ai} u_h = 0 \tag{5.142}$$

Letting $X_{des}(t_k) = X(t_k)$, (5.142) simplifies to.

$$[e^{A_{des}\delta} - S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta}] X(t_k) - e^{A_a n\delta} B_{ai} u_h = 0 \tag{5.143}$$

Using the Moore-Penrose, set

$$u_h = K_h X_i(t_k) \tag{5.144}$$

where

$$K_h = \left[(S_{ia} e^{A_a n\delta} B_{ai})^T S_{ia} e^{A_a n\delta} B_{ai} \right]^{-1} (S_{ia} e^{A_a n\delta} B_{ai})^T \\ [e^{A_{des}\delta} - S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(1-n)\delta}] \tag{5.145}$$

Since switching occurs at $t_k + (1-n)\delta$, it is desirable to write u_h in the form

$$u_h = K_h(e^{A_i(1-n)\delta})^{-1}X_i(t_k + (1-n)\delta) \tag{5.146}$$

By substituting (5.146) back into (5.140) and find

$$\begin{aligned}
&e^{A_{des}\delta}X_{des}(t_k) + e^{A_{des}(t_k+\delta)}\int_{t_k}^{t_k+\delta} e^{-A_{des}\tau}B_{des}F_{ext}(\tau)d\tau - \\
&[S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)\delta}X(t_k) + S_{ia}e^{A_a(t_k+\delta)}\int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau}B_aF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(t_k+(1-n)\delta)}\int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau}B_iF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_a n\delta}B_{in}K_h(e^{A_i(1-n)\delta})^{-1}X(t_k + (1-n)\delta) \\
&+ S_{ia}e^{A_a n\delta}K_{ai}F_{ext}(t_k + (1-n)\delta) + S_{ia}e^{A_a n\delta}B_{ai}u_p] = 0
\end{aligned} \tag{5.147}$$

Using equation (5.125) we write (5.147) as

$$\begin{aligned}
&\{e^{A_{des}\delta}X_{des}(t_k) - S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)\delta}X(t_k) - S_{ia}e^{A_a n\delta}B_{ai}K_hX(t_k)\} + \\
&e^{A_{des}(t_k+\delta)}\int_{t_k}^{t_k+\delta} e^{-A_{des}\tau}B_{des}F_{ext}(\tau)d\tau - \\
&[S_{ia}e^{A_a(t_k+\delta)}\int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau}B_aF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(t_k+(1-n)\delta)}\int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau}B_iF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_a n\delta}B_{ai}K_he^{A_it_k}\int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau}B_iF_{ext}(\tau)d\tau \\
&+ S_{ia}e^{A_a n\delta}K_{ai}F_{ext}(t_k + (1-n)\delta) + S_{ia}e^{A_a n\delta}B_{ai}u_p] = 0
\end{aligned} \tag{5.148}$$

From equations (5.139), (5.140), (5.144), and from linearity we have the choice of u_p can only be chosen to minimize portions of (5.148) containing the external force. Thus, (5.148) simplifies to

$$
\begin{aligned}
& e^{A_{des}(t_k+\delta)} \int_{t_k}^{t_k+\delta} e^{-A_{des}\tau} B_{des} F_{ext}(\tau) d\tau - \\
& [S_{ia} e^{A_a(t_k+\delta)} \int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau} B_a F_{ext}(\tau) d\tau \\
& + S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(t_k+(1-n)\delta)} \int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau} B_i F_{ext}(\tau) d\tau \\
& + S_{ia} e^{A_a n\delta} B_{ai} K_h e^{A_i t_k} \int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau} B_i F_{ext}(\tau) d\tau \\
& + S_{ia} e^{A_a n\delta} K_{ai} F_{ext}(t_k + (1-n)\delta) + S_{ia} e^{A_a n\delta} B_{ai} u_p] = 0
\end{aligned} \tag{5.149}
$$

Solving for u_p gives

$$
\begin{aligned}
u_p = & \left[(S_{ia} e^{A_a n\delta} B_{ai})^T S_{ia} e^{A_a n\delta} B_{ai} \right]^{-1} (S_{ia} e^{A_a n\delta} B_{ai})^T \\
& \{ e^{A_{des}(t_k+\delta)} \int_{t_k}^{t_k+\delta} e^{-A_{des}\tau} B_{des} F_{ext}(\tau) d\tau - \\
& [S_{ia} e^{A_a(t_k+\delta)} \int_{t_k+(1-n)\delta}^{t_k+\delta} e^{-A_a\tau} B_a F_{ext}(\tau) d\tau \\
& + S_{ia} e^{A_a n\delta} S_{ai} e^{A_i(t_k+(1-n)\delta)} \int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau} B_i F_{ext}(\tau) d\tau \\
& + S_{ia} e^{A_a n\delta} B_{in} K_h e^{A_i t_k} \int_{t_k}^{t_k+(1-n)\delta} e^{-A_i\tau} B_i F_{ext}(\tau) d\tau] \\
& - S_{ia} e^{A_a n\delta} K_{ai} F_{ext}(t_k + (1-n)\delta) \}
\end{aligned} \tag{5.150}
$$

which may be written in the form

$$u_p = \int_{t_k}^{t_k+\delta} F_p(t-\tau)F_{ext}(\tau)d\tau + H_p F_{ext}(t_k + (1-n)\delta) \tag{5.151}$$

where $F_p(t)$ is given by

$$\begin{aligned} F_p = & \left[(S_{ia}e^{A_a n\delta}B_{ai})^T S_{ia}e^{A_a n\delta}B_{ai}\right]^{-1}(S_{ia}e^{A_a n\delta}B_{ai})^T \\ & \{e^{A_{des}(t_k+\delta)}e^{-A_{des}\tau}B_{des} \\ & - [S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(t_k+(1-n)\delta)}e^{-A_i\tau}B_i \\ & + S_{ia}e^{A_a n\delta}B_{in}K_h e^{A_i t_k}e^{-A_i\tau}B_i]\} \end{aligned} \tag{5.152}$$

for $\tau \in (t_k, t_k + (1-n)\delta]$, F_p is given by

$$\begin{aligned} F_p = & \left[(S_{ia}e^{A_a n\delta}B_{ai})^T S_{ia}e^{A_a n\delta}B_{ai}\right]^{-1}(S_{ia}e^{A_a n\delta}B_{ai})^T \\ & \{e^{A_{des}(t_k+\delta)}e^{-A_{des}\tau}B_{des} - S_{ia}e^{A_a(t_k+\delta)}e^{-A_a\tau}B_a\} \end{aligned} \tag{5.153}$$

for $\tau \in (t_k + (1-n)\delta, t_k + \delta]$, and H_p is given by

$$H_p = -\left[(S_{ia}e^{A_a n\delta}B_{ai})^T S_{ia}e^{A_a n\delta}B_{ai}\right]^{-1}(S_{ia}e^{A_a n\delta}B_{ai})^T S_{ia}e^{A_a n\delta}K_{ai} \tag{5.154}$$

Chapter 6

Experiments

We now investigate the performance of the hybrid impedance and admittance controller as we change n value using the KUKA-DLR lightweight arm, shown in Figure 6.1. The arm is equipped with a force sensor at the end of the effector to measure the external forces.

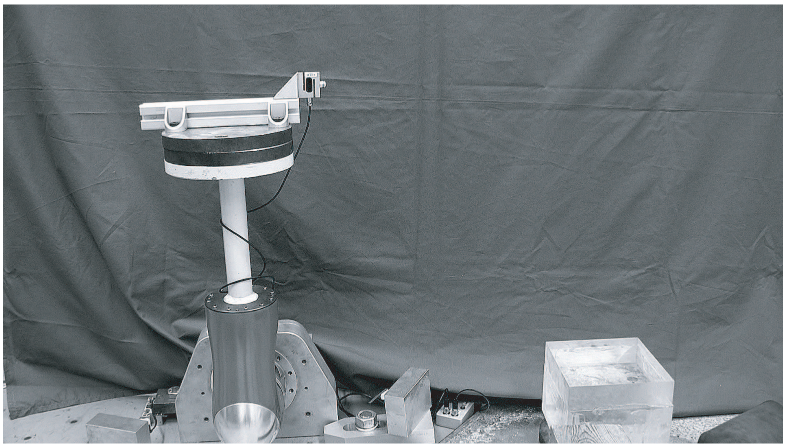

Figure 6.1: Photo of Experimental Setup

6.1 Experimental Set-up

The reference angle for the arm is shown in Figure 6.2. The large gear ratio in the drive of the joint causes the arm to act according to the flexible joint model. The joint is equipped with a force torque sensor to measure the resulting torque between the link and the motor. The model of the single-DOF flexible joint Kuka-DLR arm is assumed to be:

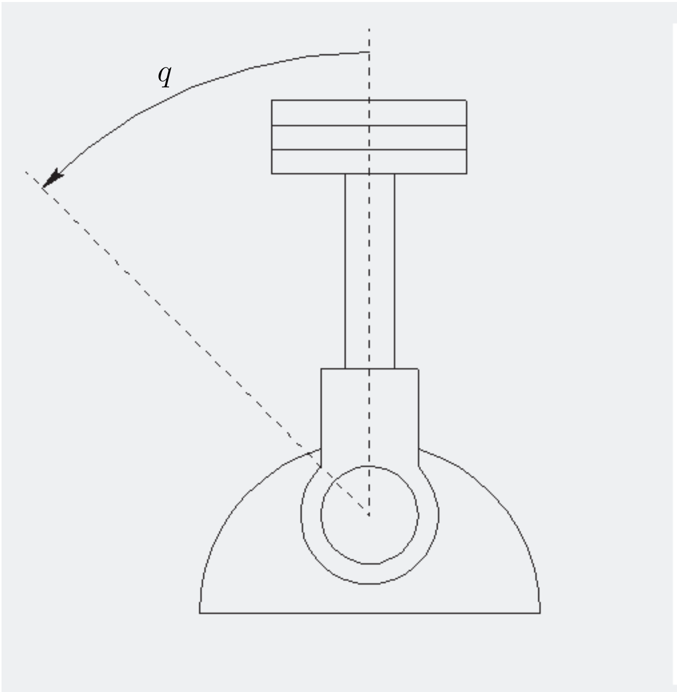

Figure 6.2: Reference Diagram of Experimental Set-up

$$m\ddot{q} + g(q) = \tau + D_m K_m^{-1}\dot{\tau} + J^T F_{ext} \tag{6.1}$$

$$B\ddot{\theta} + \tau + D_m^{-1} K_m \dot{\tau} = \tau_m \tag{6.2}$$

$$\tau = K_m(\theta - q) \tag{6.3}$$

$$\tag{6.4}$$

The parameters of the single joint of the KUKA-DLR lightweight arm are given as follows:

$$m = 3.6\ Kg\ m^2 \quad B = 1.5308\ Kg\ m^2 \quad K_m = 24000\ Nm$$
$$D_m = 12\ Nms \quad g(q) = -73\sin(q)\ Nm \tag{6.5}$$

