THREE ESSAYS IN APPLIED ECONOMIC THEORY

By

Chilei Oscar Lau

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ABSTRACT

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Chapter 1 suggests “soft debt” as a social convention that facilitates long-term reciprocal relationships. In each round of a two-player repeated game, one player develops a need for help from his counterpart, who may or may not be able to help. Unlike the existing favor trading models, the benefit and cost involved in each round, as well as the roles of potential receiver and provider of help, are randomly drawn. In addition, instead of automatically accepting help whenever it is offered by the provider, the receiver can decline the offer.

If help is rendered, a soft debt is tacitly accrued by the receiver to the provider and added to the soft debt balance between them. A player is said to follow a soft debt strategy if his decisions about offering and soliciting help depend on the entire history only through the soft debt balance. His counterpart’s expected future value of the relationship therefore also depends on the balance. This consideration creates intertemporal incentives that promote reciprocity. Soft transactions occur when help is traded between players following soft debt strategies.

Under discrete benefits, there exist stationary Markov equilibria in which the players trade as long as the debt balance does not exceed a certain limit. The first best allocation is never achieved, but all trades that do occur in the equilibria are efficient.

The model provides a unifying framework for decision making when both hard and soft transactions are available (e.g. hiring a mover versus asking a friend to help move to a new home).

Chapter 2 emphasizes that when the benefits and costs are random, the favors do not necessarily result in positive surpluses. The players then need to consider not only whether the counterparts are trustworthy, but also whether the favor is efficient and whether the expected value of future trades justify the current cost even if the counterpart is willing to cooperate. This chapter proposes a modified grim trigger strategy, called the "limited efficiency strategy," in which a player will help if and only if (i) the favor is efficient (ii) the cost incurred is below a certain threshold (iii) no one has ever defected. Multiple equilibria with trades exist.

Chapter 3 presents an axiomatic approach to extend the Expected Utility Theorem to multi-periods. The standard time-separable form is recovered when the functions describing the risk and substitution attitudes are identical. Further, I argue that the standard analysis of risk aversion neglects the agent's option to trade his lottery outcome. With trading, an agent is *not* indifferent to the timing of resolution of lotteries on consumptions. It is straightforward to characterize risk attitudes and to compare risk aversiveness between agents (even with different ex post preferences). The approach can accommodate multiple goods in each period. Time consistency is discussed.

To my wife.

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PREFACE

This is a dissertation in applied economic theory in which I consider optimal decisions over time in settings characterized by uncertainty.

The first two chapters of this dissertation study the formation reciprocal relationships using the framework of game theory, and the third chapter models how individuals make intertemporal decisions under uncertainty. The two chapters on reciprocal relationships depart from the typical models in that the payoffs for each outcome are random rather than fixed. Therefore if there is any common theme to all three chapters, it is that they are all concerned with uncertainty and time, two key elements that dramatically alter the analysis in any field of economics – and social sciences for that matter.

Even with fixed payoffs, the outcome of a game is seldom certain (even at the theoretical level) because multiple equilibria exist. Players face uncertainty because they cannot predict what strategy their counterparts would play. With random payoffs, however, there is an additional level of uncertainty. Even if a player knows exactly what his opponent's strategy is, his payoff is still uncertain.

This additional level of uncertainty is important in terms of both real world relevance and theoretical analysis. On the relevance front, individuals often face uncertain payoffs even with a given outcome. This feature is apparent in reciprocal relationships. In particular, the benefit and cost of favors vary in each interaction. In fact, as I argue in the chapters, the very reason that the players choose to cooperate through reciprocity instead of market transaction is that the situation changes every time they meet, making transaction costs prohibitively high if they were to negotiate and sign contracts every time. For standardized goods and services, often market transactions would be available in a modern economy.

Regarding theoretical analysis, the randomness of payoff means the players are not only concerned about the incidences of reciprocity, but also about the sizes of the favors involved. Again this point should be clear from real world experience. People obviously do value a huge favor (donation of kidney) and a small one (a ride) very differently.

As the title "Soft Transactions" suggests, the first chapter treats reciprocity as a form of implicit transaction, in parallel with hard transactions conducted via contracts. They are considered transactions because both players (not just the helper) must agree before the favor is made, and that after the transaction the receiver accrues a "soft debt" to the helper. The "soft price" derives from the change in soft debt position of the players.

The idea that reciprocity is a form of transaction was inspired by Ronald Coase's classical 1937 paper "The Nature of Firm," in which he argues that firms (and organization in general) are characterized by the suppression of the price mechanism. He borrows D. H. Roberston's analogy which describes firms as "islands of conscious power in this ocean of unconscious co-operation like lumps of butter coagulating in a pail of buttermilk."

While it is true that interactions within the realm of organization planning are less fluid than those that thrive in the market, I believe a clear-cut dichotomy would not be a fair description of the situation. For example, when a group of workers are assigned by the manager to certain tasks, there is a variety of way through which the workers can adjust their specific duties through repeated give-and-take interactions among themselves, i.e., soft transactions. There may not be formal negotiation and "bribing" involved, as Coase suggests as a solution in his other classical paper "The Problem of Social Cost" (1960), but the arrangements would be far from rigid. The similar argument extends to the relationships between family members, friends, neighbors, etc.

On the other hand, market transactions are not entirely fluid. To the extent that contracts are incomplete (as is always the case), there is always the possibility of unaddressed contingencies occurring. As emphasized by the relational contract literature, the prospect of repeated trading give rise to roles of trust and relationship building beyond simple and anonymous transactions. In the context of chapter, this means no transactions are absolutely hard.

Therefore, the boundary between the butter and buttermilk may be more blurred than one might expect. Since it is not a simple dichotomy between market and non-market, neither should the analytical approach be entirely different. The major advantage of the soft transaction approach is to provide a unifying framework for analyzing problems when both hard and soft transactions are available.

The second chapter, titled "Limited Efficiency Strategy in Favor Trading under Random Benefits and Costs," takes a more conventional look at reciprocal relationships. Like chapter one, payoffs are random. However, granting of favor does not require the consent of the receiver. Another major difference from Chapter 1 in terms of model set up is that in Chapter 1 the helper may not be able to help, and his ability is not observable to the receiver; whereas in Chapter 2 the helper is always able to help. This assumption enables players to identify whether the counterpart has defected, thus allowing them to follow a modified version of the grim trigger strategy. In this modified strategy, players will stop helping as soon as anyone has defected like in the grim trigger strategy, but they forgive each other for not helping if it would be inefficient to help, or if the cost drawn is beyond a certain threshold.

The third chapter is titled "Disentangling Intertemporal Substitution and Risk Aversion under the Expected Utility Theorem." Center to the proposed approach to disentangling the two attitudes is the notion of "constancy equivalent." The constancy equivalent of a (deterministic) consumption sequence is the consumption level that, if maintained throughout the lifetime, would make the individual indifferent. A lottery on consumption sequences can thus be translated into a lottery on constancy equivalents, which are scalars. The translation incorporates the individual's attitude toward intertemporal substitution. Next, the risk attitude on the constancy equivalents can be modeled in the conventional expected utility approach applicable to scalar awards. In this way, the two attitudes can be addressed in the two steps respectively. Only when the functions describing the two attitudes happen to coincide would the objective function reduce to the conventional expected utility specification.

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Soft Transactions

1.1 Introduction

When a person is asked by a friend to help him, say, move to a new home, how would he respond? He may anticipate a return of the favor in the future should he help. But would the return be expected to adequately compensate for his effort? He may be particularly concerned if there have been instances in the past when he needed help but the friend declined to help. The friend might have had excuses (e.g. time conflicts) but he cannot tell whether they were genuine. On the other hand, the friend may have helped him before, and by helping this time he will be returning past favors. But then how many past favors would it take to justify his effort? What if these past favors had been mostly trivial?

Of course both of these motives, namely anticipation of return and return of past favors, can be in play at the same time. Moreover, the friend would also have gone through similar thoughts before deciding to ask for help in the first place. Similar situations arise not just between friends, but also between family members, co-workers, neighbors, or even strangers who may become acquaintances in the future. How is reciprocity sustained in these relationships as the players pursue their self interests?

This chapter suggests "soft debt" as a social institution that provides a solution to reciprocity. The premise is that whenever help is provided, a soft debt is tacitly accrued by the beneficiary to the provider. The soft debt is then added to the soft debt balance between them. If both of them follow "soft debt strategies", i.e., they make decisions about offering and soliciting help based on this balance, then their expected future values of the relationship also depend on the balance. The more one is owed, the more likely that he will be helped and the more valuable the help is expected to be, and thus the higher his expected future value. This consideration creates intertemporal incentives that promote reciprocity.

Instead of trying to set up some defection detection and punishment mechanism, the players rely on the incentives created by the soft debt mechanism to maintain cooperation. When a player helps the other (if he is able to), he earns an increase in his expected future value. Similarly, if he has need and is helped, he incurs a drop in his expected future value. In other words, there are "soft prices" to be earned or paid. A player will "sell" a favor whenever the soft price he would earn exceeds the cost. Conversely, he will "buy" a favor whenever the soft price he would pay is less than the benefit. The benefit and cost can vary from round to round, but the players follow the same calculation. I call such voluntary favor trading without contracts "soft transactions."

The soft debt mechanism simplifies bilateral relationships by summarizing the full history between two individuals into a single number. The notion of soft debt is evident from the daily use of words like "owe," "repay," "price," "indebted" between acquaintances even when no contract is involved. The soft debt mechanism also implies that in contrast to the usual assumption, an individual who needs help do not automatically accept favor when offered, because there is a soft price to be paid for the favor. A soft transaction goes through only if both players agree.

Along with hard transactions, soft transactions constitute an unified approach to analyzing a host of economic activities. When individuals have needs (e.g. to move to a new home), often they can compare the hard (hire a mover) and soft (ask a friend for help) alternatives side-by-side.[1] Conversely when they have goods or services to offer (e.g. baby-sitting), again they can opt for either hard (get paid) or soft (help friends for "free") transactions. Both types of transactions stem from similar cost and benefit analysis mediated by prices, be them hard or soft.

This chapter starts with a general set-up for a two-player repeated stochastic game. In each round, one of the players develops a need for help from his counterpart. Call them the (soft) buyer and (soft) seller for this round respectively. The seller may or may not be able to help, and the ability to help is her private information (as in [30]). Help is rendered if only if both the buyer buys and the seller sells. The roles of buyer and seller, as well as the benefit and cost involved, are drawn randomly in each round. Suppose both players follow soft debt strategies, I show that an autarky equilibrium always exists, but the first best outcome (i.e., trade occurs if and only if the benefit exceeds the cost) can never be achieved.

Under discrete benefits, I show that stationary Markov equilibria with trades could be sustained when the players follow the "debt limit strategy", a particular form of soft debt strategy whereby players buy and sell as long as the debt balance is within a certain limit. In one version of the model, the cost-to-benefit ratio is fixed. In another version, the cost associated with each benefit is random. The cost may exceed the benefit and its realized

[1] Of course, self production is usually another alternative.

value is observable only to the seller. The debt limit depends on the the cost structure and parameters such as the discount factor and the probability that the seller is able to help. Although the first best allocation cannot be attained, all trades that do occur are efficient.

I then discuss the advantages and disadvantages of soft transactions as compared to their hard counterparts. As a result of their differences, soft transactions are popular in long-term relationships involving trading of personalized favors, while hard transactions dominate anonymous exchanges of mass-produced goods. I finally suggests some directions for further research related to soft transactions.

This chapter is closely related to the favor trading literature. [30] assume that the benefit and cost are fixed and that the ability to help is private information. He shows that favor trading could be maintained between two players when they grant favors if and only if the net number of favors granted is below a certain number. Therefore the debt limit strategy in the current paper is similar to this mechanism. He also demonstrates how cooperation is sustainable in a group of players who rarely meet each other, if indirect favors are allowed. [16] build on the bilateral model and make two relaxations: (i) the exchange rate between current and future favors is allowed to deviate from one; (ii) the balance of favors is allowed to appreciate or depreciate. They show that with these relaxations higher payoffs can be achieved. [31] presents a discrete time version of Hauser and Hopenhayn's model and allows the opportunities to offer help to arise at different rates for the two players. [18] studies the model with different discount factors. In another variation, he assumes complete information and that the players have concave utility functions instead of being risk neutral, hence favor trading is considered a form of insurance.

The model presented in this chapter differ from the favor trading literature mainly in two ways. First, the benefit and cost are random (instead of fixed), so that the players need to take into consideration the specific benefit and cost drawn in each round. In reality, people often face a new and unique situation each time they meet. Even between close partners there are variations among established routines. If the situation is the same in each round, then the players could have negotiated a contract applicable for each interaction instead of relying on reciprocity, since the cost of negotiation can be spread over the numerous interactions. Second, in the spirit of transactions, help is rendered only if both parties agree. In the above mentioned papers, only the provider unilaterally decides whether to help. However, in reality people do not automatically accept favors even when they are offered one. In experiments of ultimatum games in some primitive societies, subjects would even reject offers greater than 50% (see [13] for a survey). Even in modern economies, people would hesitate and think before asking for helps. These observations point to the fact that favors do not come free.

This chapter is also related to the broader literature on reciprocity. This includes for instance the literature on social norms and reciprocity (e.g. [19]), informal insurance (e.g. [8], [23]), relational contracts (e.g. [6], [27]) and games of imperfect public monitoring and private monitoring (e.g. [15], [4]).

The rest of the chapter is organized as follows. Section 2 presents the general model, define the various concepts related to soft debt, and introduces some general results. Section 3 and Section 4 solves for equilibria constituted by the debt limit strategy under discrete benefits. In Section 3 the cost-to-benefit ratio is assumed to be fixed. In Section 4 the cost associated with each benefit is random and its realized value (which could exceed the benefit) is observable only to the seller. Section 5 compares soft transactions against their hard counterparts. Finally, the conclusion suggests some directions for further research.

1.2 General Model

1.2.1 Game structure

Two players, player 1 and player 2, are randomly matched from a population to play an infinitely repeated game, starting in round 1. At the beginning of each round of stage game, one player is assigned the role of (soft) buyer, who needs help from the other player. The other player is assigned as the (soft) seller, who may or may not be able to help. She is called a capable seller for the round if she can render help, or an incapable seller otherwise. The ability to help is randomly drawn and observed only by the seller. No one else in the population is able to help. In each round, player i ($i = 1, 2$) is assigned as the buyer with probability $\beta_i \in (0, 1)$, which is fixed across rounds, so that $\beta_1 + \beta_2 = 1$. Conditional on being assigned the seller, player i is capable of helping with probability $\pi_i \in (0, 1)$, which is also fixed across rounds. All of these probabilities are common knowledge.

In round t, if help is rendered, the buyer (suppose player i) will receive a benefit of $b_{it} > 0$ and the (capable) seller (player j) will incur a cost of $c_{jt} > 0$. The values b_{it} and c_{jt} are drawn from the joint distribution of random variables$\left(\widetilde{b}_{it}, \widetilde{c}_{jt}\right)$. The joint distribution is independent and identically distributed (i.i.d.) across time, and is independent of whether the seller is capable. The distribution is common knowledge. For completeness, the values b_{jt} and c_{it} are zeros, i.e., any help is unilateral in each round. The role assignment, benefit and cost are revealed to both players once drawn, while the ability to help remains the seller's private information forever.[2]

After the drawing, the buyer and the capable seller simultaneously decide whether to buy and sell respectively. This is the only decision they need to make in the round. An incapable

[2] In the model in Section 4, the cost will be observed only by the seller.

seller cannot sell. The decisions are then revealed to both players. Help is rendered if and only if *both* the buyer buys and the seller sells. If help is not rendered, both players receive zero payoffs for this round. The game proceeds to the next round.

Unlike typical repeated games, the rendering of help requires the consent of not just the provider (the seller), but also the beneficiary (i.e. the buyer). The reason for this requirement is that the help, as in the case of hard transactions, may not come free, as explained in the next subsection. Even if the seller finds it profitable to sell, the buyer may not want to buy.

In the model the buyer and seller are treated as if they make decisions simultaneously. In practice, since the soft transaction occurs only if both players agree, it makes no difference if one of them initiate the offer, and then the other respond, as long as one's decisions does not depend on the other's.

The players are risk neutral with discount factor δ_i. Player i's objective in round τ is to maximize the expectation of discounted sum of lifetime payoffs *conditional* on his current information:

$$E_{i\tau} \sum_{t=1}^{\infty} \delta_i^t \left(\beta_i \widetilde{b}_{it} - \beta_j \widetilde{c}_{it} \right)$$

where $E_{i\tau}$ denotes the expectation operator conditional on player i's information set in round τ. Note that in each round t, one or both of the realized values b_{it} and c_{it} will be zero(s), depending on what role he is assigned and whether help is rendered. Both discount factors are common knowledge and reflect the chance of future meetings as well as the patience levels of individual players. This completes the specification of the stochastic game.

1.2.2 Strategy and Equilibrium Concept

Following the notion of soft debt, I focus on strategies that prescribe actions based on the soft debt balance.

