ANALYSIS OF WIND TURBINE BLADE VIBRATION AND DRIVETRAIN LOADS

By

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ABSTRACT

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The reliability of wind turbines is a major issue for the industry. Drivetrain and blade failures are common, costly and not fully understood. Designers must thus examine and understand the key parameters that influence reliability. As wind turbines increase in size, the blades are designed to be more lightweight and flexible, increasing the potential for large-displacement oscillations during operation. This necessitates the incorporation of nonlinearity in the formulation of the blade model to better understand the dynamics and stability characteristics. Also, oscillations in the blade impart dynamic loading onto the gearbox. Understanding these dynamic loads is essential for the design of reliable gears and bearings, and hence economically viable wind turbines. Traditional studies of wind turbines have focused on the aerodynamic performance of the blades, the reliability of gearbox and its components, grid connectivity and improvements in power distribution. The aspect of blade vibration from a dynamics point of view has garnered interest but not been fully developed and understood.

In this work, the partial and ordinary differential equations that govern the in-plane and outof-plane motion of a wind turbine blade subject to gravitational and aerodynamic loading are developed using Hamilton's principle and Lagrange formulations respectively. These differential equations include nonlinear terms due to nonlinear curvature and nonlinear foreshortening, as well as parametric and direct excitation at the frequency of rotation. The equations are reduced using an assumed uniform cantilevered beam mode to produce single second-order ordinary differential equations (ODE) to approximate the blade model for the case of constant rotation rate. Embedded in the ODE's are terms of a forced Mathieu equation with cubic nonlinearity. Different variations of the forced Mathieu equation are analyzed for resonances by using the method of multiple scales. The forced Mathieu equation has instabilities and resonances at multiple superharmonic and subharmonic frequencies. Second-order expansions are used to unfold the expressions that govern the amplitude of response at these critical resonances. The equations of motion (EOM's) also have regions of instability and we employ perturbation analysis to identify the stability transition curves of the system. These calculations compare well with numerical simulations for simple systems under study.

The formulation is then extended to wind turbine blades. Aerodynamic forces on the wind turbine blades are calculated using the Blade Element Momentum (BEM) theory and its extensions. From parametric studies of the blade EOM we can understand the parameter values at which superharmonic resonances become dominant. It is shown that as wind turbine blades become larger they are prone to these resonances, whose existence may not be within the scope of current design strategies. The amplitude of response at all resonances tend to become amplified for much larger blades. Both in-plane and out-of-plane responses will increase the loading at the rotor hub and consequently, increase the loads and moments on the wind turbine drivetrain.

To capture the effect of increased loading on the wind turbine drivetrain, we use a commercial software to model a 750 kW gearbox used as a part of the Gearbox Reliability Collaborative (GRC) headed by National Renewable Energy Labs (NREL). For this, we partnered with Romax Technology Ltd. to analyze the sensitivities of the load on the elements of the gearbox to variations in the input loads. Using the Romax gearbox model, we suggest methods to optimize the gear geometry to improve reliability of the drivetrain by minimizing influence of manufacturing and assembly tolerances and misalignments. We also designed novel approaches to predict gearbox vibration using the models and suggested changes that are required to improve the overall design of the gearbox (these have been implemented on newer generations of the NREL GRC gearbox).

The fundamental work formulated in this thesis can be extended to more complex models to understand other system level dynamics of interest (multi-mode interaction, multi-blade resonance, etc.). More detailed formulation of aerodynamic loads (for example by using ONERA semiempirical approach) would also improve model fidelity for predicting the influence of aerodynamic loads on blade vibration.

Copyright by VENKATANARAYANAN RAMAKRISHNAN 2017 To my parents (Ramakrishnan and Chitra) and loving sister (Lalitha).

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CHAPTER 1

INTRODUCTION

1.1 Objective

The aim of this work is to improve the understanding of wind turbine blade oscillations and how they translate to dynamic loading on the gearbox. Blade oscillations can be generated in a variety of ways. Oscillations in the blades imply dynamic loadings transferred to the gearbox. Understanding these dynamic loadings is essential to the design of reliable gears and bearings, and hence economically viable wind turbines.

In particular, this work focuses on horizontal axis wind turbines with aerodynamic and gravitational loading, and aims to

- Obtain the nonlinear dynamical equations of motion of rotating blades, including nonlinearity due to large deflections (nonlinear curvature and nonlinear foreshortening), aerodynamic and aeroelastic loading, and gravitational loading.
- 2. Reduce the partial differential equations with assumed modes. Analyze the reduced model using perturbation methods.
- 3. Understand the resonances and instabilities for the developed model. Isolate the roles of parameters and each type of loading on the dynamics.
- 4. Understand the dynamics for single blade models with in-plane deflection and out-of-plane deflection.
- 5. Interpret the resonant and unstable dynamics as input loads to the gearbox. Simulate gearbox loading with commercial gearbox software (Romax) to understand the sensitivities of bearing loads and gear loads for different operating conditions.

The approach in this thesis is to apply reduced-order modeling, and obtain analytical expressions of response characteristics as functions of parameters, so that the roles of parameters may be understood and used to guide more detailed numerical simulations. The numerous sources of external loads for an example offshore wind turbine blade are shown in figure 1.1 to showcase the complexity of these systems.

1.2 Background and Motivation

The wind industry has grown consistently at high rates every year in terms of global cumulative installed capacity (figure 1.2). This is enabled by political and public pressure behind the wind industry to boost this renewable energy sector with ambitious plans, such as 20% wind energy by 2030 in the USA [7] and similar continuously increasing targets in other global markets. As of 2013, wind energy accounts for 4% of the total energy generation mix in the United States (see figure 1.3).

Accompanying the growth in demand and installed capacity is the growth in the size of the machinery. In 2001, the global average size of installed turbines was 0.93 MW and by 2008 this metric had reached 1.53 MW [8]. Current production wind turbines are typically in the 2 - 3 MW range and active research and development is being done for offshore installation with a capacity of 5 - 13.5 MW per wind turbines. Alongside this rapidly expanding market and growing machinery is a significant issue with the reliability and maintenance of rotating machinery, and there is a focus on improving gearbox and bearing life. With the larger wind turbines, gearbox failures are too frequent.

Reliability statistics for European databases of wind turbine failures and the analysis of the Landwirtschaftskammer Schleswig-Holstein (LWK) [9] which survey up to 643 wind turbines is reproduced here from the American Wind Energy Association's annual report in figure 1.4. The statistics show the blades and drivetrain (gearbox and main shaft) have significant failure rates and also a gearbox failure yields the highest downtime. It is also one of the most expensive components to replace; with any failure comes additional costs for removal and replacement, and loss of energy



Figure 1.1: External loads on an off shore wind turbine (graphic from Jason Jonkman, NREL)



Figure 1.2: Global cumulative installed wind capacity from 1996 - 2013 [1]



Figure 1.3: Electricity generation mix during 2013 in United States [2]



Figure 1.4: Failure rate of wind turbine components and down time per failure [2]

capture. These findings have also been substantiated with data collected by other organizations and researchers (for example, [10]).

The growing market and need for improved reliability and robust designs have brought an industry-wide efforts such as the development of a new IEC standard 61400-4, and the National Renewable Energy Laboratory Gearbox Reliability Collaborative (NREL GRC) [5, 11], a large multi-year project which brings together a range of participants across the industry for analysis and testing of wind turbine gearboxes. Germanischer Lloyd, a certification body enlists a set of design specifications [12] which designers should adhere to, in order to get the turbines certified. In spite of these efforts, frequent component failures occurring in wind turbine systems well before expected lifetime of operations indicate that loadings and failure mechanisms of operating wind turbines are not yet sufficiently understood. The lack of understanding leads to potentially operating the system under extreme loading conditions that have not been accounted for. It is likely that there exist unknown dynamical loading cases.

There are a variety of dynamic loading cases. The turbine during steady operation has cyclic aeroelastic and gravitational loads. The aeroelastic loads have the capacity to cause self-excitation, much like the aeroelastic flutter observed in aircraft [13, 14]. There are sudden changes of wind

speed and direction, which lead to time varying aeroelastic blade loadings. The ratio of flutter speed to wind turbine operating speed also reduces as the blades grow in size. In [4], it is noted that as blade length increases from 5 m to 34 m the ratio drops from 6 to about 2.5. For much larger blades, there is little to no margin on flutter speed based on some preliminary calculations [6].

When winds become high, the brakes may be applied, introducing the possibility of frictional stick-slip, which has a myriad of complications in and of itself [15,16]. A yaw controller can induce yaw motions, which in turn induce inertial blade and shaft excitations. It has been noted that parked wind turbine system also exhibit unstable dynamic behavior that cause extreme loading on the blades and contribute to failure [17]. When the turbine is on idle, there can be self-induced oscillations in the standby position, which can then involve stick-slip, and clearance and impact events in the gearbox. The dynamic blades are coupled to other substructures. Coupling with the gearbox allows the gearbox modes and nonlinearities to affect the rotor and blades. Coupling between two or three blades, each of which has nearly the same modal characteristics, through the blade hub, allows for the possibility of internal resonances [18] and symmetry-breaking phenomena [19], which can lead to vibration localization [20, 21], meaning extra-large vibrations locally on the coupled structure. Finally, the total structure is also coupled with the tower, which may have nearly identical in-plane and out-of-plane modal characteristics. Any resonances or instability can amplify the loading that is transferred to the gearbox. It would be no wonder, if there are such dynamical events that are currently not catalogued, that drivetrain loads go beyond their design specifications.

The large wind-turbine industry will benefit greatly from improved understanding of the dynamical loads on the gearboxes. To better understand these dynamical loads, a careful description of the aeroelastic and gravitational loads is required, and then a careful nonlinear dynamic analysis for instabilities and resonances is needed.

In this thesis, we will focus on the role of cyclic gravitational and aerodynamic loading on the in-plane and out-of-plane vibration of an isolated blade. We also include nonlinearity associated with large deflections (nonlinear curvature and nonlinear foreshortening). Very large blades exhibit large deflections under static loads and dynamic loads (see figure 1.5). Dynamic deflections have

the capacity to be even larger. The blades are more flexible in the out-of-plane direction with deflections up to 10 m during normal operation.

Accounting for nonlinearity will enable us to accurately capture the motion of the blade during operation under specified loads and will also expose the presence of resonances other than one-to-one resonances of the structural modes. Combining nonlinearity and parametric excitation terms leads to subharmonic and superharmonic resonances. That is, such conditions can easily lead to resonances when the excitation frequency is twice or half (or some multiple or fraction) of a modal frequency. We first analyze the loadings on a single rotating blade.

Aeroelastic blade loadings are known to induce self-excited blade oscillations [22–26], which tend to involve the coupling of particular linear modes, and get particularly more serious with offshore turbines [27]. In horizontal axis wind turbines, aeroelastic loadings can be cyclic, thereby producing forced oscillations of the blades [28]. Sources of cyclic variation are the wind speed gradient, or wind shear, such that the blade loading varies with elevation, as well as tower pass-by. Furthermore, the blades endure cyclic gravitational loadings which also lead to forced oscillations. These cyclic and aeroelastic loadings can induce vibrations and instabilities. We aim to understand these dynamic responses.

There is a large body of work on modeling of rotating blades under aerodynamic and gravitational loading. Wendell [29] developed the partial differential equations of motion for a rotating wind turbine blade. Additional efforts have been made by to include gravity and pitch action by Kallosóe [30]. Caruntu [31] has developed nonlinear equations that model the flexural potential in non-uniform beam that can be extended to large flexible blades of wind turbines. Chopra et al. [32] have modeled the blades as a hinged structure and constructed equations of motion to study its dynamic behavior. Jonkman [33, 34] used the modal information of blade vibration to calculate the tip displacement in blade, the torques and moments experienced at the hub amongst other things. These calculations have been used to develop the code FAST that is used in research laboratories, industry and universities around the world as a basis for performance evaluations. Numerous commercial and research codes are also available that enable us to understand the behavior





Figure 1.5: Large static and dynamic deflections of wind turbine blade (Images taken at NREL)

of wind turbine blades and the influence of external loads on their responses. Recent academic thesis [3, 35, 36] have addressed improving aerodynamic and aeroelastic modeling of wind turbine blade in order to improve formulations of the loading of wind turbine systems.

Extensive literature also exists on analysis related to the problem of vibration of helicopter blades [37–40]. Hodges and Dowell [41] also developed the nonlinear partial differential equations for a twisted helicopter rotor blade. These rotate at much higher rotational velocities than wind turbine blades, and in a horizontal plane, unlike the wind turbine blades. Hence the gravitational influence on the vibration characteristics of the wind turbine blade would be much different. However, many features of helicopter blade dynamics, such as modeling of centrifugal effects and deflected geometry, carry over to the wind turbine blade models.

1.3 Wind Turbine Blade Modeling

Wind turbine blade modeling can be broadly classified into two categories: aerodynamic modeling and aeroelastic modeling.

1.3.1 Aerodynamic Modeling

Theodorsen's work [42] on the general theory of aerodynamic instability and mechanism of flutter laid the foundations for significant follow up work in the area. The aerodynamic forced on an oscillating airfoil with three degrees of freedom were determined. Numerous simplification and alterations of these models have been used in commercial codes to predict the loads on an airfoil structure. There are numerous other simpler aerodynamic models and a few that increase accuracy by more detailed formulations. Some of the various types of aerodynamic modeling are listed below:

- 1. Actuator disc model (momentum theory): The simplest model for wind turbine aerodynamics that provides useful results and insights. In this, we consider a rotating annular stream tube and apply conservation of momentum and Bernoulli's equation to get axial and tangential forces on annular element of fluid.
- 2. Glauert annulus momentum vortex theory: The next level of complexity is the annulus momentum vortex theory. Each radial section of the blade is analyzed independently using a 2-D airfoil data and equations are based on continuity and conservation of momentum. Corrections based on the work of Prandtl and Goldstein have been satisfactory for extending this theory to finite number of blades (average effects). This theory is most useful for basic rotor design.
- 3. Prescribed wake vortex theory: The basic concept is that the vortices that form a spiral trail behind the rotor-blade tips define the flow field around the rotor according to the Biot-Savart theorem. Each section of the rotor blade generates a lift force that is proportional to its local bound vorticity. By integrating the effects of trailing vortices over the blade, the induced flow and hence the rotor forces and moments can be determined.
- 4. Free wake vortex theory: The assumption of rigid vortex theory can be eliminated and one can attempt to find the exact path of the trailing vortices iteratively.

The blade element momentum (BEM) theory is a combination of the actuator disc model (momentum theory) and the annulus momentum vortex theory (blade element theory). It is a computationally cheaper method based on the 2D airfoil characteristics (lift and drag). BEM theory is most commonly used in wind turbine dynamic simulations due to its computational efficiency and overall accuracy. Limited success has been made with computational flow solvers based on Reynolds Averaged Navier Stokes (RANS) and other similar three-dimensional models. This is primarily due to the complexity of modeling wind turbines. Wind turbine aerodynamics are dependent on far field conditions, several rotor diameters up and down stream, while at the same time being dependent on small scale flow conditions at the blade. Coupled with body motion, the need to have fine resolution and a large domain makes these models highly computationally intensive.

BEM theory equates two methods of examining how a wind turbine operates.

- 1. The first method is to use a momentum balance on a rotating annular stream tube passing through a turbine.
- 2. The second method is to examine forces generated by the aerofoil lift and drag co-efficient at various sections along the blade (in an undeformed state) and these forces are integrated over the total length of the blade to give the overall aerodynamic loads.

BEM theory is used to develop a blade design tool by considering power requirements [43]. The rotor torque applied by the lift and drag forces (calculated using BEM) are determined and then this torque is equated to the torque applied by rotating annulus (using Bernoulli's equation) produced by wind field. No aerodynamic interactions between different blade elements are considered and the blades are assumed to be rigid.

In [44–46], BEM theory is used to optimize the blade shape and pitch control strategy. A cost function is formulated based on the aerodynamic performance of the blade (power obtained from lift and drag forces), weight of the blade and its rigidity. Aerodynamic properties are calculated using BEM and blade rigidity is determined using a simple beam theory. A simplified pitch control

system has been formulated to relate pitch angle with power. This cost function is then used to optimize the blade parameters like chord length, section area and twist.

In [47], BEM theory is used in integration with a Finite Element (FE) approach. The system model is constructed by combining an aerodynamic model (obtained using BEM) and a FE blade model. Aerodynamic model consists of a Bézier surface to calculate the lift, drag and moments caused by aerodynamics for different wind velocities and attack angles. In the model, the wind velocity is assumed to be normal to the plane of rotation and steady state flow is considered. Variation of the gust excitation is modeled by varying the angle of attack. Change in aerodynamic forces due to dynamic stall (dynamic stall is the unsteady aerodynamic effect caused by the strong vortex formation that occurs when airfoil rapidly changes the angle of attack) is neglected. The FE model of the blade consists of several sections and each section is made up of 3D elastic beam elements. Connection between the blade and hub is rigid and the twist angle variation of the blade is neglected. The blade section is assumed to be piece-wise variable along its length.

BEM theories generally give acceptable approximations to aerodynamic loads under the condition where wind is normal to the plane of rotor (i.e. the turbine is un-yawed with respect to the wind). But under yawed conditions (when wind is at certain angle to the plane of rotation) the BEM theory is less able to model the physics due to the 3D nature of the flow.

Since the basis of the BEM theory is strictly a 2D analysis, 3D effects such as physical roll-off in the lift as the blade tip is approached (tip loss) cannot be modeled analytically but can be treated by using some corrections such as Prandtl's tip loss function. Several advanced aerodynamic models are described in [48] which take into account the 3D wind flow properties.

The model described in [48] takes into account the vertical wake system produced behind the rotor. This wake is comprised of vortices that trail from each of the blade tip. Though various modifications to the BEM approaches have been used to represent these wake effects, they have not been satisfactory in determining the blade loads. An improved version models these wakes by considering the aerodynamic lag of the inflow development over the rotor disk in response to changes in blade pitch or rotor thrust. The mathematics is described in terms of ordinary differential

equations and a time constant representing the dynamic lag in the build-up of the inflow. Since the aerodynamic equations are described in terms of ordinary differential equations, it can be easily coupled with the aeroelastic models and the combined system can be solved simultaneously.

Another model described in [48] takes into account the unsteady aerodynamics caused by the blade section (stall effect) and tower shadow effect (the distortion in the wind flow when the blade passes through the tower). The model used is semi-empirical but is represented in terms of state-space formulation and hence is easy to implement. Other advanced aerodynamic models are based on solution of full field Navier-Stokes equation. But these methods are computationally very expensive and are generally implemented in CFD codes.

Our work focuses on modeling the blade vibration and related dynamic instabilities. We seek to use a simpliefied model to account for the effects of wind on the blade loading. BEM theory, in spite of its limitations provides a valid model for blade loading due to wind flow. We incorporate wind shear in our calculation of blade load as it induces a cyclic variation for every revolution.

1.3.2 Aeroelastic Modeling

Simple beam models (Euler-Bernoulli or Timoshenko) are used to model the blade sections in [43, 46, 47]. These beam models take into account the blade flexibility. The degrees of freedom involved are less, in order to make these models computationally cheaper. Typically, beam models are all based on symmetric cross-sections and hence they neglect the effect of cross-sectional properties like shear center, tension center, etc. Another draw back of these beam elements is the assumption that an initially plain cross-section remains plain under deformation. This assumption does not hold for wind turbine blades but nonetheless provide a basic foundation for studying the dynamics of wind turbine blades.

In [30], as noted before, an advanced beam model is developed by using the principle of virtual work (taking into account the potential and kinetic energy of the element). The equations of motion are derived in terms of partial differential equations (using a Lagrangian formulation), relatively small terms are neglected in a consistent manner by introducing an ordering scheme. The model

takes into account the effect of sectional properties like shear center and gravity center on the blade motion. The effects of gravitational forces and Coriolis forces are also considered. The model has full coupling between the bending and torsion modes of the blade. A pitch model is also developed to determine the pitching torque for the given value of pitch angle and consequently the effect of pitching action on the blade dynamics is also studied. An eigenvalue problem is solved using the finite difference discretization of the equations of motion to get the natural frequencies and mode shapes. However, the influence of yaw motion or tower flexibility on the blade dynamics is not considered, and the blade is assumed to be isotropic and inextensible.

Other advanced rod models are formulated [49], that takes into account the warping effect, blade extension and deformation of the blade section due to tilt. The equations of motion are derived in terms of partial differential equations using the principle of virtual work. These equations are also coupled with a warping function that takes into account the out-of-plane deformation of the cross section. A strain-displacement relationship is also formulated that takes into account the distortion of the blade section due to in-plane motion. Instead of using one element for a cross section; the cross section is defined using several quadrilateral elements. Due to the higher number of degrees of freedom, reduction methods are used to reduce the size of the model. The model is then validated with the experimental results. In this work, the blade model is developed only to validate/justify the improved rod element for WT applications, so the blade is assumed to be stationary (like a test setup); therefore the effect of centrifugal forces and gyroscopic terms is neglected.

Full finite element blade models are used to investigate the modal response in [3]. But these models are computationally very expensive. Modal analysis can be performed using these models but it is very difficult to use them for full time domain dynamic analysis.

Other efforts have been made to do experimental modal analysis to determine the natural frequencies, damping and mode shapes for blades [50]. The test setup is capable of testing full sized wind turbine blades (up to 20 m in length). The blade is excited by a hammer and the frequencies and mode shapes associated with first and second flap mode, first edge mode and first torsional mode are calculated. Similar efforts have been made at other national labs across the

globe, however, with the increasing dimensions of the wind turbine blades, these methods are restricted for use on small blades.

However, it should be noted that as wind turbine increase in size, their design will be stabilitydriven in contrast to current load driven design and analysis. Large rotating structures can have harmonic resonances and instabilities in their operating range. Operating the wind turbine under those parameters cannot be avoided, unless the design process accounts for existence of such resonances and instabilities. In fact, parked (idle) wind turbine systems also have instabilities associated with the edgewise motion of the blade and the side to side motion of the tower [17]. Hence a good aeroelastic code is essential to capture the inherent structural response properties of wind turbine blades. We develop the structural model in this thesis, by defining the motion of a single blade around the rotor at a constant speed. The variations in geometry along the blade are accounted for. Large deflections induce nonlinear phenomena, however, the associated details on the aerodynamic loads is less comprehensive, but, sufficient to capture the necessary dynamics.

1.3.3 Overview of Design Codes

Wind turbines systems are designed and analyzed using simulation tools that capable of predicting the coupled dynamic response of a system under various operating conditions. Onshore wind turbine analysis relies on the use of aero-servo-elastic codes, which incorporate wind inflow, aerodynamic, control system (servo) and structural-dynamic (elastic) models in time domain in a coupled simulation environment [33]. In recent years, a number of these codes have been expanded to include the additional dynamics pertinent to offshore support structures. Some standard design codes/softwares that are currently in use in industries that have been developed by NREL to calculate loads and forces in wind turbine structures are listed below:

• AirfoilPrep

- Developed by Windward Engineering and NREL
- Generates airfoil data files from 2D data (experimental)

- Adjusts 2D data for rotational augmentation (3D effects) and computes dynamic stall parameters
- WT Perf
 - Developed by NREL (initially developed at OSU)
 - Calculates performance of rotor geometry and airfoil data
 - Steady-state calculations (no dynamics), computes power, torque, thrust and blade-root bending moment
 - Uses BEM theory
- PreComp
 - Developed by NREL
 - Computes coupled section properties of composite blades for beam-type models
 - Inputs are blade external shape and internal lay up of composite laminas
 - Section properties like direct and cross stiffnesses, inertias, locations of shear, tension and mass centers are found directly
 - Uses modified combined laminate theory with shear flow approach
 - Underlying assumptions: straight blade, thin walled closed sections, no transverse shearing, no in-plane distortion, no warping
- AeroDyn
 - Developed by Windward Engineering and NREL
 - Computes aerodynamic loads as part of aero-elastic solution (fully coupled)
 - Uses BEM and Dynamic Wake Algorithms; hub loss, tip loss and skewed wake corrections in BEM
 - Capable of simulating turbulent wind inputs

Apart from the methods and codes listed above, there are numerous other codes that have been developed by national labs and universities for aerodynamic and aeroelastic modeling. FLEX5 by Technical University of Denmark, HAWC2 by Risø National Laboratory, DUWECS by Delft University of Technology, Alcyone and GAST by National Technical University of Athens, PHATAS by Netherlands Energy Research Foundation, to name a few have been developed in an effort to understand various aspects of wind turbine loads. Commercial code such as ADAMS/WT developed by MSC Software and GH Bladed by Garrad Hassan have also been extensively used in the industry for predicting loads experienced by the blades during operation.

One limitation of these linear frequency and time domain analyses is that they cannot capture the non-linear dynamic characteristics and transient events that are important considerations in wind turbine analysis. Each code focuses on different aspects of the model and adopts different techniques to assess loading. These can be broadly classified into three main categories: multi body simulation using rigid elements (for e.g. ADAMS/WT), finite element modeling of the blade structure (for e.g. HAWC2) and an assumed cantilever beam mode approach (for e.g. FAST). However, in our survey of these codes, most of the codes assume that the deflection of the wind turbine blades are small and consequently, linear. FAST applies foreshortening to the blade deflection, however, linearity is still assumed. This assumption becomes increasingly invalid for large deflections of wind turbine blades. Also, aerodynamic loads are applied to the static blade shape. This is a pretty valid assumption for most cases, however, literature exists that account for aerodynamic loading with deformed blade elements. The loads computed by the BEM theory suggested in this thesis should be extended for local deflections of the blade element. Aerodynamic modeling using the ONERA semi-empirical approach for dynamic stall provides a better framework for computing lift and drag forces that are dependent on the instantaneous velocity of the blade accounting for motion due to blade vibration. In [51], Bierbooms compares two semi-empirical approaches and mentions that the unsteady aerodynamic effects captured by the ONERA approach may have important influence on the fatigue life of the wind turbine and its stability.

Another aspect of code development for wind turbine loads analysis has been that blade modeling

and flow modeling theories have been developed typically in isolation. No model implements the advanced aerodynamic models in conjunction with the advanced blade models. Some investigation is also required on the effectiveness of the existing theories to model the complex non-linear phenomenon like warping, tower shadow and dynamic stall accurately. There is very little work done on the influence of tower flexibility and yawing motion on the blade dynamics, although some very simple models do exists that attempt the capture these interactions.