The parameters of the desired impedance were chosen as follows:

$$M_\theta = m \quad K_\theta = 100\ Nm \quad D_\theta = 1.4\sqrt{K_\theta M_\theta}\ Nms \quad \frac{B}{B_\theta} = 4.5 \tag{6.6}$$

The impedance controller is implemented according to

$$\begin{aligned}\tau_m = \tau_i =& BB_\theta^{-1}\{g(\overline{q}_l(\theta)) + MM_\theta^{-1}[-K_\theta(\overline{q}_l(\theta) - q_0) - D_\theta(\theta - \dot{q}_0)] \\ &+ M\ddot{q}_0 + (MM_\theta^{-1} - 1)\tau_{ext}\} + (1 - BB_\theta^{-1})\tau\end{aligned} \tag{6.7}$$

where $\overline{q}_l$ is given by

$$\overline{q}_l(\theta) = h_l^{-1}(\theta) \tag{6.8}$$

h_l is given by

$$h_l(q) = q + K_m^{-1} l(q) \tag{6.9}$$

and $l(q)$ is given by

$$l(q) = g(q) - K_\theta(q - q_0) \tag{6.10}$$

and the admittance controller is implemented according to the control law.

$$\begin{aligned}\tau_m = \tau_a = &\frac{B}{B_\theta}[-K_p\theta - K_d(\dot{\theta}) - K_T(\theta - q) - K_s(\dot{\theta} - \dot{q}) \\ &+ (K_m + K_T + K_p)K_m^{-1}g(q_d) + K_p q_d + K_d\dot{q}_d] + (1 - \frac{B}{B_\theta})\tau\end{aligned} \tag{6.11}$$

with q_d again being given by

$$M_\theta(\ddot{q}_d - \ddot{q}_0) + D_\theta(\dot{q}_d - \dot{q}_0) + K_\theta(q_d - q_0) = J^T F_{ext} \tag{6.12}$$

We see that the mapping $\overline{q}_l$ is a steady state approximation of the value of q at a given value of θ. The control gains chosen to be

$$\begin{aligned} K_p = 6732.78\ Nm \quad & K_d = 292.09\ Nms \\ K_T = -0.402\ Nm \quad & K_s = -0.00446\ Nms \end{aligned} \tag{6.13}$$

We implemented the hybrid controller described by (5.32). At the instant that the system is switched from the impedance controller to the admittance controller we consider the choice of states q_d and $\dot{q}_d$ such that the control force τ_m is continuous. Setting equations (6.7) and (6.11) equal gives

$$(K_m + K_P + K_T)K_m^{-1}g(q_d) + K_p q_d = (K_d + K_S - D_\theta)\dot{\theta}$$
$$+(K_p + K_T)\theta - K_T q - K_S \dot{q} + K_\theta(q_0 - \overline{q}_l(\theta)) + g(\overline{q}_l(\theta)) - K_d \dot{q}_d \qquad (6.14)$$

Inverting equation (6.14) may be difficult analytically, However, given $\dot{q}_d$ we find that (6.14) may be solved using the same iterative process used in section (4.2.4). To solve for $\dot{q}_d$ we wish to use a process similar to that used in section (5.3). However, the non-linearity of the control and switching complicates the minimization equations. Therefore, to simplify the process we consider the linearization of the equations about the operating point and use the linear method to find $\dot{q}_d$.

We have $\dot{q}_d$ given by

$$\dot{q}_d = u_h + u_p \qquad (6.15)$$

where u_h and u_p are given by the relations

$$\begin{aligned} u_h &= \left[(S_{ia}e^{A_a n\delta}B_{ai})^T S_{ia}e^{A_a n\delta}B_{ai}\right]^{-1}(S_{ia}e^{A_a n\delta}B_{ai})^T \\ &\quad e^{A_{des}\delta}(e^{A_i(1-n)\delta})^{-1}X_i(t_k + (1-n)\delta) \\ &\quad - S_{ia}e^{A_a n\delta}S_{ai}e^{A_i(1-n)\delta}(e^{A_i(1-n)\delta})^{-1}X_i(t_k + (1-n)\delta) \qquad (6.16) \\ u_p &= \int_{t_k}^{t_k+\delta} F_p(t-\tau)F_{ext}(\tau)d\tau + H_p F_{ext}(t_k + (1-n)\delta) \qquad (6.17) \end{aligned}$$

Note that the solution of u_p requires knowledge of the external torque at future times

$t \in [t_k + (1-n)\delta, t_k + \delta]$. We then use theorem 3.2.3 from [20] which allows us to rewrite (6.17) as

$$u_p = U_p(t_k + \delta)F_{ext}(t_k + \delta) + H_p F_{ext}(t_k + (1-n)\delta) \tag{6.18}$$

This again requires $F_{ext}(t_k + \delta)$ which is not known at time $t_k + (1-n)\delta$. However, $F_{ext}(t)$ is surjective and therefore there exists a right inverse allowing us to write

$$u_p = [\hat{U}_p(t_k + \delta) + H_p]F_{ext}(t_k + (1-n)\delta) \tag{6.19}$$

Since $\hat{U}_p(t_k + \delta)$ cannot be calculated at the time the experiment was performed, we instead show the validity of the approach by choosing a set of values constant K_{pa} where

$$u_p = K_{pa}F_{ext}(t_k + (1-n)\delta) \tag{6.20}$$

We now proceed to analyse the response of the system under different values of $\hat{H}_p$ to demonstrate the change in response and verify that there exists a time varying function that satisfies (6.20) is equivalent to (6.17).

6.2 Results for Desired Response Equal to Passive Impedance

In (5.2.2) we discussed briefly that it is possible to choose different desired behaviors for a flexible joint model based on the passive impedance control response and the rigid impedance control response. Therefore, we will analyse multiple desired behaviors the understand fully the advantages and disadvantages of each.

First we consider the desired behavior to be given by the passive impedance control response. The experimental response of the hybrid system when no external force is shown in Figure 6.3 with a switching rate of 25 ms

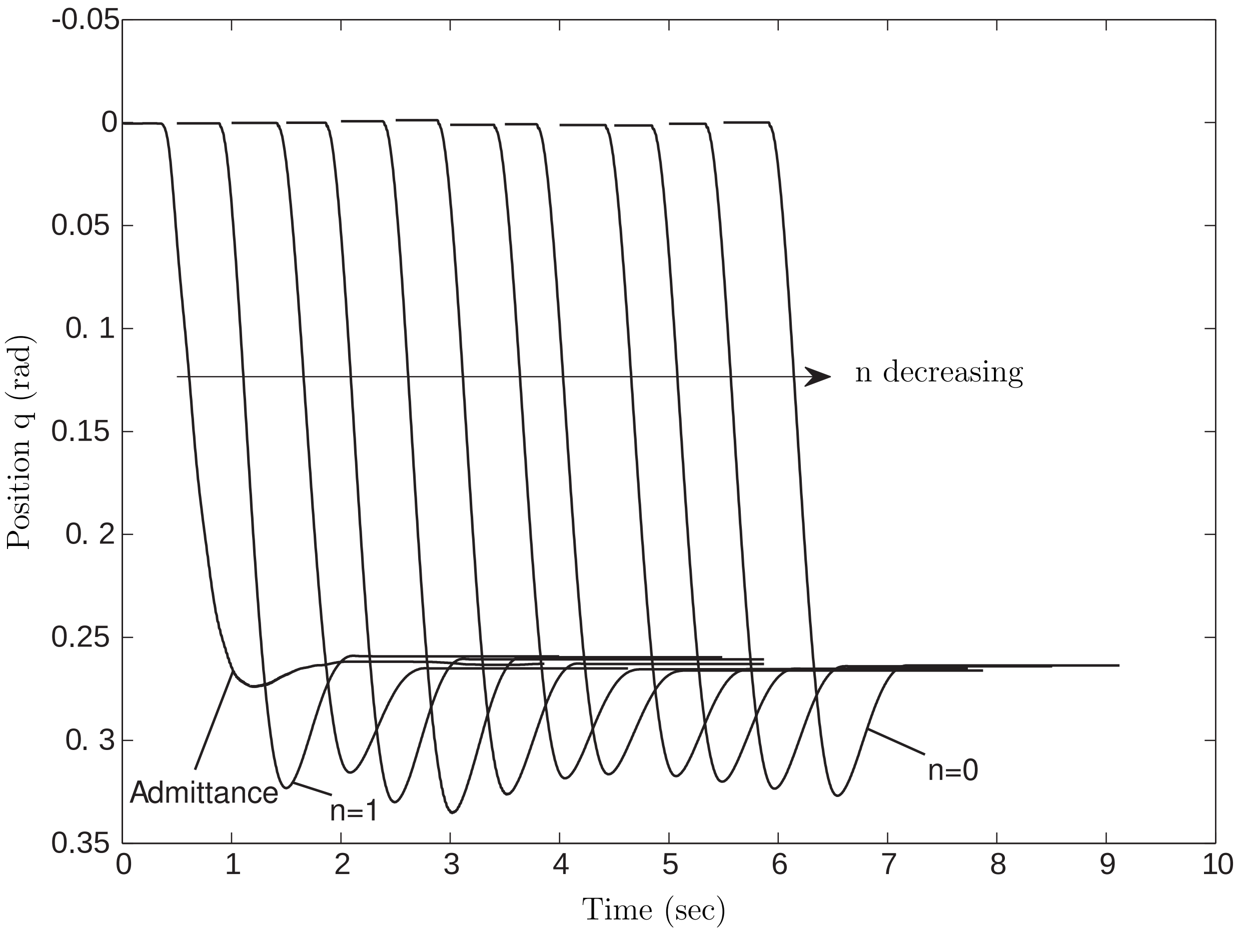

Figure 6.3: Free Space Hybrid Response for Different n values with $\delta = 25$ ms, and using the passive impedance as the desired behavior

From Figure 6.3 we see that there is little deviation between the passive impedance controlled system and the hybrid system during the free response case. This is not surprising since the passive impedance controller is used as the desired behavior for the choice of states of the switching conditions, and because gravitational feedback was tuned for the passive impedance control. To verify that there is a difference in response for different values of n in the case with no external force let us consider the application of the gravity free control in the system with gravity.