In each round, if help is rendered, then the buyer accrues a soft debt of $d(b,c)$ to the seller, where b is the benefit to the buyer and c is the cost to the seller. Assume d is invariant across time. Also assume $d > 0$ for all b and c; and d is increasing in both b and c. Then the soft debt holding can be defined as follows:

Definition 1. Player i's *soft debt holding* (i.e. net soft debt balance owed by player j to him) at the beginning of round τ is

$$D_{i\tau} = \sum_{t=1}^{\tau-1} \left[d\left(b_{jt}, c_{it}\right) - d\left(b_{it}, c_{jt}\right)\right]$$

Obviously, $D_{1\tau} = -D_{2\tau}$ and $D_{i1} = 0$. Note also that for simplicity no "interest" (positive or negative) is charged on the debt.[3]

Let $\mathcal{A} \equiv \{\text{buyer, capable seller, incapable seller}\}$ be the set of role assignments.

Definition 2. A *soft debt strategy* for player i is a mapping from the product set of the sets of current roles, benefit to the buyer, cost to the seller, and soft debt holding into the set of actions:

$$\sigma_i : \mathcal{A} \times \mathbb{R}_+^2 \times \mathbb{R} \to \{\text{trade, not trade}\}$$

In the definition, "trade" means to buy (in the case of the buyer) or to sell (in the case of the capable seller). Note that σ_i is time independent (i.e. stationary). The player's decisions

[3] Unlike the nominal interest rate in hard transactions, the interest rate here can be negative, which can reflect fading of memories.

depend on the entire history of past actions and information only through the current debt balance. That is, the whole history is condensed to the debt balance.

Definition 3. A *soft debt equilibrium* is a stationary Markov equilibrium in which both players' follow soft debt strategies.

A *soft transaction* is a trade that occurs (i.e. help is rendered) in a soft debt equilibrium.

1.2.3 General Analysis

With soft debt equilibrium defined, we can now define soft prices.

Definition 4. Let $V_1(D_1)$ and $V_2(D_2)$ be the players' expected future value in a soft debt equilibrium where D_1 and D_2 $(= -D_1)$ are their respective current debt holdings. Suppose player i is assigned as the buyer and the benefit b_i and cost c_j are drawn, then his *soft buying price* is

$$p_i^{\text{buy}} \equiv \delta_i \left[V_i(D_i) - V_i\left(D_i - d\left(b_i, c_j\right)\right)\right]$$

and as the seller player j's *soft selling price* is

$$p_j^{\text{sell}} \equiv \delta_j \left[V_j\left(D_j + d\left(b_i, c_j\right)\right) - V_j\left(D_j\right)\right]$$

The functions V_1 and V_2 are stationary because the probability distributions for role assignment, benefits and costs of helping are all i.i.d. across time. The soft price is the increase (for the seller) or decrease (for the buyer) in the expected future value of the relationship resulting from the change in the debt balance. Unlike hard prices, the soft price paid by the buyer and that received by the seller can be different. The divergence exists because in soft transactions, there is no money or other exchange media that defines a single price. This means even if the benefit exceeds the cost, mutually beneficial trade may or may not occur. This limitation may offer an explanation for the development of hard transactions to fill in the gap.

Player i will buy iff the incentive compatibility (IC) condition is met: [4]

$$\text{IC (buyer): } b_i > p_i^{\text{buy}}$$

Similarly, player j, if capable, will sell iff:

$$\text{IC (seller): } c_j < p_j^{\text{sell}}$$

Denote by $H_i(D)$ the event that player i is helped given that he is the buyer with debt holding D and the seller is capable, then:

$$\Pr(H_i(D)) = \Pr\left(\widetilde{b}_i > \widetilde{p}_i^{\text{buy}}(D) \text{ and } \widetilde{c}_j < \widetilde{p}_j^{\text{sell}}(-D)\right)$$

where

$$\widetilde{p}_i^{\text{buy}}(D) \equiv \delta_i\left[V_i(D) - V_i\left(D - d\left(\widetilde{b}_i, \widetilde{c}_j\right)\right)\right]$$

and

$$\widetilde{p}_j^{\text{sell}}(-D) \equiv \delta_j\left[V_j\left(-D + d\left(\widetilde{b}_i, \widetilde{c}_j\right)\right) - V_j(-D)\right]$$

are the random buying and selling prices respectively.

In general, the value function is composed of four components that correspond to the

[4] There is no need to check the individual rationality (IR) conditions because the "worst" expected future value cannot fall below zero (autarky). Also assume that the players do not buy or sell if they are indifferent.

following scenarios: player i is the buyer and he is helped; he is the buyer but is not helped; player j is the seller and she is helped; she is the seller but is not helped. The value function of player i at the beginning of a round can thus be formulated recursively:

$$\begin{aligned} & V_i(D) \\ = \quad & \beta_i \left\{ \begin{array}{cl} & \pi_j \Pr(H_i(D)) E\left[\widetilde{b}_i + \delta_i V_i\left(D - d\left(\widetilde{b}_i, \widetilde{c}_j\right)\right) \mid H_i(D)\right] \\ + & (1 - \pi_j \Pr(H_i(D))) \delta_i V_i(D) \end{array} \right\} \\ & + \beta_j \left\{ \begin{array}{cl} & \pi_i \Pr\left(H_j(-D)\right) E\left[-\widetilde{c}_i + \delta_i V_i\left(D + d\left(\widetilde{b}_j, \widetilde{c}_i\right)\right) \mid H_j(-D)\right] \\ + & \left(1 - \pi_i \Pr\left(H_j(-D)\right)\right) \delta_i V_i(D) \end{array} \right\} \end{aligned}$$

I further simplify the notations by defining the following conditional probability and conditional mean benefit and cost:

$P_i(D) \equiv \Pr(H_i(D))$, $\overline{b}_i(D) \equiv E\left(\widetilde{b}_i \mid H_i(D)\right)$, $\overline{c}_i(-D) \equiv E\left(\widetilde{c}_i \mid H_j(-D)\right)$. Then

$$\begin{aligned} & V_i(D) \\ = \quad & \beta_i \left\{ \begin{array}{cl} & \pi_j P_i(D) \left[\overline{b}_i(D) + \delta_i E\left[V_i\left(D - d\left(\widetilde{b}_i, \widetilde{c}_j\right)\right) \mid H_i(D)\right]\right] \\ + & \left(1 - \pi_j P_i(D)\right) \delta_i V_i(D) \end{array} \right\} \\ & + \beta_j \left\{ \begin{array}{cl} & \pi_i P_j(-D) \left[-\overline{c}_i(-D) + \delta_i E\left[V_i\left(D + d\left(\widetilde{b}_j, \widetilde{c}_i\right)\right) \mid H_j(-D)\right]\right] \\ + & \left(1 - \pi_i P_j(-D)\right) \delta_i V_i(D) \end{array} \right\} \end{aligned} \tag{1.1}$$

Upon rearranging terms,

$$V_i(D) = \frac{1}{1 - \delta_i} \left\{ \begin{array}{c} \beta_i \left\{\pi_j P_i(D) \left[\overline{b}_i(D) - \overline{p}_i^{\text{buy}}(D)\right]\right\} \\ + \beta_j \left\{\pi_i P_j(-D) \left[-\overline{c}_i(-D) + \overline{p}_i^{\text{sell}}(D)\right]\right\} \end{array} \right\} \tag{1.2}$$

where $\overline{p}_i^{\text{buy}}(D) \equiv E\left[\widetilde{p}_i^{\text{buy}}(D) \mid H_i(D)\right]$ and $\overline{p}_i^{\text{sell}}(D) \equiv E\left[\widetilde{p}_i^{\text{sell}}(D) \mid H_j(-D)\right]$, for $i = 1, 2$.

Some key terms are interpreted as follows:

- $\overline{p}_i^{\text{buy}}(D)$: player i's conditional mean buying price with debt holding D
- $\overline{p}_i^{\text{sell}}(D)$: player i's conditional mean selling price with debt holding D
- $\bar{b}_i(D) - \overline{p}_i^{\text{buy}}(D)$: player i's conditional mean buying surplus with debt holding D
- $-\bar{c}_i(-D) + \overline{p}_i^{\text{sell}}(D)$: player i's conditional mean selling surplus with debt holding D

The two conditional mean surpluses must be positive because the player will trade only if it is profitable. Equation (1.2) shows that a player's expected future value equals the discounted sum of surpluses weighted by the chances of buying and selling.

Definition 5. The *first best allocation* is achieved when help is rendered iff the benefit exceeds the cost and the seller is able to help.

For the general model, the following observations can be made (all proofs are relegated to the Appendix).

Proposition 1. *(i) An autarky soft debt equilibrium always exists. (ii) The first best allocation can never be achieved in a soft debt equilibrium.*

While part (i) of the Proposition confirms that a soft debt equilibrium always exists, part (ii) proclaims that inefficiency is inevitable in soft debt equilibria. The question is by how much trades would be limited under soft debt equilibria. The next two sections demonstrate that one possibility is to restrict trades so that the debt balance does not exceed certain debt limits. Solving for equilibria with trade would require some simplifying assumptions to the general model.

1.3 Discrete Benefit Model with Proportional Cost

Assume $\delta_1 = \delta_2 = \delta$, $\beta_1 = \beta_2 = 1/2$, $\pi_1 = \pi_2 = \pi$, $\left(\widetilde{b}, \widetilde{c}\right)$ are identically distributed for the two players, and focus on symmetric equilibria where V_1 and V_2 coincide. Consider the case where $\widetilde{b}$ follows a uniform discrete distribution, realizing each outcome of $1, .., B$ with probability $1/B$. The cost corresponding to each benefit of b is fixed at αb where$\alpha \in (0, 1)$ is constant, so the first best outcome is for the players to trade in all rounds as long as the seller is capable. Although b is not continuous, in practice the benefit can be measured in units as small as one wants.

Suppose $d(\cdot, \cdot)$ is homogenous of degree 1, then $d(b, \alpha b) = b \cdot d(1, \alpha)$, and $D_{i\tau} = \left[\sum_{t=1}^{\tau-1} \left(b_{jt} - b_{it}\right)\right] \cdot d(1, \alpha)$. Since $d(1, \alpha)$ is constant, we can use the net cumulative benefit (call it the debt holding level) as an index for the debt balance. Denote by V_k the expected future value when the debt holding level is k:

$$V_k \equiv V\left(k \cdot d(1, \alpha)\right)$$

Define $p_{k+b,k} \equiv \delta\left(V_{k+b} - V_k\right)$, $b = 1, .., L-k$ as the soft selling price received by the seller holding debt level k for providing a benefit of b (or equivalently the soft buying price paid by the buyer for a benefit of b when his debt holding is $k+b$). Note that $p_{k+b,k} = \sum_{r=0}^{b-1} p_{k+r+1,k+r}$. That is, the soft price for b units of benefits is the sum of b one-unit soft prices.

Definition 6. A *debt limit strategy* with limit $L > 0$ is a strategy whereby the player buys (if he is drawn as the buyer) and sells (if capable seller) iff debt levels in absolute terms do not exceed L both before and after the transaction.

A *debt limit equilibrium* with limit L is a soft debt equilibrium in which both players follow the debt limit strategy with limit L.

In other words, under the strategy a player will trade as long as (i) he is capable and (ii) after the trade, neither he will owe nor he will be owed more than L. Once the debt level falls outside the limit (which is off equilibrium path), autarky prevails forever. The limit L defines the range of soft debt level that the players engage in. A higher L means more opportunity for soft transactions. Therefore L can be regarded as depth of relationship.

For a strategy profile to constitute an equilibrium, the following IC and IR conditions have to be satisfied:

$$\text{IC (buyer): } p_{k,k-b} < b \text{ if } b = 1, .., L + k$$

$$\text{IC (seller): } p_{k+b,k} > \alpha b \text{ if } b = 1, .., L - k$$

$$\text{IR: } V_k > 0$$

for $k = 0, \pm 1, .., \pm L$.

The IC conditions ensure that it is profitable for the buyer to buy and the seller (if capable) to sell when the resulting debt level falls within the limit. The IR condition guarantees that the player will always stay in the relationship. Note that the IR conditions are interim IR conditions, i.e., they need to be satisfied not only ex ante (before the drawing in each round), but also ex post.

There are a total of $L\,(L+1)$ inequalities in the IC condition for buyer, the same number in the IC condition for seller, and $2L$ in the IR condition.

From Proposition 1 we already know that the first best allocation cannot be attained. The following proposition shows that when the costs are low enough, simple "debt limit strategies" would constitute equilibria with trade.

Proposition 2. *Consider any positive integer $L \leq B/2$. In the discrete benefit model with proportional cost, if*

$$\alpha < \frac{1}{2B\left(\frac{1/\delta - 1}{\pi}\right) + 1} \tag{1.3}$$

then there exists a debt limit equilibrium with limit L.

The lower B is, the higher the cost ratio α that can be sustained in an equilibrium. Conversely, a high B means an equilibrium is possible only if the α is low. This is because a high B implies a high potential limit L, which means a player may find herself owed a lot from the other. The cost ratio then needs to be low enough to entice the players to engage in the relationship.

The higher δ and π are, the higher α that can be supported. If δ is high, which means the players are patient and meet frequently, then an equilibrium can be sustained even with a high α. On the other hand, a high π means the players are able to help each other with high probability, which again make a high cost ratio affordable. The affordable cost ratio approaches 1 as δ and π both tend to 1.

For deep relationships like those between family members, δ and π are high because they meet frequently and are often able to help each other. Also, α is low because they are familiar with each other's taste, habit, information, etc., which means it takes relatively little effort to help each other. Therefore a high B, and thus L can be supported. Conversely, between strangers δ and π are low and α is high, so only a low L is possible.

It is obvious from Proposition 2 that multiple debt limit equilibria can coexist. As long as the condition on α is satisfied, all debt limit equilibria with limits smaller or equal to $B/2$ exist. In addition there is always the autarky equilibrium. At first glance the equilibrium seems indeterminate. However, a closer examination would reveal that players will always incline to reach the highest equilibrium limit ($B/2$ for even B or $B/2-1$ for odd B) supported by the cost structure. First note the following proposition.

Proposition 3. *Denote by V_k^L the expected future value in the debt limit equilibrium with limit L when the player's debt holding level is k. Then $V_k^{L'} > V_k^L$ iff $L' > L$ for all $k = 0, \pm 1, .., \pm L$.*

The intuition for the proposition is as follows. A higher debt limit would allow more trading opportunities that are not available under a lower limit. For instance, a player with debt holding level k could potentially buy a benefit up to $L' + k$ or sell a benefit up to $L' - k$ under limit L'. The range would be narrower if the limit is L instead. The difference in the scope of trade holds true in every round. Since the trades are mutually beneficial, the expected future value is higher with a higher debt limit.

In any round, the remaining candidates of equilibrium limits are bounded between the highest debt level attained thus far at the lower end, and $B/2$ at the upper end. Now suppose in the current round if the transaction goes through the debt level will exceed the highest attained level but not $B/2$. The buyer in this round would have no reservation in buying. If the seller sells, not only that the buyer will profit from the transaction, the transaction will also signify a higher limit, which means a higher expected future value (as shown in Proposition 3). If the seller does not sell, trade will not occur but there is no harm in trying to buy anyway. Conversely, the same logic applies to the (capable) seller. Being aware of each other's calculation, the players will always agree to trade within the maximum limit $B/2$.

Therefore although multiple equilibria exist, the one with the maximum limit will prevail ultimately.

1.4 Discrete Benefit Model with Random Private Cost

In this section, the discrete benefit model is modified by assuming instead of a fixed cost-to-benefit ratio, that the cost is random and only observable to the seller. In particular, for each benefit $b \in \{1, 2, .., B\}$, the corresponding cost $\widetilde{c}_b$ follows some distribution over $(\underline{c}_b, \overline{c}_b)$ where $0 < \underline{c}_b < \overline{c}_b$. The upper limit $\overline{c}_b$ can exceed b, therefore rendering help can be inefficient. The distribution of $\widetilde{c}_b$ is common knowledge. Let $\widehat{c}_b \equiv E(\widetilde{c}_b)$.

Since the actual cost is unobservable to the buyer, costs do not enter the soft debt formula. The cumulative net benefit is taken as the soft debt balance, i.e., $D_{i\tau} = \left[\sum_{t=1}^{\tau-1} \left(b_{jt} - b_{it}\right)\right]$. Like before, V_k denotes the expected future value when the soft debt holding is k, and $p_{k+b,k}$ denotes the soft price $p_{k+b,k} \equiv \delta \left(V_{k+b} - V_k\right)$.

The IC condition for buyer and the IR condition are the same as in Section 3. The IC condition for (capable) seller is:

$$\text{IC (seller): } p_{k+b,k} > \overline{c}_b \text{ if } b = 1, .., L-k,\ k = 0, \pm 1, .., \pm L$$

This IC condition ensures that the capable seller is willing to sell even if the highest cost for delivering the benefit is drawn.

Proposition 4. *Consider any positive integer $L \leq B/2$. In the discrete benefit model with random private cost, if*

$$\overline{c}_b < \frac{b\left(L + k + \frac{b+1}{2}\right) + \sum_{r=0}^{b-1} \widehat{c}_{L-k-r}}{2B\left(\frac{1/\delta - 1}{\pi}\right) + 2L + 1} \tag{1.4}$$

for $b = 1, .., 2L$, $k = -L, -L+1, .., L-b$, then there exists a debt limit equilibrium with limit L.

All transactions that occur under the equilibrium are efficient, i.e. $\overline{c}_b < b$ *for all* $b \leq 2L$*. Moreover, there always exists some distributions of* $\{\widetilde{c}_b,\ b = 1, .., 2L\}$ *that satisfy* (1.4).

