TRANSFORMATIONS OF ELECTRIC NETWORKS BY MATRICES: CASES OF INVARIANT POWER, NON-INVARIANT POWER, AND INVARIANT IMPUT-IMPEDANCE

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TRANSFORMATIONS OF ELECTRIC NETWORKS BY MATRICES: CASES OF INVARIANT POWER, NON-INVARIANT POWER, AND INVARIANT INPUT-IMPEDANCE

By

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Contents

Chapter I. Introduction

Kron published many papers and books [1] [2] about matrix treatments of electric networks. The core of his works is congruent transformations, which are explained in mathematics books [4] and the books by Guillemin [5] [6]. Congruent transformations keep quadratic forms, such as electric power in the case of electric networks, invariant, when transformation matrix is non-singular.

In this thesis the author deals with both invariant and non-invariant transformations, and defines necessary and sufficient conditions for each case and the valid range of applications of each of them. Although almost all examples are by mesh methods, the same discussions hold for node methods.

In Chapter VII transformations of invariant input-impedance are discussed. These are actually reduction of electric networks to Foster's equivalent forms. This process by usual methods [5] is very laborious and tedious. By using the matrix treatments, which are explained in Chapter VII, this process is made orderly and less tedious.

This papaer is written by the author as the thesis for Master's Degree. Professor J. A. Strelzoff taught the author matrix treatments of electric networks and made it possible for the author to work on these subjects.

To Professor J. A. Strelzoff the author presents his sincere thanks.

Chapter II. Derivation of the Generalized Transformation Equation

The equation of a electric network can be written in the matrix form of

$$[e] = [Z][i] \qquad 2.1$$

Now we change current $[i]$ into new current $[i']$ by the transformation equation

$$[i] = [A][i'] \qquad 2.2$$

Substituting 2.2 into 2.1, we get

$$[e] = [Z][A][i'] \qquad 2.3$$

Multiplying 2.3 by a matrix $[B]$, we get

$$[B][e] = [B][Z][A][i'] \qquad 2.4$$

In the process of deriving equation 2.4 we change voltage $[e]$ into new voltage $[e']$ by the transformation equation

$$[e'] = [B][e] \qquad 2.5$$

From 2.4 and 2.5 we get

$$[e'] = [B][Z][A][i'] \qquad 2.6$$

Let new impedance matrix for new voltage $[e']$ and new current $[i']$ be written $[Z']$, then

$$[e'] = [Z'][i'] \qquad 2.7$$

From 2.6 and 2.7 we get

$$[Z'] = [B][Z][A] \qquad 2.8$$

So we can conclude as following,

When current $[i]$ and voltage $[e]$ are transformed into new current $[i']$ and new voltage $[e']$ by the transformation equations 2.2 and 2.5, the new impedance matrix $[Z']$ for the new current and the new voltage is given by 2.8.

This is the generalized transformation equation, which contained congruent transformation $[C]_t[Z][C]$ as one special case.

Chapter III Necessary and Sufficient Conditions for Invariance of Power

To reach transformation equations, Kron started from the assumption that power is invariant, that is, powers at each impedance branch of new system are equal respectively to those of old system. And he got the following results.

When current $[i]$ is transformed into new current $[i']$ by the transformation equation

$$[i] = [C][i'] \qquad 3.1^{*}$$

and voltage $[e]$ is transformed by the equation

$$[e'] = [C]_t[e] \qquad 3.2^{*}$$

, then impedance $[Z]$ is transformed by the transformation equation

$$[Z'] = [C]_t[A][C] \qquad 3.3^{*}$$

* From his whole book it seems that Kron thinks that 3.1, 3.2 and 3.3 are the necessary and sufficient conditions for invariance of power. But it is not correct as explained later.

When we compare these equations with 2.2, 2.7 and 2.8, we see that

$$[A] = [C] \qquad [B] = [C]_t \qquad 3.4$$

From what Kron derived, we can see that the equations 3.4 are the necessary conditions for invariance of power. But it is not yet clear whether these are the sufficient conditions, too. Let us examine it.

Power for the old system is given by

$$P = [e]_t \{i\} \qquad 3.5$$

And power for the new system is given by

$$P = [e']_t [i'] \qquad 3.6$$

From 3.1 we get

$$[i'] = [C]^{-1} [i] \qquad 3.7$$

From 3.2 we get

$$[e']_t = \left([C]_t [e]\right)_t = [e]_t [C] \qquad 3.8$$

By substituting 3.7 and 3.8 into 3.6 we get

$$P' = [e]_t [C] [C]^{-1} [i] = [e]_t [i] = P \qquad 3.9$$

So power is invariant for this transformation. This type of transformation is called congruent transformation. It seems true that 3.4 is the sufficient condition for invariance of power, too. But this is not sufficient. In the process above we used $[C]^{-1}$ in the equation 3.7. So $[C]$ must be non-singular.

Then we can conclude that the condition 3.4 and non-singularity of the transformation matrix $[C]$ are the necessary and sufficient conditions for invariance of power.

Here there must be some comments about the statement by Pipe[7] that the transformation matrix [C](Pipe uses[A]) need not be non-singular. But he insists that only the shape of the energy equation (not the numerical value itself) is preserved under the transformation by 3.4. In order that not only the shape of the energy equation but also the numerical value of the energy are preserved, non-singularity of the transformation matrix [C] must be added.

Chapter IV Two Kinds of Transformation Matrices C .

To set up ~~canonical~~ equations of electric networks, Kron started from their primitive networks. For Example, Fig. 4.1 a) is the primitive network for Fig. 4.1 b) and c).

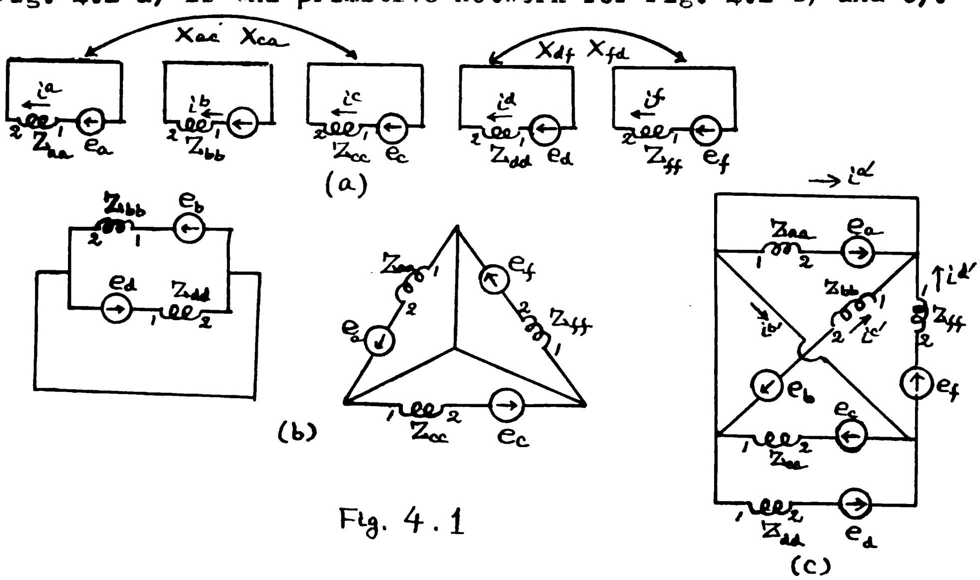

Fig. 4.1

These are all mesh networks, because they have as many meshes as number of impedances. In these networks all emf's in series with each impedances are short-circuited. So each current in each impedance is not changed by changing interconnections as a), b) and c). In this case power is invariant for these trasfoemations. Let's find the transformation matrix [C] between Fig. a) and c) by assuming reference currents indicated in the figures.

In impedance Z_{aa} $\quad i^{a} = - i^{a'} - i^{c'} - i^{d'}$

" $\quad Z_{bb}$ $\quad i^{b} = - i^{c'}$

" $\quad Z_{cc}$ $\quad i^{c} = i^{d'} + i^{f'} - i^{b'}$ $\quad$ 4.1

" $\quad Z_{dd}$ $\quad i^{d} = - i^{f'}$

" $\quad Z_{ff}$ $\quad i^{f} = - i^{d'}$

From these equations we get the trnasformation matrix C

$$[C] = \begin{array}{c} \\ a \\ b \\ c \\ d \\ f \end{array} \begin{array}{c} \begin{array}{ccccc} a' & b' & c' & d' & f' \end{array} \\ \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array} \qquad 4.2$$

The matrix 4.2 is non-singular.

Equation 4.1 and matrix 4.2 give the correspondences between old currents and new currents. And their correspondences are not only geometrical or positional * but also numerisal.

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* Geometrical or positional correspondence means that corresponding currents flows in the same impedance of the network. Nemerical correspondence means that values of these corresponding currents are equal numerically.

In some cases correspondence is only geometrical, but not numerical. Such cases are explained later.

All transformation matrices should be geometrically correspondent. But they need not be numerically correspondent.

Let's examine another simple example.