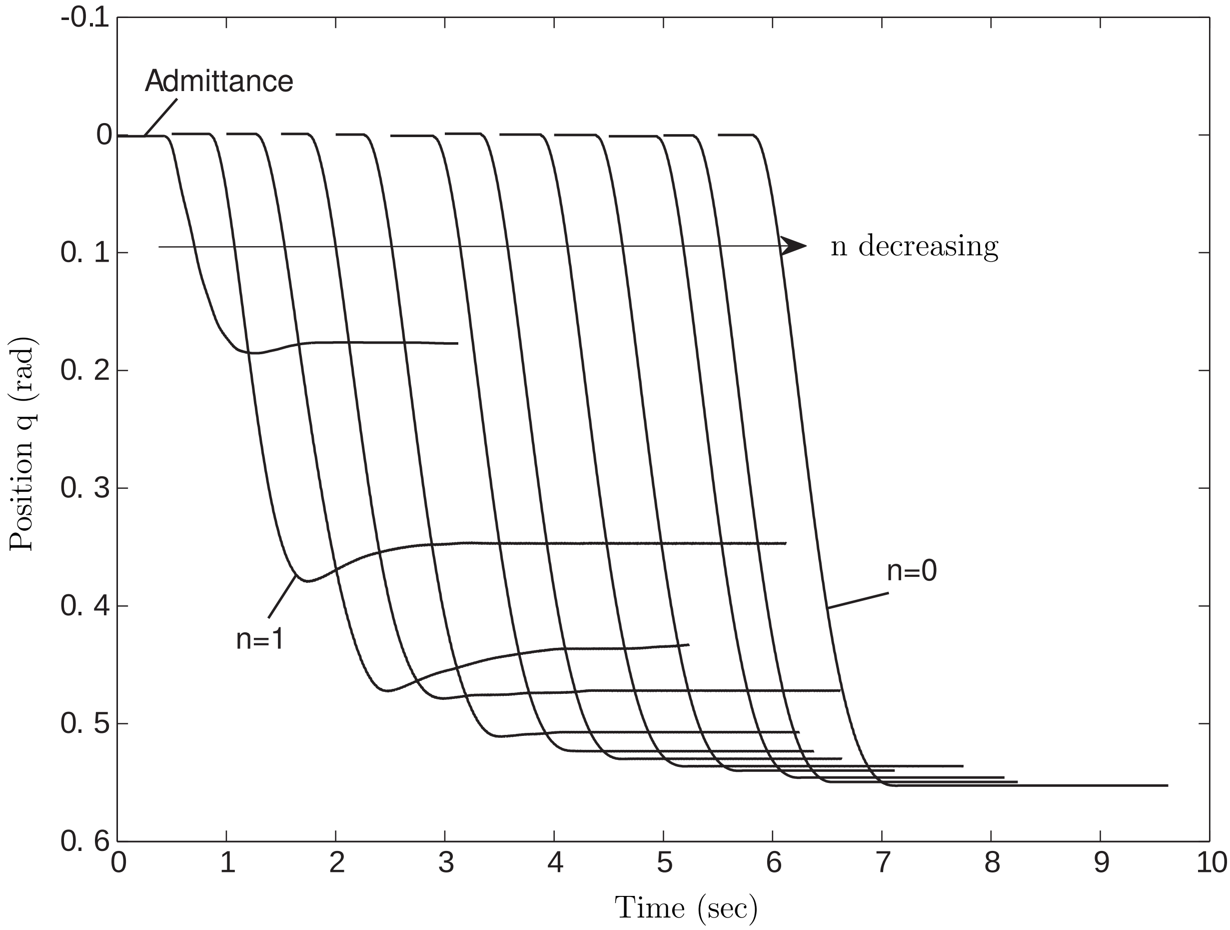

Figure 6.4: Free Space Hybrid Response for Different n values with $\delta = 25\ ms$, and using the passive impedance as the desired behavior ignoring gravity

We see from Figure 6.4 that when gravity is not considered the system produces a large steady state error when controlled by the passive impedance controller while it produces virtually no error when controlled by the admittance controller. We further see that the hybrid controller produces a steady state error between that of the admittance controller and the passive impedance controller. Furthermore, the steady state error of the hybrid controller decreases as the value of n increases giving more weight to the admittance controller. However, we find that the hybrid controller does not closely approximate the admittance controller when $n = 0$. This is due to restrictions on the position controller gains due to torque restrictions and noise in the read position and velocity.

We next consider contact with a soft environment. To this end, a padded surface is placed in the path of the end effector. Then the link is moved until the effector is in contact but there is no force. Then the reference position is commanded to move 10 degrees into the surface. Since the presence of the external force requires the particular solution of the feedback in the form of u_p in (6.20), we observe the change in response for a constant n value and a switching rate of 25 ms, and for different values of K_{pa}.

Figure 6.5 shows the response of the system for $n = 1$ (admittance control with resetting) for K_{pa} with values 0.005, 0.000, -0.001, -0.002, -0.003, -0.004, and -0.005 shown in decreasing order from right to left. We notice that there is a small change in the steady state solution as K_{pa} varies, but the change is small due to the softness of the external stiffness. We also see that as K_{pa} decreases, the rate at which the hybrid system response converges to the steady state solution increases. We may therefore infer that the changing the value of K_{pa} changes the transient response of the system. However, we are unable to compare the system to the admittance controlled system since the experiment would need to be reinitialized to change the controller, which changes the set point at which the effector contacts the surface.

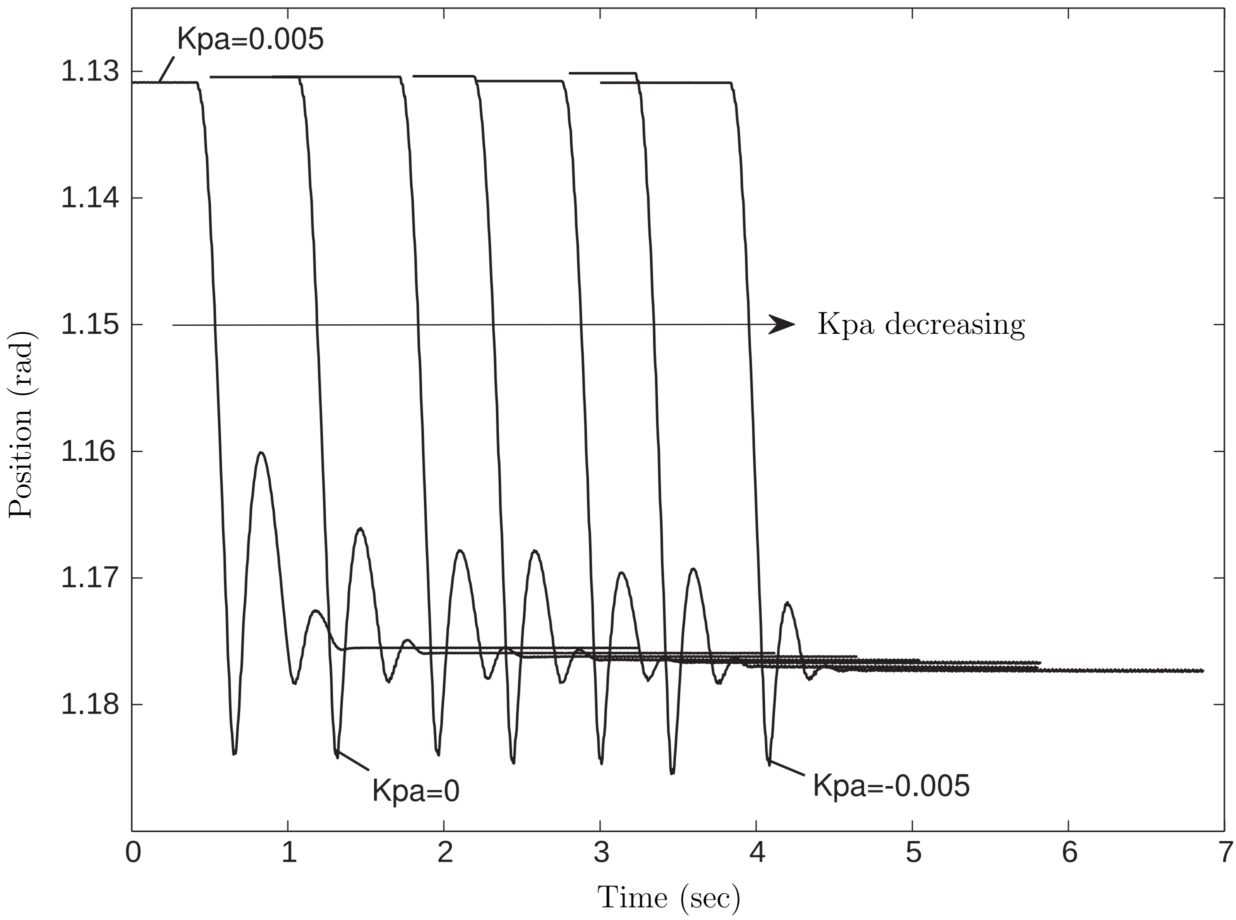

Figure 6.5: Soft contact position response for $\delta = 25\ ms$, $n = 1$, and varying K_{pa} values while using passive impedance controller for desired behavior

Figure 6.6 shows the response of the hybrid system with switching rate 25 ms using a constant value $K_{pa} = 0.005$ and varying n from 1 to 0 in decrements of 0.1 shown from left to right. We see from the figure that there is a small change in the steady state position as the value of n changes. However, some of the variation can be attributed to the slight change in initial configuration. We also notice that the rate of convergence to the steady state solution increases noticeably as n decreases. The change of rate of convergence is characteristic of the admittance controller with lower gains. From Figures