Again L measures the depth of the relationship. The more frequently the players meet (higher δ), or the more likely the seller is able to help (higher π), or the smaller the maximum benefit B (which means a lower potential L), the easier it is for the cost structure to support the equilibrium. On the other hand, the more efficient they are in helping each other (lower $\overline{c}_b$'s in general), the higher L can be sustained. Therefore the debt limit is highest in closely knit groups where members understand each other's needs well, such as the family. At the other extreme, the debt limit would be very low between strangers.

As shown in the proof, the right-hand side of (1.4) in the proposition is just $p_{k+b,k}$. Therefore (1.4) is equivalent to IC (seller). To understand the formula for the soft price, it would be easier to start with the soft price for one unit of benefit:

$$p_{k+1,k} = \frac{L + k + 1 + \widehat{c}_{L-k}}{2B\left(\frac{1/\delta - 1}{\pi}\right) + 2L + 1}$$

This is the increase in future value when the debt holding increases from k to $k+1$. The value increase for two reasons. First, the higher debt holding opens up the opportunity to receive a maximum benefit of $L + k + 1$ (which will bring his debt position to the lowest limit). Second, at the same time he owes his counterpart one less (or she owes him one more), so he will not help if the benefit drawn is $L - k$ or more (the $\widehat{c}_{L-k}$ term).

These factors are adjusted by the denominator for realizing these additional values under different possibilities in different future time. The higher B is, the lower the chance that the future benefits and costs fall within the debt limit and hence the less likely that the values can be realized soon. On the other hand, the higher δ is, either because the agents meet more frequently or they are more patient, the higher the values will be. The soft price for b units of benefit $p_{k+b,k}$ is just the summation of b number of one-unit soft price.

As in Section 3, although multiple equilibrium may exist, the players will gravitate toward the highest limit $L = \frac{B}{2}$. But note one difference between the two models. In for the constant cost ratio model, if the equilibrium with limit $L/2$ exists, then all equilibria with lower limits exist too. In the current model, the existence of an equilibrium does not guarantee that all equilibria with lower limits exist too. Whether the equilibria exist depends on whether (1.4) is satisfied by the corresponding costs.

1.5 Soft vs Hard Transactions

The availability of both hard and soft transactions begs the question of which one is chosen under what situations. The key advantage of soft transactions over the hard alternatives is their saving in transaction costs. By their very nature, soft transactions involve no (or minimal) negotiation of price and conditions. (The absence of formal contracts also means no formal record keeping and no taxes.) On the other hand, the lack of formal enforcement mechanism means that they are harder to start between strangers.

It is therefore unsurprising that soft transactions are preferred when the needs are personal and specific, for which customization would be valuable but contracting would be costly. They are well suited for interactions between acquaintances, especially when repeated interactions would further lower the costs as players learn more about each others' tastes, habits and costs.

This observation is consistent with the conditions for existence of soft debt equilibria in the previous two sections. Also, since the favors are often non-standardized, they tend to be personal services rather than tangible goods. In contrast, when the need is general and standardized, hard transactions gain the advantage by exploiting the economies of scale through mass production and routine transactions.[5]

A second factor that distinguishes between the two alternatives concerns the medium of exchange. While hard transactions enjoy the benefits of having money as the medium of exchange, soft transactions are essentially tacit barters that occur over time, which require both parties to have something that the other wants. Even if multilateral favor trading is allowed in a group through indirect favors, as in [30], trading in soft debts still cannot match the flexibility and convenience of trading in money. Therefore soft transactions are more viable when there are common interests between the parties.

As a result of these two factors, soft transactions are pervasive everywhere from the family to the neighborhood to the workplace.[6] Spouses share household duties; neighbors trade favors such as baby-sitting and house-sitting; research collaborators take turns in contributing to their projects, all probably without formal contracts. The reliance on personal contacts also suggests that soft transactions play a particularly strong role in the social fabric of developing countries. Their lack of formal records makes it difficult to compare their size with the hard sector, but a moment of casual observation would reveal that they are pervasive in essentially everyone's daily economic activities.

Soft transactions may also be advantageous in situations where silent mutual understanding is preferred to explicit agreement (e.g. tacit collusion between businesses). In some other

[5]Economists have long recognized the differences between personal interactions between acquaintances and anonymous market transactions. Adam Smith examined the two types of social interactions in *The Theory of Moral Sentiments* and *The Wealth of Nations* respectively.

[6]Although a marriage is often accompanied by a contract, the "contract" is probably far too vague to make the marriage a hard transaction.

cases, hard transactions are simply unavailable for legal and ethical reasons. For instance, researchers can contract out tasks only to a certain extent.

Soft transactions could also play a role in property rights allocation problems. For example, although auctioning would be an efficient way for a family to allocating the right to choose TV programs, it is seldom adopted in reality. Instead family members make compromises without negotiating explicit terms of compensation to the conceder, who nevertheless expects to be compensated (or have some of his soft debt offset). The same kind of tacit give-and-take reciprocity is as well practiced between friends, neighbors and coworkers, etc.

1.6 Conclusion

This chapter highlights soft debt as an incentive device that facilitates reciprocity under incomplete information. Players engaged in soft transactions perform profit calculations using soft prices just like in hard transactions. The first best allocation can never be achieved. Nevertheless, trading within certain debt limits are possible depending on the discount factor, chance of the seller being able to help, and the distribution of benefit and cost.

There are several directions in which the notion of soft transactions can be extended. First, the chapter focuses on isolated bilateral relationship and avoids mentioning soft markets. The model assumes the players are randomly matched and only the seller can help the buyer. But often the buyer would be able to choose from different sellers, and vice versa. Just like in the hard market, the soft prices will be the driving force behind the player's choices. In this setting, whether a relationship is exclusive (e.g. marriage) or non-exclusive (e.g. friends) would affect the player's decisions since breaking up an exclusive relationship can be costly. Another possibility is to consider multilateral relationship in a group, where indirect favors could be granted (as in [30]).

Second, the scope of soft transactions is inherently limited by the lack of money as a medium of exchange. Unlike hard prices, the soft price paid by the buyer and that received by the seller can be different. The divergence means that mutually beneficial trades may not occur. This limitation may offer an explanation for the rise of hard transactions to fill the gap. This approach may complement the existing literature on the interface between personal and market exchanges (e.g. [25], [1]).

Third, this chapter presumes a simple dichotomy of hard and soft transactions, while in reality there are many "hybrid" transactions. For example, an employment contract contains both the hard (employment contract) and soft (vague duties within "reasonable" bounds). The relational contract literature (e.g. [6], [27]) is pertinent to the subject since it is concerned with combining explicit and implicit incentives in repeated long-term relations. In fact, to the extent that a contract is incomplete, there is always some "softness" in it and hence there is potential for long-term relationship. In purely hard transactions which allow no room for ambiguity, if they ever exist, the parties would simply trade and part.

The presence of hybrid transactions may also help explain the puzzle that providing rewards and punishments sometimes has perverse effects (see [5] for a collection of examples). These phenomena suggest that analyses based purely on the hard elements are incomplete. For example, [14] reports that fining late parents in an Israeli day-care center actually resulted in *more* late arrivals. Soft debt may shed some light on the puzzle. Without the fine system, if the parents are late they may expect to pay a soft price in one form or another in the future. They would avoid being late if there are limited means to repay the soft debt. But with the fines in place, they may take the hard prices as substitutes for the soft prices, and therefore worry less about being late.

Lastly, as in standard neo-classical economics, this chapter takes self interest maximization as individuals' only objective. It actually even portraits what normally perceived as the most intimate relationships as long series of self-interest maximizing behaviors. But undeniably altruism plays an important role in human relationship. Evolutionary biology (e.g. [38]) provides the theoretical basis of altruism. However, the two perspectives need not be mutually exclusive. People may sacrifice for no return, especially for loved ones and sometimes even for strangers. But even between the closest relationships, there are often give-and-take interactions that are better analyzed by soft transactions. Recognizing the validity of both perspectives could potentially reconcile the different views on human nature.

1.7 Proofs

Proof of Proposition 1

Proof. (i) If one player's strategy is to never buy or sell regardless of the debt level, there will be no trade regardless of the other player's response. Therefore a best response of the the other player is to follow the same strategy. An autarky equilibrium is thus sustained. In terms of future values, this means setting V_1 and V_2 to zero invariably.

(ii) Assume on the contrary that the first best allocation is attained in a soft debt equilibrium. Suppose player 1 deviates from the existing strategy by not selling although he is able to help at a cost less than the benefit to player 2. Since player 2 cannot observe whether he could help, she would continue the existing strategy which prescribe her to help whenever she can at a cost below the benefit. Player 1 therefore can deviate and profit. The same is true for player 2. This means the soft debt equilibrium that supports the first best allocation does not exist. □

Proof of Proposition 2

Proof. I first compute the expected future values and soft prices assuming that both players follows the debt limit strategy. Next by using these results I verify that the incentive compatibility (IC) and individual rationality (IR) conditions hold. Finally, I argue that the debt limit equilibria exist using the one-shot deviation principle.

1. *Computation of expected future value and soft prices*

Suppose the debt limit has never been exceeded. Starting with a debt level between $-L$ and L in any arbitrary round, since $B \geq 2L$, the debt level after a transaction (if it occurs) can be any integer within the same range. Given that all transactions go through iff the debt level after transaction remains within $[-L, L]$ and the seller is capable, then for $-L \leq k \leq L$,

$$\begin{aligned} V_k = \quad & \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L+k}(b+\delta V_{k-b}) + \left(1-\frac{\pi}{B}(L+k)\right)\delta V_k\right] \\ + \quad & \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L-k}(-\alpha b+\delta V_{k+b}) + \left(1-\frac{\pi}{B}(L-k)\right)\delta V_k\right] \end{aligned} \tag{1.5}$$

In the first bracket the player is assigned as the buyer. The summation term in the bracket refers to the benefits and resulting future values if transaction occurs (with his resulting debt holding level falling to $k-1,\ k-2, .., \ -L$), while the next term captures the no transaction case. Similarly, in the second bracket the player is drawn as the seller, with his debt holding level rising to $k+1,\ k+2, .., \ L$ if trade goes through.

Group all V_k terms to the left hand side,

$$2\left[\frac{B(1-\delta)}{\pi}+L\delta\right]V_k=\sum_{b=1}^{L+k}(b+\delta V_{k-b})+\sum_{b=1}^{L-k}(-\alpha b+\delta V_{k+b}) \tag{1.6}$$

where $\sum_{l=1}^{0}\equiv 0$.

Iterate k forward to $k+1$, then for $-L-1\leq k\leq L-1$,

$$2\left[\frac{B(1-\delta)}{\pi}+L\delta\right]V_{k+1}=\sum_{b=1}^{L+k+1}(b+\delta V_{k-b+1})+\sum_{b=1}^{L-k-1}(-\alpha b+\delta V_{k+b+1}) \tag{1.7}$$

Subtract (1.6) from (1.7), for $-L\leq k\leq L-1$,

$$2\left[\frac{B(1-\delta)}{\pi}+L\delta\right](V_{k+1}-V_k)=(L+k+1)+\delta V_k+\alpha(L-k)-\delta V_{k+1}$$

$$\left[\frac{2B(1/\delta-1)}{\pi}+2L+1\right]\delta(V_{k+1}-V_k)=L+k+1+\alpha(L-k)$$

Recall the definition $p_{k+1,k}\equiv\delta(V_{k+1}-V_k)$,

$$p_{k+1,k}=\frac{L+k+1+\alpha(L-k)}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L+1} \tag{1.8}$$

Since $p_{k+b,k}=\sum_{r=0}^{b-1}p_{k+r+1,k+r}$ for $b=1,..,L-k$,

$$p_{k+b,k}=\frac{L+k+\frac{b+1}{2}+\alpha\left(L-k-\frac{b-1}{2}\right)}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L+1} \tag{1.9}$$

2. Verification of IC and IR

Now I show that under (1.3) the players indeed find it profitable to trade whenever the debt level will remain within L. The IC and IR conditions are restated below:

$$\text{IC (buyer): } p_{k,k-b} < b \text{ if } b = 1, .., L + k$$

$$\text{IC (seller): } p_{k+b,k} > \alpha b \text{ if } b = 1, .., L - k$$

$$\text{IR: } V_k > 0$$

By (1.9), IC (seller) can be rewritten as:

$$\alpha < \frac{L + k + \frac{b+1}{2}}{2B\left(\frac{1/\delta-1}{\pi}\right) + L + k + \frac{b+1}{2}}$$

Since k is lowest at $k = -L$ and b is lowest at $b = 1$, therefore $L+k+\frac{b+1}{2} \geq 1$. By substituting this lowest value in IC (seller), we can obtain a sufficient condition for IC (seller):

$$\alpha < \frac{1}{2B\left(\frac{1/\delta-1}{\pi}\right) + 1}$$

which is just (1.3) in the proposition.

To show that IC (buyer) is met, first note that IC (buyer) is equivalent to:

$$p_{k+1,k} < 1 \ , \ k = -L, -L + 1, .., L - 1$$

(This restated IC (buyer) is obviously a sufficient condition for the original one. For the necessity part, note that if $p_{r+1,r} \geq 1$ for any $r = -L, -L+1, .., L-1$, then the corresponding inequality for $b = 1$ and $k = r + 1$ in the original IC (buyer) will be violated.)

Use (1.8) and rearrange terms, IC (buyer) becomes:

$$\alpha < \frac{2B\left(\frac{1/\delta-1}{\pi}\right)+L-k}{L-k}$$

which must hold because $\alpha < 1$.

To verify IR, since V_k is increasing in k (as $p_{k+1,k} > 0$), it is sufficient to show that $V_{-L} > 0$.

By (1.6), when $k = -L$,

$$\begin{aligned}
2\left[\frac{B(1-\delta)}{\pi} + L\delta\right] V_{-L} &= \sum_{b=1}^{2L} \left(-\alpha b + \delta V_{-L+b}\right) \\
&= \sum_{b=1}^{2L} \left(-\alpha b + \delta V_{-L} + p_{-L+b,-L}\right) \\
\frac{2B(1-\delta)}{\pi} V_{-L} &= \sum_{b=1}^{2L} \left(-\alpha b + p_{-L+b,-L}\right)
\end{aligned}$$

which is positive according to IC (seller) for $k = -L$. Therefore IR also holds given (1.3).

3. Existence of equilibrium

The one-shot deviation principle states that a strategy profile constitutes a subgame perfect equilibrium iff there is no profitable one-shot deviation for any player at any history. Consider three cases. (i) If the debt level has ever exceeded the limit, then the other player will never buy or sell. So there can be no profitable deviations. (ii) If the debt level has never exceeded the limit but will after the transaction, then again the other player will not buy or sell, and there can be no profitable deviations. (iii) If the debt level has never exceeded the limit and will remain so after the transaction, the IC conditions above guarantee that any deviation will be unprofitable.

In conclusion, by the one-shot deviation principle, the debt limit strategies with limit $L \leq B/2$ do constitute debt limit equilibria provided that (1.3) holds. □

Proof of Proposition 3

Proof. Restate (1.5) for V_k^L:

$$\begin{aligned} V_k^L = & \quad \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L+k}\left(b+\delta V_{k-b}^L\right)+\left(1-\frac{\pi}{B}\left(L+k\right)\right)\delta V_k^L\right] \\ & + \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L-k}\left(-c_b+\delta V_{k+b}^L\right)+\left(1-\frac{\pi}{B}\left(L-k\right)\right)\delta V_k^L\right] \end{aligned}$$

where $c_b = \alpha b$

Compare it to $V_k^{L'}$, written as:

$$\begin{aligned} V_k^{L'} = & \quad \frac{1}{2}\left\{ \begin{array}{l} \quad \left[\frac{\pi}{B}\sum_{b=1}^{L+k}\left(b+\delta V_{k-b}^{L'}\right)+\left(1-\frac{\pi}{B}\left(L+k\right)\right)\delta V_k^{L'}\right] \\ + \quad \frac{\pi}{B}\sum_{b=L+k+1}^{L'+k}\left(b+\delta V_{k-b}^{L'}-\delta V_k^{L'}\right) \end{array} \right\} \\ & + \frac{1}{2}\left\{ \begin{array}{l} \quad \left[\frac{\pi}{B}\sum_{b=1}^{L-k}\left(-c_b+\delta V_{k+b}^{L'}\right)+\left(1-\frac{\pi}{B}\left(L-k\right)\right)\delta V_k^{L'}\right] \\ + \quad \frac{\pi}{B}\sum_{b=L-k+1}^{L'-k}\left(-c_b+\delta V_{k+b}^{L'}-\delta V_k^{L'}\right) \end{array} \right\} \end{aligned}$$

There are two differences between the formula for V_k^L and that for $V_k^{L'}$. First, for $V_k^{L'}$, there are two extra summation terms outside the brackets. They are summations of surpluses from trades and thus are both positive. Second, the V^L terms in the first formula are replaced by $V^{L'}$terms in the second. But we can expand the V^L and $V^{L'}$ terms again in the same fashion as above. $V^{L'}$ again has two extra positive terms over V^L. Continue the process repeatedly, the difference due to the unexpanded $V^L$and $V^{L'}$terms tend to zero because of discounting. However, $V_k^{L'}$ accumulates two extra positive terms in each iteration, therefore$V_k^{L'} > V_k^L$.