The need to improve the modeling mechanisms of blade and wind turbine systems has generated a lot of interest in the academic community and consequently various researchers have developed mathematical models to best represent the wind turbine blades and drivetrains. Sant [36] explored improving the BEM-based aerodynamic models by including the free wake vortex formulations. Weinzierl [35] also developed a BEM-based simulation tool that incorporates active flow control elements (for e.g., pitch control, trailing edge flaps, etc.) in the model. Larsen [52] proposed a dynamic stall model for capturing the forces on the wind turbine air foil. Hansen and Butterfield [53] in their review paper make note of the improved understanding of dynamics loads for HAWT's whose structural dynamics and airfoil characteristics are well understood. However, the prediction of loads under off-design (say superharmonic resonant loading) conditions still remains a formidable challenge. Blade motion under resonance has also been observed in simulations and experiments of out-of-plane motion in work done by Ishida *et al.* [54, 55].

In this thesis, we model the blade structure with large nonlinear deflections through assumed mode approximations of a beam and apply aerodynamic loads using the BEM theory. This allows for assessing dynamic response phenomenon that have not been captured by simple linear representations of blade motion and accounts for aerodynamic loading.

1.4 Beam Vibrations and the Mathieu Equation

Modeling and analysis of the dynamics of beam is one of the cornerstones towards improved understanding of the behavior of various systems ranging from the stability of bridges and buildings to nematode locomotion. Mathematical expression that detail the static and dynamic response of the beam under various loading conditions enables us to explain the fundamental phenomenon that govern the response of the more complex structures they represent under similar loading conditions.

In the presence of nonlinearities, the beam response exhibits numerous phenomenon not observed for the linear system. Harmonic resonances, hysteresis, chaos and numerous other phenomena can be characterized by the nonlinear equations of motion that are used to represent the motion of a beam under excitation. Nonlinearities occur due to numerous factors and as such in reality all physical system exhibit nonlinear behavior. Typical nonlinearities considered for modeling the vibration of beams include nonlinearity arising due to inertia, geometry, large deflection and damping [56]. The manifestation of nonlinearity in the more complex systems that the beam represent can be due to other factors, for e.g., in gear tooth modeling, backlash and deformations in the root induce nonlinearity which can be captured via a beam model. In the formulations of nonlinear analysis of beam motion, it is typical to consider weak nonlinearity, and use book keeping parameters to quantify relative magnitude of the terms. This help us arrive at an approximate analytical solution to the differential equations, thereby, enabling us to understand the influence of terms that govern the dynamics of the system.

There have been numerous studies on different formulations for modeling blade vibration. Yao [?], Sabuncu [57], Venkatesan [58], Yoo [59] and Al-Nassar [60] have analyzed the dynamics of rotating blades with some additional complexity viz., nonlinear flapping motion, pre-twisted blades, varying rotational speed, stability due to torsional excitation, etc. These studies have been motivated by the fundamental dynamics that govern the system under different conditions. The blade models with nonlinearity and excitation are represented by the Duffing oscillator or the nonlinear Mathieu equation with parametric excitation. Both these are fundamentally well understood system, but, ongoing research in this area still actively explores these systems and their characteristic responses.

In this thesis, we use the beam to model the motion of the wind turbine blade. Under steady operation and wind conditions, in which the blade rotates at a constant rate Ω , a blade takes on cyclic transverse loading due to both gravity and aerodynamic forces with effects of wind shear and tower

passing. The blade also endures gravitational axial compression (softening) when pointing up, and axial tensioning (stiffening) when pointing down. Considering these effects on a single-degree-of-freedom description of transverse deflection motivates us to study a forced Mathieu equation with cubic nonlinearity. Also, the terms intrinsic to a forced nonlinear Mathieu is embedded in the in-plane and out-of-plane EOM developed for the wind turbine blade.

There have been extensive studies on systems with parametric excitation that fit in into a minor variation of the Mathieu equation. Rhoads and Shaw [61] have studied MEMS structures with parametric amplification, in which direct excitation occurs at half the frequency of excitation of the parametric excitation. Parametric amplification was also demonstrated in experiments by Miller [62]. Other work has examined nonlinear variations of the Mathieu equation, including Van der Pol, Rayleigh oscillators and the Duffing nonlinear terms. Rand *et al.* [63–66] analyzed the dynamics and bifurcations of a forced Mathieu equation and properties of superharmonic 2:1 and 4:1 resonances. Belhaq [67] studied quasi-periodicity in systems with parametric and external excitation. Veerman [68] analyzed the dynamic response of the Van der Pol - Mathieu equation. Arrowsmith [69] and Marathe [70] studied the stability regions for the Mathieu equation. Reference [71] is an extensive compilation of the various fields of study in which the characteristics of the Mathieu equation are found and employed.

In studying the Mathieu, we omit effects of other terms specific to the rotating beam model, as our attention is drawn to this more fundamental equation in dynamical systems. Indeed, full partial differential equations (PDEs) of the rotating beam can be analyzed and modally reduced. The analysis of the full PDE is pursued through simulations to demonstrate that the underlying dynamic phenomenon is that of the forced nonlinear Mathieu equation.

Analytical solutions of the Mathieu equation is performed using perturbation techniques, such as the method of multiple scales (MMS) [72–75]. We analyze resonances of the forced nonlinear Mathieu equation, as a phenomenological representative of blade motion. The analysis reveals the existence of subharmonic and superharmonic resonances. We unfold the super- and sub-harmonic resonance cases present in the system and identify those that may be critical to wind turbines during

normal operation.

1.5 Wind Turbine Drivetrain

The rotor hub is the mechanism that is used to attach the wind turbine blades to the drivetrain. The wind turbine drive train is a huge single speed gearbox with speed multiplication in the range of 100:1. Aerodynamic forces from the wind are translated to rotational forces which amount to torque at the generator. This is the 'useful' work done by the wind turbine system. All the other loads and moments (other than the rotational torque) are transferred to the drivetrain and tower.

Typical wind turbine design codes develop the structural model with about 16 to 24 degrees of freedom [76]. Most of the dof's are associated with the bending and torsional modes of the blades and tower and the entire drivetrain is represented by 1 dof. This simplification implies that the individual elements that make up the drivetrain do not contribute to the overall dynamics of the system under study. This is however not true and only proceeds to mask our understanding of the interaction between the dynamics of the blade and the components in the gearbox. The coupling between the external low frequency excitations from the blade dynamics with the internal high frequency dynamics of the gearbox (such as gear mesh and bearing frequency) could produce undesirable loads. Also, as wind turbine blades and gearboxes increase in size the frequency gap between the internal resonances in the gearbox and external system level resonance decreases, necessitating a more detailed formulation of gearbox and drivetrain in the modeling environment for wind turbine systems. There has been extensive efforts by National Renewable Energy Labs and Romax Technology through the Gearbox Reliability Collaborative to improve industries understanding of the complexity of the loading conditions in the gearbox. This effort treats loading from blades as external. Our focus is to integrate the blade and gearbox simulations as a path way for future studies that will recognize the need for detailed system level models for design and analysis of large wind turbine systems. Some preliminary modeling efforts performed as a part of this thesis include assessment of gearbox design. In particular, detailed analysis has been performed to assess the impact of bearing clearances on gear and bearing misalignments and stresses. A simulation toolset is also used to predict the vibration of the gearbox housing under various operating conditions. Analysis from the models developed for the studies was used to perform a comprehensive assessment of factors that influence the reliability of a wind turbine gearbox. Some critical parameters include the choice of bearings, manufacturing fits, and other detailed geometry modification of gearbox internals. RomaxWIND provides solutions during steady state operation of wind turbine to provide damage and life assessment of the drivetrain. We also explored quasi-transient approaches to capture the effects of dynamic loads. However, a simple torsional model was also developed to perform some estimates of cyclic variations of torque and speed under various operating conditions. A summary of this model is also presented.

CHAPTER 2

EQUATIONS OF MOTION FOR BLADE VIBRATION

2.1 Formulation of In-Plane EOM

We formulate the equations of motion of in-plane deflection of a beam attached to a hub which rotates at a constant rate. The equation of motion of the rotating beam is obtained by applying the extended Hamilton's principle [72]. The energy formulation for an inextensible beam with large deflections, rotating about a horizontal axis with flexure is discussed in this section.

The gravitational potential energy of the rotating beam is integrated over the beam elements with the help of Figure 2.1. The height h(x,t) of an element dx, located at a point x of the beam, rotated by an angle ϕ , is given by the angular inclination $x(1 - \cos \phi)$ plus the inclined foreshortening s(y, x, t), plus the inclined transverse beam displacement, y(x, t), such that $h(x, t) = x(1-\cos \phi)+s(x,t)\cos \phi+y(x,t)\sin \phi$, where $s(y, x, t) = \int_0^x (\frac{y'^2}{2} + \frac{y'^4}{8})dx$ and the primes indicate partial derivatives with respect to x. Then the gravitational potential energy is

$$V_g = \int_0^L m(x)gh(x,t)dx = \int_0^L m(x)g[x(1-\cos\phi) + s(x,t)\cos\phi + y(x,t)\sin\phi]dx$$
(2.1)

where *L* is the length and m(x) is the mass per unit length.

If the beam is rotating at a fixed angular speed Ω , then $\phi = \Omega t$. In a wind turbine, the speed will naturally be a slowly changing dynamic variable, effected by the aerodynamic forces on the blades, and the moment on the hub including effects such as generator dynamics, gearbox dynamics, friction and control. In our resonance analyses, we restrict motion to a fixed angular speed. Any variation about the mean speed can be dealt with as an extension of this analysis.

The flexural potential energy will include geometric nonlinearity. The nonlinear curvature is given as $k = \frac{y''}{(1+y'^2)^{3/2}}$. The square of the curvature is expanded into the form $k^2 \approx y''^2(1-3y'^2)$, and inserted into the bending potential energy as


Figure 2.1: The arbitrary configuration of the position of a point P on the rotated and deflected beam. The point P corresponds to location x on the undeflected beam.

$$V_b = \int_0^L \frac{1}{2} EI(x) k^2 dx \approx \int_0^L \frac{1}{2} EI(x) {y''}^2 (1 - 3{y'}^2) dx,$$
 (2.2)

where E is the Young's modulus and I is the area moment of inertia of a cross-section of the beam.

The kinetic energy of the beam is formulated from the velocity of each element. The position of an element is $\mathbf{r} = (x - s(x, t))\mathbf{e}_r + y(x, t)\mathbf{e}_{\phi}$ in terms of the unit vectors in the direction of the undeformed, rotated beam (\mathbf{e}_r) and transverse in the ϕ direction (\mathbf{e}_{ϕ}) . Using $\dot{\mathbf{e}}_r = \dot{\phi}\mathbf{e}_{\phi}$ and $\dot{\mathbf{e}}_{\phi} = -\dot{\phi}\mathbf{e}_r$, and considering the case where $\dot{\phi} = \Omega$, the velocity is $\mathbf{v} = (-\dot{s} - \Omega y)\mathbf{e}_r + [\Omega(x-s) + \dot{y}]\mathbf{e}_{\phi}$. The contribution to the integral of the rotational energy density, $\frac{1}{2}J(x)(y' + \dot{\phi})^2$, where J(x) is the mass moment of inertia per unit length, can also be added. Then

$$T = \int_0^L \frac{1}{2} m(x) \mathbf{v} \cdot \mathbf{v} dx + \int_0^L \frac{1}{2} J(x) (y' + \dot{\phi})^2 dx$$
(2.3)

is the total kinetic energy.

The extended Hamilton's principle is now applied, such that

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W) dt = 0$$
(2.4)

under the constraint that the varied path coincides with the true path at $t = t_1$ and $t = t_2$. Constructing δT , δV , and δW , and integrating by parts to obtain a common δy term in the integrands, yields the partial differential equation and boundary conditions.

We sketch the treatment of one of these terms, $\int_{t_1}^{t_2} \delta V_g dt = \int_{t_1}^{t_2} \int_0^L m(x)g\cos\phi(\delta s)dxdt$, where $\delta s = \int_0^x (y' + {y'}^3/2)\delta y' dz$, written then as

$$\int_{t_1}^{t_2} \int_0^L m(x)g\cos\phi\left(\int_0^x (y'+{y'}^3/2)\delta y'dz\right)dxdt$$

On the nested integral, we integrate by parts with $u = (y' + {y'}^3/2)$ and $dv = \delta y' dz$, to produce

$$\int_{t_1}^{t_2} \int_0^L m(x)g\cos\phi\left((y'+{y'}^3/2)\delta y\Big|_0^x - \int_0^x (y'+{y'}^3/2)'\delta ydz\right)dxdt,$$

expanded as,

$$\int_{t_1}^{t_2} \left[\int_0^L m(x)g\cos\phi \left((y'+{y'}^3/2)\delta y \Big|_0^x \right) dx - \int_0^L m(x)g\cos\phi \left(\int_0^x (y'+{y'}^3/2)'\delta y dz \right) dx \right] dt.$$

The first term endpoint evaluations lead to an *x*-dependent term in the first integral, and an x = 0 evaluated term that can be separated out while the integration variable is renamed *z*. Integrating by parts on the second integral, the nested integral is *u*, and the rest is *dv*. Focusing in the *dx* integral inside the *dt* integral, we have

$$\begin{split} &\int_{0}^{L} m(x)g\cos\phi(y'+{y'}^{3}/2)\delta y dx - \left(\int_{0}^{L} m(z)g\cos\phi dz\right) \left[(y'+{y'}^{3}/2)\delta y\right] \\ &- \left(\int_{0}^{x} m(z)g\cos\phi dz\right) \left(\int_{0}^{x} (y'+{y'}^{3}/2)'\delta y dz\right) \Big|_{0}^{L} \\ &+ \int_{0}^{L} \left(\int_{0}^{x} m(z)g\cos\phi dz\right) \left(y'+{y'}^{3}/2\right)'\delta y dx. \end{split}$$

Evaluating the third term, the contribution is zero when x = 0. Therefore, then renaming one of the integration variables from *z* to *x*,

$$\begin{split} &\int_{0}^{L} m(x)g\cos\phi(y'+{y'}^{3}/2)\delta ydx - \left(\int_{0}^{L} m(z)g\cos\phi dz\right) [(y'+{y'}^{3}/2)\delta y] \\ &- \left(\int_{0}^{L} m(z)g\cos\phi dz\right) \left(\int_{0}^{L} (y'+{y'}^{3}/2)'\delta ydx\right) \\ &+ \int_{0}^{L} \left(\int_{0}^{x} m(z)g\cos\phi dz\right) (y'+{y'}^{3}/2)'\delta ydx. \end{split}$$

The third term has the form

$$-\int_0^L \left(\int_0^L m(z)g\cos\phi dz\right)(y'+{y'}^3/2)'\delta y dx$$

and therefore can be combined with the fourth term. As such, our integral is

$$\int_{0}^{L} m(x)g\cos\phi(y'+{y'}^{3}/2)\delta ydx - \left(\int_{0}^{L} m(z)g\cos\phi dz\right)[(y'+{y'}^{3}/2)\delta y] - \int_{0}^{L} \left(\int_{x}^{L} m(z)g\cos\phi dz\right)(y'+{y'}^{3}/2)'\delta ydx.$$

In summary, the term $\int_{t_1}^{t_2} \int_0^L m(x)g\cos\phi\delta s dx dt$ has been reworked such that the integral on the *x* variable, expressed as the above integral, contains either integral terms with a δy term or boundary terms (in this case at x = 0).

A similar analysis on all of the integrals in $\delta H = 0$ put the grand integral in the form such that inside the *t* integral, there are *x* integrals from 0 to *L* with a common δy , and then boundary terms evaluated at either x = 0 or x = L. Since δy is arbitrary, its coefficient (an integrand) must be zero, leading to a partial differential equation of motion. Similarly, at the x = L boundary, δy and $\delta y'$ are considered arbitrary, and so their coefficients are both set to zero, producing natural boundary conditions at x = L.

We assume the x = 0 boundary to be clamped, and thus impose y(0, t) = y'(0, t) = 0 as geometric boundary conditions.

To this end, we obtain an integral-partial differential equation (IPDE) of motion and boundary conditions. For the case when $\phi = \Omega t$,

$$-m(a(\ddot{s}, s, \dot{y}, x, t) + g\cos\Omega t)(y' + {y'}^{3}/2) + \int_{x}^{L} m(a(\ddot{s}, s, \dot{y}, z, t) + g\cos\Omega t)dz(y' + {y'}^{3}/2)' +mb(\ddot{y}, y, \dot{s}, x, t) + (J(x)\ddot{y}')' - (EIy'' - 3EIy''y'^{2})'' - (3EIy''^{2}y')' + f(y, \dot{y}, x, t) = mg\sin\Omega t,$$
(2.5)

where the dots are partial derivatives with respect to time, with boundary conditions y(0, t) = y'(0, t) = 0 at x = 0 and

$$-J(L)\ddot{y}' + (EIy'' - 3EIy''y'^{2} + 3EIy''^{2}y') = 0$$

$$EIy'' - 3EIy''y'^{2} = 0$$
(2.6)

at x = L, where $a(\ddot{s}, s, \dot{y}, x, t) = \ddot{s} + 2\Omega \dot{y} + \Omega^2(x-s)$, $b(\ddot{y}, y, \dot{s}, x, t) = -\ddot{y} + \Omega^2 y + 2\Omega \dot{s}$, and $f(y, \dot{y}, x, t)$ represents distributed aeroelastic loads.

Interpreting these equations of motion, the IPDE has nonlinear inertial terms due to the geometric nonlinear term $m(a(\ddot{s}, s, \dot{y}, x, t) + g \cos \Omega t)(y' + {y'}^3/2)$, along with linear inertial terms, including rectilinear, centripetal, and Coriolis effects. The IPDE has parametric excitation, with the $\cos \Omega t$ terms, as well as direct excitation $mg \sin \Omega t$ at the same frequency. These come from the gravitational potential energy, with parametric excitation involving large-deflection nonlinearity. The aerodynamic force f has cyclic components and will contribute to direct and parametric excitation terms as well. The *EI* terms include nonlinear curvature terms from large deflection. The aeroelastic $f(y, \dot{y}, x, t)$ terms have further dependence on *s* and \dot{s} , and produce additional direct or parametric excitation terms.

2.1.1 Modal Reduction of the EOM

To make the integral-partial differential equation amenable to the first level of analysis, we perform a modal reduction to project the IPDE onto a finite number of ordinary differential equations (ODEs). We use cantilevered beam modes as assumed modes and apply a Galerkin projection. To this end, we approximate the transverse deflection as

$$y(x,t) \approx \sum_{i=1}^{N} q_i(t)\psi_i(x), \qquad (2.7)$$

where N is the number of retained modes, $\psi_i(x)$ are the assumed modal functions, and $q_i(t)$ are the assumed modal coordinates.

We substitute this expression into equation (2.5), multiply by $\psi_j(x)$, and integrate over the length of the blade to obtain the j^{th} second-order ODE. To simplify the analysis, we take N = 1 and neglect the effect of J(x). Neglecting the contributions of the rotational inertia is a common approximation for the analysis of thin beams [72]. Also, we expect the dominant failure mode to be in the first resonant mode of the beam. Including higher order modes will reveal additional details but our aim is to start simple and analyze the representative system for dynamic instabilities.

Employing the aforementioned simplifications to the model, the differential equation takes the form

$$\ddot{q} + b\ddot{q}q^{2} + c\dot{q}q + d\dot{q}^{2}q + (e_{1} + e_{2}\cos\Omega t)q + (f_{1} + f_{2}\cos\Omega t)q^{3}$$

= $g\sin\Omega t \int_{0}^{L} \psi(x) dx - \int_{0}^{L} \psi(x)f(y, \dot{y}, s, \dot{s}, x, t) dx$ (2.8)

where b, c, d,
$$e_1, e_2, f_1, f_2$$
 are constant coefficients and can be evaluated according to

$$b = \int_0^L \psi(x) \left(\psi''(x) \int_x^L \int_0^z {\psi'}^2(u) \, du \, dz - {\psi'(x)} \int_0^x {\psi'}^2(v) \, dv \right) \, dx$$

$$c = \int_0^L \psi(x) \left(-2\Omega\psi(x)\psi'(x) + 2\psi''(x)\Omega \int_x^L \psi(z) \, dz \right) \, dx$$

$$d = \int_{0}^{L} \psi(x) \left(\psi'(x) \int_{0}^{x} {\psi'}^{2}(v) \, dv + {\psi''(x)} \int_{x}^{L} {\int_{z}^{L} {\psi'}^{2}(u) \, du \, dz} \right) dx$$

$$e_{1} = \int_{0}^{L} \psi(x) \left(\Omega^{2} x \psi'(x) + \Omega^{2} {\psi''(x)} (\frac{L^{2} - x^{2}}{2}) \psi(x) - (EI\psi''(x))'' \right) dx$$

$$e_{2} = \int_{0}^{L} \psi(x) \left(\psi'(x)g + g\psi''(x)(L - x) \right) dx$$

$$f_{1} = \int_{0}^{L} \psi(x) \left((-\frac{{\psi'}^{3}(x)}{2}) \Omega^{2} x + {\psi'(x)} \Omega^{2} \int_{0}^{x} \frac{{\psi'}^{2}(v)}{2} \, dv + \frac{\Omega^{2}}{2} (L^{2} - x^{2}) (\frac{{\psi'}^{3}(x)}{2})' \right)$$

$$- \left[\int_{x}^{L} \int_{0}^{z} \frac{{\psi'}^{2}}{2} \, du \, dz \right] \psi''(x) \Omega^{2} + EI\psi''(x) (3\psi'^{2}(x))' - (3EI\psi''^{2}(x)\psi'(x))' \right) dx$$

$$f_{2} = \int_{0}^{L} \psi(x) \left(-\frac{{\psi'}^{3}(x)}{2}g + \frac{(L - x)g}{2} ({\psi'}^{3}(x))' \right) dx$$

These coefficients are dependent on the assumed modal function $\psi(x)$ and the distributed parameters m(x) and EI(x). The form of $f(y, \dot{y}, s, \dot{s}, x, t)$ influences the last integral in equation (2.8), and the q terms born from it.

For example, if we approximate the system as a uniform beam, such that m(x) and EI(x) are constants, and use the first modal function of a uniform cantilevered Euler-Bernoulli beam [77], given by

$$\psi = \left[\left(\cosh(\lambda x/L) - \cos(\lambda x/L) \right) - \delta(\sinh(\lambda x/L) + \sin(\lambda x/L) \right], \tag{2.9}$$

where $\lambda = 1.87510407$ and $\delta = 0.7341$, then we obtain parameter values that produce

$$\begin{split} \ddot{q} &- 0.94\ddot{q}q^2 - 0.918\dot{q}q + 1.37\dot{q}^2q + (-2.66EIL^{-2} + 0.363L^2\Omega^2 + 5.12L\cos\Omega t)q \\ &+ (5.352EIL^{-3} + 13.05EIL^{-4} - 0.226\Omega^2 - 8.534L^{-1}\cos\Omega t)q^3 \\ &= 1.65L\sin\Omega t - \int_0^L \psi(x)f(y,\dot{y},s,\dot{s},x,t)\,dx. \end{split}$$

Including a model for aeroelastic loading via f will contribute some terms (with dependence on q, L, etc.) already present in equation (2.10) and also some new terms. Wind shear and tower passing will introduce cyclic terms in f which will contribute to the direct forcing term, and possibly other parametrically excited terms. Aeroelastic forces will also be significant on models of out-of-plane deflection. It is apparent, from inspection, that this resulting ODE in q has linear and cubic stiffness effects, with parametric excitation on both the linear and cubic terms, along with direct excitation.

In addition we include a damping term in our model to account for operational damping in the system due to resistance in blade motion in the plane of motion. Also the cross coupled terms in \ddot{q} , \dot{q} , q give rise to lot more interplay between the terms in a purely analytical case study. Since the mode shapes are fixed, the coefficients are only dependent on the geometry of the blade (beam), its material properties and the rotational frequency.

The aerodynamic force on the beam can be modeled based on the lift and drag force at each section along the length of the blade. We use the framework of the blade element momentum theory (discussed in [43]) which is widely used in preliminary calculations in the industry and academics (detailed discussion in Chapter 5).

Our intent with this formulation is to analyze this system, rebuild the approximate y(x, t) under resonant cases, and determine the resulting loads applied to the low speed shaft and hub via y(x, t)and its derivatives. In reality, blade motion is a combination of in-plane motion, out-of-plane motion and twisting of the blade structure. In the previous section, we developed the EOM for the in-plane motion. Blades are more pliable in the flap direction due to lesser rigidity in the out-of-plane mode. This is due to the fact that the wind forces on the blade that cause motion act predominantly in the out of place direction. The lift force on the airfoil causes the rotation of the blade and the drag is perpendicular to it. The component of these forces can then be split into component in the in-plane and out-of-plane directions. Most commercial design codes (including those discussed in section 1.3.3) ignore the torsion component of the blade motions due to the fact that its contribution to overall loading on the blade is minimal when compared to in-plane and out-of-plane loads. As such, a comprehensive method would include all the degrees of freedom in the modeling approach, however, we seek to isolate one mode of motion to study its effect on the blade motion and consequently, their contribution to drivetrain loads. In essence, the effect of coupled modes for the 2-dof system would not be included in the current analysis.

2.2 Formulation of Out-of-Plane EOM

We formulate the equations of motion of out-of-plane deflection of a beam attached to a hub which rotates at a constant rate similar to the process laid out in the previous section. In this section, we employ the Lagrange approach to arrive at the ordinary differential equations.

Meirovitch [72], compares the Galerkin method, which is the projection of the PDEs onto the assumed modes, to the Rayleigh-Ritz Method, which is the application of the assumed modes to the energies (for the linear case). We formulate the out-of-plane equation of motion using the Rayleigh-Ritz approach. It has been verified that the two approaches produced the same fundamental equation of motion that govern the system. Due to the process of integration and differentiation of terms, the expressions defining coefficients in the ODE for the modal coordinate q(t), appear differently in both methods, but numerical computations of the coefficients reveal that they are essentially equivalent.

Following the approach laid out in the previous section, the gravitational potential energy of the rotating beam is integrated over the beam elements with the help of Figure 2.2. The height h(x,t) of an element dx, located at a point x of the beam, rotated by an angle ϕ , is given by the angular inclination $x(1 - \cos \phi)$ plus the inclined foreshortening s(z, x, t) due to bending in the out of plane direction, such that $h(x,t) = x(1 - \cos \phi) + s(z, x, t) \cos \phi$, where the nonlinear foreshortening $s(z, x, t) = \int_0^x \left(\frac{z'^2}{2} + \frac{z'^4}{8}\right) dx$ is evaluated from geometry (see figure 2.3). The primes indicate partial derivatives with respect to x. z(x, t) is the out-of-plane deflection at each element along the length of the blade at a given time. Then the gravitational potential energy is

$$V_{g} = \int_{0}^{L} m(x)gh(x,t)dx$$

$$= \int_{0}^{L} m(x)g[x(1-\cos\phi) + s(z,x,t)\cos\phi]dx$$
(2.11)



Figure 2.2: The arbitrary configuration of the position of a point P on a beam rotating in-plane and deflecting in the out-of-plane direction. The point P corresponds to location x on the undeflected beam.

where *L* is the length and m(x) is the mass per unit length.