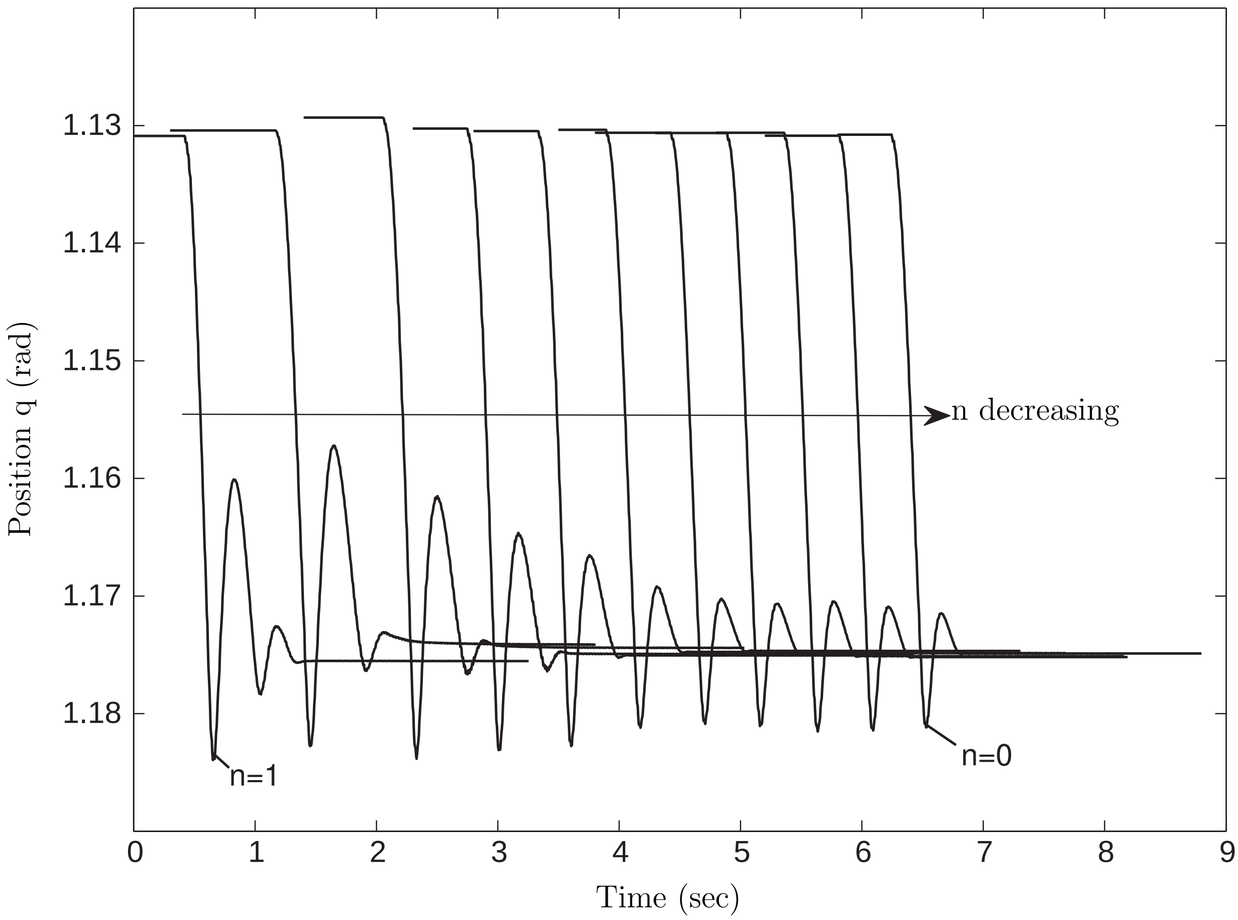

Figure 6.6: Soft contact position response for $\delta = 25\ ms$, $K_{pa} = 0.005$, and varying n while using passive impedance controller for desired behavior

6.5 and 6.6 we see that when in contact with a soft environment the hybrid system shows large overshoot and oscillations for small n which decrease as n increase, and that we may change the value of K_{pa} to achieve the desired performance for a given n value.

We now study the response when the hybrid controller is implemented during contact with a stiff environment. The experiment for stiff contact is done in the same manner as the soft contact, except that the padded surface has been removed completely leaving only the stiff surface the pad was sitting on. We observe the change in response when

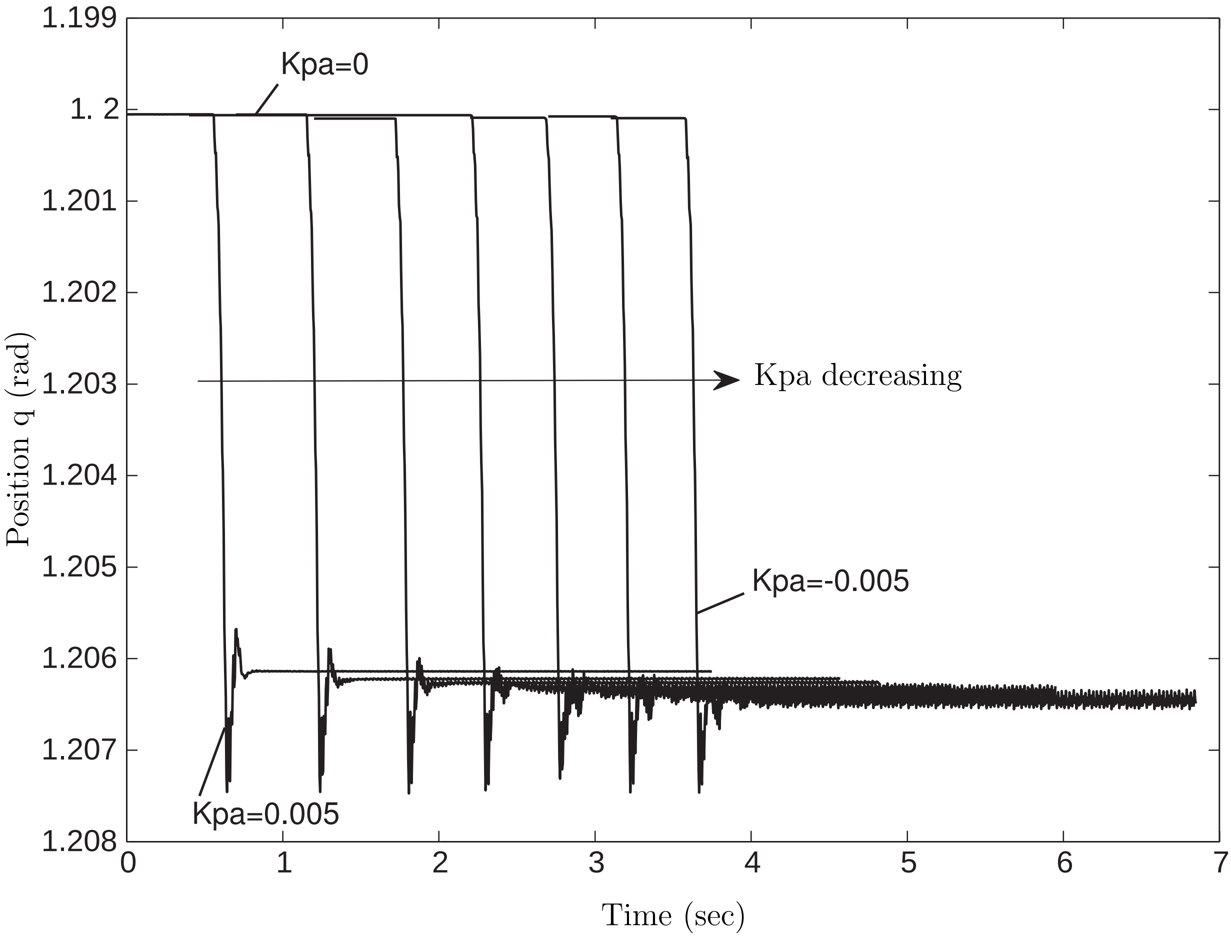

Figure 6.7: Hard contact position response for $\delta = 25\ ms$, $n = 1$, and varying K_{pa} while using passive impedance controller for desired behavior

contact is made with a constant n value and a varying K_{pa} value. Figure 6.7 shows the position response when in contact with a stiff environment for $n = 1$ (admittance control with resetting) and with K_{pa} values 0.005, 0.000, -0.001, -0.002, -0.003, -0.004, and -0.005 shown in decreasing order from right to left. We notice a slight variation in the mean steady state value, as was the case for soft contact. However, for smaller K_{pa} values the system does not reach a steady state value, and produces a high frequency vibration about a mean value instead. This is because the steady state value of the position controller is not the same value as the passive impedance controller, and the deviation between the steady state values increases as the external stiffness increases. This also explains why the oscillations were not observed for the soft and free space cases. Noting that the performance changes as K_{pa} changes allows us to again infer that there exists a varying K_{pa} value that produces the desire result.

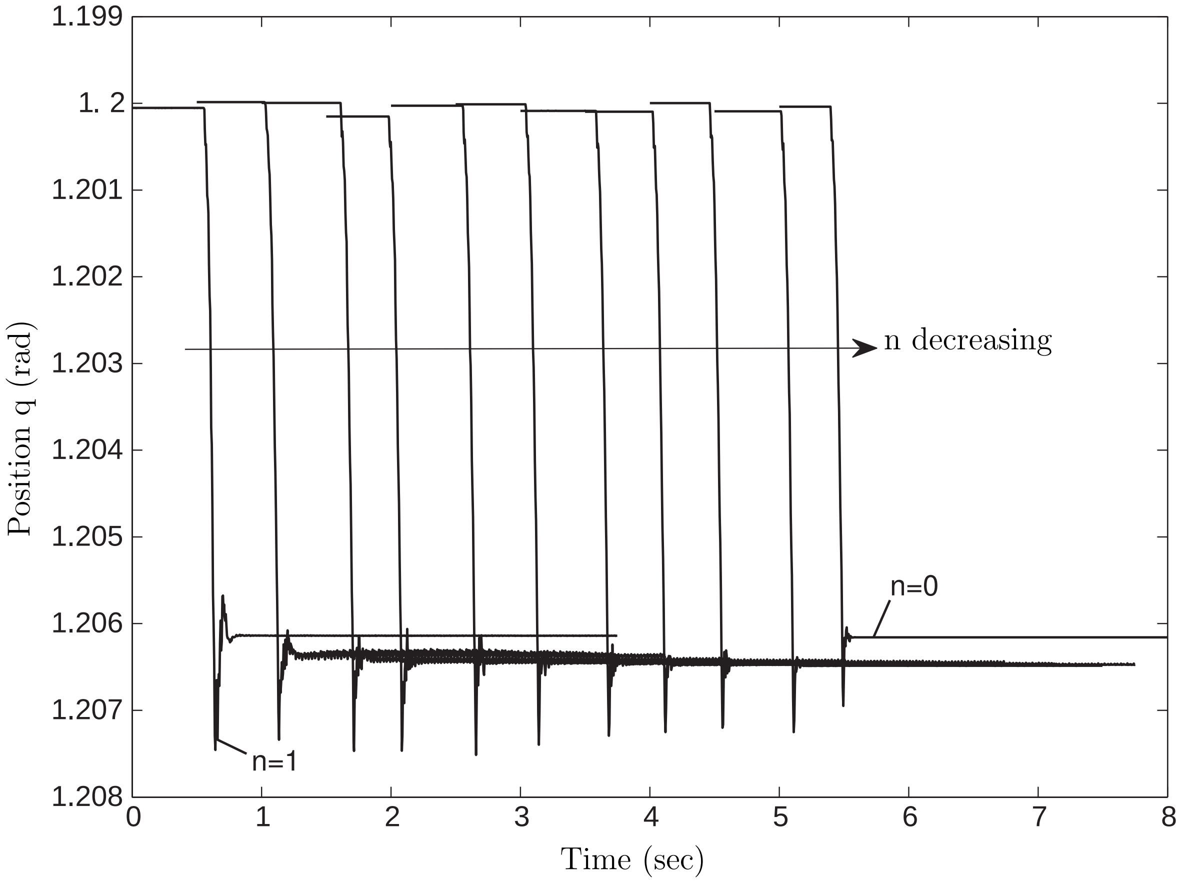

Figure 6.8: Hard contact position response for $\delta = 25\ ms$, $K_{pa} = 0.005$ and varying n while using passive impedance controller for desired behavior

Figure 6.8 shows the response of the joint angle for $K_{pa} = 0.005$, $\delta = 0.25$, and n varying from 0 to 1, with n decreasing from left to right. Similarly Figure 6.9 shows the response of the joint angle for $K_{pa} = -0.005$, $\delta = 0.25$, and n varying from 0 to 1, with