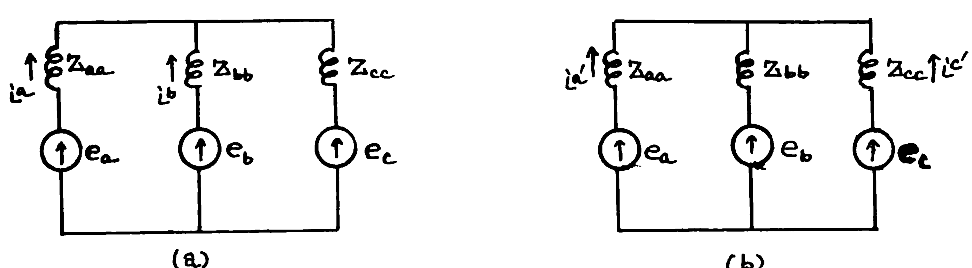


Fig. 4.2

Fig. 4.2 a) and b) are the same network, which has two independent meshes. In a) i^a and i^b are assumed as independent, while in b) $i^{a'}$ and $i^{c'}$ are assumed independent. Only difference in the difference of reference currents. Correspondences of these currents are,

In impedance Z_{aa} $\quad i^a = i^{a'}$

In impedance Z_{bb} $\quad i^b = -i^{a'} - i^{c'}$

So the transformation matrix is

$$[C] = \begin{array}{c c} & \begin{array}{cc} a' & c' \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \end{array} \qquad 4.3$$

This is a non-singular matrix. And the correspondence by

$$[i] = [C][i'] \qquad 4.4$$

is not only geometrical but also numerical.

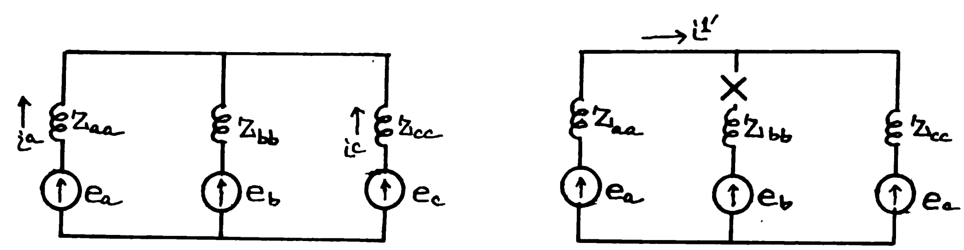
We have just examined two cases, where transformation matrices are non-singular. It may seem too early to conclude that all non-singular transformation matrices are correspondent geometrically and numerically. But this is true. We get non-singular transformation matrices, only when transformations are not accompanied by any change of interconnections of impedances. For instance. the transformation of Fig. 4.2 does not have any change of interconnections of impedances. It has only change of reference currents. So all currents at all impedamce branches are invariant at this transformation. Accordingly the transformation matrix for this case is correspondent geometrically and numerically.

Transformation matrices for tansformations between different independent reference variables of the same circuit are generally non-singular and corresponding geometrically and numerically.

It seems likely that the transformation of Fig. 4.1 is accompanied by changes of interconnection of impedances. But all impedances remain short-circuited even after the transformation, and the mumber of meshes remains the same. All currents of impedance branches remain the same. So that the transformation matrix [C] of 4.2 is non-singular and correspondent geometrically and numerically.

Here we can conclude that all transformation matrices for transformations, where the number of meshes remains the same, are non-singular and correspondent geometrically and numerically, so long as all reference variables are independent.

Let's examine the transformation of Fig. 4.3.



(a) (b)

Fig. 4.3

(a) is trasformed into (b) by opening one impedance branch Z_{bb}.
Reference currents are assumed as indicated in the figure.

Then the correspondence between old and new currents are

$$i^a \rightarrow i^{1'}$$
$$i^c \rightarrow i^{1'} \qquad 4.5$$

So the transformation matrix is

$$[C] = \begin{array}{c c} & 1' \\ a & \left[\begin{array}{c} 1 \\ -1 \end{array}\right] \\ c & \end{array} \qquad 4.6$$

$[C]$ is singular.

It is easily understood that power is not invariant in this transformation. So it seems likely that congruent trasnformation $[C]_t[Z]\,[C]$ can not be applied in this case, because Kron proved congruent transformations by assuming that power is invariant in transformations. Let's try to apply it to this case.

For Fig. 4.3 (a)

$$e_a - e_b = (Z_{aa} + Z_{bb})i^a + Z_{bb}\,i^c$$
$$e_c - e_b = Z_{bb}\,i^a + (Z_{bb} + Z_{cc})i^c \qquad 4.7$$

This is written in matrix form as below.

$$[e] = [Z]\,[i]$$

, where $[e] = \begin{bmatrix} e_a - e_b \\ e_c - e_b \end{bmatrix}$ $[Z] = \begin{bmatrix} Z_{aa} + Z_{bb} & Z_{bb} \\ Z_{bb} & Z_{bb} + Z_{cc} \end{bmatrix}$

$$[i] = \begin{bmatrix} i^a \\ i^c \end{bmatrix} \qquad 4.8$$

According to the congruent transformation, new matrices for the new network of Fig. 4.3 (b) are given as following.

$$[C]_t[Z]=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}Z_{aa}+Z_{bb} & Z_{bb}\\ Z_{bb} & Z_{bb}+Z_{cc}\end{bmatrix}=\begin{bmatrix}Z_{aa} & -Z_{cc}\end{bmatrix}$$

$$[Z']=[C]_t\,[Z]\,[C]=\begin{bmatrix}Z_{aa} & -Z_{cc}\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}Z_{aa}+Z_{cc}\end{bmatrix} \qquad 4.9$$

$$[e']=[C]_t\,[e]=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}e_a-e_b\\ e_c-e_b\end{bmatrix}=\begin{bmatrix}e_a-e_c\end{bmatrix}$$

So for the new network the equation of network $[e']=[Z'][i']$ is

$$e_a - e_c = (Z_{aa} + Z_{cc})\,i' \qquad 4.10$$

This is valid as easily seen from Fig. 4.3 (b). So $[C]_t[Z][C]$ method can be applied even for the transformations, where power is not invariant. And this fact is already proved by equation 2.8, because we did not assume invariance of power to derive equation 2.8. But some limitations are necessary for **applying** 2.8 generally,as explained later.

Here we would like to call attention to the fact that equation 4.5 does not hold numerically, but it does hold only geometrically. So "$\longrightarrow$" is used instead of "= ".there. Transformation matrix 4.6 indicates only geometrical correspondence between old and new currents. So it is not correct to use $[i]=[C][i']$ to evaluate one currents group from values of the other currents group, which are already solved.

It is interesting that congruent transformation can be applied even to cases where transformation matrix is correspondent only geometrically but not numerically.

Transformations, which change number of impedance branches, are accompanied by transformation matrices, which are singular. In these cases, power is not invariant and transformation matrices are correspondent only geometrically, but not numerically. As showed above, congruent transformation can be applied to this case. But number of impedance branches should be changed into less number. Otherwise this method cannot be applied.

Here we can conclude as following;

1. Transformation matrix for transformations, which are not accompanied by change of number of impedance branches, are non-singular.
2. Non-singular transformation matrices are correspondent geometrically and numerically.
3. Transformation matrices for transformations, which are accompanie by change of number of impedance branches (into smaller number), are singular.

4. Singular transformation matrices are correspondent only geometrically but not numerically. In this case power is not invariant.

5. But still $[C]_t[Z][C]$ method can be used for singular transformation matrices.

6. But we can not evaluate $[i]$ from $[i']$, using $[i] = [C][i']$.

Chapter V. Invariant Transformations.

As explained in Chapter III, the necessary and sufficient conditions for invariance of power are

$$[A] = [C] \qquad [B] = [C]_t \tag{5.1}$$

And $[C]$ must be non-singular.

In this case formulas of transformation between old circuit equation

$$[e] = [Z][i] \tag{5.2}$$

and new circuit equation

$$[e'] = [Z'][i'] \tag{5.3}$$

are given as following.

$$[i] = [C][i'] \tag{5.4}$$

$$[e'] = [C]_t[e] \tag{5.5}$$

$$[Z'] = [C]_t[Z][C] \tag{5.6}$$

Because $[C]$ is non-singular,

$$[i'] = [C]^{-1}[i] \tag{5.7}$$

We can evaluate $[i]$ from $[i']$, using 5.4, because 5.4 is correspondent numerically.

Most of examples in Kron's book (1) are invariant transformations, although some non-invariant transformations are mixed, without explanations. So it is not necessary to present here more such examples.

Chapter VI Non-invariant Transformations

When at least one of the necessary conditions for invariant transformations is not satisfied, the transformation is non-invariant. Power is not kept invariant at this transformation. So the sufficient condition for non-invariant transformation is ;

1. $[A] = [C]$ and $[B] = [C]_t$ are not satisfied.

or 2. Transformation matrix $[C]$ is singular.

According to which necessary condition for invariance of power is not satisfied, there are several kinds of non-invariant transformations.

(1) Cases where $[A] = [C]$ and $[B] = [C]_t$ are not satisfied.

As derived in Chapter II, when old and new variables are related by

$$[i] = [A][i'] \qquad 6.1$$

$$[e'] = [B][e] \qquad 6.2$$

, the new impedance matrix $[Z']$ is given by

$$[Z'] = [B][Z][A] \qquad 6.3$$

So long as transformation equations 6.1 and 6.2 are numerically correspondent, $[A]$ and $[B]$ can be arbitrary. Numerical corespondences of 6.1 and 6.2 mean that matrices [A] and [B] are non-singular, as explained in Chapter IV.

Let's show several examples of this case.