The flexural potential energy will include geometric nonlinearity, following [31]. The same expression was also used in the formulation of the in-plane equation of motion in section 2.1 to formulate the bending potential energy as

$$V_b = \int_0^L \frac{1}{2} EI(x) k^2 dx \approx \int_0^L \frac{1}{2} EI(x) {z''}^2 (1 - 3{z'}^2) dx,$$
 (2.12)

where *E* is the Young's modulus and *I* is the area moment of inertia of a cross-section of the beam.

The kinetic energy of the beam is formulated from the velocity of each element. The position of an element is $\mathbf{r} = (x - s(z, x, t))\mathbf{e}_r + z(x, t)\mathbf{e}_z$ in terms of the unit vectors in the direction of the undeformed, rotated beam (\mathbf{e}_r) and transverse in the ϕ direction (\mathbf{e}_{ϕ}) . Using $\dot{\mathbf{e}}_r = \dot{\phi}\mathbf{e}_{\phi}$ and $\dot{\mathbf{e}}_z = 0$, and considering the case where $\dot{\phi} = \Omega$, the velocity is $\mathbf{v} = -\dot{s}\mathbf{e}_r + \Omega(x-s)\mathbf{e}_{\phi} + \dot{y}\mathbf{e}_z$. The contribution to the integral of the rotational energy density, $\frac{1}{2}J(x)(\dot{\phi})^2$, where J(x) is the mass moment of inertia



Figure 2.3: The nonlinear foreshortening due to out-of-plane blade motion at a point along the blade at a distance x

per unit length, can also be added. Then

$$T = \int_0^L \frac{1}{2} m(x) \mathbf{v} \cdot \mathbf{v} dx + \int_0^L \frac{1}{2} J(x) \dot{\phi}^2 dx$$
(2.13)

is the total kinetic energy.

We approximate the out-of-plane deflection as

$$z(x,t) \approx \sum_{i=1}^{N} q_i(t)\psi_i(x), \qquad (2.14)$$

where *N* is the number of retained modes, $\psi_i(x)$ are the assumed modal functions, and $q_i(t)$ are the assumed modal coordinates. We substitute this expression for *z* into the kinetic and potential energy expressions, which are now $T(q_i, \dot{q}_i)$ and $V(q_i)$.

The Lagrangian function *L* is defined as L = T - V, where *T* is the total kinetic energy and *V* is the total potential energy (gravitational and flexure) of the system respectively. The generalized form of the Lagrangian equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i'$$
(2.15)

where Q_i' accounts for frictional forces and cyclic forcing functions (aerodynamic lift and drag force on the airfoils).

To simplify the analysis, we take N = 1 and neglect the effect of J(x). Neglecting the contributions of the rotational inertia is a common approximation for the analysis of thin beams [72]. Also, we expect the dominant failure mode to be in the first resonant mode of the beam.

Substituting the terms for kinetic and potential energy in equation (2.15) and constructing it as an equation in the modal coordinate q, we arrive at the differential equation of motion for the first out of plane mode for the wind turbine blade as

$$\ddot{q} + b\ddot{q}q^2 + d\dot{q}^2q + (e_1 + e_2\cos\Omega t)q + (f_1 + f_2\cos\Omega t)q^3 = Q$$
(2.16)

where b, d, e_1, e_2, f_1, f_2 are constant coefficients and can be evaluated as

$$b = \int_{0}^{L} m(x) \left(\int_{0}^{x} {\psi'}^{2} du \right)^{2} dx$$

$$d = \int_{0}^{L} m(x) \left(\int_{0}^{x} {\psi'}^{2} du \right)^{2} dx$$

$$e_{1} = \int_{0}^{L} m(x) \Omega x \left(\int_{0}^{x} {\psi'}^{2} du \right) dx + \int_{0}^{L} EI {\psi''}^{2} dx$$

$$e_{2} = \int_{0}^{L} m(x) g \left(\int_{0}^{x} {\psi'}^{2} du \right) dx$$

$$f_{1} = -\int_{0}^{L} \frac{m(x) \Omega^{2}}{2} \left(\int_{0}^{x} {\psi'}^{2} du \right)^{2} dx$$

$$+ \int_{0}^{L} \frac{m(x) \Omega^{2} x}{2} \left(\int_{0}^{x} {\psi'}^{4} du \right) dx - 6 \int_{0}^{L} EI {\psi''}^{2} {\psi'}^{2} dx$$

$$f_{2} = \frac{1}{2} \int_{0}^{L} m(x) g \left(\int_{0}^{x} {\psi'}^{4} du \right) dx$$

(2.17)

Equation (2.16) has a very similar structure to the equation (2.8) which governs the in-plane dynamics. Indeed, all the terms excepts the coupled $q\dot{q}$ appear in both expressions. The out-of-plane blade ODE has parametric excitation, with the $\cos \Omega t$ terms, as well as direct excitation Q which consists of aerodynamic forces. These come from the gravitational potential energy, with parametric excitation involving large-deflection nonlinearity. The aerodynamic force has cyclic components and will contribute to direct, and parametric, excitation terms as well. The *EI* terms are nonlinear curvature terms from large deflection. The external force terms may have further

dependence on s and \dot{s} , and may produce additional direct or parametric excitation terms. An aeroelastic formulation may also be accompanied by the appropriate fluid mechanics equation, coupled through boundary conditions.

CHAPTER 3

FORCED NONLINEAR MATHIEU EQUATION

Modeling the in-plane blade and out-of-plane motion in its full depth, i.e. including all the nonlinearities and gravitational loading, leads to a nonlinear parametrically excited equation. Our goal is to simplify the system of equations and look for analytical descriptions of representative phenomena. For this we apply reduced-order modeling to obtain analytical expressions of response characteristics as functions of parameters. We make assumed modal reductions using the cantilevered beam modes to obtain a single mode model of blade motion in the lead-lag direction. The resulting second order differential equation has elements of a parametrically and directly excited Mathieu/Duffing equation. This begs us to study it. The simplified equation thus becomes the focus of our analysis, which is performed in search of possible phenomenon related to wind turbine systems. The simplifications are relaxed in work where the analysis would be more numerical.

3.1 Perturbation Analysis: First Order

Embedded in the single mode ODE's (2.8) and (2.16) are terms of a Mathieu-Duffing equation:

$$\ddot{q} + \epsilon \hat{\mu} \dot{q} + (\omega^2 + \epsilon \gamma \cos \Omega t)q + \epsilon \alpha q^3 = F \sin \Omega t, \qquad (3.1)$$

Since this is a fundamental equation of dynamic systems, equation (3.1) takes our attention and becomes the focus of an initial study. In this chapter we only consider the influence of sinusoidally varying direct excitation of the system. For a wind turbine, both sinusoidal and constant direct excitation forces will occur. This will be treated in detail in Chapter 4.

In equation (3.1), ω plays the role of the undamped natural frequency based on the mean stiffness-to-mass ratio, $\epsilon \gamma$ is the magnitude of the parametric excitation corresponding to the cyclic variation in stiffness, *F* is the magnitude of direct cyclic forcing due to transverse gravitational and aerodynamic loading, Ω is the fundamental frequency of cyclic loadings, based on the rotation rate of the rotor, and $\epsilon \hat{\mu}$ represents linear viscous damping. Parameter α is the coefficient of cubic

nonlinearity expected due to large deformation effects, and ϵ is a small bookkeeping parameter.

We seek approximate solutions to equation (3.1) by using the method of multiple scales (MMS) [74,75]. The analysis reveals the existence of various sub-harmonic and super-harmonic resonances. We unfold these resonance cases and identify the critical ones for wind turbines during normal operation.

When the forcing in equation (3.1) is of order ϵ , the analysis will indicate a primary resonance. But, this resonance is the same as in the Duffing equation, and is not treated here. The forcing in the above expression is of order one — also known as hard forcing. This will help us unfold secondary resonances. Using MMS, we incorporate fast and slow time scales (T_0, T_1) and also variations in amplitude. This allows for a dominant solution q_0 and a variation of that solution q_1 , i.e. such that

$$q = q_0(T_0, T_1) + \epsilon q_1(T_0, T_1) + \dots$$
(3.2)

where $T_i = \epsilon^i T_0$. Then $\frac{d}{dt} = D_0 + \epsilon D_1$ and $D_i = \frac{\partial}{\partial T_i}$.

We substitute this formulation into equation (3.1) and then simplify and extract coefficients of ϵ^0 , ϵ^1 . The expression for co-efficient of ϵ^0 is

$$D_0^2 q_0 + \omega^2 q_0 = F \sin \Omega T_0.$$
 (3.3)

The solution for this is

$$q_0 = Ae^{i\omega T_0} - i\Lambda e^{i\Omega T_0} + c.c.$$
(3.4)

where $\Lambda = \frac{F}{2(\omega^2 - \Omega^2)}$, and $A = \frac{1}{2}ae^{i\beta}$.

Since equation (3.3) is a partial differential equation, the coefficient *A*, and hence *a* and β are functions of T_1 .

The expression for the co-efficient of ϵ^1 is

$$D_0^2 q_1 + \omega^2 q_1 = -\hat{\mu} D_0 q_0 - 2D_0 D_1 q_0 - \gamma q_0 \cos \Omega T_0 - \alpha q_0^3.$$
(3.5)

Substituting the solution for q_0 from equation (3.4), we expand the terms on the right hand side. We need to eliminate coefficients of $e^{i\omega T_0}$ that constitute the secular terms and would make the solutions unbounded. The solvability condition, set by equating the coefficients of $e^{i\omega T_0}$ terms to zero, leads to slow flow equations that describe the slowly varying variables *a* and β of the leading order solution given by equation (3.4).

3.1.1 Non-Resonant Case

If there is no specific relation between Ω and the natural frequency (ω) of the system, then an analysis of the solvability condition leads to the conclusion that $a \to 0$ at steady-state solution. Hence there is no effect of the nonlinear terms in the non-resonant case.

3.1.2 Superharmonic Resonances

3.1.2.1 $2\Omega \approx \omega$

If $2\Omega \approx \omega$, then the $\epsilon \alpha \cos \Omega t$ term and the nonlinearity from equation (3.5) contribute to the secular terms. We detune the frequency of excitation such that $2\Omega = \omega + \epsilon \sigma$. In this case the solvability condition can be written as

$$-2A'i\omega - \hat{\mu}Ai\omega - \alpha(3A^2\bar{A} + 6\Lambda^2A) + \frac{i\gamma\Lambda}{2}\exp i\sigma T_1 = 0$$
(3.6)

where ' denotes differentiation with respect to T_1 . Letting $A = \frac{1}{2}ae^{i\beta}$ and $\phi = \sigma T_1 - \beta$, the real and imaginary parts of equation (3.6) lead to

$$a\phi'\omega - \sigma\omega a + 3\alpha a(\Lambda^2 + \frac{a^2}{8}) + \frac{\gamma\Lambda}{2}\sin\phi = 0,$$
$$a'\omega + \frac{\hat{\mu}a\omega}{2} + \frac{\gamma\Lambda}{2}\cos\phi = 0.$$

At steady-state, a' = 0, $\phi' = 0$ and the relationship between the response amplitude and the detuning parameter σ is



Figure 3.1: Simulated response of equation (3.1) showing superharmonic resonances at orders 1/2 and 1/3; $\hat{\mu} = 0.5$, $\epsilon = 0.1$, $\alpha = 0.5$, F = 0.1, $\gamma = 3$. The frequency ratio sweeps up.

$$\sigma = 3\alpha\Lambda^2 + \frac{3}{8}\alpha a^2 \pm \left(\frac{\gamma^2\Lambda^2}{4a^2\omega^2} - \frac{\hat{\mu}^2}{4}\right)^{1/2}.$$
 (3.7)

The peak amplitude would be

$$a_p = \frac{\gamma \Lambda}{\omega \hat{\mu}}.$$
(3.8)

The corresponding value of σ_p is

$$\sigma_p = \frac{3\alpha\Lambda^2}{\omega} \bigg(1 + \frac{\gamma^2}{8\omega^2\hat{\mu}^2} \bigg).$$

Hence we can conclude the following:

- the peak value is independent of α ,
- α , Λ and $\gamma/\hat{\mu}$ affect the peak location and as they increase, $|\sigma_{2p}|$ increases,
- the sign of α determines the sign of σ_p .

Vertical tangencies in the curve of equation (3.7) can be found and shown to coincide with stability transitions, such that the unstable branch lies between the vertical tangencies.

A sample response of equation (3.1) is numerically simulated. The response curve shown in Figure 3.1 has the primary frequency and two superharmonics shown at 1/3 and 1/2 the natural frequency. Superharmonic response at this order is also noticed for a linear system (see Figure 3.2). This agrees with our calculation of peak amplitude (see equation (3.8)). When nonlinearity $\alpha = 0$, the γ term still shows up in the solvability condition (3.6), and this superharmonic resonance peak persists.

Wind turbine blades are generally designed such that the natural frequency of the rotating blade in lead-lag (in-plane) motion is below the rotational frequency. This analysis implies potential existence of superharmonic resonances which would also provide additional critical frequencies where the response of the blade would be dominant. Such responses would increase loading on the gearbox and other components and increased bending on the blades. The superharmonic resonance has also been observed in simulations and experiments on out-of-plane motion [54, 55]. The sign of the nonlinear term also has a significant effect on the response of the system. Based on the expression for α given in [78], the value can be negative for certain wind turbine blades that would lead to a softening type behavior (the resonance curve bends to the left). This would be consistent with the slightly lower frequencies associated with large oscillation of the blades.

3.1.2.2 $3\Omega \approx \omega$

If $3\Omega \approx \omega$, the cubic nonlinearity contributes to the secular terms. We detune the frequency of excitation such that $3\Omega = \omega + \epsilon \sigma$. The solvability condition will have additional terms that have $3\Omega T_0$ as an exponential argument in equation (3.5), and it takes the form

$$-2A'i\omega - \hat{\mu}Ai\omega - \alpha(3A^2\bar{A} + 6\Lambda^2 A) - i\alpha\Lambda^3 e^{i\sigma T_1} = 0.$$
(3.9)

Following the analysis done in the previous section for the 2 Ω superharmonic resonance, we substitute a polar form for *A* and separate the equation into real and imaginary parts. Using $\phi = \sigma T_1 - \beta$, we arrive at a homogenous set of equations



Figure 3.2: Simulated response of the linear case of equation (3.1) showing superharmonic resonances at orders 1/2 and 1/3; $\hat{\mu} = 0.5$, $\epsilon = 0.1$, $\alpha = 0$, F = 0.5, $\gamma = 3$

$$a\phi'\omega - \sigma\omega a + 3\alpha a(\Lambda^2 + \frac{a^2}{8}) - \alpha\Lambda^3 \sin\phi = 0$$
$$a'\omega + \frac{\hat{\mu}a\omega}{2} + \alpha\Lambda^3 \cos\phi = 0$$

in amplitude *a* and phase ϕ .

These expressions are the same as the ones we would get in the synthesis of a standard Duffing equation with hard excitation [74]. Steady state solutions are satisfied by

$$\left[\left(\frac{\mu}{2}\right)^2 + \left(\sigma - \frac{3\alpha\Lambda^2}{\omega} - \frac{3\alpha a^2}{8\omega}\right)^2\right]a^2 = \frac{\alpha^2\Lambda^6}{\omega^2}.$$

This is a quadratic expression in the detuning parameter σ . Solving for σ , we get

$$\sigma = \frac{3\alpha\Lambda^2}{\omega} + \frac{3\alpha a^2}{8\omega} \pm \left(\frac{\alpha^2\Lambda^6}{\omega^2 a^2} - \left(\frac{\mu}{2}\right)^2\right)^{1/2}.$$
(3.10)

The peak amplitude and corresponding value of σ are

$$a_p = \frac{2\alpha\Lambda^3}{\mu\omega} \tag{3.11}$$

and

$$\sigma_p = \frac{3\alpha\Lambda^2}{\omega} \left(1 + \frac{\alpha^2\Lambda^4}{2\omega^2\mu^2} \right).$$

Therefore, the peak amplitude and frequency of this resonance are affected by the nonlinearity via α and by the amplitude of excitation via Λ . The parametric excitation term does not contribute to this resonance at first order. The value of peak amplitude at this resonance is directly proportional to α (from equation (3.11)). However, numerical simulations for a linear version of our system show the persistence of this superharmonic peak (see Figure 3.2). Higher order perturbation analysis (discussed in chapter 4) reveals the existence of this resonant condition in the absence of nonlinearity.

3.1.3 Subharmonic Resonances

3.1.3.1 $\Omega \approx 2\omega$

If $\Omega \approx 2\omega$ then the terms with $i(\Omega - \omega)T_0$ as the exponent add to the secular terms. The linear parametric excitation term and the nonlinearity contribute to this resonance. The solvability condition with these added terms is

$$-2A'i\omega - \hat{\mu}Ai\omega - \alpha(3A^2\bar{A} + 6\Lambda^2 A) - \frac{\gamma\bar{A}}{2}\exp i\sigma T_1 = 0$$
(3.12)

We substitute the exponential form for A, separate the real and imaginary parts, and let $\phi = \sigma T_1 - 2\beta$, to get

$$-\frac{a\omega\phi'}{2} - \frac{\sigma\omega a}{2} + 3\alpha a(\Lambda^2 + \frac{a^2}{8}) + \frac{\gamma a}{4}\cos\phi = 0$$
$$a'\omega + \frac{\hat{\mu}a\omega}{2} + \frac{\gamma a}{4}\sin\phi = 0$$

Seeking steady state, we let $\phi' = a' = 0$, which yields

$$\frac{\sigma a}{2} - \frac{\alpha}{\omega} \left(\frac{3a^3}{8} + 3\Lambda^2 a\right) = \frac{\gamma a}{4\omega} \cos \phi$$
$$\frac{-\hat{\mu}a}{2} = \frac{\gamma a}{4\omega} \sin \phi$$

We eliminate ϕ dependence to obtain the frequency response equation as

$$\frac{\gamma^2 a^2}{16\omega^2} = \left[\frac{\sigma a}{2} - \frac{3\alpha a}{\omega} \left(\frac{a^2}{8} + \Lambda^2\right)\right]^2 + \frac{\hat{\mu}^2 a^2}{4}$$
(3.13)

For this equation we have the trivial solution a = 0 and another set of solutions of the form $a^2 = p_1 \pm (p_1^2 - q_1)^{1/2}$, where

$$p_1 = \frac{4\sigma\omega}{3\alpha} - 8\Lambda^2 \text{ and } q_1 = \frac{64\omega^2}{9\alpha^2} \left[\left(\frac{\sigma}{2} - \frac{3\alpha\Lambda^2}{\omega}\right)^2 + \frac{4\hat{\mu}^2\omega^2 - \gamma^2}{16\omega^2} \right]$$

These are similar in form to the standard Duffing subharmonic (presented in section 3.1.3.2), but the dependence on parameters differs. The nontrivial solutions for *a* are real only when $p_1 > 0$ and $p_1 \ge q$. These conditions imply that solutions will exist if

$$\Lambda^2 < \frac{\sigma\omega}{6\alpha} \text{ and } \hat{\mu}^2 \le \frac{\gamma^2}{4\omega^2}$$
 (3.14)

These are two distinct conditions on the parameters of our problem. The parametric excitation term must be sufficiently large, or the damping sufficiently small. Also, the product of the coefficients of the nonlinear cubic term and Λ^2 must be sufficiently small or detuning must be sufficiently large. However, the presence of the cubic nonlinearity is essential for this subharmonic to occur.

3.1.3.2 $\Omega \approx 3\omega$

If $\Omega \approx 3\omega$, the cubic nonlinearity contributes to the secular terms. We detune the frequency of excitation such that $\Omega = 3\omega + \epsilon\sigma$. Then $\Omega T_0 = (3\omega + \epsilon\sigma)T_0 = 3\omega T_0 + \sigma T_1$.

The solvability condition takes the form

$$-2A'i\omega - \hat{\mu}Ai\omega - \alpha(3A^2\bar{A} + 6\Lambda^2A) - i3\alpha\bar{A}^2\Lambda e^{i\sigma T_1} = 0$$
(3.15)

which is the same as that of the standard Duffing equation. We substitute the exponential form for *A*, separate the real and imaginary parts, and let $\phi = \sigma T_1 - 3\beta$ to transform this into an

autonomous system. Seeking steady state, we let $\phi' = a' = 0$. Eliminating ϕ leads to the nonlinear frequency response equation

$$\left[\frac{9\hat{\mu}^2}{4} + \left(\sigma - \frac{9\alpha\Lambda^2}{\omega} - \frac{9\alpha^2}{8\omega}\right)^2\right]a^2 = \frac{81\alpha^2\Lambda^2}{16\omega^2}a^4$$
(3.16)

The solutions of equation (3.16) are either a = 0 or $a^2 = p \pm (p^2 - q)^{1/2}$ where

$$p = \frac{8\omega\sigma}{9\alpha} - 6\Lambda^2 \text{ and } q = \frac{64\omega^2}{81\alpha^2} \left[\frac{9\hat{\mu}^2}{4}^2 + (\sigma - \frac{9\alpha\Lambda^2}{\omega})^2 \right].$$

Since q is always positive, we need p > 0 and $p^2 \ge q$. This requires that

$$\Lambda^2 < \frac{4\omega\sigma}{27\alpha} \text{ and } \frac{\alpha\Lambda^2}{\omega} \left(\sigma - \frac{63\alpha\Lambda^2}{8\omega}\right) - 2\hat{\mu}^2 \ge 0$$

which can be compared to the order-two conditions (3.14). Then

$$\frac{63\alpha\Lambda^2}{2\omega\hat{\mu}} = \frac{2\sigma}{\hat{\mu}} \pm \left(\frac{4\sigma^2}{\hat{\mu}^2} - 63\right)^{1/2}$$

defines the boundary of the region in the $\Lambda\sigma$ plane where non-trivial solutions can exist.

3.2 Parameter Dependence Discussion

We have shown the details of the first-order analysis of superharmonic and subharmonic resonance for the system of interest. At first order, the superharmonic resonance of order 1/3 is the same phenomenon as in the Duffing equation. The nonlinear parameter, α , scales the peak response, while both the nonlinear parameter and the direct excitation level affect the frequency value of the peak response. The superharmonic response of order 1/2 involves interaction between the parametric excitation and both the nonlinear parameter and the direct excitation. In fact, if the nonlinearity is not present, i.e. if $\alpha = 0$, this resonance persists. As such, a *linear system* excited both parametrically and directly at the same frequency can exhibit a superharmonic resonance. This complements linear primary resonance phenomena and linear subharmonic resonances exploited in parametric amplification [61,62].



Figure 3.3: Amplitudes of simulated response of equation (3.1) showing the effect of the parametric forcing amplitude, with $\epsilon = 0.1$, $\hat{\mu} = 0.5$, $\alpha = 0$, F = 0.5. Different curves depict $\gamma = 0.5$, 1 and 3.

There also exists a primary resonance which is the same at first order as that of the Duffing equation. The parametric excitation, when at the same frequency as the direct excitation, does not affect primary resonance amplitude at first order.

Subharmonic resonances at orders two and three exist in the first-order approximations. Ordertwo subharmonics involve interactions with the parametric excitation term, while order-three subharmonic resonance is the same as that of the Duffing equation to first order. The subharmonic resonance may not be critical to wind turbines, which motivate the study, since they operate below the natural frequency of the rotating blade.

The variation in system responses for changes in some parameters have been observed in simulations. Their behavior has been summarized below.

1. Effect of parametric forcing term: The system resonances increase with increased parametric forcing (Figure 3.3). This amplification occurs at resonances, but not between them. Beyond a certain value of γ the system goes unstable at primary resonance. When approaching this



Figure 3.4: Amplitude of simulated response of equation (3.1) showing the effect of the direct forcing amplitude, with $\epsilon = 0.1$, $\hat{\mu} = 0.5$, $\alpha = 0$, $\gamma = 3$. Different curves depict F = 0.5, 1 and 2.



Figure 3.5: Amplitude of simulated sweep-up response of equation (3.1) showing the effect of the nonlinear term, with $\epsilon = 0.1$, $\hat{\mu} = 0.5$, $\gamma = 3$, F = 0.5. Different curves depict $\alpha = 0.1$, 0.5 and 1.

instability, the primary resonance increases abruptly, which is not captured in the first order analysis. Higher order perturbation analysis performed in the following chapters details the onset of instability as we increase parametric forcing term.

- 2. Effect of direct forcing term: As expected, an increase in the direct forcing term *F* increases the overall magnitude of response evenly over the entire spectrum (Figure 3.4).
- 3. Effect of damping term: Increase in the damping decreases the overall magnitude of response over the entire spectrum, particularly at resonances. The superharmonic and primary resonance peaks scale as $1/\hat{\mu}$.
- 4. Effect of nonlinearity: An increase in nonlinearity (α) causes the response curve to bend over more significantly. This bending can induce jump instabilities as the frequency slowly varies. Also, in the presence of strong enough nonlinearity, the response amplitude of the superharmonic of order two is of a similar order of magnitude as the primary resonance, as seen in Figure 3.5.

This chapter has focused on the resonances of the forced nonlinear Mathieu equation, and not stability. The Mathieu equation is well known to have instabilities in the space of stiffness and parametric excitation parameters. These instabilities are observed when the system is simulated with certain parameter values at primary resonance. A higher-order analysis is necessary to describe stability transitions, as well as higher-order effects on resonances. The presence of superharmonic resonances may be significant for wind turbines, which are designed to operate below the lowest natural frequency. There are also numerous instabilities at superharmonic frequencies in the Mathieu parameter space. One of the main motivations to perform high order expansions is to characterize resonance and instability of the system as a function of system parameters. These can then be translated to the complete ODE for in-plane and out-of-plane motion i.e., equation (2.8) and (2.16) to establish limits for material, mass and stiffness of blades in order to not have undeniable response amplitudes at their operating ranges.

CHAPTER 4

SECOND-ORDER PERTURBATION ANALYSIS

In this chapter, we seek to explain the existence of multiple superharmonics for a linear Mathieu system with a second-order perturbation expansion. Higher-order perturbation techniques have been used extensively across various sciences to get more accurate expressions of the solutions to problems, in order to derive an understanding of the underlying physics [79, 80]. Dynamical systems have been described by higher-order perturbation expansions [81, 82] as well as other scaling techniques [83]. We extend the second-order formulations to systematically construct asymptotic solutions of the linear forced Mathieu equation at primary resonance.