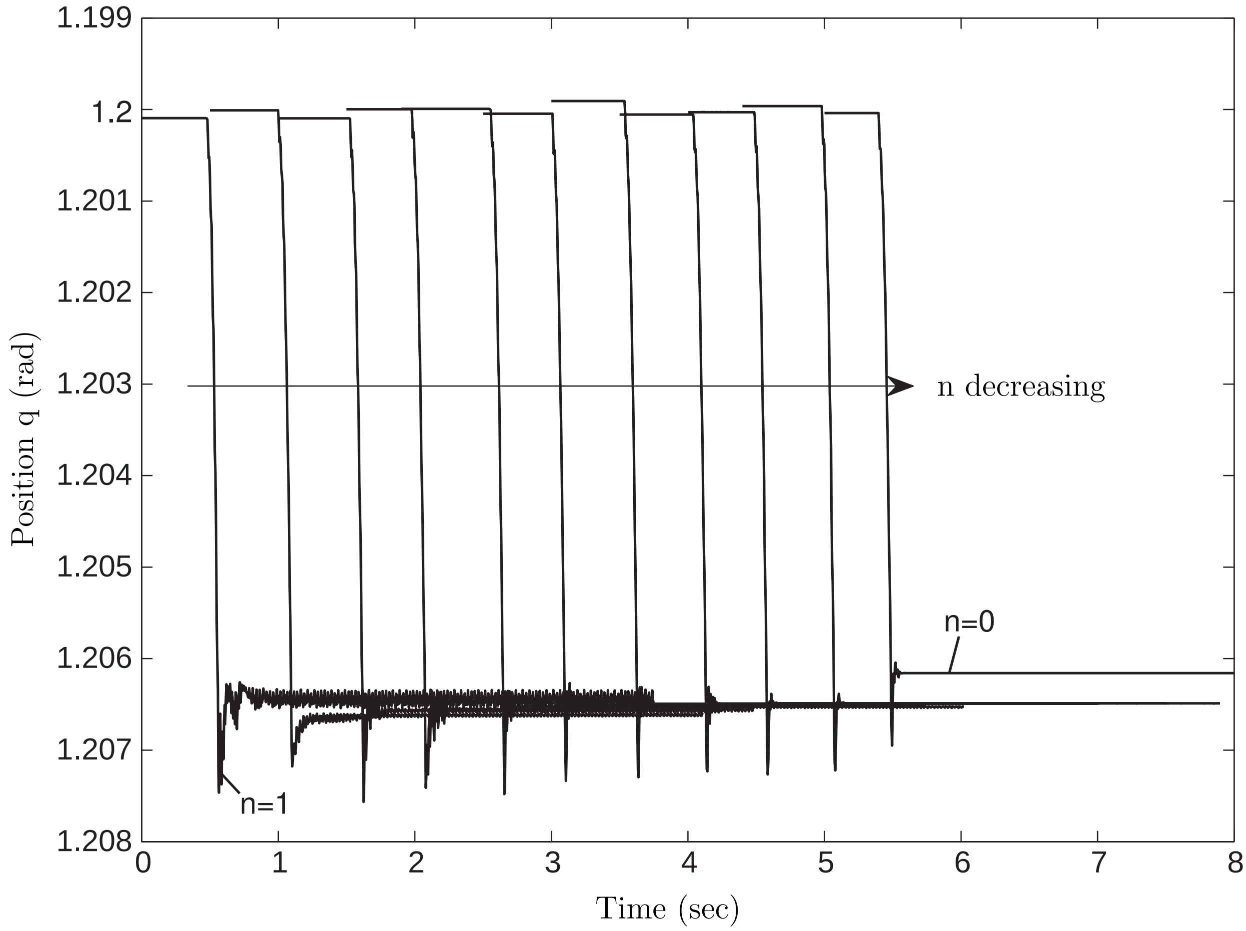

Figure 6.9: Hard contact position response for $\delta = 25\ ms$, $K_{pa} = -0.005$, and varying n while using passive impedance controller for desired behavior

n decreasing from left to right. We see from both Figures 6.8 and 6.9 that changing the value of n changes the steady state behavior of the system, and can result in a persistent excitation. We also note that the response more closely resembles the passive impedance controller as n goes to 0. Therefore, we find that we may choose a value of n to produce the best behavior for a given external stiffness.

6.3 Results for Desired Response Equal to Combination of Passive Impedance and Rigid Impedance

In the previous section we analysed experimental results with resetting conditions that were chosen to minimize the difference between the switched behavior and the ideal passive impedance behavior. However, the ideal passive impedance behavior does not exactly match the desired link behavior. Therefore, we investigate experimentally the effect of switching based on a convex combination of the passive impedance controller and the rigid joint impedance as described in Section 5.2.2.3. The experiments are preformed with γ in equation (5.111) given by

$$\gamma = (1 - n)^2 \tag{6.21}$$

Figure 6.10 shows the response of the system in free motion for varying n values with $\delta = 25\ ms$. For free motion we do not have to consider the external torque. We notice that there is a change in transient response of the hybrid controller as n is changed, and that small n values produce transient responses closer to the passive impedance controller while larger n values produce transient responses closer to the admittance controller. This is an improvement over the choice of passive impedance as the desired behavior shown in figure 6.3 since it did not change the transient response as n varied. However, we notice that there is a difference between the steady state position of the admittance control compared to both the impedance and hybrid control methods, and that the steady state response of the hybrid controller does not vary much as n changes. To investigate the steady state error effects closely, we apply the control again in free

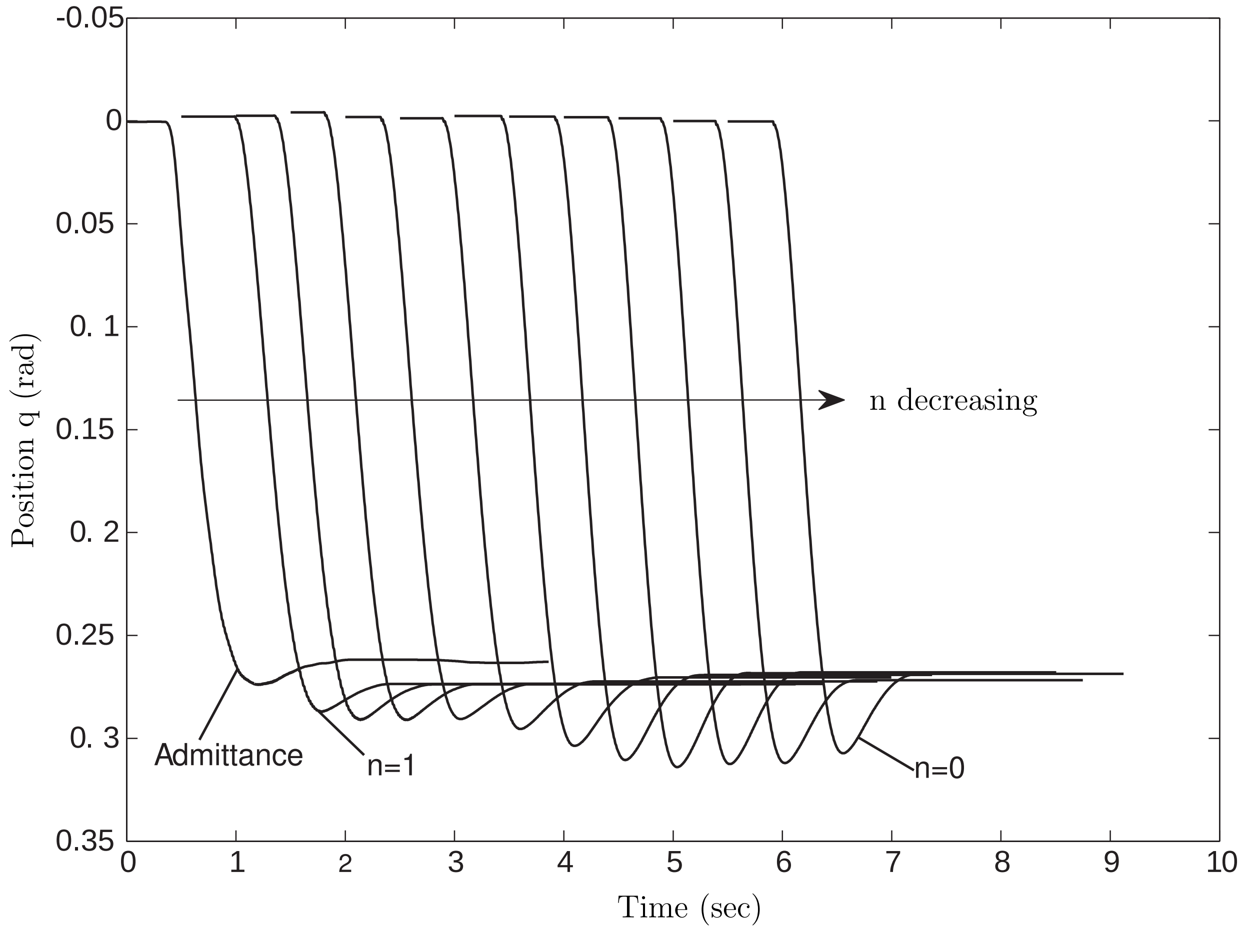

Figure 6.10: Free Space Hybrid Response for Different n values with $\delta = 25\ ms$, and using a combination of passive impedance and rigid impedance as the desired behavior

space but assume that there is no gravity present.

