ITERATIVE SOLUTIONS OF PLANE ELASTOSTATIC PROBLEMS

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ABSTRACT

ITERATIVE SOLUTIONS OF PLANE ELASTOSTATIC PROBLEMS

by Chester L. Davis

Three objectives of this thesis are: to compare the efficiency of three iterative methods of solving biharmonic-finite-difference equations, to report on ISOPEP, a system of computer programs designed for the numerical solution of biharmonic plane elastostatic problems, and to demonstrate the utility of this system of programs and the advantages and limitations of numerical solutions by examples.

Three matrix iterative methods considered are point successive overrelaxation, the alternating direction implicit method, and the cyclic Chebyshev semi-iterative method. These are compared in terms of computer storage and time required for the solution of a model problem. Numerical results indicate successive overrelaxation is best unless the mesh is refined so the number of points exceeds 350. Then the alternating direction implicit method is superior.

ISOPEP, a system of FORTRAN II subprograms for the iterative solution of plane elastostatic problems, is explained. Documentation of ISOPEP, including listings of source decks, specifications for input and the output from a sample problem, is provided.

Discrete values of the stress functions and stress components for six ISOPEP problem solutions are provided. These problems include three notched tensile specimens, an infinite plate with a square hole and a semi-infinite plate with a uniformly distributed load along a

portion of the edge. The numerical solutions of the six example problems indicate that a high speed digital computer with a large main memory provides an effective and economical means for the analysis of plane elastostatic problems. Good agreement as shown in the comparison of the numerical and explicit stress solutions for some of these problems. Use of numerical solutions for the investigation of stress concentrations is shown in several of the examples.

ITERATIVE SOLUTIONS OF PLANE ELASTOSTATIC PROBLEMS

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I. INTRODUCTION

Determination of the stresses in a plate under conditions of either plane strain or plane stress is a fundamental problem for the structural engineer. The literature of the classical theory of elasticity includes exact solutions of numerous problems. For an account of the mathematical theory see Timoshenko and Goodier (1951) and Muskhelishvili (1953). However, exact solutions are available only for those problems which have rather simple geometric shapes and boundary constraints. Finite-difference equations have been used as an alternative for the analysis of more complex practical problems. This dissertation compares several iterative finite-difference solution methods with respect to their efficiency in treating plane elastostatic problems.

The finite-difference solution of a boundary-value problem is a two-step procedure. First the partial differential equation and the associated boundary conditions are replaced by difference equations which relate the discrete values of an approximating function at a finite number of points. A regular mesh of lines is superimposed on the domain of the given boundary-value problem. A finite system of linear equations is formed by writing a difference equation for each node of the mesh. The second step is the solution of this system of simultaneous equations. It is often necessary to solve for values at a thousand or more points.

The relaxation technique was employed by Southwell (1946) for the solution of the biharmonic difference equations. Though Southwell and his colleagues solved a number of complex engineering problems,

the relaxation method has not been readily adapted for digital computer solutions.

Iterative methods have been used extensively for computer solutions of systems of equations. These methods make repeated use of simple algorithms which at each application provide an improved approximate solution at one or more of the mesh points. The exact solution is the limit of the sequence of the adjusted point values. Though there are many iterative methods, the three which seemed to offer most promise for the solution of the biharmonic finite-difference equations were: the technique of point successive overrelaxation introduced by Frankel (1950) and Young (1954), the alternating-direction implicit method of Conte and Dames (1958), and the cyclic Chebyshev semi-iterative method proposed by Griffin and Varga (1963). The comparison of these iterative schemes is the first objective of this dissertation.

Using the computer time required to solve a given problem as the measure of the efficiency of an iterative method, the numerical results obtained indicate that for the biharmonic equation there is a critical mesh spacing h^* such that the point successive overrelaxation iterative method is the best of the three methods tested for $h \geq h^*$ while for $h < h^*$ the alternating-direction implicit method is best.

Each of these methods uses a parameter for accelerating the convergence. Formulas for the bounds of optimum parameters are included for rectangular regions. For the more general case of irregular boundaries there are no convenient relationships for estimating the optimum parameters at the start of the iterative solution. The procedure for determining the accelerating parameter is different for each of the three methods. There existed the possibility that the choice of the most efficient method might be more dependent on the method used for determining the

acceleration parameter than on the performance of the iterative method. To investigate this possibility the optimum parameter for the solution of a model problem was determined on an empirical basis for each method. The results were consistent with the other comparisons of the three methods. However, this study did reveal that the machine time required could be reduced from 18% to 42% by starting the solution with an optimum parameter.

Another objective is the preparation of a set of computer programs for solution of plane elasticity problems. Six FORTRAN-II routines have been written for the Control Data Corporation 3600, and slightly modified versions have been run on an IBM 1620. The solution of a particular problem requires the preparation of a pair of routines for the given boundary conditions. The main program which provides linkage of these subroutines is called ISOPEP. A full description is provided in Appendix B.

The third objective of this dissertation is the demonstration of the utility of the ISOPEP program. Solutions in terms of stress functions and stress components are provided for six problems: (1) A square plate with uniformly distributed loads on portions of two edges, (2) A semi-infinite plate with a uniformly distributed load applied on a segment of one edge, (3) A flat-plate tensile specimen with two semi-circular notches, (4) A flat-plate tensile specimen with two V-notches. (5) A flat-plate tensile specimen with two rectangular notches, (6) An infinite plate with a square hole. Several of the problems were selected as examples of the numerical calculation of stress concentrations. Two of the problems are included for comparison of the exact and numerical solutions. The numerical solutions obtained with sufficiently small mesh intervals provide stress components which are in good agreement with values from the exact solutions.

II. THE DIFFERENTIAL EQUATIONS

Plane stress is the state of stress approximated in a thin plate which has loads applied only on the boundary and parallel to the plane of the plate. In a three-dimensional Cartesian coordinate system a state of planes stress exists if the stress components $\sigma_z, \tau_{xz}, \tau_{yz}$ are zero at every point.

Consideration of static force equilibrium under conditions of plane stress leads to the equilibrium equations.

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y &= 0 \end{aligned} \tag{1}$$

where X and Y are the components of body force per unit volume.

For plane stress the Hooke's law relationship between stress and strain is,

$$\begin{aligned} E\epsilon_x &= \sigma_x - \nu\sigma_y + E\alpha\xi \\ E\epsilon_y &= \sigma_y - \nu\sigma_x + E\alpha\xi \\ E\gamma_{xy} &= 2(1+\nu)\tau_{xy} \end{aligned} \tag{2}$$

where E is Young's modulus, ν is Poisson's ration, $\xi(x,y)$ is the difference between the local current temperature T, and the original temperature T_o, and α is the coefficient of thermal expansion.

At a point in the plate the strains are defined:

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3}$$

where u and v, the components of the displacement, are continuous functions. Since the three strain components are expressed in terms of the two functions u and v, they cannot be taken arbitrarily. By differentiating and combining Equations (3) it is possible to obtain the compatibility equation,

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad (4)$$

At each point in the plate Eq.'s (1) and (4) must be satisfied. G. B. Airy derived a single differential equation in terms of a stress function, $\phi(x,y)$, which will satisfy both the compatibility conditions and equilibrium equations. The stresses are determined by the following:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} + V, \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} + V, \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \qquad (5)$$

where $V(x,y)$ is the potential of the body forces.

The Hooke's law relationships (2) are substituted into the compatibility equation (4):

$$\frac{\partial^2}{\partial y^2}\left[\sigma_x - \nu\sigma_y + E\alpha\xi\right] + \frac{\partial^2}{\partial x^2}\left[\sigma_y - \nu\sigma_x + E\alpha\xi\right] = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} \qquad (6)$$

It is advantageous to introduce the equilibrium equations by forming the sum of the derivative of the first with respect to x and the derivative of the second with respect to y.

$$\begin{aligned} \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial X}{\partial x} &= 0 \\ \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial Y}{\partial y} &= 0 \end{aligned} \qquad (7)$$

or $$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\left[\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right].$$

Eq.'s (5) and (6) can be combined so the term containing τ_{xy} is eliminated.

$$\frac{\partial^2}{\partial y^2}\left[\sigma_x - \nu\sigma_y + E\alpha\xi\right] + \frac{\partial^2}{\partial x^2}\left[\sigma_y - \nu\sigma_x + E\alpha\xi\right] = -(1+\nu)\left[\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right]$$

This simplifies to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[\sigma_x + \sigma_y + E\alpha\xi\right] + (1+\nu)\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right] = 0.$$

In terms of the two-dimensional Laplacian operator ∇^2 this can be written

$$\nabla^2\left[\sigma_x + \sigma_y + \alpha E \xi\right] + (1+\nu)\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial Y}\right] = 0.$$

Substituting the expressions given in Equations 5 for the stresses and

assuming body forces can be expressed in terms of a potential function V(x,y) the equation can be written

$$\nabla^4\phi + E\alpha\nabla^2\zeta + (1-\nu)\nabla^2 V = 0 \tag{8}$$

If the body force potential is harmonic, the equation becomes

$$\nabla^4\phi + E\alpha\nabla^2\zeta = 0 \tag{9}$$

and if in addition temperature changes are negligible, it becomes

$$\nabla^4\phi = 0 \tag{10}$$

where in Cartesian coordinates,

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4}$$

The solution of a plane stress problem, when the body forces can be expressed in terms of a potential, thus consists of finding the stress function ϕ which satisfies the appropriate one of Eq.'s (8), (9) or (10) and the prescribed boundary conditions.

As noted by Sokolnikoff (1956), even in a thin plate the stresses vary somewhat through the thickness. The two dimensional problem formulated here, strictly speaking, applies to the averages through the thickness and this is often called a state of generalised plane stress.

For a state of plane strain instead of plane stress the governing equations have a similar formulation. The appropriate equations are obtained by replacing ν by $\nu/(1-\nu)$, E by $E/(1-\nu^2)$ and α by $\alpha(1+\nu)$ in Eq.'s (2), (5), (7), (8) and (9).

If $\bar{X}$ and $\bar{Y}$ are the components of external loads per unit area acting on the boundary and α and β are the direction angles which the normal makes with the x and y-axes respectively, then neglecting body forces

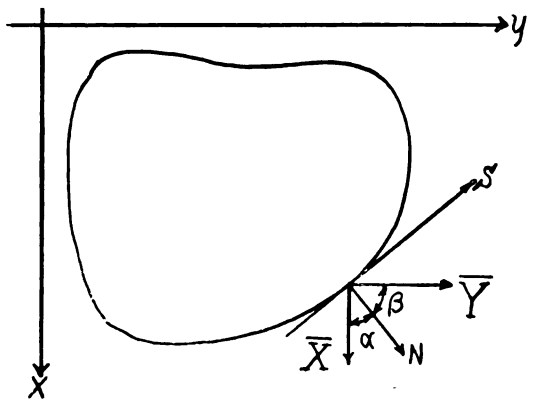


Figure 2.1 Boundary forces

$$\bar{X} = l\sigma_x + m\tau_{xy}$$
$$\bar{Y} = m\sigma_y + l\tau_{xy} \qquad (11)$$

where

$$l = \cos\alpha\,, \qquad m = \cos\beta$$

In terms of the stress function the stresses on the boundary are given by

$$\bar{X} = l\frac{\partial^2\phi}{\partial y^2} - m\frac{\partial^2\phi}{\partial x\partial y}\,, \qquad \bar{Y} = m\frac{\partial^2\phi}{\partial x^2} - l\frac{\partial^2\phi}{\partial x\partial y} \qquad (12)$$

Introducing coordinate axes s and n, tangent and normal respectively to the boundary, the boundary conditions can be written

$$\bar{X} = \frac{\partial^2\phi}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2\phi}{\partial x\partial y}\frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right)$$
$$\bar{Y} = -\frac{\partial^2\phi}{\partial x^2}\frac{dx}{ds} - \frac{\partial^2\phi}{\partial x\partial y}\frac{dy}{ds} = -\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right) \qquad (13)$$

These can be integrated along the boundary

$$\frac{\partial\phi}{\partial x} = -\int_0^s \bar{Y}\,ds + \left(\frac{\partial\phi}{\partial x}\right)_0$$
$$\frac{\partial\phi}{\partial y} = \int_0^s \bar{X}\,ds + \left(\frac{\partial\phi}{\partial y}\right)_0 \qquad (14)$$

Since

$$\frac{\partial\phi}{\partial s} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial s}$$

integrating along the boundary yields

$$\phi = \int_0^s \left[\cos\alpha\int_0^s \bar{X}\,ds + \sin\alpha\int_0^s \bar{Y}\,ds\right]ds + \left(\frac{\partial\phi}{\partial y}\right)_0(y-y_0) + \left(\frac{\partial\phi}{\partial x}\right)_0(x-x_0) + \phi_0 \qquad (15)$$

where

$$\cos\alpha = \frac{dy}{ds}\,, \qquad \sin\alpha = -\frac{dx}{ds}$$

Substituting into the normal derivative of ϕ,

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial n}$$
$$\frac{\partial\phi}{\partial n} = -\cos\alpha\int_0^s \bar{Y}\,ds + \sin\alpha\int_0^s \bar{X}\,ds + \left(\frac{\partial\phi}{\partial x}\right)_0\cos\alpha + \left(\frac{\partial\phi}{\partial y}\right)_0\sin\alpha \qquad (16)$$

Eq.'s (15) and (16) determine ϕ and $\frac{\partial\phi}{\partial n}$ at every point of the boundary in terms of the boundary stresses $\bar{X}$, $\bar{Y}$ and the constants of integration $\left(\frac{\partial\phi}{\partial x}\right)_0$, $\left(\frac{\partial\phi}{\partial y}\right)_0$ and ϕ_0. These constants of integration may be chosen arbitrarily since they do not appear in the expressions for the stresses.