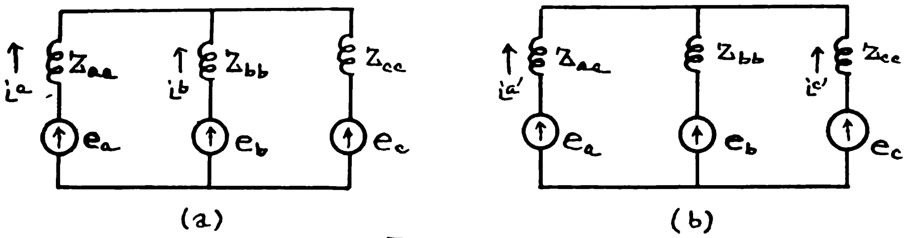


Fig. 6.1

The network of Fig. 6.1 (a) gives the following matrix equation.

$$[e] = [Z]\,[i]$$

, where

$$[e] = \begin{bmatrix} e_a - e_c \\ e_b - e_c \end{bmatrix}$$

$$[i] = \begin{bmatrix} i^a \\ i^b \end{bmatrix} \qquad 6.5$$

$$[Z] = \begin{bmatrix} Z_{aa} + Z_{cc} & Z_{cc} \\ Z_{cc} & Z_{bb} + Z_{cc} \end{bmatrix}$$

Now if independent variables [i] are changed into [i'], which are indicated in fig. 6.1 (b), then the transformation matrix [C] is

$$[C] = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} a' & c' \end{array} \\ \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \end{array} \qquad 6.6$$

Here we choose

$$[A] = [C] \quad \text{and} \quad [B] = [U] \qquad 6.7$$

, where $[U]$ is a unit matrix.

Then

$$[Z'] = [B]\,[Z]\,[A] = [Z]\,[C]$$

$$= \begin{bmatrix} Z_{aa} + Z_{cc} & Z_{cc} \\ Z_{cc} & Z_{bb} + Z_{cc} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} Z_{aa} & -Z_{cc} \\ -Z_{bb} & -(Z_{bb} + Z_{cc}) \end{bmatrix} \qquad 6.8$$

$$[e'] = [B]\,[e] = [e] \qquad 6.9$$

So finally we get as $[e'] = [Z'][i']$

$$\begin{bmatrix} e_a - e_c \\ e_b - e_c \end{bmatrix} = \begin{bmatrix} Z_{aa} & -Z_{cc} \\ -Z_{bb} & -(Z_{bb} + Z_{cc}) \end{bmatrix} \begin{bmatrix} i^{a'} \\ i^{c'} \end{bmatrix} \qquad 6.10$$

From 6.10 we get

$$i^{a'} = \frac{e_a(Z_{bb} + Z_{cc}) - e_b Z_{cc} - e_c Z_{bb}}{Z_{aa} Z_{bb} + Z_{bb} Z_{cc} + Z_{cc} Z_{aa}}$$

$$i^{c'} = \frac{-e_a Z_{bb} - e_b Z_{aa} + e_c (Z_{aa} + Z_{bb})}{Z_{aa} Z_{bb} + Z_{bb} Z_{ca} + Z_{cc} Z_{aa}} \qquad 6.11$$

Now let's use the usual C Z C method. Then

$$[C]_t [Z] [C] = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} Z_{aa} & -Z_{cc} \\ -Z_{bb} & -(Z_{bb} + Z_{cc}) \end{bmatrix}$$

$$= \begin{bmatrix} Z_{aa} + Z_{bb} & Z_{bb} \\ Z_{bb} & Z_{bb} + Z_{cc} \end{bmatrix}$$

$$[e'] = [C]_t [e] = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e_a - e_c \\ e_b - e_c \end{bmatrix}$$

$$= \begin{bmatrix} e_a - e_b \\ -e_b + e_c \end{bmatrix}$$

So finally we get as $[e'] = [Z'] [i']$

$$\begin{bmatrix} e_a - e_b \\ -e_b + e_c \end{bmatrix} = \begin{bmatrix} Z_{aa} + Z_{bb} & Z_{bb} \\ Z_{bb} & Z_{bb} + Z_{cc} \end{bmatrix} \begin{bmatrix} i^{a'} \\ i^{c'} \end{bmatrix} \tag{6.12}$$

From 6.12 we get

$$i^{a'} = \frac{e_a (Z_{bb} + Z_{cc}) - e_b Z_{cc} + e_c Z_{bb}}{Z_{aa} Z_{bb} + Z_{bb} Z_{cc} + Z_{cc} Z_{aa}}$$

$$i^{c'} = \frac{-e_a Z_{bb} - e_b Z_{aa} + e_c (Z_{aa} + Z_{bb})}{Z_{aa} Z_{bb} + Z_{bb} Z_{cc} + Z_{cc} Z_{aa}} . \tag{6.13}$$

Final results of both 6.11 and 6.13 coincide with each other as expected. To reach the final result 6.11, we used $[A] = [C]$ and $[B] = [U]$. To reach the final result 6.13, we used $[A] = [C]$ and $[B] = [C]_t$. So in the latter we must handle more operations, such as $[C]_t [Z] [C]$ and $[C]_t [e]$. In the former method we can omit these operations, so it is much easier to reach the final answer. This is so simple case that difference of amount of work involved in both methods is not so great. Let's show another more complicated case.

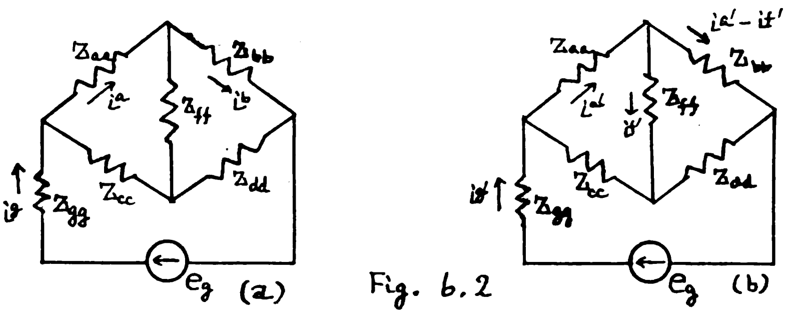


Fig. 6.2

In fig. 6.2 there are given circuits of Wheatstone's bridge. At first indèpendent currents are given as indicated in fig. (a), and then circuit equations are given as below.

$$\begin{bmatrix} e_g \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{gg} + Z_{cc} + Z_{dd} & -Z_{cc} & -Z_{dd} \\ -Z_{cc} & Z_{aa} + Z_{ff} + Z_{cc} & -Z_{ff} \\ -Z_{dd} & -Z_{ff} & Z_{bb} + Z_{dd} + Z_{ff} \end{bmatrix} \begin{bmatrix} i^g \\ i^a \\ i^b \end{bmatrix} \qquad 6.14$$

To get the balance condition of Wheatstone's bridge, we must get current in impedance Z_{ff} . So let's assume $i^{f'}$, $i^{g'}$ and $i^{a'}$ as new independent currents, as indicated in fig. (b).

Then we get the transformation matrix [C]

$$[C] = \begin{array}{c} \\ g \\ a \\ b \end{array}\!\!\begin{array}{c} \begin{array}{ccc} f' & g' & a' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \end{array} \qquad 6.15$$

Assuming [A] = [C] and [B] = [U] , we get

$$[Z'] = [Z]\,[C] = \begin{bmatrix} Z_{dd} & Z_{gg} + Z_{cc} + Z_{dd} & -(Z_{cc} + Z_{dd}) \\ Z_{ff} & -Z_{cc} & Z_{aa} + Z_{cc} \\ -(Z_{bb} + Z_{dd} + Z_{ff}) & -Z_{dd} & Z_{bb} + Z_{dd} \end{bmatrix} \qquad 6.16$$

So we get new circuit equation as below.

$$\begin{bmatrix} e_g \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{dd} & Z_{gg} + Z_{cc} + Z_{dd} & -(Z_{cc} + Z_{dd}) \\ Z_{ff} & -Z_{cc} & Z_{aa} + Z_{cc} \\ -(Z_{bb} + Z_{dd} + Z_{ff}) & -Z_{dd} & Z_{bb} + Z_{dd} \end{bmatrix} \begin{bmatrix} i^{f'} \\ i^{g'} \\ i^{a'} \end{bmatrix} \qquad 6.17$$

From 6.17 we get

$$-\,i^{f'} = \frac{e_g\,(Z_{cc}\,Z_{bb} - Z_{aa}Z_{dd})}{D} \qquad 6.18$$

, where

$$\begin{aligned} D = {} & Z_{aa}Z_{bb}Z_{cc} + Z_{aa}Z_{bb}Z_{dd} + Z_{aa}Z_{bb}Z_{gg} + Z_{aa}Z_{cc}Z_{dd} + \\ & Z_{aa}Z_{cc}Z_{ff} + Z_{aa}Z_{dd}Z_{ff} + Z_{aa}Z_{dd}Z_{gg} + Z_{aa}Z_{ff}Z_{gg} + \\ & Z_{bb}Z_{cc}Z_{dd} + Z_{bb}Z_{cc}Z_{ff} + Z_{bb}Z_{cc}Z_{gg} + Z_{bb}Z_{dd}Z_{ff} + \\ & Z_{bb}Z_{ff}Z_{gg} + Z_{cc}Z_{dd}Z_{gg} + Z_{cc}Z_{ff}Z_{gg} + Z_{dd}Z_{ff}Z_{gg} \end{aligned}$$

From 6.18 we get the well-known balance condition

$$Z_{aa}Z_{dd} = Z_{bb}\,Z_{cc} \qquad 6.19$$

Now let's try the usual $[C]_t$ $[Z]$ $[C]$ method.

$$[Z'] = [C]_t\ [Z]\ [C]$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} Z_{dd} & Z_{gg} + Z_{cc} + Z_{dd} & -(Z_{cc} + Z_{dd}) \\ Z_{ff} & -Z_{cc} & Z_{aa} + Z_{cc} \\ -(Z_{bb} + Z_{dd} + Z_{ff}) & -Z_{dd} & Z_{bb} + Z_{dd} \end{bmatrix}$$

$$= \begin{bmatrix} Z_{bb} + Z_{dd} + Z_{ff} & Z_{dd} & -(Z_{bb} + Z_{dd}) \\ Z_{dd} & Z_{gg} + Z_{cc} + Z_{dd} & -(Z_{cc} + Z_{dd}) \\ -(Z_{bb} + Z_{dd}) & -(Z_{cc} + Z_{dd}) & Z_{aa} + Z_{bb} + Z_{cc} + Z_{dd} \end{bmatrix} \qquad 6.20$$

$$[e'] = [C]_t\ [e] = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e_g \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ e_g \\ 0 \end{bmatrix} \qquad 6.21$$

Now we can set up $[e'] = [Z'] [i']$

$$\begin{bmatrix} 0 \\ e_g \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{bb} + Z_{dd} + Z_{ff} & Z_{dd} & -(Z_{bb} + Z_{dd}) \\ Z_{dd} & Z_{gg} + Z_{cc} + Z_{dd} & -(Z_{cc} + Z_{dd}) \\ -(Z_{bb} + Z_{dd}) & -(Z_{cc} + Z_{dd}) & Z_{aa} + Z_{bb} + Z_{cc} + Z_{dd} \end{bmatrix} \begin{bmatrix} i^{f'} \\ i^{g'} \\ i^{a'} \end{bmatrix} \qquad 6.22$$

From 6.22 we get the same final results as 6.18, which proves validity of $[Z]$ $[C]$ mehtod. $[Z]$ $[C]$ method does not involve operations 6.20 and 6.21, so it is much easier to get final results.