Figure 6.11 shows the response of the system in free space with varying n, n decreasing from left to right, for $\delta = 25\ ms$, and without gravity compensation. We see from the figure that there is a large change in the steady state response as n changes, with the steady state position of the hybrid controller equal to that generated by the passive impedance controller when $n = 0$ and approaching the steady state position of the admittance control as n goes to 1. However, the steady state response with the

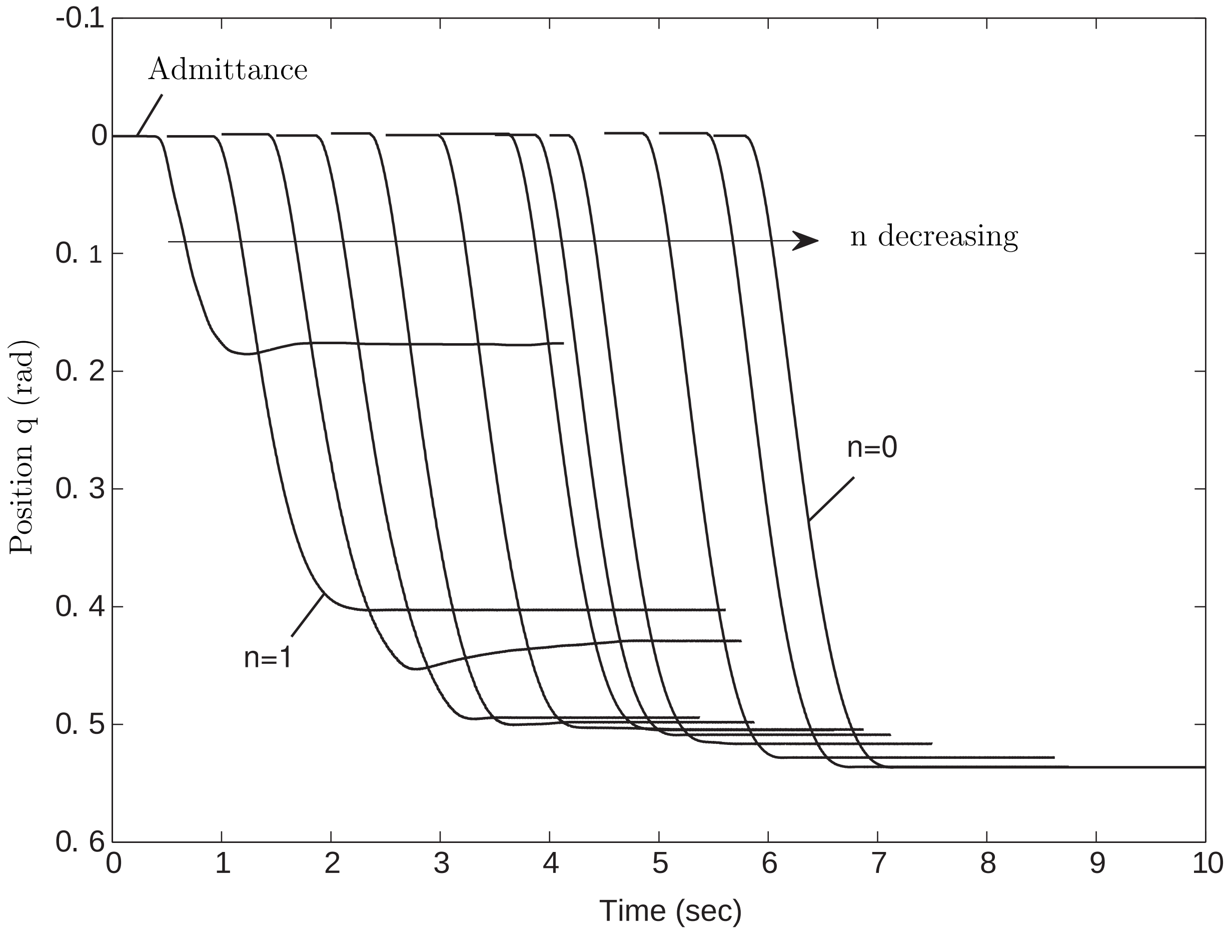

Figure 6.11: Free Space Hybrid Response for Different n values with $\delta = 25\ ms$, and using a combination of passive impedance and rigid impedance as the desired behavior while ignoring gravity

hybrid controller never reaches the steady state position of the admittance control. This is due to the low position gains, constrained by hardware limitations.

We next consider the response of the system when it comes in contact with an external surface. We first consider a soft surface which is produced by putting a padding on the rigid surface. We use the same procedure as in the previous section of moving the effector until it is in contact with the surface but the force measurement is still zero. We then send a step input command of moving 10 degrees into the surface and record

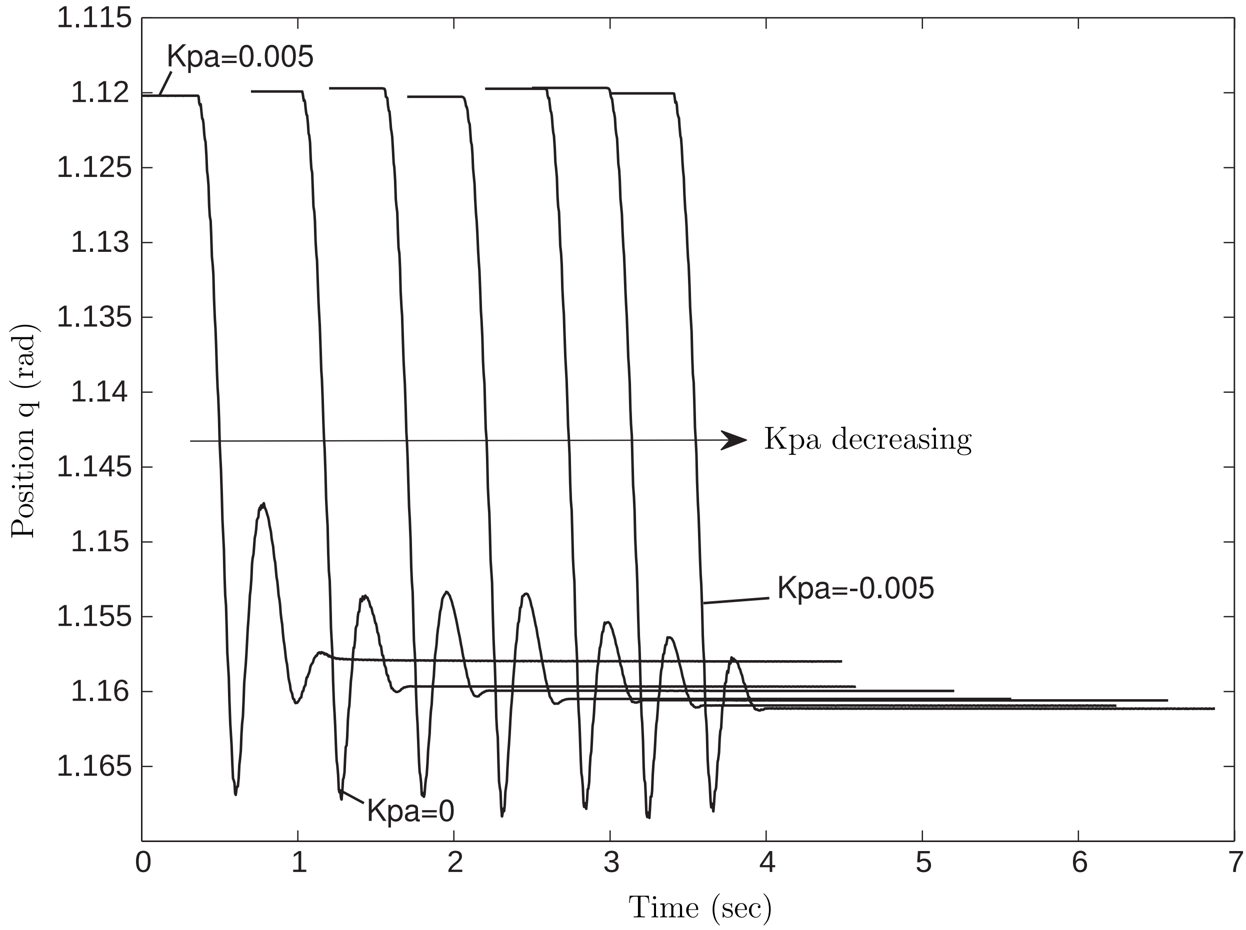

Figure 6.12: Soft contact position response for $\delta = 25\ ms$, $n = 1$, and varying K_{pa} values while using a combination of passive impedance and rigid impedance as the desired behavior

the response. Figure 6.12 shows the joint angle response for $\delta = 25\ ms$, $n = 1$, and for varying K_{pa} values. We see from the figure that the steady state response changes as K_{pa} changes. Also, the rate at which the system converges to steady state changes as K_{pa} changes. Clearly, it is possible to change the performance of the hybrid system through our choice of K_{pa}. We next investigate the change in response as n varies.

Figure 6.13 shows the response of the system for $\delta = 25\ ms$, $K_{pa} = 0.005$, and n changing in decrements of 0.1 from left to right. We see from the figure that the

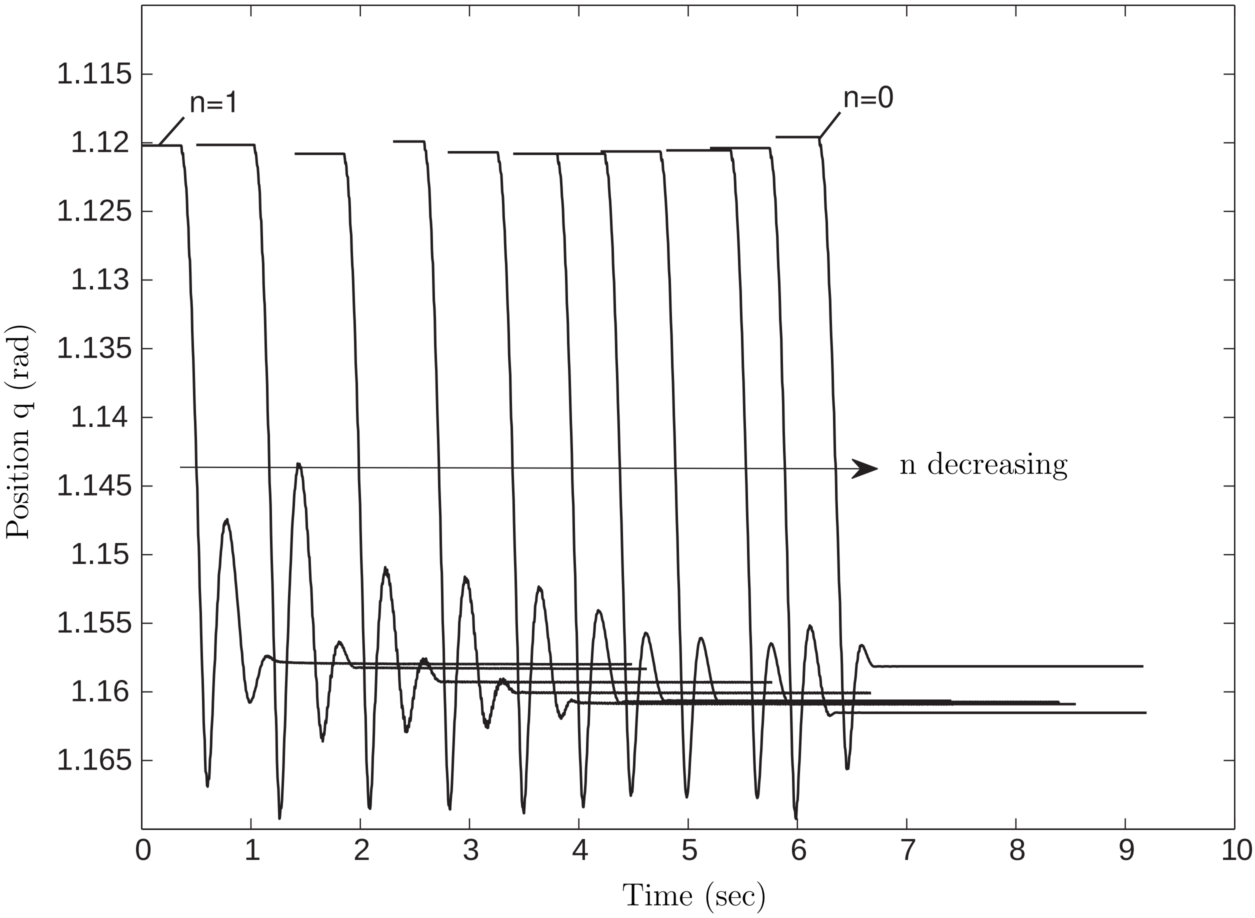

Figure 6.13: Soft contact position response for $\delta = 25\ ms$, $K_{pa} = 0.005$, and varying n while using a combination of passive impedance and rigid impedance as the desired behavior

steady state response of the hybrid system varies noticeably as n changes. Also notice that the transient response changes as n changes. However, we notice that the passive impedance control, $n = 0$, does not appear in line with the trend of decreasing n. Namely, we notice that as n is decreasing, the steady state position gradually decreases and the rate of convergence of the system gradually increases. When n changes from 0.1 to 0.0, we notice a sudden increase in the steady state position of the system and a very large decrease in the rate of convergence of the system. This is due to a constant choice of K_{pa} over all n values. We notice from chapter 5.3.1 that the value of K_{pa} should ultimately be determined by both the external stiffness, and the switching weight n and from figure 6.12 we see that the value for K_{pa} effects the steady state position. Therefore, we are able infer that varying K_{pa} and n can be used to change the behavior of the system meaning there is a selection of K_{pa} and n that will produce the best response for the soft contact case. However, we notice that the steady state response is more sensitive to the choice of K_{pa} then when the passive impedance control is used as the desired behavior. The admittance controller is again not shown as the system must be reinitialized in order to run the admittance control which changes the set point of the surface contact. We have now examined the soft contact and the free space motion of the link and seen that choosing n and K_{pa} allows for a wide range of performance form the system.