If the constants of integration can be chosen so ϕ is symmetric with respect to a line of physical symmetry for the plate, then

$$\frac{\partial\phi(x,y)}{\partial n} = 0\,, \qquad \frac{\partial}{\partial n}\left(\nabla^2\phi(x,y)\right) = 0 \qquad (17)$$

for points (x,y) on the line of symmetry, where n is the normal to the line of symmetry.

The problems herein considered will have boundary conditions as given by Equation (17) on a line of symmetry of the body and of the form

$$\frac{\partial \phi(x,y)}{\partial x} = f_1(x,y), \qquad \frac{\partial \phi(x,y)}{\partial y} = f_2(x,y) \tag{18a}$$

which follows from Equation (14) or

$$\phi(x,y) = f_3(x,y), \qquad \frac{\partial \phi(x,y)}{\partial n} = f_4(x,y) \tag{18b}$$

which follows from Eq.'s (15) and (16) on an outer boundary. The functions f_1, f_2, f_3, and f_4 are valid only on the boundary and could be expressed in terms of the single parameter s; n is the outward normal.

For problems with displacements specified on the boundaries the Navier Equations should be used rather than the biharmonic equation. This pair of coupled second order partial differential equations for plane stress conditions, obtained by replacing the stresses in Equation (1) with expressions in terms of strains and then substituting derivatives of the displacements for the strains as given in Equation (3), can be written:

$$\begin{aligned} \frac{E}{2(1-\nu^2)}\left[2\frac{\partial^2 u}{\partial x^2} + (1-\nu)\frac{\partial^2 u}{\partial y^2} + (1+\nu)\frac{\partial^2 v}{\partial x \partial y}\right] + X = 0 \\ \frac{E}{2(1-\nu^2)}\left[(1-\nu)\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2} + (1+\nu)\frac{\partial^2 u}{\partial x \partial y}\right] + Y = 0 \end{aligned} \tag{19}$$

III. DERIVATION OF THE DIFFERENCE EQUATIONS

Though analytical solutions have been obtained for certain special cases of the biharmonic boundary value problem, the use of approximate numerical methods is often necessary. Finite-difference methods are readily adapted for solving the problems with high speed computers and attention will be directed to these methods.

The governing partial differential equation is replaced by a finite-difference approximation. A rectangular mesh is superimposed on the region and the intersections of the horizontal and vertical lines inside the region are called nodes or mesh points. Boundary points occur at the intersection of the mesh lines with the boundary. It is convenient to use a uniform mesh spacing, say h. At each interior point the function $\phi(x,y)$ is replaced by an approximating function U(P), where P is an interior mesh point (x_p,y_p). The function U(P) is defined only at the mesh points. Discretization of the problem is accomplished by replacing the partial differential equation in terms of $\phi(x,y)$ by a finite system of equations in terms of U(P). The equation for U(P) is given in terms of the values of U at neighboring points. Thus the problem is reduced to solving a set of simultaneous finite-difference equations.

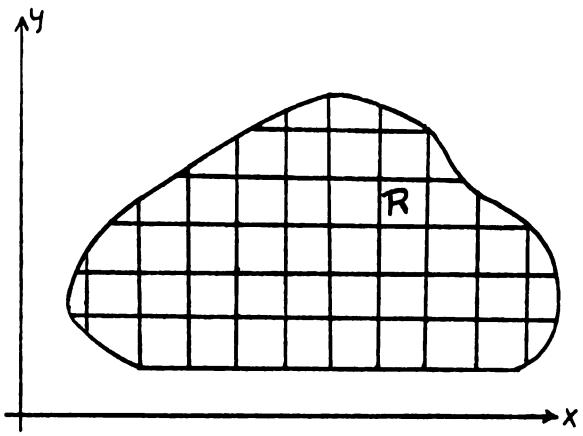


Figure 3.1 Rectangular mesh

Three different derivations of the pertinent difference equations will be considered. These are based on Taylor's series, integration and a variational formulation. Each has distinct advantages and contri-

butes to a better understanding of the problem.

Taylor Series

Let f(x,y) be a function of two variables, which is continuous in the neighborhood of the point (a,b) and has continuous partial derivatives up to order n in the neighborhood of (a,b). The Taylor's expansion of f(x,y) about the point (a,b) is given by:

$$f(x,y) = f(a,b) + (f_x(a,b) + f_y(a,b))h + (f_{xx}(a,b) + 2f_{xy}(a,b) + f_{yy}(a,b))\frac{h^2}{2!} + \cdots + R_n$$

where $$R_n = \frac{h^n}{n!}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^n f(a+\theta_1 h, b+\theta_2 h), \qquad 0 \le \theta_1, \theta_2 \le 1$$

$$X = a+h, \qquad y = b+h$$

and subscripts denote partial derivatives. In the region of the xy plane where a solution of the biharmonic equation is sought, a mesh point (x_0,y_0) will be called a regular point if the neighboring points shown in Fig. 3.2 are all interior or boundary points of the region and each point is at a distance h from the adjacent points shown.

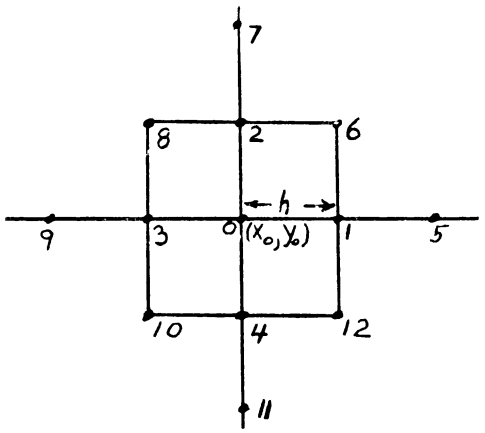


Figure 3.2 A regular point

A Taylor's expansion can be written for each of the points in the neighborhood of (x_0,y_0) and these expansions can be combined to find difference quotients corresponding to any partial derivative with respect to x or y.

Consider

$$U_1 = U_0 + (U_x)_0 h + (U_{x^2})_0 \frac{h^2}{2!} + (U_{x^3})_0 \frac{h^3}{3!} + (U_{x^4})_0 \frac{h^4}{4!} + (U_{x^5})_0 \frac{h^5}{5!} + (U^*_{x^6})_{\theta_1} \frac{h^6}{6!}$$

$$U_3 = U_0 - (U_x)_0 h + (U_{x^2})_0 \frac{h^2}{2!} - (U_{x^3})_0 \frac{h^3}{3!} + (U_{x^4})_0 \frac{h^4}{4!} - (U_{x^5})_0 \frac{h^5}{5!} + (U^*_{x^6})_{\theta_3} \frac{h^6}{6!}$$

$$U_1 + U_3 = 2U_0 + (U_{x^2})_0 \frac{2h^2}{2!} + (U_{x^4})_0 \frac{2h^4}{4!} + \frac{2h^6}{6!} M_1$$

where $(U_x^{*6})_{0,1} = U_{x^6}(x + \theta h, y)$ $0 \le \theta \le 1$, and $M_1 \le \max |U_{x^6}|_{1,3}$ along the line between points 1 and 3

$$U_5 = U_o + (U_x)_o(2h) + (U_{x^2})_o \frac{4h^2}{2!} + (U_{x^3})_o \frac{8h^3}{3!} + (U_{x^4})_o \frac{16h^4}{4!} + (U_{x^5})_o \frac{32h^5}{5!} + (U_x^{*6})_{0,5} \frac{64h^6}{6!}$$

$$U_9 = U_o - (U_x)_o(2h) + (U_{x^2})_o \frac{4h^2}{2!} - (U_{x^3})_o \frac{8h^3}{3!} + (U_{x^4})_o \frac{16h^4}{4!} - (U_{x^5})_o \frac{32h^5}{5!} + (U_x^{*6})_{0,9} \frac{64h^6}{6!}$$

$$U_5 + U_9 = 2U_o + (U_{x^2}) \frac{8h^2}{2!} + (U_{x^4})_o \frac{32h^4}{4!} + \frac{128h^6}{6!} M_2$$

where $M_2 \le |U_{x^6}|_{5,9}$. Subtracting $4(U_1 + U_3)$ from $(U_5 + U_9)$ we find

$$U_5 + U_9 - 4(U_1 + U_3) + 6U_o + \frac{136h^6}{6!} M_2 = h^4 (U_{x^4})_o$$

Thus,

$$(U_{x^4})_o = \frac{1}{h^4}\left[U_5 + U_9 + 6U_o - 4U_1 - 4U_3\right] - \frac{136h^2}{6!} M_2 \qquad (20)$$

Similarly it can be shown

$$(U_{y^4})_o = \frac{1}{h^4}\left[U_7 + U_{11} + 6U_o - 4U_2 - 4U_4\right] - \frac{136h^2}{6!} M_3 \qquad (21)$$

where $M_3 \le \max |U_{y^6}|_{7,11}$. For $\frac{\partial^4 \phi}{\partial x^2 \partial y^2}$ consider

$$U_6 = U_o + (U_x)_o h + (U_y)_o h + \left[(U_{x^2})_o + 2(U_{xy})_o + (U_{y^2})_o\right] \frac{h^2}{2!} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^3 U \frac{h^3}{3!}$$
$$+ \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^4 U_o \frac{h^4}{4!} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^5 U_o \frac{h^5}{5!} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^6 U^*_{o,6} \frac{h^6}{6!}$$

$$U_{10} = U_o - (U_x)_o h - (U_y)_o h + \left[(U_{x^2})_o + 2(U_{xy})_o + (U_{y^2})_o\right] \frac{h^2}{2!} - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^3 U_o \frac{h^3}{3!}$$
$$+ \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^4 U_o \frac{h^4}{4!} - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^5 U_o \frac{h^5}{5!} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^6 U^*_{o,10} \frac{h^6}{6!}$$

$$U_6 + U_{10} - 2U_o = \left[(U_{x^2})_o + 2(U_{xy})_o + (U_{y^2})_o\right] \frac{2h^2}{2!} + \left[(U_{x^4})_o + 4(U_{x^3y})_o + 6(U_{x^2y^2})_o\right.$$
$$\left. + 4(U_{xy^3})_o + (U_{y^4})_o\right] \frac{2h^4}{4!} + \frac{64h^6}{6!} M_4$$

where $M_4 \le \max |U_{x^n y^m}|_{6,10}$ $n + m = 6$. Similarly,

$$U_8 + U_{12} - 2U_o = \left[(U_{x^2})_o + 2(U_{xy})_o + (U_{y^2})_o\right] \frac{2h^2}{2!} + \left[(U_{x^4}) - 4(U_{x^3y})_o + 6(U_{x^2y^2})_o\right.$$
$$\left. - 4(U_{xy^3})_o + (U_{y^4})_o\right] \frac{2h^4}{4!} + \frac{64h^6}{6!} M_5$$

where $M_5 \le \max |U_{x^n y^m}|_{8,12}$, $n + m = 6$. Adding we find

$$U_6 + U_8 + U_{10} + U_{12} - 4U_o = 2\left[(U_{x^2})_o \frac{2h^2}{2!} + (U_{x^4})_o \frac{2h^4}{4!} + (U_{y^2}) \frac{2h^2}{2!} + (U_{y^4}) \frac{2h^4}{4!}\right]$$
$$+ (U_{x^2y^2})_o h^4 + \frac{64h^6}{6!} M_6$$

where $M_6 \le \max |U_{x^n y^m}|$, $n + m = 6$ along all lines through the mesh point.