Sometimes circuit equations are set up, but some terminals of emf's are not accessible, so these emf's can not be measured. In such a case we must change emf's, which appear in circuit equations, into other emf's, which are accessible. Let's Present one example of such a case.

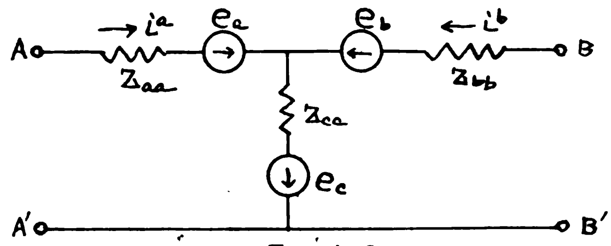


Fig. 6.3

For the network of fig. 6.3 circuit equations are given as below.

$$\begin{bmatrix} e_{AB} \\ e_{AA'} \end{bmatrix} = \begin{bmatrix} e_a - e_b \\ e_a + e_c \end{bmatrix} = \begin{bmatrix} Z_{aa} & -Z_{bb} \\ Z_{aa} + Z_{cc} & Z_{cc} \end{bmatrix} \begin{bmatrix} i^a \\ i^b \end{bmatrix} \qquad 6.23$$

To use 6.23, no load voltages between terminals A-B and A-A' must be measured. But it is possible to measure no load voltages between A-A' and B-B'. Then voltages must be transformed into measurable no load voltages between A-A' and B-B'.

$$\begin{array}{lll} \text{No load voltage between A-A'} & e_{AA'} = e_a + e_c & \\ \text{" \qquad\qquad\qquad B-B'} & e_{BB'} = e_b + e_c & \end{array} \qquad 6.24$$

The relations between old and new emf's are

$$\begin{bmatrix} e_{AA'} \\ e_{BB'} \end{bmatrix} = \begin{bmatrix} e_a + e_c \\ e_b + e_c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e_a - e_b \\ e_a + e_c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e_{AB} \\ e_{AA'} \end{bmatrix} \qquad 6.25$$

So transformation matrix [B] is

$$[B] = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \qquad 6.26$$

Then if assumed [A] = [U],

$$[Z'] = [B]\,[Z] = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} Z_{aa} & -Z_{bb} \\ Z_{aa} + Z_{cc} & Z_{cc} \end{bmatrix}$$

$$= \begin{bmatrix} Z_{aa} + Z_{cc} & Z_{cc} \\ -Z_{cc} & Z_{bb} + Z_{cc} \end{bmatrix} \qquad 6.27$$

So we get new circuit equation form 6.25 and 6.27.

$$\begin{bmatrix} e_a + e_c \\ e_b + e_c \end{bmatrix} = \begin{bmatrix} Z_{aa} + Z_{cc} & Z_{cc} \\ -Z_{cc} & Z_{bb} + Z_{cc} \end{bmatrix} \begin{bmatrix} i^a \\ i^b \end{bmatrix} \qquad 6.28$$

This equation contains only measurable emf's, so we can evaluate currents by solving 6.28.

This is the case where [A] is a unit matrix and [B] is trnasformation matrix between old and new emf's.

Sometimes we find that a given circuit equation is not adequate, because some of the emf's used in the equation is not accessible and cannot be measured, or because we want to use other independent currents than those used in the given circuit equation. In such a case we must transform both emf's and currents used into other independent emf's and currents, which are suitable to our purposes. Then old and new variables must be related in some way. These relations can be expressed by

$$[i] = [A][i'] \qquad 6.29$$

$$[e'] = [B][e] \qquad 6.30$$

Generally $[A]$ and $[B]$ are not related by the relation $[B] = [A]_t$ as is the case with congruent transformations. Let's show one such example.

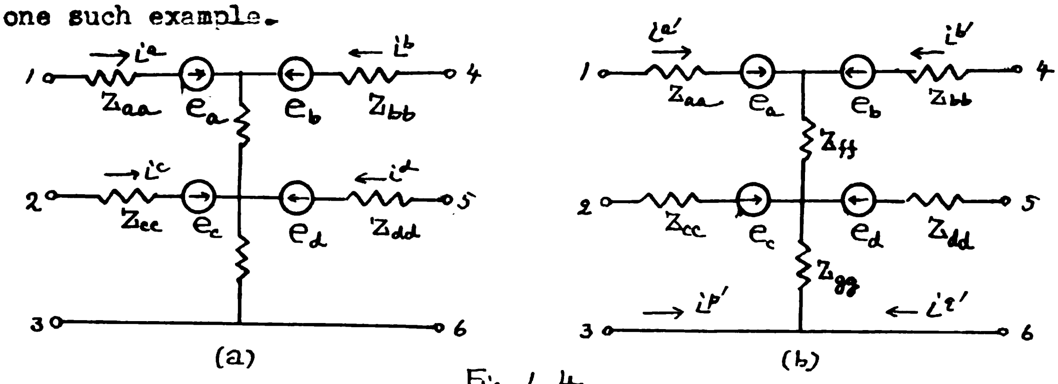


Fig. 6.4

For the network shown in fig. 6.4 (a) a circuit equation is set up as below.

$$\begin{bmatrix} e_a - e_b \\ e_c - e_d \\ e_a \\ e_d \end{bmatrix} = \begin{bmatrix} Z_{aa} & -Z_{bb} & 0 & 0 \\ 0 & 0 & Z_{cc} & -Z_{dd} \\ Z_{aa} + Z_{ff} + Z_{gg} & Z_{ff} + Z_{gg} & Z_{gg} & Z_{gg} \\ Z_{gg} & Z_{gg} & Z_{gg} & Z_{dd} + Z_{gg} \end{bmatrix} \begin{bmatrix} i^a \\ i^b \\ i^c \\ i^d \end{bmatrix} \qquad 6.31$$

But four currents, which are indicated in fig 6.4 (b), are wanted. And four no load voltages, which are used in 6.31, cannot be measured, but four no-load voltages between 1-2, 4-5, 2-3, and 5-6 can be measured. These four currents, which are wanted, and four measurable no-load voltage are assumed as new reference variables. Then relations between old and new currents are,

$$i^a = i^{a'}$$

$$i^b = i^{b'}$$

$$i^c = -i^{a'} - i^{p'}$$

$$i^d = -i^{b'} - i^{q'}$$

So the transformation matrix [A] for currents is,

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \qquad 6.32$$

Measurable four no-load voltages are,

$$\begin{aligned} e_{1-2} &= e_a - e_c \\ e_{4-5} &= e_b - e_d \\ e_{2-3} &= e_c \\ e_{5-6} &= e_d \end{aligned} \qquad 6.33$$

The relation between new and old voltages is

$$\begin{bmatrix} e_a - e_c \\ e_b - e_d \\ e_c \\ e_d \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_a - e_b \\ e_c - e_d \\ e_a \\ e_d \end{bmatrix} \qquad 6.34$$

So the transformation matrix [B] for voltages is,

$$[B] = \begin{bmatrix} 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 6.35$$

Impedance matrix for new variables is given by [B] [Z] [A], that is,

$$[Z]\,[A] = \begin{bmatrix} Z_{aa} & -Z_{bb} & 0 & 0 \\ 0 & 0 & Z_{cc} & -Z_{dd} \\ Z_{aa} + Z_{ff} + Z_{gg} & Z_{ff} + Z_{gg} & Z_{gg} & Z_{gg} \\ Z_{gg} & Z_{gg} & Z_{gg} & Z_{dd} + Z_{gg} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$
= \begin{bmatrix} Z_{aa} & -Z_{bb} & 0 & 0 \\ -Z_{cc} & Z_{dd} & -Z_{cc} & Z_{dd} \\ Z_{aa} + Z_{ff} & Z_{ff} & -Z_{gg} & -Z_{gg} \\ 0 & -Z_{dd} & -Z_{gg} & -(Z_{dd} + Z_{gg}) \end{bmatrix}
$$

$$
[B]\,[Z]\,[A] = \begin{bmatrix} Z_{aa} + Z_{cc} + Z_{ff} & Z_{ff} & Z_{cc} & 0 \\ Z_{ff} & Z_{bb} + Z_{dd} + Z_{ff} & 0 & Z_{dd} \\ -Z_{cc} & 0 & -(Z_{cc} + Z_{gg}) & -Z_{gg} \\ 0 & -Z_{dd} & -Z_{gg} & -(Z_{dd} + Z_{gg}) \end{bmatrix} \tag{6.36}
$$

So the new circuit equation, which is suitable to our purposes, is given from 6.34 and 6.36 as below.

$$
\begin{bmatrix} e_a - e_c \\ e_b - e_d \\ e_c \\ e_d \end{bmatrix} = \begin{bmatrix} Z_{aa} + Z_{cc} + Z_{ff} & Z_{ff} & Z_{cc} & 0 \\ Z_{ff} & Z_{bb} + Z_{dd} + Z_{ff} & 0 & Z_{dd} \\ -Z_{cc} & 0 & -(Z_{cc} + Z_{gg}) & -Z_{gg} \\ 0 & -Z_{dd} & -Z_{gg} & -(Z_{dd} + Z_{gg}) \end{bmatrix} \begin{bmatrix} i^{a'} \\ i^{b'} \\ i^{p'} \\ i^{t'} \end{bmatrix} \tag{6.37}
$$

We can check easily validity of 6.37 by inspecting fig. 6.4 and 6.37.

