

AN ANALYSIS OF CONTENT KNOWLEDGE AND COGNITIVE ABILITIES AS
FACTORS THAT ARE ASSOCIATED WITH ALGEBRA PERFORMANCE

By

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Educational Psychology and Educational Technology – Doctor of Philosophy

2017

ABSTRACT

AN ANALYSIS OF CONTENT KNOWLEDGE AND COGNITIVE ABILITIES AS FACTORS THAT ARE ASSOCIATED WITH ALGEBRAIC PERFORMANCE

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The current study investigated college students' content knowledge and cognitive abilities as factors associated with their algebra performance, and examined how combinations of content knowledge and cognitive abilities related to their algebra performance. Specifically, the investigation examined the content knowledge factors of computational fluency, numeracy skills, fraction knowledge, understanding of equivalence, and algebraic reasoning skills, and the cognitive abilities of spatial visualization, crystallized intelligence, and fluid intelligence. A multiple regression analysis found that while controlling for gender, the highest math course taken, and the number of years since an algebra course, fraction knowledge and the spatial visualization ability of spatial imagery were statistically significant predictors of algebra performance along with the control variable identifying whether or not participants had taken at least one calculus course. In addition, cluster analysis identified six content knowledge and cognitive ability profiles, with varying levels of both content knowledge and cognitive abilities observed across the six clusters. The six profiles – characterized as *Low All*, *Moderate-Low All*, *Moderate-High MASMI*, *Moderate-Low Spatial*, *Moderate-High All*, and *High Spatial* – varied somewhat in terms of their algebra performance scores. In particular, the participants in the *High Spatial* cluster group and participants in the *Moderate-High All* cluster group had similarly high algebra performance scores, which were significantly higher than performances scores observed for participants in the other cluster groups. Additionally, the participants in the other cluster groups exhibited similar low algebra performance scores to each other except for participants in

the in the *Moderate-Low Spatial* and *Low All* cluster groups. Participants in the *Moderate-Low Spatial* cluster group had significantly higher algebra performance scores than participants in the *Low All* cluster group. The differences in algebra performance scores among cluster groups suggested that the observation of higher algebra performance occurred when participants had strong spatial visualization skills, strong fluid intelligence skills, and high content knowledge or when participants had strong fraction knowledge, numeracy skills, algebraic reasoning skills, and spatial imagery skills.

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To Mama Hawkins whose simple words of encouragement inspired a little girl to achieve more than she ever dreamed.

ACKNOWLEDGEMENTS

It has taken me more years than I would have liked to finish graduate school, but the journey has been at times hard and difficult, but mostly fun and enjoyable because of the people around me. It is these people that I would like to acknowledge and give my heartfelt thanks. First, to my advisor Lisa, I say thank you for taking me in and nurturing me the last year. I know our research interests are quite different, but working with you has pushed me to be a better researcher and scholar. To my undergraduate research assistants Maggie, Kathryn, and Harmony, thanks for your tireless efforts in collecting and coding data. Without your help, I would never have completed my dissertation. To the people in the Center for Ethics and Humanities in the Life Sciences, I say thank you for generously giving your resources, respecting my time, and your endless words of encouragement and support. To my graduate school friends and colleagues DaSha, Chiharu, Rani, William, Emily, and Kristy, I say thank you for your listening ears and endless words of advice and support. Talking with you made me feel less of an imposter, and showed me I was not alone in this journey. Lastly, to my parents Patricia and Thaddeus, I say thank you for being my guiding light. You always brought me back when I felt myself drifting away. Even more thanks to my mother who has been my sounding board throughout this whole dissertation process. Sometimes it was your random and weird comments that sparked ideas and regenerated my thinking when it had fizzled out.

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CHAPTER 1: Introduction

In a society that places emphasis on students' ability to reason through quantitative situations in work, school, and daily life, it is important to understand students' mathematical development. One pivotal component of mathematical development is students' understanding of algebra. It provides students with the tools in which to reason mathematically about real-life situations (Lacampagne, Blair, & Kaput, 1995; National Research Council, 1998; RAND Mathematics Study Panel, 2003). In addition, the completion of an algebra course is seen as a necessary requirement for learning higher level mathematics, having higher education opportunities, and getting technically skilled jobs (Kaput, 1998; R. P. Moses & Cobb Jr., 2001; National Mathematics Advisory Panel, 2008; Vogel, 2008). Yet, students encounter many obstacles as they learn algebra and exhibit difficulty comprehending a range of algebraic concepts (Booth, 1988; French, 2002; National Council of Teachers of Mathematics, 1988).

The literature on the teaching and learning of algebra is substantial and centered on student learning and instructional approaches. It helps to explain the areas with which students have difficulty (e.g. Booth, 1984; French, 2002; Kuchemann, 1978; National Council of Teachers of Mathematics, 1988; Stacey & MacGregor, 1997b), the reasons for these difficulties (e.g. Godino, Neto, Wilhelmi, Ake, & Etchegaray, 2015; National Mathematics Advisory Panel, 2008; Sfard & Linchevski, 1994), and the different ways to help students understand algebra (e.g. Bednarz, Kieran, & Lee, 1996; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Nathan, Kintsch, & Young, 1992; Stacey & MacGregor, 2000). Results from the National Assessment of Educational Progress (NAEP) studies provide evidence that the focus on student learning and instructional approaches have increased average students' scores in algebra (Kloosterman, 2016; Kloosterman & Lester, 2004; Kloosterman & Lester Jr., 2007; Kloosterman, Moher, & Walcott,

2016; National Center for Educational Statistics, 2012, 2015). In fact, Kloosterman (2016) stated that current 4th and 8th grade students are better prepared now to take a formal algebra course than 9th grade students were in 1990. Nevertheless, the progress of improvement for 4th and 8th grade students in algebra has slowed since 2005 and there have been no gains in 12th grade students' algebra performance since 2009 (Kloosterman, 2016; Perez, Roach, Creager, & Kloosterman, 2016). These findings seem to imply that there is a limit to how much improvement can occur when you focus on student learning and instructional approaches.

One possible reason for this limitation is the narrow focus of the research on the teaching and learning of algebra, and in particular, the narrow viewpoint research has taken on student learning. Researchers assumed that students' weak content knowledge led to weak algebra performance. Thus, examinations of student learning focused on identifying students' errors and misconceptions as well as the fundamental mathematical concepts necessary for solving algebra problems (e.g. Booth, 1984; Ketterlin-Geller & Chard, 2011). This viewpoint on student learning ignores the fact that the student is more than what content knowledge they learn. Students are also a dynamic system that develops based on many factors that interact with one another, and these factors could be the mental, behavioral, and biological aspects of individual functioning as well as the social, cultural, or physical nature of the environment (Bergman, Magnusson, & El-Khoury, 2003). Content knowledge as the sole determinant of student learning neglects the consideration of other aspects of individual functioning such as other factors like cognitive abilities, working memory, executive functioning, motivation, and personality, which have been shown to play a role in students learning (e.g. Bailey et al., 2014; Spinath, Spinath, Harlaar, & Plomin, 2006). Therefore, it begs the question what might be some other factors of individual functioning that associated with students' algebra performance.

Research on the factors that predict mathematics achievement holds promise for helping us better understand and investigate additional factors of individual functioning that may shape students' algebra performance. Of the many individual functioning predictors of mathematics achievement, cognitive abilities is frequently identified as the strongest predictor (Colom & Flores-Mendoza, 2007; Gagne & St. Pere, 2002; Hofer, Kuhnle, Kilian, & Fries, 2012; Karbach, Gottschling, Spengler, Hegewald, & Spinath, 2013; Kriegbaum, Jansen, & Spinath, 2014; Kyttälä & Lehto, 2008; Lu, Weber, Spinath, & Shi, 2011; Weber, Lu, Shi, & Spinath, 2013). This could be because cognitive abilities may limit and/or completely prevent students from being able to process the information presented to them. For instance, some students can take away from a course enough knowledge to use it in real life situations and others cannot (Anderson, Brubaker, Alleman-brooks, & Duffy, 1985; A. L. Brown, Campione, Reeve, Ferrara, & Palincsar, 1991; Erlwanger, 1973; Loveless, 2008; Stylianides & Stylianides, 2007). This difference could be because students receive and process information differently. Some process new information by connecting it to knowledge that they already have. Others take it in as isolated facts they need to remember for a test, and then forget it. Some may argue that making connections versus isolated facts depends on the instructional approaches used, but the student still determines the reception of that information. For example, the teacher may present the connections between Lesson A and Lesson B multiple times throughout a course session in many different ways, and still have some students who get it and others who do not. Because of these differences, it stands to reason that students' cognitive abilities may influence their mathematics achievement, and may possibly be another factor that shapes students' algebra performance.

Some research has shown that an association existed between students' general cognitive abilities, content knowledge, and algebra performance (Fuchs et al., 2012, 2016; Geary, Hoard,

Nugent, & Rouder, 2015; Lee, Ng, & Ng, 2009; Lee, Ng, Bull, Pe, & Ho, 2011), but a number of factors limited the finding from this research. With these limitations, it is hard to say how students' cognitive abilities and content knowledge relate to their algebra performance. Therefore, in this study I sought to examine how students' cognitive abilities and content knowledge related to their algebra performance by identifying what specific skills, understandings, and/or abilities were associated with algebra performance as well as how varying combinations of these characteristics of students relate to algebra performance. Knowing these specific details provides a deeper understanding of the factors associated with students' algebra performance.

CHAPTER 2: Literature Review

Many have recognized that there are numerous factors besides content knowledge and instructional experiences that relate to student performance (e.g. Wang, Haertel, & Walberg, 1990, 1993). What these factors are depends upon what perspective you take. The perspective of this study is the holistic-interactionist perspective (Bergman et al., 2003; Magnusson, 2003). The holistic –interactionist perspective emphasizes the importance of the individual in the study of development because all individuals do not function and develop in the same way. It views the individual as an integrated organism that has mental, behavioral, and biological factors that influence functioning and development. By taking the holistic-interactionist perspective, I acknowledge that there are other factors besides content knowledge and instructional experiences that influence the way students perform, and assume that students are different with respect to their algebra performance because there are differences in their functioning and development. Differences in the way an individual functions and develops could be because of many different factors, but I contend that the differences in students' cognitive abilities and content knowledge changes the way students perform. Although there could be many other factors beyond cognitive abilities and content knowledge that change the way students perform (e.g. students' motivation or personality), cognitive abilities are the most consistent and strongest predictors of students' general academic achievement and mathematics achievement (Colom & Flores-Mendoza, 2007; Gagne & St. Pere, 2002; Hofer et al., 2012; Karbach et al., 2013; Kriegbaum et al., 2014; Kyttälä & Lehto, 2008; Lu et al., 2011; Weber et al., 2013). Thus, understanding how cognitive abilities and content knowledge combine to predict algebra performance is an important first step in understanding the complexity of variables that are associated with algebra performance. In the following sections, I give an overview of the research on the relation between cognitive abilities,

content knowledge, and algebra performance. Then I identify the specific forms of cognitive abilities and content knowledge investigated and how a relation exists between them and students' algebra performance.

Cognitive Abilities, Content Knowledge, and Algebra Performance

Some researchers would agree that general mathematics performance depends on both content knowledge and general cognitive abilities (e.g. D. C. Geary, 2004; von Aster & Shalev, 2007). This claim is based on prior research that establishes their effect both separately and together (G. Brown & Quinn, 2007; Bull, Espy, & Wiebe, 2008; De Smedt et al., 2009; Fuchs, Geary, Compton, Fuchs, Hamlett, & Bryant, 2010; Fuchs, Geary, Compton, Fuchs, Hamlett, Seethaler, et al., 2010; Kroesbergen, Van Luit, & Aunio, 2012; Siegler et al., 2012). Much of this research centered on the elementary grades. There is little research on mathematics performance beyond the elementary grades (Caviola, Mammarella, Lucangeli, & Cornoldi, 2014; Fuchs, Geary, Compton, Fuchs, Hamlett, Seethaler, et al., 2010; Jordan et al., 2013; Krajewski & Schneider, 2009; Lefevre et al., 2010; Seethaler, Fuchs, Star, & Bryant, 2011; Vukovic et al., 2014; Ye et al., 2016). The focus on the elementary grades was most likely because of the desire to understand how mathematical competency develops in order to identify precursors to mathematics difficulties and learning disabilities. The assumption was that if research could identify the weaknesses in content knowledge and general cognitive abilities from the onset then interventions can be designed to help alleviate mathematical difficulties and support mathematics learning disabilities (Cowan & Powell, 2014; Hecht & Vagi, 2010; Hornung, Schiltz, Brunner, & Martin, 2014; Passolunghi & Lanfranchi, 2012; Peng et al., 2016). The few studies that examined mathematics performance in higher grade levels have also found that mathematics

performance depended upon both content knowledge and cognitive abilities (Cirino, Tolar, Fuchs, & Huston-Warren, 2016; Geary et al., 2015).

Researchers defined content knowledge in terms of basic numerical competencies such as counting, computational strategies, quantity comparison, and studied general cognitive abilities such as intelligence, working memory, processing speed, and executive functioning (Geary, 2011; C. M. Irwin, 2013; Krajewski & Schneider, 2009; Passolunghi & Lanfranchi, 2012; Passolunghi, Lanfranchi, Altoe, & Sollazzo, 2015). The relation of content knowledge and cognitive abilities on mathematics performance varied depending upon the type of mathematics performance investigated. Prior research suggested that cognitive abilities and content knowledge were associated with students' mathematics performance for calculations (e.g. Östergren & Träff, 2013; Peng et al., 2016), word problems (e.g. Cowan & Powell, 2014; Hecht & Vagi, 2010), fractions (Jordan et al., 2013; Ye et al., 2016), algebra (e.g. Fuchs et al., 2016; Lee, Ng, Bull, Pe, & Ho, 2011), and general mathematics achievement (e.g. Chu, Van Marle, & Geary, 2016; Geary, 2011). The research findings suggested that mathematical performance is a complex association between content knowledge and general cognitive abilities where each context of mathematics learning comes with its own constellation of important content knowledge and cognitive abilities that facilitate its development.

The few research studies that have looked at algebra performance in relation to general cognitive abilities and content knowledge have shown similar results to the studies for the different types of mathematics learning in the elementary grades. For instance, Fuchs et al. (2012) and Fuchs et al. (2016) investigated the connection between general cognitive abilities and content knowledge on algebra performance via students pre-algebraic knowledge. They defined pre-algebraic knowledge in terms of Pillay, Wilss, and Boulton-Lewis (1998)'s model of

algebra development. Students were within the pre-algebra stage of development when they understood the relational meaning of equivalence (i.e. both sides are the same value) in nonstandard equations (e.g. $7 = 3 + 4$), recognized unknowns and variables in equations and expressions, and understood the concept of concatenation (i.e. $3x$ means 3 *times* x). Findings from the path analysis demonstrated that second grade word problem solving skills, calculation skills, approximate representation of numerical magnitudes, nonverbal reasoning, working memory, and attentive behavior had both direct and indirect effects on fourth grade students' pre-algebra knowledge. In addition, second and third grade calculation skills and word problem solving skills mediated the indirect effects of these second grade skills.

Instead of pre-algebraic knowledge, Lee et al. (2009) and Lee et al. (2011) focused on the contributions of general cognitive abilities and content knowledge on algebraic word problems. Specifically, they examined the effects of working memory, executive functioning, computational fluency, pattern recognition, problem formation, and problem representation. Lee and colleagues found an association between working memory and algebraic word problems. In particular, working memory was important for problem formation and problem representation. They also found that executive functioning had no bearing on algebraic word problems solving performance, but pattern recognition and computational fluency did.

In an attempt to understand a more complex form of algebra performance, Geary et al. (2015) studied knowledge of the coordinate plane, fluency and accuracy in evaluating algebraic expressions, and memory for algebraic equations in relation to students' acuity of the approximate number system (ANS) and memory for addition facts. While controlling for parental education, sex, reading achievement, speed of numeral processing, fluency of symbolic number processing, intelligence, and the central executive component of working memory, they

found that ANS acuity was related to knowledge of the coordinate plane and fluency in evaluating algebraic expression. On the other hand, memory for addition facts related to memory of algebraic equations only. While not considering how the control variables related to these algebraic topics, the researchers did examine the external validity of these algebraic topics as predictors of general algebra achievement. Algebra achievement was participant's performance on questions concerning solving for x , systems of equations, factoring, determining equation slope, and concept questions such as the definition for a vertical line. Results indicated that knowledge of the coordinate plane, fluency and accuracy in evaluating expressions, and memory for algebraic equations were all significant predictors of algebraic achievement in both separate and simultaneous regression analyses.

Even though these studies provided some evidence of the relation between cognitive abilities, content knowledge, and algebra performance, conclusions about students' algebra performance are restricted by the age of the studies population and the measures for algebra performance. Using elementary grade populations meant that the results from these few studies did not cover a full range of algebra problems, cognitive abilities, nor types of content knowledge. This leaves questions concerning how cognitive abilities and content knowledge may associate with students' algebra performance after formal algebra instruction. Thus, in this study I extended the research by considering how cognitive abilities and content knowledge related to the performance of students who have had formal algebra instruction. Additionally, given the differences in the knowledge and abilities of an elementary population versus those with formal algebra instruction, I also extended the research by considering other cognitive abilities and types of content knowledge that are relevant to algebra performance. In the following sections, I identify what these abilities and skills are and how algebra performance relates to them.

Cognitive Abilities Related to Algebra Performance

Research on content knowledge, cognitive abilities, and algebra performance has examined only a few cognitive abilities such as working memory, executive functioning, and processing speed. This is a limited perspective when there are many more types of cognitive abilities. Carroll (1993) defined cognitive ability as “any ability that concerns some class of cognitive tasks” (pg. 10), and cognitive tasks are “any task in which correct or appropriate processing of mental information is critical for performance” (pg. 10). In this sense, cognitive abilities are skills needed to perform certain types of tasks appropriately. Research suggested that cognitive abilities fall within a three stratum hierarchical model called the Carroll-Horn-Cattell (CHC) theory (Ackerman & Lohman, 2006; McGrew, 2009; McGrew & Wendling, 2010). Stratum I consisted of about 70 primary/narrow cognitive abilities. Each of these narrow abilities concerns different cognitive processes that are a greater specialization of the broad abilities found in Stratum II, and represent specific skills acquired through experience and learning (Carroll, 1993). Stratum II consisted of the nine broad abilities of fluid reasoning/intelligence, comprehension knowledge/crystallized intelligence, short-term memory, visual processing, auditory processing, long-term memory retrieval, processing speed, decision/reaction time, read and writing, and quantitative reasoning. The abilities in Stratum II are broad domains of behavior representative of an individual person with emphasis on the process (i.e. skills for reasoning, memory, and learning), content (i.e. information they know and perceive), and manner of response (i.e. speediness of response) (Carroll, 1993). Stratum III consisted of the general ability factor g , which is the idea that all cognitive abilities are independent factors of one main construct like general intelligence.

The research on cognitive abilities and general mathematics achievement has identified specific cognitive abilities that predict educational success in math. Some of these cognitive abilities were the broad abilities of crystallized intelligence, fluid intelligence, and processing speed (Floyd, Evans, & McGrew, 2003; Keith, 1999; McGrew & Hessler, 1995; McGrew & Wendling, 2010; Taub, Keith, Floyd, & McGrew, 2008) as well as the narrow abilities of phonological processing, working memory, and perceptual speed (McGrew & Wendling, 2010; Proctor, 2012). From these, the three cognitive abilities most related to algebra were (1) crystallized intelligence, (2) fluid intelligence, and (3) spatial abilities (Floyd et al., 2003; McGrew & Wendling, 2010; Proctor, 2012; Taub et al., 2008; Tolar, Lederberg, & Fletcher, 2009). Below is a discussion of both theory and research that suggest how each of these three cognitive abilities relates to algebra performance.

Fluid intelligence. Fluid intelligence is the ability to solve novel tasks that cannot be performed automatically using mental operations such as identifying relations, drawing inferences, concept formation, concept recognition, extrapolating, etc. (Horn, 1989; McGrew, 2009; McGrew & Evans, 2004). Research has shown that fluid intelligence related to many different types of mathematics achievement such as the measurement of students' ability to solve a range of calculations problems (i.e. calculation skills) and the measurement of students' ability to analyze and solve problems by comprehending what the problem is asking, recognizing relevant information, and choosing the appropriate strategy for calculations (i.e. problem solving) (Calderón-Tena, 2016; Floyd et al., 2003; Geary, 2011; Keith, 1999; McGrew & Wendling, 2010; Proctor, 2012). This cognitive ability related to algebra performance because the mental operations of fluid intelligence are similar to some algebraic reasoning skills. For example, the algebraic reasoning skill of generalization and functional thinking involve the

mental operations of drawing inferences, identifying relations, and extrapolation. Specifically, generalization requires that students draw inferences from a given set of information and transform that information into another form. This usually occurs when students are asked to write an algebraic equation in order to find a solution for a pattern of numerical values (Radford, 2006; Rivera, 2010; Rivera & Becker, 2008; Steele, 2008). In addition, functional thinking is about identifying the relations between quantities (Kaput, 1995; Usiskin, 1999), and understanding how one quantity changes because of changes in the another. Thus, functional thinking requires that students must identify the relationship between two quantities from a small set of given values and be able to apply it for any value not given. Given that both generalization and functional thinking require the use of the fluid intelligence mental operations of extracting and applying information, fluid intelligence may be a cognitive ability related to students' algebra performance.

Research also suggested that fluid intelligence was associated with students' understanding of algebra because it facilitated students' ability to connect natural language to mathematical symbols. Herscovics (1988) and Stacey and MacGregor (1997b) mentioned that translating from natural language into algebraic symbolism is an area of difficulty for students. A reason for this difficulty was that students were not able to process the meaning of the words in conjunction with how mathematical symbols were used (Clement, 1982; MacGregor & Stacey, 1993). The connection between the natural language and mathematical symbols may require that students develop a conceptual organization for the mathematical symbols, which includes the individual symbol's meaning as well as the meaning behind combinations of symbols. Students' ability to process the connection between the meaning of the words and the use of mathematical symbols determines the formation of this conceptual organization. For example, Chesney et al.

(2014) investigated students' conceptual organization of addition facts in relation to their understanding of the equal sign. They found those who mentally organized their addition facts around equivalent values were more likely to have a better interpretation of the equal sign. Since students' poor interpretation of the equal sign is linked to poor algebraic performance (Kieran, 1988, 1992; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Knuth, Stephens, McNeil, & Alibali, 2006; Seo & Ginsburg, 2003), the results from Chesney et al. (2014) demonstrated that students' fluid intelligence may be related to students' algebra performance because this difficulty highlights students' inability to perform the mental operation of concept formation.

Crystallized intelligence. Crystallized intelligence is the ability to apply the breadth and depth of acquired knowledge from cultural, educational and life experiences (Floyd, Evans, & McGrew, 2003; McGrew, 2009; McGrew & Evans, 2004; Proctor, 2012), which emphasizes the importance of cultural, societal, and everyday knowledge. This is an important idea that has already been shown to have some bearing on students' understanding of numbers and operations (Baranes, Perry, & Stigler, 1989; T. N. Carraher, Carraher, & Schliemann, 1985; T. N. Carraher, Carraher, & Schliemann, 1987; Keith, 1999; Schliemann & Carraher, 2002). Carraher, Carraher, and Schliemann (1985) in their study about how Brazilian children solved mathematics in and out of school found that children of street vendors were better able to solve math problems when they were presented in an out of school context rather than an in school context. Also, Baranes, Perry, and Stigler (1989) found that when math problems were presented in an out of school context using numbers that were meaningful to the context, students performed better on number matched content types of problems than the school type problems. This suggested that participants' understanding of numbers and operations was dependent upon their cultural, societal, and everyday knowledge.

Additionally, measures of crystallized intelligence rely heavily on language, which have a big influence on students' understanding of mathematical concepts as well as their algebra performance (Barton, Fairhall, & Trinick, 1998; MacGregor & Stacey, 1997; Philipp, 1992). For instance, one particular concept that is an area of difficulty is the concept of variable. The main reason for students' trouble with the concept of variable was the many different uses of variables and how it is presented to students (Kieran, 1988; MacGregor & Stacey, 1997; Philipp, 1992). Philipp (1992) and MacGregor and Stacey (1997) both recognized the importance and influence language based teaching materials had on students' understanding of the concept of variable. MacGregor and Stacey (1997) demonstrated that the initial teaching practice of using letters as abbreviated words and letters contributed to the misinterpretations students had with variables. Likewise, Philipp (1992) theorized that a lack of discussion in mathematics leads to syntactically but semantically weak understanding of concepts. He used two discussion activities on the concept of variable to demonstrate that students can develop a better understanding of variables by talking about the many different ways variables are used. Since interactions with teachers and teaching materials rely on language, this research would suggest that language impacts students' understanding of the concept of variable, and in turn their algebra performance.

Spatial abilities. There are many different types of spatial abilities, but there is no broadly accepted definition that defines all spatial abilities. Some researchers have attempted to categorize the broad set of abilities referred to as spatial abilities based on whether or not spatial tasks involve a single object (i.e. *intrinsic*) or group of objects (i.e. *extrinsic*) and the movement of these objects in space (i.e. *dynamic* or *static*) (Mix & Cheng, 2012; Uttal et al., 2012). However, most researchers simply define spatial abilities based on a specific type of ability being assessed. For instance, spatial abilities have been defined in terms of disembedding, spatial

visualization, mental rotation, spatial perception, and perspective taking (Mix & Cheng, 2012; Uttal et al., 2012). In the current work, I focus on the type of spatial ability found to be associated with algebra performance (Chrysostomou et al., 2013; Logan, 2015; Terao, Koedinger, Sohn, Qin, & Anderson, 2004; Tolar et al., 2009), namely spatial visualization, which refers to the mental representation and transformation of an object or groups of objects in a 2-dimensional and 3-dimensional plane.

The empirical findings on the relationship between spatial abilities, broadly defined, and mathematics achievement are mixed, with some research suggesting that spatial abilities are important for mathematical development (e.g. Gunderson, Ramirez, Beilock, & Levine, 2012; Proctor et al., 2005; Skagerlund & Träff, 2016), and others suggesting the opposite (e.g. Floyd et al., 2003; McGrew & Wendling, 2010). The inconsistency of the connection between spatial abilities and mathematics achievement may be because of the cognitive ability tests used to measure spatial abilities. For instance, measures of *visual-spatial thinking*, which is the ability to create, recognize, and transform visual images, typically use multiple forms of spatial abilities including visualization, spatial relations, closure speed, visual memory, spatial scanning, etc. (Floyd et al., 2003; McGrew, 2009; McGrew & Evans, 2004; Proctor, 2012). It is possible that including all these spatial abilities as a measure of *visual-spatial thinking* washes out the relation that any one of the spatial abilities may have with general mathematics achievement. This is evident by a detailed analysis of working memory that suggested the spatial component of working memory was predictive of general mathematics achievement (Kyttälä & Lehto, 2008; Reuhkala, 2001), and visual spatial representations were important for solving word problems (Hegarty & Kozhevnikov, 1999; van Garderen, 2006).