Substituting from above

$$(U_{x^2y^2})_o = \frac{1}{h^4}[U_6 + U_8 + U_{10} + U_{12} + 4U_o - 2(U_1 + U_2 + U_3 + U_4)] - \frac{72h^2}{6!}M_6 \quad (22)$$

Hence

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$$

is approximated by

$$(U_{x^4})_o + 2(U_{x^2y^2})_o + (U_{y^4})_o = \frac{1}{h^4}[U_5 + U_7 + U_9 + U_{11} + 2(U_6 + U_8 + U_{10} + U_{12}) - 8(U_1 + U_2 + U_3 + U_4) + 20U_o] + R_\eta = 0 \quad (23)$$

where $$R_\eta \leq \frac{26}{45}h^2M_6$$

Kantorovich and Krylov (1964) give $R_\eta \leq \frac{10}{9}h^2M$ where M is obtained by replacing all the derivatives of higher order in Eq.'s (20), (21) and (22) by their maximum absolute values on the mesh lines joining point 0 and the surrounding points.

The remainder R_η provides a bound for the discretization error, that is, the difference between $\phi(x,y)$ and the truncated series approximation $U(x,y)$ at the nodal points. This indication of the order of the approximation is a major advantage of the Taylor's series method for deriving the finite difference equations.

Integration Technique

The use of integrals for the derivation of finite difference equations has not been used as extensively as the other methods. Varga (1962) provides a general introduction to the method and indicates it has been used in nuclear reactor design computer codes for several years. Use of this method for the biharmonic equation is made by Griffin and Varga (1963).

The satisfaction of the partial differential equation (8) at every point of a region R is equivalent to the satisfaction of the integral equation

$$\iint_R \nabla^2(\nabla^2\phi + BV + C\xi)\,dx\,dy = 0 \quad (24)$$

where $B = (1-\nu)$, $C = \alpha E$, for every arbitrarily chosen subregion A_i. Hence, for any arbitrarily chosen subregion A_i bounded by γ_i, Green's theorem gives

$$\iint_{A_i} \nabla^2(\nabla^2\phi + BV + C\xi)\,dx\,dy = \oint_{\gamma_i} \frac{\partial}{\partial n}(\nabla^2\phi + BV + C\xi)\,ds = 0 \qquad (25)$$

Discretization is accomplished by substituting a set of discrete quantities U_i, ($i = 1,2,3,\text{----}m$) for the values of the continuous function $\phi(x,y)$ at the mesh points corresponding to $i = 1,2,3,\text{----}m$. A major advantage of this method is the simplicity of the treatment of irregular mesh spacings. In Fig. 3.3 the mesh spacing to a point P_i adjacent to a general interior point P_o is h_{oi}. The subregion about point P_o has sides S_{oi}, $i = 1,2,3,4$.

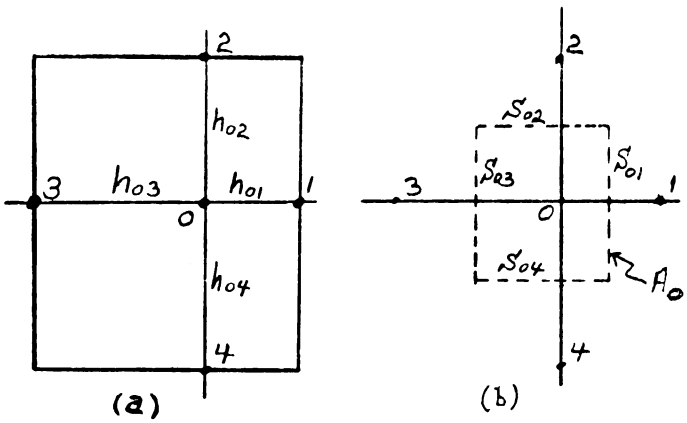


Figure 3.3
(a) Unequal mesh spacings
(b) Subregion A_o

$$S_{01} = S_{03} = \tfrac{1}{2}(h_{02} + h_{04})$$

$$S_{02} = S_{04} = \tfrac{1}{2}(h_{01} + h_{03})$$

Central difference quotients are used to approximate the line integral in Equation (25) about the subregion A_o of Fig. 3.3 (b)

$$\sum_{i=1}^{4} (\nabla^2 U_i - \nabla^2 U_o)\frac{S_{oi}}{h_{oi}} = -\sum_{i=1}^{4}\left[B(V_i - V_o) + C(\xi_i - \xi_o)\right]\frac{S_{oi}}{h_{oi}} \qquad (26)$$

where $\nabla^2 U_i$ is the value of $\nabla^2\phi$ at point P_i. Approximate values of $\nabla^2 U_i$ and $\nabla^2 U_o$ are obtained by using Green's theorem again.

$$\iint_{A_i} \nabla^2 U_i\,dx\,dy = \oint_{\gamma_i} \frac{\partial U_i}{\partial n}\,ds \qquad (27)$$

The discrete value of $\nabla^2 U_o$ is taken such that

$$\nabla^2 U_o = \frac{1}{A_o}\oint_{\gamma_o} \frac{\partial U_o}{\partial n}\,ds = \frac{1}{A_o}\sum_{i=1}^{4}(U_i - U_o)\frac{S_{oi}}{h_{oi}} \qquad (28)$$

$$A_o = (S_{01})(S_{02}) = (h_{02} + h_{04})(h_{01} + h_{03})/4$$

At point P_1 as given in Fig. 3.2

$$\nabla^2 U_1 = \left[(U_5 - U_1)\frac{s_{1,5}}{h_{1,5}} + (U_6 - U_1)\frac{s_{1,6}}{h_{1,6}} + (U_0 - U_1)\frac{s_{1,0}}{h_{1,0}} + (U_{12} - U_1)\frac{s_{1,12}}{h_{1,12}}\right]\frac{1}{A_1}$$

where $A_1 = (s_{1,6})(s_{1,5})$

For the general case

$$\sum_{i=1}^{4}(\nabla^2 U_i - \nabla^2 U_0)\frac{s_{0,i}}{h_{0,i}} = \sum_{i=1}^{4}\left[\left\{\frac{1}{A_i}\sum_{j=1}^{4}(U_{i,j} - U_i)\frac{s_{i,j}}{h_{i,j}}\right\} - \frac{4}{A_0}(U_i - U_0)\frac{s_{0,i}}{h_{0,i}}\right] \qquad (29)$$

If the mesh spacing is constant, Equation (29) reduces to the form of Equation (23).

The evaluation of the right side of Equation (23) requires 12 additions and 3 multiplications. (When the mesh spacing is uniform the multiplication by h^{-2} is not necessary.) The evaluation of the right side of Equation (29) requires 29 additions and 25 multiplications if the ratios $\frac{s_{i,j}}{h_{i,j}}$ are computed once and stored. For the IBM 1620 the time required for 10 floating point additions is equivalent to that required for one multiplication. Thus on the 1620 Equation (23) could be evaluated for approximately 7 mesh points in the same time required to evaluate the right hand side of Equation (29) for a single mesh point. Any iterative method of solution of the finite-difference equations requires the evaluation of one of these expressions at every interior node at least once during each iteration. Several hundred iterations may be required. The rate of convergence of an iterative scheme which uses Equation (29) would have to be 7 times the rate of convergence of a scheme which uses Equation (23) for the same number of mesh points before Equation (29) would be preferred. In some problems it is possible to reduce the number of mesh points significantly through the use of arbitrary mesh spacing. Regardless of whether it is possible to establish the superiority of Equation (29) over Equation (23) for use at

every point, Equation (29) provides an excellent method of handling irregular boundaries and the changing of mesh size from one subregion to another within R.

Variational Formulation

Application of the variational method for deriving finite-difference equations can be found in Courant and Hilbert (1953) and Forsythe and Wasow (1960). Engeli, Ginsburg, Rutishauser and Steifel (1959) show its use in deriving biharmonic difference equations.

The variational formulation of difference equations for plane elastostatic problems is especially convenient for problems given in terms of two displacement functions. Griffin (1965) has shown the advantages of this approach. The basis for this derivation is the Principle of Stationary Potential Energy which states: From the set of continuously differentiable displacement distributions which satisfy the given boundary conditions of an elastic body, the displacement distribution which actually occurs is the one which makes the potential energy stationary.

For a plate subjected to plane stress, take X,Y as the body forces per unit volume and $\bar{X},\bar{Y}$ the surface forces per unit area. Then the strain energy V, per unit volume is

$$V = \frac{E}{2(1-\nu^2)}\left[\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y + \frac{(1-\nu)}{2}\gamma_{xy}^2\right]$$

The strain energy per unit volume can be expressed in terms of displacements (u,v) if these are continuously differentiable functions of (x,y) by substituting

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$V = \frac{E}{2(1-\nu^2)}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\nu\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{(1-\nu)}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] \tag{30}$$

A change or variation of total strain energy will occur with any arbi-

trary variation of the displacements δu, δv,

$$\delta V_T = \delta \iiint_R^1 V dx dy dz = \iint_R \delta V dx dy$$

where for convenience the z-dimension is taken as one and it is assumed that V does not vary with z.

The virtual work δW_{ext} done by the external forces under the virtual displacements δu, δv is given by

$$\delta W_{ext} = \iint_R (X \delta u + Y \delta v) dx dy + \oint_S (\bar{X} \delta u + \bar{Y} \delta v) ds$$

The potential energy is defined

$$Q = \iint_R V dx dy - \iint_R (Xu + Yv) dx dy - \oint_S (\bar{X} u + \bar{Y} v) ds \qquad (31)$$

Applying the principle of stationary potential energy and considering body forces and surface forces constant $\delta Q = 0$

$$\delta Q = \iint_R \delta V dx dy - \iint_R (X \delta u + Y \delta v) dx dy - \oint_S (\bar{X} \delta u + \bar{Y} \delta v) ds \qquad (32)$$

This states that the change in strain energy will be equal to the work done on the body by external forces for any arbitrary virtual displacements $\delta u, \delta v$. This will be true only if u,v are the actual elastic displacements produced by the external forces.

The usual procedure in the calculus of variations is to consider the conditions imposed on the integrand of Equation (32) and thus derive a linear partial differential equation of the form

$$L\{q\} = 0 \qquad (33)$$

where $q=q(u,v)$. Also additional boundary conditions arise which are called natural boundary conditions. Instead of finding the differential equation of the form of Equation (33) and then introducing finite differences the variational form of Equation (32) will be solved numerically. There are two major advantages of this approach. First the function V which satisfies Equation (32), will automatically satisfy the natural boundary conditions so that further consideration of the

natural boundary conditions is not necessary. Second, the discretization of the problem leads to a linear system of equations that has a coefficient matrix which is symmetric and positive definite. These relationships are important because they are part of the criteria for the convergence of iterative methods for solving linear systems of equations. (See Appendix A)

For a plate in the region R of the xy plane and unit thickness the total potential energy is

$$Q = \frac{E}{2(1-\nu^2)} \iint_R \left[\nu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 + (1-\nu)\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} + \frac{(1-\nu)}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] dx\,dy - \iint_R (Xu + Yv)\,dx\,dy - \oint_S (\bar{X}u + \bar{Y}v)\,ds \qquad (34)$$

The stationary value of potential energy is a minimum for stable equilibrium under specified boundary conditions.

A rectangular mesh and its dual are imposed on the region R. The dual mesh lines (indicated with dash lines) are parallel to and spaced halfway between the lines of the primary mesh.