The regular Mathieu equation is well known to have instability regions in the parametric forcing amplitude-frequency parameter space, such that the fixed point at the origin can be stable or unstable depending on system parameters. The stability wedges of the linear Mathieu equation can be studied by applying Floquet theory [71] with harmonic balance solutions of periodic transitional motions [75], and also by a higher-order perturbation expansion [74].

For lightly damped cases, these regions of instability originate near frequency ratios $\Omega/\omega = 2/N$, *N* being an integer (see for example reference [75]). The dependence of the instability region on the strength of damping has been further explored in detail in [84]. There have been numerous studies on systems with parametric excitation with nonlinear variations of the Mathieu equation, for example with added Van der Pol, Rayleigh, and Duffing nonlinear terms, revealing interesting dynamics (e.g. [64, 85]).

The introduction of direct forcing to the Mathieu equation turns it into a system that does not directly align it with Floquet theory. However, for the linear Mathieu equation, the solution can be expressed as a sum of homogenous and particular solutions, given as $q = q_h + q_p$, where, q_h satisfies $\ddot{q}_h + 2\epsilon\mu\dot{q}_h + (\omega^2 + \epsilon\gamma\cos\Omega t)q_h = 0$ and q_p satisfies $\ddot{q}_p + 2\epsilon\mu\dot{q}_p + (\omega^2 + \epsilon\gamma\cos\Omega t)q_p = F_0 + F\sin\Omega t$. The q_h term, from the Floquet theory, has the various instabilities that originate due to parametric excitation. The q_p part of the solution, approximated by perturbation techniques, can have resonance conditions as well as instabilities. This implies that the instability properties known about the linear Mathieu equation apply to the system under study in this work. Some of the resonance conditions obtained through solving the particular solution coincide with the instability property of the linear Mathieu as will be explained further.

In this chapter, we analyze equation (3.1) under hard forcing with two orders of multiple-scales expansion in order to capture superharmonic resonance at one-third for the linear system with hard forcing. We extend our second-order perturbation analysis for a weakly forced system in order to describe a parametric amplification at primary resonance, and to capture information regarding stability transition curves. Additional sections also provide commentary on the influence of direct forcing terms and nonlinearity towards to response of the system for various parameter ranges.

4.1 Second-Order Perturbation Analysis – Hard Forcing

Figure 3.3 shows the simulated response of the system in equation (3.1) for a system with O(1) excitation. Based on first-order perturbation analysis, the existence of superharmonic resonance at order one-half is expected for a system with direct and parametric excitation at the same frequency. Numerical simulations confirm this, but also show a peak at a third of the natural frequency. We thus proceed to conduct the second-order perturbation analysis of equation (3.1) with the specific goal of describing the superharmonic resonance of order 1/3.

Equation (3.1) is written with hard excitation and sinusoidally varying direct force as shown below:

$$\ddot{q} + 2\epsilon\mu\dot{q} + (\omega^2 + \epsilon\gamma\cos\Omega t)q = F\sin(\Omega t + \theta).$$
(4.1)

Employing MMS, we incorporate three time scales (T_0, T_1, T_2) , and allow for a dominant solution q_0 and small variations q_1 and q_2 , such that

$$q = q_0(T_0, T_1, T_2) + \epsilon q_1(T_0, T_1, T_2) + \epsilon^2 q_2(T_0, T_1, T_2) + \dots$$
(4.2)

where
$$T_i = \epsilon^i t$$
. Then $\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2$, where $D_i = \frac{\partial}{\partial T_i}$

We substitute this into equation (3.1) and then simplify and balance the coefficients of ϵ^0 , ϵ^1 and ϵ^2 :

$$O(1): \quad D_0^2 q_0 + \omega^2 q_0 = F \sin \Omega T_0$$

$$O(\epsilon): \quad D_0^2 q_1 + \omega^2 q_1 = -2\mu D_0 q_0 - 2D_0 D_1 q_0 - \gamma q_0 \cos \Omega T_0$$

$$O(\epsilon^2): \quad D_0^2 q_2 + \omega^2 q_2 = -2D_0 D_1 q_1 - (D_1^2 + 2D_0 D_2) q_0$$

$$-2\mu (D_0 q_1 + D_1 q_0) - \gamma q_1 \cos \Omega T_0$$
(4.3)

Solving the O(1) equation, we arrive at the solution of q_0 as

$$q_0 = A e^{i\omega T_0} - i\Lambda e^{i\Omega T_0} + c.c.$$
(4.4)

where $\Lambda = \frac{Fe^{i\theta}}{2(\omega^2 - \Omega^2)}$, $A = \frac{1}{2}ae^{i\beta}$, and *c.c.* means complex conjugate, and where ω differs from Ω (no primary resonance).

The coefficient *A*, and hence *a* and β , are functions of T_1 and T_2 . Substituting this into the $O(\epsilon)$ of equation (4.3), we arrive at the expression

$$D_0^2 q_1 + \omega^2 q_1 = -2\mu (Ai\omega e^{i\omega T_0} + \Lambda \Omega e^{i\Omega T_0}) - 2D_1 Ai\omega e^{i\omega T_0} -\frac{\gamma}{2} (A e^{i(\omega + \Omega)T_0} + \bar{A} e^{i(\Omega - \omega)T_0} - i\Lambda e^{2i\Omega T_0}) + c.c.$$

$$(4.5)$$

Here at first-order, $\Omega \approx \omega/2$ and $\Omega \approx 2\omega$ lead to secular terms. This was studied in the first-order expansions in [86]. But, $\Omega \approx \omega/3$ is not yet revealed as a combination that would lead to secular terms. As we are seeking that specific resonance condition, we proceed to find the solution of q_1 of the nonresonant case of equation (4.5).

4.1.1 Superharmonic Resonances at $\omega/3$

The solvability condition at $O(\epsilon)$ for the non-resonant condition is $-2\mu Ai\omega - 2i\omega D_1 A = 0$. Hence, the particular solution for q_1 is

$$q_{1} = -\frac{2\mu\Lambda\Omega}{(\omega^{2} - \Omega^{2})}e^{i\Omega T_{0}} + \frac{\gamma A}{2(\Omega^{2} + 2\omega\Omega)}e^{i(\Omega + \omega)T_{0}} + \frac{\gamma A}{2(\Omega^{2} - 2\omega\Omega)}e^{i(\Omega - \omega)T_{0}} + \frac{i\gamma\Lambda}{2(\omega^{2} - 4\Omega^{2})}e^{2i\Omega T_{0}} + c.c.$$

$$(4.6)$$

We substitute the solutions for q_0 and q_1 into the $O(\epsilon^2)$ expression in equation (4.3) and arrive at

$$D_{0}^{2}q_{2} + \omega^{2}q_{2} = -2\left(\frac{i\gamma(\Omega+\omega)D_{1}A}{2(\Omega^{2}+2\omega\Omega)}e^{i(\Omega+\omega)T_{0}} + \frac{i\gamma(\Omega-\omega)D_{1}\bar{A}}{2(\Omega^{2}-2\omega\Omega)}e^{i(\Omega-\omega)T_{0}}\right)$$

$$-(D_{1}^{2}Ae^{i\omega T_{0}} + 2i\omega D_{2}Ae^{i\omega T_{0}}) - 2\mu\left(-\frac{2i\Omega^{2}\mu\Lambda}{(\omega^{2}-\Omega^{2})}e^{i\Omega T_{0}} + \frac{i(\Omega+\omega)\gamma A}{2(\Omega^{2}+2\omega\Omega)}e^{i(\Omega+\omega)T_{0}}\right)$$

$$+\frac{i(\Omega-\omega)\gamma\bar{A}}{2(\Omega^{2}-2\omega\Omega)}e^{i(\Omega-\omega)T_{0}} - \frac{\Omega\gamma\Lambda}{(\omega^{2}-4\Omega^{2})}e^{2i\Omega T_{0}} + D_{1}Ae^{i\omega T_{0}}\right) - \gamma\left(-\frac{2\mu\Lambda\Omega}{(\omega^{2}-\Omega^{2})}e^{i\Omega T_{0}}\right)$$

$$+\frac{\gamma A}{2(\Omega^{2}+2\omega\Omega)}e^{i(\Omega+\omega)T_{0}} + \frac{\gamma\bar{A}}{2(\Omega^{2}-2\omega\Omega)}e^{i(\Omega-\omega)T_{0}} + \frac{i\gamma\Lambda}{2(\omega^{2}-4\Omega^{2})}e^{2i\Omega T_{0}}\right) + \frac{i\gamma\Lambda}{2(\omega^{2}-4\Omega^{2})}e^{2i\Omega T_{0}}\right)$$

$$+c.c.$$

We recognize that the terms in the differential equation for q_2 involve q_0 and q_1 differentiated over different time scales. Only the $\gamma q_1 \cos \Omega T_0$ term, i.e. the last term in equation (4.7), has interaction between the response and excitation. Expanding that term produces the exponential term $e^{3i\Omega T_0}$. This is secular when $3\Omega \approx \omega$, giving rise to the 1/3 superharmonic.

While a superharmonic resonance of order 1/3 is apparent, we note that the analysis does not reveal a subharmonic resonance of $\Omega \approx 3\omega$ in this linear forced Mathieu equation.

The solvability conditions up to $O(\epsilon^2)$ for $3\Omega \approx \omega$ are

$$O(\epsilon): -2\mu\omega Ai - 2D_1 Ai\omega = 0$$

$$O(\epsilon^{2}): -D_{1}^{2}A - 2i\omega D_{2}A - 2\mu D_{1}A - \frac{\gamma^{2}A}{4(\Omega^{2} + 2\omega\Omega)} - \frac{\gamma^{2}A}{4(\Omega^{2} - 2\omega\Omega)} - \frac{i\gamma^{2}A}{4(\omega^{2} - 4\Omega^{2})}e^{i\sigma T_{1}} = 0$$
(4.8)

where σ is the detuning parameter defined by $3\Omega = \omega + \epsilon \sigma$.

From this it is clear that the forced linear Mathieu equation has a one-third superharmonic, which required a second-order analysis to be captured. Furthermore, as evidenced from figure 3.3 the response is ϵ order lower than the peak at one-half. Analytical expressions for amplitude derived in later sections also corroborate this. In order to get an expression for A we need to solve the solvability conditions at $O(\epsilon)$ and $O(\epsilon^2)$ together. We do this by recombining the time scales in a process called reconstitution [74]. Eliminating D_1A and D_1^2A from the $O(\epsilon^2)$ equation, using our expression for Λ , and substituting $\Omega = \frac{\omega + \epsilon \sigma}{3}$, we can show that the resulting solvability conditions come from a multiple-scales expansion of

$$-2i\omega\frac{dA}{dt} - 2\epsilon i\omega\mu A + \epsilon^2 \left(\mu^2 A + \frac{9\gamma^2 A}{70\omega^2} - \frac{81i\gamma^2 F}{320\omega^4}e^{i\sigma\epsilon t}\right) = 0$$
(4.9)

We seek a solution in the form $A = (X + iY)e^{i\sigma\epsilon t}$, with real X and Y. We enforce this solution in equation (4.9), separate imaginary and real parts and cancel the common exponential term to obtain

$$2\omega \frac{dX}{dt} + 2\epsilon \sigma \omega Y - 2\epsilon \mu \omega X - \epsilon^2 \mu^2 Y - \epsilon^2 \frac{9\gamma^2 Y}{70\omega^2} + \epsilon^2 \frac{81\gamma^2 F}{320\omega^4} \cos\theta = 0$$
(4.10)

$$2\omega \frac{dY}{dt} + 2\epsilon \sigma \omega X - 2\epsilon \mu \omega Y + \epsilon^2 \mu^2 X + \epsilon^2 \frac{9\gamma^2 X}{70\omega^2} = 0$$
(4.11)

We have to solve for X and Y to get relations between γ , F, μ , ϵ , σ and ω . In a standard unforced Mathieu analysis, we would then seek a solution of the form $(X, Y) = (x, y)e^{kt}$ which, through k, would describe the stability of the zero solution. However, this is not appropriate with direct

forcing [74], as the origin is not a solution for the above set of equations. Instead, we can seek equilibrium solutions for X and Y equations (4.10) and (4.11).

Alternatively, we can write A in polar coordinates, i.e. we let $A = \frac{1}{2}ae^{i\beta}$ in equation (4.9), and separate real and imaginary parts to get

$$a\omega\dot{\beta} + \epsilon^2 \frac{\mu^2 a}{2} + \epsilon^2 \frac{9\gamma^2 a}{140\omega^2} + \epsilon^2 \frac{81\gamma^2 F}{320\omega^4} \sin(\sigma\epsilon t + \theta - \beta) = 0$$
(4.12)

$$-\omega\dot{a} - \epsilon\omega\mu a - \epsilon^2 \frac{81\gamma^2 F}{320\omega^4} \cos(\sigma\epsilon t + \theta - \beta) = 0$$
(4.13)

We make the system of equations (4.12) and (4.13) autonomous by letting $\sigma \epsilon t - \beta = \phi$ and correspondingly $\sigma \epsilon - \dot{\beta} = \dot{\phi}$ to get to expressions for \dot{a} and $\dot{\phi}$. For steady state solutions we set $\dot{a} = \dot{\phi} = 0$, and square and add the two equations to get

$$\left(a\sigma\epsilon + \frac{\epsilon^2\mu^2 a}{2\omega} + \frac{\epsilon^29\gamma^2 a}{140\omega^3}\right)^2 + (\epsilon\mu a)^2 = \left(\frac{\epsilon^281\gamma^2 F}{320\omega^5}\right)^2 \tag{4.14}$$

Equation (4.14) has the solution

$$a = \sqrt{\frac{q}{p}} \tag{4.15}$$

where

$$p = \left(\sigma + \frac{\epsilon \mu^2}{2\omega} + \frac{\epsilon^9 \gamma^2}{140\omega^3}\right)^2 + \mu^2$$

$$q = \left(\frac{\epsilon 81 \gamma^2 F}{320\omega^5}\right)^2$$
(4.16)

Hence, we have an expression for the amplitude of the system as a function of all its parameters.
Since p and q are positive, a solution exists over the entire parameter space. To find
$$a_{max}$$
 we differentiate a^2 from expression (4.15) with respect to σ and set $\frac{da^2}{d\sigma} = 0$. This yields

$$2\left(\sigma_{max} + \frac{\epsilon\mu^2}{2\omega} + \frac{\epsilon9\gamma^2}{140\omega^3}\right)a^2 = 0$$

From this we get the value of σ_{max} and the corresponding value of a_{max} as

$$\sigma_{max} = -\frac{\epsilon \mu^2}{2\omega} + \frac{\epsilon 9 \gamma^2}{140\omega^3}$$

$$a_{max} = \frac{\epsilon 81 \gamma^2 F}{320 \mu \omega^5}$$
(4.17)

The response amplitude is $a = 2\sqrt{X^2 + Y^2}$, and its stability can be ascertained from equations (4.10) and (4.11). Noting that these equations are in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}$, the equilibrium solution $\mathbf{x}_e = -\mathbf{A}^{-1}\mathbf{F}$ can be determined from the determinant and trace of \mathbf{A} . We find that if $\mu > 0$, the response at this resonance is stable.

4.1.2 Simulations and Discussions

Figure 4.1 shows a numerical plot of equation (4.14) showing the variation of amplitude with respect to the detuning parameter. The maximum amplitude in the plot is $\hat{a}_{max} = 0.23$ at $\hat{\sigma}_{max} = -0.07$, which agrees with the values obtained by using equations (4.17). We can approximate the total amplitude of oscillation by adding the amplitudes of the free and forced oscillation components, i.e. compute $|q| \approx |a| + |2\Lambda|$, which is approximate because of phase effects, and compare the response to the numerical simulations shown in figure 3.3. For the parameters shown in figure 3.3 (noting that $\hat{\mu} = 2\mu$, and figure 3.3 involved forcing amplitude at *F* compared to the current ϵF), $\hat{a}_{max} = 0.46$ is calculated from equation (4.17), which corresponds to the rise above the primary resonance curve. This is consistent with the peak seen at $\omega = 1/3$ for $\gamma = 3$ in figure 3.3.

Further numerical simulations were carried out varying the parameters that would make the system go unstable at primary resonance. However, if a system could operate below $\Omega \approx \omega$ excitation, such as with wind turbines, this instability would not be encountered. Figure 4.2 shows one such scenario. We notice the existence of multiple peaks. These can be correlated to either a harmonic frequency or the seeds of instability wedges of the unforced Mathieu equation, which occur near frequency ratios, $\Omega/\omega = 2/N$ (N = 1, 2, ...) in the frequency response curve [75]. It is well known that these instability wedges become more slender and weaker as we increase N in a typical Mathieu plot, and especially weaker in the presence of damping. As we can see from figure



Figure 4.1: Numerical evaluation of equation (4.14) shows the amplitude versus detuning curve. Graph shown is generated using $\mu = 0.25$, $\epsilon = 0.1$, F = 0.5, $\gamma = 3$, and $\theta = 0$. $a_{max} = 0.2278$ occurs at $\sigma = -0.07$.

4.2, the response near a ratio of 2/3 is distinct in the sense that the amplitude increases abruptly. This can be attributed to the presence of an instability wedge, such that the amplitude values are not steady-state, but are increasing during the time interval of the sweep. The lower-frequency instability wedges (1/2, 1/3, 2/5, 2/7, etc.) are not penetrated in this simulation. The responses at the other frequency ratios are due resonance phenomena (1/2, 1/3, etc.). In some cases, there exists both a resonance curve and an instability wedge. Future work will explore parameters that will make the system operate in such critical zones.

Figure 4.3 shows the response of the system when excited at frequencies higher than the natural frequency. We can clearly see the subharmonic instability occurring at twice the natural frequency, as there is an instability wedge at $\Omega = 2\omega$. The figure is plotted with weak parametric excitation to show the existence of a superharmonic at order 2. The superharmonic resonance conditions exists even for this case, but the magnitude of response in comparison to primary resonance and order 2 subharmonic is relatively small.



Figure 4.2: Simulated response of the linear case of equation (4.1) showing multiple superharmonic resonances $\mu = 0.05$, $\epsilon = 0.1$, F = 0.5, $\gamma = 7$, and $\theta = 0$. For the isolated superharmonic resonance at $\Omega/\omega \approx 1/3$, $a_{max} = 2.477$ by (4.17).



Figure 4.3: Simulated response of (4.1) showing subharmonic resonance at 2; $\mu = 0.05$, $\epsilon = 0.1$, F = 0.5, $\theta = 0$, and $\gamma = 0.5$. (Low parametric excitation is chosen so that the system doesn't show unstable response at primary resonance. For the given set of values, system is unstable at subharmonic resonance.)

4.2 Weak Excitation—Primary Resonance Amplification

In this section we perform second order analysis to uncover resonance conditions and instability wedges under soft excitation ($O(\epsilon)$ forcing). Our primary interest is in describing primary resonance amplification observed in simulations. However, we will take note of other resonances that may arise. We focus our attention back to the original equation (4.18) restated as

$$\ddot{q} + 2\epsilon\mu\dot{q} + (\omega^2 + \epsilon\gamma\cos\Omega t)q = \epsilon F\sin(\Omega t + \theta)$$
(4.18)

We follow the second-order expansion analysis done in the previous section to arrive at expressions for coefficients of ϵ^0 , ϵ^1 and ϵ^2 as

$$O(1): \quad D_0^2 q_0 + \omega^2 q_0 = 0$$

$$O(\epsilon): \quad D_0^2 q_1 + \omega^2 q_1 = -2\mu D_0 q_0 - 2D_0 D_1 q_0 - \gamma q_0 \cos \Omega T_0 + F \sin(\Omega T_0 + \theta)$$

$$O(\epsilon^2): \quad D_0^2 q_2 + \omega^2 q_2 = -2D_0 D_1 q_1 - (D_1^2 + 2D_0 D_2) q_0 - 2\mu (D_0 q_1 + D_1 q_0)$$

$$-\gamma q_1 \cos \Omega T_0$$

$$(4.19)$$

The solution of the O(1) equation is

$$q_0 = A(T_1, T_2)e^{i\omega T_0} + c.c.$$
(4.20)

where $A = \frac{1}{2}ae^{i\beta}$.

The differential equation for q_1 is then rewritten by substituting the solution for q_0 into the $O(\epsilon)$ equation as

$$D_0^2 q_1 + \omega^2 q_1 = -2i\omega D_1 A e^{i\omega T_0} - 2\mu i\omega A e^{i\omega T_0} -\gamma (\frac{A}{2} e^{i(\omega + \Omega)T_0} + \frac{\bar{A}}{2} e^{i(\Omega - \omega)T_0}) - \frac{iF}{2} e^{i(\Omega T_0 + \theta)} + c.c.$$
(4.21)

4.2.1 Weakly Forced Linear Mathieu Equation: Resonance Cases

We have three cases for equation (4.21) which can contribute secular terms:

- 1. No specific relationship between Ω and ω
- 2. $\Omega \approx \omega$
- 3. $\Omega \approx 2\omega$

The treatment of these cases will be laid out in the following sections. In each case, the secondorder expansions will give us the slow time scale variations of the amplitude A of our fast time scale solution q_0 . A second-order expansion will involve solvability conditions obtained by eliminating secular terms at both $O(\epsilon)$ and $O(\epsilon^2)$.

4.2.2 Case 1: No Resonance at $O(\epsilon)$

When there is no specific relationship between the forcing frequency Ω and the natural frequency ω , we remove secular terms by letting $-2i\omega D_1 A - 2\mu i\omega A = 0$, and solve for the particular solution using the remaining terms in differential equation (4.21), noting that we treat *A* as constant with respect to the independent variable T_0 . As such,

$$q_1 = \frac{\gamma A}{2(\Omega^2 + 2\omega\Omega)} e^{i(\Omega + \omega)T_0} + \frac{\gamma \bar{A}}{2(\Omega^2 - 2\omega\Omega)} e^{i(\Omega - \omega)T_0} + \frac{iFe^{i\theta}}{2(\Omega^2 - \omega^2)} e^{i\Omega T_0} + c.c.$$
(4.22)

We substitute the solutions for q_0 and q_1 into the equation (4.19) at $O(\epsilon^2)$. After expanding the resulting q_2 differential equation, we seek to identify terms that can contribute to a resonance condition. Although there were no resonances specified at $O(\epsilon)$, new resonances can be identified at $O(\epsilon^2)$. When there is not a specified relationship between Ω and ω at either $O(\epsilon)$ or $O(\epsilon^2)$, i.e. no resonances are specified, secular terms are removed and the resulting solvability conditions from both $O(\epsilon)$ and $O(\epsilon^2)$ are

$$O(\epsilon): -2i\omega D_1 A - 2\mu i\omega A = 0$$

$$O(\epsilon^2): -D_1^2 A - 2D_2 Ai\omega - 2\mu D_1 A - \frac{\gamma^2 A}{4(\Omega^2 + 2\omega\Omega)} - \frac{\gamma^2 A}{4(\Omega^2 - 2\omega\Omega)} = 0$$
(4.23)

Following references [74, 75], these equations can be analyzed for their fixed points and stability.

The removal of the secular terms in equation (4.19) at $O(\epsilon^2)$ leaves a differential equation for q_2 as

$$D_{0}^{2}q_{2} + \omega^{2}q_{2} = -2\left(i(\Omega + \omega)D_{1}Ke^{i(\Omega+\omega)T_{0}} + i(\Omega - \omega)D_{1}Le^{i(\Omega-\omega)T_{0}} + i\Omega D_{1}Ne^{i\Omega T_{0}}\right) - \left(D_{1}^{2}Ae^{i\omega T_{0}} + 2i\omega D_{2}Ae^{i\omega T_{0}}\right) - 2\mu\left(i(\Omega + \omega)Ke^{i(\Omega+\omega)T_{0}} + i(\Omega - \omega)Le^{i(\Omega-\omega)T_{0}} + i\Omega Ne^{i\Omega T_{0}} + D_{1}Ae^{i\omega T_{0}}\right) - \frac{\gamma}{2}\left(Ke^{i(2\Omega+\omega)T_{0}} + Ke^{i\omega T_{0}} + Le^{i(2\Omega-\omega)T_{0}} + Le^{-i\omega T_{0}} + Ne^{i2\Omega T_{0}} + N\right) + c.c.$$

$$(4.24)$$

where
$$K = \frac{\gamma A}{2(\Omega^2 + 2\omega\Omega)}$$
, $L = \frac{\gamma \overline{A}}{2(\Omega^2 - 2\omega\Omega)}$, and $N = \frac{iFe^{i\theta}}{2(\Omega^2 - \omega^2)}$, for the nonresonant case.

But there are also specific combinations of Ω and ω appearing at second order that could lead to resonance conditions in the system, and thereby add terms to the solvability condition at $O(\epsilon^2)$. These can be identified by examining the remaining terms of the $O(\epsilon^2)$ equation given by equation (4.24). We arrive at the following candidates:

a. $\Omega \approx \omega/2$

b. $\Omega \approx \omega$

c.
$$\Omega \approx 2\omega$$

The first case was treated with a first-order multiple scales analysis in [86], in which hard excitation induced secular terms at the $O(\epsilon)$ expansion. Similarly, in this case, the excitation, this time at $O(\epsilon)$, induced secular terms one order down at $O(\epsilon^2)$. Hence, similar results may be
expected, and so this case is not emphasized in this paper. The latter two cases actually produce secular terms at $O(\epsilon)$, and are therefore cannot be treated within Case 1, since it is a nonresonant case at $O(\epsilon)$.

Also, in this analysis, using weak forcing, we do not capture the linear superharmonic at order 1/3, as with weak forcing this superharmonic is buried "one-order lower". (If we were to continue our analysis further, i.e., do a third-order MMS expansion, we will encounter that resonance term.) The increasingly higher-order superharmonics show up at an increasingly lower order of ϵ . These superharmonics will occur at frequency ratios of 1/n, n being an integer. If the Mathieu system were weakly forced we would have to extend the perturbation analysis to n^{th} order expansions to capture corresponding effects. If a hard-forced Mathieu equation were considered, the order of MMS expansion to get a 1/n, 2/n order superharmonic would be $(n - 1)^{th}$. However, for higher orders of ϵ , the effect of superharmonic resonances become increasingly negligible.