When there arise a round where the drawing reveals that a transaction would push the debt level over k for the first time but remain below k'. Each player will be better-off by trading given the other player will trade too. Their future value for the same debt level will be higher, and they will gain from the transaction. The equilibrium with the highest limit dominates all the rest. □

Proof of Proposition 4

Proof. Like in the proof for Proposition 2, I first compute the expected future values and soft prices assuming that both players follows the debt limit strategy, and then verify the IC and IR conditions. In verifying the IC conditions, I also show that all transactions that occur are efficient. Lastly I confirm that (1.4) is always met by some distributions of costs.

1. Computation of expected future value and soft prices

Suppose the debt limit has never been exceeded. Given that all transactions go through if the debt after transaction falls within $[-L, L]$ and the seller is able to help, then for $-L \leq k \leq L$,

$$\begin{aligned} V_k = \quad & \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L+k}(b+\delta V_{k-b}) + \left(1-\frac{\pi}{B}(L+k)\right)\delta V_k\right] \\ + \quad & \frac{1}{2}\left[\frac{\pi}{B}\sum_{b=1}^{L-k}(-\widehat{c}_b+\delta V_{k+b}) + \left(1-\frac{\pi}{B}(L-k)\right)\delta V_k\right] \end{aligned}$$

Compared to the V_k in (1.5) of the proof of Proposition 1, the expected cost αb is replaced by $\widehat{c}_b$.

Following similar steps as in Proposition 1, we get:

$$p_{k+1,k} = \frac{L + k + 1 + \widehat{c}_{L-k}}{2B\left(\frac{1/\delta - 1}{\pi}\right) + 2L + 1} \tag{1.10}$$

Again, since $p_{k+b,k} = \sum_{r=0}^{b-1} p_{k+r+1,k+r}$ for $b = 1, .., L-k$,

$$p_{k+b,k} = \frac{b\left(L + k + \frac{b+1}{2}\right) + \sum_{r=0}^{b-1} \widehat{c}_{L-k-r}}{2B\left(\frac{1/\delta - 1}{\pi}\right) + 2L + 1} \tag{1.11}$$

2. Verification of IC and IR

Now I show that under (1.4) the players indeed find it profitable to trade whenever the debt level will remain within L. For $k = 0, \pm 1, .., \pm L$, the IC and IR conditions are restated as follows:

$$\text{IC (buyer): } p_{k,k-b} < b \text{ if } b = 1, .., L+k$$

$$\text{IC (seller): } p_{k+b,k} > \overline{c}_b \text{ if } b = 1, .., L-k$$

$$\text{IR: } V_k > 0$$

Note that the right-hand side of (1.4) in the proposition is just $p_{k+b,k}$. Therefore (1.4) is equivalent to IC (seller).

I now show that IC (buyer) is guaranteed by IC (seller). First note again that IC (buyer) is equivalent to:

$$p_{k+1,k} < 1 \ , \ k = -L, -L+1, .., L-1$$

Use (1.10) and rearrange terms, IC (buyer) becomes:

$$\widehat{c}_{L-k} < 2B\left(\frac{1/\delta - 1}{\pi}\right) + L - k$$

Substitute b for $L-k$, IC (buyer) can be rewritten as:

$$\widehat{c}_b < 2B\left(\frac{1/\delta - 1}{\pi}\right) + b \text{ ,} b = 1, .., 2L$$

Therefore showing that all transactions that occur are efficient (i.e. $\bar{c}_b < b$) is more than sufficient to prove that IC (buyer) holds.

I will prove the efficiency by induction. From (1.4) and (1.10), pick $k = L - b$ and $b = 1$,

$$\bar{c}_1 < p_{L,L-1} = \frac{2L + \widehat{c}_1}{2B\left(\frac{1/\delta-1}{\pi}\right) + 2L + 1} \implies \bar{c}_1 < \frac{2L}{2B\left(\frac{1/\delta-1}{\pi}\right) + 2L} < 1$$

So all transactions of single-unit benefits are expected to be efficient.

Next, assume efficiency ($\bar{c}_r < r$) holds for $r = 1, .., b-1$, where $b \leq 2L$, then from IC (seller) and (1.11),

$$\bar{c}_b < p_{L,L-b} \quad = \quad \frac{b\left(2L + \frac{1-b}{2}\right) + \sum_{r=1}^{b} \widehat{c}_r}{2B\left(\frac{1/\delta-1}{\pi}\right) + 2L + 1}$$

But

$$\sum_{r=1}^{b} \widehat{c}_r \leq \sum_{r=1}^{b} \bar{c}_r < \sum_{r=1}^{b-1} b + \bar{c}_b = \frac{b\,(b-1)}{2} + \bar{c}_b$$

(The second step holds as a result of the above assumption.) Combine the two inequalities and rearrange terms,

$$\bar{c}_b < \frac{2Lb}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L} < b$$

Therefore efficiency holds for transactions involving benefit b if it holds for transactions of all lower benefits. By induction, all transactions that occur are efficient. IC (buyer) is satisfied.

IR can be verified in similar manner as in Proposition 2.

Applying the one-shot deviation principle as in the proof of Proposition 2, the debt limit equilibrium is shown to exist.

3. Existence of distribution of costs that support the equilibrium

To show that (1.4) can always be satisfied by some set of $\{\bar{c}_b, \widehat{c}_b,\ b = 1, .., 2L\}$ so that the equilibrium exists, consider the case where $\bar{c}_b = \bar{c}$ and $\widehat{c}_b = \bar{c}/2$ for all b (for example, $\widetilde{c}_b$ is uniformly distributed over $(0, \bar{c})$ for all b). Since $p_{k+b,k}$ is increasing in b, if (1.4) is satisfied for $b = 1$, then it is for all b. This condition simply requires

$$\bar{c} < \frac{L+k+1+\bar{c}/2}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L+1} \Leftrightarrow \bar{c} < \frac{L+k+1}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L+\frac{1}{2}}$$

Since $k \geq -L$, picking a $\bar{c}$ smaller than $\frac{1}{2B\left(\frac{1/\delta-1}{\pi}\right)+2L+\frac{1}{2}}$ guarantees (1.4) is met. $\square$

Chapter 2

Limited Efficiency in Favor Trading under Random Benefits and Costs

2.1 Introduction

Market transactions represent only a part of the economic activities. A large part of economic activities are performed through long-term cooperation with relatives, friends, coworkers, etc without explicit contracts. The existing literature on favor trading often assume fixed payoffs for each outcome. In particular, the benefit to the beneficiary and the cost to the helper are constant. However, in real world situations the benefits and costs often vary from time to time. The favors involved in informal exchanges tend to be idiosyncratic. In fact, if the interactions are so immutable such that the benefits and costs always stay constant, then there is no reason why the players would not negotiate a formal contract for the service. The one-time negotiation cost could then be spread out over multiple transactions. In modern society, if the favors are standardized goods or services then people could probably acquire them in the market at lower costs.

When the benefits and costs are constant (with the former being greater), favors will be granted in every round when both players follow the grim trigger strategy (i.e. help as long as both have always helped before, not help otherwise) given that they are patient enough. The maximum social welfare is achieved. When the benefits and costs are random, however, the grim trigger strategy would contribute to inefficiency in two ways: occurrence of inefficient trades and loss of efficient trades. The first source of efficiency occurs since the realized cost could exceed the realized benefit from time to time. The second source occurs because when a player considers whether to help, he needs not only to consider the trustworthiness of his counterpart, but also to weigh the cost against the expected future value of the relationship. If a player is drawn a cost so large (e.g. kidney transfer) such that it exceeds the future expected value of the relationship, he will simply refuse help, hence terminating the relationship and depriving both players from trades in the future.

In view of these two sources of inefficiency, it is reasonable to postulate that the players would modify the grim trigger strategy in order to avoid the inefficiency. The most obvious solution would be to forgive the counterpart for not helping if any of the two situations mentioned happens. The first modification is straightforward: the counterpart will be forgiven for not helping if the cost exceeds the benefit. The second modification requires more deliberation. The future value of the relationship is not fixed, but rather it depends on how much the other player values the relationship. The higher a player values the relationship, the higher his counterpart's value will be, and vice versa. This is because first, the chance of the counterpart rendering help is higher; second, conditional on helping, the benefit of the favor will be greater if the benefit is positively related to the cost. The interdependence of the values of relationship give rise to multiple equilibria. A relationship with high values can be thought of as one having high trust levels.

These observations lead to the idea of "limited efficiency strategy," a modification of the grim trigger strategy whereby the players offer help if and only if (i) the favor is efficient (ii) the cost incurred is below a certain threshold (iii) no one has ever defected. The question is how the thresholds are determined. In the model specified with uniform distributions of cost and surplus, it is shown that subgame perfect equilibria with trading exist (given the parameters of the distribution permit) when both players follow the limited efficiency strategy. There are two features of the equilibria that are worth mentioning. First, the first best outcome can be achieved in but one of the equilibria. Second, in equilibria the thresholds of the players can be different even though all parameters are symmetric. Third, the players may have thresholds that are lower than their respective future values of relationship. In other words, players may forgive each other for not helping even when the cost is below his future value.

This chapter is closely related to the favor trading literature. [30] assume that the benefit and cost are fixed and that the ability to help is private information. He shows that favor trading could be maintained between two players when they grant favors if and only if the net number of favors granted is below a certain number. Therefore the debt limit strategy in the current chapter is similar to this mechanism. He also demonstrates how cooperation is sustainable in a group of players who rarely meet each other, if indirect favors are allowed. [16] build on the bilateral model and make two relaxations: (i) the exchange rate between current and future favors is allowed to deviate from one; (ii) the balance of favors is allowed to appreciate or depreciate. They show that with these relaxations higher payoffs can be achieved. [31] presents a discrete time version of Hauser and Hopenhayn's model and allows the opportunities to offer help to arise at different rates for the two players. [18] studies the model with different discount factors. In another variation, he assumes complete information and that the players have concave utility functions instead of being risk neutral, hence favor trading is considered a form of insurance.

This chapter is also related to the broader literature on reciprocity. This includes for instance the literature on social norms and reciprocity (e.g. [19]), informal insurance (e.g. [8], [23]), relational contracts (e.g. [6], [27]) and games of imperfect public monitoring and private monitoring (e.g. [15], [4]).

In the following, Section 2 develops the game structure and strategy, Section 3 analyze the model with specific distribution assumptions. Section 4 concludes and suggests further extensions.

2.2 Game Structure and Strategy

Two infinitely-lived, risk-neutral players are randomly matched. They take turn to develop needs for favor in alternate periods. Only the other player is able to help. The benefit to the recipient and cost to the helper are random in each period. The distributions of benefits and costs are known to both players. At the beginning of each period, the benefit and cost are drawn and observed by both players. Then the potential helper decides whether to help. In the next period, the process repeats with the roles of the players reversed.

A player's strategy only needs to address the decision rule on rendering help because when it is his turn to need help, he has no decision to make. When deciding whether to help, a plausible condition for rendering help is that the favor is efficient. The first best solution is achieved if both players help whenever it is efficient. However, a strategy that prescribes help based solely on the efficiency criterion is not individually rational because the cost may exceed the value of maintaining the relationship. Even if failing to help would trigger termination of the relationship, the maximum cost a player is willing to bear is the net value of maintaining the relationship. Knowing that the counterpart has a threshold for tolerance of cost, a player's best response is either to accept the counterpart's tolerance level or terminate the relationship. The relationship can be sustained if both players accept the others threshold.

The above reasoning leads to the "limited efficiency strategy". It prescribes that when it is the player's turn to consider whether to help, he will help if and only if (i) the favor is efficient and (ii) the cost is below a certain threshold determined by the value of the relationship. Once any one of the players has defected even when the favor satisfy the two criteria above, then never help regardless of the benefit and cost.

Suppose both agents have a discount factor of $\delta < 1$. The discount factor reflects not just impatience, but the fact that the chance of future meeting is smaller than 1. The favor, if rendered, will bring a benefit of b_i to the receiver player i and cost c_j to the helper player j. Player i's threshold (maximum cost he is willing to bear) is denoted by $\hat{c}_i$.

Player i's objective in period τ is:

$$\max \mathbb{E} \sum_{t=\tau}^{\infty} \delta^{t-\tau} \left(b_i^t - c_i^t \right)$$

Further assume that the distributions of benefits and costs are the same for both players and are time invariant (so that the time subscript can be ignored). Consider player 1's decision when player 2 needs help. The value of maintaining the relationship is:

$$
\begin{aligned}
V_1 &= \left(\delta + \delta^3 + ...\right) \Pr\left(b_1 > c_2 \text{ and } c_2 < \hat{c}_2\right) \mathbb{E}\left(b_1 \mid b_1 > c_2 \text{ and } c_2 < \hat{c}_2\right) \\
&\quad - \left(\delta^2 + \delta^4 + ...\right) \Pr\left(b_2 > c_1 \text{ and } c_1 < \hat{c}_1\right) \mathbb{E}\left(c_1 \mid b_2 > c_1 \text{ and } c_1 < \hat{c}_1\right) \\
&= \frac{\delta}{1-\delta^2} \left\{ \begin{array}{c} \Pr\left(b_1 > c_2 \text{ and } c_2 < \hat{c}_2\right) \mathbb{E}\left(b_1 \mid b_1 > c_2 \text{ and } c_2 < \hat{c}_2\right) \\ -\delta \Pr\left(b_2 > c_1 \text{ and } c_1 < \hat{c}_1\right) \mathbb{E}\left(c_1 \mid b_2 > c_1 \text{ and } c_1 < \hat{c}_1\right) \end{array} \right\}
\end{aligned}
$$

Individual rationality implies that player 1 will not tolerate any cost above V_1, which means:

$$\hat{c}_1 \leqq V_1$$

Since a player has no incentive to help as long as the counterpart is willing to forgive, a player's tolerance level is also his counterpart's forgiveness level. If both players adopt the limited efficiency strategy, subgame perfect equilibrium where cooperation is sustained exists if $\hat{c}_1 \leq V_1$ and $\hat{c}_2 \leq V_2$.

To be more concrete, consider specific distributional assumptions on the benefits and costs. Suppose that in each period, the cost is uniformly distributed on $(\underline{c}, \overline{c})$ where $\underline{c} > 0$. The surplus $s \equiv b - c$ is also uniformly distributed on $(\underline{s}, \overline{s})$. Assume that $\underline{s} < 0 < \overline{s}$, so that the favors are not necessarily efficient. Finally assume that c and s in each period are independently drawn. Under these assumptions, a higher cost of a favor tends to result in a higher benefit. The following results are readily obtained:

$$\Pr\left(b_i > c_j \text{ and } c_j < \hat{c}_j\right) = \frac{\overline{s}}{\overline{s} - \underline{s}} \cdot \frac{\hat{c}_j - \underline{c}}{\overline{c} - \underline{c}}$$

$$\mathbb{E}\left(b_i \mid b_i > c_j \text{ and } c_j < \hat{c}_j\right) = \frac{\underline{c} + \hat{c}_j + \overline{s}}{2}$$

$$\mathbb{E}\left(c_i \mid b_j > c_i \text{ and } c_i < \hat{c}_i\right) = \frac{\underline{c} + \hat{c}_i}{2}$$

Now the value of maintaining the relationship for, say player 1 is:

$$\begin{aligned} V_1\left(\hat{c}_1, \hat{c}_2; \delta, \underline{c}, \overline{c}, \underline{s}, \overline{s}\right) &= \left(\delta + \delta^3 + ...\right) \frac{\overline{s}}{\overline{s} - \underline{s}} \cdot \frac{\hat{c}_2 - \underline{c}}{\overline{c} - \underline{c}} \cdot \frac{\underline{c} + \hat{c}_2 + \overline{s}}{2} \\ &\quad - \left(\delta^2 + \delta^4 + ...\right) \frac{\overline{s}}{\overline{s} - \underline{s}} \cdot \frac{\hat{c}_1 - \underline{c}}{\overline{c} - \underline{c}} \cdot \frac{\underline{c} + \hat{c}_1}{2} \\ &= \frac{1}{A}\left[\left(\hat{c}_2 - \underline{c}\right)\left(\underline{c} + \hat{c}_2 + \overline{s}\right) - \delta\left(\hat{c}_1 - \underline{c}\right)\left(\underline{c} + \hat{c}_1\right)\right] \end{aligned}$$

$$\text{where } A \equiv \frac{2\left(1 - \delta^2\right)}{\delta} \cdot \frac{\overline{s} - \underline{s}}{\overline{s}} \cdot \left(\overline{c} - \underline{c}\right)$$

Subgame perfect equilibrium in which cooperation is sustained exists if $\hat{c}_1 \leqq V_1$and $\hat{c}_2 \leqq V_2$.

2.3 Analysis

To begin, consider the case where the thresholds are common for the two players, i.e., $\hat{c}_1 = \hat{c}_2 = \hat{c}$, then

$$V_1 = \frac{1}{A}\left(\hat{c} - \underline{c}\right)\left[\left(1 - \delta\right)\left(\hat{c} + \underline{c}\right) + \overline{s}\right]$$

$V_1$is increasing and convex in $\hat{c}$. This is because as $\hat{c}$ increases, not only the chance that help will be rendered is higher, but the benefit conditional on helping is also higher.