We now wish to examine contact with a stiff contact surface. The contact with the stiff contact is done the same way as in the previous section. The effector is bought into contact with the surface such that the force sensor continues to read 0 external force. Then the system is given a step into of 10 degrees into the surface and the response is measured. Figure 6.14 shown the response curves for stiff contact with $n = 1$, $\delta = 25\ ms$, and K_{pa} varying. We see from the figure that there is an increase in high frequency vibration as K_{pa} decreases. Furthermore, the high frequency vibration persists causing the system to never reach a steady state equilibrium. This behavior is also present when the passive impedance control is used as the desired dynamics. We now proceed to investigate the effect of varying n for a given K_{pa} value.

Figures 6.15 and 6.16 show the response of the hybrid controlled system for $\delta =$

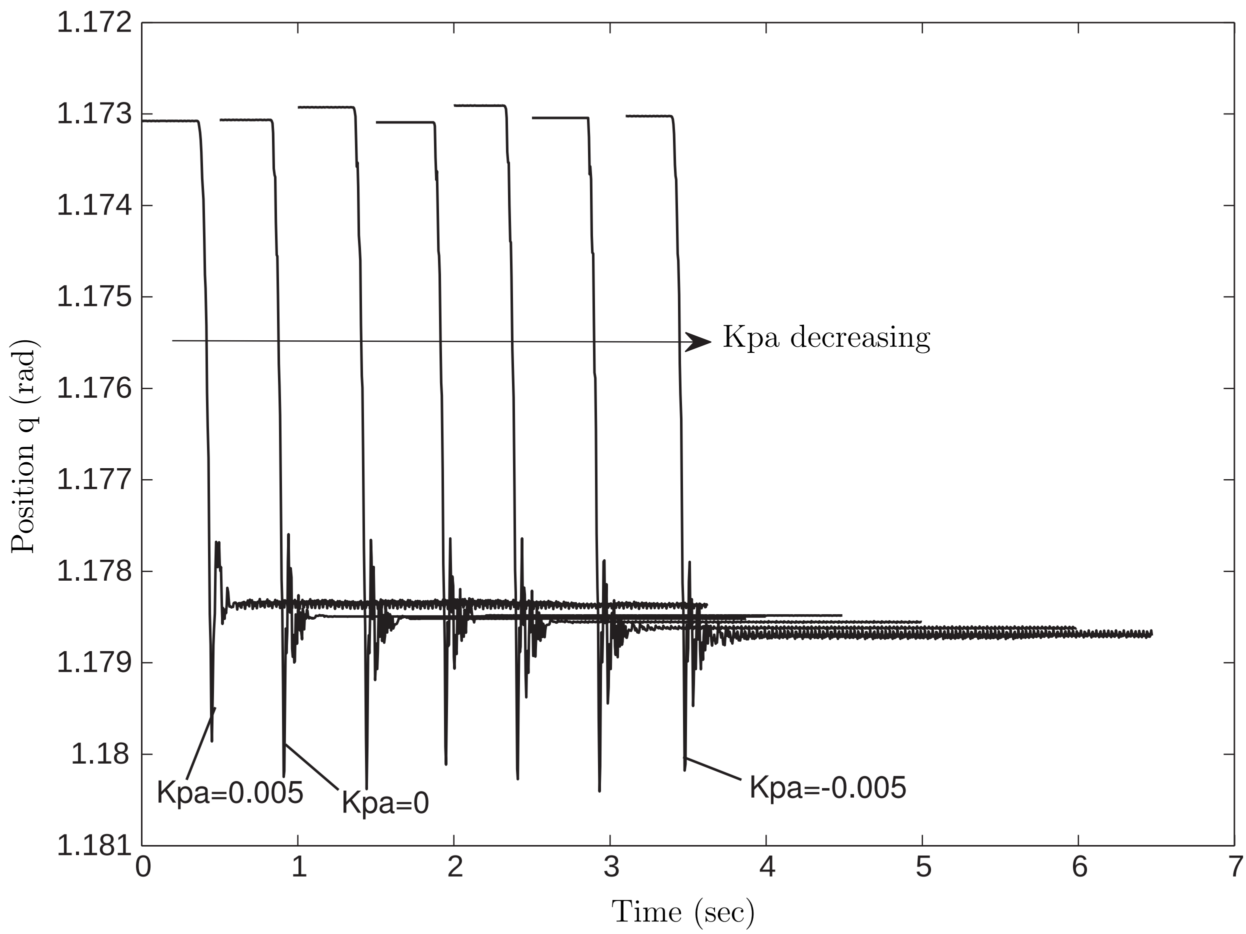

Figure 6.14: Hard contact position response for $\delta = 25\ ms$, $n = 1$, and varying K_{pa} while using a combination of passive impedance and rigid impedance as the desired behavior

25 ms, n decreasing by 0.1 with each curve from right to left, and $K_{pa} = 0.005$ and $K_{pa} = -0.005$ respectively. From figure 6.15 we see that the steady state behavior changes drastically as n changes from 0.2 to 0.1. Furthermore, we see that the high frequency behavior decreases as n decreases. However, in figure 6.16 we notice that the large change in steady state behavior occurs when n changes from 0.3 to 0.2, a larger n value than in figure 6.15. However, we also see in figure 6.16 that the higher frequency behavior of the system is of larger magnitude for larger n values. Figures 6.8

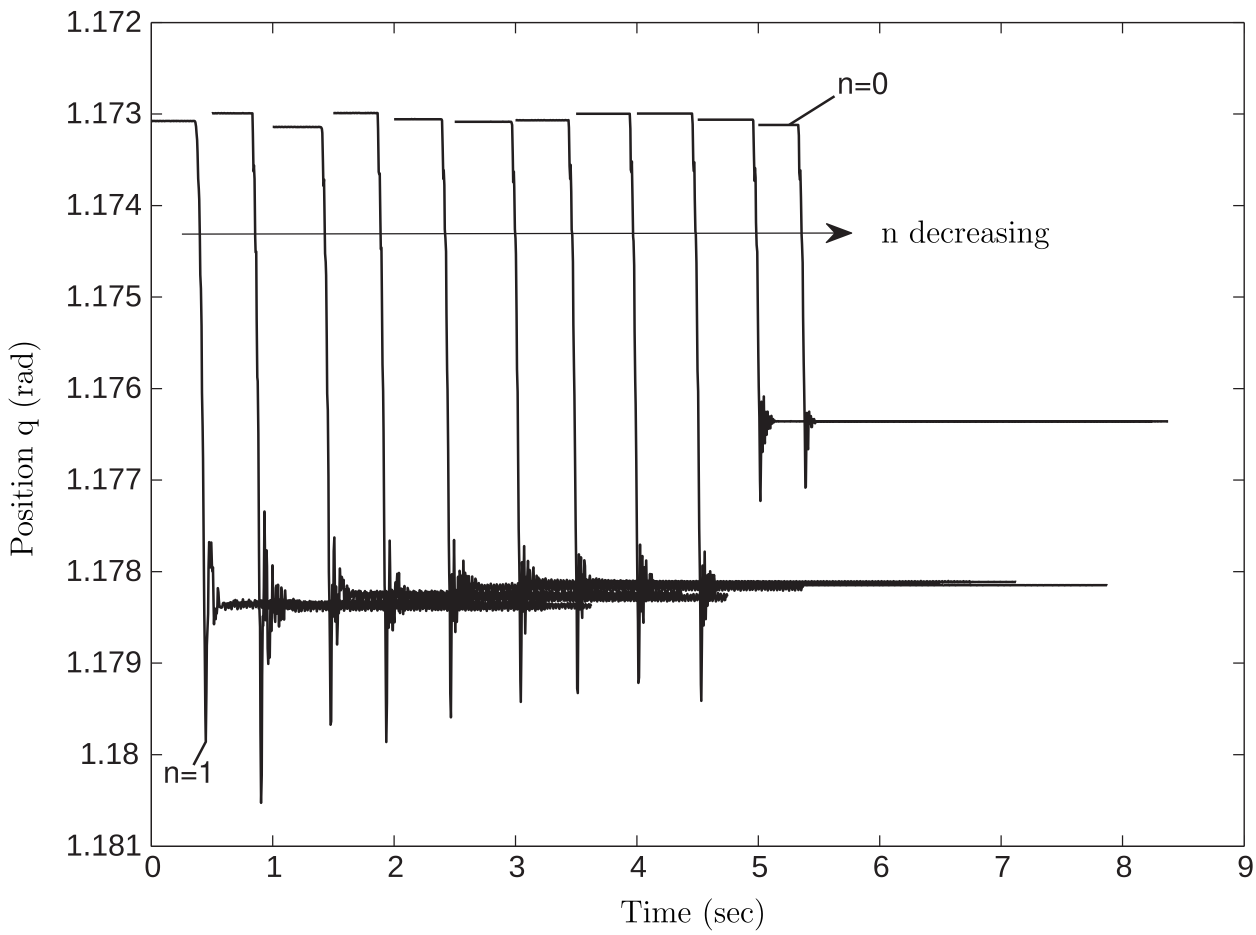

Figure 6.15: Hard contact position response for $\delta = 25\ ms$, $K_{pa} = 0.005$ and varying n while using a combination of passive impedance and rigid impedance as the desired behavior

and 6.9 which use the passive impedance control show smaller deviations in steady state behavior and smaller effects of the high frequency motion for larger n. However, we see that the high frequency motion has higher magnitude at steady state when using the passive impedance control as the desire behavior.