4.2.3 Case 2: Primary Resonance and Parametric Amplification

When $\Omega \approx \omega$, i.e. letting $\Omega = \omega + \epsilon \sigma$ in equation (4.21), the solvability condition is

$$O(\epsilon): \qquad -2i\omega D_1 A - 2\mu i\omega A - \frac{iFe^{i\theta}}{2}e^{i\sigma T_1} = 0 \qquad (4.25)$$

The particular solution for q_1 then becomes

$$q_1 = \frac{\gamma A}{2(\Omega^2 + 2\omega\Omega)} e^{i(\Omega + \omega)T_0} + \frac{\gamma \bar{A}}{2(\Omega^2 - 2\omega\Omega)} e^{i(\Omega - \omega)T_0} + c.c.$$
(4.26)

We substitute the solution for q_0 and q_1 into the equation (4.19) at $O(\epsilon^2)$. After expanding the right-hand side of the resulting q_2 differential equation, we seek to identify secular terms at this resonance condition.

For the case of primary resonance, we let $\Omega = \omega + \epsilon \sigma$ in equation (4.19) and eliminate secular terms to get

$$O(\epsilon^{2}): -D_{1}^{2}A - 2D_{2}Ai\omega - 2\mu D_{1}A - \frac{\gamma^{2}A}{4(\Omega^{2} + 2\omega\Omega)} - \frac{\gamma^{2}A}{4(\Omega^{2} - 2\omega\Omega)} - \frac{\gamma^{2}\bar{A}}{4(\Omega^{2} - 2\omega\Omega)}e^{2i\sigma T_{1}} = 0$$
(4.27)

Rewriting the $O(\epsilon)$ equation in equation (4.25), we have

$$D_1 A = -\mu A - \frac{F e^{i\theta} e^{i\sigma T_1}}{4\omega}$$

We compute $D_1^2 A$ from the above expression as

$$D_1^2 A = \mu^2 A + \frac{\mu F e^{i\theta}}{4\omega} e^{i\sigma T_1} - \frac{i\sigma F e^{i\theta}}{4\omega} e^{i\sigma T_1}.$$

We substitute these into the $O(\epsilon^2)$ expression in equation (4.27) to arrive at

$$-2D_{2}Ai\omega - \mu^{2}A - \frac{\mu F e^{i\theta}}{4\omega} e^{i\sigma T_{1}} + \frac{i\sigma F e^{i\theta}}{4\omega} e^{i\sigma T_{1}} - 2\mu \left(-\mu A - \frac{F e^{i\theta} e^{i\sigma T_{1}}}{4\omega}\right) -\frac{\gamma^{2}A}{4(\Omega^{2} + 2\omega\Omega)} - \frac{\gamma^{2}A}{4(\Omega^{2} - 2\omega\Omega)} - \frac{\gamma^{2}\bar{A}}{4(\Omega^{2} - 2\omega\Omega)} e^{2i\sigma T_{1}} = 0$$

$$(4.28)$$

Following the approach in [74], the $O(\epsilon)$ solvability condition from equation (4.27) and the $O(\epsilon^2)$ solvability condition from equation (4.28) result from a multiple-scales expansion of

$$-2i\omega\frac{dA}{dt} + \epsilon \left(-2\mu i\omega A - \frac{iFe^{i\theta}}{2}e^{i\sigma\epsilon t}\right) + \epsilon^{2}\left(\mu^{2}A + \frac{\gamma^{2}\bar{A}}{4\omega^{2}}e^{2i\sigma\epsilon t} + \frac{\gamma^{2}A}{6\omega^{2}} + \frac{i\sigma Fe^{i\theta}}{4\omega}e^{i\sigma\epsilon t} + \frac{\mu Fe^{i\theta}}{4\omega}e^{i\sigma\epsilon t}\right) = 0$$

$$(4.29)$$

Transforming the system to polar coordinates leads to steady-state equations in *a* and $\phi = \sigma T_1 - \beta$. The steady-state equations include $\sin(\phi), \cos(\phi)$ and $\sin(2\phi), \cos(2\phi)$ terms, which make it more difficult to solve than other familiar examples.

Instead we follow the solution procedure given in [74]. We seek a solution in the form $A = (X + iY)e^{i\sigma\epsilon t}$, with real X and Y. We enforce this solution in equation (4.29), separate imaginary and real parts and cancel the common exponential term to obtain

$$2\omega \frac{dX}{dt} + 2\epsilon\mu\omega X - (2\epsilon\omega\sigma + \epsilon^2\mu^2 - \frac{\epsilon^2\gamma^2}{12\omega^2})Y = \epsilon\alpha, \qquad (4.30)$$

$$2\omega \frac{dY}{dt} + (2\epsilon\omega\sigma + \epsilon^2\mu^2 + \frac{5\epsilon^2\gamma^2}{12\omega^2})X + 2\epsilon\mu\omega Y = \epsilon\delta, \qquad (4.31)$$

where $\epsilon \alpha = -\frac{\epsilon F}{2} \cos \theta + \frac{\epsilon^2 F \sigma}{4\omega} \cos \theta + \frac{\epsilon^2 F \mu}{4\omega} \sin \theta$ and $\epsilon \delta = -\frac{\epsilon F}{2} \sin \theta + \frac{\epsilon^2 F \sigma}{4\omega} \sin \theta - \frac{\epsilon^2 F \mu}{4\omega} \cos \theta$.

At steady state $\dot{X} = \dot{Y} = 0$. Hence from equations (4.31) and (4.30) we get

$$2\epsilon\mu\omega X + -(2\epsilon\omega\sigma + \epsilon^{2}\mu^{2} - \frac{\epsilon^{2}\gamma^{2}}{12\omega^{2}})Y = \epsilon\alpha,$$

$$(2\epsilon\omega\sigma + \epsilon^{2}\mu^{2} + \frac{5\epsilon^{2}\gamma^{2}}{12\omega^{2}})X + 2\epsilon\mu\omega Y = \epsilon\delta.$$
(4.32)

4.2.3.1 Parametric Amplification

We solve (4.32) for X and Y, and then write the square of the total amplitude $r^2 = (X^2 + Y^2)$ as

$$r^{2} = \frac{\left[2\omega\mu\alpha + \delta\left(2\omega\sigma + \epsilon\mu^{2} - \frac{\epsilon\gamma^{2}}{12\omega^{2}}\right)\right]^{2}}{\left[(2\omega\mu)^{2} + \left(2\omega\sigma + \epsilon\mu^{2} - \frac{\epsilon\gamma^{2}}{12\omega^{2}}\right)\left(2\omega\sigma + \epsilon\mu^{2} + \frac{5\epsilon\gamma^{2}}{12\omega^{2}}\right)\right]^{2}} + \frac{\left[2\omega\mu\delta - \alpha\left(2\omega\sigma + \epsilon\mu^{2} + \frac{5\epsilon\gamma^{2}}{12\omega^{2}}\right)\right]^{2}}{\left[(2\omega\mu)^{2} + \left(2\omega\sigma + \epsilon\mu^{2} - \frac{\epsilon\gamma^{2}}{12\omega^{2}}\right)\left(2\omega\sigma + \epsilon\mu^{2} + \frac{5\epsilon\gamma^{2}}{12\omega^{2}}\right)\right]^{2}}$$

$$(4.33)$$

Equation (4.33) provides information *related* to the amplitude of the primary resonance as a function of parameters, including the detuning parameter σ and the phase angle θ between the parametric and direct excitation terms. Note that we had $A = \frac{1}{2}ae^{i\beta} = X + iY = re^{i\phi}$, and as such the response amplitude is a = 2r. Since α and δ depend on θ , the response amplitude is also dependent on θ .

Looking, for example, at the special case of $\theta = 0$, we have

$$r_{0}^{2} = \frac{\left[\left(2\omega\sigma + \epsilon\mu^{2} - \frac{\epsilon\gamma^{2}}{12\omega^{2}}\right)\left(\frac{-\epsilon F\mu}{4\omega}\right) - \left(\frac{F}{2} - \frac{\epsilon F\sigma}{4\omega}\right)(2\omega\mu)\right]^{2}}{\left[\left(2\omega\mu\right)^{2} + \left(2\omega\sigma + \epsilon\mu^{2} - \frac{\epsilon\gamma^{2}}{12\omega^{2}}\right)\left(2\omega\sigma + \epsilon\mu^{2} + \frac{5\epsilon\gamma^{2}}{12\omega^{2}}\right)\right]^{2}} + \frac{\left[\left(2\omega\mu\right)\left(\frac{-\epsilon F\mu}{4\omega}\right) + \left(\frac{F}{2} - \frac{\epsilon F\sigma}{4\omega}\right)\left(2\omega\sigma + \epsilon\mu^{2} + \frac{5\epsilon\gamma^{2}}{12\omega^{2}}\right)\right]^{2}}{\left[\left(2\omega\mu\right)^{2} + \left(2\omega\sigma + \epsilon\mu^{2} - \frac{\epsilon\gamma^{2}}{12\omega^{2}}\right)\left(2\omega\sigma + \epsilon\mu^{2} + \frac{5\epsilon\gamma^{2}}{12\omega^{2}}\right)\right]^{2}}\right]^{2}}$$

$$(4.34)$$

In the limit as $\gamma \to 0$, $\sigma \to 0$ and for small ϵ (such that we can neglect second order ϵ terms), then in equation (4.34) $r_0 = \frac{F}{2\omega\mu}$. With $r_0 = a/2$, $\epsilon\mu = \zeta\omega$ and recognizing that the excitation in equation (4.18) is ϵF , this amounts to $a = \frac{\epsilon F}{2\zeta\omega^2}$, which is consistent with the forced linear oscillator at resonance.

Equation (4.34) provides a predicted response amplitude as a function of system parameters for the 1:1 primary resonance of the linear forced Mathieu equation with damping. Figure 4.4 shows the variation of amplitude as a function of parametric excitation γ for a given set of parameter values. The response amplitude predictions agree with numerical simulations of the linear Mathieu equation given by equation (3.1).

Figure 4.4 shows how the response amplitude (a = 2r) at primary resonance varies with increasing γ for various values of σ . The response amplitude is nearly uniform for a range of small values of parametric excitation. In this range of γ , there is no significant parametric amplification, and the response levels at $\gamma = 0$ in figure 4.4 are consistent with those of simple forced linear oscillators. For larger values of γ , the steady-state amplitude increases hyperbolically with increasing γ and becomes unbounded when γ reaches a critical value, γ_{cr} . The extension or stretch in the amplitude is known as *parametric amplification*.

Figure 4.6 shows the variation in the amplitude of the response versus the detuning parameter, σ . In this case, the peak occurs at $\sigma = -0.01$, and the associated peak amplitude for the given



Figure 4.4: Variation of maximum amplitude of vibration at primary resonance as a function of parametric excitation γ for various values of σ . (Left) Excitation frequency is detuned to be slightly less than natural frequency. (Right) Excitation frequency is detuned to be slightly more than natural frequency. Parameters are F = 5, $\mu = 0.1$ and $\epsilon = 0.1$.



Figure 4.5: (Left) Numerical simulations show that the predicted parametric resonance can have errors when the amplification becomes large as γ reaches critical values. (Right) Graphed is zoomed in to show that numerical and analytical results match for moderate levels of parametric excitation.

parameter set is a = 2r = 10 for the case of $\gamma = 0$ (no parametric excitation).

Inspection of equation (4.33), including the special case of equation (4.34) for $\theta = 0$, reveals that denominator *D* is a fourth-degree polynomial in σ , and the numerator *N* is quadratic in σ . Therefore, to find the value of σ_{max} for the maximum amplitude of the resonant response from $d(r^2)/d\sigma = (N'D - D'N)/D^2 = 0$ involves finding roots of a fifth-degree polynomial in the



Figure 4.6: (Left) The numerical evaluation of equation (4.34) showing the amplitude versus the detuning. Graph shown was generated using $\mu = 0.25$, $\epsilon = 0.1$, F = 5, γ ranges from 0 to 4 in 0.5 increments; as γ increases *a* increases. (Right) The numerical evaluation of equation (4.34) for $\gamma = 3$ showing the amplitude versus the detuning at various orders of ϵ expansion. Note that the curves with $O(\epsilon)$ (dotted line) and $O(\epsilon^2)$ (dashed line) expansions are visibly erroneous.

numerator. So we will not present an analytical expression for σ_{max} or r_{max} .

However, Figures 4.4 and 4.6 suggest that the peak value may come close to $\sigma = 0$, at least for some parameter sets with small values of γ (this deviates as γ increases). An expression for r^2 at $\sigma = 0$ for this approximation of the peak value can be expressed. As such,

$$r_{00}^{2} = \frac{\left[-\left(\epsilon\mu^{2} - \frac{\epsilon\gamma^{2}}{12\omega^{2}}\right)\frac{\epsilon F\mu}{4\omega} - \omega\mu F\right]^{2} + \left[\left(\frac{-\epsilon F\mu^{2}}{2}\right) + \frac{F}{2}\left(\epsilon\mu^{2} + \frac{5\epsilon\gamma^{2}}{12\omega^{2}}\right)\right]^{2}}{\left[(2\omega\mu)^{2} + \left(\epsilon\mu^{2} - \frac{\epsilon\gamma^{2}}{12\omega^{2}}\right)\left(\epsilon\mu^{2} + \frac{5\epsilon\gamma^{2}}{12\omega^{2}}\right)\right]^{2}}$$
(4.35)

The amplifier gain can be approximated by taking the ratio of amplitudes r_{00} as predicted by equation (4.35) and the associated value for the simple harmonic oscillator, i.e. $a_h = \frac{F}{2\mu\omega^2}$. In the gain expression, the force level *F* cancels out. For general phases, the expression r^2/a_h can be evaluated at $\sigma = 0$ and plotted versus θ to show how the amplifier gain depends on the phasing of direct and parametric excitations, as done with previous works on subharmonic parametric amplification [61, 62].

Since this response is predicted with small-parameter asymptotic theory, it is likely that the

predicted amplitude values in the large-response range have some error. This is shown in the plots of figure 4.5. Next, we will see how the unbounded response relates to stability.

4.2.3.2 Stability of Primary Resonance

Equations (4.31) and (4.30) can be rewritten in the form

$$2\omega \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F},\tag{4.36}$$

where, in the case of $\theta = 0$,

$$\mathbf{x} = \begin{bmatrix} X \\ Y \end{bmatrix}, \mathbf{F} = \begin{bmatrix} -\frac{\epsilon F}{2} + \frac{\epsilon^2 F \sigma}{4\omega} \\ -\frac{\epsilon^2 F \mu}{4\omega} \end{bmatrix}$$

and
$$\mathbf{A} = \begin{bmatrix} -2\epsilon\mu\omega & 2\epsilon\omega\sigma + \epsilon^2\mu^2 - \frac{\epsilon^2\gamma^2}{12\omega^2} \\ -2\epsilon\omega\sigma - \epsilon^2\mu^2 - \frac{5\epsilon^2\gamma^2}{12\omega^2} & -2\epsilon\mu\omega \end{bmatrix}$$

The equilibrium $\mathbf{x}_0 = \mathbf{A}^{-1}\mathbf{F}$ was found previously, and its amplitude *r* was studied. Since equation (4.36) is linear, dynamics of *X* and *Y*, and hence the leading-order slowly varying response amplitude *A*, about the equilibrium, and hence the stability, are governed by matrix **A**. In addition, the homogeneous solution has its own equilibrium at the origin. The homogeneous solution is that for which F = 0 in equation (4.18), and consequently $\mathbf{F} = \mathbf{0}$ in equation (4.36). Thus, matrix **A** simultaneously governs the stability of the homogeneous and particular solutions of equation (4.18). The stability is determined by the trace (*T*) and determinant (*D*) of **A**, which are

$$T = -4\epsilon\mu\omega$$

$$D = 4\epsilon^{2}\omega^{2}(\mu^{2} + \sigma^{2}) + 4\epsilon^{3}\mu^{2}\omega\sigma + \frac{2\epsilon^{3}\gamma^{2}\sigma}{3\omega} + \epsilon^{4}\mu^{4} - \frac{5\epsilon^{4}\gamma^{4}}{144\omega^{4}}$$
(4.37)

Since $\epsilon > 0, \mu > 0$ and $\omega > 0$ the trace T < 0. Therefore, the stability is governed by the determinant *D*. The instability transition is given by parameters for which D = 0. Close examination of equation (4.34) shows that the denominator of the r^2 expression is in fact D^2 ,



Figure 4.7: Stability wedge for the linear forced Mathieu equation with damping at primary resonance $((\omega + \epsilon \sigma)^2 = \delta = N^2/4, N = 2$ for traditional Mathieu equation). Figure shows comparison of stability of numerical system response/amplitude and the analytically calculated stability boundary.

consistent with the denominator of A^{-1} . Thus, the instability criterion for the forced response coincides with the approach of r^2 to infinity, and with the instability of x_h of the unforced Mathieu equation. Solving D = 0 for γ yields the critical parametric excitation as

$$\gamma_{cr} = \sqrt{\frac{24\omega^2}{5\epsilon}(2\omega\sigma + \epsilon\mu^2) \pm \frac{12\omega^2}{5\epsilon}\sqrt{9(2\omega\sigma + \epsilon\mu^2)^2 + 5(2\mu\omega)^2}}$$
(4.38)

The calculated value of γ_{cr} is the value of γ in figure 4.4 for which the amplitude becomes unbounded for specific values of detuning parameter. The increase in amplitude until that point is a result of parametric amplification, analytically approximated by equation (4.34). The analytically predicted values of γ_{cr} are plotted versus σ in figure 4.7. Numerical simulations were also checked for stability in this parameter space, and stable (x's) and unstable (dots) events are plotted along with the predicted stability boundary from equation (4.38) with good agreement.

4.2.3.3 Summary of Primary Parametric Resonance

We have seen that the linear forced Mathieu equation has a homogeneous and particular solution. The homogeneous solution has the behavior previously studied traditionally by Floquet theory [71], perturbation analysis [74], or Floquet-motivated harmonic balance [75]. The particular solution is a forced response with the potential for a primary resonance, which was analyzed in this section by multiple scales. A second-order analysis shows that the primary resonance is stretched, or amplified, by the parametric excitation. Given the parameter set, as the level of parametric excitation γ grows, the resonance amplitude is at first hardly affected by small γ , but then begins to grow quickly for larger γ , and approaches infinity at a critical γ_{cr} . The bounded resonance response is stable. As the response becomes infinite at γ_{cr} , the response destabilizes. At the same time, the homogeneous part of the response also becomes unstable, as the parameters cross into the unstable part of the Arnold tongue.

4.2.3.4 Cautionary Note on Second-Order Expansions in ϵ

It may be tempting to expand r^2 into powers of ϵ consistent with the level of analysis for the reconstituted differential equation, for example to powers of ϵ^2 in a second-order analysis, as in [81]. This would lead us to an expression for r^2 from a geometric series expansion:

$$r_{approx}^{2} = \frac{F^{2}}{16\omega^{2}(\mu^{2} + \sigma^{2})} + \epsilon \frac{\left(F^{2}\gamma^{2}\sigma - 24F^{2}\mu^{2}\omega^{2}\sigma - 12F^{2}\omega^{2}\sigma^{3}\right)}{192\omega^{5}(\mu^{2} + \sigma^{2})^{2}} + \epsilon^{2}\left(\left(35F^{2}\gamma^{4}\mu^{2} - 120F^{2}\gamma^{2}\mu^{4}\omega^{2} + 3F^{2}\gamma^{4}\sigma^{2} - 264F^{2}\gamma^{2}\mu^{2}\omega^{2}\sigma^{2} + 1440F^{2}\mu^{4}\omega^{4}\sigma^{2}\right) - 48F^{2}\gamma^{2}\omega^{2}\sigma^{4} + 1008F^{2}\mu^{2}\omega^{4}\sigma^{4} + 144F^{2}\omega^{4}\sigma^{6}\right)\right) \left| \left(9216\omega^{8}\left(\mu^{2} + \sigma^{2}\right)^{3}\right) + O(\epsilon^{3})\right| \right|$$

where all terms up to $O(\epsilon^2)$ are kept in the analysis. However, terms of order ϵ^3 and higher are dropped from the expansion of equation (4.34) in ϵ . In the limit as $\epsilon \to 0$, $\sigma \to 0$, equation (4.39) is consistent with the common forced harmonic oscillator.

Quantitative predictions for r^2 values computed from equation (4.39) can be quite erroneous. This can be seen by comparing the values for amplitude obtained by equation (4.34) and (4.39) to the values obtained numerically in figure 3.3 (recalling that $\hat{\mu} = 2\mu$, and figure 3.3 involved forcing amplitude at *F* compare to the current ϵF). For example, for $\mu = 0.25$, $\epsilon = 0.1$, F = 5, $\gamma = 3$ and $\sigma = 0$ the amplitude obtained from figure 3.3 at primary resonance ($\Omega \approx \omega$) is |q| = 15. The analytical calculations from equations (4.34) and (4.39) for amplitude are |q| = 13.95 and $|q_{approx}| = 13.3$ respectively (insight from figure 4.6 suggests some error due to $\sigma = 0$ not σ_{max}). Furthermore, the transition to unbounded responses as predicted by the denominator of the simplified equation (4.39) deviates from that plotted in figure 4.7 from equation (4.38) and the stability data from simulations. Thus the truncated expression including up to $O(\epsilon^2)$ gives an erroneous result.

This indicates that the process of truncating to the 2^{nd} power of ϵ in effort to achieve ϵ -expansion consistency with the multiple scales expansion is incorrect. Instead, it seems that the reconstituted equation (4.29) has information of a second-order expansion, and the approximate steady-state solution (4.34) uses all of this information. The truncated expression (4.39) loses information and is more quickly divergent from the true solution. Thus, both the results of the amplitude expression and the stability boundary indicate that we need to consider all powers of ϵ while solving the reconstituted equation (4.29). We follow the same approach while solving for response amplitude and stability boundaries at other resonance conditions.

4.2.4 Case 3: Subharmonic Resonance of Order Two

This case was examined in the previous first-order multiple scales analysis [86]. It is possible that a second-order analysis may improve the asymptotic expressions somewhat, and so we sketch its analysis. When $\Omega \approx 2\omega$, i.e. $\Omega = 2\omega + \epsilon\sigma$, the solvability condition for equation (4.21) is

$$O(\epsilon): -2i\omega D_1 A - 2\mu i\omega A - \frac{\gamma \bar{A}}{2}e^{i\sigma T_1} = 0$$
(4.40)

The particular solution of equation (4.21) for q_1 then becomes

$$q_{1} = \frac{\gamma A}{2(\Omega^{2} + 2\omega\Omega)}e^{i(\Omega + \omega)T_{0}} + \frac{iF}{2(\Omega^{2} - \omega^{2})}e^{i\Omega T_{0}} + c.c.$$
(4.41)

Applying to equation (4.19) at $O(\epsilon^2)$, and removing secular terms, leaves us with a solvability condition as

$$O(\epsilon^{2}): -D_{1}^{2}A - 2D_{2}Ai\omega - 2\mu D_{1}A - \frac{\gamma^{2}A}{4(\Omega^{2} + 2\omega\Omega)} = 0.$$
(4.42)

We combine equations (4.40) and (4.42) and reconstitute to get an ODE

$$-2i\omega\frac{dA}{dt} + \epsilon \left(-2\mu i\omega A - \frac{\gamma\bar{A}}{2}e^{i\sigma\epsilon t}\right) + \epsilon^2 \left(\mu^2 A + \frac{\gamma\sigma\bar{A}}{4\omega}e^{i\sigma\epsilon t} - \frac{\gamma^2 A}{32\omega^2}\right) = 0$$
(4.43)

We follow the steps laid out in the previous section to arrive at the expressions for trace (T) and determinant (D) for order 2 subharmonic, from the \dot{X} and \dot{Y} that govern the stability of the forced resonant response, and likewise for the homogeneous solution. The trace and determinant are given by

$$T = -4\epsilon\omega\mu$$

$$D = (2\epsilon\omega\mu)^{2} + \left(2\epsilon\sigma\omega + \epsilon^{2}\mu^{2} - \frac{3\epsilon^{2}\gamma^{2}}{32\omega^{2}}\right)^{2} - \left(\frac{\epsilon\gamma}{2} - \frac{\epsilon^{2}\sigma\gamma}{4\omega}\right)^{2}$$
(4.44)

The expression for *D* is quadratic in γ^2 which when equated to zero gives the stability boundaries of the order-2 subharmonic under hard excitation. These agree with detailed analyses in the existing literature [74,75] and we do not present the details of this analysis. The expressions for the resonance under weak excitation also appear at an order lower than the subharmonic found using first-order expansions with hard excitation in [86].

4.3 Effect of a Direct Forcing Term *F*₀

The previous section we have considered a sinusoidal forcing term and parametric excitation at the same frequency. The motivation for the study of this equation arises from the simplifications of the wind turbine blade equation of motion formulation. The mean wind loading on a blade represents a constant direct forcing term F_0 whose magnitude with respect to the harmonic forcing is dependent on the direction of motion of the blade (in plane versus out of plane). In this section, we look at the fundamental dynamics of equation (3.1) with the constant force term. As such, the Mathieu equation with constant loading is

$$\ddot{q} + 2\epsilon\mu\dot{q} + (\omega^2 + \epsilon\gamma\cos\Omega t)q = F_0. \tag{4.45}$$

We follow the second-order procedure established in sections 4.1 and 4.2 to arrive at expressions for amplitude of resonance at primary and other harmonic resonance conditions as a function of system parameters. The multiple scales expansions of equation (4.45) are

$$O(1): \quad D_0^2 q_0 + \omega^2 q_0 = F_0$$

$$O(\epsilon): \quad D_0^2 q_1 + \omega^2 q_1 = -2\mu D_0 q_0 - 2D_0 D_1 q_0 - \gamma q_0 \cos \Omega T_0$$

$$O(\epsilon^2): \quad D_0^2 q_2 + \omega^2 q_2 = -2D_0 D_1 q_1 - (D_1^2 + 2D_0 D_2) q_0$$

$$-2\mu (D_0 q_1 + D_1 q_0) - \gamma q_1 \cos \Omega T_0$$
(4.46)

The solution of the O(1) equation for q_0 is

$$q_0 = A e^{i\omega T_0} + \kappa + c.c. (4.47)$$

where $\kappa = \frac{F_0}{2\omega^2}$.