As for algebra performance, there have been a few studies that identified a relation with the specific spatial ability of *spatial visualization* (Chrysostomou et al., 2013; Logan, 2015; Terao et al., 2004; Tolar et al., 2009). Spatial visualization is the ability to imagine, manipulate, or transform mental images and identify how they would appear under different conditions (McGrew & Evans, 2004; Mix & Cheng, 2012; Tartre, 1990), and is only one type of spatial ability. Measurements of spatial visualization abilities involve mental rotation and mental transformation tasks (Tartre, 1990). Mental rotation tasks measure the ability to be able to determine if more than one objects are the same by mentally rotating them while mental transformation tasks measure the ability to identify the transformation of object by either mentally putting it together or taking it apart (Tartre, 1990). Studies on spatial visualization and algebra performance have shown that in a structural equation model it had a direct effect on algebra achievement (Tolar et al., 2009), was related to student performance on numeracy and algebraic reasoning (Chrysostomou et al., 2013), and solving algebra word problems activated the visuospatial regions of the brain (Terao et al., 2004). Spatial visualization was theorized to be important for algebra performance because in general, elements of mathematics inherently have a visuospatial component (Logan, 2015; Terao et al., 2004) and specifically, because algebra requires the ability to represent functional relationships graphically and manipulate visual-spatial representations mentally (Tolar et al., 2009).

Content Knowledge Related to Algebra Performance

Researchers investigated student learning to determine the reasons why students had difficulty learning algebra. Some of this research concentrated on eliciting student thinking while solving algebra problems in order to identify students' errors and misconceptions about algebra (Booth, 1984; Greenes & Rubenstein, 2008; Kuchemann, 1978; National Council of Teachers of

Mathematics, 1988; M. Russell, O'Dwyer, & Miranda, 2009; Stacey & MacGregor, 1997b; Welder, 2012). Others tried to determine which foundational mathematical concepts were necessary for solving algebra problems (Britt & Irwin, 2008; K. C. Irwin & Britt, 2005; Ketterlin-Geller & Chard, 2011; Ketterlin-Geller, Gifford, & Perry, 2015; National Mathematics Advisory Panel, 2008; Schifter, 1999; Stacey & MacGregor, 1997a; Wu, 2001). From these investigations, multiple researchers have proposed many different types of content knowledge needed for learning algebra. The content knowledge needed for learning algebra consisted of things that student should be able to do (Ketterlin-Geller & Chard, 2011; Ketterlin-Geller et al., 2015; Kieran, 1988, 1992; Schifter, 1999; Stacey & MacGregor, 1997a; Wu, 2001) as well as the necessary understandings needed to do those things (Blanton & Kaput, 2005; Herbert & Brown, 1997; Herscovics, 1988; Jacobs et al., 2007; Kaput, 1999; Kieran, 2004; M. Russell et al., 2009; Stacey & MacGregor, 1997b; Welder, 2012). Theory and empirical evidence, taken together, point to the connection between content knowledge and students' algebra performance as being dependent upon four key types of content knowledge: (1) understanding numbers and operations, (2) proficiency with fractions, (3) understanding equivalence, and (4) algebraic reasoning. Below is a discussion about the empirical evidence that demonstrated the relation between these types of content knowledge and algebra performance.

Understanding of numbers and operations. Multiple researchers suggested that an understanding of numbers and operations is important for students' understanding of algebra (Britt & Irwin, 2008; K. C. Irwin & Britt, 2005; Ketterlin-Geller & Chard, 2011; Ketterlin-Geller et al., 2015; National Mathematics Advisory Panel, 2008; Schifter, 1999; Stacey & MacGregor, 1997a). An understanding of numbers and operations involves being able to understand place value, understand numerical magnitudes, to compose/decompose numbers, understand

mathematical operations, as well as understand the properties of numbers. Stacey and MacGregor (1997a) proposed that an understanding of numbers helps when students fail to remember the rules of symbolic manipulation. Students can use their knowledge about how to manipulate numbers to figure out how to work with algebraic expressions. K. C. Irwin and Britt (2005) theorized that the ability to generalize mental operational strategies and use them effectively to solve different numerical problems represents algebraic thinking. It was algebraic thinking because students were using the number as variables instead of letters. An investigation into the development of students' generalized mental strategies demonstrated that those who had instructional experiences that promoted generalized mental strategies exhibited the use of algebraic operational strategies more often in comparison to others who had not. In a second study, Britt and Irwin (2008), showed that after developing their mental strategies students were able to transfer this knowledge to literal symbols. Given that students were able to demonstrate the same skills for numbers and operations with letters as variables suggested that their understanding of numbers and operations was associated with their algebra performance.

Proficiency with fractions. Fractions are mathematical quantities that represent parts of a whole, and researchers theorized that students' understanding of how to operate and use fractions relates to their algebra performance. Researchers have theorized various reasons for the connection between fractional knowledge and algebra performance. Wu (2001) and the National Mathematics Advisory Panel (2008) suggested that students' knowledge about fractions provides a stepping stone for their understanding of rules of symbolic manipulation used in algebra. Empson, Levi, and Carpenter (2011) suggested that fraction knowledge related to students' algebra performance because solving problems involving operation on and with fractions uses the same understanding of numbers and operations that are necessary for algebra. Additionally,

Kilpatrick and Izsak (2008) claimed that some big ideas used by fractions are also important for algebra. Specifically, an understanding of fractions helps to develop students' multiplicative structures and understanding of the distributive property, which are important ideas for working with algebraic equations and expressions. Even with these reasons, only a few studies have shown that proficiency with fractions does impact algebra performance (e.g. G. Brown & Quinn, 2007; Siegler et al., 2012). G. Brown and Quinn (2007) investigated students' proficiency with fractions in relation to their success in algebra. Proficiency, defined by this study, was as the ability to understand fractional concepts and manipulate fractions for accurate computation without the aid of a calculator. Success in algebra was student tests scores from an algebra course. A Pearson Correlation Coefficient calculation determined that there was a significant relationship between the two test scores. In addition, Siegler et al. (2012) found that fraction knowledge measured 5 or 6 years before algebra instruction was a stronger predictor of later algebra achievement than other types of mathematical knowledge, general intelligence, working memory, and family income and education.

Understanding of equivalence. Mathematics is heavily depended upon the use of symbols, and one symbol that demonstrated a relation to students' algebra performance is the equal sign. The equal sign represents equivalence, which is the understanding that two quantities on each side of the equals sign are the same. Equivalence is seen as a common issue related to algebra because misunderstanding equivalence makes it difficult to understand, remember, or apply algebraic processes and principles (Seo & Ginsburg, 2003). Additionally, the misunderstanding of equivalence limits students' understanding of why the process of doing the same operation on both sides of the equation balances the equation (Kieran, 1988), limits their understanding of the transformations used to solve algebraic equations (Knuth et al., 2005), and

limits their ability to perceive algebraic expressions as mathematical objects rather than processes (Kieran, 1992). Additionally, researchers have demonstrated that students' understanding of equal sign as well as how misinterpretations of this symbol led to difficulties with algebra. In particular, Knuth, Stephens, McNeil, and Alibali (2006) examined middle school students' understanding of the equal sign in relation to their performance on solving algebraic equations. Results demonstrated that participants with a relational understanding of equivalence (i.e. the equal sign means the same as) were more likely to solve algebraic equations correctly regardless of grade level and mathematics ability.

Algebraic Reasoning. Some researchers found that the emphasis on teaching symbolism before the application of mathematical knowledge to word problems was counter to how students naturally approached problems (Godino, Aké, Gonzato, & Wilhelmi, 2014; Godino et al., 2015; Nathan & Koedinger, 2000; Nathan & Koellner, 2007; Nathan & Petrosino, 2003; Sfard & Linchevski, 1994). The researchers realized that students used verbal reasoning skills before symbolism, which allowed students to use their mathematical knowledge to solve algebra problems. As such, some researchers claimed that there are reasoning skills related to algebra. They suggested that the underdevelopment of these reasoning skills makes some students unsuccessful. Kaput (1998, 2000, 2008) stated that there are five forms of reasoning that constitute the reasoning skills needed for algebra: (1) the expression and use of generalizations based on arithmetic and quantitative reasoning, (2) reasoning with and acting upon symbols given the rules of manipulation, (3) the understanding of mathematical structures as objects rather than process based on generalizations built from arithmetic and quantitative reasoning, (4) the understanding of the relationship between two or more quantities that vary, and (5) the use of mathematic symbols as a language to express situational behaviors.

Beyond the validation studies for early algebra education (e. g. Bastable & Schifter, 2008; M. L. Blanton & Kaput, 2004, 2011; D. W. Carraher, Martinez, & Schliemann, 2008; Cooper & Warren, 2011; Lannin, 2003, 2005; Moss & McNab, 2011; S. J. Russell et al., 2011), there has not been much research that has examined the connection between students' algebra performance and their algebraic reasoning skills. However, one particular study evaluated the effect of sustained and comprehensive early algebra instruction. The study was in fact the first step in trying to determine the influence of algebraic reasoning skills on students' understanding of algebraic concepts. Blanton et al. (2015) implemented a yearlong intervention focused on developing students' algebraic reasoning alongside the traditional arithmetic focused elementary mathematics curriculum. Results demonstrated greater improvement in the understanding of algebraic concepts and practices of intervention students as compared to nonintervention students. The researchers suggested that traditional arithmetic-focused elementary mathematics curriculums alone are not enough to prepare students in the future for algebra. The results also suggested even though some basic mathematical content knowledge like understanding numbers and operations and fractions may be associated with algebra performance, algebraic reasoning skills are necessary for strong algebra performance. Thus, it is important to include algebraic reasoning skills as factors of students' algebra performance.

Current Study

When trying to understand why students perform differently from expectations, past algebra research assumed that it was because of students' content knowledge and the instruction that they experienced (e.g. Ketterlin-Geller & Chard, 2011; Ketterlin-Geller et al., 2015; Nathan & Koedinger, 2000; Sfard & Linchevski, 1994). Researchers conducted studies that examined students' understanding of multiple mathematical concepts (e.g. variables and equivalence,

Knuth, Alibali, McNeil, Weinberg, & Stephens, 2011), but little attention was given to other factors like cognitive abilities, which are the underlying cognitive processes that can affect what a person knows and understands. Cognitive abilities are another aspect of the individual that changes the ways students perform. Thus, in this study I examined both cognitive abilities (i.e. crystalized intelligence, fluid intelligence, spatial visualization) and content knowledge (i.e. understanding numbers and operations, fractions, equivalence, algebraic reasoning) in order to determine the different factors and combinations of factors that have a relation with students' algebra performance.

Rationale for design. In particular, I studied the cognitive abilities and content knowledge of undergraduate students. The population is particularly important because previous cognitive studies have always focused on student who are just beginning to study algebra, and rarely considered those who are expected to have a good grasp of the mathematical content (Kieran, 1990, 2006). This is particularly problematic because research has shown that algebraic misconceptions can linger well after formal algebra instruction (e.g. Bernardo et al., 1994; McNeil & Alibali, 2005b; Triguero & Ursini, 2003). For instance, multiple studies demonstrated that even though undergraduate students understand the equal sign as a relational symbol, an unsophisticated understanding still exists, and when this unsophisticated understanding is activated students perform in similar ways to beginning algebra students (Chesney & Mcneil, 2014; Chesney, McNeil, Brockmole, & Kelley, 2013; McNeil & Alibali, 2005b; McNeil, Rittle-Johnson, Hattikudur, & Peterson, 2010). This performance is especially salient under timed conditions. Since the present study has time constraints it is possible that undergraduate students will demonstrate a less sophisticated understanding of equivalence that would relate to their algebra performance. Similarly, prior research has shown that even though undergraduate

students have a conceptualization of variables, they lack a rich conceptualization (Trigueros & Jacobs, 2014; Trigueros & Ursini, 2003; Weinberg, Dresen, & Slater, 2016) and use them incorrectly (Akgün, 2011; Bernardo & Okagaki, 1994; Clement, 1982; Clement, Lochhead, & Monk, 1981).

With algebra having a heavy emphasis on both equivalence and variables, having these sorts of difficulties could influence performance. Thus, studying undergraduate students should provide an understanding of how cognitive abilities and content knowledge relate to algebra performance in similar ways that studying beginning algebra students would. It is also possible that students' cognitive abilities may have helped them overcome these obstacles. Therefore, by studying college students, I am able to determine if students' performance is heavily reliant on content knowledge or if cognitive abilities could have a supportive impact on students' performance when content knowledge is limited.

Another aspect that is different about this study is that in investigating cognitive abilities and content knowledge I am taking both a variable-oriented and person-oriented approach to algebra performance. Both of these approaches allow me to examine algebra performance in different ways. The person-oriented approach acknowledges the person as a complex system with many factors of influence, and seeks to answer questions concerning the differences among individuals (Bergman et al., 2003; Laursen & Hoff, 2006; Magnusson, 2003). This approach allows me to understand common patterns of knowledge and examine how these patterns relate to performance. It can describe what types of students do well or poorly. For example, a cognitive obstacle students faced in algebra has been the difference in meaning and interpretations given to operations and symbols in arithmetic as compared to their use in algebra (Booth, 1984; Kieran, 1990). The person-oriented approach can identify if those who do well

have strong vocabulary, reading skills or some other aspect of individual functioning. In contrast, the variable-oriented approach assumes that each individual will perform in similar ways and tries to determine how the different variables independently predict algebra performance, on average for the entire sample (Bergman et al., 2003; Laursen & Hoff, 2006; Magnusson, 2003). For the same cognitive obstacle mentioned above, the variable-oriented approach revealed that much of the difficulty students faced can be traced back to the type of instruction they had received (e.g. McNeil, 2008; McNeil et al., 2006), and changing that instructional experience helped (e.g. Chesney et al., 2014; McNeil, Fyfe, Petersen, Dunwiddie, & Brletic-Shipley, 2011). Separately, these two approaches may reveal two different stories, but taken together they provide new insights into the connection between content knowledge, cognitive abilities, and algebra performance. Specifically, these new insights could possibly provide new avenues for teaching algebra as well as new ways to prepare students for learning algebra.

Research questions and hypotheses. Much of what we know about the variations in students' algebra performance has come from research that has examined how students solve algebra problems (e.g. Booth, 1984; Kuchemann, 1978). The focus of this research was on how the individual students made sense of the problems in order to solve them. This research has provided a wealth of information that has led to new understandings about student knowledge and the influence of instructional practices (Booth, 1984; Kieran, 1990, 2006). Nevertheless, recent research into the teaching and learning of algebra has focused less on the differences across individual students and more on the general tendencies among groups of people. This approach may have contributed to the slowdown in progress research was making in improving students' algebra performance as evident from the recent analysis of NAEP scores (Kloosterman, 2016; Perez et al., 2016). The current study returns the focus back on the differences across

individuals in order to determine what other factors besides content knowledge link to students' algebra performance. Specifically, this study allowed me to answer the following questions:

RQ 1: Which forms of content knowledge and cognitive abilities most strongly predict algebra performance? One goal of this study is to determine if other factors besides content knowledge predict algebra performance. Prior research on the factors associated with mathematics achievement have demonstrated that both content knowledge and general cognitive abilities (Fuchs, Geary, Compton, Fuchs, Hamlett, Seethaler, et al., 2010; Fuhs, Hornburg, & McNeil, 2016) predicted mathematics achievement; therefore, I expect that both students' content knowledge and cognitive abilities will predict their algebra performance. It is uncertain which specific skills will indicate a significant relation to students' algebra performance, but I hypothesize that the assessments of fluid intelligence, algebraic reasoning, equivalence, and numeracy will be particularly important predictors. These skills are particularly important predictors of algebra performance because they are measures of abilities that allow students to reason algebraically. For example, Britt and Irwin (2008) and K. C. Irwin and Britt (2005) connected students' numeracy skills to their ability to generalize with variables, which are an important part of algebra (e.g. Booth, 1984, 1988; Knuth et al., 2005). With fluid intelligence involving the same general skills as algebraic reasoning, it also seems to represent a more general reasoning ability that might facilitate algebra performance. In addition, equivalence is important because those who think in relational terms have the necessary skills to reason algebraically (M. Stephens, 2007; M. Stephens & Wang, 2008). Moreover, researchers theorized that algebraic reasoning skills are important for learning algebra because they represent different ways of thinking based on the different approaches to teaching algebra (Bednarz et al., 1996; Kaput, 1998, 2000, 2008), and Kloosterman (2016) suggested that even when algebraic content

knowledge was lacking, reasoning skills helped to bolster 4th and 8th grade students' algebra performance.

RQ 2a: What combination of content knowledge and cognitive abilities naturally occur in students who have studied algebra? Person-oriented analyses describe groups or types of individuals that share particular traits, attributes, or relations among variables (Laursen & Hoff, 2006; Magnusson, 2003). These groups can help researchers identify specific reasons why certain students perform the way that they do. By grouping people who are similar in content knowledge and cognitive abilities together and examining their algebra performance as a function of their grouping, one can gain a more natural picture to how different combinations of predictors are associated with algebra performance. Current studies suggested that linked to mathematics performance are combinations of content knowledge and cognitive abilities, so I hypothesize that it is possible that a mixture of content knowledge and cognitive abilities or content knowledge and cognitive abilities by themselves may characterize groups of participants. For my mixture hypothesis, there could be combinations of skills and abilities defined by participants' performance scores that could be high or low for fluid intelligence, algebraic reasoning, equivalence, and numeracy because they measure skills related to algebraic reasoning. Given that there is a lack of research that used a person-oriented approach, this hypothesis just speculates about what could possible occur.

RQ 2b: How do students with these different content knowledge and cognitive abilities profiles perform in algebra? It is impossible to know exactly what profiles will emerge, but I theorize that there will be the typical high and low performance groups characterized by strengths and weaknesses in both content knowledge and cognitive abilities or in one over the other that will have different algebra performance. Additionally, there could be unique groups,

which would defy expectations. For example, an expected low content knowledge group might perform well because they have strong cognitive abilities. Similarly, a group expected to do well (e.g. high content knowledge) may have low performance because they have weak cognitive abilities. Overall, I hypothesize that participants' cognitive abilities will preclude better algebra performance over having a good grasp of content knowledge.

CHAPTER 3: Method

Participants

Participants were ($N = 141$) undergraduate students at Michigan State University from all school levels (34 freshman, 29 sophomore, 30 junior, 37 senior, 11 5+ year senior) and a variety of majors (27% Biological Sciences, 19.1% Communication Arts & Sciences, 12.1% Social Sciences, 9.2% Business & Management, 8.5% Engineering, 8.5% Education, 6.4% Environmental Sciences, 3.5% Fine Arts & Letters, 2.1% Physical & Mathematical Sciences, 2.8% Undecided, 0.7% Unknown). The mean age of participants was 20.28 years ($SD = 1.98$). The majority of participants were female students (110 female, 31 male). The distribution of race/ethnicity in the sample was as follows: 8.5% Asian, 22% Black/African American, 1.4% Hispanic/Latino, 61.7% White, and 6.4% Multiracial.

Assessments and Measures

In the sections below, I described each of the assessments and measures used in the current study. There are three assessments for cognitive abilities, five assessments for content knowledge, and a single outcome measure for algebra performance. Additionally, see Table 1 for the number items, time limit, and calculator usage of each assessments and measures.

Cognitive abilities. In the literature, the most common test of human cognitive abilities is the *Woodcock Johnson Tests of Cognitive Abilities* (WJ), which is a licensed restricted assessment in its fourth edition that measures all nine of the broad cognitive abilities as defined by the CHC theory of cognitive abilities (Mather & Wendling, 2014). Due to the restricted nature of the WJ, this study will be using alternative assessments for measuring participants' cognitive abilities. The assessments for cognitive abilities consisted of a single measure to assess fluid and crystallized intelligence, and two measures to assess spatial visualization.

Fluid and crystallized intelligence. To assess fluid and crystallized intelligence this study used the Reynolds Adaptable Intelligence Test (RAIT). The RAIT is a good alternative to the WJ because 1) it is a standardized assessment that does not require a license in psychology, 2) it measures fluid and crystallized intelligence in a similar way to the WJ, 3) it allows for group or individual administration, and 4) it has standardized scores for participants 10-75 years old. It consists of seven subtests for the measurement of fluid intelligence, crystallized intelligence, and quantitative intelligence. Each subtest has a time limit that allows the RAIT to be a “power” test instead of a “speeded” test which means that the time limits were set based on 95% of the participants being able to get the same number of answers correct regardless if they were given a time limit or not. For each subtest, raw scores were the number of questions correctly answered, which converted into T-scores with a mean of 50 and a standard deviation of 10. There was no penalty for incorrect answers. A combination of subtests yielded scaled scores or indexes for fluid intelligence, crystallized intelligence, quantitative intelligence, total intelligence (i.e. fluid and crystallized combined), and total battery intelligence (i.e. all three combined). The indexes are scaled scores set to a mean of 100 and a standard deviation of 15. This study only used the subtests for the Fluid Intelligence Index and the Crystallized Intelligence Index.

Fluid intelligence index (FII). The FII consisted of the subtest of Sequences (SEQ) and Nonverbal Analogies (NVA), which measure deductive reasoning using nonverbal reasoning tasks. The summation of the scaled scores for each subtest yields the FII scaled score. The median reliability for SEQ and NVA are .86 and .89 respectively, and the FII has a median composite reliability of .93. The SEQ subtest asked participants to complete a series of pictures that denoted a change progression by picking which image went next. Participants had 10 minutes to complete the section. The NVA subtest asked participants to complete an analogy of

the format ____ is to ____ as ____ is to ____ using images instead of words. Participants got three images and had to choose the fourth that completed the relationship. NVA had a time limit of 7 minutes.

Crystallized intelligence index (CII). Three subtests comprised the CII. All three subtests assessed crystallized intelligence by using verbal reasoning tasks that invoked inductive reasoning. The three subtests are General Knowledge (GK), Odd Word Out (OWO), and Word Opposites (WO). They have median reliabilities of .84, .83, and .81 respectively, and the CII has a median composite reliability of .93. The GK subtest measured common cultural knowledge, reasoning skills, and classification skills by asking participants to categorize the names of well-known people into one of six categories like politics, military, religion, arts, and sciences. Participants only had 3 minutes to make these categorizations. OWO assessed vocabulary and verbal reasoning by having participants choose the one word that did not belong in the set. The set contained a group of five words where four words had a conceptual link to one another and one word did not. Participants had 5 minutes to complete this subtest. The last subtest WO also assessed vocabulary and verbal reasoning by having participants choosing the one word out of five that had the opposite meaning of the target word. Participants had 5 minutes to complete this subtest.

Spatial visualization. In order to measure spatial visualization, this study used two assessments that are similar to how WJ-IV measures spatial visualization. The measures were the *Measure of the Ability to form Spatial Mental Imagery* (MASMI) and the *Measure of the Ability to Rotate Mental Images* (MARMI) (Campos, 2009, 2012), which are companion measures of spatial visualization that used a net of an unfolded cube with different symbols on each side. In the WJ-IV, the measurement for spatial visualization used (a) a spatial relations task where

participants have to determine which pieces put together form a complete shape, and (b) block rotation, which requires the participants to identify the same 3-dimensional shape when rotated. The MASMI is the match for the spatial relations task because it asked participants to mentally reassemble the cube in order to identify which cubes had the correct symbol for the left and right side of the target cube. Although different from the WJ-IV, the MASMI is measuring a similar construct because each task asked participants to form a mental image of a shape and answer questions about that shape. The MARMI is the match for the block rotation because each task had participants identify blocks that are the same when given a different rotation. In both assessments, participants have to reassemble the cube in their mind in order to answer the assessment questions. Each question presents the same unfolded cube reassembled and rotated differently. Each assessment had 23 questions with four options. Of these four options, two options were correct and two were incorrect. Participants had to pick the two options that they thought were correct for each question. There was a penalty for incorrect responses where participants lost one point from the total amount of correct responses for each incorrect response. The internal sample reliabilities for MASMI and MARMI are .82 and .93 respectively.

Content knowledge. The assessments for content knowledge measured participants' performance on a number of tasks that related to the important skills and concepts of understanding of numbers and operations, fractions, equivalence, and algebraic reasoning. See Appendix C for complete details about all content knowledge assessments, and see Table 2 for Cronbach's Alpha internal reliabilities for each assessment based on the given sample.

Understanding of numbers and operations. An understanding of numbers and operations is demonstrated by: (1) an understanding of place value and magnitude, (2) the ability to compose and decompose numbers, (3) grasping the meaning of the operations, (4) the ability

to use and understand the properties of distributive, commutative, associative, and (5) automatic recall of addition, subtraction, multiplication, and division facts (Ketterlin-Geller & Chard, 2011; Ketterlin-Geller et al., 2015; National Mathematics Advisory Panel, 2008). The current study used two assessments (i.e. computational fluency and numeracy) to measure these different skills. Together both assessments measured each of the different ways to demonstrate an understanding of numbers and operations given above. The computational fluency assessment covers the an understanding of place value and recall of math facts (i.e. points 1 and 5) while the numeracy assessment covers the understanding of magnitude, the ability to compose and decompose numbers, and the ability to use the properties of numbers and operations (i.e. points 1, 2, 3, 4).

Computational fluency. This measure demonstrated an understanding of numbers and operations by assessing students' understanding of place value and their recall of addition, subtraction, multiplication, and division facts. It was called computational fluency to describe the fact that it asked participants to solve a number of computational math problems within a given time period. To measure computational fluency, this study used the Curriculum Based Measurement (CBM) Computational Fluency Assessment, which is a different type of assessment from those used by others (see Chapter 5 for further discussion). The CBM-Computational Fluency Assessment is a time based assessment that allows for teachers to quickly and efficiently assess how accurately students can solve addition, subtraction, multiplication, and division problems (Wright, 2013). The computational fluency assessment considers accuracy to be the correct number of digits per problem. For example, in the given problem $6220 + 3545$ the correct answer is 9765 which means that a participant can receive an accuracy score of 4 or less for this problem. The participant would receive a score of 4 if they

gave 9765 as their answer, but would receive a score of 3 if they gave 9755 because the second five is an incorrect digit. The correct digit coding is a more nuanced way of examining participants' computational fluency. It gives them credit for being able to add, subtract, multiple, and divide numbers while also allowing for the occasional error. For this assessment, participants received a worksheet that included 16 items for addition, subtraction, multiplication, and division math facts. The worksheet focused on multi-digit calculations with regrouping and no remainders. Participants solved problems involving 2-6 digit numbers. They had 3 minutes to complete as many as they could without the use of a calculator.

Numeracy. Numeracy is the ability to reason with numbers and numerical concepts by using knowledge about mathematical relationships. Numeracy skills include flexibility and efficiency with strategies, an understanding of algorithms, and using reasoning skills to calculate instead of performing prescribed steps (Harris, 2011). This assessment consisted of 24 multiple-choice questions taken from the *As Close as It Gets* activities from Harris (2014). *As Close as It Gets* activities are multiple-choice questions with answer choices that do not have the correct answer. Such questions require students to consider the numbers in the problem to inform which strategy to use as well as determining the reasonableness of an answer choice based on the magnitudes of the numbers. These types of questions measure place value concepts, magnitude, the ability to compose and decompose numbers, meanings of operations, and an understanding of the properties of numbers. The 24 questions of this assessment consisted of six problems for each mathematical operation. In addition, the questions included problems with whole numbers, decimals, and fractions. To encourage students to use their numeracy skills rather than algorithms, participants had only 4 minutes to complete the assessment without the aid of a

calculator. The time limit calculations were a 10-second time limit for each question and a 1-minute time limit for each operation.