The maximum cost that player 1 is willing to incur is limited by V_1. For $\hat{c} \leq V_1$,

$$\begin{aligned} \hat{c} &\leq \frac{1}{A}\left(\hat{c}-\underline{c}\right)\left[\left(1-\delta\right)\left(\hat{c}+\underline{c}\right)+\overline{s}\right] \\ \Rightarrow 0 &\leq \left(1-\delta\right)\hat{c}^2+\left(\overline{s}-A\right)\hat{c}-\left[\left(1-\delta\right)\underline{c}+\overline{s}\right]\underline{c} \\ \Rightarrow \hat{c} &\geq \frac{1}{2\left(1-\delta\right)}\left[A-\overline{s}+\sqrt{\left(A-\overline{s}\right)^2+4\left(1-\delta\right)\left[\left(1-\delta\right)\underline{c}+\overline{s}\right]\underline{c}}\right] \end{aligned}$$

Define the expression on the right-hand-side as $\hat{c}^*$, which is the minimum common thresholds for cooperation to sustain. The higher $\hat{c}$ is, the better off both agents are. For $\hat{c}^*$to exist, it needs to be smaller than $\overline{c}$. Since $V_1$is increasing in $\hat{c}$, the condition is equivalent to $V_1\left(\overline{c}\right) \geq \overline{c}$.

To see how $\hat{c}^*$varies with the parameters, put $\underline{c}=0$ for simplicity. Then $A \equiv \frac{2\left(1-\delta^2\right)}{\delta} \cdot \frac{\overline{s}-\underline{s}}{\overline{s}} \cdot \overline{c}$ and $\hat{c}^* = \frac{A-\overline{s}}{1-\delta}$. It is straightforward to verify that:

$$\frac{\partial \hat{c}^*}{\partial \delta} = -\frac{1}{\left(1-\delta\right)^2}\left[\frac{2\left(1-\delta\right)^2}{\delta^2}\frac{\overline{s}-\underline{s}}{\overline{s}}\overline{c}+\overline{s}\right] < 0$$

$$\frac{\partial \hat{c}^*}{\partial \overline{c}} = 2\left(1+\frac{1}{\delta}\right)\frac{\overline{s}-\underline{s}}{\overline{s}} > 0$$

$$\frac{\partial \hat{c}^*}{\partial \overline{s}} = 2\left(1+\frac{1}{\delta}\right)\left(\frac{\underline{\mathrm{s}}}{\overline{s}^2}\right)\overline{c}-\frac{1}{1-\delta} < 0$$

$$\frac{\partial \hat{c}^*}{\partial \underline{\mathrm{s}}} = -2\left(1+\frac{1}{\delta}\right)\frac{\overline{c}}{\overline{s}} < 0$$

Therefore the minimum common threshold for cooperation is lower when the discount factor is higher (the players are more patient or they meet with higher probability), the maximum cost in the uniform distribution is lower, or when the distribution of surplus move upward.

So far it is assumed that the players have the same threshold. But it is not necessarily the case in equilibrium even though the two players are identical. When $\hat{c}_1$ and $\hat{c}_2$ are allowed to be different, again the maximum cost that player 1 is willing to incur to maintain relationship with player 2 equals the value of the relationship:

$$\begin{aligned}
\hat{c}_1 &\leq \frac{1}{A}\left[\left(\hat{c}_2-\underline{c}\right)\left(\underline{c}+\hat{c}_2+\overline{s}\right)-\delta\left(\hat{c}_1-\underline{c}\right)\left(\underline{c}+\hat{c}_1\right)\right] \\
\Rightarrow 0 &\geq \delta\hat{c}_1^2+A\hat{c}_1-\left(\hat{c}_2-\underline{c}\right)\left(\underline{c}+\hat{c}_2+\overline{s}\right)-\delta\underline{c}^2 \\
\Rightarrow \hat{c}_1 &\leq \frac{1}{2\delta}\left(-A+\sqrt{A^2+4\delta\left[\left(\hat{c}_2-\underline{c}\right)\left(\underline{c}+\hat{c}_2+\overline{s}\right)+\delta\underline{c}^2\right]}\right)
\end{aligned}$$

Denote the expression on the right-hand-side as $\hat{c}_1^*\left(\hat{c}_2\right)$, which refers to player 1's threshold (maximum cost he is willing to incur) given that player 2's threshold.

To study the properties of $\hat{c}_1^*$, first consider the boundaries. When $\hat{c}_2=\underline{c}$,

$$\begin{aligned}
\hat{c}_1^* &= \frac{1}{2\delta}\left(-A+\sqrt{A^2+4\delta^2\underline{c}^2}\right) \\
&= \frac{1}{2\delta}\left(-A+\sqrt{\left(A+2\delta\underline{c}\right)^2-4\delta\underline{c}}\right) \\
&< \underline{c}
\end{aligned}$$

Since $\underline{c}$ is the lower bound of the support of the distribution for cost, the threshold for player 1 does not exist. In other words, if player 2 chooses the minimum cost as her threshold, then player 1 will also set his threshold at the minimum cost. Of course there will be no exchange of favors at all. Then how large $\hat{c}_2$ has to be in order to motivate player 1 to set his threshold above $\underline{c}$? When $\hat{c}_1^* = \underline{c}$,

$$A\underline{c} = (\hat{c}_2 - \underline{c})(\underline{c} + \hat{c}_2 + \overline{s})$$

$$\Rightarrow \hat{c}_2^2 + \overline{s}\hat{c}_2 - \left(\underline{c}^2 + \underline{c}\overline{s} + A\underline{c}\right) = 0$$

$$\begin{aligned}\Rightarrow \hat{c}_2 &= \frac{1}{2}\left(-\overline{s} + \sqrt{\overline{s}^2 + 4\left(\underline{c}^2 + \underline{c}\overline{s} + A\underline{c}\right)}\right) \\ &= \frac{1}{2}\left(-\overline{s} + \sqrt{(\overline{s} + 2\underline{c})^2 + 4A\underline{c}}\right) \\ &> \underline{c}\end{aligned}$$

Next consider the shape of $\hat{c}_1^*$. Not surprisingly, $\hat{c}_1^*$ is increasing in $\hat{c}_2$. It can be shown that:

$$\frac{d\hat{c}_1}{d\hat{c}_2} = \frac{2\hat{c}_2 + \overline{s}}{\sqrt{\Delta}} > 0$$

where $\Delta \equiv A^2 + 4\delta\left[(\hat{c}_2 - \underline{c})(\underline{c} + \hat{c}_2 + \overline{s}) + \delta\underline{c}^2\right]$

Incidentally, $\hat{c}_1^*$is either always concave or always convex, depending on the parameters. It can be shown that:

$$\frac{d^2\hat{c}_1}{d\hat{c}_2^2} = \frac{2}{\sqrt{\Delta}} - \frac{2\delta\,(2\hat{c}_2 + \overline{s})^2}{\Delta^{\frac{3}{2}}} = \frac{2}{\sqrt{\Delta}}\left[1 - \frac{\delta\,(2\hat{c}_2 + \overline{s})^2}{\Delta}\right]$$

so $\frac{d^2\hat{c}_1}{d\hat{c}_2^2} > 0$ if and only if

$$A^2 + 4\delta\left[\left(\hat{c}_2 - \underline{c}\right)\left(\underline{c} + \hat{c}_2 + \overline{s}\right) + \delta\underline{c}^2\right] > \delta\left(2\hat{c}_2 + \overline{s}\right)^2$$

$$\Leftrightarrow \frac{A^2}{\delta} + 4\delta\underline{c}^2 > \left(\overline{s} + 2\underline{c}\right)^2$$

For the special case where $\hat{c}_1^*$is linear, $\frac{d^2\hat{c}_1}{d\hat{c}_2^2} = 0$, which means

$$\Delta = \delta\left(2\hat{c}_2 + \overline{s}\right)^2$$

so

$$\frac{d\hat{c}_1}{d\hat{c}_2} = \frac{1}{\sqrt{\delta}} > 1$$

The graphical analysis is depicted in Figure 2.1. The bottom and the top panels represent common threshold and different threshold respectively. In the top panel, the top right corner enclosed by the $\hat{c}_1^*\left(\hat{c}_2\right)$ and $\hat{c}_2^*\left(\hat{c}_1\right)$ curves are the set of cooperative equilibria.

2.4 Conclusion

This chapter lays out a simple model of exchange of favor when the future benefits and costs are random. In particular, a trigger strategy that takes both efficiency of the favor and individual rationality into consideration is proposed. There are a number of directions for further investigation. First, the uniform distribution is adopted in this chapter for its tractability. It would however be interesting to see how thin-tailed distributions would change the results. In particular, as a result the first best solution may not an equilibrium outcome.

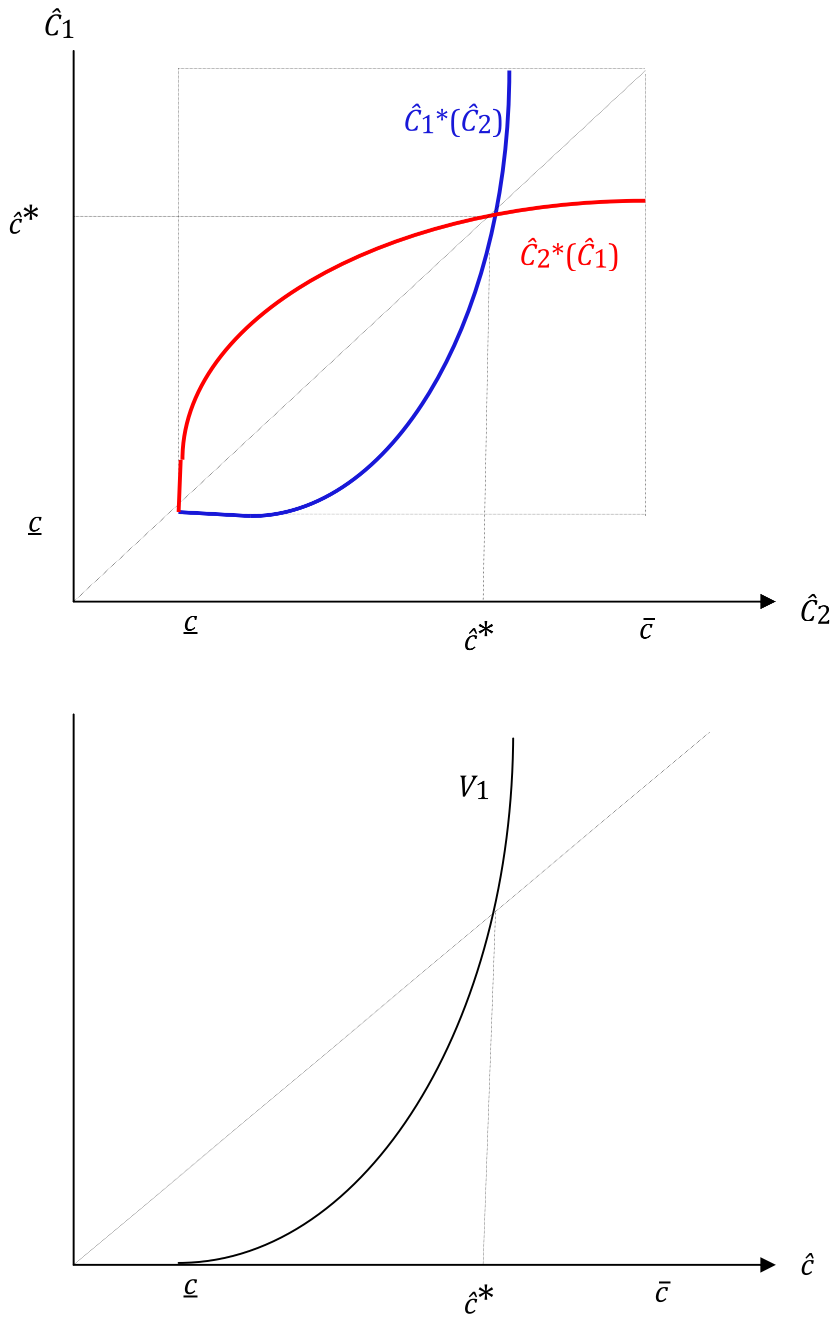

Figure 2.1: Limited Efficiency Strategy
(For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this dissertation.)

Second, the chapter does not attempt to refine the cooperative equilibria. Which point would the players settle on may depend on the particular sequence of realized benefits and costs. Trust between the players may build up more easily if the earlier draws are favorable (low costs with high benefits). *Ex ante* identical relationships may work out or fall apart with different realizations of costs and benefits sequences. Third, the players are identical *ex ante* in the present model. Changing the parameters on the distributions for one player will affect not only his threshold, but his counterpart's as well. In addition, introduction of agents with different preferences will also induce strategic responses from the players.

Chapter 3

Disentangling Intertemporal Substitution and Risk Aversion under the Expected Utility Theorem

3.1 Introduction

To model intertemporal choice under uncertainty within the expected utility (EU) framework, the conventional approach is to simply take expectation of the lifetime utility function under certainty. This approach applies not only to the standard time-separable utility, but also those preferences that embody habit formation (e.g. [9], [7]). A disturbing feature of this approach is that the agent's attitudes toward intertemporal substitution (variation aversion) and risk aversion are often entangled. For example, under the familiar time-separable power utility form, the coefficient of relative risk aversion is tied to the reciprocal of the elasticity of intertemporal substitution. There is no reason for the two attitudes to be necessarily correlated, let alone being tied quantitatively. Why could not an agent be, for instance, variation-loving and risk-averse at the same time? Using survey data, [3] find no significant relationship, either statistically or economically, between the two attitudes.

The problem arises because under the standard approach the preference under certainty not only describes the agent's attitude toward substitution across time but also dictates the risk attitude, so there is no separate control for the risk attitude. [21] provide a way to control for the risk attitude separately under EU. To extend the Arrow-Pratt concept of risk aversion from scalar variables to multi-variables, they transform the utility function under certainty with a (usually concave) real-valued function before taking expectation. A more concave transformation means higher risk aversion, so that the risk aversiveness can be altered with the intertemporal substitution attitude intact. Recent researches by [11], [39] and [20] apply the Kihlstrom-Mirman approach to study asset pricing.

Section 2 of this chapter proposes an alternative EU approach by introducing the notion of *constancy equivalent*. The constancy equivalent of a (deterministic) lifetime consumption sequence is defined as the scalar consumption level that, if held constant throughout the lifetime, will make the agent indifferent. When all possible outcomes (i.e. consumption sequences) in a lottery are translated into their constancy equivalents, the lottery on consumption sequences becomes a lottery on the scalar constancy equivalents, to which the familiar EU approach for single good can then be applied. The agent's attitudes toward intertemporal substitution and risk aversion are encoded in the two steps independently, and therefore disentangled.[1] I show that the objective function resulting from this approach ex-

[1]By "disentangled" I mean having separate control parameters for the two attitudes. It does not mean that the agent's risk attitude toward gambles is not influenced by his intertemporal attitude. Actually, as I will show in Section 3, there is no way that an agent's risk attitude can be isolated from his intertemporal attitude in general.

ists whenever the standard assumptions of the expected utility theorem (EUT) are satisfied. The conventional time-separable specification is nested as a special case when the functions describing the two attitudes are identical. Since the constancy equivalent - on which the risk attitude is defined - is a cardinal scalar, this approach provides an intuitive interpretation of the risk attitude (e.g. along the line of risk premium) as in the case of single good. When the ex post preference is homothetic, the constancy equivalent coincides with the "least concave" representation introduced by [10] and studied by [22] in a follow-up of their 1974 paper.

Risk aversion in multi-dimensions under the constancy equivalence approach is analyzed in Section 3. I argue that the standard analysis of risk aversion is inadequate because it presumes that the agent will just consume whatever outcome he has drawn in a lottery. In other words, it ignores the fact that the agent is able to trade the outcome along the price line to his most preferred affordable bundle. With trading, the distinction between a gamble on consumptions and one on incomes is irrelevant as long as the the consumptions are worth exactly the incomes. I also contend that contrary to common claims, an agent is *not* indifferent to the timing of resolution of lotteries on consumptions under the EU framework. I extend the definitions for the classification of risk attitudes (averse, neutral, loving), the properties of absolute and relative risk aversion (increasing, constant, decreasing), and comparison of risk aversiveness across agents (including those with different ex ante preferences) to multi-dimensions. I show that when the ex post preference is homothetic, it is straightforward to characterize agents with the above properties.

Current researches focus more on non-EU approaches in disentangling the two attitudes. [12] and [40], building on [26], propose non-EU recursive preferences to separate the two attitudes. [12] express concern on the EU approach of [21] for problems revolving around time consistency and inclusion of past consumptions in making current decisions. The same criticism applies to the constancy equivalence EU framework. I address these concerns in Section 4.

Section 5 suggests a generalization of the constancy equivalence approach to accommodate multiple goods in each period. Section 6 concludes.

3.2 Two-Step Approach of Disentanglement

The current period is τ. Consider an agent who was born in period 0, and will live until period T (can be ∞) under uncertainty. There is a single consumption good c. The conventional objective function is formed by adding a conditional expectation operator in front of the lifetime utility function used in non-random models. For example,

$$E_\tau \sum_{t=\tau}^{T} \beta^{t-\tau} u(c_t)$$

where $\{c_t\}_{t=\tau}^{T}$ is a random consumption sequence, u is the period utility function, β is the discount factor, and E_τ represents the expectation operator conditional on all information available to the agent in period τ. The utility function under uncertainty remains time-separable. This specification is simple, tractable, and intuitive, which explains its rise to popularity since being adopted by [32]. The profession seem to have accepted that the conventional specification is a straightforward result of applying the EUT. However, both attitudes toward intertemporal substitution and risk aversion are dictated by the concavity of u, resulting in the entanglement of the two attitudes.