All examples in this article are the case where the condition $[A]=[C]$ and $[B]=[C]_t$ are not satisfied.

Here it should be emphasized that both [A] and [B] are non-singular and give numerical correspondences between old and new variables. Sometimes [A] or [B] is a unit matrix, which is apparently non-singular. Cases, where [A] or [B] is singular, are treated in the following article.

(2) Cases where transformation matrix [C] is singular.

In Chapter IV we explained that transformations, which changes number of impedance branches, are generally accompanied by singular transformation matrices. And singular matrices relate old and new variables only geometrically but not numerically. In Chapter III we expalained that power is not invariant even when $[C]_t[Z][C]$ transformationis used, if [C]is singular. Let's show examples.

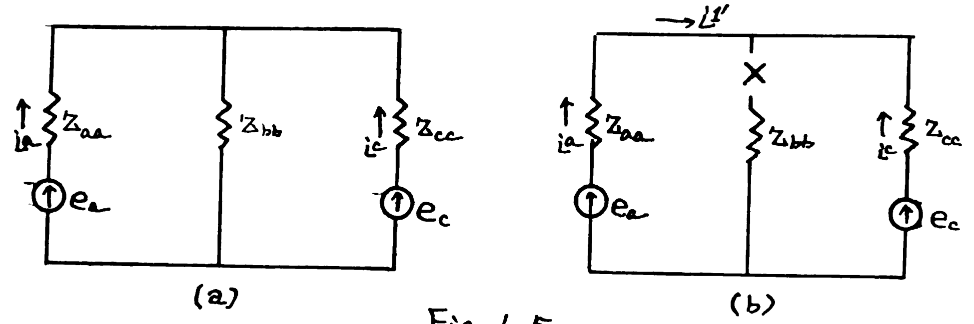


Fig. 6.5

The circuit equation for figure 6.5 (a), which is the same as figure 4.3, is

$$\begin{bmatrix} e_a \\ e_c \end{bmatrix} = \begin{bmatrix} Z_{aa} + Z_{bb} & Z_{bb} \\ Z_{bb} & Z_{bb} + Z_{cc} \end{bmatrix} \begin{bmatrix} i^a \\ i^c \end{bmatrix} \qquad 6.38$$

In figure 6.5 (b) branch of Z_{bb} is broken, and one new current $i^{1'}$ is enough to describe the circuit behaviors. Then the correspondence between old and new currents is

$$\begin{aligned} i^a &\rightarrow i^{1'} \\ i &\rightarrow -i^{1'} \end{aligned} \qquad 6.39$$

This is a geometrical correspondence but not numerical. So it is numerically wrong when we write $[i] = [C]\,[i']$ as below.

$$\begin{bmatrix} i^a \\ i^c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} i^{1'} \qquad 6.40$$

So it seems wrong to use the C Z C method for this case, because we use numerically wrong equation $[i] = [C]\,[i']$. But it was already shown in Chapter IV that the $[C]_t[Z]\,[C]$ method is valid even for this case. Let's try here once more.

$$[Z]\,[C] = \begin{bmatrix} Z_{aa} + Z_{bb} & Z_{bb} \\ Z_{bb} & Z_{bb} + Z_{cc} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} Z_{aa} \\ -Z_{cc} \end{bmatrix} \qquad 6.41$$

$$[Z'] = [C]_t[Z][C] = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} Z_{aa} \\ -Z_{cc} \end{bmatrix} = \begin{bmatrix} Z_{aa} + Z_{cc} \end{bmatrix} \qquad 6.42$$

$$[e'] = [C]_t[e] = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} e_a \\ e_c \end{bmatrix} = [e_a - e_c] \qquad 6.43$$

So we get as the new circuit equation,

$$e_a - e_c = (Z_{aa} + Z_{cc})\, i^{1'} \qquad 6.44$$

This is evidently correct for the circuit of figure 6.5 (b).

Next let's try the case where [A] = [C] and [B] is a unit matrix. From 6.41 we get the result at once. That is,

$$\begin{bmatrix} e_a \\ e_c \end{bmatrix} = \begin{bmatrix} Z_{aa} \\ -Z_{cc} \end{bmatrix} [i^{1'}] \qquad 6.45$$

Expanded we get,

$$\begin{aligned} e_a &= Z_{aa}\, i^{1'} \\ e_c &= -Z_{cc}\, i^{1'} \end{aligned} \qquad 6.46$$

This is evidently incorrect for the circuit of figure 6.5 (b).

Next let's try the case where $[B] = [C]_t$ and [A] is a unit matrix.

$$[C]_t[Z] = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} Z_{aa} + Z_{bb} & Z_{bb} \\ Z_{bb} & Z_{bb} + Z_{cc} \end{bmatrix} = [Z_{aa} \quad - Z_{cc}] \qquad 6.47$$

From 6.43 and 6.47 we get,

$$[e_a - e_c] = [Z_{aa} - Z_{cc}]\begin{bmatrix} i^a \\ i^c \end{bmatrix} \qquad 6.48$$

That is,

$$e_a - e_c = Z_{aa}\, i^a - Z_{cc}\, i^c \qquad 6.49$$

This result is not wrong, but it is not adequate to solve the circuit of figure 6.5 (b), because it contained two currents for the one-mech circuit.

Let's examine the case where a 3-meshes circuit is transformed into a 2-meshes circuit as shown in figure 6.6.

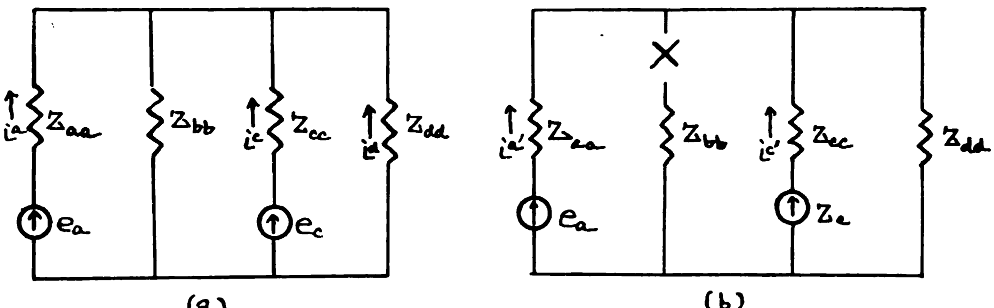


Fig. 6.6

The circuit equation for figure 6.6 (a) is

$$\begin{bmatrix} e_a \\ e_c \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{aa} + Z_{bb} & Z_{bb} & Z_{bb} \\ Z_{bb} & Z_{bb} + Z_{cc} & Z_{bb} \\ Z_{bb} & Z_{bb} & Z_{bb} + Z_{dd} \end{bmatrix} \begin{bmatrix} i^a \\ i^c \\ i^d \end{bmatrix} \qquad 6.50$$

In figure 6.6 (b) branch of Z_{bb} is broken and new currents are assumed as shown in the figure. Then geometrical corresponcences between old and new currents are given by

$$\begin{bmatrix} i^a \\ i^c \\ i^d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} i^{a'} \\ i^{c'} \end{bmatrix} \qquad 6.51$$

Equation 6.51 does not correspond numerically. From 6.51 we get a singular transformation matrix [C] .

$$[C] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad 6.52$$

Then

$$[Z]\,[C] = \begin{bmatrix} Z_{aa} + Z_{bb} & Z_{bb} & Z_{bb} \\ Z_{bb} & Z_{bb} + Z_{cc} & Z_{bb} \\ Z_{bb} & Z_{bb} & Z_{bb} + Z_{dd} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} Z_{aa} & 0 \\ 0 & Z_{cc} \\ -Z_{dd} & -Z_{dd} \end{bmatrix} \qquad 6.53$$

$$[Z'] = [C]_t\,[Z]\,[C] = \begin{bmatrix} Z_{aa} + Z_{dd} & Z_{dd} \\ Z_{dd} & Z_{cc} + Z_{dd} \end{bmatrix} \qquad 6.54$$

$$[e'] = [C]_t[e] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} e_a \\ e_c \\ 0 \end{bmatrix} = \begin{bmatrix} e_a \\ e_c \end{bmatrix} \qquad 6.55$$

So we get as $[e'] = [Z'][i']$

$$\begin{bmatrix} e_a \\ e_c \end{bmatrix} = \begin{bmatrix} Z_{aa} + Z_{dd} & Z_{dd} \\ Z_{dd} & Z_{cc} + Z_{dd} \end{bmatrix} \begin{bmatrix} i^{a'} \\ i^{c'} \end{bmatrix} \qquad 6.56$$

This result is evidently correct for the circuit of figure 6.6 (b).