Fractions. G. Brown and Quinn's (2006) 25-question fractions assessment was a guide for the development of the fraction assessment for this study. The assessment used questions from previous research and questions devised by the researchers. It assessed conceptual knowledge and computational fluency aimed at developing an understanding of rational numbers. There were six categories of questions; algorithmic operations, application of basic fraction concepts in word problems, elementary algebraic concepts, specific arithmetic skills prerequisite for algebra, comprehension of the structure of rational numbers, and computational fluency.

In this study, I adapted this assessment to include twelve questions covering all of the six categories. I kept two questions from each of the six categories. The algorithm operations category asked participants to find the difference of fractions and write a mixed number as an improper fraction. The word problem category asked participants to solve two word problems. The elementary algebraic concepts category question asked participants to solve two algebraic equations involving fractions. As part of the arithmetic skills prerequisite for algebra category, participants wrote a fraction in form of a sum and evaluated the value of fraction divided by zero. The questions from comprehension of the structure of rational numbers asked participants to order fractions and to compare the values of fractional quotients. For the computational fluency category, one question asked participants to find the sum of a complex fraction equation, and the second asked participants to find the remaining fractional component needed to equal one. Participants got 10 minutes to complete as many questions as they could without the aid of a calculator.

Equivalence. The equivalence assessment evaluated participants' understanding of equivalence. Using prior research Matthews et al. (2012) created an equivalence assessment compiled from three different ways of measuring students' understanding of equivalence; open equation solving items (e.g. $8 + 4 = __ + 5$), equation structure items (e.g. is $3 + 5 = 5 + 3$ true or false), and equal sign definition items. Matthews et al.'s (2012) assessment was for participants in grades 2-6. Since the participants in this study were much older, it was not feasible to use the same assessment. Nevertheless, the design of this assessment used similar items.

This equivalence assessment consisted of six items; two open equation items, two equation structure items, and two equal sign definition items. All questions on this assessment required some form of explanation for participants' answers, so participants received full credit (e.g. 2 points) for problems if and only if they provided an appropriate explanation. Students' understanding of equivalence can be separated into two distinct views; a *relational* view and an *operational* view (Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; Baroody & Ginsburg, 1983; Kieran, 1981; Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008; Knuth et al., 2005, 2006; MacGregor & Stacey, 1999; McNeil, 2007; McNeil & Alibali, 2005a). The relational view of equivalence is the general idea that the equal sign represents the relationship that the two quantities separated by the symbol are the same. The operational view of equivalence is the general idea that the equal sign means the answer comes next or to apply the operation to all the numbers. An appropriate explanation had answers that expressed a relational understanding, which used terms such as "the same as", "can mean two numbers are the same", "same as the other number", or "the value on one side is the same as the value on the other side". Otherwise, participants received partial credit (e.g. 1 point) for either having the correct answer and a non-

relational explanation or an incorrect answer and a relational explanation. Participants received no credit for incorrect answers and non-relational explanations. Participants received 5 minutes to complete the assessment without the aid of a calculator.

Algebraic reasoning. The algebraic reasoning assessment evaluated participants' algebraic reasoning skills. The assessment consisted of two items for each of the five forms of algebraic reasoning. The five forms of algebraic reasoning are generalization, functional thinking, modeling, symbolic manipulation, and structure sense (Kaput, 1998, 2000, 2008). The items were questions used in prior research or were researcher designed based on common errors identified from prior research. Generalization is the ability to abstract out common relationships in order to apply rules to any particular instance such as writing equations from patterns. The generalization items in this assessment asked participants to write an equation or an expression that represented a functional relationship based on information presented in tabular form. Functional thinking is the ability to understand the relationship between two or more quantities such as being able to describe the relationship in multiple forms (e.g. symbolically or in words). The functional thinking items in this assessment required the participants to determine the relationship between two or more items using a table format. Then participants had to describe the relationship using words, an equation, and/or an expression. Modeling is the ability to represent real world situations with mathematics such as translating a written relationship into symbolic form. The modeling items in this assessment required participants to translate a written relationship into an equation. Symbolic manipulation is the ability to understand the rules that govern how to work with symbols such as knowing how to combine like terms, knowing how to balance an equation, and being able to evaluate symbolic solutions for accuracy. In this assessment, symbolic manipulation items required participants to write an equivalent expression

for a given expression. Structure sense is the ability to look at the structure of a problem and use that understanding to help solve the problem. The structure sense items in this assessment asked participants to solve equations that have cancellable elements that make finding the answer easier. The ability to see cancellable elements of a problem to make problem solving easier rather than just solving the problem is an indicator of structure sense, and participants received credit for these questions if and only if they demonstrated cancellation. Participants received 10 minutes to complete the assessment without the aid of a calculator.

Algebra performance. The algebra performance assessment measured participants' current algebra performance. Included in this assessment were the following topics: systems of equations, functions, solving equations, inequalities, graphing, exponents, factoring, complex numbers, polynomial division, and logarithms. The assessment consisted of 20 questions. The items were multiple-choice items with five answers choices. Participant had 20 minutes to complete the entire assessment. Questions for this assessment were from the *National Assessment of Educational Progress* (NAEP) mathematics released item database ($n = 9$) and supplemented by researcher-designed questions ($n = 11$). The NAEP release items are algebra content questions from the years of 1990-2013. The questions range in difficulty from medium to hard. This assessment used NAEP questions because multiple panels of experts had reviewed them, and the questions were pilot tested (National Center for Educational Statistics, 2016). The supplemental questions covered additional topics not found in the NAEP item release database. The inclusion of these items gave a good sample of questions that covered a range of topics found in college algebra courses.

Participant demographic survey. The participant demographic survey collected demographic information about each participant (see Appendix C). The survey asked

participants to identify their age, gender, race/ethnicity, school level, and major in school. The survey also asked about past mathematics education. Since the participants were college students, time and level of algebraic exposure are confounding variables. These are confounding variables because memory decays over time, especially for information that is not used on a regularly basis (Santrock, 2008), and as evident from practice effects the more exposure that a person has to something the better that can become (Bullard, Griss, Greene, & Gekker, 2013). Given these circumstances, there was a need to establish the length of time since the participants have taken a math course or studied algebra as well as the highest level of mathematics taken. So the survey had each participant identify the types of math courses taken in high school and college as well as the years taken. Variables for time since the last algebra based course taken and the highest level of mathematics taken were calculated and used as covariates in data analysis.

Data Collection Procedures

Participants completed all assessments and measures in paper and pencil format during a 2-hour individually administrated session. Since there are number of measures for this research study, there existed the possibility of order effects. Order effects are the differences in participants' responses due to the order in which the assessments and measures occurred. The types of order effects that could possible show up in this research study are fatigue effects and carryover effects. Fatigue affects occur when the data collection procedure is long, repetitive, or uninteresting. In order to deal with fatigue all assessments had time limits of no longer than 20 minutes (see Table 1). Additionally, researchers told participants that they could take a break at any time and asked participants if they wanted a break when they looked like they were tired. Carryover effects occur when participants' performance on one assessment influence their

performance on another. In order to deal with the carryover effects, each participant got a pre-assigned a randomized order (see Table 3). To create the testing orders, I first grouped the assessments into equally timed first hour and second hour of testing. The groups each equaled out to be 55 minutes of testing. Additionally, I made sure to separate any assessments that I thought would contribute to a carryover effect. For example, I separated MARMI and MASMI so that participants would know that they were two separate assessments even though they looked similar. Orders 1 and 3 consisted of the testing order of the first hour group and then the second hour group while orders 2 and 4 had the second hour group first then the first hour group. Within each hour group the tests were randomized across each of the four testing order, which made sure that no assessment or measure occurred in the same testing order (e.g. if fractions came first in order 1 it did not come first in order 3).

Trained undergraduate research assistants or I gave each research session. Each session began with participants given time to read the information and consent form, and ask questions. Before signing the consent, the participant got a short summary of the consent form that explained what was going happen during the research session that participation was voluntary, and that they got compensation upon completion. The summary allowed the researchers to make sure that participant understood what the research process was and understood the information on the information and consent form. Then participants got the assessments and measures in the order indicated by their assigned order. The researcher read aloud the printed directions for each assessment and measure from the front cover. For complete details on the research protocols, see Appendix D.

Data Analysis

In this section, I briefly describe the analytical plan for answering each research question. The results section provides a more detailed description of each analysis.

Research question 1. The first research question asked whether the prediction of participants' algebra performance depended upon the degree to which they mastered the content and/or the strength of their cognitive abilities. To answer this question, I conducted a multiple regression analysis with participants' scores on the algebra performance assessment as the outcome measure and participant scores on the assessments of computational fluency, numeracy, equivalence, algebraic reasoning, MARMI, MASMI, crystallized intelligence, and fluid intelligence were predictor variables. Anticipating different lengths of time since participants had done the types of problems within the algebra performance assessment, I added to the regression model a control variable for the number of years since an algebra course. I also anticipated that there would be individual level variability in the degree to which participants had continued to take higher-level mathematics courses, so I also included a control variable indicating highest math course taken. Due to the variety of higher-level math courses taken by the sample population (e.g. one or more algebra courses, pre-calculus/trigonometry, or multivariate calculus), I created a dichotomous variable that indicated if each participant's highest math course was at or above Calculus 1. A control variable for gender was also included in the regression model due to the differences found for mathematics performance (e.g. Else-Quest, Hyde, & Linn, 2010; Hyde, Fennema, & Lamon, 1990) and the uneven representation of gender in this sample population.

Research question 2a. A multiple regression analysis assumes a linear relationship between algebra performance, content knowledge, and cognitive abilities, but development does

not necessarily occur in a linear fashion (Bergman et al., 2003). Thus, the second research question asked whether there existed groups of people who perform similarly on these assessments of content knowledge and cognitive abilities. I used cluster analysis to group participants, which identified groups of people that were closely related to one another but distinct from other groups. The cluster variables were participant scores on the cognitive assessments of crystallized intelligence, fluid intelligence, MARM, and MASMI, and the content knowledge assessments of computational fluency, numeracy, fractions, equivalence, and algebraic reasoning.

Analyses were conducted using a two-step procedure where I performed both a hierarchical (Ward's Method) and a nonhierarchical (k-means) cluster analysis. Lastly, I examined differences in the clustering variables for the final cluster solution by performing a multivariate analysis of variance (MANOVA) on cluster variables by cluster membership and examined basic demographic differences across clusters using Chi-square Tests of Independence and a one-way analysis of variance (ANOVA).

Research question 2b. The third question asked whether the groups found from the cluster analysis varied in algebra performance. I used a one-way analysis of variance (ANOVA) to examine these differences. The outcome variable was participant scores on the algebra performance assessment, and the independent variable was the cluster memberships. This ANOVA analysis did not include any control variables because they were categorical variables, which cannot be included as covariates. Therefore, I also used a separate multiple regression analyses to determine if there was a difference in algebra performance by cluster group while accounting for the control variables. The regression model included participant scores on the algebra performance assessment as the outcome measure, and the predictor variables were the

different demographic characteristics of race/ethnicity, major, highest math course taken, and years since an algebra course as well as planned contrasts representing cluster membership.

CHAPTER 4: Results

Preliminary Analyses

See Table 2 and Table 4 for descriptive statistics, bivariate correlations, and internal reliabilities for all assessments. Participants' average performance scores for all assessments were at or above 50% with the exception for algebra performance. The low scores on the algebra performance measure could be because of test difficulty. After completing the assessment multiple participants would remark on how hard the assessment was or how long it has been since they had seen certain types of problems. On the other hand, item level analysis of the content knowledge assessments (See Table 5) suggested that the low performance scores of participants were because participants made errors while taking the assessments or they did not have a complete understanding of the assessed mathematical content. This is evident by the fact that the majority of participants only received partial credit for most item categories as well as the low means of certain items on the fraction, equivalence, algebraic reasoning, and algebra performance assessments (See Appendix E). Another possibility is that the time limits for the assessment may not have provided enough time for the participants to complete the assessments. Moreover, there are a number of limitations that could have attributed to participants' low performance scores. For more in depth discussion of the impact of these limitations, see the limitation section in Chapter 5.

Bivariate correlations show that, as expected, all content knowledge assessments had statistically significant correlations with each other at $p < 0.05$, but did not have extremely high correlations (i.e. $r > 0.70$). All Pearson's correlations were below 0.70 with the correlations between fractions and numeracy, fractions and algebraic reasoning, and crystallized and fluid intelligence being close to 0.70 at .626, .651, and .633 respectively. There were also statistically

significant correlations between all content knowledge and cognitive abilities assessments with the outcome measure of algebra performance between .366 and .657, with most around .400 to .500.

Additionally, I conducted analyses on the randomized orders and gender to determine if the necessity of control variables for these issues in subsequent analyses. Multivariate analysis of variance (MANOVA) showed that there were no statistically significant differences between randomized orders on all assessments, Wilks's λ (30, 376.381) = .780, p = .319, and therefore randomized orders were not included as controls in any subsequent analyses. In order to determine gender differences, I performed both an Independent Samples T-Test and a Mann-Whitney U Test for those measures that violated the assumptions for the t-test. The assumptions for the t-test assume that the data has no significant outliers, is approximately normally distributed, and that there is homogeneity of variance between the groups. To assess for outliers and normality, I inspected a boxplot for outliers and a Normal Q-Q Plot for the distribution of the data. There were only three measures (i.e. MARMI, crystallized intelligence, and fluid intelligence) that violated assumptions of normality with no outliers and normality assumptions and thus required the use of the Mann-Whitney U Test; all other measures met these assumptions. In addition, all measures except for algebra performance met the last assumption of equality of variance as assessed by Levene's Test.

An independent samples t-test indicated that there was a statistically significant difference in scores for males and females for numeracy, with males scoring higher than females (see Table 6). There were no significant differences in the assessments of computational fluency, fractions, equivalence, algebraic reasoning, MASMI, and algebra performance. The Mann-Whitney U Test revealed that there were no statistical difference in scores for males and females

for the assessments of MARMI, crystallized intelligence, and fluid intelligence (See Table 6). Since there was a gender difference for one of the assessments, gender was also included as a control variable in the following analyses.

Regression Analysis

To determine which forms of content knowledge and/or cognitive abilities predicted algebra performance, a multiple regression analysis was performed (RQ1). Participants' scores on the algebra performance assessment was the outcome measure, and participant scores on the assessments of computational fluency, numeracy, equivalence, algebraic reasoning, MARMI, MASMI, crystallized intelligence, and fluid intelligence were predictor variables. In addition, there were control variables for the number of years since an algebra course, the highest math course taken, and gender.

Preliminary test for assumptions. Tests for all assumptions for multiple regression analysis confirmed the validity of the regression model. The first assumption was independence of observations. There was independence of observations as determined by a Durbin-Watson statistic of 2.229; a value close to two indicates an independence of observations. The next assumption was that there are linear relations between the dependent variable and independent variables both separately and collectively. There was linearity as assessed by partial regression plots and a plot of studentized residuals against predicted values. For the partial regression plots, no violation of linearity occurs when the plots visually show the points falling in somewhat of a straight line. All the partial plots visually showed somewhat of a straight line with Highest Math – Calculus or Above, Fractions, and MASMI showing a defined upward sloping line and all others a straight horizontal line.

The third tested assumption was that of constant error variance. The usual evaluation of constant variance is a visual inspection of the scatterplot of the studentized residuals against the predicted values. The scatterplot must show that the variance along the line of best fit remains similar as you move along the x-axis. A visual examination of the scatterplot seemed to meet the assumption of constant variance, but there was still a level of uncertainty; therefore, a statistical inference test also determined the assumption of constant variance. The Breusch-Pagan Test for Heteroscedasticity uses a chi-square test statistic to test a null hypothesis of constant variance against the alternative hypothesis of no constant variance. A large chi-square value means that there were no constant variance and returns a small p-value. In this regression model, there was constant variance, $\chi^2(1) = 10.149, p = .603$. The next assumption was that there are no issues of multicollinearity. An examination of correlation coefficients and tolerance values assessed multicollinearity. Correlation coefficients should not have values above 0.70, which demonstrates a high correlation between variables, and tolerance values should be greater than 0.1. For all predictor variables, there were no correlation coefficients above 0.70 or tolerance values below 0.1.

Another assumption was that there are no significant outliers, influential points or leverage points. There were no outliers as assessed by not having any studentized deleted residuals at ± 3 . Cook's distance values measured the influence of each observation by calculating the amount the data changes by deleting the observed value. The optimal Cook's distance values are those less than 1.0. In this regression model, all Cook's values were less than 1.0. Leverage points are extremely low or high values of the predictor that may exert undue influence on the statistical analysis. Safe leverage points are those less than 0.20, but risky leverage points occur between 0.20 and 0.50. All leverage points were safe except one, which

had a value of 0.27. Even though this one participant had a risky leverage point, the regression model included this participant because all other tests for unusual points deemed this participant within acceptable range. The last assumption test was that the data in the regression model is approximately normal. There was normality as assessed by the visual examination of a Q-Q Plot. For the data to be approximately normal, the data points should fall closely along the diagonal line of the Q-Q Plot. The Q-Q Plot showed that the data points follow the line closely enough to be approximately normally distributed.

Regression model. The multiple regression model statistically significantly predicted algebra performance, $F(12, 128) = 12.368, p < .001$. The predictors explained about 50% of the variation in algebra performance, $R^2 = .537$, Adjusted $R^2 = .494$ (see Figure1). The only significant predictors of algebra performance were participant scores on fractions and MASMI, and having taken at least one calculus course (see Table 7). The squared semi partial correlations showed that fraction scores accounted for 5.8% of the variation in algebra performance. MASMI scores accounted for 2.2% of the variation, and having taken at least one calculus course accounted for 1.8% of the variance in algebra performance. Thus, the results of this regression model suggested that better algebra performance is more likely for those having taken at least one calculus course, those with strong fraction knowledge, and those with strong spatial imagery ability.

Cluster Analysis

The use of cluster analysis in this study provided information about the different participant profiles as a function of content knowledge and cognitive abilities. The profiles highlighted the different combinations of content knowledge and cognitive abilities that naturally occur within participants who have studied algebra (RQ2a). Since both the presence of outliers

and the scale of the clustering variables affect measures of similarity, I made adjustments for each of these issues before identifying cluster groups. By not checking for outliers and adjusting the scale for each of the clustering variables, the measure of similarity would be unduly influenced by the variable with the largest standard deviation, which would mask the influence of any other clustering variable (Hair & Black, 2000; Hair, Black, Babin, & Anderson, 2009). I used Grubbs' (1969) outlier test to identify any potential outliers for each clustering variable. The test only identified a single outlier for the crystallized intelligence assessment. Since the participant had only that one outlier, I changed the crystallized intelligence score to the same score as its closest neighbor in order to retain the participant. After adjusting for outliers, I calculated standardized z-scores to change all cluster variable raw scores to the same scale with a total sample mean of 0 and standard deviation of 1.

The first step in the analysis identified the number of cluster groups within the sample. Specifically, I conducted a hierarchical clustering procedure called Ward's Method, with squared Euclidean distance as the measure of similarity, to develop different numbers of cluster solutions. Ward's method systematically combines clusters by joining the clusters that minimizes the within-cluster variance (Hair & Black, 2000; Hair et al., 2009; Mooi & Sarstedt, 2011; Tan, Steinbach, & Kumar, 2005). The generated cluster solutions range from individual cases in a cluster group by themselves to all cases in one single cluster group. An inspection of the agglomeration schedule for large differences in the fusion coefficients can aid in the identification of possible cluster solutions. Large differences in the fusion coefficient suggest that moving to a smaller number of clusters combines more disparate clusters or groups of participants. Table 8 shows the agglomeration schedule for the first nine cluster solutions where large differences in the fusion coefficients start to occur. The changes in the fusion coefficients

were large, but how similar or different those changes are can pinpoint viable cluster solutions. As would be expected, a very large change in the fusion coefficient occurred when the entire sample was one cluster; the change from 1 to 2 clusters resulted in a difference in fusion coefficient of 445.352. The most similar changes in the fusion coefficients were the differences between 5-to 6-cluster solution, 6-to 7- cluster solutions, and 7-to 8-cluster solution; the differences in their fusion coefficients were within 3-5 points of each other. On the other hand, the most dissimilar fusion coefficient differences were between 4-to 5-cluster solution and 5-to 6- cluster solutions as well as 7-to 8- cluster solutions and 8-to 9- cluster solutions; their fusion coefficient were more than 10 points apart. Based on these differences, it was possible that a viable cluster solution could involve anywhere between 4 to 8 cluster groups.

The next step in the analysis was to narrow down the number of possible cluster solutions. To do this I performed a multivariate analysis of variance (MANOVA) of all clustering variables for the 4-8 cluster solutions. The MANOVA provided means, standard deviations, and partial eta squared values for all cluster variables as well as the sample sizes of each cluster group within each possible cluster solution. See Table 9 for information on the variance explained and sample sizes for the 4-8 cluster solutions. An examination of all this information helped to narrow down the number of possible cluster solutions. First, I examined the sample size of each cluster group for unreasonable numbers. There is no general rule for what is a reasonable and unreasonable sample size, so I considered a sample sizes less than 10% of the total sample size as unreasonable. Using a 10% cutoff value, takes into consideration the use of the final cluster solution in further data analysis where small sample sizes would be problematic. For this criteria, any cluster solution that had cluster group sample sizes smaller

than 14 would not be a viable solution. This eliminated both the 7- and 8-cluster solutions, which had cluster group sample sizes of 10 and 13.

Next, I considered partial eta squared values. Partial eta squared values are the proportion of variance explained by the cluster groups on the cluster variables when accounting for the effect the cluster variables have on each other (Richardson, 2011). Richardson (2011) stated that partial eta squared measures are comparable to the Cohen's d measure of effect size which suggest a good partial eta squared value would be at or above 0.40. All cluster solutions had partial eta squared value below 0.40 except for the 8-cluster solution, but they increased when going from a 4 to a 5 to a 6 cluster solution. In particular, in going from the 4 to 5 clusters there was a substantial increase in the variance explained for numeracy (i.e. $\Delta = .178$) and large increases (e.g. $\Delta > .050$) for algebraic reasoning and MASMI, and in going from 5 to 6 clusters there was a substantial increase for MASMI ($\Delta = .176$) and a large increase for fluid intelligence. Since there was some improvement of the variance explained for each of these cluster solutions, I did not eliminate the 4-, 5-, or 6-cluster solutions as possible final cluster solutions.

The last thing that I considered was the changes in the means of each cluster variable when moving from one cluster solution to the next. With Ward's hierarchical procedure, cluster groups combined based on preexisting cluster groups, so it is possible to consider the theoretical and practical implications for the separation or addition of cluster groups. Going from a 4-cluster solution to a 5-cluster solution split apart a cluster group from the 4-cluster solution that had performance scores above the mean for all cluster variables except for MARMI, which was below the mean. Splitting this cluster group into 2 clusters for the 5-cluster solution resulted in one cluster with scores above the mean on numeracy, algebraic reasoning, and MASMI and the other cluster had scores at or below the mean. This suggests that there may be potentially

important differences in these three variables that were lost within a 4-cluster solution. In addition, going from a 5-cluster solution to a 6-cluster solution split apart a cluster group from the 5-cluster solution that had performance scores below the mean for all cluster variables into two clusters in the 6-cluster solution differentiated by their performance on MASMI. One cluster had a MASMI performance score below the mean while the other had a MASMI performance score above the mean. The separation of cluster groups produced high/low MASMI scores or high/low MASMI scores in conjunction with high/low scores on other variables (i.e. numeracy and algebraic reasoning). The identification of this shared but different characteristic may support my hypothesis that both content knowledge and cognitive abilities are associated with algebra performance, as well as that there are combinations of content knowledge and cognitive abilities related to changes in algebra performance. In addition, MASMI was one of the significant predictors found in the regression analysis, which determining if there are differences in algebra performance for these cluster groups may corroborate the finding from the regression analysis. On the other hand, it is possible that the 4-cluster solution had enough differentiation between groups for changes in algebra performance. Thus, I conducted the next step in the cluster analysis using the 4-, 5-, and 6-cluster solutions.

The third step was to find a final cluster solution by rerunning the cluster analysis using a nonhierarchical clustering procedure called k-means. The k-means analysis partitions the sample into a specified number of clusters, using the variable means from the hierarchical cluster analyses, and iteratively moves cases into and out of clusters in order to maximize the homogeneity within the cluster and the differences between cluster groups (Hair & Black, 2000; Hair et al., 2009; Mooi & Sarstedt, 2011; Tan et al., 2005). The k-means analysis has an advantage over hierarchical (Ward's) because with Ward's there is no switching of cases once

combined. By not switching cases once combined, there is an increased chance that the final cluster groups obtained with Ward's method have combined less homogenous cases together due to the order of the cases in the dataset. Another MANOVA analysis provided the necessary information for choosing a final cluster solution.

As shown in Table 10, all cluster solutions had reasonable sample sizes for each cluster group, and moderate to large amount of variance explained. There were some fluctuations in the variance explained. Specifically, the variance explained for equivalence, fluid intelligence, and algebraic reasoning decreased slightly when adding another cluster group. On the other hand, the variance explained for MASMI increased substantially with each addition of a cluster group; the MASMI variance increased by .123 when going from 4 to 5 clusters and increased by .102 when going from 5 to 6 clusters. In addition, the changes in means for each cluster variable were similar to the changes found with Ward's where the defining characteristic of change was MASMI scores. In particular, going from a 4 to 5 clusters splits apart a cluster with just a low MASMI score into two groups with either a high or a low MASMI score and a low MARMI score. Similarly, going from a 5 to 6 clusters splits a cluster group with low scores on all cluster variables into two groups with low scores on the cluster variables with either a high or a low MASMI score. As mentioned above, this distinction based on MASMI performance scores, which the 4-cluster solution suppresses, was valuable information so the best choice for the final cluster solution was either the 5-cluster solution or the 6-cluster solution. Given that both cluster solutions provided conceptually different cluster groups and there were moderate to large amounts of variance explained, the final cluster solution selected was the 6-cluster solution. The final 6-cluster solution demonstrated statistically significant differences between cluster groups on all cluster variables, Wilks's $\lambda(45, 571.205) = .017, p < .001$, partial $\eta^2 = .559$. See Table 10

for the raw score means, standard deviations, and partial eta squared values of the final cluster solution, and Figure 2 for a graphical representation of the average standardized means for each cluster variable by cluster group. This final 6-cluster solution is described in greater detail in the next section.

The last step was to validate the stability and replicability of the final cluster solution using the double-split cross-validation procedure. The guidelines for the double-split cross-validation procedure are: (1) split the sample into two equal halves, (2) perform the two-step (Ward's followed by K-means) cluster analysis on both halves, (3) combine the two datasets by reassigning similar cases into the same cluster group, (4) conduct a nearest neighbor analysis and (5) compare your assignment with the nearest neighbor analysis. By doing a nearest neighbor analysis, which reassigns half of the cases in your dataset to the most similar profile of the most similar case in the other, you can determine how well you did in cluster classification. Cohen's kappa, which is a measurement of interrater reliability ($\kappa > 0.60$ indicates acceptable replicability), was then used as a way to check the stability and reliability of the cluster solution. For the 6-cluster solution, Cohen's kappa was 0.536. This is a moderate level of agreement, which indicates that the cluster solution is somewhat stable and replicable. The lack of strong stability and replicability could be due to number of cluster variables used for this particular sample size, which is a limitation of this study.