Substituting this into the $O(\epsilon)$ of equation (4.46), we arrive at the expression

$$D_{0}^{2}q_{1} + \omega^{2}q_{1} = -2D_{1}Ai\omega e^{i\omega T_{0}} - 2\mu(Ai\omega e^{i\omega T_{0}}) -\frac{\gamma}{2}(Ae^{i(\omega+\Omega)T_{0}} + \bar{A}e^{i(\Omega-\omega)T_{0}} - \kappa e^{i\Omega T_{0}}) + c.c.$$
(4.48)

We can immediately identify two values of Ω which contribute secular terms ($\Omega \approx \omega$ and $\Omega \approx 2\omega$). For the case when there is not a specific relation between the frequencies of parametric excitation and natural frequency, the Ω equation assumes the form

$$D_{0}^{2}q_{2} + \omega^{2}q_{2} = -2\left(i(\Omega + \omega)D_{1}Ke^{i(\Omega+\omega)T_{0}} + i(\Omega - \omega)D_{1}Le^{i(\Omega-\omega)T_{0}} + i\Omega D_{1}Ne^{i\Omega T_{0}}\right) - \left(D_{1}^{2}Ae^{i\omega T_{0}} + 2i\omega D_{2}Ae^{i\omega T_{0}}\right) - 2\mu\left(i(\Omega + \omega)Ke^{i(\Omega+\omega)T_{0}} + i(\Omega - \omega)Le^{i(\Omega-\omega)T_{0}} + i\Omega Ne^{i\Omega T_{0}} + D_{1}Ae^{i\omega T_{0}}\right) - \frac{\gamma}{2}\left(Ke^{i(2\Omega+\omega)T_{0}} + Ke^{i\omega T_{0}} + Le^{i(2\Omega-\omega)T_{0}} + Le^{-i\omega T_{0}} + Ne^{i2\Omega T_{0}} + N\right) + c.c.$$

$$(4.49)$$

where $K = \frac{\gamma A}{2(\Omega^2 + 2\omega\Omega)}$, $L = \frac{\gamma \overline{A}}{2(\Omega^2 - 2\omega\Omega)}$, and $N = \frac{\gamma F_0}{4(\Omega^2 - \omega^2)}$, for the nonresonant case. There is no contribution from F_0 to the secular terms at $O(\epsilon^2)$ in the non-resonant case.

The resonant cases and the associated secular terms for two orders of multiple scale expansions are given below (also referred to as solvability conditions).

<u>Case 1</u>: $\Omega = \omega + \epsilon \sigma$

$$O(\epsilon): -2i\omega D_1 A - 2\mu i\omega A - \frac{\kappa\gamma}{2}e^{i\sigma T_1} = 0$$

$$O(\epsilon^2): -D_1^2 A - 2D_2 Ai\omega - 2\mu D_1 A - \frac{\gamma^2 A}{4(\Omega^2 + 2\omega\Omega)} - \frac{\gamma^2 A}{4(\Omega^2 - 2\omega\Omega)} = 0$$
(4.50)

<u>Case 2</u>: $\Omega = 2\omega + \epsilon \sigma$

$$O(\epsilon): -2i\omega D_1 A - 2\mu i\omega A - \frac{\gamma \bar{A}}{2}e^{i\sigma T_1} = 0$$

$$O(\epsilon^2): -D_1^2 A - 2D_2 Ai\omega - 2\mu D_1 A - \frac{\gamma^2 A}{4(\Omega^2 + 2\omega\Omega)} = 0$$

$$(4.51)$$

Case 3: $\Omega = \omega/2 + \epsilon \sigma$

$$O(\epsilon): -2i\omega D_1 A - 2\mu i\omega A = 0$$

$$O(\epsilon^{2}): -D_{1}^{2}A - 2D_{2}Ai\omega - 2\mu D_{1}A - \frac{\gamma^{2}A}{4(\Omega^{2} + 2\omega\Omega)} - \frac{\gamma^{2}A}{4(\Omega^{2} - 2\omega\Omega)} - \frac{\kappa\gamma^{2}}{4(\Omega^{2} - 2\omega\Omega)}e^{i\sigma T_{1}} = 0$$
(4.52)



Figure 4.8: Amplitudes of simulated responses of equation (4.45) showing unstable response at subharmonic resonance due to increase of the parametric forcing amplitude; $\epsilon = 0.1, \mu = 0.25, F_0 = 2$. Different curves depict $\gamma = 0.5$ and 1.



Figure 4.9: Amplitudes of simulated responses of equation (4.45) showing the occurrence of superharmonic resonances at high parametric forcing amplitude; $\epsilon = 0.1$, $\mu = 0.25$, $F_0 = 2$. Different curves depict $\gamma = 2$ and 4.



Figure 4.10: Trend of amplitudes of simulated responses of equation (4.45) at different critical frequencies. Order 2 subharmonic goes unstable at significantly lower values of γ and hence is not shown.

Other superharmonic resonance conditions are revealed at higher orders of expansion. For example, the third-order expansion reveals resonance cases at $\Omega \approx \omega/3$ and $\Omega \approx 2\omega/3$. The existence of resonances and instability zones is governed by the solutions to the solvability conditions for specific combinations of Ω and ω . The set of equations for the various cases above are set up in the same way as were the systems of equations for solvability conditions for the harmonically forced equation (following the procedure detailed in sections 4.1 and 4.2). Next, we take an example case and sketch out the solution to verify the predictions with numerical plots.

4.3.1 Primary Resonance

After reconstituting the solvability conditions at orders ϵ and ϵ^2 in Equations (4.50), we have

$$-2i\omega\frac{dA}{dt} + \epsilon \left(-2\mu i\omega A - \frac{\gamma\kappa}{2}e^{i\sigma\epsilon t}\right) + \epsilon^2 \left(\mu^2 A + \frac{\kappa\gamma\sigma}{4\omega}e^{i\sigma\epsilon t} - \frac{i\mu\kappa\gamma}{4\omega}e^{i\sigma\epsilon t} + \frac{\gamma^2 A}{6\omega^2}\right) = 0.$$
(4.53)

We let $A = (X + iY)e^{i\epsilon\sigma t}$ and then split the equation (4.53) into its imaginary and real parts following the process established in previous sections to arrive at

$$-2\omega\frac{dX}{dt} + \epsilon(-2\mu\omega X + 2\omega\sigma Y) + \epsilon^2(\mu^2 Y - \frac{\gamma\kappa\mu}{4\omega} + \frac{\gamma^2 Y}{6\omega^2}) = 0, \qquad (4.54)$$

$$2\omega \frac{dY}{dt} + \epsilon (2\mu\omega Y + 2\omega\sigma X - \frac{\gamma\kappa}{2}) + \epsilon^2 (\mu^2 X + \frac{\gamma\kappa\sigma}{4\omega} + \frac{X\gamma^2}{6\omega^2}) = 0.$$
(4.55)

At steady state $\dot{X} = \dot{Y} = 0$. Hence rearranging equations (4.31) and (4.30) we get

$$-2\epsilon\mu\omega X + (2\epsilon\omega\sigma + \epsilon^{2}\mu^{2} + \frac{\epsilon^{2}\gamma^{2}}{6\omega^{2}})Y = \frac{\epsilon^{2}\kappa\mu\gamma}{4\omega}$$

$$(4.56)$$

$$(2\epsilon\omega\sigma + \epsilon^{2}\mu^{2} + \frac{\epsilon^{2}\gamma^{2}}{2})X + 2\epsilon\mu\omega Y = \frac{\epsilon\gamma\kappa}{4\omega} - \frac{\epsilon^{2}\kappa\gamma\sigma}{4\omega}$$

 $r^2 = (X^2 + Y^2)$ as

$$r^{2} = \frac{\left[\left(\frac{\gamma\kappa}{2} - \frac{\epsilon\kappa\gamma\sigma}{4\omega}\right)\left(2\omega\sigma + \epsilon\mu^{2} + \frac{\epsilon\gamma^{2}}{6\omega^{2}}\right) - \left(\frac{\epsilon\kappa\mu\gamma}{4\omega}\right)(2\omega\mu)\right]^{2}}{\left[\left(2\omega\mu\right)^{2} + \left(2\omega\sigma + \epsilon\mu^{2} + \frac{\epsilon\gamma^{2}}{6\omega^{2}}\right)^{2}\right]^{2}} + \frac{\left[\left(\frac{\epsilon\kappa\mu\gamma}{4\omega}\right)\left(2\omega\sigma + \epsilon\mu^{2} + \frac{\epsilon\gamma^{2}}{6\omega^{2}}\right) + \left(\frac{\gamma\kappa}{2} - \frac{\epsilon\kappa\gamma\sigma}{4\omega}\right)(2\omega\mu)\right]^{2}}{\left[\left(2\omega\mu\right)^{2} + \left(2\omega\sigma + \epsilon\mu^{2} + \frac{\epsilon\gamma^{2}}{6\omega^{2}}\right)^{2}\right]^{2}}$$

$$(4.57)$$

From the expression for r^2 for the case of primary resonance (given by equation (4.57)) it is clear that at two orders of expansion for the differential equation prescribed by (4.45) there is no instability realized. This is due to the fact that the expression for the determinant as given by the denominator of expression for r in equation (4.57) is always positive. The values for amplitude at primary resonance as obtained from equation (4.57) are in agreement with the numerical simulations of equation (4.45). The analytical prediction for amplitude at $\gamma = 1$ from equation (4.57) is in agreement with the values from numerical simulations as shown via figure 4.10.

4.3.2 Subharmonic Resonance

We follow the same procedure to show the existence of instability at the sub-harmonic resonance for equation (4.45). The expression for amplitude r^2 and the corresponding trace and determinant for the reconstituted equations from the solvability conditions at various orders of expansion are as follows:

$$r^{2} = \frac{\left[\left(-\frac{\epsilon\kappa\gamma\sigma}{4\omega}\right)\left(2\omega\sigma + \frac{\gamma}{2} + \epsilon\mu^{2} + \frac{\epsilon\gamma^{2}}{6\omega^{2}}\right) - \left(\frac{\epsilon\kappa\mu\gamma}{4\omega}\right)(2\omega\mu)\right]^{2}}{\left[\left(2\omega\mu\right)^{2} + \left(2\omega\sigma + \epsilon\mu^{2} + \frac{\epsilon\gamma^{2}}{6\omega^{2}}\right)^{2} - \left(\frac{\gamma}{2}\right)^{2}\right]^{2}} + \frac{\left[\left(\frac{\epsilon\kappa\mu\gamma}{4\omega}\right)\left(2\omega\sigma - \frac{\gamma}{2} + \epsilon\mu^{2} + \frac{\epsilon\gamma^{2}}{6\omega^{2}}\right) - \left(\frac{\epsilon\kappa\gamma\sigma}{4\omega}\right)(2\omega\mu)\right]^{2}}{\left[\left(2\omega\mu\right)^{2} + \left(2\omega\sigma + \epsilon\mu^{2} + \frac{\epsilon\gamma^{2}}{6\omega^{2}}\right)^{2} - \left(\frac{\gamma}{2}\right)^{2}\right]^{2}}$$

$$(4.58)$$

$$T = -4\epsilon\mu\omega$$

$$D = (2\omega\mu)^2 + \left(2\omega\sigma + \epsilon\mu^2 + \frac{\epsilon\gamma^2}{6\omega^2}\right)^2 - \left(\frac{\gamma}{2}\right)^2$$
(4.59)

From equation (4.59) we can compute a critical value for parametric resonance, beyond which the system will be unstable at the $\Omega \approx 2\omega$ subharmonic, as

$$\gamma_{cr} = \left(\frac{18\omega^2}{\epsilon} \left(\frac{1}{4} - \frac{\epsilon}{3\omega^2} (2\omega\sigma + \epsilon\mu^2)\right) + \sqrt{\left(\frac{1}{4} - \frac{\epsilon}{3\omega^2} (2\omega\sigma + \epsilon\mu^2)\right)^2 - \frac{\epsilon^2}{9\omega^4} (4\mu^2\omega^2 + 2\omega\sigma + \epsilon\mu^2)}\right)^{1/2}$$
(4.60)

Beyond the critical value, the system response would be unstable.

4.3.3 Superharmonic Resonance

We cannot reveal instabilities with the case of $\Omega = \omega/2$ superharmonic at this level. The expressions for amplitude at this level are in accord with two orders of expansion, but, to analyze the critical values at which the system would become unstable, we would have to perform higher order expansions. As such, the instability wedges at these frequencies are very slender and would not be excited for "reasonable" values of system parameters. As we can see in figure 4.9, if we keep increasing the value of parametric excitation, lower order superharmonics also show-up in the response plots. However, if we were to continue increasing γ at a certain point, the system could hit unstable wedges corresponding to various frequencies and the response would become unstable. In this work, we restrict analysis to the dominant harmonics that are observed for reasonable system values.

The response predictions for systems with direct excitation in combination with the sinusoidal excitation, exacerbate the response at the resonance conditions thus increasing the likelihood of resonant depending on the relative magnitude of the terms. We showcased the comparison of amplitudes at primary resonance for this condition to check the accuracy of predictions. By superposition, the response of equation (3.1) would be a combination of the response of equations (4.18) and (4.45), thus increasing the likelihood of large amplitude oscillations.

4.4 Effect of Nonlinearity on Mathieu Resonances and Instabilities

Typically while analyzing the Mathieu equation we look for stability transition curves in the $\epsilon - \delta$ (i.e. magnitude of parametric forcing - frequency) space. The transition curves are the Arnold tongues/Mathieu wedges. In our analysis of the weakly forced linear Mathieu equation, we sought to reconstruct the Arnold tongue in a similar space and study the effect of direct forcing on the system. Addition of nonlinearity will impact the stable and unstable solutions arrived at through the linear analysis. From [75] we know that there are supercritical and subcritical bifurcations that take place for an unforced nonlinear Mathieu equation as we vary parameters in the $\epsilon - \delta$ space.

Here we apply second-order multiple scales to the nonlinear Mathieu equation (3.1). Inserting the expansion (4.2) into equation (3.1) where $\hat{\mu} = 2\mu$, and equating coefficients of like powers of ϵ yields

$$O(1): \quad D_0^2 q_0 + \omega^2 q_0 = 0$$

$$O(\epsilon): \quad D_0^2 q_1 + \omega^2 q_1 = -2\mu D_0 q_0 - 2D_0 D_1 q_0 - \gamma q_0 \cos \Omega T_0 - \alpha q_0^3 + F \sin \Omega T_0$$

$$O(\epsilon^2): \quad D_0^2 q_2 + \omega^2 q_2 = -2D_0 D_1 q_1 - (D_1^2 + 2D_0 D_2) q_0 - 2\mu (D_0 q_1 + D_1 q_0)$$

$$-\gamma q_1 \cos \Omega T_0 - 3\alpha q_0^2 q_1$$
(4.61)

The solution of O(1) equation is

$$q_0 = A(T_1, T_2)e^{i\omega T_0} + c.c.$$
(4.62)

We substitute this into the $O(\epsilon)$ equation and identify other resonance conditions to eliminate secular terms and seek the solution of q_1 .

The equation for q_1 obtained by substituting the solution for q_0 into $O(\epsilon)$ equation is

$$D_{0}^{2}q_{1} + \omega^{2}q_{1} = -2i\omega D_{1}Ae^{i\omega T_{0}} - 2\mu i\omega Ae^{i\omega T_{0}} - \alpha (A^{3}e^{3i\omega T_{0}} + 3A^{2}\bar{A}e^{i\omega T_{0}}) -\gamma (\frac{A}{2}e^{i(\omega+\Omega)T_{0}} + \frac{\bar{A}}{2}e^{i(\Omega-\omega)T_{0}}) - \frac{iF}{2}e^{i\Omega T_{0}} + c.c.$$
(4.63)

We have three cases for equation (4.63) which can contribute towards resonance condition.

- 1. No specific relation between Ω and ω
- 2. $\Omega \approx \omega$
- 3. $\Omega \approx 2\omega$

<u>Case 1</u>: When there is no specific relationship between the forcing frequency Ω and the natural frequency ω we equate the secular terms to zero, such that

$$-2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \bar{A} = 0$$

and solve the remaining ODE in equation (4.63) to get the particular solution, by treating A as constant with respect to the independent variable T_0 , as

$$q_{1} = \frac{\alpha A^{3}}{8\omega^{2}}e^{3i\omega T_{0}} + \frac{\gamma A}{2(\Omega^{2} + 2\omega\Omega)}e^{i(\Omega + \omega)T_{0}} + \frac{\gamma \bar{A}}{2(\Omega^{2} - 2\omega\Omega)}e^{i(\Omega - \omega)T_{0}} + \frac{iF}{2(\Omega^{2} - \omega^{2})}e^{i\Omega T_{0}} + c.c.$$

$$(4.64)$$

<u>Case 2</u>: For the second condition of when $\Omega \approx \omega$ i.e. $\Omega = \omega + \epsilon \sigma_1$ (σ_1 being the detuning parameter for this case) from equation (4.63),

$$-2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \overline{A} - \frac{iF}{2}e^{i\sigma_1 T_1} = 0$$

forms the solvability condition. The particular solution for q_1 then becomes

$$q_1 = \frac{\alpha A^3}{8\omega^2} e^{3i\omega T_0} + \frac{\gamma A}{2(\Omega^2 + 2\omega\Omega)} e^{i(\Omega + \omega)T_0} + \frac{\gamma \bar{A}}{2(\Omega^2 - 2\omega\Omega)} e^{i(\Omega - \omega)T_0} + c.c.$$
(4.65)

<u>Case 3</u>: And finally, when $\Omega \approx 2\omega$ i.e. $\Omega = 2\omega + \epsilon \sigma_2$ (σ_2 being the detuning parameter for this case)

$$-2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \overline{A} - \frac{\gamma \overline{A}}{2}e^{i\sigma_2 T_1} = 0$$

forms the solvability condition. The particular solution for q_1 then becomes

$$q_1 = \frac{\alpha A^3}{8\omega^2} e^{3i\omega T_0} + \frac{\gamma A}{2(\Omega^2 + 2\omega\Omega)} e^{i(\Omega + \omega)T_0} + \frac{iF}{2(\Omega^2 - \omega^2)} e^{i\Omega T_0} + c.c.$$
(4.66)

We now substitute the appropriate solutions for q_0 and q_1 into the expression (4.61) at $O(\epsilon^2)$. After expanding the terms in the expression for the q_2 ODE, we seek to identify terms that can contribute to a resonance condition.

For Case 1 when there is no relation between Ω and ω , the expression for q_2 is:

$$D_{0}^{2}q_{2} + \omega^{2}q_{2} = -2\left(i(\Omega + \omega)D_{1}Ke^{i(\Omega+\omega)T_{0}} + i(\Omega - \omega)D_{1}Le^{i(\Omega-\omega)T_{0}} + 3i\omega D_{1}Me^{3i\omega T_{0}} + i\Omega D_{1}Ne^{i\Omega T_{0}}\right) - \left(D_{1}^{2}Ae^{i\omega T_{0}} + 2i\omega D_{2}Ae^{i\omega T_{0}}\right) - 2\mu\left(i(\Omega + \omega)Ke^{i(\Omega+\omega)T_{0}} + i(\Omega - \omega)Le^{i(\Omega-\omega)T_{0}} + 3i\omega Me^{3i\omega T_{0}} + i\Omega Ne^{i\Omega T_{0}} + D_{1}Ae^{i\omega T_{0}}\right) - \frac{\gamma}{2}\left(Ke^{i(2\Omega+\omega)T_{0}} + Ke^{i\omega T_{0}} + Le^{i(2\Omega-\omega)T_{0}} + Le^{-i\omega T_{0}} + Me^{i(\Omega+3\omega)T_{0}} + Me^{i(-\Omega+3\omega)T_{0}} + Ne^{i2\Omega T_{0}} + N\right) - 3\alpha\left(A^{2}Ke^{i(\Omega+3\omega)T_{0}} + A^{2}Ke^{i(\Omega-\omega)T_{0}} + 2A\bar{A}Ke^{i(\Omega+\omega)T_{0}} + A^{2}Le^{i(\Omega+\omega)T_{0}} + A^{2}Le^{i(\Omega-3\omega)T_{0}} + A^{2}Me^{5i\omega T_{0}} + \bar{A}^{2}Me^{i\omega T_{0}} + 2A\bar{A}Me^{3i\omega T_{0}} + A^{2}Ne^{i(\Omega+2\omega)T_{0}} + A^{2}Ne^{i(\Omega+2\omega)T_{0}} + A^{2}Ne^{i(\Omega+2\omega)T_{0}} + C.$$

$$(4.67)$$

where
$$K = \frac{\gamma A}{2(\Omega^2 + 2\omega\Omega)}$$
, $L = \frac{\gamma A}{2(\Omega^2 - 2\omega\Omega)}$, $M = \frac{\alpha A^3}{8\omega^2}$, $N = \frac{iF}{2(\Omega^2 - \omega^2)}$.

Following the analysis done in [74] for the unforced, linear case, we write the solvability conditions together for various resonance cases. Once we enlist the solvability conditions for each of the resonance case identified, we look for the fixed points and stability for each case separately.

For the case of no specific relationship between ω and Ω

$$O(\epsilon): -2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \bar{A} = 0$$

$$O(\epsilon^{2}): -D_{1}^{2}A - 2D_{2}Ai\omega - 2\mu D_{1}A - \frac{3\alpha^{2}A^{3}\bar{A}^{2}}{8\omega^{2}} - \frac{\gamma^{2}A}{4(\Omega^{2} + 2\omega\Omega)} - \frac{\gamma^{2}A}{4(\Omega^{2} - 2\omega\Omega)} = 0$$
(4.68)

There are various other combinations of Ω and ω appearing at second-order that could lead to resonance conditions in the system. Examining the $O(\epsilon^2)$ equation given by equation (4.67) for the general case we arrive at the following combinations:

- 1. $\Omega \approx 3\omega$
- 2. $\Omega \approx \omega/2$

3. $\Omega \approx 4\omega$

The other resonance combinations identified at $O(\epsilon)$ i.e.

4.
$$\Omega \approx \omega$$

5.
$$\Omega \approx 2\omega$$

lead to slightly different expressions for an ODE in q_2 based on the expression of q_1 computed for each case and given in equations (4.65) and (4.66). We examine the resulting expression for solvability conditions at $O(\epsilon)$.

For each of these conditions, the corresponding equations at $O(\epsilon)$ and $O(\epsilon^2)$ are given below.

 $\Omega\approx 3\omega$

$$O(\epsilon): \quad -2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \bar{A} = 0$$

$$O(\epsilon^{2}): -D_{1}^{2}A - 2D_{2}Ai\omega - 2\mu D_{1}A - \frac{3\alpha^{2}A^{3}\bar{A}^{2}}{8\omega^{2}} - \frac{\gamma^{2}A}{4(\Omega^{2} + 2\omega\Omega)} - \frac{\gamma^{2}A}{4(\Omega^{2} - 2\omega\Omega)} - \frac{i3\alpha F\bar{A}^{2}}{2(\Omega^{2} - \omega^{2})}e^{i\sigma T_{1}} = 0$$
(4.69)

 $\Omega \approx \omega/2$

$$O(\epsilon): -2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \bar{A} = 0$$

$$O(\epsilon^{2}): -D_{1}^{2}A - 2D_{2}Ai\omega - 2\mu D_{1}A - \frac{3\alpha^{2}A^{3}\bar{A}^{2}}{8\omega^{2}} - \frac{\gamma^{2}A}{4(\Omega^{2} + 2\omega\Omega)} - \frac{\gamma^{2}A}{4(\Omega^{2} - 2\omega\Omega)} - \frac{\gamma Fi}{4(\Omega^{2} - \omega^{2})}e^{i\sigma T_{1}} = 0$$
(4.70)

 $\Omega\approx 4\omega$

$$O(\epsilon): \quad -2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \bar{A} = 0$$

$$O(\epsilon^{2}): -D_{1}^{2}A - 2D_{2}Ai\omega - 2\mu D_{1}A - \frac{3\alpha^{2}A^{3}\bar{A}^{2}}{8\omega^{2}} - \frac{\gamma^{2}A}{4(\Omega^{2} + 2\omega\Omega)}$$

$$-\frac{\gamma^{2}A}{4(\Omega^{2} - 2\omega\Omega)} - \frac{3\alpha\gamma\bar{A}^{3}}{2(\Omega^{2} - 2\omega\Omega)}e^{i\sigma T_{1}} - \frac{\gamma\alpha\bar{A}^{3}}{16\omega^{2}}e^{i\sigma T_{1}} = 0$$

$$(4.71)$$

 $\Omega\approx\omega$

$$O(\epsilon): -2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \overline{A} - \frac{iF}{2}e^{i\sigma T_1} = 0$$

$$O(\epsilon^{2}): -D_{1}^{2}A - 2D_{2}Ai\omega - 2\mu D_{1}A - \frac{3\alpha^{2}A^{3}\bar{A}^{2}}{8\omega^{2}} - \frac{\gamma^{2}A}{4(\Omega^{2} + 2\omega\Omega)} - \frac{\gamma^{2}\bar{A}}{4(\Omega^{2} - 2\omega\Omega)} - \frac{\gamma^{2}\bar{A}}{4(\Omega^{2} - 2\omega\Omega)}e^{2i\sigma T_{1}} = 0$$
(4.72)

 $\Omega\approx 2\omega$

$$O(\epsilon): -2i\omega D_1 A - 2\mu i\omega A - 3\alpha A^2 \bar{A} - \frac{\gamma \bar{A}}{2} e^{i\sigma T_1} = 0$$

$$O(\epsilon^2): -D_1^2 A - 2D_2 A i\omega - 2\mu D_1 A - \frac{3\alpha^2 A^3 \bar{A}^2}{8\omega^2} - \frac{\gamma^2 A}{4(\Omega^2 + 2\omega\Omega)}$$

$$-\frac{3\alpha\gamma A \bar{A}^2}{2(\Omega^2 + 2\omega\Omega)} e^{i\sigma T_1} = 0$$
(4.73)

The σ 's above are the detuning parameter in each case respectively.

In this analysis, using weak forcing, we do not capture the linear superharmonic at order 1/3. This happens because of the way the terms and the bookkeeping parameter ϵ appear in the equations. The terms are multiplied at varying orders, thus burying the effect of that superharmonic "one-order lower". If we were to continue our analysis further we will encounter that resonance term.

The solutions for the set of equations given for each resonance case above would give us the slow time scale variations of the amplitude A of our fast time scale solution q. In order to arrive at a solution for A we re-combine the terms at $O(\epsilon)$, $O(\epsilon^2)$ into a single ODE and seek its solution. To arrive at this juncture would require elimination of D_1A terms from $O(\epsilon^2)$ equation.