Next let us examine the case where $[A] = [C]$ and $[B]$ is a unit matrix. From 6.53 we at once as the new circuit equation,

$$\begin{bmatrix} e_a \\ e_c \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{aa} & 0 \\ 0 & Z_{cc} \\ -Z_{dd} & -Z_{dd} \end{bmatrix} \begin{bmatrix} i^{a'} \\ i^{c'} \end{bmatrix} \qquad 6.57$$

This result is evidently incorrect for the circuit of figure 6.6 (b).

Let us examine the case where $[B] = [C]_t$ and $[A]$ is a unit matrix.

$$[C]_t[Z] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} Z_{aa} + Z_{bb} & Z_{bb} & Z_{bb} \\ Z_{bb} & Z_{cc} + Z_{bb} & Z_{bb} \\ Z_{bb} & Z_{bb} & Z_{bb} + Z_{dd} \end{bmatrix}$$

$$= \begin{bmatrix} Z_{aa} & 0 & -Z_{dd} \\ 0 & Z_{cc} & -Z_{dd} \end{bmatrix} \qquad 6.58$$

From 6.55 and 6.58 we get as the new circuit equation,

$$\begin{bmatrix} e_a \\ e_c \end{bmatrix} = \begin{bmatrix} Z_{aa} & 0 & -Z_{dd} \\ 0 & Z_{cc} & -Z_{dd} \end{bmatrix} \begin{bmatrix} i^{a'} \\ i^{c'} \\ i^{d'} \end{bmatrix} \qquad 6.59$$

That is,

$$\begin{aligned} e_a &= Z_{aa}\, i^{a'} - Z_{dd}\, i^{d'} \\ e_c &= Z_{cc}\, i^{c'} - Z_{dd}\, i^{d'} \end{aligned} \qquad 6.60$$

This result is not wrong, but is not adequate to solve the circuit of figure 6.6 (b), because it contains more numbers of reference currents than numbers of independent meshes.

From two examples above we may conclude for the case of singular transformation matrix [C] as following.

(i) The method of " [e'] = $[C]_t$ [e] , [i] = [C] [i'] and [Z'] = $[C]_t$[Z] [C] " is valid.

(ii) The method of "$[e'] = [e]$, $[i] = [C][i']$ and $[Z'] = [Z][C]$" is wrong.

(iii) The method of "$[e'] = [C]_t[e]$, $[i'] = [i]$ and $[Z'] = [C]_t[Z]$ " is not wrong, but is not adequate for solving the problems, because it contains more currents than numbers of independent meshes.

It seems rather strange to reach the right results of (i) and (iii), even when we use the numerically wrong relation $[i] = [C][i']$. Let us consider the reason for it.

In the method (iii) we use $[e'] = [C]_t[e]$, but do not use $[i] = [C][i']$. And it is easily understood from the two examples above that $[e'] = [C]_t[e]$ is numerically correct, while $[i] = [C][i']$ is numerically wrong. In figure 6.5 (b) the new mesh emf' e' is

$$e' = e_a - e_c \qquad 6.61$$

, while

$$[C]_t[e] = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} e_a \\ e_c \end{bmatrix} = [e_a - e_c] \qquad 6.62$$

From 6.61 and 6.62 we see that $[e'] = [C]_t[e]$ is numerically correct. Another thing, which should be emphasized, is the fact that all emf's, which appear in matrix equations, are mesh-emf's. Mesh-emf's mean that they are emf's which act around closed meshes. When we choose new emf's for transformed circuits, we must choose mesh-emf's. The transformation $[e'] = [C]_t [e]$ gives automatically mesh-emf's, as new emf's $[e']$, because $[B] = [C]_t$ is a transformation matrix between old and new mesh-currents *. And even when

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* Here it should be stated that branch-currents can be considered as mesh-currents, so long as these branch-currents are independent. (8)

$[i] = [C][i']$ is correspondent only geometrically, $[e'] = [C]_t [e]$ is correspondent geometrically and numerically. This is the reason why method (iii) gives correct results, while method (ii) gives wrong results.

When we applied method (ii) to figure 6.5, we got 6.45, which is wrong. In 6.45 emf's are not mesh-emf's, but branch-emf's or open emf's. So it is impossible for them to give correct circuit equations, which are actually the Second Kirchhoff's law.

Pre-multiplication of [Z][C] by $[C]_t$ is equivalent to transforming of open emf's into mesh-emfs, and it sets up the Second Kirchhoff's Law correctly. So method (i) gives correct results.

It is not necessary that [A] = [C] and $[B] = [C]_t$ are satisfied, to reach correct results. For instance, in method (iii) $[B] = [C]_t$ and [A] is a unit matrix. We can use any transformation matrices [A] and [B] , so long as [i] = [A] [i'] gives geometrical correspondences between old and new independent currents and [e'] = [B] [e] gives geometrical and numerical correspondences between new and old independent mesh-emf's. Let us apply the case where $[B] \neq [A]_t$ to figure 6.6. We assume that new currents are the same as shown in figure 6.6 (b), but new emf's are those acting meshes of $Z_{aa} - Z_{cc}$ and $Z_{aa} - Z_{dd}$. Then [e'] = [B] [e] is

$$\begin{bmatrix} e_a - e_c \\ e_a - 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} e_a \\ e_c \\ 0 \end{bmatrix} \qquad 6.63$$

[Z][A] = [Z][C] is given by 6.53. Then

$$[Z'] = [B][Z][A] = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} Z_{aa} & 0 \\ 0 & Z_{cc} \\ -Z_{dd} & -Z_{dd} \end{bmatrix}$$

$$= \begin{bmatrix} Z_{aa} & -Z_{cc} \\ Z_{aa} + Z_{dd} & Z_{dd} \end{bmatrix} \qquad 6.64$$

From 6.63 and 6.64 we get as the new circuit equation,

$$\begin{bmatrix} e_a - e_c \\ e_a \end{bmatrix} = \begin{bmatrix} Z_{aa} & -Z_{cc} \\ Z_{aa} + Z_{dd} & Z_{dd} \end{bmatrix} \begin{bmatrix} i^{a'} \\ i^{c'} \end{bmatrix} \qquad 6.65$$

It is easily seen from figure 6.6 (b) that 6.65 is correct. Here it shold be mentioned that we must be careful in setting up $[e'] = [B][e]$ as shown in 6.63. That is, when there is no emf in some branches, some simbol (here we use 0) must be substituted to get the right [B].

Now we can conclude about transformations by singular matrices as folling.

(i) The method of " $[e'] = [C]_t[e]$, $[i] = [C][i']$ and $[Z'] = [C]_t[Z][C]$ " can be used correctly.

(ii) The method of " $[e'] = [B][e]$, $[i] = [A][i']$ and $[Z'] = [B][Z][A]$ " can be used correctly, so long as all emf's and currents assumed are independent mesh-emf's and independent mesh-currents.*

(iii) The method of " $[i] = [C][i']$ and $[Z'] = [Z][C]$"
is not correct.

(iv) In all cases powers are not invariant.

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* Branch-currents can be considered as mesh-currents, so long as these branch-currents are independent. (8)

Chapter VII Transformations of Invariant Input-Impedance.

So far "Invariance of Power" has meant that not only input-power at terminals but also all powers consumed or stored in all impedances are invariant. But at some practically very important applications of circuit transformations only input-power at terminals and consequently input-impedance are kept invariant. Transformations between equivalent circuits are the case. So far transformation matrices contain only integers as their elements, mostly ± 1 or 0. But transformation matrices of invariant input-impedance can have non-integer elements as shown later.

Here it is not attempted to discuss whole aspects of transformations of invariant input-impedance, but one interesting example is to be explained; that is a new method of reduction of networks to their Foster's Forms by means of matrices.

As Guillemin explained very nicely in his book, any dissipationless network can be reduced to four equivalent canonic forms, which have least number of elements. Two of them are Foster's forms and other two are Cauer's forms. Reductions to Cauer's forms are easier than to Foster's forms, because a **modular** determinant (or its minor) must be solved, to reduce to Foster's forms, but it is not necessary for Cauer's forms.

Here let us try other approach to Foster's forms.

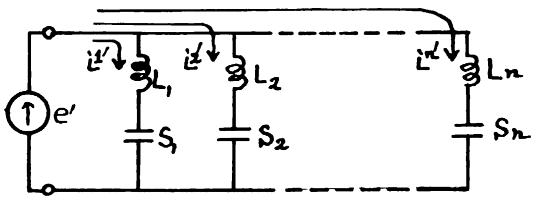


Fig. 7. 1

Figre 7.1 shows one of Foster's forms. Independent currents are assumed as shown in the figure. Then the circuit equation is,

$$\begin{bmatrix} e' \\ e' \\ \cdot \\ e' \end{bmatrix} = \begin{bmatrix} L_1\lambda + \frac{S_1}{\lambda} & 0 \cdot \cdot \cdot \cdot \cdot & 0 \\ 0 & L_2\lambda + \frac{S_2}{\lambda} \cdot \cdot \cdot & 0 \\ \cdot \cdot \cdot \cdot \cdot \cdot & \cdot \cdot \cdot \cdot \cdot \cdot & \cdot \cdot \\ 0 & 0 \cdot \cdot & L_n\lambda + \frac{S_n}{\lambda} \end{bmatrix} \begin{bmatrix} i^{1'} \\ i^{2'} \\ \cdot \\ i^{n'} \end{bmatrix} \qquad 7.1$$

, where $\lambda = \frac{d}{dt}$, $\frac{1}{\lambda} = \int dt$.