Cluster labels. I labelled each cluster group to reflect the level of each clustering variable, with a particular emphasis on labeling clusters based on the more extreme values (high or low based on sample averages). Since I used standardized z-scores instead of raw scores to create the cluster groups, extremely high and low means for the clustering variables were the variables that had standardized means that were greater than 1 and less than -1, but the

standardized means that fall between -1 and 1 are moderately high and low. Also included are further descriptions of each cluster group focusing on raw scores instead of standardized scores in order to compare this sample's performance scores across different studies. In particular, I interpreted all content knowledge scores using the NAEP Achievement Levels for grade 12, see Table 12 for a description of each achievement level. To interpret spatial visualization performance scores, I used previously published means, which were 21.49 and 22.49 for MASMI (Campos, 2009, 2012) and 8.90 for MARMI (Campos, 2012). Additionally, I used percentile ranks to interpret how well the participants in each cluster group did on the standardized intelligence measures.

The first cluster group ($n = 21$, 14.89%) had participants with low average standardized scores on all clustering variables with seven out of the nine being extremely low. Specifically, the participants performed extremely low on numeracy, fractions, equivalence, algebraic reasoning, MASMI, crystallized intelligence, and fluid intelligence, and had low average performance scores on computational fluency and MARMI. Given these low average scores on all assessments and their extremely low average standardized scores on seven out of nine clustering variables, the label for this cluster group was *Low All*. With these scores, the participants in this cluster group had statistically significantly lower performance scores than participants in the other cluster groups on all clustering variables except for algebraic reasoning and computational fluency. The participants' scores in the *Low All* cluster group were similar to the *Moderate-High MASMI* cluster groups' participants' scores for computational fluency and the *Moderate-Low All* and *Moderate-High MASMI* cluster groups' participants' performance scores for algebraic reasoning. In terms of raw scores, the participants in this cluster group averaged between 20-45% accuracy on all content knowledge variables, averaged less than 10

correct problems for both spatial visualization measures, were within the 21st percentile for crystallized intelligence, and were within the 37th percentile for fluid intelligence. Their content knowledge accuracy performance placed them below basic level using the NAEP achievement level classification. This means that participants in the *Low All* cluster group were not able to solve the most basic of mathematical problems. In comparison to previously published means for the measures of spatial visualization, the participants in this cluster group were well below average for both their MARMI and MASMI scores, which were more than 7 points below the published means (e.g. 21.49 vs. 7.00, 22.49 vs. 7.00, and 8.90 vs. 1.14).

The second cluster group ($n = 22$, 15.60%) had participants with moderately low average standardized scores on all clustering variables, with none being extremely low. Thus, the label for this cluster group was *Moderate-Low All*. Even though *Low All* and *Moderately Low-All* have participants with low average standardized scores on all clustering variables, they were statistically significantly different on all clustering variables except for all the cognitive variables and algebraic reasoning. In raw score performance, the participants in this cluster group averaged between 45-50% accuracy on all content knowledge variables, averaged less than one correct problem for MARMI, averaged 11 correct problems for MASMI, was within the 37th percentile for crystallized intelligence, and was within the 47th percentile for fluid intelligence. NAEP achievement level classifications suggested that participants in the *Moderate-Low All* cluster group were below the basic level for algebraic reasoning and equivalence, at or above the basic level for computational fluency, and at or above the proficient level for numeracy and fractions. This means that participants in this cluster group could not solve the most basic algebraic reasoning and equivalence problems, but could solve basic computational fluency problems as well as apply and integrate mathematical concepts for numeracy and fractions.

Similar to the participants in the *Low All* cluster group, the participants in the *Moderate-Low All* cluster group was also well below average for spatial visualization; both means were more than 10 points below the published means.

The third cluster group ($n = 27$, 19.15%) had participants with moderately high average standardized scores on all clustering variables except for the MARMI and MASMI variables, which were moderately low average standardized scores. Thus, the label for this cluster group is *Moderate-Low Spatial*. The participants in this cluster group averaged 60-78% accuracy on all content knowledge variables, averaged about two correct problems for MARMI, averaged 20 correct problems for MASMI, was within the 63rd percentile for crystallized intelligence, and was within the 77th percentile for fluid intelligence. Some of their content knowledge scores were at or above the proficient level and others were at or above the advanced level. Specifically, the participants in the *Moderate-Low Spatial* cluster group were at or above proficient for computational fluency, numeracy, equivalence, and algebraic reasoning as well as at or above the advanced level for fractions. Even with moderately low spatial visualization scores, the participants in this cluster group had similar performance scores on all content knowledge variables as the participants in the sixth cluster group (i.e. *High Spatial*) that had extremely high spatial visualization scores. For spatial visualization comparisons, the participants in the *Moderate-Low Spatial* cluster group were below the published mean averages; both the MASMI and MARMI scores were at least two points below average.

The fourth cluster group ($n = 19$, 13.48%) had participants with moderately low average standardized scores for all clustering variables except for one. Specifically, the average standardized scores for MASMI was moderately high while all other clustering variables were moderately low. Given the distinctive moderately high average standardized score on MASMI,

the label for this cluster group was *Moderate-High MASMI*. Even with a single clustering variable with a moderately high average standardized score, the participants in *Moderate-High MASMI* cluster group had similar performance scores to the participants in the *Moderate-Low All* cluster group on the content knowledge variables. The content knowledge raw scores were also similar to the participants in the *Moderate-Low All* cluster group with 40-63% accuracy. In particular, they were below basic for computational fluency and equivalence, at or above basic for fractions and algebraic reasoning, and at or above proficient for numeracy. As for the cognitive variables, the participants in *Moderate-High MASMI* cluster group had similar performance scores to the participants in the *Moderate-Low Spatial* cluster group on everything but MASMI. Their average crystallized intelligence standardized score was within the 47th percentile and their average fluid intelligence score was within the 70th percentile. Additionally, their average MARMI raw score was below average with more than three points below the published mean. Conversely, on MASMI their average raw score was well above average at 14 or more points above the published means.

The fifth and largest cluster group ($n = 37$, 26.24%) had participants with moderately high average standardized scores on all clustering variable with no extremely low or high values; thus I labeled this cluster group *Moderate-High All*. The participants in this cluster group had raw score values at or above average for all clustering variables. The averages for their content knowledge scores were between 70-89%, which means that their performance was at or above both the proficient and advanced levels. Specifically, their scores were at or above proficient on computational fluency and equivalence, and were at or above advanced for numeracy, fractions, and algebraic reasoning. Additionally, their MASMI scores were more than 13 points above the published means. The only performance score that was not at or above average was MARMI.

Even though the participants in the *Moderate-High All* cluster group had an average MARMI scores that was the second highest among all cluster group averages, it was still about two points below the published average mean. In overall performance across all clustering variables, the participants in the *Moderate-High All* cluster group outperformed all other cluster groups' participants by having the highest raw scores on five out of nine times.

The sixth cluster group ($n = 15$, 10.64%) had participants with moderately high average standardized scores for all cluster variables with two of those scores being extremely high. Specifically, while the scores for MASMI and MARMI were extremely high, the average standardized scores on all content knowledge and intelligence cluster variables were moderately high, which were similar to the performance scores of participants in the *Moderate -High All* cluster group. Given that the two extremely high scores were measures of spatial visualization, the label for this cluster group was *High Spatial*. The raw score performance for the participants in this cluster group was 62-86% accuracy for all content knowledge variables, more than 50% accuracy for MARMI, more than 90% accuracy for MASMI, was within the 75th percentile for crystallized intelligence, and was within the 90th percentile for fluid intelligence. Similar to the *Moderate-High All* cluster group, this cluster group had content knowledge scores at or above both the proficient and advanced levels, but unlike the *Moderate-High All* group, their scores on algebraic reasoning were at or above the proficient level rather than at or above the advanced level. MASMI scores were also well above average like the *Moderate-High All* group, but the participants in the *High Spatial* group outperformed them on the MARMI with performance scores more than 20 points above the published means.

Demographic characteristics. In addition to the differences in clustering variables, I also considered differences in the demographic variables. I performed a Chi-square Test of

Independence to determine if there was a difference in the proportion of students in each cluster group based on demographic characteristics. Given the number of participants and the variety of categories for some of the demographic variables, most demographic variables failed to meet the expected count assumption for Chi-square analysis, so for all the categorical demographic variables that were not already dichotomous I recoded them into dichotomous variables. For instance, the majority of students in this study majored in the natural sciences, so I reclassified the majors as STEM (Science, Technology, Engineering, and Mathematics) and Non-STEM (e.g. social sciences, fine arts business, etc.). In addition, the majority of participants in the sample were White, so I reclassified race/ethnicity to White and Non-White. I also changed school levels from a five classification to a dichotomous classification where freshmen and sophomores categorized as lowerclassmen and juniors, seniors, and 5+ seniors as upperclassmen.

As shown in Table 13, Chi-square analyses revealed that there was no statistically significant difference for gender [$\chi^2(5) = 4.715, p = .460$] and school level [$\chi^2(5) = 3.042, p = .693$], but there were for race/ethnicity [$\chi^2(5) = 14.356, p = .013$], major [$\chi^2(5) = 21.364, p = .001$], and highest math course taken [$\chi^2(5) = 32.954, p < .001$]. To identify the difference in cluster membership for each of the demographic variables I used the adjusted standardized residuals, which were the difference in the observed frequency and expected frequency. Large residuals values over 2.0, both positive and negative, identified any associations (Laerd Statistics, 2016). The race/ethnicity difference in cluster membership was driven by the more than expected number of Non-White participants in the *Low All* group (residual = 2.9). The statistically significant difference in cluster membership for majors was because of the less than expected number of STEM participants in the *Low All* cluster group (residual = 2.5), less than expected number of STEM participants in the *Moderate-Low All* cluster group (residual = 2.7) ,

and the more than expected number of STEM participants in the *Moderate-High All* cluster group (residual = 3.4). The difference in cluster membership for highest math course taken was because of the less than expected number of Calculus participants in the *Low All* cluster group (residual = 4.2) as well as more than expected number of Calculus participants in the *Moderate-High All* cluster group (residual = 4.0).

The only demographic variable that was continuous was the number of years since an algebra course; therefore, a one-way Analysis of Variance was conducted to determine if the number of years since an algebra course differed by cluster groups. There were some outliers for the *Moderate-Low Spatial* ($n = 1$), *Moderate-Low All*, ($n = 4$), and *High Spatial* ($n = 1$) cluster groups that were changed to the value of the nearest neighbor instead of being removed in order to keep the composition of each cluster group. There was also a lack of homogeneity of error variance as identified by Levene's Test of Homogeneity of Variances ($p = .003$), but the data was normally distributed by inspection of Normal Q-Q Plot. Since there was a lack of constant variance a one-way Welch ANOVA revealed that the number of years since an algebra course was statistically significantly different between cluster groups, Welch's $F(5, 55.650) = 12.392$, $p < .001$. The number of years increased from the *Moderate-Low All* ($M = 2.73$, $SD = .985$) to the *Low All* ($M = 3.76$, $SD = 2.143$), *Moderate-High MASMI* ($M = 3.74$, $SD = 1.968$), *Moderate-Low Spatial* ($M = 4.11$, $SD = 2.225$), *High Spatial* ($M = 4.93$, $SD = 1.280$), and *Moderate-High All* ($M = 5.08$, $SD = 1.479$) cluster groups in that order. Games-Howell post-hoc analysis showed that the statistically significant difference in cluster membership was because of the *Moderate-Low All*, *High Spatial*, *Moderate-High All* cluster groups. The participants in the *Moderate-Low All* cluster group had more than two years difference between the participants in the *High Spatial* and the *Moderate-High All* cluster groups, which mean that the *Moderate-Low All* cluster group

had more participants who have had an algebra course more recently as compared to the *High Spatial* and the *Moderate-High All* cluster groups. This could be because they have taken a college algebra course or they recently graduated from high school.

Cluster Membership and Algebra Performance

The cluster memberships outlined above identified participant profiles with different combinations of strengths and weaknesses in content knowledge and cognitive abilities. Each profile had certain skills, understandings, and/or abilities that participants were good at, which could have a different association with their algebra performance (RQ2b). To determine if cluster membership made a difference in algebra performance, I conducted a one-way Analysis of Variance (ANOVA). Before analysis, I checked to ensure the data meet all assumptions such as no outliers, linearity, and homogeneity of variance. There were no outliers as assessed by inspecting a boxplot. The inspection of a Normal Q-Q Plot determined that algebra performance scores were normally distributed. In addition, there was a violation of homogeneity of variance as assessed by Levene's Test of Homogeneity of Variances ($p = .004$), so a Welch ANOVA was performed instead. As shown in Table 14 and Figure 3, analysis revealed that there was statistically significant difference in algebra performance as a function of cluster membership, $F(5, 56.603) = 18.896, p < .001$. In particular, the participants in the *High Spatial* and *Moderate-High All* cluster groups had similar algebra performance scores to each other but were significantly different from the algebra performance scores of participants in the rest of the cluster groups. Additionally, the participants in the *Moderate-High MASMI* and *Moderate-Low All* cluster groups had similar algebra performance scores to the participants in the *Moderate-Low Spatial* and *Low All* cluster groups, but the participants in the *Moderate-Low Spatial* and *Low All* cluster groups had different algebra performance scores from each other.

In describing the cluster groups, Chi-squared analyses indicated statistically significant differences in cluster membership for some demographic variables. The previous ANOVA analysis did not account for any of the demographic variables because they were categorical instead of continuous variables, so I performed a multiple regression analysis to determine if cluster membership made a difference in algebra performance after controlling for demographic characteristics. In this multiple regression analysis, the outcome measure was participants' raw score on the algebra performance measure, and the predictor variables were the dichotomous demographic variables already shown to have statistically significant association with cluster membership (i.e., highest math taken, race/ethnicity, major, and the continuous variable of years since last algebra course). The demographic variables that were not statistically significant (i.e. gender and school level) were not included because they were not a differences in cluster membership that could be attributed to differences in algebra performance. In addition, I used planned contrasts to compare cluster groups while controlling for these demographic variables.

Planned contrast variables are an extended form of dummy coding that allows for researchers to examine mean differences between groups (Davis, 2010). They are categorical variables like cluster membership, but instead of the values denoting cluster membership they were coded to denote the cluster groups being compared. For example, for the comparison of clusters groups 1 and 3, the planned contrast variable would be (1, 0, -1, 0, 0, 0). Cluster groups 1 and 3 have the values of 1 and -1 because they are the comparison groups. In addition, each contrast was orthogonal so that each contrast was not affected by other contrasts (Davis, 2010). To construct orthogonal contrasts, the sum of the cluster membership values should equal to zero as well as the summation of the cross product of each contrast to another contrast. The advantage of using orthogonal contrasts is that it accounts for issues of suppression effects when comparing

groups within a regression model; however, it also restricts cluster group comparisons, which makes it difficult to do pairwise comparisons.

In this model, I created five planned contrasts, which examined group comparisons based on theoretical implications for the differences in cluster groups, which the differences in clustering variables identified. Specifically, the first three contrasts examined differences in cluster groups based on participants' levels of content knowledge to address the question of whether or not there were differences between cluster groups whose participants had different levels of content knowledge or similar levels of content knowledge. The last two contrasts investigated differences in participants' cognitive abilities given participants' similar levels of content knowledge (e.g., whether or not there were differences in algebra performance for cluster groups that had participant performance that varied for spatial visualization and fluid intelligence abilities but had similar levels of content knowledge). I describe the five contrasts in more detail below in the presentation of the findings.

See Table 15 for the results of the multiple regression analysis. The multiple regression model statistically significantly predicted algebra performance, $F(9, 131) = 12.273, p < .001$, and met all assumptions. There was independence of observations as determined by a Durbin-Watson statistic of 2.129. There was linearity as assessed by partial regression plots and a plot of studentized residuals against predicted values. There was somewhat of a straight horizontal line for all plots except for Highest Math Taken-Calculus or Above and the orthogonal contrast between participants within the low and high content knowledge cluster groups, which showed an upward sloping line. The Breusch-Pagan Test for Heteroscedasticity demonstrated constant variance, $\chi^2(1, N = 140) = 11.213, p = .261$. There were no issues of multicollinearity as assessed by all correlation coefficients below 0.70 and all tolerance values greater than 0.10.

There were no outliers as assessed by not having any studentized deleted residuals at ± 3 , and no influential or leverage points as assessed by no Cook's distances above 1 and no leverage values greater than 0.2. Lastly, the Normal Q-Q Plot showed that the data points follow the line closely enough to be approximately normally distributed. The predictors explained 42% of the variance in algebra performance, $R^2 = .457$, Adjusted $R^2 = .420$. The only statistically significant demographic predictor was the highest math course taken. Taking at least one calculus course predicted a 1.652 point increase in algebra performance, which suggests that the increased exposure to mathematics improves algebra performance. As for group comparisons, only two of the five contrasts were statistically significantly different.

The first contrast investigated the differences between participants' algebra performance scores of those within low content knowledge cluster groups (i.e. *Low All*, *Moderate-Low All*, and *Moderate-High MASMI*) and the high content knowledge cluster groups (i.e. *Moderate-High All*, *High Spatial*, and *Moderate-Low Spatial*). The most basic assumption when examining group difference is that better performance is associated with being more knowledgeable. The comparison of participants within these cluster groups helped to provide evidence for this basic assumption. Even though this contrast was a given, it was included in the regression model because differences in content knowledge was one of the main distinctive features of cluster membership, and it was assumed that it would explain a good portion of the variance in algebra performance. As expected, results revealed that there was a statistically significant difference between participants' algebra performance scores for those in the low content knowledge cluster groups and the high content knowledge cluster groups, which being in the high content knowledge cluster groups predicted a 5.179 point increase in participants' algebra performance scores.

The next two contrasts compared the differences in algebra performance between participants in the cluster groups with similar levels of content knowledge. Specifically, I compared the participants in all of the high content knowledge cluster groups to each other, and the participants in all of the low content knowledge groups to each other. Rules for orthogonal contrasts dictated that to compare the participants of three cluster groups I would need to combine two cluster groups together. Orthogonal contrasts also allowed for the direct comparison of the participants of the two combined cluster groups. Therefore, the choice of which two cluster groups to combine had to make sense theoretically.

I chose to combine the participants in the *High Spatial* and *Moderate-Low Spatial* cluster groups and the participants in the *Moderate-Low All* and *Moderate-High MASMI* cluster groups. The reason I chose these particular cluster groups was because both pairs of cluster groups had participants with similar content knowledge and crystallized intelligence scores but different spatial visualization and fluid intelligence scores. The only differences between the participants in the cluster group pairs were their levels of content knowledge. Participants in the *High Spatial* and *Moderate-Low Spatial* cluster groups had moderately high levels of content knowledge while participants in the *Moderate-Low All* and *Moderate-High MASMI* cluster groups had moderately low levels of content knowledge. By comparing, the participants in these cluster groups (i.e. *High Spatial* vs. *Moderate-Low Spatial* and *Moderate-Low All* vs. *Moderate-High MASMI*), I could make possible conclusions about how content knowledge might interact with cognitive abilities, and in particular did the differences in participants' spatial visualization and fluid intelligence abilities support their algebra performance irrespective of their level of content knowledge.

Thus, for the investigation of the differences in algebra performance scores of participants with similar levels of content knowledge, I combined the two cluster groups that I wanted to do a pairwise comparison for later on. In particular, I compared the participants in *Moderate-High All* cluster group to the combined performance of participants in the *Moderate-Low Spatial* and *High Spatial* cluster groups as well as the participants in the *Low All* cluster group to the combined performance of the participants in the *Moderate-Low All* and *Moderate-High MASMI* cluster groups. The analysis showed that there was no statistically significant difference in algebra performance scores for participants for either level of content knowledge. Specifically, the *Moderate-High All* cluster group had participants' algebra performance scores that were similar to the algebra performance scores of participants in the *Moderate-Low Spatial* and *High Spatial* cluster groups combined. In addition, the algebra performance scores of participants in the *Low All* cluster group were similar to the algebra performance scores of the participants in the *Moderate-Low All* and *Moderate-High MASMI* cluster groups combined. This suggests that even with differences in the performance scores for content knowledge and cognitive abilities, participants with similar levels of content knowledge had similar algebra performance scores.

The fourth contrast compared cluster groups based on participants with similar levels of content knowledge but different levels of spatial visualization and fluid intelligence. Specifically, the contrast examined the differences between the participants in the *Moderate-Low Spatial* and *High Spatial* cluster groups. The comparison of the participants in these two cluster groups helped to determine if cognitive abilities supported algebra performance irrespective of moderately high content knowledge, since their participants had similar levels of moderately-high content knowledge but differed in terms of spatial abilities and fluid intelligence. Analysis

indicated that the participants in the *Moderate-Low Spatial* group had statistically different algebra performance scores from the participants in the *High Spatial* cluster group. In particular, being in the *High Spatial* cluster group increased participants' algebra performance scores by 1.340 points. This suggests that having stronger spatial visualization and fluid intelligence abilities was associated with higher algebra performance scores for those students who already have moderately high content knowledge.

Lastly, the fifth planned contrasts also compared cluster groups who had participants with similar performance scores on the content knowledge variables but were different on spatial visualization and fluid intelligence. Instead of examining the differences between two cluster groups with participants with moderately high content knowledge this contrast examined the differences between two cluster groups with participants with moderately low content knowledge. In particular, the contrasts compared the differences between participants in the *Moderate-Low All* and the *Moderate-High MASMI* cluster groups. Thus, the comparison of *Moderate-Low All* and *Moderate-High MASMI* helped to clarify if participants' differences in spatial visualization and fluid intelligence related to their algebra performance given the similarity between their moderately low content knowledge scores. Results indicated that there were no statistically significant differences between the participants in the *Moderate-Low All* and *Moderate-High MASMI* cluster groups. This suggests that for students with moderately low content knowledge stronger spatial visualization and fluid intelligence skills are not enough to show higher algebra performance scores.

Even though the multiple regression analysis did not allow for the same pairwise comparisons as the one-way ANOVA, the planned contrasts indicated that the analyses shared similar results. In particular, participants in the low content knowledge cluster groups (i.e. *Low*

All, *Moderate-Low All*, and *Moderate-High MASMI*) had similar algebra performance scores. Participants in the high content knowledge cluster groups (i.e. *Moderate-High All*, *High Spatial*, and *Moderate-Low Spatial*), on the other hand, had varying levels of algebra performance scores. In particular, the participants in the *Moderate-Low Spatial* group had different algebra performance scores than the participants in the *Moderate-High All* and *High Spatial* cluster groups, but had similar algebra performance scores to the participants in the low content knowledge cluster groups of *Moderate-Low All* and *Moderate-High MASMI*. In contrast, participants' spatial visualization scores were not classified based on just high or low scores for the all spatial visualization variables. Instead, there was a high spatial group (i.e. *High Spatial* and *Moderate-High All*), a low spatial group (i.e. *Low All* and *Moderate-Low All*), and a mixed spatial group (i.e. *Moderate-Low Spatial* and *Moderate-High MASMI*). The low and high spatial groups had participants with similar performance scores for both MASMI and MARMI, but the mixed spatial group had participants with similar MARMI scores but different MASMI scores. Unlike the content knowledge groups, there was no statistically significant difference in algebra performance for participants within these groups, but there was similarity in participant performance scores across cluster groups. Specifically, both mixed spatial cluster groups had participants with similar algebra performance to the participants in the *Moderate-Low All* cluster group, but only participants in the *Moderate-High MASMI* cluster group had similar algebra performance scores to participants in the *Low ALL* cluster group.

CHAPTER 5: Discussion

Previous research on improving students' algebra performance has focused on instructional approaches and students' content knowledge (e.g. Blanton et al., 2015; Ketterlin-Geller, Gifford, & Perry, 2015; Sfard & Linchevski, 1994). The assumption was that better instructional practices and math skills would more likely lead to better performance. Even though changes in instructional approaches and students' content knowledge have been somewhat effective in improving students' algebra performance, there is still more to understand (Kloosterman, 2016; Kloosterman & Lester, 2004; Kloosterman & Lester Jr., 2007). Prior research on pre-algebra knowledge demonstrated that domain general and domain specific factors such as cognitive abilities and content knowledge related to pre-algebra performance, but there is a lack of corroboration for these findings with investigations focused on algebra performance. Thus, this study aimed to extend the research on algebra performance by examining the connection students' cognitive abilities and content knowledge may have on algebra performance. This study also extends previous research by taking a person-oriented approach, which allows for the identification of different constellations of skills associated with algebra performance. The research findings provide a more nuanced understanding of algebra performance as well as new insights into how different combinations of skills, understandings, and/or abilities related to algebra performance.

Predictors of Algebra Performance

Similar to prior research, the current study suggested that both content knowledge and general cognitive abilities are associated with algebra performance. However, contrary to my hypothesis, the skills related to algebra performance were not fluid intelligence, algebraic reasoning, equivalence, and numeracy. I assumed that given their seemingly close relation to

algebra they would be significant predictors. Instead, the regression analysis revealed that fraction knowledge and spatial imagery better predicted algebra performance along with the control variable for calculus as the highest math course taken. The finding regarding prior calculus course completion also seemed to indicate that exposure to higher-level mathematics may improve algebra performance because algebraic concepts are integrated within higher-level mathematics. The reason that fractions and spatial imagery were positively associated with algebra performance could be because they each have something to do with understanding relationships. In particular, fraction knowledge helps with working with quantitative relationships while spatial imagery helps with identifying the relationships between knowns and unknowns variables.

Fraction are another form of numbers that can speak to the value of discrete (e.g. the number of objects) and continuous (e.g., length, area, volume) quantities. Reasoning with these values requires an understanding of the relationship between numbers and operations, which prior research has connected to algebra performance. Siegler et al. (2012) theorized that fraction knowledge is important for algebra because understanding fractional magnitudes help with estimating values of unknowns and evaluating the reasonableness of algebraic equations. As Siegler et al. (2012) explained, when given the equation $\frac{3}{4}x = 6$, understanding fractions allows you to see that the value of the unknown is slightly larger than six because you understand how multiplication works with fractions. Similarly, linear functions use fractions to denote the ratio between the independent and dependent variable, which when multiplied by a value of the independent variable determines a value for the dependent variable. By understanding how multiplication works with fractions, students can also understand the multiplicative relationship between the independent and dependent variables of a linear function.

Spatial imagery focuses on the mental transformation of images such as putting things together or taking them apart. This is also similar to understanding relationships because Terao et al. (2004) found that visuospatial regions of the brain activated when participants constructed an equation from a problem statement, which requires an understanding of the relationships between the known and unknown elements of a problem. Similarly, Tolar et al. (2009) suggested that spatial visualization, which includes spatial imagery, was linked to problem solving abilities. Problem solving abilities also involve identifying the relationship between knowns and unknowns. Moreover, Chrysostomou et al. (2013) found that those who had higher spatial imagery had better algebraic reasoning achievement, and connected to the algebraic reasoning skills of functional thinking, generalization, and modelling is the ability to understand relationships between quantities as well as known and unknown variables.

My current research findings are both consistent and inconsistent with prior research. The consistency was mostly for the fact that I examined both content knowledge and general cognitive abilities as predictors of algebra performance. Only one particular study, Tolar et al. (2009), closely resembled my current study. Both studies investigated content knowledge and cognitive abilities as factors associated with algebra performance of college students, and defined algebra performance in terms of symbolic algebra. Also consistent with Tolar et al. (2009) was the fact that past algebra education and spatial visualization were directly connected with algebra performance. As much as the current study and Tolar et al. (2009) had their similarities in sample population and some findings, there were inconsistencies in the design that were in turn reflected in the results. In particular, Tolar et al. (2009) examined only four factors while the current study examined nine different factors. The addition of more content knowledge and cognitive ability factors may help to explain the differences between the two studies in the

relation between computational fluency and algebra performance. Computational fluency was no longer directly associated with algebra performance; instead, fraction knowledge took precedent. In addition, the results indicated that a specific form of spatial visualization connected to algebra performance instead of a composite measure.