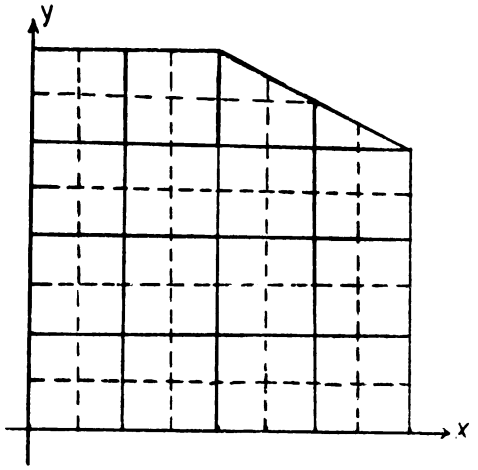


Figure 3.4 Dual meshes

The variational problem Equation (32) is given in terms of a continuous displacement distribution u(x,y) and v(x,y). This is replaced by a problem in which the displacements have discrete values u_i and v_i at the mesh points. The potential energy of each mesh point is approximated, and the sum taken over all mesh points represents the total potential energy $Q(u_i, v_i)$. The problem is reduced to finding the unknown displacements u_i and v_i at each node which will make $Q(u_i, v_i)$

stationary. This requires

$$\frac{\partial Q}{\partial u_i} = 0 \,, \qquad i = 1, 2, 3, \ldots\ldots n$$
$$\frac{\partial Q}{\partial v_i} = 0 \,, \qquad i = 1, 2, 3, \ldots\ldots m \tag{35}$$

where n is the number of nodes at which u_i is unknown and m the number where v_i is unknown. Equations (35) represents a system of n+m linear difference equations.

Engeli, Stiefel et al (1959) show that a variety of quadratic functions may be used to approximate the potential energy. The polygons formed by mesh lines and dual lines will be called cells. A primary mesh point will occur at one vertex of a cell. A regular interior point will be the common point of four adjacent cells as shown in Fig. 3.5. The potential energy integral (34) is approximated for each cell under the assumption that the functions u and v and their derivatives are uniform over the cell. The functions take the discrete values at the node P_0. The derivatives are approximated for cell β by

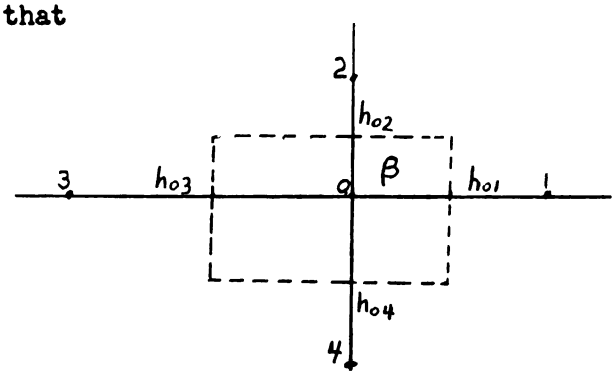


Figure 3.5 Cell β

$$\frac{\partial u}{\partial x} = \frac{u_1 - u_0}{h_{01}} \,, \qquad \frac{\partial u}{\partial y} = \frac{u_2 - u_0}{h_{02}}$$

The approximation for the potential energy is

$$Q_{021} = \frac{E}{2(1-\nu^2)}\left[\nu\left(\frac{u_1-u_0}{h_{01}} + \frac{v_2-v_0}{h_{02}}\right)^2 + (1-\nu)\left\{\left(\frac{u_1-u_0}{h_{01}}\right)^2 + \left(\frac{v_2-v_0}{h_{02}}\right)^2\right\} + \frac{(1-\nu)}{2}\left(\frac{u_2-u_0}{h_{02}} + \frac{v_1-v_0}{h_{01}}\right)^2\right]\frac{h_{01}h_{02}}{4} - \left(\bar{X}u_0 + \bar{Y}v_0\right)\frac{h_{01}h_{02}}{4} - \left(\bar{\bar{X}}u_0 + \bar{\bar{Y}}v_0\right)\frac{h_{01}+h_{02}}{2} \tag{36}$$

The last term is included only if P_0 is a boundary point. At a boundary point specifed derivatives of u and v are introduced in Eq. (36).

The potential energy at the node P_0 is the sum of the potential energy in the four cells adjacent to P_0

$$Q_o = Q_{o21} + Q_{o23} + Q_{o34} + Q_{o41}$$

and the total potential energy is the sum of the potential energy in all cells

$$Q = \sum Q_{ijk}$$

The difference equations are obtained from Equation (35).

Consider all the contributions to the potential energy Q which involve the displacements at a single mesh point u_o, v_o. There are twelve cells as shown in Fig. 3.6 which would use u_o, v_o at a regular interior mesh point.

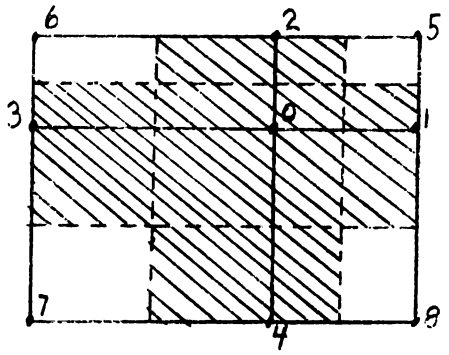


Figure 3.6 Twelve Cells for P_o

$$\frac{\partial Q}{\partial u_o} = \frac{E}{2(1-\nu^2)}\left[(1-\nu)\frac{h_{o1}+h_{o3}}{2}\left(\frac{u_o-u_2}{h_{o2}} + \frac{u_o-u_4}{h_{o4}}\right) + (h_{o2}+h_{o1})\left(\frac{u_o-u_1}{h_{o1}} + \frac{u_o-u_3}{h_{o3}}\right) - \frac{(1+\nu)}{4}(v_5+v_7-v_8-v_6)\right] + X_o A_o = 0 \tag{37}$$

where $A_o = \frac{1}{4}[h_{o1}h_{o2} + h_{o2}h_{o3} + h_{o3}h_{o4} + h_{o4}h_{o1}]$

$$\frac{\partial Q}{\partial v_o} = \frac{E}{2(1-\nu^2)}\left[(1-\nu)\frac{h_{o2}+h_{o4}}{2}\left(\frac{v_o-v_1}{h_{o1}} + \frac{v_o-v_3}{h_{o3}}\right) + (h_{o1}+h_{o3})\left(\frac{v_o-v_2}{h_{o2}} + \frac{v_o-v_4}{h_{o4}}\right) - \frac{(1+\nu)}{4}(u_5+u_7-u_6-u_8)\right] + Y_o A_o = 0 \tag{38}$$

Note the relationship to the Navier Equation (19). When the solution of the finite-difference equations (37) and (38) is obtained, stresses can be approximated using an area weighted average of the stresses over all cells which have one node as a common vertex. The stress components in terms of displacements are

$$\begin{aligned}
\sigma_x &= \frac{E}{(1-\nu^2)}\left[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right] \\
\sigma_y &= \frac{E}{(1-\nu^2)}\left[\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right] \\
\tau_{xy} &= \frac{E}{2(1+\nu)}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]
\end{aligned} \tag{39}$$

None of the problem solutions included in this dissertation use Equations (37) and (38).

A simple example considered by Timoshenko and Goodier (1951) will illustrate several details of the solution. Given a unit square plate which is subjected to boundary loads as shown in Fig. 4.1, find a numerical solution which approximates the stress function $\phi(x,y)$ on the region of the x,y plane occupied by the plate.

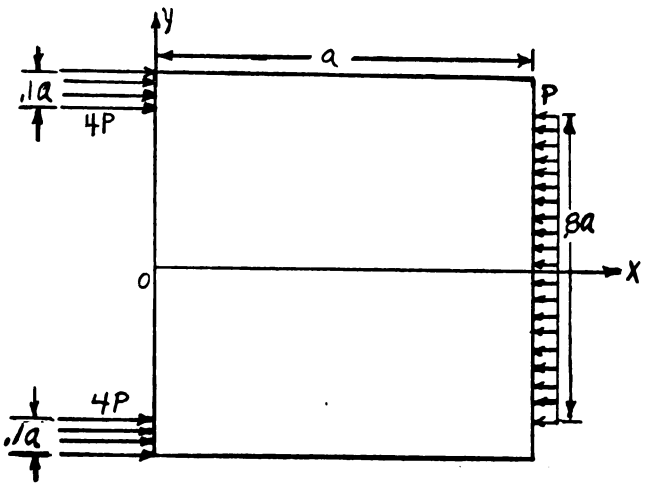


Figure 4.1 The model problem

The boundary conditions are of the form given in Equation (12), but are readily transformed into the form of Equation (18) which leads to the problem statement

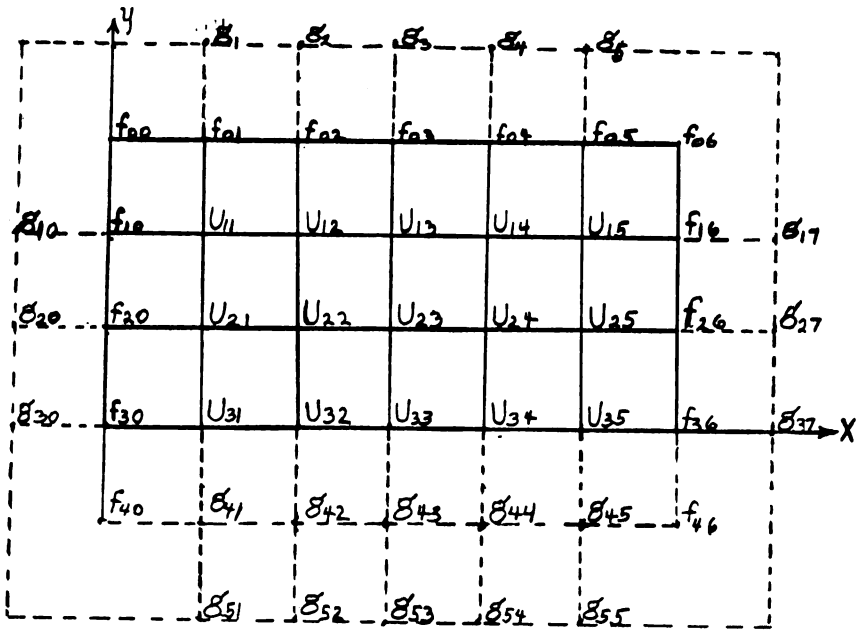


Figure 4.2 Mesh for the model problem

$$\nabla^4 \phi(x,y) = 0 \qquad (40)$$

for $0 < x < a$, $-\frac{1}{2}a < y < \frac{1}{2}a$

subject to boundary conditions

$$\left.\begin{aligned} \phi(x,y) &= f(x,y) && \text{for x,y on} \\ \frac{\partial \phi(x,y)}{\partial n} &= \eta(x,y) && \text{the boundary} \end{aligned}\right\} \qquad (41)$$

A simple way to introduce the boundary derivative is to solve for a point one mesh space outside of the boundary in terms of the derivative and the point inside the boundary on the same mesh line, eg,

$$g_1 = U_{11} + \eta_{01}(2h),$$

where U_{ij} is the approximation to the stress function $\emptyset$. If the constants of integration are selected so Equations 17 are satisfied then, by symmetry,

$$g_{4i} = U_{2i} \text{ and } g_{5i} = U_{1i} \qquad i=1,2,3,---5$$

The partial differential equation (40) is replaced by Equation (23) at each interior node.

$$U_{11} = 1/20\left[8(U_{12}+U_{21}+f_{10}+f_{01})-2(U_{22}+f_{02}+f_{00}+f_{20})-(U_{31}+U_{13}+8_1+8_{10})\right],$$

$$U_{12} = 1/20\left[8(U_{11}+U_{13}+U_{22}+f_{02})-2(f_{03}+f_{01}+U_{21}+U_{23})-(U_{32}+U_{14}+8_2+f_{10})\right],$$

etc.