This equation has a diagonal or normal forms. So the inductance matrix [L] is,

$$[L] = \begin{bmatrix} L_1 & 0 & 0 \cdots\cdots 0 \\ 0 & L_2 & 0 \cdots\cdots 0 \\ 0 & 0 & L_3 \cdots\cdots 0 \\ \cdots & \cdots & \cdots\cdots\cdots \\ \cdots & \cdots & \cdots\cdots\cdots \\ 0 & 0 & 0 \cdots\cdots L_n \end{bmatrix} \qquad 7.2$$

The susceptance matrix [S] is,

$$[S] = \begin{bmatrix} S_1 & 0 & 0 \cdots\cdots 0 \\ 0 & S_2 & 0 \cdots\cdots 0 \\ 0 & 0 & S_3 \cdots\cdots 0 \\ \cdots & \cdots & \cdots\cdots\cdots \\ 0 & 0 & 0 \cdots\cdots S_n \end{bmatrix} \qquad 7.3$$

7.2 and 7.3 have diagonal or normal forms.

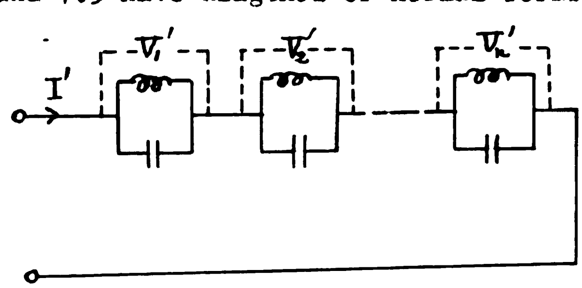


Fig. 7.2

Figure 7.2 shows the other form of Foster's equivalent circuits. Independent junction voltages are assumed as shown in the figure. Then the circuit equation is,

$$\begin{bmatrix} I' \\ I' \\ \cdot \\ \cdot \\ I' \end{bmatrix} = \begin{bmatrix} C_1\lambda + \frac{\Gamma_1}{\lambda} & 0 & \cdots & 0 \\ 0 & C_2\lambda + \frac{\Gamma_2}{\lambda} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & C_n\lambda + \frac{\Gamma_n}{\lambda} \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \\ \cdot \\ V_n' \end{bmatrix} \qquad 7.4$$

This circuit equation has diagonal or normal admittance matrix. Then the capacitance matrix [C] is,

$$[C] = \begin{bmatrix} C_1 & 0 & 0 & \cdots & 0 \\ 0 & C_2 & 0 & \cdots & 0 \\ 0 & 0 & C_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & C_n \end{bmatrix} \qquad 7.5$$

The reciprocal inductance matrix $[\Gamma]$ is,

$$[\Gamma] = \begin{bmatrix} \Gamma_1 & 0 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & 0 & \cdots & 0 \\ 0 & 0 & \Gamma_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \Gamma_n \end{bmatrix} \qquad 7.6$$

7.5 and 7.6 have diagonal or normal forms, too.

When a dissipationless network is given, we can easily write down its matrices of [L] and [S] by means of mesh method, and its matrices of [C] and [Γ] by means of node method. And if we can diagonalize these matrices in such a way as transformed circuit equations have the forms of 7.1 or 7.4, then we can write down its Foster's equivalent circuit as shown above.

Let us assume that a given two-terminal network has following [L] and [S] .

$$[L] = \begin{bmatrix} L_{11} & L_{12} \cdots & L_{1n} \\ L_{21} & L_{22} \cdots & L_{2n} \\ \cdots & \cdots & \cdots \\ L_{n1} & L_{n2} \cdots & L_{nn} \end{bmatrix} \qquad 7.7$$

$$[S] = \begin{bmatrix} S_{11} & S_{12} \cdots & S_{1n} \\ S_{21} & S_{22} \cdots & S_{2n} \\ \cdots & \cdots & \cdots \\ S_{n1} & S_{n2} \cdots & S_{nn} \end{bmatrix} \qquad 7.8$$

Then the next step is to find such a transformation matrix [C] as

$[C]_t [L] [C]$ gives a diagonal form.

$[C]_t [S] [C]$ gives a diagonal form.

Now we can see that the problem is simultaneous reduction of two matrices [L] and [S] to their diagonal forms. Guillemin treats very nicely simultaneous diagonalization of two matrices in his book.(10) But we can not apply to this case directly what is explained in his book. Let us explain necessary procedures.

Using [L] and [S] of 7.7 and 7.8,

$$\left|\lambda[L] + [S]\right| = 0 \qquad 7.9$$

has generally n roots of λ.

7.9 can be expanded as below.

$$\begin{vmatrix} \lambda L_{11} + S_{11} & \lambda L_{12} + S_{12} & \cdots & \lambda L_{1n} + S_{1n} \\ \lambda L_{21} + S_{21} & \lambda L_{22} + S_{22} & \cdots & \lambda L_{2n} + S_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda L_{n1} + S_{n1} & \lambda L_{n2} + S_{n2} & \cdots & \lambda L_{nn} + S_{nn} \end{vmatrix} = 0 \qquad 7.10$$

And 7.10 is the determinant of n set of homogeneous equations below.

$$\left.\begin{aligned} (\lambda L_{11} + S_{11})x_1 + (\lambda L_{12} + S_{12})x_2 + \cdots + (\lambda L_{1n} + S_{1n})x_n &= 0 \\ (\lambda L_{21} + S_{21})x_1 + (\lambda L_{22} + S_{22})x_2 + \cdots + (\lambda L_{2n} + S_{2n})x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (\lambda L_{n1} + S_{n1})x_1 + (\lambda L_{n2} + S_{n2})x_2 + \cdots + (\lambda L_{nn} + S_{nn})x_n &= 0 \end{aligned}\right\} \qquad 7.11$$

7.11 has solutions when 7.9 and 7.10 are satisfied. So 7.11 has n sets of solutions for n roots of λ, which satisfy 7.9 and 7.10. But these n sets of solutions for 7.11 are not unique, but only proportionalities among each components x_1, x_2, $x_3 \cdots\cdot$ are determined. If cofactors of the determinant 7.10 for the sth root of λ's are denoted by $\overset{s}{K}_{ik}$, then one set of solution is given by

$$\ell_{ks} = \frac{\overset{s}{K}_{ik}}{\sqrt{(\overset{s}{K}_{i1})^2 + (K_{i2})^2 + \cdot\ \cdot\ \cdot + (K_{in})^2}} \qquad 7.12$$

(for k=1,2, n)

, where the index i is arbitrary but must, of course, be the same for all sets of ℓ's.

Then

$$\overset{s}{x}_k = p_k \ell_{ks} \text{ (for k = 1, 2, n)} \qquad 7.13$$

satisfy 7.11, where p_s is a arbitrary constant. 7.11 can be expressed by using matrix as below.

$$(\lambda[L] + [S])\cdot[x] = 0 \qquad 7.14$$

, where $[x] = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$

Substituting 7.13 into 7.14, we get

$$\lambda[L]\{(p_s \ell_{ls})\} = -[S]\{(p_s \ell_{ls})\} \qquad 7.15$$

There are n equations of this form for n λ's. And these can be combined into one matrix equation by defining a new matrix, which is called a modal matrix. The modal matrix for this case is,

$$[\mathcal{L}] = \begin{bmatrix} \ell_{11} & \ell_{12} \cdot \cdot \cdot \cdot & \ell_{1n} \\ \ell_{21} & \ell_{22} \cdot \cdot \cdot \cdot & \ell_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot & \cdot \\ \ell_{n1} & \ell_{n2} \cdot \cdot \cdot \cdot & \ell_{nn} \end{bmatrix} \qquad 7.16$$

,where the first column elements are given by 7.12 for s = 1. Then n equations of the form of 7.15 can be combined into

$$[L][\mathcal{L}][p][\Lambda] = -[S][\mathcal{L}][p] \qquad 7.17$$

, where $[\Lambda]$ is the diagonal matrix with n latent roots of λ as diagonal elements, and [p] is the diagonal matrix with arbitrary constnat p_1, p_2, ... p_n as diagonal elements. 7.17 is pre-multiplied on both sides by transpose of $[\mathcal{L}][p]$, then

$$[p]_t[\mathcal{L}]_t[L][\mathcal{L}][p][\Lambda] = -[p]_t[\mathcal{L}]_t[S][\mathcal{L}][p] \qquad 7.18$$

In 7.18 $[p]_t[\mathcal{L}]_t[L][\mathcal{L}][p]$ and $[p]_t[\mathcal{L}]_t[S][\mathcal{L}][p]$ are symmetrical, because [L] and [S] are symmetrical. But the left-hand side of 7.18 is not symmetrical due to post-multiplication by $[\Lambda]$, which is a diagonal matrix, if $[p]_t[\mathcal{X}]_t[L][\mathcal{X}][p]$ is not a diagonal matrix. So only one possibility is that both $[p]_t[\mathcal{X}]_t[L][\mathcal{X}][p]$ and $[p]_t[\mathcal{X}]_t[S][\mathcal{X}][p]$ must be diagonal matrix (11), in order for 7.18 to be correct. That is

$$[p]_t[\mathcal{X}]_t[L][\mathcal{X}][p] = [D_1] \qquad 7.19$$

$$[p]_t[\mathcal{X}]_t[S][\mathcal{X}][p] = [D_2] \qquad 7.20$$

, where $[D_1]$ and $[D_2]$ are diagonal matrices.

From 7.19 and 7.20 we know that

$$[C] = [\mathcal{X}][p] \qquad 7.21$$

is the matrix which diagonalizes both $[L]$ and $[S]$ simultaneously. But $[C]$ is not unique, because it contains the diagonal matrix $[p]$, which has arbitrary diagonal elements.