The current study was also inconsistent with prior research in terms of the specific factors theorized to be connected to algebra performance. While past research would consider content knowledge as basic numerical competencies (e.g. Geary, 2011; Passolunghi & Lanfranchi, 2012), the current study considered content knowledge in similar ways to the algebra readiness research (e.g. Ketterlin-Geller & Chard, 2011; Ketterlin-Geller et al., 2015). This perspective allowed for the identification of additional potential content knowledge factors not previously considered by those who studied the predictors of algebra performance. For example, researchers studying the predictors of algebra performance have not considered fraction knowledge. Their focus has been on students' understanding of whole numbers and their properties as well as calculation and word problem solving skills (e.g. Fuchs et al., 2012, 2016). This is mostly because their research populations are in the elementary grades that have not begun to learn about fractions. The lack of consideration given to fractions neglects the fact that the entire number system, which includes whole numbers, fractions, and decimals, lays the foundation for learning algebra as well as many other mathematical subjects.

Another inconsistency was that most studies about the predictors of algebra performance examined grade levels without formal algebra instruction. By studying the earlier grades on pre-algebra knowledge, researchers hoped to find the foundational mathematics skills that would support the learning of algebra, and alleviate mathematical learning difficulties (e.g, Caviola et al., 2014; Vukovic et al., 2014; Ye et al., 2016). The present study also wanted to determine what

skills would help to alleviate mathematical learning difficulties, but instead of investigating the foundational skills before instruction, this study investigated the foundational skills after instruction. Unlike previous studies, I aimed to identify which existing skills would predict current algebra performance, and more specifically which existing skills currently support strong algebra performance given that time has elapsed since formal instruction. By taking a top down approach rather than the bottom up approach, I hoped to determine which developed skills might alleviate mathematical learning difficulties.

The current findings suggested that, after formal algebra instruction, both fraction knowledge and spatial imagery statistically significantly predicted algebra performance. Unfortunately, very few studies have examined fractional knowledge and spatial imagery in relation to algebra performance. Some have made theoretical claims about fraction knowledge and spatial imagery (e.g. Kilpatrick & Izsak, 2008; Mix & Cheng, 2012; National Mathematics Advisory Panel, 2008; Wu, 2001), but few have proven it with empirical evidence. The present study adds to this literature by showing that even when considering other forms of mathematical content knowledge (e.g. computational fluency, understanding of equivalence, numeracy) and cognitive abilities (e.g. crystallized and fluid intelligence), fraction knowledge and spatial imagery stand out as important factors related to algebra performance. Thus, it may be prudent for teachers to focus on developing students' fractional knowledge and spatial imagery skills in preparation for formal algebra instruction.

Cognitive Abilities and Content Knowledge Profiles and Algebra Performance

Previous research has shown that content knowledge and cognitive abilities are associated with algebra performance. It has even identified which forms of content knowledge and cognitive abilities have a strong association. What previous research does not explain is how

different combinations of content knowledge and cognitive abilities may change algebra performance. Thus, the present study sought to identify these different combinations of content knowledge and cognitive abilities and their relation to algebra performance. Specifically, I conducted a cluster analysis to identify groups of participants with similar performance on all predictors within the group, but had different performance on all predictors across groups. The subsequent one-way ANOVA analyses determined the differences between each group, and interpretations of these differences explained the different combinations of skills, knowledge, and/or abilities that can contribute to changes in algebra performance.

The results of the cluster analysis indicated that within the current sample there were six cluster groups with different levels of content knowledge and cognitive ability performance. The speculation about the types of groups that would emerge held true for the idea that there would be separate groups based on strengths and weaknesses for content knowledge or cognitive abilities, but did not for the idea that there would be defined groups characterized by certain types of skills sets. The lack of defined skill sets could be because the 6-cluster solution suppressed any skill set differences. The sample size of the current study limits the number of possible cluster solutions. With more clusters, it may be possible to find skill set differences, but with the current study, the more salient differences were for overall performance on both content knowledge and cognitive ability measures and spatial visualization. Specifically, there were two groups with low performance on all variables (i.e. *Low All* and *Moderate-Low All*), one group with high performance on all variables (i.e. *Moderate-High All*), and three groups that had distinguishable differences in spatial visualization performance in comparison to content knowledge and intelligence (i.e. *Moderate-Low Spatial*, *Moderate-High MASMI*, *High Spatial*). Unlike the results of the regression analysis, fractions did not appear to be a key factor in

differentiating among clusters. Fractions may not have contributed to the creation of the cluster groups because there were larger differences in participants' spatial visualization performance scores in which to group cases than there were for fractions. In addition, not captured by the cluster analysis is the fact that fractions have a higher association with algebra performance such that even when accounting for its association with other predictors it still stood out, and because the clusters were not based on its association with algebra performance fraction knowledge did not stand out when clustering participants.

Differences in clustering variable performance suggested that cluster group spatial visualization differences can vary regardless of the level of content knowledge and/or intelligence. This was evident by the content knowledge and/or intelligence performance scores similarities for the participants in cluster groups with differences in spatial visualization performance scores. For example, the participants in the *Moderate-Low All* group and *Moderate-High MASMI* group had different performance scores on both measures of spatial visualization, yet they had similar moderately low content knowledge performance scores (see Figure 4). The participants in the *Moderate-Low Spatial* group and *High Spatial* group also had different performance scores on both measures of spatial visualization, but they had similar moderately high content knowledge performance scores (see Figure 5). Moreover, the participants in the *Moderate-High All* and *High Spatial* groups had different performance scores on both measures of spatial visualization, but had similar performance scores on all content knowledge and intelligence variables (see Figure 6). Also highlighted was the fact that the cluster group with the highest participants' raw score averages for five out of nine clustering variables was the *Moderate-High All* cluster group, which is not one of the cluster groups with distinctive participants' performance scores for spatial visualization. These findings would

suggested that even though there were identifiable variations in spatial visualization performance scores, the participants in the cluster groups were mostly different based on their overall low and high performance scores on both content knowledge and cognitive abilities measures.

As for algebra performance, interpretations of both the one-way ANOVA and multiple regression analysis suggested that the same differences between cluster groups existed even when controlling for demographic characteristics. With the similarities in results for both the multiple regression and one-way ANOVA analysis, I used the ANOVA post hoc analysis results as the primary for interpreting the relation between cluster membership and algebra performance, and the results of the multiple regression analysis were a supplement. The results indicated that similar to previous research both content knowledge and cognitive abilities were associated with algebra performance. As shown in Table 14 and Figure 2, the highest algebra performance scores were for those participants in clusters groups with both high content knowledge and high cognitive abilities, and lower algebra performance scores accompanied lower cognitive abilities or content knowledge performance scores.

Additionally, the current findings suggested that higher algebra performance scores occurred for participants with higher levels of overall content knowledge as indicated by the difference in algebra performance scores for participants within the *Moderate-High All* and *Moderate-High MASMI* cluster groups (see Figure 7). These two cluster groups shared similar performance scores on the spatial visualization variables and fluid intelligence, but were different on all content knowledge variables and crystallized intelligence. Thus, the difference in algebra performance scores suggested that higher content knowledge and stronger crystallized intelligence scores demonstrated higher algebra performance scores.

The research finding also suggested that there were specific combinations of content knowledge and cognitive abilities that were associated with higher algebra performance scores. In particular, the combination of strong spatial visualization abilities, strong fluid intelligence, and high content knowledge was associated with higher algebra performance scores as found by the comparing participants within the *High Spatial* and *Moderate-Low Spatial* cluster groups (see Figure 5), the participants within the *High Spatial* and *Moderate-High All* cluster groups (see Figure 6), and the participants within the *Moderate-High MASMI* and *Moderate-Low All* cluster groups (see Figure 7). In addition, stronger fraction, numeracy, algebraic reasoning, and spatial imagery skills appear to be associated with higher algebra performance scores as seen by comparing participants within the *Moderate-High All* cluster group and the participants in the *Moderate-Low Spatial* cluster group (see Figure 8).

The participants in the *High Spatial* and *Moderate-Low Spatial* cluster groups had similar performance scores for all content knowledge variables, but had different performance scores on spatial visualization and fluid intelligence (see Figure 5). In particular, the participants in the *High Spatial* cluster group had higher mean averages for spatial visualization and fluid intelligence. Their difference in algebra performance scores showed that the *High Spatial* cluster group's participants had a statistically significant higher scores on algebra performance than the *Moderate-Low Spatial* cluster groups' participants, which suggested that the stronger spatial visualization and fluid intelligence abilities of the participants in the *High Spatial* cluster group might have contributed to better scores on algebra performance.

Additionally, participants within the *High Spatial* and *Moderate-High All* were different only on the measures of spatial visualization (see Figure 6). The similarity between their algebra performance scores suggested that spatial visualization skills alone are not enough to

demonstrate higher algebra performance scores. On the other hand, the participants within the *Moderate-High MASMI* and *Moderate-Low All* cluster groups were very similar to the participants within the *Moderate-Low Spatial* and *High Spatial* cluster groups in differences in clustering variables (see Figure 4 and Figure 5). They too were only different for their scores on spatial visualization and fluid intelligence, but the participants in these cluster groups demonstrated low levels of content knowledge instead. The lack of difference in their algebra performance scores suggests that for participants with low levels of content knowledge, stronger spatial visualization, and fluid intelligence abilities may not be enough to yield higher algebra performance scores. Taken together the results of these three comparisons suggest that it is the combination of stronger spatial visualization, fluid intelligence abilities, and moderately high levels of content knowledge that may best support higher algebra performance scores rather than any of these skills by themselves.

Moreover, the participants in the *Moderate-High All* cluster group and the *Moderate-Low Spatial* cluster group had similar performance scores on all content knowledge and cognitive ability assessments except for numeracy, fractions, algebraic reasoning, and MASMI (see Figure 8). The statistically significant higher algebra performance scores for the participants in the *Moderate-High All* cluster group over the *Moderate-Low Spatial* cluster group suggested that stronger skills in numeracy, fractions, algebraic reasoning, and MASMI might have contributed to the difference in algebra performance.

These research findings confirmed my hypothesis that better cognitive abilities (e.g., fluid intelligence, spatial visualization) may support better algebra performance scores, but the fact that the better cognitive abilities occurred in addition to high content knowledge was surprising. This was a surprising finding because I would logically assume that better cognitive abilities

would support better algebra performance scores irrespective of level of content knowledge. One possible explanation would be that the relative differences in content knowledge are more salient at this level of knowledge. In particular, the *High Spatial* cluster group's advanced achievement level in numeracy and fractions may have contributed to the differences in algebra performance. Unfortunately, the information provided by the current study does not allow for definite conclusions on this matter, but the statistically significant difference in algebra performance for *Moderate-High All* and *Moderate-Low Spatial* suggested that this assumption could be true.

Even though it is beyond the scope of this study to uncover why or how these different combinations of skills, understandings, and/or abilities relate to algebra performance, the results did provide a more nuanced understanding of how the factors of content knowledge, spatial visualization, and intelligence could contribute to differences in algebra performance. The understanding was that there are two circumstances associated with higher algebra performance scores: students who had strong spatial visualization skills, strong fluid intelligence skills, and high content knowledge or students who had strong fraction knowledge, numeracy skills, algebraic reasoning skills, and spatial imagery skills.

Variable-Oriented vs. Person-Oriented: What do the differences mean?

Most studies about the predictors of algebra performance have taken a variable-oriented approach that emphasizes the independent relation of each predictor variable on the average performance of the whole sample. Few have considered a person-oriented approach that emphasizes individual patterns of development. Research that has come close to the person-oriented approach were the studies that focused on understanding common errors and misconceptions of specific topics based on group level performances (e.g. Booth, 1988;

Kuchemann, 1978). These studies were individual interviews that drew conclusions about the whole sample instead of evaluating individual patterns of development for different combinations of skills, knowledge, and/or abilities related to algebra performance.

Investigating individual patterns of development highlights how combinations of skills, knowledge, and/or abilities can differentially predict algebra performance that the variable-oriented approach may obscure. For instance, findings from the regression analysis suggested that fraction and MASMI scores predicted algebra performance scores while the cluster analysis suggested that a noticeable difference in algebra performance occurs when you have strong content knowledge in addition to strong spatial visualization abilities and fluid intelligence. Additionally, the cluster analysis suggested that other skills that related to algebra performance were numeracy and algebraic reasoning. The only findings shared between the regression analysis and cluster analysis were the importance of taking at least one calculus course and spatial visualization. In both analyses those who had taken a calculus course had higher performance scores compared to those who had only taken at least one algebra course, trigonometry, or pre-calculus course, and strong spatial imagery abilities were defining factors for high algebra performance.

The differences found between the cluster and regression analysis highlight the fact that there are certain patterns of relations among the clustering variables that multiple regression may or may not detect. The multiple regression analysis has limited capacity to identify these relationships. The regression model does not automatically account for these unless it is a predetermined addition to the model in terms of an interaction term. Their detection depends upon whether or not there is sufficient power within the model to detect statistically significant associations, which depends upon sample size and the number of predictors (Hair et al., 2009).

Too many or too little in sample size and the number of predictors can change the outcome of the multiple regression. While cluster analysis may bring our attention to the sorts of relationships missed by regression analysis, it is a more subjective research methodology. It requires the researcher to make decisions based on theoretical implications guided by statistical outcomes. This means that generalizability of the research findings depends on researchers coming to the similar conclusions. This may be difficult because each researcher has his or her own theoretical framework for identifying clusters. Nevertheless, the information gleaned from the cluster analysis gives new insight to algebra performance given the limitations of the multiple regression analysis.

Limitations

There are a number of limitations for this study. The first limitation would be for the sample population, which are college students whose exposure to mathematics goes beyond algebra. Most prior research on algebra performance has examined grade levels without formal algebra instruction or those who are just beginning to learn algebra. It is possible that the exposure to additional math topics altered the association between algebra performance and the content knowledge variables, and would alter the findings from this study. Therefore, it is necessary to replicate this study with beginning algebra students in order to be able to generalize.

Similarly, the current study's small sample size is also a limitation of the study that could alter the findings. The plan was to conduct the study with 200 participants, which apriori power analyses suggested would be sufficient for about 80% - 95% power depending on the statistical analysis. Additionally, these power analysis were made with the assumption that the present study would have a medium effect size (i.e. $f^2 = .15$ or $f = .25$) (Faul, Erdfelder, Lang, & Buchner, 2007). The actual sample size for the current study was 141 participants, which is

smaller the targeted number of participants. Even though I was not able to run 200 participants, the calculated effect sizes for the multiple regression and ANOVA analyses were large (i.e. $f^2 > .35$ or $f > .40$) instead of the assumed medium effect (Faul et al., 2007). For the multiple regression analysis to determine the strongest predictors of algebra performance the calculated effect size was 1.159, and the one-way ANOVA and multiple regression analysis used to determine the differences in algebra performance based on cluster membership had calculated effect sizes of 0.834 and 0.845 respectively. The larger effect sizes meant that I was able to achieve 95% power with a smaller sample size, so the multiple regression and ANOVA analyses had sufficient power with only 141 participants instead of the targeted 200. The only analysis that the small sample size may have affected was the cluster analysis. The recommended sample size for cluster analysis is 2^m , where m is the number of clustering variables. In the present study, there were 9 clustering variables, so the recommended sample size was $2^9 = 512$. The current sample size of 141 is much smaller than the suggested 512. This smaller sample size would have limited the number of cluster groups found, the number of participants within each cluster group, changed the general make-up of the types of clusters groups found, as well as possibly changed the differences in algebra performance between cluster groups. In the current study, the biggest impact that the small sample size had was on the number of clusters found. As shown in Table 9, the amount of explained variance in the cluster variables was higher with more cluster groups, which suggested that there was more differentiation in participant performance scores on the cluster variables that may denote more types of participants. This differentiation could have signified more combinations of content knowledge and cognitive abilities that may relate to participants' algebra performance scores. Nevertheless, the results found in the current study

have given the first indication that there are certain combinations of content knowledge and cognitive abilities that are associated with better algebra performance scores.

Another limitation is that each content knowledge assessment had a time limit, which may or may not have been enough for all participants. Very few participants finished each assessment, even though pilot testing showed that all assessments could be finished within the given period. This means that the score of zero for incorrect and blank answers may not be an accurate representation of the knowledge participants had.

In addition to the time limit, most content knowledge measures were also researcher designed measures that had some design flaws. One particular design flaw was the issues of counterbalancing problem types. Not all assessments made sure that participants were able to see all problem types no matter if they finished or not. Specifically, the numeracy and algebra performance assessments had issues with counterbalance. For numeracy, the problem was that all addition problems and subtraction problems came before all the multiplication problems and division problems. This was an issue when participants focused on using algorithms instead of their knowledge of the properties of numbers and operations to answer the problem because the focus on doing the algorithms rarely got them beyond the addition or the subtraction problems within three minutes, and they missed the multiplication and division problems. Unlike the numeracy assessment, I tried to counterbalance the algebra performance assessments, but the procedure that I used did not provide true randomization. The assessment ended up with all the factoring and exponent questions at the end of the tests, which some students did not reach. A better way to counterbalance would be to make sure that one question out from each item category would end up in each half of the test in a randomized order.

Another design flaw was for the types of questions used to assess the mathematical content knowledge. This is an issue because there are many different ways to measure the mathematical content knowledge. For instance, Tolar et al. (2009) measured computational using the ‘number facility’ subtest of the ETS Kit of Factored-Referenced Cognitive Tests, which measured computational fluency with both multi-digit calculation problems and problems that had participants determine if the suggested answer was correct for the given problem. I on the other hand only measured computational fluency with multiple multi-digit calculation problems, which could possibly account for the differences in the results for computational fluency. Similarly, Hecht, Close, and Santisi (2003) and Hecht and Vagi (2010) measured fraction knowledge with assessments for computing fractions, estimating fractions, word problems, comparing fractions, and identifying fractions while I based my fraction assessment off of the assessment designed by G. Brown and Quinn (2007). Their assessment measured fraction knowledge for algorithmic applications, word problems, elementary algebraic concepts, arithmetic skills, structure of rational numbers, and computational fluency. The inconsistency of assessments between research studies makes it difficult to know if the questions used are an accurate representation of the mathematical content knowledge. All that guides us is whether or not the internal consistency of the assessment has an acceptable Cronbach’s Alpha value of 0.70 or better, but this only provides evidence that the questions we are using are measuring the same construct, which may or may not be the same given another set of questions.

The algebra performance assessment was also a limitation of this study. This study defined algebra performance in similar ways to Tolar et al. (2009), whose definition is the ability to solve algebra problems using pre-learned symbolic manipulation algorithms. The heavy emphasis on symbolic algebra may not be an accurate representation of what algebra is. The

choice for focusing on symbolic algebra was to make sure that it did not overlap with the algebraic reasoning assessment. If I could change the algebra performance assessment, I would probably put more emphasis on application type questions. I think this would have shown participants ability to both demonstrate symbolic knowledge as well as reasoning skills. The only downside to the application questions would be that I would have to eliminate algebraic reasoning as a predictor variable in multiple regression analysis because of possible issue with being highly related to the outcome measure; however, it would not be a problem with a cluster analysis, and is worth considering.

Implications

Although more work is needed to have a better understanding of why factors such as fraction knowledge and spatial imagery (a.k.a. MASMI) predict algebra performance, their statistical significance in the regression model suggest that, in preparation for formal algebra instruction, it may be useful for students to develop their fraction knowledge and spatial imagery skills. Spatial imagery development may be as simple as encouraging students to draw more pictures when trying to solve problems or using more pictorial based problems solving method such as the Singapore Model Method (e.g. Lee & Ng, 2009). On the other hand developing fraction knowledge can be more challenging. Fractions are a mathematical concept with which students already have difficulty (e.g. G. Brown & Quinn, 2006; Peck & Matassa, 2016). Some researchers have attributed this difficulty to the lack of personal understanding that teachers have for fractions (e.g. Siegler et al., 2012), so developing students' fraction knowledge may involve developing not only students fractional content knowledge but teachers as well.

Just like with the results found from the multiple regression analysis, more work is needed to understand completely how content knowledge and cognitive abilities profiles are associated

with algebra performance. In particular, the why and how the different combinations of content knowledge and cognitive abilities support algebra performance; however, the current findings have certain implication for educators. One implication is that educators may scaffold the development of content knowledge with the use of instructional practices that take advantage of the spatial imagery and fluid intelligence of students. For example, the use of more discussions about students' problem solving strategies that include visual depictions of different strategies emphasizes both spatial imagery and fluid intelligence. The different strategies make apparent the different logical steps used to arrive at the same answer, and promote the use of fluid intelligence skills by getting student to synthesize across methods as to why they both work. The use of visual depictions of the solution method gets students to use their spatial imagery skills and makes apparent how students think about the relationship between the known and unknown variables.

Another implication is that the elementary grades are a place to develop the skills of necessary for strong algebra performance. In particular, the development of the skills of fractions and numeracy already occur in the elementary grades while algebraic reasoning and spatial imagery can be (e.g. Cooper & Warren, 2011; Lannin, 2003; Lee & Ng, 2009; Moss & McNab, 2011). In fact, the identification of these sets of skills as factors that can improve algebra performance provided credence to the recommendation to teach algebraic concepts starting in the elementary grades (Blanton et al., 2015; A. Stephens, Blanton, Knuth, Isler, & Gardiner, 2015). Additionally, with more research we can also identify which content knowledge and cognitive abilities relates to particular algebraic topics, which may point towards a way to provide extra help for those who do not have mathematics learning disabilities.

Conclusion

The present study investigated content knowledge and cognitive abilities as factors associated with algebra performance. This association was determined by conducting a multiple regression and cluster analysis. The two approaches allowed the examination of algebra performance from both variable-oriented and person-oriented approaches. The variable-oriented approach (i.e. multiple regression analysis) revealed the independent predictors of algebra performance, while the person-oriented approach (i.e. cluster analysis) revealed how different patterns of content knowledge and cognitive abilities related to algebra performance. Each perspective brings out its own unique understanding of algebra performance that has expanded our understanding of what factors are associated with it. The finding from the current study proposed that fraction knowledge and spatial imagery are additional predictors of algebra performance not covered by previous research, which makes necessary more research in order to understand how they connect with the other factors already known. Additionally, the claim that algebra performance is associated with both content knowledge and general cognitive abilities was supported through not only with the addition of fraction knowledge and spatial imagery as predictors, but with different combinations of content knowledge and cognitive abilities between participants in the cluster groups that may have contributed to differences in their algebra performance. More research is needed to corroborate these findings but they are an important first step into understanding how content knowledge and cognitive abilities extend our understanding of the complex array of factors associated with algebra performance.

APPENDICES

Appendix A: Tables

Table 1

Basic Information about Assessments and Measures

Tests	Time (mins)	Calculator	Number of Items
Cognitive Abilities			
Fluid Intelligence			
Nonverbal Analogies	7	N	52
Sequences	10	N	43
Crystallized Intelligence			
General Knowledge	3	N	47
Odd Word Out	5	N	40
Word Opposites	5	N	40
Spatial Visualization			
MASMI	10	N	23
MARMI	10	N	23
Content Knowledge			
Computational Fluency	3	N	16
Numeracy	4	N	24
Fractions	10	N	12
Equivalence	5	N	6
Algebraic Reasoning	10	N	7
Algebra Performance	20	Y	20

Note. Y = Yes; N = No; MASMI = Measure of the Ability to Form Spatial Mental Images; MARMI = Measure of the Ability to Rotate Mental Images.

Table 2

Descriptive Statistics for Assessments and Measures

Assessments & Measures	<i>M</i>	<i>SD</i>	α	Minimum Score	Maximum Score
Control Variables					
Gender	—	—	—	—	—
Calculus	—	—	—	—	—
Years since algebra course	4.20	2.112	—	0	14
Content Knowledge					
Computational Fluency	31.57	10.19	0.80	0	55
Numeracy	16.86	4.98	0.87	0	24
Fractions	8.30	2.70	0.76	0	12
Equivalence	6.11	2.70	0.73	0	11
Algebraic Reasoning	12.73	4.49	0.85	0	22
Cognitive Abilities					
MARMI	5.99	10.89	0.82	-46	46
MASMI	25.55	15.30	0.93	-46	46
Crystallized Intelligence ^a	102.22	12.39	— ^b	35	185
Fluid Intelligence ^a	108.18	11.12	— ^b	35	171
Algebra Performance	7.64	3.79	0.76	0	20

Note. MASMI = Measure of the Ability to Form Spatial Mental Images; MARMI = Measure of the Ability to Rotate Mental Images.

^a Standardized assessments with standard scores of $\mu = 100$, $\sigma = 15$. ^b Sample internal reliabilities not calculated because assessments were standardized.

^{**} $p < 0.01$. ^{*} $p < 0.05$.

Table 3

Randomized Testing Orders

Test Order	Group Order			
	1	2	3	4
1	Fractions	Numeracy	Algebra Performance	MASMI
2	Computational Fluency	RAIT	Equivalence	Algebraic Reasoning
3	MARMI	MASMI	Computational Fluency	RAIT
4	Equivalence	Algebraic Reasoning	MARMI	Numeracy
5	Algebra Performance	MARMI	Fractions	Computational Fluency
6	RAIT	Fractions	Algebraic Reasoning	MARMI
7	Numeracy	Algebra Performance	MASMI	Equivalence
8	Algebraic Reasoning	Computational Fluency	Numeracy	Fractions
9	MASMI	Equivalence	RAIT	Algebra Performance

Note. MASMI = Measure of the Ability to Form Spatial Mental Images; MARMI = Measure of the Ability to Rotate Mental Images.

Table 4

Bivariate Correlations for Assessments and Measures

Assessments & Measures	1	2	3	4	5	6	7	8	9	10	11	12
Control Variables												
1. Gender	—											
2. Calculus	-.143	—										
3. Years since algebra course	-.300**	.281**	—									
Content Knowledge												
4. Computational Fluency	-.045	.312**	.128	—								
5. Numeracy	-.219**	.394**	.239**	.399**	—							
6. Fractions	-.118	.415**	.218**	.496**	.626**	—						
7. Equivalence	-.035	.418**	.171*	.433**	.475**	.536**	—					
8. Algebraic Reasoning	-.082	.419**	.180*	.363**	.567**	.651**	.517**	—				
Cognitive Abilities												
9. MARMi	-.113	.238**	.184*	.202*	.316**	.293**	.300**	.267**	—			
10. MASMI	.014	.350**	.150	.216**	.446**	.474**	.432**	.499**	.475**	—		
11. Crystallized Intelligence ^a	-.004	.228**	.222**	.377**	.482**	.605**	.529**	.482**	.264**	.412**	—	
12. Fluid Intelligence ^a	.010	.353**	.210*	.350**	.535**	.527**	.589**	.540**	.428**	.585**	.633**	—
Algebra Performance	-.110	.471**	.200*	.443**	.470**	.657**	.459**	.544**	.336**	.504**	.411**	.425**

Note. MASMI = Measure of the Ability to Form Spatial Mental Images; MARMi = Measure of the Ability to Rotate Mental Images.