The system of equations can be written in matrix form

$$\underline{A}\,\underline{U} = \underline{F} \tag{42}$$

where

$$\underline{F} = \begin{bmatrix}
.4(f_{01} + f_{10})- .1(f_{00}+f_{02}+f_{20}) - .05(g_1 + g_{10}) \\
.4\ f_{02}-.1(f_{03}+f_{01})-.05\ g_2 \\
.4\ f_{03}-.1\ (f_{02}+f_{04})-.05\ g_3 \\
.4\ f_{04}-.1\ (f_{03}+f_{05})-.05\ g_4 \\
.4\ (f_{05}+f_{16}) - .1(f_{04}+f_{06}+f_{26})-.05\ (g_5 + g_{17}) \\
\hline
.4\ f_{20} - .1\ (f_{10}+f_{30}) -.05\ (g_{20} + f_{01}) \\
-.05\ f_{02} \\
-.05\ f_{03} \\
-.05\ f_{04} \\
.4\ f_{26}-.1(f_{16}+f_{36}) -.05\ (f_{05}+ g_{27}) \\
\hline
.4\ (f_{30}+g_{41})-.1(f_{20}+f_{40}+g_{42}) -.05\ (g_{30} + g_{51}) \\
.4\ g_{42} \\
.4\ g_{43} \\
.4\ g_{44} \\
.4\ (g_{45} +f_{36})-.1(f_{26}+f_{46}+g_{44}) -.05\ (g_{55} + g_{37})
\end{bmatrix}
\qquad
\underline{U} = \begin{bmatrix}
U_{11} \\
U_{12} \\
U_{13} \\
U_{14} \\
U_{15} \\
\hline
U_{21} \\
U_{22} \\
U_{23} \\
U_{24} \\
U_{25} \\
\hline
U_{31} \\
U_{32} \\
U_{33} \\
U_{34} \\
U_{35}
\end{bmatrix}$$

and

$$
\underline{A} = \left[\begin{array}{rrrrr|rrrrr|rrrrr}
1 & -.4 & .05 & 0 & 0 & -.4 & .1 & 0 & 0 & 0 & .05 & 0 & 0 & 0 & 0 \\
-.4 & 1 & -.4 & .05 & 0 & .1 & -.4 & .1 & 0 & 0 & 0 & .05 & 0 & 0 & 0 \\
.05 & -.4 & 1 & -.4 & .05 & 0 & .1 & -.4 & .1 & 0 & 0 & 0 & .05 & 0 & 0 \\
0 & .05 & -.4 & 1 & -.4 & 0 & 0 & .1 & -.4 & .1 & 0 & 0 & 0 & .05 & 0 \\
0 & 0 & .05 & -.4 & 1 & 0 & 0 & 0 & .1 & -.4 & 0 & 0 & 0 & 0 & .05 \\
\hline
-.4 & .1 & 0 & 0 & 0 & 1 & -.4 & .05 & 0 & 0 & -.4 & .1 & 0 & 0 & 0 \\
.1 & -.4 & .1 & 0 & 0 & -.4 & 1 & -.4 & .05 & 0 & .1 & -.4 & .1 & 0 & 0 \\
0 & .1 & -.4 & .1 & 0 & .05 & -.4 & 1 & -.4 & .05 & 0 & .1 & -.4 & .1 & 0 \\
0 & 0 & .1 & -.4 & .1 & 0 & .05 & -.4 & 1 & -.4 & 0 & 0 & .1 & -.4 & .1 \\
0 & 0 & 0 & .1 & -.4 & 0 & 0 & .05 & -.4 & 1 & 0 & 0 & 0 & .1 & -.4 \\
\hline
.05 & 0 & 0 & 0 & 0 & -.4 & .1 & 0 & 0 & 0 & 1 & -.4 & .05 & 0 & 0 \\
0 & .05 & 0 & 0 & 0 & .1 & -.4 & .1 & 0 & 0 & -.4 & 1 & -.4 & .05 & 0 \\
0 & 0 & .05 & 0 & 0 & 0 & .1 & -.4 & .1 & 0 & .05 & -.4 & 1 & -.4 & .05 \\
0 & 0 & 0 & .05 & 0 & 0 & 0 & .1 & -.4 & .1 & 0 & .05 & -.4 & 1 & -.4 \\
0 & 0 & 0 & 0 & .05 & 0 & 0 & 0 & .1 & -.4 & 0 & 0 & .05 & -.4 & 1
\end{array}\right]
$$

The original boundary value problem, Eq.'s (40) and (41), has been replaced by the problem of solving a system of linear algebraic equations as given in matrix form by Equation (42). Direct methods have been used for solving systems of linear equations. Recent reviews of these methods are found in Fox (1963) and Forsythe and Wasow (1960). Faddeeva (1959) provides a detailed account of a variety of methods. Direct methods for problems which require a large number of mesh points, several hundred or even several thousand for example, are rarely used for two reasons. First, if Gaussian elimination is used it is necessary to store all the elements of matrix $\underline{A}$. For N mesh points this requires the storage of N^2 elements. Iterative schemes require the storage of only a small integer multiple of N such as 3N or 5N. The second

complication arises from the round-off error in the direct solution of some systems of linear equations. Iterative solutions have the advantage of being self-correcting and round-off error is minimized.

If the non-zero elements in $\underline{A}$ are sparse and arranged in bands parallel to the main diagonal the storage requirements for a direct solution may be substantially reduced. Cornock (1954) gives an improved method for the direct solution of the biharmonic equation. Equation (42) can be partitioned into submatrices as indicated by the dotted lines. The submatrices have a more convenient form if a scalar factor of 20 is introduced.

$$20\,\underline{A}\,\underline{U} = 20\,\underline{F}$$

$$20\,\underline{A}\,\underline{U} = \begin{bmatrix} \underline{M} & \underline{B} & \underline{I} \\ \underline{B} & \underline{M} & \underline{B} \\ \underline{I} & \underline{B} & \underline{M} \end{bmatrix} \begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \\ \underline{U}_3 \end{bmatrix} = \begin{bmatrix} \underline{F}_1 \\ \underline{F}_2 \\ \underline{F}_3 \end{bmatrix} \qquad (43)$$

where $\underline{I}$ is the nxn unit submatrix, n is the number of interior mesh points in each row. The $\underline{U}_k$ are the column submatrices composed of elements U_{ij} along one row of grid points. The $\underline{F}_k$ are the corresponding column submatrices of 20 $\underline{F}$.

$$\underline{M} = \begin{bmatrix} 20 & -8 & 1 & 0 & 0 \\ -8 & 20 & -8 & 1 & 0 \\ 1 & -8 & 20 & -8 & 1 \\ 0 & 1 & -8 & 20 & -8 \\ 0 & 0 & 1 & -8 & 20 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} -8 & 2 & 0 & 0 & 0 \\ 2 & -8 & 2 & 0 & 0 \\ 0 & 2 & -8 & 2 & 0 \\ 0 & 0 & 2 & -8 & 2 \\ 0 & 0 & 0 & 2 & -8 \end{bmatrix}$$

For the general case of a rectangular region of n columns and p rows in the mesh, the submatrices $\underline{M}$, $\underline{B}$, $\underline{I}$ are all nxn if the difference equations are written for successive mesh points starting at U_{11} and sweeping across the first row then from left to right across successive rows. Cornock

shows how to determine the elements of a matrix $\underline{E}$ such that

$$\underline{E}\,\underline{A}\,\underline{U} = \underline{E}\,\underline{F} = \underline{G}$$

reduces to the form

$$\begin{bmatrix} \underline{C}_{11} & \underline{C}_{12} & 0 \\ \underline{C}_{21} & \underline{C}_{22} & 0 \\ \underline{C}_{31} & \underline{C}_{32} & \underline{I} \\ - & - & \ddots \\ - & - & \\ \underline{C}_{p1} & \underline{C}_{p2} & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \\ \underline{U}_3 \\ - \\ - \\ \underline{U}_p \end{bmatrix} = \begin{bmatrix} \underline{G}_1 \\ \underline{G}_2 \\ \underline{G}_3 \\ - \\ - \\ \underline{G}_p \end{bmatrix} \tag{44}$$

The system

$$\begin{bmatrix} \underline{C}_{11} & \underline{C}_{12} \\ \underline{C}_{21} & \underline{C}_{22} \end{bmatrix} \begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \end{bmatrix} = \begin{bmatrix} \underline{G}_1 \\ \underline{G}_2 \end{bmatrix}$$

is solved by direct elimination. With $\underline{U}_1$ and $\underline{U}_2$ known, $\underline{U}_3 - - - \underline{U}_p$ can be found by back - substitution.

One alternative to direct elimination is the "relaxation" technique of Southwell (1946). An initial solution is assumed at the points of the grid superimposed on the plate. Using difference equations a new value is computed at each point and residuals are determined at each point. The largest residual is identified and the initial guess is modified systematically so all residuals are reduced to zero. This procedure has not been used much for digital computer solutions, because it is most effective when the succeeding modifications are judged by a skilled practitioner. To date the logic of these decisions has not been efficiently programmed. However, some aspects of this type of decision making have been incorporated in direct search methods as discussed by Hooke and Jeevis (1961).

As a second alternative, better adapted to computer solution, consider an iterative method. Equation (42) can be written

$$(\underline{A} - \underline{I})\,\underline{U} + \underline{I}\;\underline{U} = \underline{F}$$

or $$\underline{U} = (\underline{I} - \underline{A})\,\underline{U} + \underline{F} \qquad (45)$$

An initial vector approximation $\underline{U}^{(o)}$ to the solution $\underline{U}$ is selected and a sequence of successive vector iterates $U^{(m)}$ are calculated using

$$\underline{U}^{(m+1)} = (\underline{I} - \underline{A})\underline{U}^{(m)} + \underline{F} \qquad m > o \qquad (46)$$

This scheme is known as the Richardson iterative method (or method of simultaneous displacements, point Jacobi, or point total-step method.) See Varga (1960). It requires all elements of the vector iterate $U^{(m)}$ for the computation of $\underline{U}^{(m+1)}$. Other iterative methods introduce the new values of the elements of the vector as they are determined, and these are used in the computation of successive elements. Methods of the latter type use only half the storage required for Richardson's method.

Iterative methods are characterized by the repeated application of a computational scheme which yields an approximation to the exact answer as a limit of the sequence of successive vector iterates. A basic question which must be answered affirmatively for any useful method is; does the sequence of vector iterates converge? To answer this an error for each vector iterate is defined

$$\underline{E}^{(m)} = \underline{U}^{(m)} - \underline{U} \qquad m \geq o$$

Subtracting Equation (45) from (46) we find

$$\underline{E}^{(m+1)} = (\underline{I} - \underline{A})\,\underline{E}^{(m)}$$

Repeated application of this relationship, starting with the initial vector $\underline{U}^{(o)}$ gives

$$\underline{E}^{(m)} = (\underline{I} - \underline{A})^{m}\;\underline{E}^{(o)} \qquad m > o \qquad (47)$$

Considering a single element $E_i^{(m)}$ of $\underline{E}^{(m)}$, if the $\lim_{m\to\infty} U_i^{(m)} = U_i$ it is necessary that $\lim_{m\to\infty} U_i^{(m)}$ and $\lim_{m\to\infty} E_i^{(m)}$ exist and $\lim_{m\to\infty} E_i^{(m)} = 0$. This condition will hold for all elements only if

$$\lim_{m \to \infty} (\underline{I} - \underline{A})^m \ \underline{E}^{(o)} = \underline{0} \tag{48}$$

for any arbitrary vector $\underline{E}^{(o)}$. $\underline{0}$ is the null vector of k elements. Equation (48) is valid then only if

$$\lim_{m \to \infty} (\underline{I} - \underline{A})^m = \underline{0} \tag{49}$$

where $\underline{0}$ is a square null matrix.

Milne (1953) shows that Equation (49) is the necessary and sufficient condition for convergence of an iterative method and this is assured if all the eigenvalues of the matrix $\underline{M} = \underline{I} - \underline{A}$ are less than one in absolute value.

Windsor (1957) has shown by derivation of the eigenvalues of $\underline{M}$ for a rectangular plate that Richardson's method is not convergent for the biharmonic equation. In the remainder of this section iterative methods will be reviewed and those which are especially useful for solving the biharmonic equation will be identified.

A. Point - Iterative Methods

The general form of Equation (43) for the biharmonic difference equation on a rectangular region with n columns and p rows of interior mesh points is of the form

$$\underline{A}\,\underline{U} = \underline{F} \tag{50}$$

where A is a kxk sparse, non-singular matrix, $\underline{U}$ and $\underline{F}$ are column vectors of k components and $k = np$.

The derivation of the matrix form of some iterative methods is simplified if we take

$$\underline{A} = \underline{G} + \underline{D} + \underline{H} \tag{51}$$

where $\underline{G}$ is a strictly lower triangular matrix with the elements a_{ij} $(i > j)$ below the diagonal of $\underline{A}$ and all other elements zero. $\underline{D}$ is formed of the diagonal elements a_{ii} of $\underline{A}$ and $\underline{H}$ is a strictly upper

triangular matrix with the elements a_{ij} $(i < j)$, $\underline{G}$, $\underline{D}$ and $\underline{H}$ are kxk matrices.