We sketch the treatment of one of the resonant cases listed. We consider the terms for the primary resonance case $\Omega \approx \omega$ given in equation (4.72). From the $O(\epsilon)$ equation we get,

$$D_1 A = -\mu A + \frac{3\alpha i A^2 \bar{A}}{2\omega} - \frac{F e^{i\sigma T_1}}{4\omega}$$

and

$$D_1\bar{A} = -\mu\bar{A} - \frac{3\alpha i\bar{A}^2A}{2\omega} - \frac{Fe^{-i\sigma T_1}}{4\omega}$$

We compute $D_1^2 A$ from the above expressions as

$$D_1^2 A = \mu^2 A - \frac{6\alpha i\mu A^2 \bar{A}}{\omega} - \frac{9\alpha^2 A^3 \bar{A}^2}{4\omega^2} + \frac{F\mu}{4\omega} e^{i\sigma T_1} - \frac{Fi\sigma}{4\omega} e^{i\sigma T_1} - \frac{3\alpha FiA\bar{A}}{4\omega^2} e^{i\sigma T_1} - \frac{3\alpha FiA^2}{8\omega^2} e^{-i\sigma T_1} + \frac{3\alpha FiA\bar{A}}{8\omega^2} e^{-i\sigma T$$

We substitute these in the $O(\epsilon^2)$ expressions in equation (4.72) to arrive at

$$-2D_{2}Ai\omega + \left(-\mu^{2}A + \frac{6\alpha i\mu A^{2}\bar{A}}{\omega} + \frac{9\alpha^{2}A^{3}\bar{A}^{2}}{4\omega^{2}} - \frac{F\mu}{4\omega}e^{i\sigma T_{1}} + \frac{Fi\sigma}{4\omega}e^{i\sigma T_{1}} + \frac{3\alpha FiA\bar{A}}{4\omega^{2}}e^{i\sigma T_{1}} \right) + \frac{3\alpha FiA^{2}}{8\omega^{2}}e^{-i\sigma T_{1}}\right) - 2\mu\left(-\mu A + \frac{3\alpha iA^{2}\bar{A}}{2\omega} - \frac{Fe^{i\sigma T_{1}}}{4\omega}\right) - \frac{3\alpha^{2}A^{3}\bar{A}^{2}}{8\omega^{2}} - \frac{\gamma^{2}A}{4(\Omega^{2} + 2\omega\Omega)} - \frac{\gamma^{2}\bar{A}}{4(\Omega^{2} - 2\omega\Omega)} - \frac{\gamma^{2}\bar{A}}{4(\Omega^{2} - 2\omega\Omega)}e^{2i\sigma T_{1}} = 0$$

$$(4.74)$$

It can easily be shown that the $O(\epsilon)$ solvability condition from equation (4.72) and the $O(\epsilon^2)$ solvability condition from equation (4.74) result from a multiple-scales expansion of

$$-2i\omega\frac{dA}{dt} + \epsilon \left(-2\mu i\omega A - 3\alpha A^2 \bar{A} - \frac{iF}{2}e^{i\sigma\epsilon t}\right) + \epsilon^2 \left(\mu^2 A + \frac{3\alpha FiA^2}{8\omega^2}e^{-i\sigma\epsilon t} + \frac{\gamma^2 \bar{A}}{4\omega^2}e^{2i\sigma\epsilon t} - \frac{9\alpha i\mu A^2 \bar{A}}{\omega} + \frac{\gamma^2 A}{6\omega^2} + \frac{15\alpha^2 A^3 \bar{A}^2}{32\omega^2} + \frac{3\alpha FiA\bar{A}}{4\omega^2}e^{i\sigma\epsilon t} + \frac{Fi\sigma}{4\omega}e^{i\sigma\epsilon t} + \frac{F\mu}{4\omega}e^{i\sigma\epsilon t}\right) = 0$$

$$(4.75)$$

Transforming the system to polar coordinates, leads to steady state equations in *a* and $\phi = \sigma T_1 - \beta$. The steady state equations include $\sin(\phi), \cos(\phi)$ and $\sin(2\phi), \cos(2\phi)$ terms, which make it more difficult to solve than other familiar examples.

We follow the solution procedure given in [74] and detailed in the previous chapter. We seek a solution in the form $A = (B_r + iB_i)e^{i\sigma\epsilon t}$, with real B_r and B_i . We enforce this solution in equation (4.75), separate real and imaginary parts and cancel the common exponential term to obtain,

$$2\omega \frac{dB_{i}}{dt} + \epsilon \left(2\mu\omega B_{i} + 2\omega\sigma B_{r} - 3\alpha (B_{r}^{3} + B_{r}B_{i}^{2})\right) + \epsilon^{2} \left(\mu^{2}B_{r} - \frac{3\alpha F}{4\omega^{2}}B_{r}B_{i} + \frac{\gamma^{2}}{4\omega^{2}}B_{r}\right) \\ + \frac{\gamma^{2}}{6\omega^{2}}B_{r} + \frac{9\alpha\mu}{\omega}(B_{i}^{3} + B_{r}^{2}B_{i}) + \frac{15\alpha^{2}}{32\omega^{2}}(B_{r}^{5} + B_{i}^{4}B_{r} + 2B_{i}^{2}B_{r}^{3}) + \frac{F\mu}{4\omega}\right) = 0,$$

$$-2\omega \frac{dB_{r}}{dt} + \epsilon \left(-2\mu\omega B_{r} + 2\omega\sigma B_{i} - 3\alpha (B_{i}^{3} + B_{i}B_{r}^{2})\right) + \frac{\epsilon F}{2} \left(1 + \epsilon \left(\frac{3\alpha}{4\omega^{2}}(B_{r}^{2} - B_{i}^{2})\right) + \frac{\sigma}{2\omega^{2}}\right) + \epsilon^{2} \left(-\frac{\gamma^{2}B_{i}}{4\omega^{2}} + \mu^{2}B_{i} + \frac{\gamma}{6\omega^{2}}B_{i} - \frac{9\alpha\mu}{\omega}(B_{r}^{3} + B_{i}^{2}B_{r}) + \frac{15\alpha^{2}}{32\omega^{2}}(B_{i}^{5} + B_{r}^{4}B_{i})$$

$$+ 2B_{r}^{2}B_{i}^{3}\right) = 0.$$

$$(4.76)$$

As stated in the previous section, we have to solve for B_r and B_i to get relations between γ , F, μ , ϵ , σ and ω . The fifth degree terms still pose a challenge. If we proceed with enforcing polar coordinate forms to the B_i , B_r terms in the equation then we are faced with $\sin(\phi)$, $\cos(\phi)$ and $\sin(2\phi)$, $\cos(2\phi)$ terms.

Similar analysis has been done for all of the resonant conditions that exist in the nonlinear system. Here, as was in the case with the linear system, we are left with a differential equation with higher order polynomial terms. The origin is not a solution for this set of equation as we have direct forcing.

Having obtained these equations, we look for stability characteristics based on system parameters. This will require a lot more analysis before we can draw conclusions on stabilities and the parameters that dictate behavior. The influence of parametric excitation on a linear version of equations (4.76) and (4.77) has been demonstrated in the previous section. The completed form of the expressions for the nonlinear system will show the explicit dependence on the amplitude on parametric excitation and the nature of the stability of the numerous operating points that arise as steady-state solutions of the system. The second order analysis will also provide expressions for the local and global stability of the system. The current work, lays the foundation for constructing the stability transition curves and analyzing parameter zones where in the solution would either Table 4.1: Stability Wedge and Resonance Chart. R_1 : Resonance identified at 1^{*st*} order of MMS expansion. R_2 : Resonance identified at 2^{*nd*} order of MMS expansion. W_2 : Instability wedge (Arnold tongue) expression can be found at second order of MMS expansion. -: Known resonance case/ Instability not uncovered up to two orders of expansion

Forcing (Ω)	O(1) Forcing	$O(\epsilon)$ Forcing	No Forcing
2ω	R_1, W_2	<i>R</i> ₂	R_1, W_2
ω	R_1, W_2	R_1, W_2	R_2, W_2
$2\omega/3$	-	-	-
$\omega/2$	R_1, W_2	R_1, W_2	-
$2\omega/5$	-	_	_
$\omega/3$	R_2, W_2	-	-

destabilize or be resonant. Both would lead to sustained oscillation in the system thereby causing increased loading in our wind turbine system which has been modeled using these equations. Table 4.1 lists the frequency ratios at which resonances or wedges have been identified.

4.5 Summary

The main findings reported in this chapter show that a linear forced Mathieu equation has a homogenous and particular solution. When direct and parametric excitations are of the same frequency, the system can exhibit superharmonic resonances, subharmonic resonances, and primary resonance. A second-order perturbation analysis uncovers superhamonics at orders 1/3 and 1/2, and subharmonic instability at order 2. It also reveals a primary resonance parametric amplification, which becomes unbounded at a critical value of parametric excitation level γ_{cr} , at which the homogenous solution also becomes unstable. When the direct forcing is a constant load, the system can have superhrmonic, subharmonic, and primary resonances.

CHAPTER 5

APPLICATION TO WIND TURBINE BLADES

5.1 Introduction

The most essential feature of the wind turbines are the blades, which harness the power from the wind. The blades are expected to be in operation through the entire design life and often undergo extreme loading. Environmental (wind and other), electrical (grid) and soil conditions (foundation) affect loading, durability and operation of wind turbines and hence should be taken into account in the design process [12]. Each external environmental condition is typically classified into a normal or extreme operating condition with typical occurrence intervals stated for design purpose. For example, certifications of wind turbine blades consider operation of the equipment not only in normal and turbulent wind conditions but also in extreme conditions such as gusts and direction changes. Despite these efforts wind turbine blade failures are common and their maintenance requires significant capital expenditure. As these blades grow larger in size, the dynamic loads experienced is not well understood. In chapters 2, 3 and 4 we developed the equations of motion and perturbation analysis necessary to capture and predict the behavior of the system of equations that define the dynamic motion of these systems. In this chapter, we extend the application to actual data from wind turbine blades.

5.2 Numerical Simulations of ODE's

In this section, we address the response of the system described by equations (2.8) and (2.16). These describe the vibration motion of the wind turbine blade and account for nonlinearities and variations in geometry along the length of the blade. We dealt with a few terms in these equations that would dominate the response of the system. These were obtained by assuming small modal coordinates (i.e. $q = \epsilon \hat{q}$) and eliminating terms that have higher orders of ϵ . Since the cross-coupled terms contribute at least a 2^{nd} or 3^{rd} power of ϵ they can be safely ignored in the calculations of



Figure 5.1: Response amplitude of system described by equation (2.8) that described in-plane motion of the blade. Values used in simulation $b = 0.1, d = 0.1, e_1 = 1, e_2 = 0.3, f_1 = 0.1, f_2 = 0.1, F = 1$. To account for damping we add a $2\mu\dot{q}$ term with $\mu = 0.25$

the resonances and stabilities for the forced nonlinear Mathieu representation of this system.

Figure 5.1 shows the response of the system described by the ODE in equation (2.8). The superharmonic responses at orders 1/3 and 1/2 are those that have been captured in the simplification of this ODE. Thus, the fundamental behavior of the system is governed by the forced nonlinear Mathieu equation. This also holds true for the out-of-plane resonance case. We then proceed to quantify motion for actual wind turbine blade parameters. Before we do this, we need a representative aerodynamic force that accounts for external loads due to the wind. We apply the BEM theory to arrive at an expression for aerodynamic load. Commercial codes like FAST have also been used to assess blade vibration, tip deflection, etc.

5.3 Aerodynamic Forces on the Blade

We estimate the aerodynamic loads under the following assumptions:

• Constant angle of attack

- Steady wind
- Constant rate of rotation
- Linear wind shear
- Uniform wind direction

Blade element momentum theory gives a reasonably good first estimates for aerodynamic loading on the blades for analysis from a dynamic system perspective. The magnitude of forces computed using this method are only used to get a coarse estimate as it relies on two key assumptions; (i) there are non interactions between the different blade elements and (ii) forces on the blade elements are determined by the lift and drag forces on the undeformed blade. More detailed models accounting for dynamic stall or wake vortices may contribute additional terms and would have to be considered depending on the problem definition. However, for resonance analysis, the BEM theory provides reasonably good lift and drag forces along the length of the blade.

The lift force (L) and drag force (D) per unit length is:

$$L = \frac{1}{2}\rho c W^2 C_L$$

$$D = \frac{1}{2}\rho c W^2 C_D$$
(5.1)

where *c* is chord length, ρ is air density, *W* is relative velocity at blade, C_L and C_D are coefficients of lift and drag. Since we are interested in the forces acting normal to and tangential to the plane of rotation (out-of-plane and in-plane respectively) the lift and drag forces can be projected in these directions (shown as F_N and F_T in figure 5.2). The aerodynamic loads for out-of-plane and in-plane blade motion can thus be written as:

$$F_N = L\cos\phi + D\sin\phi$$

$$F_T = L\sin\phi - D\cos\phi$$
(5.2)

where L and D are from equation (5.1).



Figure 5.2: The local forced on an aerosol structure of the blade, reproduced from [3]

The information on the coefficients of lift and drag are required to calculate the in-plane and out-of-plane loading and these depend on the angle of attack, α . Further, from figure 5.3, we can write the relation between mean free stream wind velocity U_{∞} and angle of attack as

$$\tan\phi = \frac{(1-a)U_{\infty}}{(1+a')\omega r}$$
(5.3)

where a and a' are axial and tangential induction factors. From equations (5.1), (5.2) and (5.3) we can write expression for aerodynamic load at every section of the blade and compute the aerodynamic loading over the entire blade. Typically, estimation of a and a' requires an initial guess to calculate the aerodynamic forces and iterating through the solution at every step until a and a' do not change beyond a defined tolerance. This standard procedure is described in detail in [43]. We follow the procedure laid out there for an example blade in the next section.

The initial estimate for the aerodynamic forces from the BEM theory can be improved by adding blade solidity ratio, wind shear, tip loss correction factors, etc. We incorporate wind shear into this formulation. Wind shear is the variation of mean wind speed as we move vertically along the height of the wind turbine. The mean wind speed can vary significantly along the length of the blade as the blade rotates through the rotor plane. The mean wind speed at certain height is given



Figure 5.3: Velocities at the rotor plane, reproduced from [3]

with respect to the mean wind speed at height 10 m above ground by

$$U = U_{10} \left(\frac{h}{h_{10}}\right)^{\kappa}$$

where κ is based on the terrain and typically varies from 0.01 to 0.7 depending on the location of wind turbine installations. This curve can also be approximated by simpler function for each blade as

$$U(h) = U_{base} + h * w^+$$

where w^+ is the approximate rise in wind speed for an increase in height *h*. Tower shadowing also causes a cyclic excitation for each tower pass. There are numerous models that quantify the effect of tower shadow on the wind turbine blade for every pass (for e.g., Powles' model that is implemented in GH Bladed, Blevin's model and the JET wake model). The simplest models look at modeling the loss of wind speed and these can be written as a summation of sinusoidal functions at the frequency of rotation and multiples of frequency. Since we are modeling only one blade, the frequency of tower pass is the same as the frequency of rotation. This will be altered appropriately once we have coupled blades. The total aerodynamic load on the blade is compute as $Q' = \int_0^L Q_{aero} dx$, where Q_{aero} is the load per unit length.



Figure 5.4: Screenshot of NuMAD blade geometry, reproduced from [4]

5.3.1 Sandia 100 *m* Blade

In order to compute the total aerodynamic load we must know the exact geometry of the blade. We go through an example calculation of the loads on Sandia National Lab's 100 m blade. These blades are currently in development to study the scope of usage of large offshore wind turbine blades and the data is used extensively for research and development. A NuMAD model of the blade used is shown in figure 5.4. NuMAD (Numerical Manufacturing And Design) is an open-source software tool written in Matlab that enables users to create a three-dimensional model of a wind turbine blade. The tool manages all blade information including airfoils, materials, and material placement.

As we can see from figure 5.4, the airfoil shape and geometry changes significantly as we go along the length of the blade. There are 34 sections at which information about blade geometry, material, profile, etc., is provided for this blade as tabulated in tables 5.1 and 5.2.

For a given given wind speed and blade velocity, the angle of attack/incidence on an airfoil changes. The Sandia 100 m blade has various airfoil profiles along the length of the blade as indicated in table 5.1. Figure 5.5 shows the coefficients of lift and drag for two sample airfoils in

Station Number	Blade Fraction	Chord (m)	Twist (deg)	Airfoil Description
1	0	5.694	13.308	Cylinder
2	0.005	5.694	13.308	Cylinder
3	0.007	5.694	13.308	Transition (99.25%)
4	0.009	5.694	13.308	Transition (98.5%)
5	0.011	5.694	13.308	Transition (97.75%)
6	0.013	5.694	13.308	Ellipse (97%)
7	0.024	5.792	13.308	Ellipse (93.1%)
8	0.026	5.811	13.308	Ellipse (92.5%)
9	0.047	6.058	13.308	Transition (84%)
10	0.068	6.304	13.308	Transition (76%)
11	0.089	6.551	13.308	Transition (68%)
12	0.114	6.835	13.308	Transition (60%)
13	0.146	7.215	13.308	Transition (51%)
14	0.163	7.404	13.177	Transition (47%)
15	0.179	7.552	13.046	Transition (43.5%)
16	0.195	7.628	12.915	DU99-W-405
17	0.222	7.585	12.133	DU99-W-405 (38%)
18	0.249	7.488	11.35	DU99-W-350 (36%)
19	0.276	7.347	10.568	DU99-W-350 (34%)
20	0.358	6.923	9.166	DU97-W-300
21	0.439	6.429	7.688	DU91-W2-250 (26%)
22	0.52	5.915	6.18	DU93-W-210 (23%)
23	0.602	5.417	4.743	DU93-W-210
24	0.667	5.019	3.633	NACA-64-618 (19%)
25	0.683	4.92	3.383	NACA-64-618 (18.5%)
26	0.732	4.621	2.735	NACA-64-618
27	0.764	4.422	2.348	NACA-64-618
28	0.846	3.925	1.38	NACA-64-618
29	0.894	3.619	0.799	NACA-64-618
30	0.943	2.824	0.28	NACA-64-618
31	0.957	2.375	0.21	NACA-64-618
32	0.972	1.836	0.14	NACA-64-618
33	0.986	1.208	0.07	NACA-64-618
34	1	0.1	0	NACA-64-618

Table 5.1: Blade fraction align the station, chord length, twist and airfoil description for the Sandia 100m blade



Figure 5.5: Variation of coefficient of lift and drag for the NACA 64 airfoil and DU30 airfoil

the blade. The blade mass is 114,172 kg, maximum rotational speed is set to 7.44 rpm and the rated wind speed is 11.3 m/s. Using the aeroelastic code FAST (models available for download at [6]), the tip deflection, flap-wise and edge-wise moments at the rotor hub can be estimated for various operating conditions. It is typical to check for loading under most naturally occurring wind conditions as defined by IEC-61400. These are EWM (extreme wind model), EOG (extreme operating gust), ETM (extreme turbulent model), EDC (extreme direction change), EWS (extreme wind shear), etc. We use the loading conditions derived from the FAST model as aerodynamic loading for our ODE in the out-of-plane direction. From the values predicted for the moment at the root, we calculate the forces acting on the blade necessary to produce the given moment.

This will give rise to a distributed loading information on the wind turbine blade due to wind. This in combination with gravity are the most dominant loads on the wind turbine blades. We only focus on routine operational ranges, as it is in our interest to investigate potential resonances away from the natural frequency of the blade (viz., super- and sub-harmonic resonances) for standard loading. This would lead to abnormal loads on the entire structure of the wind turbine and all the machinery components leading to premature failure.

We use the information from the tables and curves prescribed for the Sandia 100 m blade and compute the lift and drag forces at various sections of the blade.

5.4 Response of Wind Turbine Blades

The response of different wind turbine blades are studied. Data from the Sandia 9 m CX blade, NREL 23.5 m GRC blades, a commercial 40 m and 63 m blade, and a concept 100 m blade are used to study the influence of direct and parametric excitation on out-of-plane blade response. Based on analysis in [54, 87], the out-of-plane motion has significant response to cyclic load variations.

We compute the coefficients of equation (2.16) restated below based on the expression given in equation (2.17).

$$\ddot{q} + \mu \dot{q} + b \ddot{q} q^2 + d \dot{q}^2 q + (e_1 + e_2 \cos \Omega t) q + (f_1 + f_2 \cos \Omega t) q^3 = \int_0^L (C_L \cos \phi + C_D \sin \phi) \frac{1}{2} \rho W^2 c \, dr$$

• Linear co-efficient of q: $e_1 = \frac{12.36EI}{mL^3} + 1.193\Omega^2$

• Nonlinearity:
$$f_1 = -\frac{121.32EI}{mL^5} + \frac{.926\Omega^2}{L^2}$$

 $f_2 = \frac{39.66}{L^3}$

• Cross-coupled terms:
$$b = d = \frac{4.59}{L^2}$$

- Parametric excitation: $e_2 = \frac{15.41}{L}$
- Direct excitation: *Qaero*

The values of the coefficients of the equation of motion that describe the out-of-plane motion of the blade are dependent on blade elasticity and moment of inertia EI, mass M and length L. The computation of the aerodynamic terms is dependent on the evaluation of lift and drag coefficients for each blade. The forces are then integrated along the length of the blade to obtain the net force. The aerodynamic term essentially bottles down to three terms: a constant term, a cyclically varying term.

We then simulate the response of the 23.5 m blade shown in figure 5.6. The response of the 23.5 m blade does not show any superharmonic resonances. Commercial blade data for other wind turbine systems in operation were collected (for e.g. the 40 m blade) and the responses of



Figure 5.6: Amplitude response of the NREL GRC blade, blade properties from [5]

those blades also showed no harmonic response. We investigated the condition under which the resonance and instability conditions are exhibited from the simplified analysis in Chapter 4. We notice that as the magnitude of the the parametric excitation increases in relation to the direct forcing the system starts to exhibit the dynamic behavior inherent in the Mathieu equation analyzed with "choice parameters". The relative magnitude of the terms in the ODE for an example system can be tuned to demonstrate this behavior, but, for a wind turbine blade, the blade properties govern the relative magnitude.

Ishida et al. [55] have shown the effect of varying the magnitudes direct and sinusoidal forcing for for a rotating beam. The constant component and the cyclic varying term of Q have equal contributions to the magnitude of the response at the superharmonic resonances. Figure 5.7 shows a sample response of a 60 m blade with constant and cyclic variation in loads. We can see that as the blades get larger, the superharmonic resonances continue to persist. Wind turbine blades are typically designed to operated well below the natural frequency. Let us consider two examples of blades we have studied and qualitatively study the likelihood of blades operating under superharmonic resonance conditions. For the NREL GRC 23 m blade, the lowest natural frequency


Figure 5.7: Response of a linear system with cyclic loading (left) and constant loading (right), based on parameter from a 60 m blade.

is 1.72 Hz. The rotational speed of the blade is limited to 22 rpm for rated power. This translates to about 1/5 of the natural frequency of the rotating blades. The Sandia 100 m blade has the lowest natural frequency of 0.49 Hz and a the rotation is set to 7.44 rpm for rated power. This translates to about 1/4 of the natural frequency of the blade. If the rotor speed were to slightly increase due to other phenomenon, these could operate near the 1/3 superharmonic.

In the formulations of out-of-plane equations of motion, we have only considered one blade with one projected mode. If we were to include higher modes then the system becomes less stiff. Also, we could include other blades to account for blade-to-blade dynamic interaction. We see from the preliminary calculations that the larger blades tend to operate close to the superharmonic captured in our analysis. There are other resonances that occur for example at much lower frequencies. However, the response amplitude increase due to them, is orders of magnitude lower than those captured in the second-order multiple scales expansions. Also, blades tend to be operated at speeds above their design recommendation due to various reasons (stall regulated blades, slow pitch control, unexpected wind flow, system requirements, etc.). The work in this thesis should guide blade designers and wind turbine manufacturers towards avoiding operation under superharmonic resonance conditions to protect the blades. The increased vibratory loads on blades have a direct impact on the reliability of components in the gearbox. We will address this in the next chapter.

The relative magnitude of the terms in the differential equation of motion are critical towards triggering responses that lead to instability of increased amplitudes at harmonic frequencies. The amplitudes of direct and parametric responses are especially of critical importance as has been seen with analysis of the in-plane equation of motion. Hence, in our ODE given by equation (2.16), the magnitude of e_2 relative to e_1 and Q determine the onset or presence of instabilities and resonances. We plot the ratio of the magnitude of parametric excitation to the square of the natural frequency for three wind turbine blades of lengths 23.5 m, 63 m and 100 m (see figure 5.8). The blade material properties are obtained from data sheets for each wind turbine blade (an example data sheet for the 100*m* blade is shown in Table 5.2, where, flap stiffness is EI_{zz} , edge stiffness is EI_{yy} , and torsional stiffness is GI_{xx} and x, y, z are axes in the axial, flapwise, and edgewise directions, E and G are Young's and shear moduli, and I is second moments of area. As the ratio of e_2/e_1 increases, the response of the system at the harmonics steadily increases. After a certain value this will breach the stability boundary at these resonances and will start to exhibit unstable response (similar to the stability transition in the traditional Mathieu equation). The magnitude of the parametric to direct forcing is also critical to ensure that the system response is stable. An increase in the ratio of the parametric to direct forcing will also result in an increase in the amplitude of blade motion at these harmonic frequencies leading to possibility of amplification of amplitude of response and eventually unstable behavior. Designers, with appropriate choice of airfoil that has a direct impact on the lift and drag forces on the system, can tune the response to be away from these resonances, provided they are aware of these phenomena.