^a Standardized assessments with standard scores of $\mu = 100$, $\sigma = 15$.

** $p < 0.01$. * $p < 0.05$.

Table 5

Analysis of Item Categories for All Content Knowledge Assessments

Assessment Items Categories	Descriptive Statistics		Total Points	Percent Correct		
	<i>M</i>	<i>SD</i>		Full Credit	Partial Credit	No Credit
Computational Fluency						
Addition	11.23	2.517	16	10.6	89.4	0.0
Subtraction	8.43	3.514	15	13.5	86.5	0.0
Multiplication	8.47	4.633	17	7.1	85.1	7.8
Division	3.44	2.237	7	8.5	75.9	15.6
Numeracy						
Addition	5.10	1.110	6	48.2	51.8	0.0
Subtraction	4.64	1.091	6	24.8	75.2	0.0
Multiplication	3.77	1.843	6	24.8	67.4	7.8
Division	3.35	2.208	6	25.5	55.4	19.1
Whole Numbers	4.65	1.459	6	41.8	58.2	0.0
Decimals	8.08	1.848	10	29.8	70.2	0.0
Fractions	4.13	2.507	8	11.3	82.3	6.4
Fractions						
Algorithmic Operations	1.59	.633	2	66.7	25.5	7.8
Word Problems	1.74	.516	2	77.3	19.1	3.5
Algebraic Concepts	1.68	.552	2	72.3	23.4	4.3
Arithmetic Skills	.98	.788	2	29.8	38.3	31.9
Rational Number	1.38	.692	2	49.6	38.3	12.1
Computational Fluency	.94	.791	2	28.4	37.6	34.0
Equivalence						
Interpretation	1.98	1.017	3	38.3	49.6	12.1
Structure	1.76	1.242	4	8.5	71.6	19.9
Open Equation	2.38	1.251	4	25.5	67.4	7.1
Algebraic Reasoning						
Functional Thinking	1.55	1.485	4	14.2	51.0	34.8
Generalization	2.43	1.431	4	23.4	56.0	20.6
Modeling	.55	.671	2	9.9	34.8	55.3
Symbolic Manipulation	5.97	1.912	8	15.6	81.6	2.8
Structure Sense	2.23	1.155	4	17.7	73.1	9.2
Algebra Performance						
Systems of Equations	.91	.774	2	25.5	39.7	34.8
Functions	1.14	.713	2	33.3	47.5	19.1
Solving Equations	1.31	.698	2	44.7	41.8	13.5
Inequalities	.72	.658	2	11.3	48.9	39.7
Graphing	.98	.751	2	27.0	44.0	29.1
Exponents	.42	.611	2	6.4	29.1	64.5
Factoring	.84	.733	2	19.9	44.0	36.2
Complex Numbers	.28	.468	2	0.7	27.0	72.3
Polynomial Division	.49	.683	2	10.6	27.7	61.7
Logarithms	.55	.659	2	9.2	36.9	53.9

Table 6

Summary of Independent Samples T-Test and Mann-Whitney U Test Analysis for Gender Differences

	Independent Samples T-Test			Mann-Whitney				
	Mean	Standard		Mean Rank				
Assessment	Difference ^a	Error	<i>t</i> (139)	Female	Male	<i>U</i>	<i>z</i>	<i>p</i>
		Difference		(<i>n</i> = 110)	(<i>n</i> = 31)			
Content Knowledge								
Computational Fluency	-1.092	2.077	-.526					.600
Numeracy	-2.621	.991	-2.645					.009
Fraction	-.767	.546	-1.403					.163
Equivalence	-.227	.549	-.413					.680
Algebraic Reasoning	-.883	.913	-.967					.335
Cognitive Abilities								
MARMI				68.55	79.71	1435.00	-1.346	.178
MASMI	.502	3.122	.161					.872
Crystallized Intelligence				70.28	73.55	1626.00	-0.393	.694
Fluid Intelligence				71.09	70.69	1695.50	-0.047	.962
Algebra Performance	-1.001	.843	-1.188					.242

Note. MASMI = Measure of the Ability to Form Spatial Mental Images; MARMI = Measure of the Ability to Rotate Mental Images.

^aMean Difference calculated by subtracting the mean scores of Males from the mean scores of Females.

Table 7

Summary of Multiple Regression Analysis

Variable	B	SE _b	β	<i>p</i>	r_{partial}	r^2_{partial}
Intercept	1.876	2.897		.518		
Gender	-.229	.601	-.025	.704	-.023	.001
Calculus	1.286	.575	.163	.027	.134	.018
Years Since Algebra Course	.011	.119	.006	.925	.006	< .001
Computational Fluency	.051	.027	.136	.063	.113	.013
Numeracy	-.028	.065	-.037	.671	-.026	.001
Fraction	.553	.138	.393	< .001	.241	.058
Equivalence	.043	.118	.030	.716	.022	< .001
Algebraic Reasoning	.087	.074	.103	.241	.071	.005
MARMI	.023	.025	.067	.349	.057	.003
MASMI	.051	.021	.204	.015	.148	.022
Crystallized Intelligence	-.005	.027	-.016	.857	-.011	< .001
Fluid Intelligence	-.027	.033	-.080	.405	-.050	.003

Note. B = unstandardized regression coefficient; SE_b = standard error of unstandardized regression coefficient; β = standardized regression coefficient; r_{partial} = semi partial correlation; r^2_{partial} = unique variance of predictors; MARMI = Measure of the Ability to Rotate Mental Images; MASMI = Measure of the Ability to Form Spatial Mental Images.

Table 8

Ward's Method Agglomeration Schedule for Clusters 1-9

Number of Clusters	Coefficient	$\Delta_{\text{coefficient}}$
1	1260.000	—
2	814.648	445.352
3	731.239	83.409
4	666.827	64.412
5	610.524	56.303
6	572.588	37.936
7	539.223	33.365
8	508.884	30.340
9	489.169	19.715

Note. $\Delta_{\text{coefficient}}$ = absolute difference in coefficients.

Table 9

Indices for Ward's Method Cluster Solutions

Cluster Solutions	Variance Explained									Sample Sizes
	CF	N	F	E	AR	MARMI	MASMI	CII	FII	
4	0.271	0.450	0.640	0.554	0.428	0.616	0.427	0.399	0.450	21, 38, 63, 19
5	0.294	0.628	0.649	0.555	0.496	0.623	0.501	0.420	0.472	21, 38, 27, 36, 19
6	0.301	0.628	0.651	0.555	0.497	0.651	0.677	0.426	0.525	21, 21, 27, 17, 36, 19
7	0.373	0.652	0.651	0.556	0.513	0.652	0.734	0.493	0.525	21, 21, 17, 17, 36, 19, 10
8	0.530	0.655	0.663	0.560	0.526	0.656	0.735	0.517	0.525	21, 21, 17, 17, 23, 19, 10, 13

Note. CF = computational fluency; N = numeracy; F = fractions; E = equivalence; AR = algebraic reasoning; MARMI = Measure of the Ability to Rotate Mental Images; MASMI = Measure of the Ability to Form Spatial Mental Images; CII = crystalized intelligence index; FII = fluid intelligence index

Table 10

Indices for K-Means Cluster Solutions

Cluster Solutions	Variance Explained									Sample Sizes
	CF	N	F	E	AR	MARMI	MASMI	CII	FII	
4	0.313	0.514	0.600	0.521	0.539	0.639	0.483	0.423	0.502	23, 44, 51, 23
5	0.354	0.536	0.638	0.512	0.588	0.657	0.606	0.424	0.490	23, 43, 23, 43, 18
6	0.397	0.558	0.654	0.526	0.579	0.683	0.708	0.432	0.526	21, 22, 27, 19, 37, 15

Note. CF = computational fluency; N = numeracy; F = fractions; E = equivalence; AR = algebraic reasoning; MARMI = Measure of the Ability to Rotate Mental Images; MASMI = Measure of the Ability to Form Spatial Mental Images; CII = crystalized intelligence index; FII = fluid intelligence index

Table 11

Raw Score Mean, Standard Deviation, F-Statistic, and Partial Eta Squared for Cluster Variables by Cluster Groups

Variable	Cluster Group						$F(5,135)$	η^2
	Low All	Mod. Low All	Mod. Low Spatial	Mod. High MASMI	Mod. High All	High Spatial		
Total n	21	22	27	19	37	15		
Computational Fluency								
M	21.62 ^d	29.09 ^{bc}	34.11 ^{ab}	24.37 ^{cd}	39.43 ^a	34.27 ^{ab}	17.799	.397
SD	3.918	5.740	10.207	6.898	9.257	8.884		
Numeracy								
M	10.24 ^d	14.95 ^c	17.00 ^{bc}	15.11 ^c	21.35 ^a	19.80 ^{ab}	34.130	.558
SD	2.644	3.359	3.893	4.202	2.761	3.468		
Fractions								
M	4.29 ^d	7.27 ^c	9.37 ^b	6.74 ^c	10.43 ^a	10.27 ^{ab}	50.934	.654
SD	1.875	1.830	1.275	2.023	1.345	1.486		
Equivalence								
M	2.57 ^c	4.68 ^b	7.52 ^a	5.05 ^b	7.73 ^a	8.00 ^a	29.941	.526
SD	1.469	1.524	1.626	2.248	2.077	2.299		
Algebraic Reasoning								
M	7.57 ^d	10.18 ^d	12.74 ^{bc}	10.58 ^{cd}	17.22 ^a	15.33 ^{ab}	37.185	.579
SD	3.059	2.889	3.096	3.746	2.573	2.469		
MARMI								
M	1.14 ^{cd}	-2.36 ^d	2.44 ^{bc}	5.16 ^{bc}	6.81 ^b	30.40 ^a	58.250	.683
SD	4.693	4.489	5.515	5.747	7.222	8.990		
MASMI								
M	7.00 ^d	11.41 ^{cd}	19.81 ^c	36.95 ^{ab}	36.16 ^b	42.00 ^a	65.535	.708
SD	6.550	8.798	11.533	6.169	8.719	3.761		
Crystallized Intelligence								
M	88.71 ^d	95.23 ^{cd}	105.19 ^{ab}	99.21 ^{bc}	110.81 ^a	109.53 ^a	20.514	.432
SD	9.012	9.507	9.319	7.406	10.298	8.700		
Fluid Intelligence								
M	95.43 ^c	98.64 ^c	110.89 ^b	107.79 ^b	114.78 ^{ab}	119.33 ^a	29.903	.526
SD	9.453	6.091	6.247	9.265	7.307	9.131		

Table 11 cont'd

Note. All F values are significant at $p < .001$. Uncommon superscripts indicate means that are statistically significantly different at $p < .05$. MARMi = Measure of the Ability to Rotate Mental Images; MASMI = Measure of the Ability to Form Spatial Mental Images.

Table 12

NAEP Achievement Level Descriptions

Achievement Levels	Cut Score		Description
	Points	Percentage	
Basic	141	47	The ability to solve problems that are the direct application of mathematical concepts and procedures.
Proficient	176	57	The mastery of mathematical concepts demonstrated by the appropriate application of concepts and procedures to solve and analyze problems.
Advanced	216	72	The ability to use mathematical knowledge to solve unfamiliar and challenging problems, make mathematical justifications, make justifiable generalization, and use appropriate mathematical language and notation.

Note. Cut score is out of 300 possible points.

Table 13

Frequency Counts and Percentages of Demographic Characteristics by Cluster Group

Demographic Characteristic	Cluster Groups						Total	χ^2
	Low All (n = 21)	Mod. Low All (n = 22)	Mod. Low Spatial (n = 27)	Mod. High MASMI (n = 19)	Mod. High All (n = 37)	High Spatial (n = 15)		
Highest math course taken ^a								
Calculus	5 (3.5)	11 (7.8)	17 (12.1)	11 (7.8)	34 (24.1)	13 (9.2)	91 (64.5)	32.954**
No Calculus	16 (11.3)	11 (7.8)	10 (7.1)	8 (5.7)	3 (2.1)	2 (1.4)	50 (35.5)	
Gender ^a								
Female	16 (11.3)	19 (13.5)	23 (16.3)	16 (11.3)	25 (17.7)	11 (7.8)	110 (78.0)	4.715
Male	5 (3.5)	3 (2.1)	4 (2.8)	3 (2.1)	12 (8.5)	4 (2.8)	31 (22.0)	
Majors ^a								
STEM	4 (2.8)	4 (2.8)	11 (7.8)	9 (6.4)	25 (17.7)	9 (6.4)	62 (44.0)	21.364*
Non-STEM	17 (12.1)	18 (12.8)	16 (11.3)	10 (7.1)	12 (8.5)	6 (4.3)	79 (56.0)	
Race/Ethnicity ^a								
White	7 (5.0)	14 (9.9)	13 (9.2)	14 (9.9)	28 (19.9)	11 (7.8)	87 (61.7)	14.356*
Non-White	14 (9.9)	8 (5.7)	14 (9.9)	5 (3.5)	9 (6.4)	4 (2.8)	54 (38.3)	
School Level								
Lowerclassmen (F, So)	8 (5.7)	8 (5.7)	15 (10.6)	9 (6.4)	15 (10.6)	8 (5.7)	63 (44.7)	3.042
Upperclassmen (Jr., Sr., 5+)	13 (9.2)	14 (9.9)	12 (8.5)	10 (7.1)	22 (15.6)	7 (5.0)	78 (55.3)	

Note. F = Freshmen. So = Sophomore. Jr. = Junior. Sr. = Senior 5+. = 5+ year Senior.

* $p < .05$. ** $p < .001$.

Table 14

Raw Score Mean, Standard Deviation, F-Statistic, and Partial Eta Squared for Algebra Performance by Cluster Groups

Variable	Cluster Group						$F(5,56.603)$	η^2
	Low All	Mod. Low All	Mod. Low Spatial	Mod. High MASMI	Mod. High All	High Spatial		
Total n	21	22	27	19	37	15		
Algebra Performance								
M	4.33 ^c	5.64 ^{bc}	7.81 ^b	5.58 ^{bc}	10.30 ^a	10.93 ^a	18.896	.410
SD	2.153	2.300	2.936	2.735	3.650	3.218		

Note. All F values are significant at $p < .001$. Uncommon superscripts indicate means that are statistically significantly different at $p < .05$.

Table 15

Summary of Multiple Regression Analysis for Demographic Characteristics and Cluster Contrasts

Variable	b	SE _b	β	<i>p</i>	r_{partial}	r^2_{partial}
Intercept	7.148	.867		< .001		
Calculus	1.652	.621	.209	.009	.171	.029
Years Since Algebra Course	-.076	.146	-.038	.604	-.033	.001
Race/Ethnicity	-.079	.535	-.010	.883	-.009	< .001
Major	-.671	.563	-.088	.236	-.077	.006
MLA, MHM, LA – MHA, HS, MLS ^a	5.719	.870	.502	< .001	.423	.179
MLS, HS – MHA ^a	.348	.457	.053	.447	.049	.0024
LA – MLA, MHM ^a	.410	.550	.051	.457	.048	.0023
MLS – HS ^a	1.340	.480	.191	.006	.180	.032
MLA – MHM ^a	.150	.466	.021	.749	.021	< .001

Note. MHA = moderate high all; HS = high spatial; MLS = moderate low spatial; MHM = moderate high MASMI; MLA = moderate low all; LA = low all; b = unstandardized regression coefficient; SE_b = standard error of unstandardized regression coefficient; β = standardized regression coefficient; r_{partial} = semi partial correlation.

^aPlanned contrasts for cluster membership comparisons going from cluster group (s) A to cluster group(s) B.

Appendix B: Figures

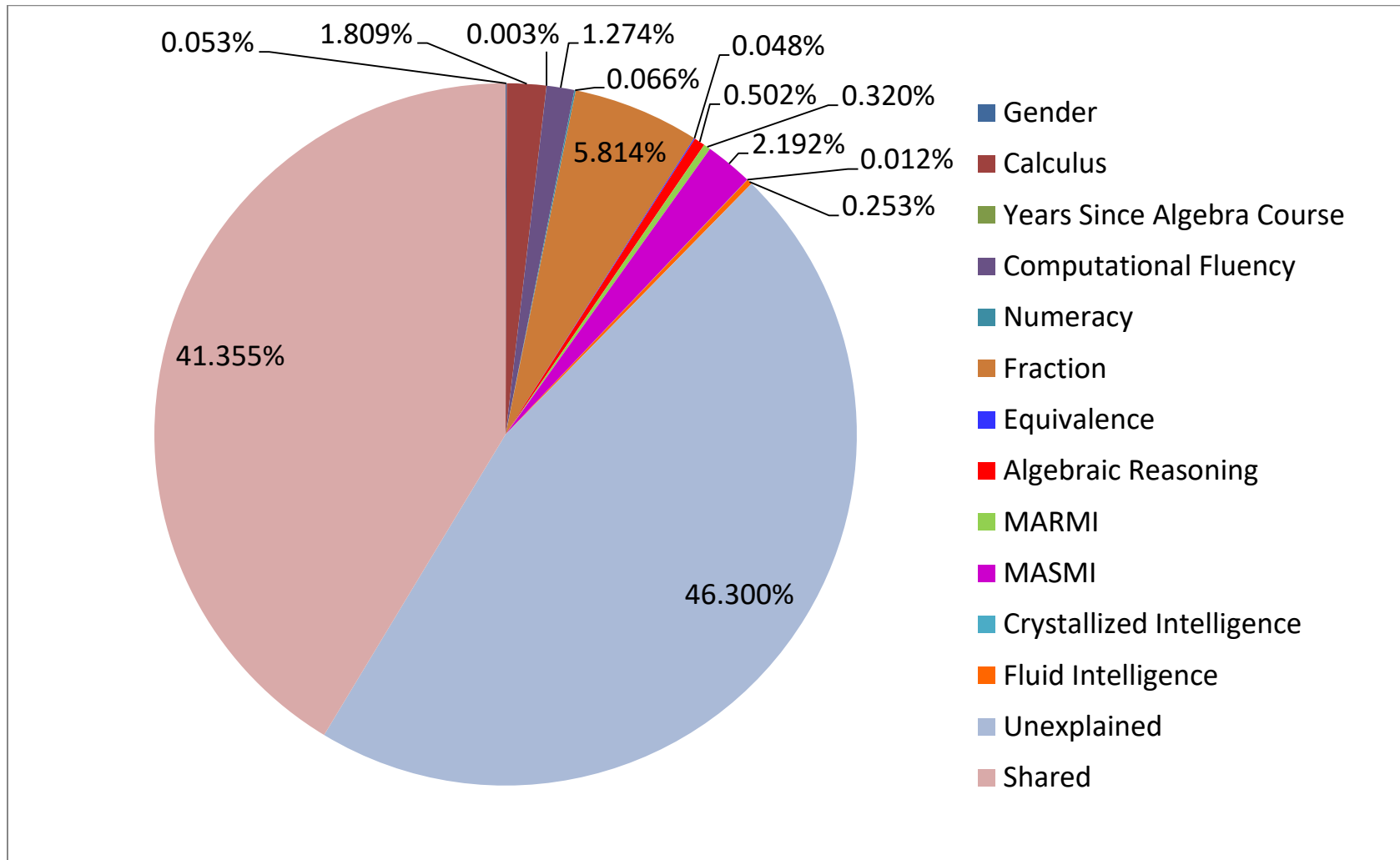


Figure 1. Variance explained in algebra performance scores by predictor variables in multiple regression analysis.

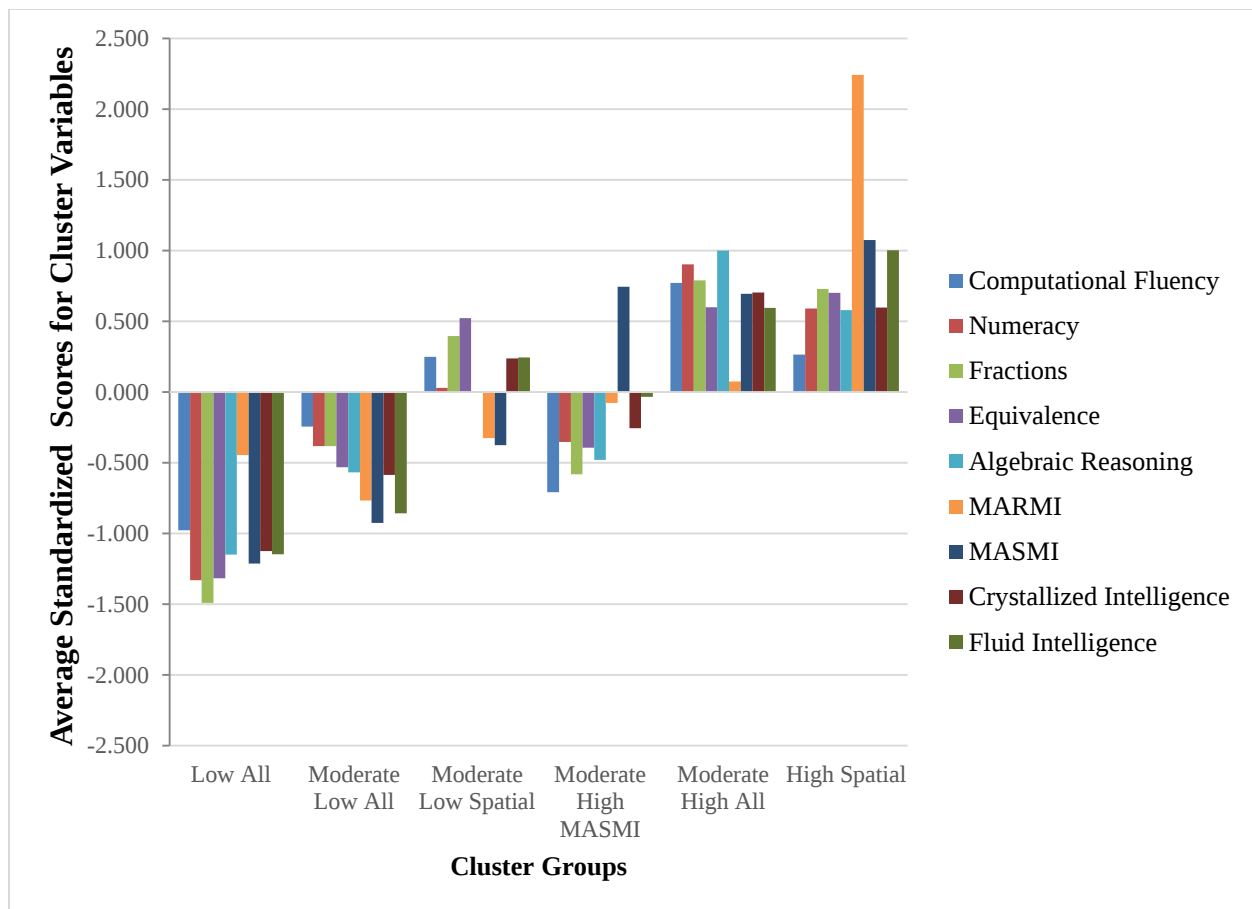


Figure 2. Results of six-cluster solution, showing the average standardized scores on all clustering variables for each cluster group.

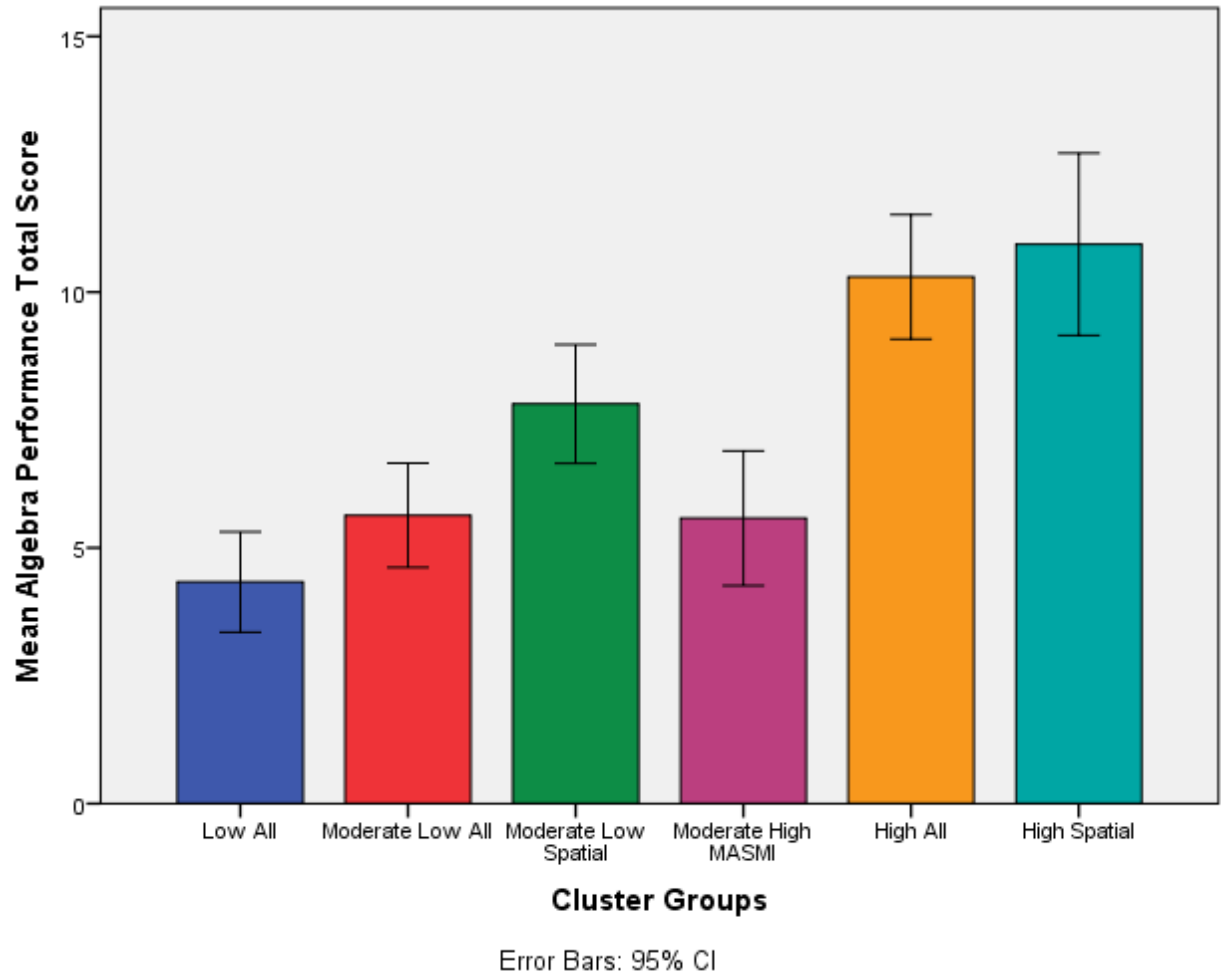


Figure 3. Mean algebra performance scores for participants in each cluster group.