Richardson's Method

$$\begin{aligned} \underline{D}\,\underline{U} &= \underline{F} - (\underline{G} + \underline{H})\,\underline{U} \\ \underline{U}^{(m+1)} &= \underline{D}^{-1}\,\underline{F} - \underline{D}^{-1}\,(\underline{G} + \underline{H})\,\underline{U}^{(m)} \end{aligned} \tag{52}$$

where $\underline{D}^{-1}$ is the inverse of $\underline{D}$, a diagonal matrix with elements $1/a_{ii}$.

For the model problem, $\underline{D}=\underline{I}$, n=5, p=3 the point value is

$$\begin{aligned} U_{ij}^{(m+1)} = F_{ij} + .4[U_{i-1,j}^{(m)} + U_{i,j-1}^{(m)} + U_{i+1,j}^{(m)} + U_{i,j+1}^{(m)}] - .1[U_{i-1,j-1}^{(m)} + U_{i+1,j-1}^{(m)} + U_{i-1,j+1}^{(m)} \\ + U_{i+1,j+1}^{(m)}] - .05[U_{i-2,j}^{(m)} + U_{i,j-2}^{(m)} + U_{i+2,j}^{(m)} + U_{i,j+2}^{(m)}], \\ 1 \le i \le n, \quad 1 \le j \le p. \end{aligned} \tag{53}$$

Though the introduction of exterior mesh points may not be the best way to account for the normal derivative boundary condition one advantage of this approach is that the term F_{ij} will be automatically included when the column subscript of any term is not within the range 1 to n and when the row subscript of any term is not within the range 1 to p. Hence the F_{ij} can be dropped from Equation (53).

Gauss-Seidel Method

Varga (1962) identifies this also as the Liebmann method, point single step method and the method of successive displacements. The main difference between this and the Richardson method is the immediate introduction of the adjusted point values into a single vector iterate. The matrix form illustrates how this is accomplished.

$$\begin{aligned} (\underline{D} + \underline{G})\;\underline{U} &= \underline{F} - \underline{H}\;\underline{U} \\ \underline{U}^{(m+1)} &= (\underline{D}+\underline{G})^{-1}\,\underline{F} - (\underline{D}+\underline{G})^{-1}\,\underline{H}\,\underline{U}^{(m)} \end{aligned} \tag{54}$$

For the example problem, the point values are computed with

$$\begin{aligned} U_{ij}^{(m+1)} = F_{ij} + .4[U_{i-1,j}^{(m+1)} + U_{i,j-1}^{(m+1)} + U_{i+1,j}^{(m)} + U_{i,j+1}^{(m)}] - .1[U_{i-1,j-1}^{(m+1)} + U_{i+1,j-1}^{(m+1)} \\ + U_{i-1,j+1}^{(m)} + U_{i+1,j+1}^{(m)}] - .05[U_{i-2,j}^{(m+1)} + U_{i,j-2}^{(m+1)} + U_{i+2,j}^{(m)} + U_{i,j+2}^{(m)}] \end{aligned} \tag{55}$$

Here too, the F_{ij} will be automatically included if point values have been provided on the boundary and on the first exterior mesh line. Starting at U_{11} and sweeping along columns successively, the required (m+1) iterate values will be available as specified by Equation (55). Boundary values computed initially remain fixed. Exterior point values can be recomputed at the end of each iteration.

This method is convergent for the biharmonic difference equation but the rate of convergence is so slow that it has limited usefulness. For a problem with 66 interior mesh points 1098 Gauss-Seidel iterations satisfied the same convergence criteria as 224 iterations of the successive overrelaxation method.

Successive Overrelaxation

This is also known as the extrapolated Liebmann method, Parter (1959), systematic overrelaxation and the extrapolated Gauss-Seidel. Young (1954) proposed an acceleration of the convergence of the Gauss-Seidel method based on an examination of the changes introduced in a given vector iterate and then introducing a multiple of the change at each point. The point value obtained by the Gauss-Seidel method will be designated $\bar{U}_i^{(m+1)}, 1 \leq i \leq k$.

Then

$$U_i^{(m+1)} = U_i^{(m)} + \omega\{\bar{U}_i^{(m+1)} - U_i^{(m)}\} = (1-\omega)U_i^{(m)} + \omega\bar{U}_i^{(m+1)} \qquad (56)$$

The quantity ω is the relaxation factor. For overrelaxation, $1 < \omega < 2$. When $\omega = 1$ the method is Gauss-Siedel. The matrix representation must account for the prior application of Equation (56) at all preceding points. Hence,

$$D\bar{U}^{(m+1)} + \underline{G}\underline{U}^{(m+1)} = \underline{F} - H\underline{U}^{(m)} \qquad (57)$$

$$U^{(m+1)} = U^{(m)} + \omega(\bar{U}^{(m+1)} - \underline{U}^{(m)}) \qquad (58)$$

Eliminating $\bar{U}^{(m+1)}$ between Eq.'s (57) and (58) we find the matrix form of the successive overrelation method

$$(\underline{D}+\omega\underline{G})\,\underline{U}^{(m+1)} = [(1-\omega)\underline{D} - \omega\underline{H}]\,U^{(m)} + \omega\underline{F} \tag{59}$$

The component U_{ij} of the vector iterate at a point where $1 \leq i \leq n, 1 \leq j \leq p$ is given by

$$U_{ij}^{(m+1)} = U_{ij}^{(m)} + \omega\left[F_{ij} + .4(U_{i-1,j}^{(m+1)} + U_{i,j-1}^{(m+1)} + U_{i+1,j}^{(m)} + U_{i,j+1}^{(m)}) - .1(U_{i-1,j-1}^{(m+1)} + U_{i+1,j-1}^{(m+1)} + U_{i-1,j+1}^{(m)} + U_{i+1,j+1}^{(m)}) - .05(U_{i-2,j}^{(m+1)} + U_{i,j-2}^{(m+1)} + U_{i+2,j}^{(m)} + U_{i,j+2}^{(m)}) - U_{ij}^{(m)}\right] \tag{60}$$

The rate of convergence of Equation (59) depends on the value of ω. The optimum value of ω is given by the formula

$$\omega_b = \frac{2}{1+\sqrt{1-\rho^2}} \tag{61}$$

where ρ is the spectral radius of the point Jacobi method.

$$\lambda^2_m = \|\underline{E}^{(m)}\| / \|\underline{E}^{(m-1)}\| \tag{62}$$

and $$\rho = \lim_{m\to\infty} \lambda_m$$

Any norm of the error vector $\underline{E}$ could be used. One readily computed is

$$\|\underline{E}^{(m)}\| = \sum_{i=1}^{k} |E_i^{(m)}| \qquad \text{(See Appendix A)}$$

A common procedure for the determination of the relaxation factor is to set it initially to one; and, after a number of iterations, say 100, use Eq.'s (62) and (61) for calculating a new ω. Subsequently ω can be recomputed every ten or twenty iterations. Forsythe and Wasow (1961) pp. 368-372 describe two alternative approaches and indicate that the determination of a good estimate of ω_b early in the computation is an investigation in which there is continuing interest.

The relationship between the estimated spectral radius and the number of iterations required for convergence of the model problem of Fig. 4.1, p.20 is given in Fig. 4.3.

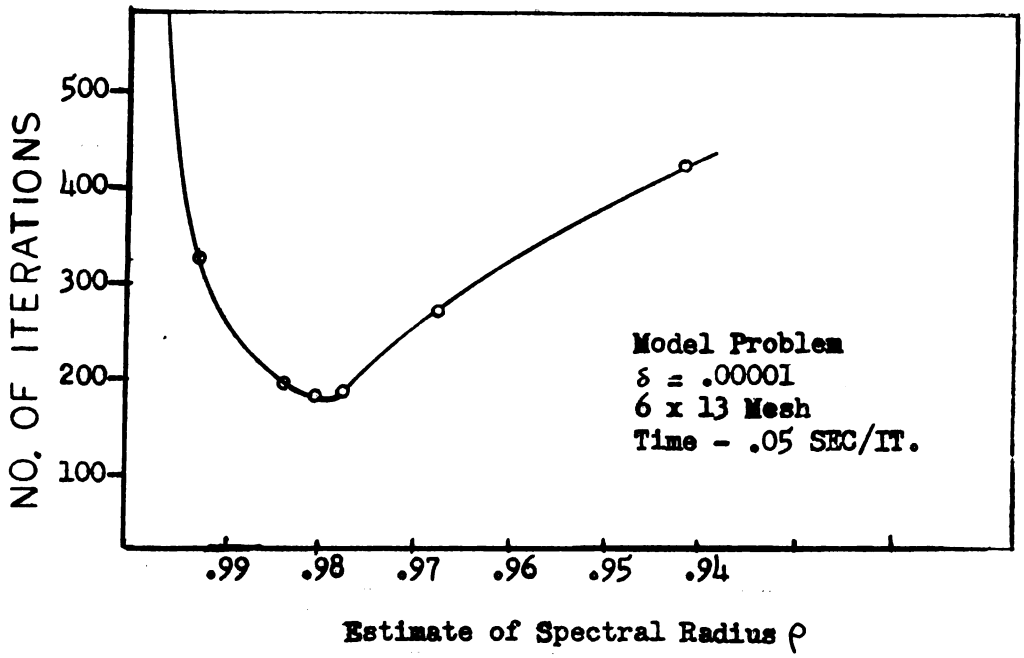


Figure 4.3 Successive overrelaxation

Number of iterations vs. ρ

B. Methods of Block Iteration

The iterative methods considered thus far used an explicit formula for the calculation of each component of the vector iterate. Is it possible to use direct methods to find a block of components of the vector iterate? Consider the model problem in terms of submatrices, Equation (43). Taking arbitrary values for the column submatrices $\underline{U}_1$ and $\underline{U}_3$, the components of $\underline{U}_2$ can be determined by solving

$$\underline{M}\,\underline{U}_2^{(m+1)} = \underline{F}_2 - \underline{B}\,\underline{U}_1^{(m)} - \underline{B}\,\underline{U}_3^{(m)} \qquad (63)$$

where $\underline{U}_2 = \{U_{21}, U_{22}, U_{23}, U_{24}, U_{25}\}$. Since $\underline{M}$ is an (nxn) matrix, substantially smaller than $\underline{A}$ which is (kxk) the direct solution of Equation (63) will not require unreasonable blocks of computer memory and is an acceptable procedure. Using this method the point values of the vector iterate are not determined explicitly one at a time; instead n components are determined simultaneously. Hence, this procedure is called the simultaneous displacement method and is classified as implicit.

The matrix form of certain block methods assures faster average rates of convergence. See Appendix A, Section 5.

Arms, Gates and Zondek (1956) and Keller (1958) have investigated block methods using the components of the solution vector on a line as the basis for partitioning the matrix $\underline{A}$. For the biharmonic difference equation, "two line" schemes and the alternating-direction implicit method have been studied and appear to have advantages over other block methods. See Parter (1961A)

The Alternating-Direction Implicit Method

Peaceman and Rachford (1955) found that the rate of convergence of a "line" method could be substantially improved if after sweeping all rows using the simultaneous displacement method and a relaxation factor, the next sweep of all the mesh points was made by columns.

Conte and Dames (1958) derived a convergent, alternating-direction iterative method for solving the biharmonic equation. This method is similar to the alternating-direction method for solving Laplace's equation proposed by Douglas and Rachford (1956). The derivatives in the biharmonic equations are replaced using central difference approximations

$$\frac{\partial^4\phi}{\partial x^4} \approx [\underline{G}\underline{U}]_{ij} = \delta_x^4 U_{ij} = U_{i,j-2} - 4U_{i,j-1} + 6U_{i,j} - 4U_{i,j+1} + U_{i,j+2}$$

$$\frac{\partial^4\phi}{\partial y^4} \approx [\underline{H}\underline{U}]_{ij} = \delta_y^4 U_{ij} = U_{i-2,j} - 4U_{i-1,j} + 6U_{i,j} - 4U_{i+1,j} + U_{i+2,j}$$

$$2\frac{\partial^4\phi}{\partial x^2 \partial y^2} \approx [\underline{P}\underline{U}]_{ij} = 2\delta_x^2\delta_y^2 U_{ij} = 2\{4U_{ij} - 2[U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1}] + U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1} + U_{i-1,j-1}\}$$

$$\nabla^4\phi \approx \delta_x^4 U_{ij} + 2\delta_x^2\delta_y^2 U_{ij} + \delta_y^4 U_{ij}$$

As shown for the model problem, Equation (42), introduction of the difference equations reduces the problem to the solution of a linear

system in k unknowns

$$\underline{A}\ \underline{U} = \underline{F}$$

which can be written

$$(\underline{H} + \underline{G} + \underline{P})\ \underline{U} = \underline{F}$$

where $[\underline{H}\ \underline{U}]_{ij}$, $[\underline{G}\ \underline{U}]_{ij}$ and $[\underline{P}\ \underline{U}]_{ij}$ respectively represent the components of $\underline{H}\ \underline{U}$, $\underline{G}\ \underline{U}$ and $\underline{P}\ \underline{U}$ at the mesh point (i,j).