Another way of viewing the entanglement is as follows. For convenience, ignore discounting and consider discrete probability distributions only. Given a lottery, the agent would be indifferent if any subsequence of consumption in one outcome is swapped with the corresponding subsequence in any other outcome with equal probability. For instance, in a two-period life, the agent would be indifferent between a lottery of consumption outcomes $(H, H), (L, L)$ with equal probabilities, and another lottery of $(H, L), (L, H)$ also with equal probabilities, where $H > L$. Compared to the first lottery, the second one has higher variation, but is less risky. The entanglement manifest itself in the fact that the two factors exactly offset with each other, leaving the agent indifferent.

Before starting to disentangle the two attitudes under the EUT, it is helpful to restate what the EUT exactly says. When applied to temporal consumption decisions, the outcomes are *sequences* of consumptions $\{c_t\}_{t=0}^{T} \equiv \mathbf{c}$, rather than a single atemporal good. Preferences are defined on the space of lotteries, each of which specifies a probability distribution of consumption sequences. The EUT claims that if the preference satisfies the axioms of completeness, transitivity, continuity and independence, then it can be represented by the expected utility form. Specifically, there exists a real valued function $z : \mathbb{R}_{+}^{T+1} \to \mathbb{R}$ (the von Neumann-Morgenstern utility function, or vNM function for short) defined on the space of consumption sequences such that the preference can be represented by

$$E_{\mathcal{T}} z(\mathbf{c}) \tag{3.1}$$

The agent's attitudes toward intertemporal substitution and risk aversion are both embedded in z, which is unique up to increasing linear transformations.[2]

Note that the EUT is concerned only with the existence of z; it tells us nothing about *how*

[2]In other words, $\tilde{z}$ is another vNM function if and only if $\tilde{z}(\mathbf{c}) = az(\mathbf{c}) + b$ where a and b are scalar constants and $a > 0$.

to postulate z. While the conventional specification does take an expected utility form, it is not prescribed by the EUT. In fact, by simply adding the conditional expectation operator, one inadvertently tie the two attitudes together. The tie is not a result of the EUT itself, but a consequence of misguided postulation of z. Conceptually, none of the EUT axioms (or any combination of them) seems to necessarily lead to the tie.

3.2.1 Constancy Equivalent

Instead of attempting to postulate z directly, I use a two-step approach. This approach is based on a simple idea as follows. We already know well how to apply the EUT to model the risk attitude toward scalar outcomes. When the outcomes are consumption sequences, we can first translate each of the consumption sequences into a cardinal scalar index that measures its desirability, then apply the EUT to this index in the familiar fashion. The two steps are described in details below.

In the first step, consider the outcomes in deterministic environment. The preference over consumption sequences under certainty (the ex post preference) describes the agent's attitude towards intertemporal substitution and time discounting. For each outcome, define its constancy equivalent as the consumption level that, if maintained throughout lifetime, will leave the agent indifferent.

Definition 7. Given an ex post preference $\succsim$ defined on $\mathbb{R}_+^{T+1} \equiv \left\{\mathbf{c} \in \mathbb{R}^{T+1} : c_t \geq 0 \text{ for } t = 0, 1, ..., T\right\}$, for any consumption sequence $\mathbf{c} \in \mathbb{R}_+^{T+1}$, its *constancy equivalent* $\bar{c} \in \mathbb{R}_+$ is the consumption level that leaves the agent indifferent between $\mathbf{c}$ and the constant consumption sequence $\bar{\mathbf{c}} \equiv \{\bar{c}\}_{t=0}^{T}$, i.e. $\bar{\mathbf{c}} \sim \mathbf{c}$. Also, call $\bar{\mathbf{c}}$ the *constancy equivalent sequence* of $\mathbf{c}$.

In the special case of single period, the constancy equivalent is simply the consumption itself. The next lemma deals with the existence and uniqueness of constancy equivalents.

Lemma 1. *If the ex post preference* $\succsim$ *on* $\mathbb{R}_+^{T+1}$ *is complete, transitive, continuous and monotone, then there exists a unique* $\bar{c}$ *for each* $\mathbf{c}$*.*

Proof. See the proof of Proposition 3.C.1 in [29]. □

The above conditions also guarantee the existence of a continuous utility function $U : \mathbb{R}_+^{T+1} \to \mathbb{R}$ that represents $\succsim$. Therefore,

$$U(\overline{\mathbf{c}}) = U(\mathbf{c})$$

The existence and uniqueness of constancy equivalents is guaranteed for most commonly used preferences. An exception is the lexicographic preference, which is not continuous. Constancy equivalent may also fail to exist if ad hoc restrictions on the space of consumption sequences are imposed. For example, if the preference represented by $U(c_0, c_1) = \sqrt{c_0} + \sqrt{c_1}$ is defined only on $\{(c_0, c_1) : c_0 \geq 1, c_1 \geq 0\}$, then any sequence with $\sqrt{c_0} + \sqrt{c_1} < 2$ has no constancy equivalent.

When the ex post preference is homothetic, $\bar{c}$ (as a function of $\mathbf{c}$) is equivalent to the "least concave representation" introduced by [10], and applied by [22] to compare risk aversiveness across wealth levels (See the proof of their Proposition 1). U^* is a least concave representation of the ex post preference if, whenever U is a concave representation of the ex post preference, $U = h(U^*)$ where h is concave and increasing. Proposition 1 in [22] postulates that if the ex post preference is continuous, convex, monotonic, and homothetic, then U^* exists and is homogenous of degree one. Obviously, $\bar{c}$ is also homogenous of degree one in $\mathbf{c}$ under these conditions.

Although U may take many different forms, the following time-separable form is often specified:

$$U(\mathbf{c}) = \sum_{t=0}^{T} \beta^t u(c_t) \tag{3.2}$$

Using Definition 1, the constancy equivalent of $\mathbf{c}$ is given by $\overline{c}$ such that

$$\overline{c} = u^{-1}\left(\frac{1-\beta}{1-\beta^{T+1}} \sum_{t=0}^{T} \beta^t u(c_t)\right) \tag{3.3}$$

3.2.2 Constancy Equivalence EUT

In the second step we move to the stochastic environment. Given a lottery of consumption sequences, from a consequentialist standpoint the agent is indifferent if the consumption sequence in each outcome is replaced by its constancy equivalent sequence. By virtue of this parity, we can convert the problem of choosing between lotteries of consumption sequences into one of choosing between lotteries of constancy equivalents. The following proposition shows that the preference can be represented in a form reminiscent of the scalar case whenever the EUT axioms are satisfied.

Proposition 5 (Constancy Equivalence EUT)**.** *If a preference on the space of lotteries of consumption sequences (i.e. the ex ante preference) satisfies the completeness, transitivity, continuity and independence axioms, then there exists a* constancy equivalence von Neumann-Morgenstern utility function $v : \mathbb{R}_+ \to \mathbb{R}$ *such that the ex ante preference in period* τ *is represented by:*

$$E_{\mathcal{T}} v(\overline{c}) \tag{3.4}$$

where $\overline{c}$ *denotes the constancy equivalent given by the (continuous and monotonic) ex post preference represented by* U*.*

Proof. Since the ex ante preference satisfies the EUT axioms, there exists a vNM function $z : \mathbb{R}_+^{T+1} \to \mathbb{R}$ such that the ex ante preference is represented by $E_{\mathcal{T}} z(\mathbf{c})$.

Now define $\overline{z}$ as the same function as z but with domain restricted to the space of lotteries on constant sequences (i.e. sequences that maintain the same amount of consumptions throughout the lifetime) only. Since the agent is indifferent between a lottery and its constancy equivalent counterpart (i.e. the lottery with each outcome $\mathbf{c}$ replaced by its constancy equivalent sequence $\overline{\mathbf{c}}$), the ex ante preference can be represented by $E_{\mathcal{T}} \overline{z}(\overline{\mathbf{c}})$. (Note that the existence and uniqueness of constancy equivalents is guaranteed by the EUT axioms together with the continuity and monotonicity of the ex post preference.)

Finally, define $v : \mathbb{R}_+ \to \mathbb{R}$ such that $v(\overline{c}) \equiv \overline{z}(\overline{\mathbf{c}})$. □

We can now model the risk attitude in multi-periods by the concavity of v much in the same way as we model the risk attitude toward scalar outcomes. The proposition and v are so named to distinguish them from the original dynamic EUT and z as in (3.1). The function z is determined jointly by U and v. The function v is defined on constancy equivalents, whereas z on consumption sequences. Like z, v is unique up to increasing linear transformations.

If all lotteries are degenerate, then the conditional expectation operator is redundant, so for any increasing v the objective function (3.4) is reduced to $\overline{c}$, representing the same preference as U. On the other hand, if there is only a single period, (3.4) is reduced to the familiar atemporal version of the EUT.

If U takes the time-separable form (3.2), then z is defined jointly by u, v and β. Any outcome of consumption sequence is translated by u and β to its constancy equivalent, which is then mapped through v. The agent's variation aversion and risk aversion (toward constancy equivalents) are controlled by u and v respectively. In the special case where $u = v$ (or any increasing linear transformation of v), (3.4) is equivalent to the familiar conventional specification (just substitute (3.3) into (3.4)). Note that even when U is time-separable, the objective function is not in general.

Corresponding to the notions of certainty equivalent and risk premium in the static setting, we can define their counterparts in the dynamic environment.

Definition 8. Given a lottery on consumption sequences, define its *certainty-constancy equivalent* $\widehat{c}$ as the non-random scalar such that the agent is indifferent between the lottery and a degenerate constant-level consumption sequence $\widehat{\mathbf{c}} \equiv \{\widehat{c}\}_{t=0}^{T}$.

Under the general form of dynamic EUT (3.1), $z(\widehat{\mathbf{c}}) = E_{\mathcal{T}} z(\mathbf{c})$. Under the constancy equivalence EUT (3.4), $\widehat{c}$ is simply the certainty equivalent of the random $\overline{c}$. In other words, $v(\widehat{c}) = E_{\mathcal{T}} v(\overline{c})$.

The scalar $\widehat{c}$ can be viewed as a lottery's "summary score". The agent chooses whichever lottery that yields the highest $\widehat{c}$.

Definition 9. Given a lottery on consumption sequences, the *constancy equivalence variation premium* $\mathbf{p}^{v} \equiv E_{\mathcal{T}}(\mathbf{c}) - E_{\mathcal{T}}(\overline{\mathbf{c}})$, the *constancy equivalence risk premium* $\mathbf{p}^{r} \equiv E_{\mathcal{T}}(\overline{\mathbf{c}}) - \widehat{\mathbf{c}}$, and the *constancy equivalence risk-variation premium* $\mathbf{p}^{rv} \equiv E_{\mathcal{T}}(\mathbf{c}) - \widehat{\mathbf{c}}$.

Therefore, $\mathbf{p}^{v}$ is the sequence of period-by-period differences between the expected value of consumption under the lottery and that of the variation-free (but random) $\overline{\mathbf{c}}$. Intuitively,

$\mathbf{p}^v$ is the vector of expected values of consumptions that the agent is willing to sacrifice at most in order to convert the outcomes to their constancy equivalents, thereby getting rid of variations within any outcome of consumption sequence. Similarly, $\mathbf{p}^r$ is the vector of expected values of consumptions that the agent is willing to give up at most in order to further get rid of uncertainty. $\mathbf{p}^{rv}$ is simply the sum of the two: the vector of expected values to be sacrificed in order to get rid of both variation and uncertainty.

Figure 3.1 illustrates the constancy-equivalence EUT for the simple case of two periods (0 and 1) and a lottery of two outcomes (A and B) with equal probabilities. The bottom panel shows the indifference curves (IC_A and IC_B) associated with the two outcomes under certainty. The constancy equivalents $\overline{c}_A$ and $\overline{c}_B$ are given by the intersection points $\overline{A}$ and $\overline{B}$ between the indifference curves and the 45° line. The constancy equivalents are then mapped to $v(\overline{c}_A)$ and $v(\overline{c}_B)$ in the top panel, from where we can find $E_0 v(\overline{c})$ and thus the certainty-constancy equivalent $\widehat{c}$. The preference under certainty is shaped by U in the bottom panel, whereas the risk attitude toward $\overline{c}$ is dictated by v in the top panel.

The agent is indifferent between the lottery (A, B), the lottery $(\bar{A}, \bar{B})$, and the degenerate and constant-level sequence denoted by $\widehat{C}$. M is the midpoint of A and B, whereas $\overline{M}$ is the midpoint of $\bar{A}$ and $\bar{B}$. So $\mathbf{p}^v$ is the vector pointing from $\overline{M}$ to M, $\mathbf{p}^r$ from $\widehat{C}$ to $\overline{M}$, and $\mathbf{p}^{rv}$ from $\widehat{C}$ to M.

We can also see from Figure 3.1 why the conventional specification is recovered when $u = v$ (assuming the time-separable form (3.2) for U). If u and v coincides, then we can replace v by u in the top panel. $E_0 v(\overline{c})$ becomes $E_0 u(\overline{c}) = \frac{1}{2} u(\overline{c}_A) + \frac{1}{2} u(\overline{c}_B)$, which equals $\frac{1}{2}[u(c_{A0}) + \beta u(c_{A1})] + \frac{1}{2}[u(c_{B0}) + \beta u(c_{B1})]$ by definition of constancy equivalent. Thus the conventional specification is recovered.

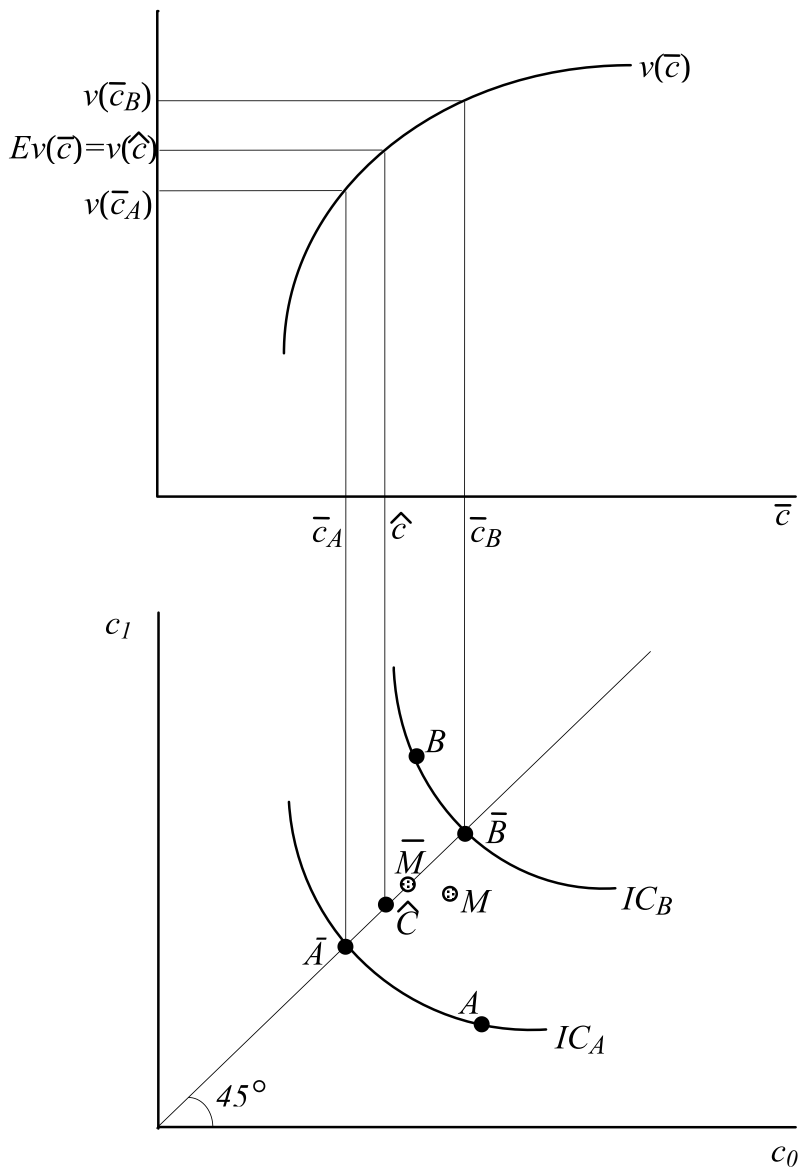

Figure 3.1: Constancy Equivalence EUT (without trading)

The framework above is constructed based on constancy equivalents. In fact, one could well translate the consumption sequences into any arbitrary scalar equivalents. For instance, one could have translated them into equally desirable sequences that grow by say 10% per period and used the period-0 consumption level as the scalar index. Graphically, it means replacing the 45° line in Figure 3.1 by a steeper line. In general, given a $(T+1)$-dimensional vector $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, .., \alpha_T)$ where $\alpha_t \geq 0$ for $t = 0, 1, .., T$ with at least one $\alpha_t > 0$, the $\boldsymbol{\alpha}$-*equivalent* of a consumption sequence $\mathbf{c}$, denoted by $\overline{c}^{\boldsymbol{\alpha}}$, is defined by $U(\overline{c}^{\boldsymbol{\alpha}}\boldsymbol{\alpha}) = U(\mathbf{c})$. The $\boldsymbol{\alpha}$-*equivalence vNM function*, denoted by $v^{\boldsymbol{\alpha}}$, is defined by $v^{\boldsymbol{\alpha}}(\overline{c}^{\boldsymbol{\alpha}}) = v(\overline{c})$. Definitions 2 and 3 can be similarly generalized. Constancy equivalence represents the special case where $\boldsymbol{\alpha} = (1, 1, .., 1)$, in which $\overline{c}^{\boldsymbol{\alpha}}$ becomes $\overline{c}$, and $v^{\boldsymbol{\alpha}}$ becomes v. It will be noted in Section 3 that when U is homothetic, the analysis of risk aversion is invariant to the choice of $\boldsymbol{\alpha}$.