So the next problem is to choose from 7.21 the proper matrix $[C]$, which keeps the input-impedance invariable. This proper diagonalizing matrix be designated by $[F]$. Then the relation between old current $[i]$ and new current $[i']$ is given by

$$[i] = [F][i'] \qquad 7.22$$

It is easily seen from figure 7.1 that the following relation must exist between old and new currents for invariance of input-impedance. That is

$$i = i^{1'} + i^{2'} + \cdots + i^{n'} \qquad 7.23$$

,where i is the input-current of the original network, which is counted as the first mesh-current.

From 7.22 and 7.23 we can see that [F] must have the form as below.

$$[F] = \begin{bmatrix} 1 & 1 \cdots\cdots & 1 \\ f_{21} & f_{22} \cdots\cdots & f_{2n} \\ \cdots & \cdots\cdots & \cdots \\ f_{n1} & f_{n2} \cdots\cdots & f_{nn} \end{bmatrix} \qquad 7.24$$

For [C] given by 7.21 to be identical to [F] given by 7.24, [p] must be the diagonal matrix as below.

$$[p] = \begin{bmatrix} 1/\ell_{11} & 0 \cdots\cdots & 0 \\ 0 & 1/\ell_{12} \cdots\cdots & 0 \\ \cdots & \cdots\cdots & \cdots \\ 0 & 0 & 1/\ell_{1n} \end{bmatrix} \qquad 7.25$$

Then

$$[F] = [\mathcal{L}]\,[p]$$

$$= \begin{bmatrix} \ell_{11} & \ell_{12} & \cdots & \ell_{1n} \\ \ell_{21} & \ell_{22} & \cdots & \ell_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix} \begin{bmatrix} 1/\ell_{11} & 0 & \cdots & 0 \\ 0 & 1/\ell_{12} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1/\ell_{1n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \ell_{21}/\ell_{11} & \ell_{22}/\ell_{12} & \cdots & \ell_{2n}/\ell_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \ell_{n1}/\ell_{11} & \ell_{n2}/\ell_{12} & \cdots & \ell_{nn}/\ell_{1n} \end{bmatrix} \qquad 7.26$$

From 7.12 we get the following relation.

$$\ell_{ms}/\ell_{ns} = \overset{s}{K}_{im} / \overset{s}{K}_{in} \qquad 7.27$$

, where $\overset{s}{K}_{i\ell}$ is cofactor of the determinant 7.10 for the sth root of n λ's. So [F] can be expressed as below.

$$[F] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \overset{1}{K}_{i2}/\overset{1}{K}_{i1} & \overset{2}{K}_{i2}/\overset{2}{K}_{i1} & \cdots & \overset{n}{K}_{i2}/\overset{n}{K}_{i1} \\ \cdots & \cdots & \cdots & \cdots \\ \overset{1}{K}_{in}/\overset{1}{K}_{i1} & \overset{2}{K}_{in}/\overset{2}{K}_{i1} & & \overset{n}{K}_{in}/\overset{n}{K}_{i1} \end{bmatrix} \qquad 7.28$$

7.26 or 7.28 gives the transformation matrix [F] , which reduces [L] and [S] of a given network to their diagonal forms and keeps the input-impedance invariant.

Here it should be explained that the new voltage matrix obtained after the transformation by [F] has the form of the left-hand side of 7.1. The given network is a two-terminal passive network, so its voltage matrix has the form as below.

$$[e]_t = \begin{bmatrix} e & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad 7.29$$

Then the new voltage [e'] is

$$[e'] = [F]_t[e] = \begin{bmatrix} e \\ e \\ \cdot \\ \cdot \\ \cdot \\ e \end{bmatrix} \qquad 7.30$$

So we can see that the transformed circuit equation has the form of 7.1, which is the circuit equation of one of Foster's equivalent circuits. Now we can say that we have reduced the given network to one of its Foster's equivalent circuits. 7.28 is the better formula of the transformation matrix than 7.26, because calculation of 7.12 is not necessary for 7.28.

Let us derive here some very useful relations. Putting 7.19 and 7.20 into 7.18, we get

$$[D_1][\Lambda] = -[D_2] \qquad 7.31$$

Three matrices contained in 7.31 are diagonal. So we get from 7.31,

$$d'_{ss}\lambda_s = -d^2_{ss} \qquad 7.32$$

, where d'_{ss} and d^2_{ss} are diagonal elements of $[D_1]$ and $[D_2]$ and λ_s is sth latent root of equation 7.10.
Using 7.31 or 7.32 we can derive $[D_1]$ or $[D_2]$ from each other.

It is easily seen from 7.32 that λ must be negative or zero, because d'_{ss} and d^2_{ss} must be both positive*, for the equivalent network to be physically realizable.
That λ's are negative or zero was anticipated from the fact that $[L]$ and $[S]$ are both positive definite.(12)

We started from equation 7.9, where λ is attached to $[L]$. If $[L]$ is singular, some of λ's are infinity. But λ's must be finite. In this case λ must be attached to $[S]$ instead of $[L]$; that is

$$\left|[L] + \lambda[S]\right| = 0 \qquad 7.33$$

At least either of $[L]$ or $[S]$ is non-singular, because these are matrices for solving electric networks.
According to which of $[L]$ or $[S]$ is non-singular, we choose 7.9 or 7.33. Otherwise the procedures remain

* Including zero.

the same as above.

All procedures of reducing a given network to its Foster's equivalent form shown in figure 7.2 are the same as explained so far in this chapter. But in this case the original matrix equation of a given network must be written by the node method, using capacitances and reciprocal inductances. And capacitance matrix $[C]$ and reciprocal inductance matrix $[\Gamma]$ are diagonalized simultaneously. That is 7.9 and 7.33 is substituted by

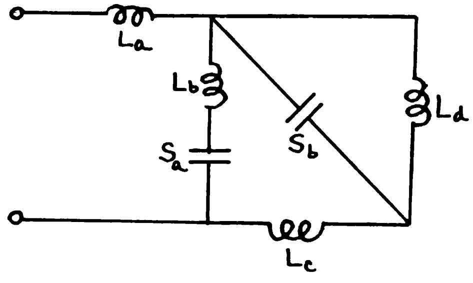
$$\left|\lambda[C] + [\Gamma]\right| = 0 \quad \text{or} \quad \left|[C] + \lambda[\Gamma]\right| = 0 \qquad 7.34$$

, according to which of $[C]$ or $[\Gamma]$ is non-singular. The form of 7.12 remains the same. 7.24 remains the same, because in this case 7.22 is replaced by

$$V = V_1' + V_2' + \cdot \cdot + V_n' \qquad 7.35$$

So we can use the same 7.26 or 7.28 for this case, using n roots of λ for 7.34.

Example.



$L_a = 1$ $S_a = 1$

$L_b = 2$ $S_b = 2$

$L_c = 3$

$L_d = 4$

Fig. 7.3

Let us reduce the network of figure 7.3 into the form of resonant components in parallel.

$$[L] = \begin{bmatrix} L_a + L_b & -L_b & 0 \\ -L_b & L_b + L_c & 0 \\ 0 & 0 & L_d \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad 7.36$$

$$[S] = \begin{bmatrix} S_a & -S_a & 0 \\ -S_a & S_a + S_b & -S_b \\ 0 & -S_b & S_s \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \qquad 7.37$$

$$\left|\lambda[L] + [S]\right| = \begin{vmatrix} 3 & +1 & -2 & -1 & 0 & \\ -2 & -1 & 5 & +3 & -2 & \\ & 0 & -2 & & 4 & +2 \end{vmatrix} = 0 \qquad 7.38$$

Solving 7.38 we get

$$\lambda_1 = 0, \quad \lambda_2 = -1.068, \quad \lambda_3 = -0.341 \qquad 7.39$$

Cofactors of 7.38 for these λ's are

$$\begin{array}{lll} \overset{1}{K}_{11} = 2 & \overset{1}{K}_{12} = 2 & \overset{1}{K}_{13} = 2 \\ \overset{2}{K}_{11} = 1.32 & \overset{2}{K}_{12} = 2.585 & \overset{2}{K}_{13} = -2.272 \\ \overset{3}{K}_{11} = -3.176 & \overset{3}{K}_{12} = 0.202 & \overset{3}{K}_{13} = 0.636 \end{array} \qquad 7.40$$

Substituting these values into 7.28, we get

$$[F] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.96 & -0.0637 \\ 1 & -1.72 & -0.20 \end{bmatrix} \qquad 7.41$$

This is the transformation matrix.

Then

$$[L'] = [F]_t [L] [F] = \begin{bmatrix} 8.0 & 0 & 0 \\ 0 & 26.21 & 0 \\ 0 & 0 & 3.435 \end{bmatrix} \qquad 7.42$$

And

$$[S'] = [F]_t [S] [F] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 28.0 & 0 \\ 0 & 0 & 1.169 \end{bmatrix} \qquad 7.43$$

From 7.42 and 7.43 we can write at once the Foster's equivalent form as shown in figure 7.4.

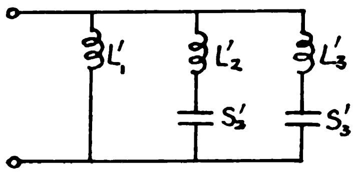


Fig. 7. 4

$L'_1 = 7.0 \qquad S'_1 = 0$

$L'_2 = 26.21 \qquad S'_2 = 28.0$

$L'_3 = 3.435 \qquad S'_3 = 1.169$

Using equations 7.31 or 7.32, 7.43 can be derived from 7.39 and 7.42 as below.

$$[S'] = -\begin{bmatrix} 8.0 & 0 & 0 \\ 0 & 26.21 & 0 \\ 0 & 0 & 3.435 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1.068 & 0 \\ 0 & 0 & -0.341 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 28.0 & 0 \\ 0 & 0 & 1.169 \end{bmatrix} \qquad 7.44$$

This is the same as 7.43.

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