For the alternating direction implicit method, Equation (43) is replaced by a pair of matrix equations

$$(r\underline{H} + \underline{I})\ \underline{U} = (\underline{I} - r\underline{G} - r\underline{P})\ \underline{U} + r\underline{F}$$

$$(r\underline{G} + r\underline{P} + \underline{I})\ \underline{U} = (\underline{I} - r\underline{H})\ \underline{U} + r\underline{F}$$

where r is any positive scalar. The iterative scheme in the form proposed by Peaceman and Rachford would appear

$$(r_{m+1}\ \underline{H} + \underline{I})\ \underline{U}^{(m+\frac{1}{2})} = (\underline{I} - r_{m+1}\ \underline{G} - r_{m+1}\ \underline{P})\ \underline{U}^{(m)} + r_{m+1}\ \underline{F} \qquad (64)$$

$$(r_{m+1}\ \underline{G} + r_{m+1}\ \underline{P} + \underline{I})\ \underline{U}^{(m+1)} = (I - r_{m+1}\ \underline{H})\ \underline{U}^{(m+\frac{1}{2})} + r_{m+1}\ \underline{F} \qquad (65)$$

The first sweep of the mesh is by rows and only Equation (64) is solved for the vector iterate $\underline{U}^{(m+\frac{1}{2})}$. This is an implicit method since all the components of $\underline{U}^{(m+\frac{1}{2})}$ in one row are determined by (64). It is necessary to retain all the components of $\underline{U}^{(m)}$ while solving for $\underline{U}^{(m+\frac{1}{2})}$. During the second sweep Equation (65) is solved one column at a time for the components of $\underline{U}^{(m+1)}$. The solution along one row or one column is obtained by direct elimination. Conte and Dames use a factorization technique, which is well adapted for the case of Fig.4.2, where there are, at most, five unknowns along a mesh line.

The Douglas-Rachford method is a variant obtained by changing (65) to a form which does not contain $\underline{H}$. Substituting for $\underline{H}\ \underline{U}^{(m+\frac{1}{2})}$ from (64) we obtain

$$(r_{m+1}\ \underline{G} + r_{m+1}\ \underline{P} + \underline{I})\ \underline{U}^{(m+1)} = 2\ \underline{I}\ \underline{U}^{(m+\frac{1}{2})} - (\underline{I} - r_{m+1}\ \underline{G} - r_{m+1}\ \underline{P})\ \underline{U}^{(m)}$$

Conte and Dames use a simplified form not containing $\underline{P}$

$$U^{(m+1)} = U^{(m+\frac{1}{2})} - r_{m+1}\left[\underline{G}\,\underline{U}^{(m+1)} - \underline{G}\,\underline{U}^{(m)}\right] \tag{66}$$

The use of Equation (66) for the column solution provides a simpler computational procedure than the one associated with Equation (65).

The determination of optimum acceleration parameters requires consideration of another form of the system of linear equations. This is found by combining Eq. (64) and (66) into a single equation.

$$\underline{U}^{(m+1)} = \underline{Rr}_{m+1}\,\underline{U}^{(m)} + \underline{Sr}_{m+1}\,\underline{F} \tag{67}$$

where $\underline{Rr} = (\underline{I} + r\underline{G})^{-1}\left[(r\underline{H} + \underline{I})^{-1}\ (\underline{I} - r\underline{G} - r\underline{P}) + r\underline{G}\right]$

$$\underline{Sr} = (\underline{I} + r\underline{G})^{-1}\ (r\underline{H} + \underline{I})^{-1}$$

The difference between the m^{th} vector iterate $\underline{U}^{(m)}$ and the solution vector $\underline{U}$ is the error vector $\underline{E}^{(m)}$. It can be shown (see Appendix A) that

$$\underline{E}^{(m+1)} = \underline{Rr}_{m+1}\,\underline{E}^{(m)} \tag{68}$$

If $\underline{U}^{(o)}$ is the initial, arbitrarily selected vector iterate then $\underline{E}^{(o)} = \underline{U}^{(o)} - \underline{U}$ and

$$\underline{E}^{(m)} = \underline{Rr}_m \cdot \underline{Rr}_{m-1} \cdots\cdots \underline{Rr}_2 \cdot \underline{Rr}_1\,\underline{E}^{(o)}$$

For a convergent iterative scheme $\underline{E}^{(m)} \to \underline{0}$ as $m \to \infty$, and convergence of this method can be accelerated by the choice of r_m for each iteration. However, the relationship of this iterative method to others considered is less complicated for the special case when a single value is assigned to the scalar r. Then

$$\underline{E}^{(m)} = (\underline{Rr})^m\,\underline{E}^{(o)} \tag{69}$$

and the eigenvalues of the Conte-Dames matrix $\underline{Rr}$ provide the basis for determining the optimum acceleration parameters. See Fig. 4.4 which shows the relationship between the acceleration parameter r and the number of iterations required for convergence of the model problem.

Fig. 4.1, p. 20.

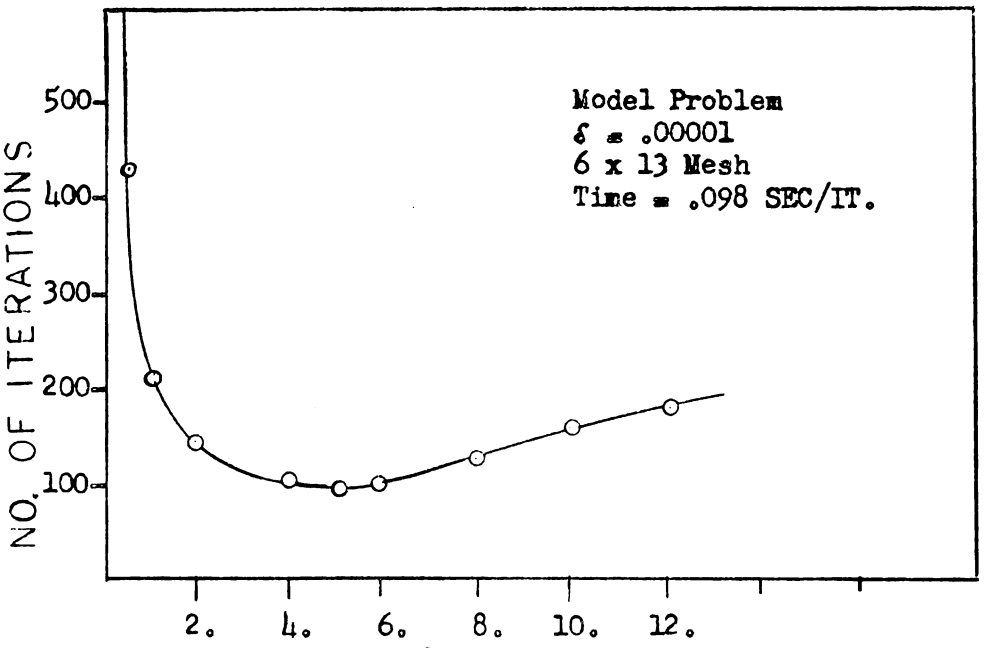


Figure 4.4 Alternating direction implicit method

Number of iterations vs. r

Choice of optimum values of r_m for a square plate has been considered by Conte and Dames (1958) and Fairweather and Mitchell (1964). For this particular geometry the eigenfunctions of Equation (68) can be expanded in the form

$$E_{ij}^{(m)} = A \sin(p\pi ih) \sin(q\pi jh), \quad (p, q = 1, 2, \cdots, k-1)$$

Consider an amplification factor

$$\lambda_{pq} = \frac{E_{ij}^{(m+1)}}{E_{ij}^{(m)}}$$

If this is substituted into Equation (68) we find

$$(16\, r_{m+1})\, \lambda_{p,q} = \frac{(1 - 16\, r_{m+1}\, S_p^2\, S_q^2)^2}{1 + 16 r_{m+1}\,(S_p^4 + S_q^4) + 256\, r_{m+1}^2\, S_p^4 S_q^4} \qquad (70)$$

where $S_p = \sin \frac{p\pi h}{2}$, $S_q = \sin \frac{q\pi h}{2}$, h is the mesh interval.

The error associated with the initial vector iterate can be

expanded in the form

$$E_{ij}^{(0)} = \sum_{p,q=1}^{h-1} C_{p,q}^{(0)} \sin(p\pi ih)\sin(q\pi jh),$$

and the error vector at the m^{th} iteration can be written

$$E_{ij}^{(m)} = \sum_{p,q=1}^{h-1} C_{p,q}^{(m)} \sin(p\pi ih)\sin(q\pi jh),$$

where

$$C_{p,q}^{(m)} = \prod_{l=1}^{m} (16r)_l \lambda_{p,q} C_{p,q}^{(0)}$$

It follows from Equation (70) that $0 \leq (16r)_l \lambda_{p,q} \leq 1$ for all p and q if r_l is positive. Hence, after m iterations each component of the error decreases by a factor

$$\prod_{l=1}^{m} (16r)_l \lambda_{p,q}$$

if r_l is positive. The minimum value of $(16r)_l \lambda_{p,q}$ found from Equation (70), is zero and occurs when

$$16r = \frac{1}{s_p^2 s_q^2}$$

This indicates that an appropriate choice of the r_l can be made so all components of the error vector will vanish and the exact solution can be obtained. Rather than attempt to find this optimum r_l, a less complicated procedure consists of choosing the r_l which optimize the rate of convergence of the method.

Consider a slightly different expression for an amplification factor

$$16r\,\bar{\lambda}_{p,q} \equiv \frac{(1 - 16r s_p^2 s_q^2)^2}{(1 + 16r s_p^2 s_q^2)^2}$$

Since

$$S_p^4 + S_q^4 \geq 2 S_p^2 S_q^2$$

we find from Equation (70)

$$|16r\bar{\lambda}| \geq |16r\lambda|$$

for all p and q. The analysis of the problem of finding an optimum set of acceleration parameters is treated by Douglas and Rachford (1956). The factor by which the error is decreased after m iterations is

$$z_t(S_p^2 S_q^2, 16r_l) = \prod_{l=1}^{t} \bar{\lambda}_{p,q}(16r_l) = \prod_{l=1}^{t} \left[\frac{(1/16r_l - S_p^2 S_q^2)}{(1/16r_l + S_p^2 S_q^2)}\right]^2$$

It is necessary to find the set of r_l ($l = 1, 2, \text{——} t$) giving a maximum value of z_t which is as small as possible for ($p,q = 1, 2, 3$ ——k-1). The Douglas-Rachford solution treats this as a Chebyshev minimax problem, while Conte & Dames recommend a set of acceleration parameters of the form

$$16r_l = \alpha^{(1-l)}, \qquad l = 1, 2, \text{——}, t$$

where $0 < \alpha < 1$. This permits the determination of an upper bound on z_t

$$z_t(S_p^2 S_q^2, 16r_l) \leq P_t(\alpha) \equiv \left[\frac{1-\alpha^{1/2}}{1+\alpha^{1/2}} e^{-\frac{\alpha^{3/2}}{1-\alpha}}\right]^4 \qquad (71)$$

The formal procedure, outlined by Conte and Dames, starts with a choice of $P_t(\alpha)$ which permits the determination of the number of cycles of t double sweeps of the mesh required for the selected reduction factor. However, the choice of $P_t(\alpha)$ is subject to the empirical observation that best results are obtained for $\alpha < 0.2$. The number of iterations, t, per cycle is computed from

$$t \geq 1 + \frac{4 \log(\sin \frac{\pi h}{2})}{\log \alpha} \qquad (72)$$

Then

$$r_l = \frac{\alpha^{(1-l)}}{16}, \qquad l = 1, 2, 3, \text{---}, t \qquad (73)$$

gives the value of r_ℓ for the ℓth iteration in the cycle.