3.2.3 Explicit Functional Forms

The functions U and v are flexible in assuming different forms, including those embodying habit formation. However, for illustration, I use the popular time-separable form (3.2) for U, and specify the following familiar power functional forms for u and v. Suppose u takes the constant elasticity of intertemporal substitution (CEIS) form:

$$u(c) = \begin{cases} c^{1-\sigma}/(1-\sigma) & \text{if} \quad \sigma \neq 1 \\ \log(c) & \text{if} \quad \sigma = 1 \end{cases}$$

where $1/\sigma$ is the elasticity of intertemporal substitution for consumption sequences under certainty. Meanwhile, v takes the form of constant relative risk aversion (CRRA):

$$v\left(\overline{c}\right)=\begin{cases}\overline{c}^{1-\rho}/\left(1-\rho\right) & \text{if} \quad \rho\neq 1\\ \log\left(\overline{c}\right) & \text{if} \quad \rho=1\end{cases}$$

where ρ is the coefficient of relative risk aversion for lotteries of constancy equivalents.

The attitudes toward intertemporal substitution and risk aversion are determined independently by σ and ρ respectively. Suppose σ, $\rho\neq 1$, then[3]

$$\overline{c}=\left[\frac{1-\beta}{1-\beta^{T+1}}\left(\sum_{t=0}^{T}\beta^{t}c_{t}^{1-\sigma}\right)\right]^{\frac{1}{1-\sigma}}$$

and the objective function (3.4) becomes (dropping the positive constant $\left(\frac{1-\beta}{1-\beta^{T+1}}\right)^{\frac{1-\rho}{1-\sigma}}$)

$$E_{\tau}\left[\frac{1}{1-\rho}\left(\sum_{t=0}^{T}\beta^{t}c_{t}^{1-\sigma}\right)^{\frac{1-\rho}{1-\sigma}}\right] \tag{3.5}$$

(3.5) is identical to the objective function in Section 4.1 of [20]. The familiar conventional specification for power utility form is recovered when $\sigma=\rho$. To the extent that σ and ρ are different, using the conventional objective function would cause significant departure from observed data, and thus lead to empirical issues. Also notice that the objective function is not time-separable, unless $\sigma=\rho$.

If the agent is risk-neutral toward $\overline{c}$ ($\rho=0$), then (3.5) becomes

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[3] Also, $\hat{c}=\left\{E_{\tau}\left\{\left[\frac{1-\beta}{1-\beta^{T+1}}\left(\sum_{t=0}^{T}\beta^{t}c_{t}^{1-\sigma}\right)\right]^{\frac{1-\rho}{1-\sigma}}\right\}\right\}^{\frac{1}{1-\rho}}$.

$$E_\tau \left[\left(\sum_{t=0}^{T} \beta^t c_t^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \right]$$

One may wonder why the objective function does not reduce to the conventional time-separable form even when the agent is risk-neutral. The answer is that the agent cares about the full-life consumption sequence *as a whole*. If the objective function were time-separable such that past consumptions can be dropped, then the intertemporal relationship between past and future consumptions are ignored. I will return to this point in Section 4.

3.3 Risk Aversion

This section extends the concepts of risk aversion and comparative risk aversiveness from single to multiple dimensions. With single dimension, an agent is said to be risk-averse (risk-loving) if his vNM function on the good is concave (convex). The concavity implies that, at any consumption level, the agent is willing to give up a certain amount (up to the risk premium) in terms of expected consumption to avoid facing a lottery. [2] and [33] introduce the notions of absolute and relative risk aversion measures to track changes in risk aversiveness for the same agent at different consumption levels. Furthermore, they propose a way to compare risk aversiveness between different agents at the same consumption level. They show that the following four statements are equivalent: (i) the vNM function of agent 1 is a concave transformation of that of agent 2, (ii) the absolute risk aversion measure of agent 1 is greater than that of agent 2 at any consumption level,[4] (iii) agent 1 is willing to pay a higher risk premium than agent 2 at the same consumption level in order to avoid a

[4]Given a vNM function v, the absolute risk aversion is defined as $-\frac{v''(c)}{v'(c)}$.

gamble, and (iv) agent 1's certainty equivalent of any gamble is higher than agent 2's.

It becomes more tricky to define and analyze risk aversion when it comes to multivariates. Consider two multivariate utility functions U_1 and U_2 representing the same ordinal (ex post) preference. [21] take their objective functions under uncertainty as $E(U_1)$ and $E(U_2)$ respectively, and define U_1 as more risk averse than U_2 if $U_1 = k(U_2)$ where k is concave and increasing. They show that the condition implies agent 1 demands a higher risk premium than agent 2 endowed with the same consumption bundle for gambles in any positive direction, and that agent 1 will invest less in a risky asset in a two-period setting. They also formulate the two-dimensional counterpart of the Arrow-Pratt absolute risk aversion measure and conjectured the measures for higher dimensions.

In their follow-up 1981 paper, they extend the comparison of risk aversiveness at different wealth levels to multi-dimensions. As mentioned in Section 2.1, if the ex post preference is homothetic (along with some regularity conditions), then it has a "least concave" representation (call it U^*) in the sense of [10]. $U = h(U^*)$ is an increasing / constant / decreasing absolute (relative) risk averse representation if h is an increasing / constant / decreasing absolute (relative) risk averse function of single variable. In a two-period setting, the fraction of wealth invested in a risky asset decreases (increases) with the wealth if U is an increasing (decreasing) relative risk averse representation and the elasticity of substitution of the the ordinal preference is uniformly greater than one. The direction of change of the fraction reverses if the elasticity of substitution is less than one. The authors did not propose a way to characterize an agent as being risk-averse / neutral / loving.

3.3.1 Risk Aversion with Trading

The analysis of risk aversion proposed here differs from Kihlstrom-Mirman in two major ways. First, under the constancy equivalence framework, risk aversion is defined on the cardinal scalar $\bar{c}$. Second, here after lottery the agent is allowed to trade his consumption bundles to the optimal along the price line. I begin with the latter point.

The standard approach to analyzing risk aversion neglects the fact that agents can trade and optimize his consumption bundle. For instance, when an agent faces a draw between $\mathbf{c}$ and $\mathbf{c}'$ with equal chances, the standard analysis assumes that he will consume, depending on the outcome, either $\mathbf{c}$ or $\mathbf{c}'$ after the draw. However, if the agent could trade his outcome ($\mathbf{c}$ or $\mathbf{c}'$) along the price vector to the optimal bundle according to his U, there is no reason for him to stick with the outcome after the draw. In almost all economic models trading does take place. In addition, in experiments subjects are typically offered gambles on monetary rewards they can decide to spend any time, rather than gambles on fixed consumption values. The no-trade premise implicitly assumed in the standard analysis is valid only in extreme situation, such as when the agent is a prisoner jailed in solitary (so he cannot trade his allocated consumptions with others) and the goods are perishable (so he cannot even save). To assess the true impact of the lottery on the agent, and hence his attitude toward risks, the consumption bundle after the lottery should be optimized according to U. The optimization is irrelevant in the single-dimension case because there is only one good.

With trading, $\bar{c}$ should be derived from optimized consumption bundles, and v accordingly defined on the optimized $\bar{c}$. Since the choice set of consumption bundles depends on the price vector, the information set in the objective function (3.4) includes price information (and prior in their distributions if future prices are uncertain). For simplicity, Figure 3.1 in

Section 2.2 is constructed without trading. Now Figure 3.2 illustrates how the constancy equivalence framework with trading differs from that without trading. Instead of settling at A or B after the lottery, the agent will trade along the price lines set by $\mathbf{p}$ to the optimized A^* or B^*. The constancy equivalents are given by $\overline{c}_A^*$ and $\overline{c}_B^*$. Since the agent has the option to trade, he is (weakly) better off. The shape of v depicts the risk attitude toward *optimized* $\overline{c}$.

With trading, the distinction between a gamble on consumptions and that on incomes is irrelevant as long as the the consumptions are worth exactly the incomes, because the agent can always trade the income for the consumptions. It is often claimed (e.g. [12], [40]) that the agent is indifferent to the timing of resolution of uncertainty in consumptions under the EU framework. Such a claim is based on the premise of absence of trading and saving. Under that situation, the timing indeed does not matter from a consequentialist's point of view because the agent will consume the outcome he draws anyway; the only difference being when he will learn of his future consumption levels. However, when trading is available the timing does matter. When the uncertainty of a gamble is resolved, the agent has only the remaining life to optimize over. So the earlier the resolution, the more choice variables there are for him to optimize over. Therefore the later the uncertainty is resolved, the (weakly) worse off is the agent. That is, the older the agent gets the more risk averse he becomes. Actually if a gamble is forthcoming, the agent would optimize ex ante in anticipation by trading in contingency goods before the gamble. In the rest of this section, in order to standardize the measure of risk aversion, I assume that the agent faces a lottery only in period 0, so the full-life consumption sequence can be optimized after the lottery.

With trading, the risk attitude is defined on the wealth to be spent over his lifetime, as in

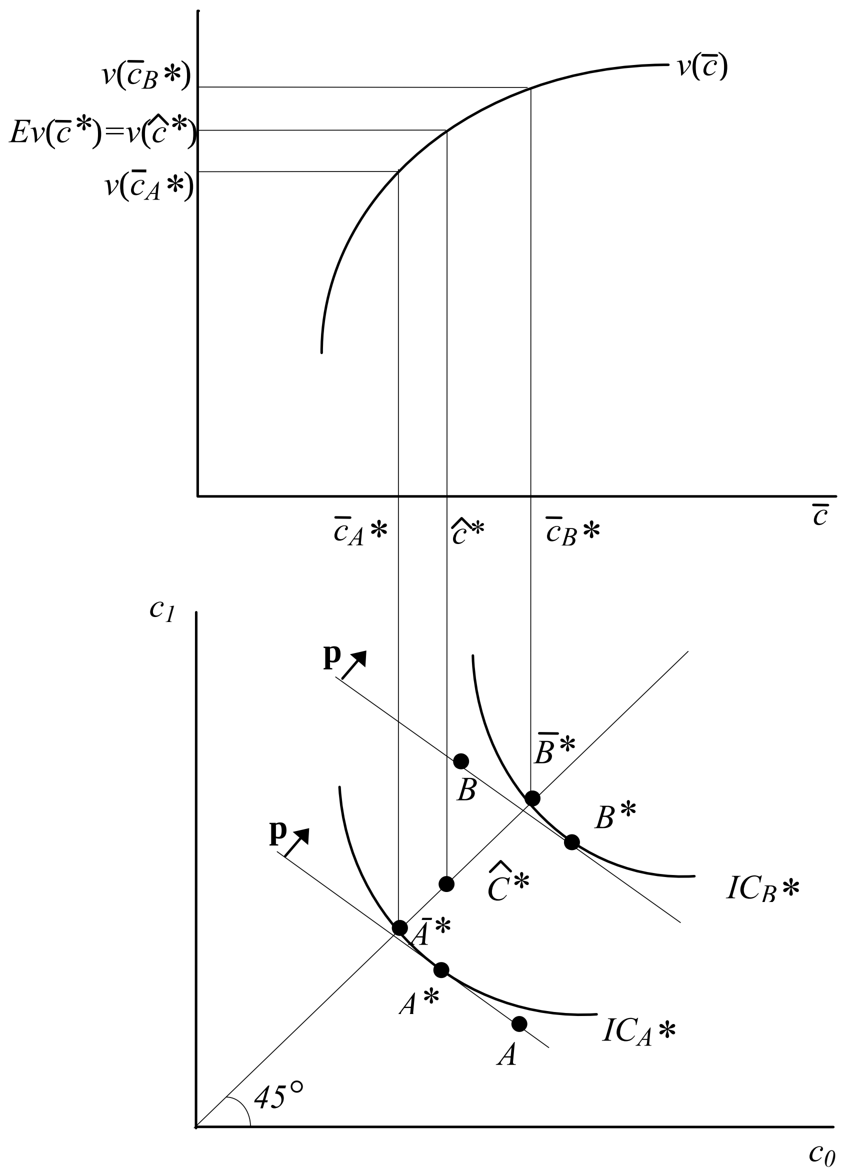

Figure 3.2: Constancy Equivalence EUT (with trading)

[36]. Denote the wealth by $w \in \mathbb{R}_+$, the price vector by $\mathbf{p} \in \mathbb{R}_+^{T+1}$, and a corresponding maximizer of the agent's utility under certainty by $\mathbf{c}_{w;\mathbf{p}}$, i.e., $\mathbf{c}_{w;\mathbf{p}} \in \arg \max_{\mathbf{c} \in \mathbb{R}_+^{T+1}} U(\mathbf{c}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{c} \leq w$. Define the indirect vNM function as follows.

Definition 10. The *indirect vNM function* $\tilde{v}$ is defined as:

$$\tilde{v}(w; \mathbf{p}) \equiv v\left(\bar{c}_{w;\mathbf{p}}\right)$$

where $\bar{c}_{w;\mathbf{p}}$ is given by $U\left(\bar{\mathbf{c}}_{w;\mathbf{p}}\right) = U\left(\mathbf{c}_{w;\mathbf{p}}\right)$.

$\bar{\mathbf{c}}_{w;\mathbf{p}}$ is implicitly defined by U, therefore the risk attitude is shaped by U as well as v. Hence the risk attitude is influenced by the intertemporal substitution attitude in general. Also note that while the maximizer $\mathbf{c}_{w;\mathbf{p}}$ may not be unique, $\bar{\mathbf{c}}_{w;\mathbf{p}}$ is.

With trading, the agent's risk attitude depends partly on $\mathbf{p}$. In one-dimension, the price is irrelevant because there is only one good; the indirect vNM function is identical to the vNM function itself. With multi-dimensions, $\mathbf{p}$ has to be taken into consideration when characterizing an agent's risk attitude. This makes sense because the values of the outcomes depend on their prices. In the following definitions, a characterization of risk attitude is established only if it holds true all possible $\mathbf{p}$.

Definition 11. An agent is said to be *risk-averse / neutral / loving* if his $\tilde{v}$ is concave / linear / convex in w for all $\mathbf{p}$.

Definition 12. An agent is said to exhibit *increasing / constant / decreasing absolute (relative) risk aversion* if his ARA (RRA) is increasing / constant / decreasing in w for all $\mathbf{p}$, where (with the following derivatives taken with respect to w):

$$ARA(w;\mathbf{p}) \equiv -\frac{\tilde{v}''(w;\mathbf{p})}{\tilde{v}'(w;\mathbf{p})} \quad \text{and} \quad RRA(w;\mathbf{p}) \equiv -\frac{\tilde{v}''(w;\mathbf{p})}{\tilde{v}'(w;\mathbf{p})} \cdot w$$

Definition 13. Agent 1 is said to be *more risk-averse* than agent 2 if for all $\mathbf{p}$ there exists a concave and increasing $k^{\mathbf{p}} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{v}_1(w;\mathbf{p}) = k^{\mathbf{p}}(\tilde{v}_2(w;\mathbf{p}))$.

3.3.2 Homothetic Ex Post Preference

In order to satisfy the above definitions, some "discipline" has to be imposed on U. In particular, the homotheticity of U proves to be a convenient property that allows the characterizations of risk attitude to hold for all $\mathbf{p}$. If U is homothetic, after the gamble is resolved the agent's optimized consumptions in every period will expand or contract in proportion to his change in wealth. The expansion or contraction is also proportional to the change in $\bar{c}$. Therefore the gamble on wealth is equivalent to one on $\bar{c}$ with proportional outcomes. The following lemma links $\tilde{v}$ with v under homotheticity of U.

Lemma 2. *If U is homothetic, then*

$$\tilde{v}(w,\mathbf{p}) = v(\lambda^{\mathbf{p}} w) \tag{3.6}$$

where $\lambda^{\mathbf{p}} > 0$ is determined by U and $\mathbf{p}$ and is invariant to w.

Proof. Given $\mathbf{p}$, for any w and $w' > 0$, consecutive use of the homothetic property of U yields

$U\left(\frac{w'}{w}\bar{\mathbf{c}}_{w;\mathbf{p}}\right) = U\left(\frac{w'}{w}\mathbf{c}_{w;\mathbf{p}}\right) = U\left(\mathbf{c}_{w';\mathbf{p}}\right) = U\left(\bar{\mathbf{c}}_{w';\mathbf{p}}\right).$

Comparing the first and last terms, it follows that $\frac{w'}{w}\bar{\mathbf{c}}_{w;\mathbf{p}} = \bar{\mathbf{c}}_{w';\mathbf{p}}$ because both consumption bundles are constant sequences.