Careful attention needs to be paid to the dynamics characteristics of the large wind turbine blades. As we see from figure 5.8, the response characteristics of the wind turbine system are altered as the blades become larger. The 100 m blade under study at Sandia National Laboratory is an "upscaled" version of the 23.5 m blade. If we define the scaling factor k as the ratio of the lengths of the blades, then the other critical properties are scaled as follows: mass $\rightarrow k^3$, aerodynamic loads $\rightarrow k^3$, power $\rightarrow k^2$ and gravitational force $\rightarrow k^4$. This implies that we pay a huge penalty in terms

Span Fraction	Flap Stiffness	Edge Stiffness	Torsional Stiffness	Mass per Length
(-)	$(N-m^2)$	$(N-m^2)$	$(N-m^2)$	(kg/m)
0	3.22E+11	3.22E+11	1.68E+11	5.74E+03
0.005	2.88E+11	2.88E+11	1.50E+11	5.09E+03
0.007	2.50E+11	2.52E+11	1.29E+11	4.44E+03
0.009	2.12E+11	2.15E+11	1.09E+11	3.78E+03
0.011	1.75E+11	1.80E+11	8.97E+10	3.15E+03
0.013	1.62E+11	1.63E+11	7.98E+10	2.88E+03
0.024	1.45E+11	1.54E+11	7.22E+10	2.91E+03
0.026	1.31E+11	1.41E+11	6.47E+10	2.68E+03
0.047	1.04E+11	1.24E+11	5.01E+10	2.33E+03
0.068	7.99E+10	1.04E+11	3.53E+10	1.98E+03
0.089	7.05E+10	9.37E+10	2.54E+10	1.84E+03
0.114	5.58E+10	8.34E+10	1.50E+10	1.63E+03
0.146	4.97E+10	8.76E+10	9.27E+09	1.68E+03
0.163	4.76E+10	9.85E+10	7.66E+09	1.74E+03
0.179	4.17E+10	1.05E+11	5.89E+09	1.76E+03
0.195	3.88E+10	1.05E+11	4.41E+09	1.81E+03
0.222	3.36E+10	1.03E+11	3.85E+09	1.79E+03
0.249	2.85E+10	9.91E+10	3.22E+09	1.76E+03
0.276	2.31E+10	6.65E+10	2.67E+09	1.58E+03
0.358	1.45E+10	5.48E+10	1.72E+09	1.43E+03
0.439	8.96E+09	3.37E+10	1.10E+09	1.26E+03
0.52	5.38E+09	2.18E+10	6.90E+08	1.11E+03
0.602	3.11E+09	1.44E+10	4.32E+08	9.35E+02
0.667	1.80E+09	1.14E+10	2.94E+08	7.79E+02
0.683	1.55E+09	1.07E+10	2.64E+08	7.40E+02
0.732	1.01E+09	8.72E+09	2.10E+08	6.07E+02
0.764	7.36E+08	7.41E+09	1.86E+08	5.04E+02
0.846	3.55E+08	4.90E+09	1.29E+08	3.59E+02
0.894	2.08E+08	3.66E+09	9.78E+07	2.90E+02
0.943	8.10E+07	1.74E+09	4.51E+07	2.10E+02
0.957	4.29E+07	9.07E+08	2.58E+07	1.56E+02
0.972	1.92E+07	4.19E+08	1.15E+07	1.20E+02
0.986	5.13E+06	1.19E+08	3.03E+06	7.91E+01
1	5.84E+03	6.22E+04	2.57E+03	6.55E+00

 Table 5.2: 100m blade structural properties



Figure 5.8: Ratio of parametric excitation to the square of natural frequency as a function of length of the blade

of the loads and mass for increase in power of the order of two. The increased mass and loading coupled with large diameters of blades exposed to different wind velocities can lead to significant challenges in capturing the physics that govern the flow of wind around the turbine blades and consequently, our estimates of loads transferred to the wind turbine blades will be incorrect. Also, as blades get larger there is a reducing factor between the blade rotation speed and the flutter speed (as seen in figure 5.9).

Hence, even though there are significant benefits towards designing larger wind turbine structures, care must be taken to perform a detailed analysis during the concept and design phase to account for all the loading conditions. As blades are designed to be longer, the ratios become more conducive towards increased response at superharmonic resonances and this will have a direct impact on the reliability of wind turbine systems. Flutter onset also has increased possibilities. Repair and maintenance for these mega structures will become increasing expensive. Accurate



Figure 5.9: Flutter speed as a function of blade length. Data obtained from [6]

modeling of external and internal loads of the system will enable us to set constraints on system parameters to prevent wind turbines from having undesirable response and potential failures.

CHAPTER 6

DRIVETRAIN DYNAMICS

Vibrations in the blades are transferred to the gearbox via the rotor hub. Any drastic changes in loading of the blades either due to wind gusts or due to increased vibration response due to resonance phenomenon is thus transferred directly to the main shaft of the gearbox. This necessitates accommodation of blade vibration loads while designing and certifying gearboxes for wind turbines. The common practice is to model these elements for dynamic responses separately as the critical frequencies are magnitudes apart. However, increasing failures in gearbox components has ushered the industry towards gaining a complete understanding of all the dynamic loads the drivetrain would experience during operation. In this chapter, we look at two modeling approaches for wind turbine drivetrains.

6.1 Modeling of the Gearbox Reliability Collaborative (GRC) Gearbox

First, we look at a detailed drivetrain model built in RomaxWind, a commercial computer-aided engineering software that is used for comprehensive analysis, optimization and certification of wind turbine drivetrains. Some of the early validation work of the RomaxWind model of the NREL GRC 750 kW wind turbine drivetrain is included in this chapter as well.

The GRC is a program run by the NREL National Wind Technology Centre (NWTC) to determine the reasons for premature failure that are common in gearboxes in wind turbines. The GRC focuses on three areas of research: drivetrain modeling and analysis, full scale dynamometer testing and field testing. A model of the GRC gearbox is shown in Figure 6.1, with some salient features labeled. Figure 6.2 shows the schematic of the GRC gearbox with various speed stages and components labeled. The gearbox has been modeled in the RomaxWind software, a virtual product development and simulation environment for the design and analysis of wind turbine gearboxes, bearings and drivetrains. The gearbox is a single stage planetary with two parallel helical gear stages - a typical configuration for medium sized gearboxes in wind turbines. The gearbox is a



Figure 6.1: RomaxWind model of GRC gearbox



Figure 6.2: Schematic showing GRC gearbox layout and component nomenclatures

speed increaser, with approximately 1:81 ratio, and is designed for high speed shaft speeds of 1200 rpm and 1800 rpm depending on the running configuration of the two speed (4/6 pole) generator.

Two 750 kW gearboxes that have had some redesign from the original arrangement are considered for modeling and testing. They have been modified to allow a comprehensive instrumentation

Mode Number	Test Frequency (Hz)	Simulation Frequency (Hz)	% Error
1	317	338	6.21
2	341	359	5.01
3	359	387	7.23
4	394	422	6.64
5	415	464	10.56
5	445	494	9.92

Table 6.1: Validation of gearbox housing natural frequencies

package (over 150 channels). Measurements of tooth root strains on the ring gear, raceway strains on the planet bearings, deflection of the planet carrier, temperatures near the bearings, input and output torque and speed, amongst other operating values are recorded. We use these measurements to validate the model of the drivetrain in RomaxWind. There are two aspects to having a reliable model for predictive computation; (a) have a comprehensive model to capture all the necessary physics to predict response of a system and (b) validate the model at component and system level to ensure that the behavior of the model emulates the real drivetrain it represents. We used measurements from accelerometers of a vibration tests with impulse excitations to check if the frequency and modal response of the gearbox housing matches those predicted by the simulation. Figure 6.3 shows a snapshot of the test set up and the corresponding simulation in RomaxWind. The gearbox housing was instrumented with ~300 accelerometers to capture the mode shape and we use the measurements to qualitatively compare the mode shapes. From table 6.1, we can see that for the first six modes the natural frequencies are at about 10% error. This is a sufficient level of validation as the exact material properties, joints in the structure represented by finite element structures and boundary conditions are not exactly captured.

The RomaxWind drivetrain model is built from the engineering drawings provided by NREL GRC. Key features of the model include beam finite element representation of shafts, solid finite element representation of gearbox housing, gear blanks, planet carrier and torque arms and 6DOF spring connections for (elastomeric) trunnion mounts. The gears and bearings are modeled with semi-analytical formulations that take account of important factors such as misalignment, area of contact under load, micro-geometry, radial and axial clearances or preload and material properties.



Figure 6.3: Comparing the 1st natural frequency and modal response of the gearbox housing. Figure shows the test set-up and test result (319 Hz) on the left and RomaxWind model and simulation result (338 Hz) on the right

Static non-linear analysis can be performed for prescribed loading conditions and the global deflections are solved simultaneously with the contact mechanics for the gears and bearings. Static analysis is considered sufficient as the inertia terms are orders of magnitude lower than the forces due to the loads carried by the gearbox. Thus the effect of the whole system behavior on contact elements is captured. We use some of these metric as measures to ensure our model is validated.

Designers typically use these models for achieving good alignment of the system under the loads, calculating gear and bearing contact stress and life, optimizing the micro-geometry of the gears for increased life and transmission error and predicting the gear vibration magnitudes, as well as for many other purposes.



Figure 6.4: RomaxWind model of GRC gearbox installed in wind turbine with a load case example: Input Torque (Mx) = 325 kNm, Off Axis Moment = -100 kNm, Rotor Weight = -1500 kN

6.2 Case Study - Influence of Carrier Bearing Clearances on Gear and Bearing Alignment and Stresses

Figure 6.4 shows a model used for the purpose of simulating the GRC gearbox as installed in the dyno. The load case shown with the figure includes a significant off-axis moment as well as loading from rotor mass (wind turbine blades and hub). Large off-axis moments are common in wind turbines, under conditions such as wind direction change, yaw error (putting the blades askew to the wind), yawing of the machine and wind shear. Most dynamometer tests only load the drivetrain with torque and it is important to understand that the loading and deflections of the gearbox may be markedly different in the two installations; particularly for three point mounting arrangement (main bearing and two torque arm supports).

For the given load case a non-linear static analysis has been performed and the contact and stress in the planetary stage is calculated at a certain carrier rotation. Planet positions are chosen at 0° , 120° and 240° and subsequent results focus on the planet/sun gear mesh and the planet and carrier bearings are shown in table 6.2.

The load as well as clearance in the main bearings can misalign the carrier to the housing and

Planet Gear	Maximum contact	Misalignment	Face load distribution
Number	stress (MPa)	$F\beta X (\mu m)$	factor KH β
1	1223	173	1.72
2	1146	103	1.25
3	1197	57.2	1.35

Table 6.2: Metrics on the Quality of Gear Contact

ring gear and affect the gear contact (note the main bearing is a spherical roller and provides little tilt stiffness).

6.2.1 Time-varying Misalignment and Stress

When the planet carrier rotates, the planets move with the carrier and subsequently the planet load share and planetary gear misalignment vary with time. This results in increased vibration and reduced life, increasing the probability of failure for the planetary gear set. By performing the analysis at incremental rotations of the carrier the time varying behavior can be examined and figures 6.5 and 6.6 chart the planet load share and sun/planet gear misalignment respectively. The time varying misalignment and load sharing has significant effects on the planet bearings, a component that is known to have high failure rates in wind turbines and are expensive to replace.

6.2.2 Optimizing the Design to Improve Contact, Stress and Life

A design analysis needs to be performed to understand the influences of misalignments on the planet gears durability. If the misalignment can be reduced then the time varying contact stress and face load distribution can be reduced. This will in turn decrease vibration and reduce fatigue on the planet gears and planet bearings. For many wind turbine gearbox designs, the alignment of the planetary stage is sensitive to the magnitude of the clearances in the planet carrier bearings. For this model, with the gearbox installed in the dyno, a full factorial study has been performed where the radial internal clearance of the upwind and downwind planet carrier bearings are varied and the misalignment and face load distribution factors ($F_{\beta X}$ and $K_{h\beta}$ [88]) are calculated for each planet. The full factorial study (see figure 6.7) demonstrates how sensitive the misalignment of the sun and



Figure 6.5: Planet load share in kNm vs. carrier rotation in degrees



Figure 6.6: Sun/Planet gear misalignment in microns vs. carrier rotation in degrees

Design Study, Full Factorial

Design Study, Full Factorial



Figure 6.7: Results of full factorial study for upwind and downwind planet carrier bearing's radial internal clearance (shown as PLCA RIC and PLCB RIC in figure): (left) Sun/Planet misalignment; (right) Sun/Planet load distribution.

planet is to these bearing settings and designers can use such models to optimize the parameters of the bearings for improved performance and life.

The best clearance specification would be the subject of a full design review. However, a brief example is provided here where the carrier clearances are set to an optimal value so that the planet/sun misalignments are consistent with position. For this design and load case the optimum settings are 50 μ m and 250 μ m on the upwind and downwind bearings respectively (indicated by markers on Figure 6.7). With the optimized clearances a similar mean stress is achieved; the maximum stress is reduced by approximately 5% and the time varying amplitude is reduced by a factor of four. The gears are still misaligned, yet this misalignment is now consistent with rotation (plots not shown) and can be corrected with lead slope adjustments.

6.2.3 Redesigning to Reduce Sensitivity

The improved carrier bearing clearance setting leads to lower vibrations levels in the components of the gearbox. But, typical manufacturing tolerances on bore of the carrier and planetary alignments are of the order of 100 microns. This would mean that even though we can specify clearances

Case	Maximum load	Face load distribution
	per unit length (N/mm)	factor $K_{h\beta}$
(a)	1131	1.72
(c)	921	1.11

Table 6.3: Metrics for the quality of gear contact for cases shown in figure 6.9

while designing the gearbox, in practice, we may not achieve optimal settings. A change of a few microns affects the load sharing significantly. This prompts the use of other design decisions to reduce the sensitivity of the model to bearing clearances. For this system, we study the effect of change in the bearing package along with a corresponding micro-geometry optimization of the planet gear set. We replace the cylindrical roller bearing of the carrier with tapered roller bearings. Micro-geometry corrections on the planet gear set are also made to correct gear misalignments and improve the face load distribution. Figure 6.9 shows the improvement in the gear contact at the face of a planet gear (which had extreme loading pattern in the original case) with a changed bearing package and changes in the lead slope. Similar improvements are noticed for the contact patches on other planet gears as well. Thus, an overall reduction in planet gear misalignments and a corresponding improvement in face loading can be obtained. The improved design is quite robust and small errors in planet position due to assembly don't affect the load sharing patterns. The proposed changes in clearance and bearing configuration also improves the loading patterns on the bearing elements. We show the variation in maximum contact stress of the downwind planet bearing for the two cases shown in figure 6.9. The variations in maximum stress per revolution is reduced by about 70% and the maximum stress is also reduced by about 6%, both of which would lead to lesser fatigue loading of the bearings.

Thus, we have used the RomaxWind model of the drivetrain, to assess loading of the drivetrain under different input loads at the rotor hub, quantified sensitivities of the model to different parameters in the gearbox and drivetrain design, and suggested methods to improve/optimize the design of components in a wind turbine drivetrain to improve reliability. We continue to explore the usage of these models for improving the predictive capability.



Figure 6.8: Sun/Planet load distribution for (top) original case; (bottom) case with tapered roller bearing with +100 μ m preload and after micro-geometry optimization

6.3 Prediction of Wind Turbine Gearbox Vibration

In this section, we use the RomaxWind model to predict the vibration levels at various location on the gearbox housing. This was specifically used to improve our confidence in the model, but, at the same time was a necessary step to understand the usage of this model to benchmark standard failure modes and design gearboxes to meet noise and vibration targets throughout their warranty. Impact tests and dyno running tests were performed on the gearbox mounted on the dyno at NREL. They are necessary to tune housing material properties and bushing stiffness to attenuate any undesirable frequency response. We seek to use this data to characterize the vibration spectrum of a healthy gearbox and for fault diagnostics. There are numerous condition monitoring techniques



Figure 6.9: Downwind planet bearing maximum contact stress with carrier rotation for (top) original case; (bottom) case with tapered roller bearing with +100 μ m preload and after micro-geometry optimization

that use vibration signatures measured realtime from operating machinery to diagnose or detect potential issues or defects with machine components [89]. Advanced signal processing techniques are employed to determine the root cause of certain frequencies in the spectrum and trace it back to component malfunction.

In figure 6.10, we compare measurements for vibration response of sensors at various locations between test and RomaxWind model. It is clear that the trend for sensitivity of sensor position towards the response amplitude correlates well, i.e., we are able to accurately identify the sensor that detects the maximum amplitude for the two cases shown. The location of the sensor is crucial to the response it records due to structural excitations. The precise location should be known and a finite element node is the exact same location should be "tagged" in the RomaxWind model for accurate analysis.

Capability to predict vibration response accurately in the RomaxWind model serves as an important assessment tool in the design phase of the gearbox development process. This model will be used to optimize the sensor location for diagnostics and also to design the gearbox for low vibration. For design improvements, requirements such as limitations on housing deflection at certain locations can be checked. While for condition monitoring, depending on the whole system's dynamic response, the choice of sensor location for vibration based condition monitoring can be decided to get the best output. However, vibration amplitudes are difficult to predict from simulation tools across different frequencies and care should be taken to model the structural flexibility of the entire wind turbine system as accurately as possible to get a representative frequency response.

6.4 Redesigning the Wind Turbine Gearbox: For Improved Reliability

The NREL GRC gearbox phase 3 redesign was performed to assess an existing design of the gearbox and suggest candidate design changes from the conceptual review amongst other requirements. The specific task performed as a part of this dissertation, performed in collaboration with Romax Technology Inc., was to provide an exploration of the effects of the design details on the loads and deflections within the gearbox. This involved use of the RomaxWind model of the



Figure 6.10: Normalized amplitude of vibration at different sensor location for High speed shaft (HSS) and Intermediate speed shaft (IMS) gear excitations

gearbox, which has been validated against many measurement on the gearboxes from phase 1 and 2. Candidate design changes arising from the concept design review relating to the planetary stage were evaluated. Some specific recommendations made are discussed in this section.

6.4.1 Tapered Roller Bearing for Carrier and Planet

Tapered roller bearing offer more load-carrying and load-distributing ability. The bearing suggested by NREL GRC supplier was assessed in "X" (force direction on pins make an X) and "O" (force direction on pins make an O) arrangements and even though the "O" offered more stiffness,



Figure 6.11: RomaxWind model with carrier and planet gear tapered roller bearing in "X" (marked at location of bearing by rectangles) and "O" (marked at location of bearings by circle) configurations

it is not recommended due to difficulty in assembly for the carrier (figure 6.11). The driving factor for choosing a taper roller bearing is to eliminate clearance and increase stiffness, hence improving planetary stage gear mesh load distributions. As the preload is increased the ring gear mesh misalignments become closer together (see figure 6.12). As clearance is increased the misalignments diverge. If the misalignments are all close together then it is possible to correct with lead slope modification. A set of misalignments that are different at each planet position indicates some significant misalignment between the carrier and the housing or the sun.

The carrier cylindrical roller bearings should be replaced with taper roller bearings. This decreases the misalignment between the carrier and the housing. The planet cylindrical roller bearings should be replaced with integral gear/outer race taper roller bearings. This reduces the misalignment due to the moment from the axial component of the mesh forces on the planet, improves the load share between the upwind and downwind bearing rows and removes the possibility of fretting between the outer race and the gear blank. The selection of a bearing with significantly greater load capacity increases the system life calculation significantly as the planet bearings are



Figure 6.12: Mesh misalignment vs. preload for initial tapered roller bearing selection

the lowest life component.

6.4.2 Interference Fit Planet Pins

If the planet pin are designed to have a tight fit, then the carrier will also have a stiffening effect. A comparison between the baseline, and models with carrier tapered roller bearings, ring gear stiffening, carrier stiffening, planet pin interference fits is shown in figure 6.13. It can be seen that increasing the ring gear stiffness has not had a significant effect on mesh misalignment. The increase in carrier stiffness has had a moderate effect because this reduces the misalignment due to torsional wind-up of the carrier. However, the most dramatic effect is found by interference fitting the pins into the carrier. This has a greater stiffening effect in the carrier and consequently would be easier to modify in the design of the gearbox.



Figure 6.13: Mesh misalignment for increasing ring and carrier stiffness with interference fit pins for planetary gear contacts

6.4.3 Decreased Spline Crowing

The sun spline showed damage around the center contact region once the above design changes were implemented. Reducing the crowning on the external spline would increase the contact area and should help negate the damage in the center contact region. A potential knock-on effect is an increase in the tilt stiffness of the spline and hence a worsening of planetary gear misalignment and load share. However, in the current model for the load cases studied, the misalignments on the planetary gear set were not affected due to changes in spline crown. We simulate changes in spline crown from the baseline value of 173 μ m to 50%, 75% and 87.5% of it's value. Figure 6.14 shows the deflection distribution of the spline teeth. The purpose of having the flexible spline joint here is to allow the sun to float in the planetary stage. Decreasing the spline crown will increase the tilt stiffness of the spline and reduce the amount of sun float. However, this is deemed to be acceptable tradeoff for the reduction in peak center load on the spline.



Figure 6.14: Deflection distribution on tooth on horizontal axis for decreasing crown of (top left) $173\mu m$, (top right) $86.5\mu m$, (bottom left) $43.25\mu m$, (bottom right) $21.625\mu m$

6.4.4 Other Changes and Recommendations

There are other more detailed design changes that will have an impact on the reliability of the gearbox design. In this section we explored a few design modification in an existing gearbox with constrains on overall geometry and parts being used. However, future work in gearbox redesign for the NREL GRC project or otherwise can explore other design changes listed below (though some recommendations are specific to the layout of the GRC gearbox) and its implications on loading and misalignments.

- The ability to modify the carrier taper roller bearing preload
- The ability to modify the planet taper roller bearing preload
- A method for adjusting the stiffness of the ring gear
- An optimization of gear micro-geometry
- An optimization of bearing preload settings
- The ability to change out the high speed pinion
- An improved design of the high speed shaft bearing system

- An improved design of the hollow shaft taper roller bearing support and lock-nut system
- An improved design of the oil feed rings
- · A review of the assembly and disassembly procedure and proposed methods for improvements
- Improvements to the gearbox lubrication system and sealing, and the including of a instrumentation methodology to allow measurement of lubrication effectiveness

6.5 Torsional Model of Wind Turbine Drivetrain

Since RomaxWind is geared towards detailed component static analysis, in order to perform transient analysis a simple matlab model was built in collaboration with Romax Technology Inc. in order to capture the dynamic interaction between the wind turbine blades, drivetrain and generator. A schematic of the torsional model is shown in figure 6.15. The model has been used to characterize and study different drivetrain configurations. Some preliminary studies with this model involved comparing the loads on drivetrain components of a medium speed drivetrain (new concept) vs. a high speed drivetrain (existing concept). The moments transmitted and the associate drivetrain deflections in the rotational axis are also captured.

6.6 Extension to Loading from Blade Vibration

In the previous sections, it has been shown that the loading in the gearbox and pressure at the roller elements of bearings are highly sensitive to the design of the system. Moreover, the deflection during operation of the machinery causes variations in the loads experienced and the quality of contact in the gear tooth. It is hence clear that any blade vibration beyond designed levels will cause changes in the loading experienced in the gearbox via coupling through the rotor hub. However, the increased loads at superharmonic resonances cannot be easily implemented in RomaxWind model. For starters, we would need the detailed geometry of multiple gearboxes to assess the impact of increased vibration loads for different sizes of blades. As this is not feasible within the scope of



Figure 6.15: 1-D torsional model of a simple wind turbine gearbox, courtesy of Romax Technology

this thesis, the torsional drivetrain model developed can be used for such studies. The aerodynamic loads computed using the BEM theory in chapter 5 can also be used as inputs to this model.

The solution of the IPDE that are used to model the blade vibration would be considered. We would seek solution to the moments that are transferred at the root of the blade. Existing commercial codes such as FAST are already able to do this. Our analysis would account for significant nonlinearity in the blades (which tend to be more longer and flexible) and associated harmonic resonances. The dominant blade frequency and the associated harmonics typically occur at much lower frequencies than the gear and bearing related frequencies. From torsional models (as shown in figure 6.15), simulations that relate to the influence of gear mesh stiffness, component inertias and mount damping to overall system behavior can be studied. Additionally, we can include torsional fluctuations at the rotor hub that are of the order of blade non linear dynamics and assess its impact.

CHAPTER 7

CONCLUSION AND FUTURE WORK

7.1 Conclusion

This work considered the cyclic loads on a wind turbine blade and how they affect blade vibration and drivetrain dynamics. The work first provided integral-partial differential equations and reduced ordinary differential equations of motion that model the in-plane and out-of-plane dynamics of a wind turbine blade. Nonlinear effects (due to curvature and foreshortening) and cyclic loading (due to gravity and aerodynamics) have been considered.

The single mode reduction led to the parametrically and directly forced nonlinear ODE equation (2.8). This motivated the investigation of a forced nonlinear Mathieu equation as a paradigm in dynamical systems research. Furthermore, we found that when typical wind turbine parameters are included, equation (2.8) may not have the convenience of a small parameter. So as a preliminary study, we examined the small parameter nonlinear forced Mathieu equation (in which the direct and parametric excitations occur at the same frequency, out of phase by 90 degrees, and which includes a cubic nonlinearity), with the expectation that the phenomena uncovered may carry over to wind turbines. Perturbation analysis of equation (3.1) revealed the presence of multiple super- and subharmonic resonances. The superharmonic resonance of order 1/2 persists for the linear Mathieu equation with direct forcing. Some resonance and instability phenomena were uncovered using simulations. Second-order perturbation methods provided the analytical framework to understand system behavior better. Primary resonance and multiple superharmonic and subharmonic resonance cases for system with hard and weak excitations, direct forcing and nonlinearity were studied. The presence of superharmonic resonances may be significant for wind turbines, which are designed to operate below the lowest natural frequency. Resonances can be amplified beyond linear predictions if there is simultaneous superharmonic and primary excitation as well.

We further extended the analytical framework to real-world example wind turbine systems.

Turbines with blade diameters ranging from 9 m to 100 m were simulated. The supporting aerodynamic theories to assess aero loads were also benchmarked and developed. The phenomenon of harmonic resonance and instability is likely to occur for larger wind turbines and this is shown via a systematic analysis.

To understand the effect of unknown dynamic loading conditions on a gearbox, we have analyzed the NREL GRC gearbox model. As a case study, the sensitivity to design and tolerances was performed. Detailed models within the RomaxWind software tool have been developed with focus on determining the quality of the function of the planetary gear stages, and predictions of planet load distributions have been validated against the test data. Key design drivers are discussed, such as the quality of alignment at the gears and bearings and the loads and stresses seen on these components. It is illustrated that small clearances, such as in the carrier bearings, can have a large effect on the performance of the design and a study shows how to identify and reduce time varying misalignment and contact stresses which leads to lower vibration, lower fatigue and a more reliable product. This model will be used to demonstrate the effect of superharmonic and subharmonic resonances on gearbox dynamics and reliability of the wind turbine system.

Improving the reliability of wind turbines is important for the industry and for reducing the cost of energy. The NREL GRC program is one example of industry, university and government research labs working together to improve the understanding of the gearbox and exploring and disseminating the reasons for premature failure. Other such projects and government initiatives around the globe point to a pressing need towards understanding issues that are critical to the proper functioning and reliability of wind turbine structures.

7.2 Future Work

The foundations laid in this thesis can be extended in multiple directions. Some of them listed below:

- Develop model for coupled in-plane, out-of-plane and twist motion of the blade.
- Second order perturbation analysis for complete nonlinear partial differential equation of wind turbine blades.
- Use the model of the beam to validate stable/unstable operating zones by experimenting with a cantilever beam attached to a shaker. The attachment will allow for inclined orientation to mimic parametric forcing.
- Use semi-empirical aerodynamic theory to get a more representative expression for forcing and incorporate it into the calculations.
- Use calculated moments at the root of the blade as input to the NREL GRC gearbox model in Romax and arrive at more realistic loading scenarios within the gearbox.

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