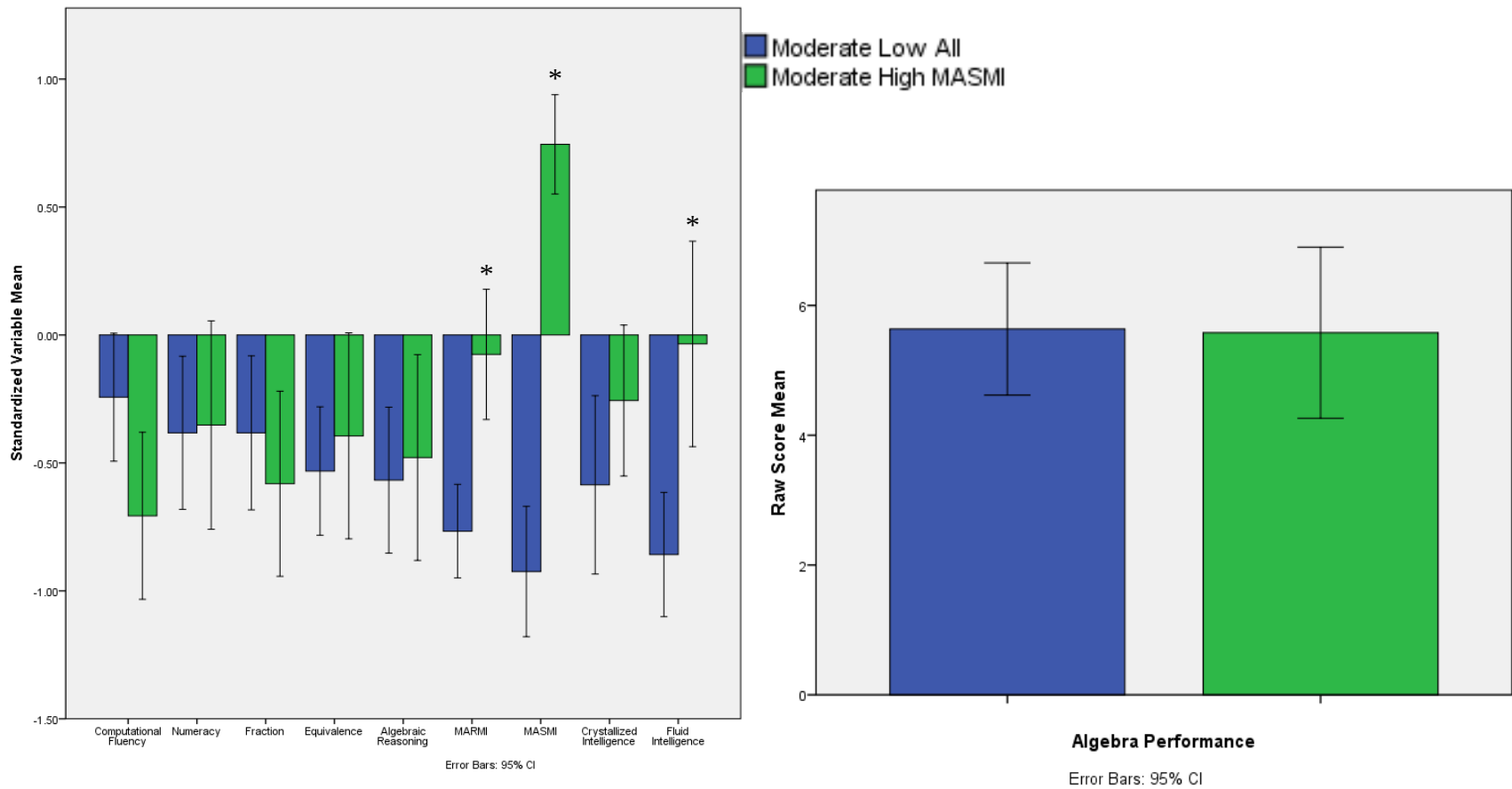


Figure 4. Cluster variable scores and algebra performance score comparison of participants in the Moderate Low All and Moderate High MASMI cluster groups.

* statistically significant difference at $p < .05$.

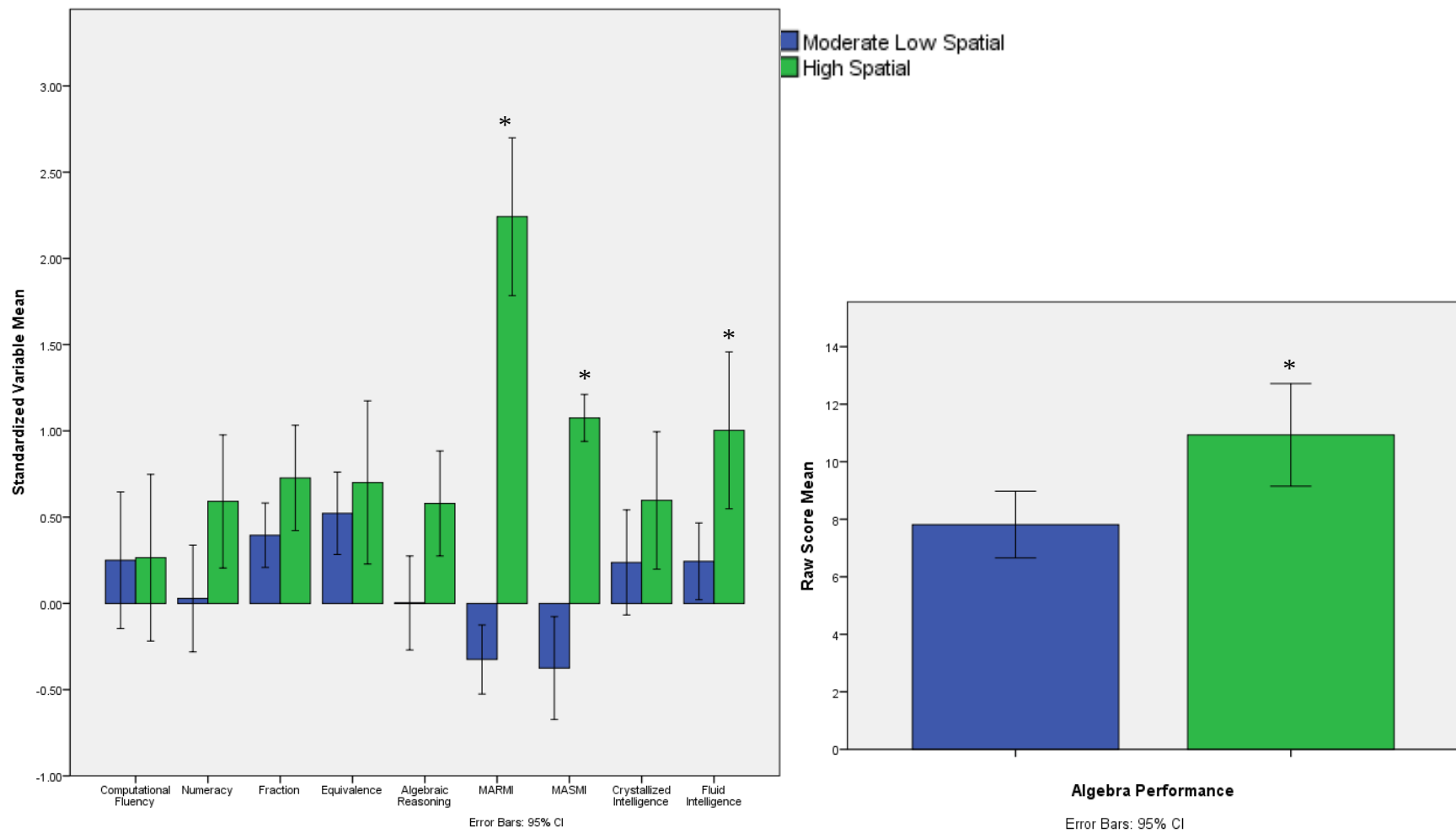


Figure 5. Cluster variable scores and algebra performance score comparison of participants in the Moderate Low Spatial and High Spatial cluster groups.

* statistically significant difference at $p < .05$.

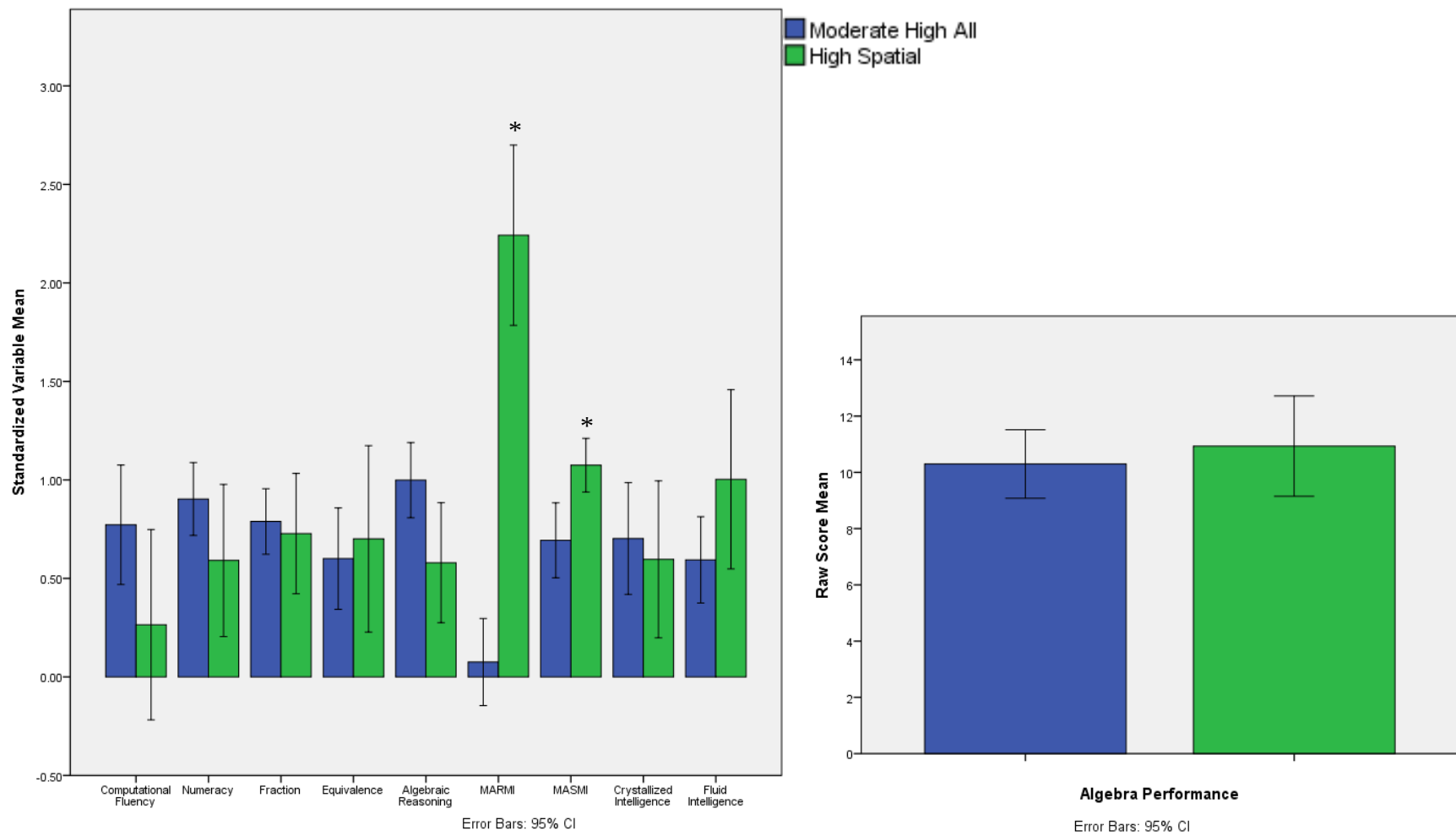


Figure 6. Cluster variable scores and algebra performance score comparison of participants in the Moderate High All and High Spatial cluster groups.

* statistically significant difference at $p < .05$.

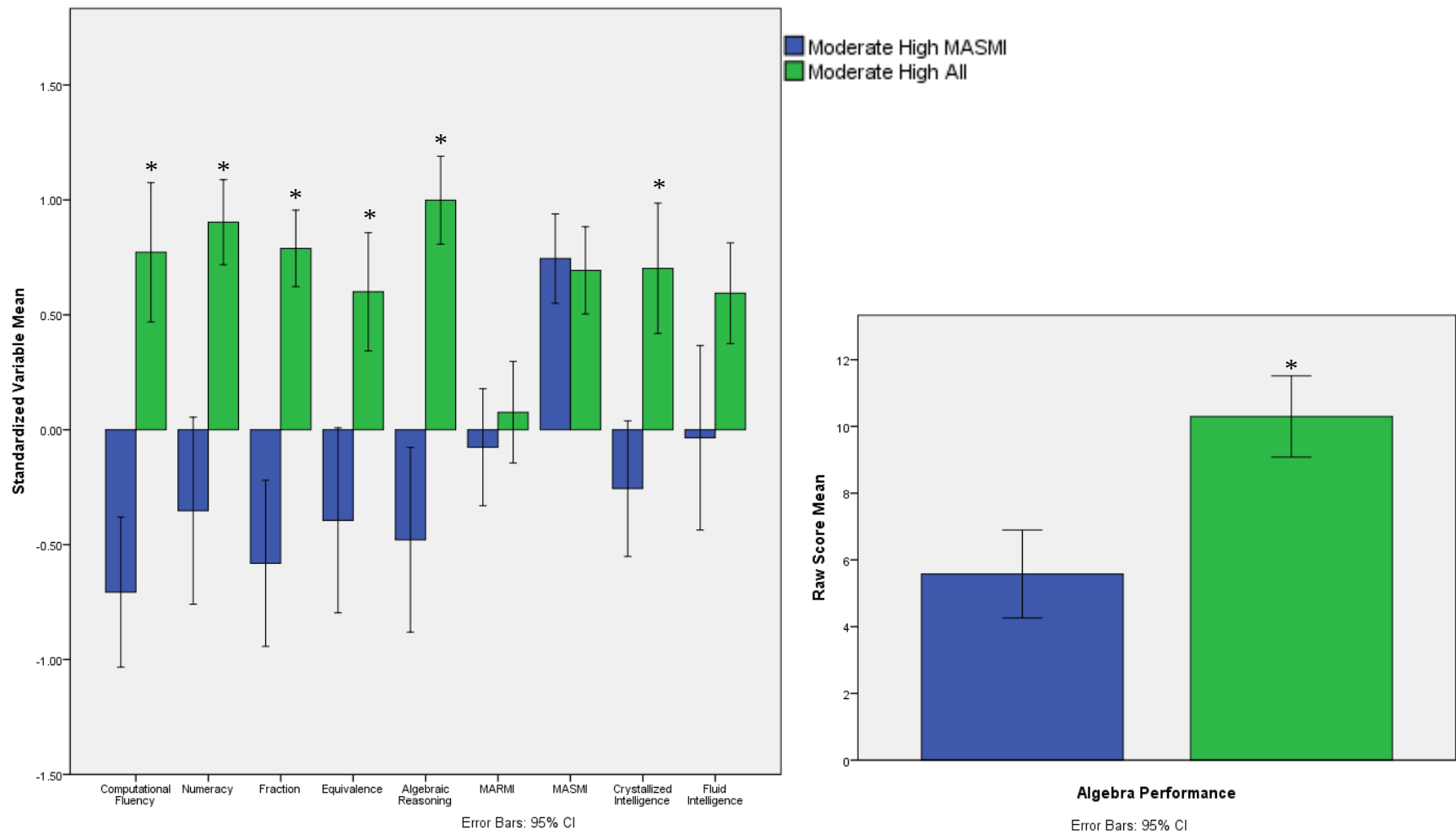


Figure 7. Cluster variable scores and algebra performance score comparison of participants in the Moderate High MASMI and Moderate High All cluster groups.

*statistically significant difference at $p < .05$.

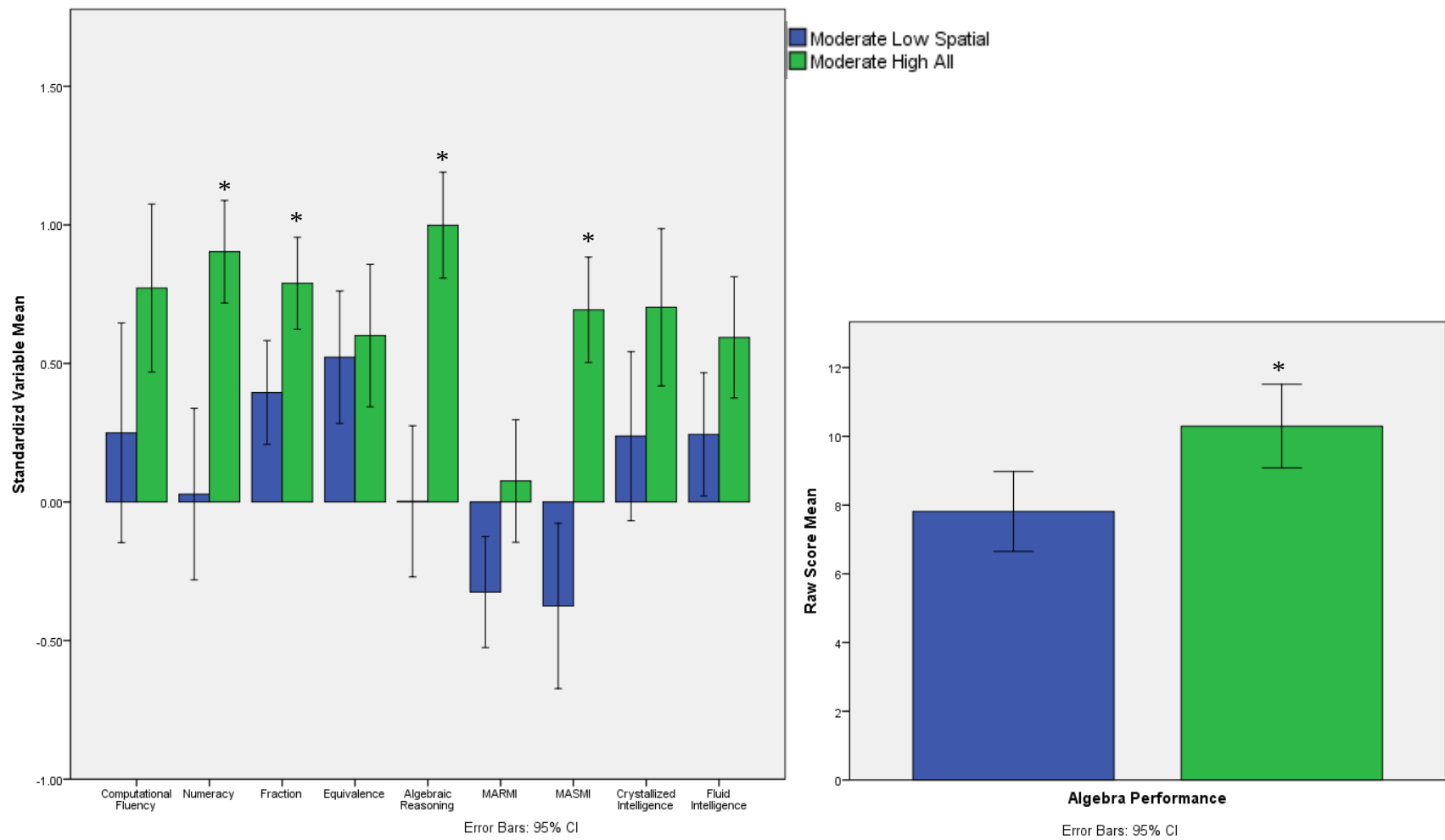


Figure 8. Cluster variable scores and algebra performance score comparison of participants in the Moderate Low Spatial and Moderate High All cluster groups.

* statistically significant difference at $p < .05$.

Appendix C: Assessments and Measures

Computational Fluency Assessment

1. $6220 + 3545$

2. $104 \div 4$

3. $4352 - 2311$

4. 21×13

5. $444 - 387$

6. 36×48

7. $27531 + 18515$

8. $432 \div 54$

9. 785×37

10. $43 + 58 + 28$

11. $488 \div 8$

12. $9101 - 7247$

13. $630 \div 15$

14. $93245 - 71378$

15. 2615×11

16. $811 + 168 + 237 + 753$

Numeracy Assessment

Addition and Subtraction

1. $1,504 + 6.2$

- a. 1,560 b. 1,100
c. 1,510 d. 1,504.62

2. $-7.2 + 6.9$

- a. -14 b. 0
c. 1 d. 14

3. $11.98 + 6.02$

- a. 7 b. 18
c. 1,800 d. 1,000

4. $73 + 7.3$

- a. 140 b. 150
c. 80 d. 70

5. $782 - 83$

- a. 700 b. 600
c. 800 d. 750

6. $3,012 - 2,998$

- a. 0 b. 100
c. 1,000 d. 1,999

7. $609 - 0.69$

- a. 550 b. 500
c. 600 d. 609

8. $0.25 - 0.12$

- a. 0.05 b. 0.12
c. 0.5 d. 0

9. $\frac{4}{5} + \frac{9}{8}$

- a. 2 b. 1
c. 3 d. 4

10. $2\frac{1}{3} + \frac{1}{4}$

- a. 2.5 b. 2
c. 1 d. 3

11. $\frac{8}{6} - \frac{6}{8}$

- a. 0 b. 0.5
c. $\frac{8}{4}$ d. $\frac{9}{4}$

12. $5\frac{1}{5} - 2\frac{1}{2}$

- a. $\frac{1}{2}$ b. $1\frac{1}{2}$
c. $2\frac{1}{2}$ d. $3\frac{1}{2}$

Multiplication and Division

13. 15×9

- a. 150 b. 200
c. 900 d. 1,500

15. 0.9×5

- a. 0 b. 2
c. 5 d. 45

17. $1,602 \div 99$

- a. 0 b. 1
c. 10 d. 16

19. $8.2 \div 10$

- a. 1 b. 4
c. 8 d. 82

21. $\frac{1}{2} \times \frac{15}{16}$

- a. 0 b. $\frac{1}{10}$
d. $\frac{1}{2}$ d. 1

23. $2 \div \frac{49}{100}$

- a. 0 b. $\frac{1}{2}$
f. $\frac{3}{4}$ d. 4

14. 26×16

- a. 50 b. 400
c. 800 d. 4,000

16. 0.3×9

- a. 0.1 b. 0.5
c. 2 d. 3

18. $4,942 \div 49$

- a. 0.5 b. 10
c. 100 d. 1,000

20. $61 \div 5.9$

- a. 1 b. 10
c. 15 d. 20

22. $\frac{7}{8} \times \frac{8}{4}$

- a. 0 b. 1
e. 2 d. $2\frac{1}{2}$

24. $\frac{11}{12} \div 3$

- a. 0 b. $\frac{1}{3}$
d. 1 d. 3

Fractions Assessment

Category I: Algorithmic Operations

1. Subtract $\frac{3}{5}$ from 8.
2. Write $3\frac{5}{6}$ as an improper fraction.

Category II: Application of Basic Fraction Concepts in Word Problems

1. If you have a half ball of string and each kite needs an eighth of a ball, how many kites can you fly?
2. Adrian has conquered only 6 giants in his new video game, Giant Trouble, but it is only two-fifths of the giants that he must conquer. How many giants are there in the new video game?

Category III: Elementary Algebraic Concepts

1. Solve $x + \frac{1}{3} = 7$.
2. $\frac{1}{3} \times a = ?$

Category IV: Specific Arithmetic Skills that are Prerequisite for Algebra

1. Write $5\frac{2}{7}$ as a sum.
2. Find $\frac{18}{0}$.

Category V: Comprehension of the Structure of Rational Numbers

1. Write fractions $\frac{4}{7}, \frac{5}{9}, \frac{3}{5}$ in order from least to greatest.
2. The quotient of $\frac{1}{2} \div \frac{1}{3}$ is greater than ($>$) or less than ($<$) $\frac{1}{2}$?

Category VI: Computational Fluency

1. Find the sum $\frac{7+5}{3+5} + \frac{\frac{5}{6}}{\frac{5}{3}}$.
2. In an election, candidate A got $\frac{1}{3}$ of the votes, candidate B got $\frac{9}{20}$ of the votes, and candidate C got $\frac{2}{15}$ of the votes. What fraction of the votes did candidate D get?

Equivalence Assessment

Equal Sign Interpretation

1. What does the equal sign (=) mean? Can it mean anything else?
2. What does the equal sign mean in this statement? 1 dollar = 100 pennies

Equation Structure

1. Is the number that goes in the blank the same number in the following two number sentences? Yes, No, How do you know? $2 \times \underline{\quad} = 58$ $8 \times 2 \times \underline{\quad} = 8 \times 58$
2. Find a number that can go in each blank. Can another number go in these blanks? Explain. $8 + 2 + \underline{\quad} = 10 + \underline{\quad}$

Open Equation

1. Fill in the blanks with the value that makes the following number sentences true.
 - a. $4,436 + 2,897 = \underline{\quad} + 3,000$
 - b. $3,901 - 2,012 = 3,889 - \underline{\quad}$
2. Place the four numbers $n - 1, n + 1, m + 3, m + 1$ in the following boxes so that the number sentence is always true. $\square + \square = \square + \square$

Algebraic Reasoning Assessment

Functional Thinking & Generalization

1.

Number of sets	Number of red Tiles	Number of green tiles	Total number of tiles
1	1	2	3
2	2	4	6
3	3	6	9
4	4	8	12
5	5	10	15
			33
	12		
		40	

Using the information in the table answer:

- Identify the relationship between the number of red tiles and any given number of sets (x).
- Identify the relationship between the number of green tiles and any given number of sets (x).
- Identify the relationship between the total number of tiles and any given number of sets (x).

2.

	Number of Squares	Number of Vertices
	1	4
	2	7
	3	10
	4	_____
	10	_____
	100	_____

- Suppose you are given the pattern of squares shown above. How would you describe the relationship between the number of squares and the number of vertices in words?
- How would you represent the relationship between the number of squares and the number of vertices using algebra?

Modeling

1. Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who ordered cheesecake, there are five people who ordered strudel." Let C represent the number of cheesecakes and S represent the number of strudels ordered.
2. Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at a certain university." Use S for the number of students and P for the number of professors.

Structure Sense

1. Solve for x .
 - a. $\left(\frac{1}{4} - \frac{x}{x-1}\right) - x = 6 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$
 - b. $1 - \frac{1}{x+2} - \left(1 - \frac{1}{x+2}\right) = \frac{1}{110}$
2. Solve the following number sentences.
 - a. $237 + 89 - 89 + 267 - 92 + 92 = ?$
 - b. $217 - 59 + 59 + 62 - 28 - 28 = ?$

Symbolic Manipulation

1. For each example, write an equivalent expression.
 - a. $4h + t =$ _____
 - b. $u + 5 + 6 + 5 + u =$ _____
 - c. $4(n + 5) =$ _____
 - d. $p + 0.05p =$ _____
 - e. $5(e + 2) =$ _____
 - f. $x - x + 2 =$ _____
 - g. $(15 + 10x) + (35 + 5x) =$ _____
 - h. $3x + 4 + 6(x + 5) =$ _____

Algebra Performance Assessment

Systems of Equations

$$x + 2y = 17$$

$$x - 2y = 3$$

1. The graphs of the two equations shown above intersect at the point (x, y) . What is the value of x at the point of intersection?

A. $3\frac{1}{2}$

B. 5

C. 7

D. 10

E. 20

(NAEP Question ID: 2005-12M3 #12 M095201)

$$\begin{cases} x + 2y = 1 \\ 2x - y = 7 \end{cases}$$

2. In the solution of the system of equations above, what is the value of x ?

A. - 1

B. 2

C. 3

D. 4

E. 5

(NAEP Question ID: 2005-12M4 #11 M053201)

Functions

1. If $f(x) = \frac{2x+1}{3}$ and $g(x) = 2x^2 + 2$, then $f(g(2)) =$

A. 3

B. 5

C. 7

D. $7\frac{5}{9}$

E. $16\frac{2}{3}$

(NAEP Question ID: 1992-12M5 #20 M025401)

2. Yvonne has studied the cost of tickets over time for her favorite sports team. She has created a model to predict the cost of a ticket in the future. Let C represent the cost of a ticket in dollars and y represent the number of years in the future. Her model is as follows.

$$C = 2.50y + 13$$

Based on this model, how much will the cost of a ticket increase in two years?