Therefore, we conclude that using a combination of rigid joint impedance and passive impedance control to find the desired behavior produces better results for larger n in free space than choosing the passive impedance control as the desired behavior. How-

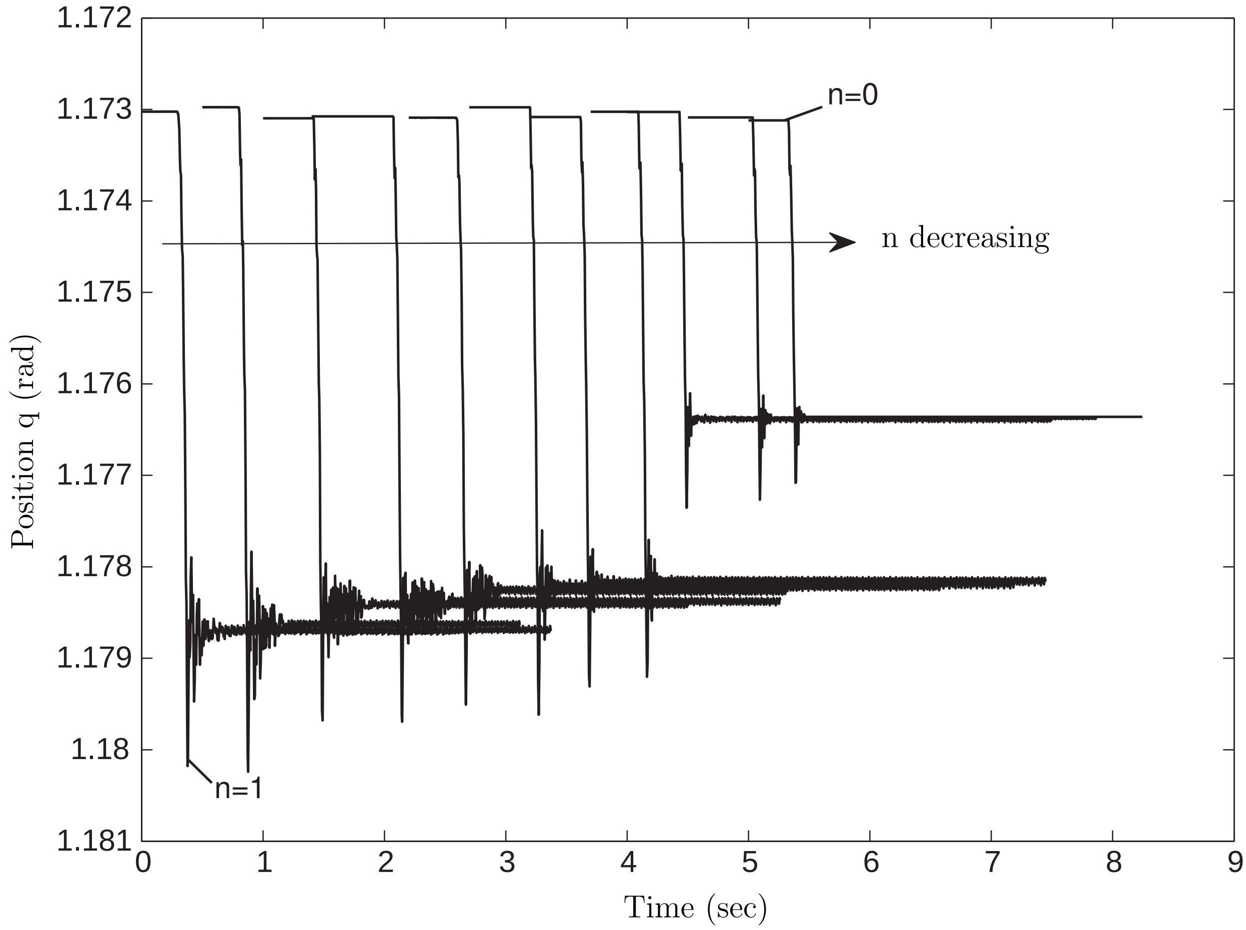

Figure 6.16: Hard contact position response for $\delta = 25\ ms$, $K_{pa} = -0.005$ and varying n while using a combination of passive impedance and rigid impedance as the desired behavior

ever, choosing the desired impedance control as the desired behavior is less sensitive to the chosen value of K_{pa} as the external stiffness increases. Even with the comparable behavior we find that we may choose a pair n and K_{pa} to produce the same or better performance of the passive impedance control for all external stiffness values, and better performance than the admittance control for stiffer external stiffness. Although we are unable to match the admittance control performance in free space due to hardware restrictions.

Chapter 7

Conclusion

We have presented a hybrid switching method as originally proposed by Ott et al. for a single degree of freedom system [19]. The method uses a duty cycle as a parameter design to interpolate performance characteristics and stability of the impedance control and the admittance control. We then expand upon the previous work by providing a new stability theorem which includes a wider range of possible systems, and produced new switching conditions to improve improve the performance of the switched system. We then generalized the theory for general system modelled by the rigid joint model, linear single degree-of-freedom system modelled with a flexible joint, and a class of non-linear single degree-of-freedom systems modelled with a flexible joint. Through simulations and experiments we find that we are able to combine robustness of the impedance control in stiff contact and accuracy of the admittance control. However, the the combination is limited by the quality of the position control and becomes numerically difficult to calculate for the flexible system as non-linearities are added. All analysis of robustness here is done based on simulation and experimentation and more analytic methods of analysis are an important goal of future work. Through experimentation and simulation we have shown that it is possible to change the performance of the controller for a given stiffness by changing the duty cycle allowing us to choose one which we believe to be best. However, since the performance of the hybrid control it based on the external stiffness and thus a method of adaptive approximation of the external stiffness is also included as a future research goal.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] A. Albu-Schaffer and G. Hirzinger. State feedback controller for flexible joint robots: A globally stable approach implemented on DLR's light-weight robots. In *Intelligent Robots and Systems, 2000.(IROS 2000). Proceedings. 2000 IEEE/RSJ International Conference on*, volume 2, pages 1087–1093. IEEE, 2000.

[2] C. An and J. Hollerbach. Dynamic stability issues in force control of manipulators. In *Robotics and Automation. Proceedings. 1987 IEEE International Conference on*, volume 4, pages 890–896. IEEE, 1987.

[3] R.J. Anderson and M.W. Spong. Hybrid impedance control of robotic manipulators. *Robotics and Automation, IEEE Journal of*, 4(5):549–556, 1988.

[4] G. Cheng, S.H. Hyon, J. Morimoto, A. Ude, G. Colvin, W. Scroggin, and S.C. Jacobsen. CB: A humanoid research platform for exploring neuroscience. In *Humanoid Robots, 2006 6th IEEE-RAS International Conference on*, pages 182–187. IEEE, 2007.

[5] T. Das and R. Mukherjee. Shared-sensing and control using reversible transducers. *Control Systems Technology, IEEE Transactions on*, 17(1):242–248, 2009.

[6] S. Eppinger and W. Seering. Understanding bandwidth limitations in robot force control. In *Robotics and Automation. Proceedings. 1987 IEEE International Conference on*, volume 4, pages 904–909. IEEE, 1987.

[7] FR Gantmacher. *The Theory of Matrices.* Chelsea, 1959.

[8] N. Hogan. Impedance control: an approach to manipulation: parts 1, 2, 3. *Journal of Dynamic Systems, Measurement and Control*, 107:1–24, 1985.

[9] R.A. Horn and C.R. Johnson. *Matrix analysis.* Cambridge university press, 1990.

[10] D.A. Lawrence. Impedance control stability properties in common implementations. In *Robotics and Automation, 1988. Proceedings., 1988 IEEE International Conference on*, pages 1185–1190. IEEE, 1988.

[11] D. Liberzon and A.S. Morse. Basic problems in stability and design of switched systems. *Control Systems Magazine, IEEE*, 19(5):59–70, 1999.

[12] GJ Liu and AA Goldenberg. Robust hybrid impedance control of robot manipulators. In *Robotics and Automation, 1991. Proceedings., 1991 IEEE International Conference on*, pages 287–292. IEEE, 1991.

[13] W.S. Lu and Q.H. Meng. Impedance control with adaptation for robotic manipulations. *Robotics and Automation, IEEE Transactions on*, 7(3):408–415, 1991.

[14] M.T. Mason. Compliance and force control for computer controlled manipulators. *Systems, Man and Cybernetics, IEEE Transactions on*, 11(6):418–432, 1981.

[15] W.S. Newman. Stability and performance limits of interaction controllers. *Journal of dynamic systems, measurement, and control*, 114:563, 1992.

[16] K. Ogata. *Discrete-time control systems*, volume 1. Prentice-Hall, 1995.

[17] C. Ott. *Cartesian impedance control of redundant and flexible-joint robots*. Springer Verlag, 2008.

[18] C. Ott, O. Eiberger, W. Friedl, B. Bauml, U. Hillenbrand, C. Borst, A. Albu-Schaffer, B. Brunner, H. Hirschmuller, S. Kielhofer, et al. A humanoid two-arm system for dexterous manipulation. In *Humanoid Robots, 2006 6th IEEE-RAS International Conference on*, pages 276–283. IEEE, 2006.

[19] C. Ott, R. Mukherjee, and Y. Nakamura. Unified Impedance and Admittance Control. In *Robotics and Automation (ICRA), 2010 IEEE International Conference on*, pages 554–561. IEEE, 2010.

[20] J.R. Partington. *Linear operators and linear systems: an analytical approach to control theory*. Cambridge University Press, 2004.

[21] P. Peleties and R. DeCarlo. Asymptotic stability of m-switched systems using Lyapunov-like functions. In *American Control Conference, 1991*, pages 1679–1684. IEEE, 1991.

[22] M. Pelletier and M. Doyon. On the implementation and performance of impedance control on position controlled robots. In *Robotics and Automation, 1994. Proceedings., 1994 IEEE International Conference on*, pages 1228–1233. IEEE, 1994.

[23] M.H. Raibert and J.J. Craig. Hybrid position/force control of manipulators. *Journal of Dynamic Systems, Measurement, and Control*, 102(127):126–133, 1981.

[24] J. Roy and L.L. Whitcomb. Adaptive force control of position/velocity controlled robots: theory and experiment. *Robotics and Automation, IEEE Transactions on*, 18(2):121–137, 2002.

[25] H. Seraji. Adaptive admittance control: An approach to explicit force control in compliant motion. In *Robotics and Automation, 1994. Proceedings., 1994 IEEE International Conference on*, pages 2705–2712. IEEE, 1994.

[26] B. Siciliano and O. Khatib. *Handbook of Robotics*. Springer-Verlag New York Inc, 2008.

[27] T. Valency and M. Zacksenhouse. Accuracy/robustness dilemma in impedance control. *Journal of dynamic systems, measurement, and control*, 125:310, 2003.