Thus $\frac{\bar{c}_{w;\mathbf{p}}}{w} = \frac{\bar{c}_{w';\mathbf{p}}}{w'}$. Since w and w' are arbitrarily chosen, the ratio is independent of wealth. Define the ratio as $\lambda^{\mathbf{p}} > 0$ and substitute $\bar{c}_{w;\mathbf{p}} = \lambda^{\mathbf{p}} w$ into the definition of $\tilde{v}$, the result follows. □

Applying Lemma 2, the following two propositions follow.

Proposition 6. *If U is homothetic, then the concavity / linearity / convexity of v implies that the agent is risk-averse / neutral / loving.*

Proof. By Lemma 2, $\tilde{v}(w;\mathbf{p}) = v(\lambda^{\mathbf{p}} w)$ where $\lambda^{\mathbf{p}} > 0$. If v is concave, then for any $s \in (0,1)$, $\tilde{v}\left(sw + (1-s)w';\mathbf{p}\right) = v\left(\lambda^{\mathbf{p}}\left(sw + (1-s)w'\right)\right) = v\left(s(\lambda^{\mathbf{p}} w) + (1-s)\left(\lambda^{\mathbf{p}} w'\right)\right) > sv(\lambda^{\mathbf{p}} w) + (1-s)v\left(\lambda^{\mathbf{p}} w'\right) = s\tilde{v}(w;\mathbf{p}) + (1-s)\tilde{v}\left(w';\mathbf{p}\right)$, the inequality holds because v is concave. So $\tilde{v}$ is also concave in w for any $\mathbf{p}$. Following similar steps, the proofs for concavity and linearity are obvious. □

Proposition 7. *Given that U is homothetic, if $\overline{ARA}$ ($\overline{RRA}$) is an increasing / constant / decreasing function, where*

$$\overline{ARA}(x) \equiv -\frac{v''(x)}{v'(x)} \quad \textit{and} \quad \overline{RRA}(x) \equiv -\frac{v''(x)}{v'(x)} \cdot x$$

then the agent exhibits increasing / constant / decreasing absolute (relative) risk aversion.

Proof.

$$ARA(w;\mathbf{p}) \equiv -\frac{\tilde{v}''(w;\mathbf{p})}{\tilde{v}'(w;\mathbf{p})} = -\frac{v''(\lambda^{\mathbf{p}} w)}{v'(\lambda^{\mathbf{p}} w)} = -\lambda^{\mathbf{p}} \cdot \frac{v''(w)}{v'(w)} = \lambda^{\mathbf{p}} \overline{ARA}(w)$$

, the second step holds because U is homothetic.

Next, $RRA(w;\mathbf{p}) = ARA(w;\mathbf{p}) \cdot w = \lambda^{\mathbf{p}} \overline{ARA}(w) \cdot w = \lambda^{\mathbf{p}} \overline{RRA}(w)$

Since $\lambda^{\mathbf{p}} > 0$, ARA (RRA) follows the increasing / constant / decreasing property of $\overline{ARA}$ ($\overline{RRA}$) in w. □

Next I turn to comparison of risk aversiveness between agents. I start with the simpler case where the agents have identical U. Note that homotheticity of U is not required for the following proposition.

Proposition 8. *Suppose agent 1 and agent 2 have identical U (not necessarily homothetic). If $v_1(x) = k(v_2(x))$ where k is concave and increasing , then agent 1 is more risk-averse than agent 2.*

Proof. Using the definition of indirect vNM functions and the relationship between v_1 and v_2, $\tilde{v}_1(w;\mathbf{p}) = v_1(\bar{c}_{w;\mathbf{p}}) = k(v_2(\bar{c}_{w;\mathbf{p}})) = k(\tilde{v}_2(w;\mathbf{p}))$ for any $\mathbf{p}$. Note that $\bar{c}_{w;\mathbf{p}}$ is common to both agents because they have identical U. □

When the agents have different ex post (ordinal) preferences, [21] argue that comparison of risk aversiveness toward gambles in consumptions is not possible. Their reasoning is as follows. If agent 1 prefers $\mathbf{c}$ to $\mathbf{c}'$ and agent 2 prefers $\mathbf{c}'$ to $\mathbf{c}$ ex post, then ex ante agent 1 prefers $\mathbf{c}$ with certainty to any gamble between $\mathbf{c}$ and $\mathbf{c}'$, while agent 2 prefers any gamble between $\mathbf{c}$ and $\mathbf{c}'$ to $\mathbf{c}$ with certainty. Agent 1 seems to act more risk-aversely than agent 2. But this situation will reverse if the agents are to compare a gamble of $\mathbf{c}$ and $\mathbf{c}'$ with a degenerate $\mathbf{c}'$, rendering the comparison of risk aversiveness between them impossible. As the authors point out, however, the comparison is possible along the approach of [36] if the gamble is on the one-dimensional wealth. When trading is allowed, wealth is the relevant prize in lotteries, so the comparison can be made.

The homotheticity of the ex post utility function again plays a key role in the following proposition. But now we also require the vNM functions to be homogenous.[5]

Proposition 9. *Suppose U_1 and U_2 of agents 1 and 2 are both homothetic (but not necessarily identical, even ordinally), and their vNM functions are homogenous. If $v_1(x) = k(v_2(x))$ where k is concave and increasing, then agent 1 is more risk-averse than agent 2.*

Proof. Using Lemma 2 and the relationship between v_1 and v_2, for any $\mathbf{p}$, $\tilde{v}_1(w, \mathbf{p}) = v_1\left(\lambda_1^{\mathbf{p}} w\right) = k\left(v_2\left(\lambda_1^{\mathbf{p}} w\right)\right) = k\left(v_2\left(\lambda^{\mathbf{p}} \lambda_2^{\mathbf{p}} w\right)\right)$ where $\lambda^{\mathbf{p}} \equiv \dfrac{\lambda_1^{\mathbf{p}}}{\lambda_2^{\mathbf{p}}}$.

Suppose v_2 is homogenous of degree r, continuing with the above expression, $\tilde{v}_1(w, \mathbf{p}) = k\left((\lambda^{\mathbf{p}})^r v_2\left(\lambda_2^{\mathbf{p}} w\right)\right) = k\left((\lambda^{\mathbf{p}})^r \tilde{v}_2(w; \mathbf{p})\right) = k^{\mathbf{p}}\left(\tilde{v}_2(w; \mathbf{p})\right)$ where $k^{\mathbf{p}}(x) \equiv k\left((\lambda^{\mathbf{p}})^r x\right)$.

Since k is concave and increasing, it follows from the proof of Proposition 2 that $k^{\mathbf{p}}$ is also concave and increasing. □

At the end of Section 3, the constancy equivalence framework is generalized to allow arbitrary scalar equivalents. It can be shown that all the propositions in this section still hold when any scalar equivalent is used.

3.4 Time Consistency

If the constancy equivalence framework is applied to multiple goods in atemporal settings, then there is no concern for time consistency. When it is applied to temporal environments, time consistency needs to be addressed. [11] reports that "many specialists in the field feel that 'bad' things will happen if we use the expected utility approach [to distangle risk aversion from the elasticity of substitution]: Plans may be time inconsistent or the past may

[5] The proposition still holds if the homogeneity of the vNM functions is replaced by the less stringent condition that for any positive constant $\lambda > 0$, $v(\lambda x) = f(\lambda) v(x)$ where $f(\lambda) > 0$. But I adhere to the homogeneity condition anyway due to its popularity.

play an important role in current choices." Some clarifications would be helpful to clarify the concerns. As [11] explains, "time consistent preferences assume that the agent uses the same function each period". By adhering to the same objective function throughout the lifetime, time consistency (in the sense of [17]) under EU is automatically guaranteed.

To be time consistent, past consumptions are included in the objective function. But history dependence itself should not be considered a drawback. As [24, p. 292] reckons, "one cannot claim a high degree of realism for such a postulate [that the consumption in the first period has no effect on the preference in the remaining future, and conversely], because there is no clear reason why complementarity of goods could not extend over more than one time period." In addition, [26] treat history independence as a special case; [12, p. 951] deem history dependence sensible in principle. Also, specifications involving endogenous habit formation (e.g. [9]) are history dependent.

An agent with time consistent preference cares about the full-life consumption sequence beginning with period 0 (when he was born) as a whole, not just the remaining life, or else inconsistency would occur. For instance, if the agent in period 0 cared about intertemporal substitution between c_0 and future consumptions, then why is c_0 forgotten in period 1? Conversely, if the agent in period 0 foresaw that he will disregard c_0 in the next period, why would he care about c_0 now? If c_0 was low, for instance, a variation-averse agent would be less averse to lotteries that entail relatively low level but also less risky future consumption.

In the atemporal environment with multiple goods, no utility function is complete without including all goods because the substitution relationship between various goods need to be taken account of. Even for goods whose consumption levels are fixed, they should be included in the utility function. For the same reason, no objective function in a temporal setting is complete without incorporating the full-life consumption. It is possible to "ignore" past consumptions in recursive formulations of preference (e.g. [28], [12] and [40]) only because the aggregator function that combines the current consumption and the future utility is additively separable (or a monotonic transformation of a additively separable function).

Under the constancy equivalence EU framework, the objective function is always

$$E\left[v\left(\bar{c}\right) \mid I_{\mathcal{T}}\right]$$

throughout the lifetime, where $I_{\mathcal{T}}$ represents the information set available to the agent in the current period, including the consumption history. It should be clear that the only source of change for the agent's preference is due to changes in $I_{\mathcal{T}}$. The agent continuously updates his plan with new information as uncertainties are resolved. But a consistent agent never updates the objective function itself. Except for rare situations, $I_{\mathcal{T}}$ has to change with time, for at least consumption history and some other random events relevant to the agent's decisions must resolve as history unfolds. The "rare" situation occurs when some consumption subsequences are predetermined, and the agent is isolated from new information. Perhaps the closest example is again a prisoner jailed in solitaire.

[12, pp. 950-952] express concern that under the Kihlstrom-Mirman approach, the usual discounting assumption implies that the more distant consumption in the past has *stronger* influence in the decision making. The same concern applies to the constancy equivalence EU framework. Recent studies that follow the Kihlstrom-Mirman approach employ different tactics to address the issue. [11] considers overlapping generations of finitely lived agents, in which past choices play a smaller role relative to their role in an infinite horizon model. [39] proposes a utility function that exhibits temporal risk aversion for asset prices, under which the dependence on more distant consumptions in the past does not necessarily increase. [20] studies a forward looking agent who ignores past consumptions, and hence is time inconsistent, but follows the consistent planning approach of [37] to takes account of "future disobedience". Furthermore, the agent's past choices were made not in isolation, but with future uncertainty taken into consideration. Therefore it may not be appropriate to take past consumption as given and measure their influence to the agent's risk attitude toward today's gamble.

3.5 Extension to Multiple Goods across Time

The constancy equivalence EUT framework so far deals with single good in multiple periods, or multiple goods in single period. This section generalizes the framework to accommodate multiple goods (e.g. leisure and consumption, durable and non-durable goods) in multiple periods. In general, suppose there are N goods and the preference under certainty is represented by $U : \mathbb{R}_+^{N\times(T+1)} \to \mathbb{R}$. For any non-random sequence of consumption bundles $\mathbf{x} = \{x_{1t}, x_{2t}, ..., x_{Nt}\}_{t=0}^{T}$ where x_{nt} denotes the consumption of good n in period t, define its constancy equivalent $\bar{x}$ by the following relationship:

$$U(\mathbf{x}) = U(\overline{\mathbf{x}})$$

where $\overline{\mathbf{x}} = \{\bar{x}, \bar{x}, ..., \bar{x}\}_{t=0}^{T}$. In other words, the agent is indifferent if the consumption level of each good in each period becomes $\bar{x}$ invariably. Given that the EUT axioms hold, the agent ranks the lotteries over the sequences of multiples goods by $E_{\mathcal{T}} v(\bar{x})$, or equivalently by the certainty-constancy equivalent $\hat{x}$ defined by $v(\hat{x}) = E_{\mathcal{T}} v(\bar{x})$.

In picking the units of measurements for the different goods, it is advisable to make "sensible" choices such that the equality of number of units of all goods would seem plausible.[6] The advantage of picking commensurate units of measurement is twofold. First, it helps to avoid reaching constraints such as maximum number of hours available for leisure. Second, typically the utility function is intended to describe trade-off relationship between goods for the regions of *realistic* consumption levels only.

There remains the question of specifying U. In light of the theme of this chapter, one would be concerned with the entangling of attitudes toward *intertemporal* substitution (across time) and *intratemporal* substitution (across contemporaneous goods). For example, a typical functional form involving two goods – non-durables (c) and durables (d) – is:

$$U(\mathbf{c}, \mathbf{d}) = \sum_{t=0}^{T} \beta^t \left[\gamma \log c_t + (1-\gamma) \log d_t\right]$$

Under this preference, in the absence of discounting, it does not matter which c and d are paired in each period, as long as the entire (unordered) set$\{c_t, d_t\}_{t=0}^{T}$ remains the same. For instance, in a two-period setting, a sequence of $\{(c_H, d_H), (c_L, d_L)\}$ is just as desirable as $\{(c_H, d_L), (c_L, d_H)\}$. The intratemporal substitution relationship between c and d is therefore mixed with the intertemporal relationship.

To address the problem, a three-step approach in the same spirit as the two-step approach in Section 2 can be applied to disentangle the three attitudes (intertemporal, intratemporal, and risk) successively. First, for each outcome, convert the consumption bundle $\{x_{1t}, x_{2t}, ..., x_{Nt}\}$ in each period t into its *period-specific constancy equivalent* $\bar{x}_t$ given by

[6] For example, if the typical worker earn about \$40,000 by working for about 2,000 hours (i.e. 6,760 leisure hours) per year, then one could choose to measure leisure in hours and income consumption in say, six dollars.

$P(\bar{x}_t, \bar{x}_t, ..., \bar{x}_t) = P(x_{1t}, x_{2t}, ..., x_{Nt})$, where P dictates the intratemporal attitude. (Here I assume that the intratemporal attitude is the same across time, so that we need only one P.) Next, translate each consumption sequence $\mathbf{x}$ into its constancy equivalent $\bar{x}$ such that $Q(\bar{x}, \bar{x}, ..., \bar{x}) = Q(\bar{x}_0, \bar{x}_1, ..., \bar{x}_T)$, where Q encodes the intertemporal attitude as well as discounting. So U is jointly defined by P and Q. Finally, apply the constancy equivalence EUT to $\bar{x}$.

As an illustration, assume constant elasticities $1/\theta$ and $1/\sigma$ for intratemporal and intertemporal substitutions respectively, and constant relative risk aversion of coefficient ρ for lotteries in $\bar{x}$:[7]

$$\begin{aligned} P(x_{1t}, x_{2t}, ..., x_{Nt}) &= \sum_{n=1}^{N} \delta_n \frac{x_{nt}^{1-\theta}}{1-\theta} \\ Q(\bar{x}_0, \bar{x}_1, ..., \bar{x}_T) &= \sum_{t=0}^{T} \beta^t \frac{\bar{x}_t^{1-\sigma}}{1-\sigma} \\ v(\bar{x}) &= \frac{\bar{x}^{1-\rho}}{1-\rho} \end{aligned}$$

then it is straightforward to show that the objective function takes the form of (with the positive constant $\left(\frac{1-\beta}{1-\beta^{T+1}}\right)^{\frac{1-\rho}{1-\sigma}}$ dropped):

$$E_\tau \left\{ \frac{1}{1-\rho} \left[\sum_{t=0}^{T} \beta^t \left(\sum_{n=1}^{N} \delta_n x_{nt}^{1-\theta} \right)^{\frac{1-\sigma}{1-\theta}} \right]^{\frac{1-\rho}{1-\sigma}} \right\}$$

Also, the certainty-constancy equivalent

[7] $\delta_n > 0$ denotes the weight of good n's utility in each period, with $\sum_{n=1}^{N} \delta_n = 1$. The functions take the log form when the relevant parameter (θ, σ or ρ) is 1.

$$\hat{x} = \left\{ E_\tau \left\{ \left[\frac{1-\beta}{1-\beta^{T+1}} \sum_{t=0}^{T} \beta^t \left(\sum_{n=1}^{N} \delta_n x_{nt}^{1-\theta} \right)^{\frac{1-\sigma}{1-\theta}} \right]^{\frac{1-\rho}{1-\sigma}} \right\} \right\}^{\frac{1}{1-\rho}}$$

3.6 Conclusion

There has been a growing body of evidence, both theoretical and empirical, that calls the EU framework into doubt.[8] This chapter suggests taking another look at the EU framework for the temporal environment by attempting to dispute three common practices: (1) equating the EU approach with adding an expectation operator to the ordinal utility function; (2) analyzing risk aversion without taking trading into consideration; and (3) regarding the EU approach to disentangling as inherently susceptible to issues involving time consistency.

The constancy approach is axiomatically constructed, flexible, and intuitive to interpret. It allows straightforward characterization of an agent's risk attitude in multi-dimension when the ex post preference is homothetic. It can also be extended to accommodate multiple goods across time.

[8] See [35] for a literature review and [34] for a more recent critique.

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