- A. \$5
- B. \$8
- C. \$13
- D. \$18
- E. \$26

(NAEP Question ID: 2005-12M12 #17 M130101)

Solving Equations

1. If $d = 110$ and $a = 20$ in the formula $d = \frac{a}{2}(2t - 1)$, then $t =$

- A. $\frac{15}{22}$
- B. $\frac{15}{8}$
- C. 5
- D. $\frac{111}{20}$
- E. 6

(NAEP Question ID: 1990-12M9 #12 M030231)

2. If $\frac{1}{a} + \frac{4}{7} = \frac{1}{2}$, then $a = (?)$.

- A. $\frac{3}{5}$
- B. -14
- C. $\frac{-3}{5}$
- D. 14
- E. $\frac{14}{3}$

Inequalities

1. What are all values of x such that $x^2 + 7x + 6 \geq 0$?

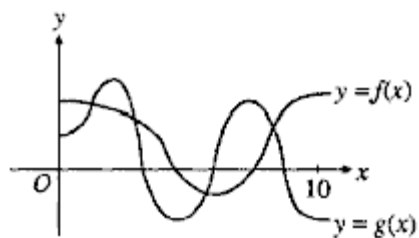
- A. $-6 \leq x \leq -1$
- B. $-6 \leq x \leq 1$
- C. $1 \leq x \leq 6$
- D. $x \leq -6$ or $x \geq -1$
- E. $x \leq 1$ or $x \geq 6$

(NAEP Question ID: 2005-12M4 #16 M019401)

2. For which values of w is $1.25 \leq \frac{l}{w} \leq 2.5$ if $l = 4$?

- A. $w \geq 1.6$ or $w \leq 3.2$
- B. $1.6 \leq w \leq 3.2$
- C. $w \geq 0.16$ or $w \leq 0.30$
- D. $0.16 \leq w \leq 0.30$
- E. $5 \leq w \leq 10$

Graphing

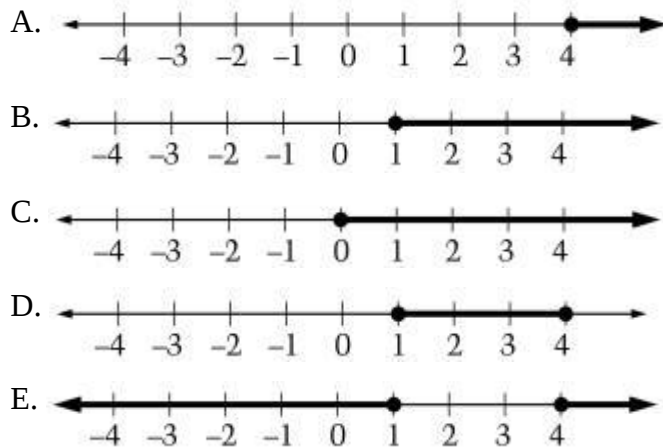


1. The graphs of $y = f(x)$ and $y = g(x)$ for $0 \leq x \leq 10$ are shown in the figure above. For how many values of x is the product $f(x)g(x) = 0$ for $0 \leq x \leq 10$?

- A. Two
- B. Four
- C. Five
- D. Six
- E. Seven

(NAEP Question ID: 1990-12M9 #19 M030931)

2. Which of the following is the graph of $|2x - 5| \geq 3$?



(NAEP Question ID: 2005-12M3 #15 M011831)

Exponents

1. For what value of x is $8^{12} = 16^x$?

- A. 3
- B. 4
- C. 8
- D. 9
- E. 12

(NAEP Question ID: 1992-12M7 #6 M057701)

2. $121^{\frac{3}{2}} =$

- A. 1331
- B. 1728
- C. 2197
- D. 4096
- E. 1452

Factoring

1. Factor $28n^4 + 16n^3 - 80n^2$. One of the factors is (?).

A. $7n + 10$

B. $n - 2$

C. $n + 5$

D. $4n^2$

E. $n - 4$

2. $m^2 + 10m + 14 + b$ is a perfect square. Find b .

A. 0

B. 7

C. -4

D. 11

E. 21

Complex Numbers

1. If $i = \sqrt{-1}$ then $\frac{1}{6+4i} = ?$

A. $\frac{6-4i}{52}$

B. $\frac{6+4i}{52}$

C. $\frac{6+4i}{20}$

D. $\frac{6-4i}{20}$

E. $\frac{6-4i}{32}$

2. Let $i = \sqrt{-1}$. After expanding and simplifying $(2 + i)^3 = ?$

A. $8 + i^3$

B. $8 - i$

C. $2 + 11i$

D. $6 + 3i$

E. $10 + 11i$

Polynomial Division

1. If $6x^3 + 5x - 8$ is divided by $x - 2$, the remainder is?

A. 50

B. -64

C. -8

D. 0

E. -16

2. $(x^3 + 12x^2 + 47x + 60) \div (x + 5) = ?$

A. $x^2 + 17x + 132 + \frac{720}{x+5}$

B. $x^2 + 17x - 38 - \frac{50}{x+5}$

C. $x^2 + 7x + 12$

D. $x^3 + 7x^2 + 12x$

E. $x^2 + 7x - 31 + \frac{215}{x+5}$

Logarithms

1. Which one is equivalent to $\log_4(16x^6)$?

A. $4 + 6 \log_4 x$

B. $12 \log_4 x$

C. $2 + 6 \log_4 x$

D. $6 \log_4 x$

E. $2 \log_4 x$

2. Solve $\log_x 16 = 2$ for x . $x = (?)$.

A. $\frac{2}{16}$

B. 4

C. 256

D. 32

E. 8

Participant Demographic Survey

Please answer the following questions.

Name (please print): _____

Age: _____

Gender: ☐ Female ☐ Male ☐ Other: _____

Race/Ethnicity (check all that apply):

☐ American Indian/Alaska Native ☐ Asian ☐ Black/African American

☐ Hispanic/Latino ☐ Native Hawaiian/Other Pacific Islander ☐ White

☐ Other: _____

School Level:

☐ Freshman ☐ Sophomore ☐ Junior ☐ Senior ☐ 5+ year Senior

Major: _____

List all mathematics courses taken in high school and the year in which it was taken.

<u>High School Mathematics Course</u>	<u>Year Taken</u>
a) _____	_____
b) _____	_____
c) _____	_____
d) _____	_____
e) _____	_____

List all mathematics courses taken in college and the year in which it was taken.

<u>College Mathematics Course</u>	<u>Year Taken</u>
a) _____	_____
b) _____	_____
c) _____	_____
d) _____	_____
e) _____	_____

Appendix D: Research Protocols

Starting Procedures

1. As the participant comes in pass out the participant information and consent form and say, *“Hi my name is [your name}. Welcome to the Factors of Algebra Performance study. Thank you for coming. Here is the Participant Information and Consent form. This form explains this study in detail. Take a minute to read it.”*
2. Give the participant a few minutes to read, and then say, *“Today I am going to be asking you to do a number of different tasks. Some of them will be mathematical in nature and some will not. Some questions will be easy and some will be hard. Each task will have a time limit. Most people do not get to all the questions nor do they get them all right. We just want you to do your best. Participation is voluntary and you may quit at any time. You may also take a break when you need to. After the completion of the study, you will receive \$15 dollars as a thank you for participating. Are there any questions?”* Answer any questions.
3. Then say, *“If you are willing to continue please fill out and sign the second page of the Participant Information and Consent Form that I have pass out to you, and hand it back to me. You may keep the first page.”* Collect the signed form.
4. Then say, *“Thank you for agreeing to participate in this study. Let’s get started with the first task.”* Use the given order for the group and begin with the first task.

Computational Fluency

1. Distribute the computational fluency packet to the student, and say, *“Now we are going to be doing another math task. Follow along with me as I read the directions aloud. The sheets in front of you are math facts. There are several types of problems on the sheet. Some are addition, some are subtraction, some are multiplication, and some are division. Look at each problem carefully before you answer it. When I say, begin, turn to the next page and begin answering the problems. Start on the first problem on the left on the top row [point]. Work across then go to the next row. If you cannot answer a problem, make an ‘X’ on it and go to the next one. If you finish one page, go to the next one. You will have only three minutes to complete all problems. It is okay if you do not complete all the problems. Just try to do your best. Are there any questions?”* Answer any questions.
2. Then say, *“If there are not any [more] questions [pause], then you may begin.”* Start the timer. Participants get 3 minutes to complete each worksheet.
3. After three minutes have passed say, *“Pencil down. Time is up and we need to move on to the next task”* and collect the computational fluency worksheet.

Numeracy

1. Distribute the numeracy packet to the student, and say, *“Now we will be doing another math task. Follow along with me as I read the directions aloud. When I say, ‘Begin’ turn to the next page, and begin with the first problem. Pay close attention to all the signs indicating positive and negative numbers. For each problem, the exact answer is not listed, so choose the answer that is as close to the exact answer as possible by circling your answer. You will only get 4 minutes to complete the whole task, so work as quickly as you can. It is okay if you do not complete all problems. Just try to do your best. Again, the exact answer is not listed, so choose an answer that is closest to the exact answer as possible. Are there any questions?”* Answer any questions.
2. Then say, *“If there are not any [more] questions [pause], then you may begin.”* Start the timer. Participants get 4 minutes to complete the task.
3. After 4 minutes have passed say, *“Pencil down. Time is up and we need to move on to the next task.”* and collect the numeracy packets.

Fractions

1. Distribute the fraction packet to the student, and say, *“Now we will be doing another math task. Follow along with me as I read the directions aloud. When I say, ‘Begin’ turn to the next page, and begin with the first problem. Make sure to show all your work in the space provided. If you need more space, you may use scratch paper. If you use scratch paper, please label your work by writing the problem number next to it. After you solve each problem, circle your answer. Some of these problems will be easy and some will be hard. If you do not know how to solve a problem, guess or estimate to the best of your ability or go on to the next one until you have completed as many as you can or are told to stop. You will only get 10 minutes to complete all the problems. It is okay if you do not complete all problems. Just try to do your best. Are there any questions?”* Answer any questions.
2. Then say, *“If there are not any [more] questions [pause], then you may begin.”* Start the timer. Participants get 10 minutes to complete the task.
3. After 10 minutes have passed say, *“Pencil down. Time is up and we need to move on to the next task.”* and collect the fraction packets.

Equivalence

1. Distribute the equivalence packet to the student, and say, *“Now we will be doing another math task. Follow along as I read the directions aloud. When I say, ‘Begin’ turn to the next page, and begin with the first problem. Make sure to show all your work in the space provided. If you do not know how to solve a problem, guess or estimate to the best of your ability or go on to the next one until you have completed as many as you can or are told to stop. You will only get 10 minutes to complete all the problems. It is*

okay if you do not complete all problems. Just try to do your best. Are there any questions?" Answer any questions.

2. Then say, *"If there are not any [more] questions [pause], then you may begin."* Start the timer. Participants get 10 minutes to complete the task.
3. After 10 minutes have passed say, *"Pencil down. Time is up and we need to move on to the next task."* and collect the equivalence packets.

Algebraic Reasoning

1. Distribute the algebraic reasoning packet to the student, and say, *"Now we will be doing another math task. Follow along as I read the directions aloud. When I say, 'Begin' turn to the next page, and begin with the first problem. Pay close attention to all the signs indicating positive and negative numbers. Make sure to show all your work in the space provided. If you need more space, you may use scratch paper. If you use scratch paper, please label your work by writing the problem number next to it. Some of these problems will be easy and some will be hard. If you do not know how to solve a problem, guess or estimate to the best of your ability or go on to the next one until you have completed as many as you can or are told to stop. You will only get 10 minutes to complete all the problems. It is okay if you do not complete all problems. Just try to do your best. Are there any questions?"* Answer any questions.
2. Then say, *"If there are not any [more] questions [pause], then you may begin."* Start the timer. Participants get 10 minutes to complete the task.
3. After 10 minutes have passed say, *"Pencil down. Time is up and we need to move on to the next task."* and collect the algebraic reasoning packets.

Algebra Performance

1. Distribute the algebraic reasoning packet to the student, and say, *"Now we will be doing another math task. Follow along as I read the directions aloud. When I say, 'Begin' turn to the next page, and begin with the first problem. Pay close attention to all the signs indicating positive and negative numbers and exponentials. You may use the calculator or scratch paper to help you solve the problems. If you use scratch paper, please label your work by writing the problem number next to it. After you solve each problem, make sure to circle your answer. Some of these problems will be easy and some will be hard. If you do not know how to solve a problem, guess or estimate to the best of your ability or go on to the next one until you have completed as many as you can or are told to stop. You will only get 20 minutes to complete all the problems. It is okay if you do not complete all problems. Just try to do your best. Are there any questions?"* Answer any questions.
2. Then say, *"If there are not any [more] questions [pause], then you may begin."* Start the timer. Participants get 20 minutes to complete the task.

3. After 20 minutes have passed say, ***“Pencil down. Time is up and we need to move on to the next task.”*** and collect the algebra performance packets.

MASMI: Measure of the Ability to form Spatial Mental Imagery

1. Distribute the MARMI/MASMI packet to the student, and say, ***“Now we will be doing another cognitive task. Follow along as I read the directions aloud.”*** Use one of the MASMI copies to read the directions on the first page.
2. Then say, ***“Are there any questions?”*** Answer any questions.
3. Then say, ***“If there are not any [more] questions [pause], then you may begin.”*** Start the timer. Participants get 10 minutes to complete the task.
4. After 10 minutes have passed say, ***“Pencil down. Time is up and we need to move on to the next task.”*** and collect the MARMI/MASMI packets.

RAIT: Reynolds Adaptable Intelligence Test

1. Distribute the test booklets and answer sheets to all students, and say, ***“Now we will be doing another cognitive task. Here is your test booklet and answer sheet. Go ahead and read the instructions on the front cover of the test booklet. Do not turn to then next page until I tell you to. Then write your date of birth in the correct spaces on the answer sheets. Do not worry about filling out the rest. Let me know when you are finished.”***
2. After the student has read the instructions on the front cover of the test booklet and filled out the answer sheet, say, ***“We will not do all the sections in the test booklet. I will let you know which sections we are going to complete. We will do one section at a time to, so please pay close attention to which page numbers that I give so that you can complete the right section. It will only take 30 minutes to complete the all the sections. Are there any questions about the instructions on the cover of the test booklet?”***
Answer any questions.
3. Then say, ***“If there are not any [more] questions, then please turn to page 3 in your test booklet. We will start with the first section in the test booklet. Take a moment to read the example for section 1. Let me know when you are finished.”***
4. Once the student has read the example, say, ***“Do you understand what to do for section 1? [wait for affirmation] You will have only 3 minutes to complete this section. Please do not mark in your test booklet. Mark all your answers on your answer sheet in the first section on left labeled GK. If you finish before time is up, you may review your answer for section 1 only. Otherwise, let me know when you are finished so we can move on to the next section. Are there any questions?”*** Answer any questions.
5. Then say, ***“If there are not any [more] questions [pause], then you may begin.”*** Start the timer. Participants get 3 minutes to complete section1.
6. After 3 minutes have passed say, ***“Pencil down. Time is up and we need to move on to the next section.”***

7. Then say, *“Please turn to page 7 in your test booklet. This should be section 2. Take a moment and read the example for section 2. Let me know when you are finished”*
8. Once the student has read the example, say, *“Do you understand what to do for section 2? [wait for affirmation] You will have only 7 minutes to complete this section. Please do not mark in your test booklet. Mark all your answers on your answer sheet in the middle section labeled NVA. If you finish before time is up, you may review your answer for section 2 only. Otherwise, let me know when you are finished so we can move on to the next section. Are there any questions?”* Answer any questions.
9. Then say, *“If there are not any [more] questions [pause], then you may begin.”* Start the timer. Participants get 7 minutes to complete section 2.
10. After 7 minutes have passed say, *“Pencil down. Time is up and we need to move on to the next section.”*
11. Then say, *“Please turn to page 35 in your test booklet. This should be section 3. Take a moment and read through the example for section 3. Let me know when you are finished.”*
12. Once the student has read the example, say, *“Do you understand what to do for section 3? [wait for affirmation] You will have only 10 minutes to complete this section. Please do not mark in your test booklet. Mark all your answers on your answer sheet in the section on the right labeled SEQ. If you finish before time is up, you may review your answer for section 3 only. Otherwise, let me know when you are finished so we can move on to the next section. Are there any questions?”* Answer any questions.
13. Then say, *“If there are not any [more] questions [pause], then you may begin.”* Start the timer. Participants get 10 minutes to complete section 3.
14. After 10 minutes have passed say, *“Pencil down. Time is up and we need to move on to the next section.”*
15. Then say, *“Please turn to page 67 in your test booklet. This should be section 6. Take a moment and read through the example for section 6. Let me know when you are finished.”*
16. Once the student has read the example, say, *“Do you understand what to do for section 6? [wait for affirmation] You will have only 5 minutes to complete this section. Please do not mark in your test booklet. Mark all your answers on your answer sheet in the third section from the left section labeled OWO on side 2. If you finish before time is up, you may review your answer for section 6 only. Otherwise, let me know when you are finished so we can move on to the next section. Are there any questions?”* Answer any questions.
17. Then say, *“If there are not any [more] questions [pause], then you may begin.”* Start the timer. Participants get 5 minutes to complete section 6.
18. After 5 minutes have passed say, *“Pencil down. Time is up and we need to move on to the next section.”*

19. Then say, *“Please turn to page 70 in your test booklet. This should be section 7. Take a moment and read through the example for section 7.”*
20. Once the student has read the example, say, *“Do you understand what to do for section 7? [wait for affirmation] You will have only 5 minutes to complete this section. Please do not mark in your test booklet. Mark all your answers on your answer sheet in the last section labeled WO on side 2. If you finish before time is up, you may review your answer for section 7 only. Otherwise, let me know when you are finished so we can move on to the next section. Are there any questions?”* Answer any questions.
21. Then say, *“If there are not any [more] questions [pause], then you may begin.”* Start the timer. Participants get 5 minutes to complete section 1.
22. After 5 minutes have passed say, *“Pencil down. Time is up. This was the last section for this task. Make sure you have filled out your date of birth on your answer sheet before handing it back.”*

Participant Demographic Survey

1. Say, *“We are now at the end of the study. You have completed all the tasks. Now I would like to ask you to fill out this survey. [pass out survey to the participant] The survey will ask you a number of demographic questions and about your past mathematics education. Please answer all questions to the best of your ability. If you come to a question that you do not wish to answer, please feel free to skip it and move on to the next one. Once you have completed the survey, place hand it back to me. I will then give you your appreciation gift and you are free to go. Are there any questions?”* Answer any questions.
2. Collect the complete surveys. As participants hand in their survey, say, *“Thank you again for participating in the Factors of Algebra Performance Study. Have a great day!”*

Appendix E: Item Level Analyses

Item Level Analysis for Fraction Assessment

Problems	<i>M</i>	<i>SD</i>	Total Points
Algorithmic Operations			
1. Subtract $\frac{3}{5}$ from 8.	.84	.364	1
2. Write $3\frac{5}{6}$ as an improper fraction.	.74	.438	1
Word Problems			
1. If you have a half ball of string and each kite needs an eighth of a ball, how many kites can you fly?	.92	.269	1
2. Adrian has conquered only 6 giants in his new video game, Giant Trouble, but it is only two-fifths of the giants that he must conquer. How many giants are there in the new video game?	.82	.389	1
Algebraic Concepts			
1. Solve $x + \frac{1}{3} = 7$.	.79	.406	1
2. $\frac{1}{3} \times a = ?$	.89	.318	1
Arithmetic Skills			
1. Write $5\frac{2}{7}$ as a sum.	.43	.497	1
2. Find $\frac{18}{0}$.	.55	.500	1
Rational Number			
1. Write fractions $\frac{4}{7}, \frac{5}{9}, \frac{3}{5}$ in order from least to greatest.	.62	.488	1
2. The quotient of $\frac{1}{2} \div \frac{1}{3}$ is greater than ($>$) or less than ($<$) $\frac{1}{2}$?	.76	.429	1
Computational Fluency			
1. Find the sum $\frac{7+5}{3+5} + \frac{\frac{5}{6}}{\frac{5}{3}}$.	.51	.502	1
2. In an election, candidate A got $\frac{1}{3}$ of the votes, candidate B got $\frac{9}{20}$ of the votes, and candidate C got $\frac{2}{15}$ of the votes. What fraction of the votes did candidate D get?	.43	.497	1

Item Level Analysis for Equivalence Assessment

Problems	<i>M</i>	<i>SD</i>	Total Points
Equal Sing Interpretation			
1. What does the equal sign (=) mean? Can it mean anything else?	1.39	.800	2
2. What does the equal sign mean in this statement? 1 dollar = 100 pennies	.59	.494	2
Equation Structure			
1. Is the number that goes in the blank the same number in the following two number sentences? Yes, No, How do you know? $2 \times \underline{\quad} = 58$ $8 \times 2 \times \underline{\quad} = 8 \times 58$	.99	.815	2
2. Find a number that can go in each blank. Can another number go in these blanks? Explain. $8 + 2 + \underline{\quad} = 10 + \underline{\quad}$	1.02	.882	2
Open Equation			
1. Fill in the blanks with the value that makes the following number sentences true.			
a. $4,436 + 2,897 = \underline{\quad} + 3,000$	.79	.411	1
b. $3,901 - 2,012 = 3,889 - \underline{\quad}$	.57	.497	1
2. Place the four numbers $n - 1, n + 1, m + 3, m + 1$ in the following boxes so that the number sentence is always true. $\square + \square = \square + \square$	.77	.762	2

Item Level Analysis for Algebraic Reasoning Assessment

Problems	<i>M</i>	<i>SD</i>	Total Points
Functional Thinking & Generalization			
1a. Identify the relationship between the number of red tiles and any given number of sets (x).	.72	.449	1
1b. Identify the relationship between the number of green tiles and any given number of sets (x).	.74	.438	1
1c. Identify the relationship between the total number of tiles and any given number of sets (x).	.70	.459	1
2. Table: 4 squares = ____ vertices	.62	.486	1
2. Table: 10 squares = ____ vertices	.41	.494	1
2. Table: 100 squares = ____ vertices	.35	.478	1
2a. Suppose you are given the pattern of squares shown above. How would you describe the relationship between the number of squares and the number of vertices in words?	.17	.377	1
2b. How would you represent the relationship between the number of squares and the number of vertices using algebra?	.26	.442	1
Modeling			
1. Write an equation using the variables <i>C</i> and <i>S</i> to represent the following statement: "At Mindy's restaurant, for every four people who ordered cheesecake, there are five people who ordered strudel." Let <i>C</i> represent the number of cheesecakes and <i>S</i> represent the number of strudels ordered.	.10	.300	1
2. Write an equation using the variables <i>S</i> and <i>P</i> to represent the following statement: "There are six times as many students as professors at a certain university." Use <i>S</i> for the number of students and <i>P</i> for the number of professors.	.45	.499	1
Structure Sense			
1. Solve for <i>x</i> .			
a. $\left(\frac{1}{4} - \frac{x}{x-1}\right) - x = 6 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$	.36	.482	1
b. $1 - \frac{1}{x+2} - \left(1 - \frac{1}{x+2}\right) = \frac{1}{110}$	.29	.456	1
2. Solve the following number sentences.			
a. $237 + 89 - 89 + 267 - 92 + 92 = ?$	.87	.343	1
b. $217 - 59 + 59 + 62 - 28 - 28 = ?$	.71	.456	1

Item Level Analysis for Algebraic Reasoning Assessment

Problems	<i>M</i>	<i>SD</i>	Total Points
Symbolic Manipulation			
1. For each example, write an equivalent expression.			
a. $4h + t =$ _____	.23	.420	1
b. $u + 5 + 6 + 5 + u =$ _____	.82	.389	1
c. $4(n + 5) =$ _____	.91	.290	1
d. $p + 0.05p =$ _____	.65	.478	1
e. $5(e + 2) =$ _____	.93	.258	1
f. $x - x + 2 =$ _____	.89	.318	1
g. $(15 + 10x) + (35 + 5x) =$ _____	.72	.452	1
h. $3x + 4 + 6(x + 5) =$ _____	.84	.371	1

Item Level Analysis for Algebra Performance Assessment

Problems	M	SD	Total Points
System Equations			
1. The graphs of the two equations shown above intersect at the point (x, y) . What is the value of x at the point of intersection?	.37	.484	1
2. In the solution of the system of equations above, what is the value of x ?	.54	.500	1
Functions			
1. If $f(x) = \frac{2x+1}{3}$ and $g(x) = 2x^2 + 2$, then $f(g(2)) =$	.77	.425	1
2. Yvonne has studied the cost of tickets over time for her favorite sports team. She has created a model to predict the cost of a ticket in the future. Let C represent the cost of a ticket in dollars and y represent the number of years in the future. Her model is as follows. $C = 2.50y + 13$ Based on this model, how much will the cost of a ticket <u>increase</u> in two years?	.38	.486	1
Solving Equations			
1. If $d = 110$ and $a = 20$ in the formula $d = \frac{a}{2}(2t - 1)$, then $t =$	.74	.442	1
2. If $\frac{1}{a} + \frac{4}{7} = \frac{1}{2}$, then $a = (?)$.	.57	.496	1
Inequalities			
1. What are all values of x such that $x^2 + 7x + 6 \geq 0$?	.26	.438	1
2. For which values of w is $1.25 \leq \frac{l}{w} \leq 2.5$ if $l = 4$?	.46	.500	1
Graphing			
1. The graphs of $y = f(x)$ and $y = g(x)$ for $0 \leq x \leq 10$ are shown in the figure above. For how many values of x is the product $f(x)g(x) = 0$ for $0 \leq x \leq 10$?	.55	.499	1
2. Which of the following is the graph of $ 2x - 5 \geq 3$?	.43	.496	1
Exponents			
1. For what value of x is $8^{12} = 16^x$?	.12	.327	1
2. $121^{\frac{3}{2}} =$	.30	.459	1
Factoring			
1. Factor $28n^4 + 16n^3 - 80n^2$. One of the factors is $(?)$.	.57	.497	1
2. $m^2 + 10m + 14 + b$ is a perfect square. Find b .	.27	.445	1

Item Level Analysis for Algebra Performance Assessment

Problems	<i>M</i>	<i>SD</i>	Total Points
Complex Numbers			
1. If $i = \sqrt{-1}$ then $\frac{1}{6+4i} = ?$	.07	.258	1
2. Let $i = \sqrt{-1}$. After expanding and simplifying $(2 + i)^3 = ?$	.21	.411	1
Polynomial Division			
1. If $6x^3 + 5x - 8$ is divided by $x - 2$, the remainder is?	.13	.335	1
2. $(x^3 + 12x^2 + 47x + 60) \div (x + 5) = ?$	.36	.482	1
Logarithms			
1. Which one is equivalent to $\log_4(16x^6)$?	.14	.350	1
2. Solve $\log_x 16 = 2$ for x . $x = (?)$.	.41	.494	1

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