As an example of the method consider a 20 x 20 grid on which it is desirable to reduce the initial error by a factor 10^{-6}. Selecting $\alpha = 0.2$ makes $P_t \approx 0.01$. The number of cycles required is determined from

$$(P_t)^n = 10^{-6}$$

Thus, $n = 3$ is the number of cycles. The number of iterations per cycle is

$$t \geq 1 + \frac{4 \log \sin \pi/40}{\log (.2)} \approx 7.35$$

or $t = 8$ and

$$r_\ell = \frac{(0.2)^{1-k}}{16}, \qquad k = 1, 2, \ldots, 8.$$

Factorization Technique

The difference equation (64) can be written in the form

$$U_{i-2,j}^{(m+1/2)} - 4U_{i-1,j}^{(m+1/2)} + (6 + \frac{1}{r_{m+1}}) U_{i,j}^{(m+1/2)} - 4U_{i+1,j}^{(m+1/2)} + U_{i+2,j}^{(m+1/2)} = F_{Rj} \qquad (74)$$

where

$$F_{Rj} = (\frac{1}{r_{m+1}} - 14) U_{ij}^{(m)} + 4(U_{i+1,j}^{(m)} + U_{i-1,j}^{(m)}) + 8(U_{i,j+1}^{(m)} + U_{i,j-1}^{(m)}) - 2(U_{i+1,j+1}^{(m)} + U_{i-1,j+1}^{(m)} + U_{i-1,j-1}^{(m)} + U_{i+1,j-1}^{(m)}) - U_{i,j+2}^{(m)} - U_{i,j-2}^{(m)}$$

Similarly, difference equation (66) can be written in the form

$$U_{i,j-2}^{(m+1)} - 4U_{i,j-1}^{(m+1)} + (6 + \frac{1}{r_{m+1}}) U_{i,j}^{(m+1)} - 4U_{i,j+1}^{(m+1)} + U_{i,j+2}^{(m+1)} = F_{ci} \qquad (75)$$

where

$$F_{ci} = \frac{1}{r_{m+1}} U_{ij}^{(m+1/2)} + U_{i,j-2}^{(m)} - 4U_{i,j-1}^{(m)} + 6U_{i,j}^{(m)} - 4U_{i,j+1}^{(m)} + U_{i,j+2}^{(m)}$$

Using (74) at all mesh points in a row leads to a "quidiagonal" system of linear equations of the form

$$\begin{bmatrix} C_1 & -4 & 1 & & & & & & & \\ -4 & C_2 & -4 & 1 & & & & & & \\ 1 & -4 & C_3 & -4 & 1 & & & & & \\ & 1 & -4 & C_4 & -4 & 1 & & & & \\ & & - & - & - & - & - & & & \\ & & & - & - & - & - & - & & \\ & & & & - & - & - & - & - & \\ & & & & & 1 & -4 & C_{n-1} & -4 & \\ & & & & & & 1 & -4 & C_n \end{bmatrix} \begin{bmatrix} U_{i1} \\ U_{i2} \\ U_{i3} \\ U_{i4} \\ - \\ - \\ - \\ U_{i,n-1} \\ U_{i,n} \end{bmatrix} = \begin{bmatrix} F_{R1} \\ F_{R2} \\ F_{R3} \\ F_{R4} \\ \\ \\ \\ F_{Rn-1} \\ F_{Rn} \end{bmatrix}$$

where the C's all have the value $(6 + \frac{1}{r_{m+1}})$. A system of linear equations of the same general form is obtained when (75) is applied at all mesh points in a single column. The factorization technique is an efficient direct method of solving a quidiagonal system of equations. Take

$$w_1 = C_1, \qquad w_q = C_q - g_{q-2} - d_q B_{q-1}$$

$$B_0 = 0, \quad B_1 = -4/C_1, \qquad Bq = -(4 + d_q g_{q-1})/w_q, \quad B_n = 0$$

$$g_0 = 0, \quad g_1 = 1/C_1, \qquad g_q = 1/w_q, \quad g_n = g_{n-1} = 0$$

$$d_q = -4 - B_{q-2}$$

$$h_0 = 0, \; h_1 = F_{R1}/C_1, \; h_q = \left[F_{Rq} - h_{q-2} - d_q h_{q-1} \right] / w_q, \; 2 \leq q \leq n$$

Starting with the specified initial values the n values of h are computed. Then the values of U_q are computed by back substitution

$$U_n = h_n$$

$$U_q = h_q - B_q U_{q+1} - g_q U_{q+2}, \qquad q = n-1, n-2, \text{-----}, 1$$

The alternating-direction method was initially applied to the solution of parabolic and elliptic partial differential equations of the second order. For Laplace's equation with sufficiently small mesh spacing it provides a significant increase in the rate of convergence per iteration over the point successive overrelaxation method. This advant-

age must be weighed against the requirement for the double computation in a two sweep scheme and the need for storage of the $(m+\frac{1}{2})$ vector iterate. Compared with recently developed block successive overrelaxation methods, superiority of the rate of convergence is not rigorously established. Varga (1962) notes that the convergence of the Peaceman-Rachford and similar alternating direction methods has been established only for rectangular regions. Though there has been some success in applying alternating direction implicit methods to more general regions, the convergence in the general case is yet to be justified. There is no general theory for the determination of optimum convergence parameters except for rectangular regions. These same limitations apply to the solution of the biharmonic equation and in addition Keller (1961) demonstrated for several block methods that corresponding biharmonic schemes converge more slowly than Laplace schemes.

Semi-iterative Methods

A system of linear equations

$$\underline{A}\,\underline{U} = \underline{F},$$

can be solved by an iterative method of the form

$$\underline{U}^{(m+1)} = \underline{M}\,\underline{U}^{(m)} + \underline{F} \qquad m \geq 0 \tag{76}$$

if $\underline{A} = \underline{I} - \underline{M}$ is a positive-definite nxn matrix. See Appendix A. As $m \to \infty$, $\underline{U}^{(m)}$ converges to the unique solution of the system of equations. A semi-iterative method uses an algebraic combination of solution vector iterates $\underline{U}^{(m)}$ as a means of increasing the rate of convergence.

Starting with an initial estimate $\underline{U}^{(o)}$, the error vector associated with the vector iterate $\underline{U}^{(m)}$ is given by

$$\underline{E}^{(m)} = \underline{U}^{(m)} - \underline{U} = \underline{M}^{m}\ \underline{E}^{(o)}$$

A linear algebraic combination of the vector iterates $\underline{U}^{(m)}$ is introduced

$$\underline{V}^{(m)} = \sum_{j=o}^{m} p_j(m)\, \underline{U}^{(j)} \qquad m \geq o \tag{77}$$

where the coefficients $p_j(m)$ are selected so that each $\underline{V}^{(m)}$ is a weighted average of the $\underline{U}^{(j)}$ and a better approximation to the solution vector $\underline{U}$ than $\underline{U}^{(m)}$. For the special case $\underline{U}^{(o)} = \underline{U}$ it is necessary that

$$\sum_{j=o}^{m} p_j(m) = 1 \tag{78}$$

Then $\underline{V}^{(m)} = \underline{U}$ for all $m \geq 0$. The requirement of Equation (78) will be imposed on the constants $p_j(m)$ for any arbitrary choice of $\underline{U}^{(o)}$.

The error vector associated with $\underline{V}^{(m)}$ is denoted $\underline{E}''^{(m)}$

$$\underline{E}''^{(m)} = \underline{V}^{(m)} - \underline{U} = \sum_{j=o}^{m} p_j(m)\, \underline{U}^{(j)} - \underline{U} \tag{79}$$

Since the constants $p_j(m)$ must satisfy Equation (78)

$$\begin{aligned}
\underline{E}''^{(m)} &= \sum_{j=o}^{m} p_j(m)\, \underline{U}^{(j)} - \Big(\sum_{j=o}^{m} p_j(m)\Big)\underline{U} \\
\underline{E}''^{(m)} &= \sum_{j=o}^{m} p_j(m)\big[\underline{U}^{(j)} - \underline{U}\big] \\
\underline{E}''^{(m)} &= \sum_{j=o}^{m} p_j(m)\, \underline{E}^{(j)} \\
\underline{E}''^{(m)} &= \Big(\sum_{j=o}^{m} p_j(m)\, \underline{M}^{j}\Big)\underline{E}^{(o)}
\end{aligned} \tag{80}$$

If we introduce a polynomial in a component of $\underline{U}$, defined by

$$P_m(u) = \sum_{j=o}^{m} p_j(m)\, u^j \qquad m \geq 0 \tag{81}$$

then we can write (80) in the form

$$\underline{E}''^{(m)} = P_m(\underline{M})\, \underline{E}^{(o)}$$

where $P_m(\underline{M})$ is a polynonial in the matrix $\underline{M}$. The condition imposed by Equation (78) on $\sum p_j(m)$ requires $P_m(1) = 1$.

Using the definitions of matrix and vector norms given in Appendix

A, we can write

$$\|E^{(m)}\| = \|P_m(M)\,E^{(0)}\| \le \|P_m(M)\|\,\|E^{(0)}\| \qquad m \ge 0$$

For $\|P_m(M)\| < 1$ the average rate of convergence of the semi-iterative method for m iterations is defined as

$$R\left[P_m(M)\right] \equiv \frac{-\ln\|P_m(M)\|}{m}$$

If the polynomial is selected so $P_m(u) = u^m$, the vector $V^{(m)}$ is identically $U^{(m)}$ and the average rate of convergence can be readily simplified.

$$R\left[P_m(M)\right] = \frac{-\ln\|P_m(M)\|}{m} = \frac{-\ln\|M^m\|}{m} = R\left[M^m\right]$$

and, as shown in Appendix A, it is equivalent to the convergence rate of the basic iterative formulation of the problem. The average rate of convergence of $V^{(m)}$ will be optimised by finding the minimum of $\|P_m(M)\|$ under the restriction $P_m(1) = 1$. See Varga (1962) Chap. 5.

Chebyshev Semi-iterative Methods

Golub and Varga (1961) identified a polynomial which satisfies the requirements for optimising the average rate of convergence. They used

$$P_m(u) = \frac{C_m\left[\frac{2u-(b+a)}{b-a}\right]}{C_m\left[\frac{2-(b+a)}{b-a}\right]}, \qquad m \ge 0$$

where $C_m(z) = \begin{cases} \cos(m \cos^{-1} z), & -1 \le z \le 1, \quad m \ge 0 \\ \cosh(m \cosh^{-1} z), & z \ge 1, \quad m \ge 0 \end{cases}$

are the Chebyshev polynomials

The Chebyshev semi-iterative method for the problem specified in Equation (76) has the form

$$V^{(m+1)} = \omega_{m+1}\left[M\,V^{(m)} + F - V^{(m-1)}\right] + V^{(m-1)} \qquad (82)$$

where (See Varga (1962) p.138)

$$\omega_1 = 1, \quad \omega_{m+1} = \frac{2\, C_m(1/\rho)}{C_{m+1}(1/\rho)}, \qquad m > 0$$

and ρ is the spectral radius of $\underline{M}$. Note should be made of the absence of the vector iterate $\underline{U}^{(m)}$. It is not used in the computation. The vector iterate $\underline{V}^{(m+1)}$ is formed directly from the preceding vector iterates $\underline{V}^{(m)}$ and $\underline{V}^{(m-1)}$. It is necessary to store both $\underline{V}^{(m)}$ and $\underline{V}^{(m-1)}$. Spectral radius is defined in Appendix A, Sec.2.

Method for a Cyclic Matrix

Consider a slightly different approach to the solution of Equation (70). Let

$$\underline{U} = \underline{M}\,\underline{Y} + \underline{F}$$

$$\underline{Y} = \underline{M}\,\underline{U} + \underline{F}$$

or equivalently

$$\underline{W} = \underline{N}\,\underline{W} + \underline{G} \tag{83}$$

where

$$\underline{W} = \begin{bmatrix} \underline{U} \\ \underline{Y} \end{bmatrix}, \qquad \underline{N} = \begin{bmatrix} \underline{0} & \underline{M} \\ \underline{M} & \underline{0} \end{bmatrix}, \qquad \underline{G} = \begin{bmatrix} \underline{F} \\ \underline{F} \end{bmatrix}.$$

If $\underline{M}$ is an nxn convergent matrix, Equation (83) has a unique solution $\underline{W}$ and the subvectors $\underline{U}$ and $\underline{Y}$ are equal to the solution of Equation (76). Since the solution of (83) would require the manipulation of (2nx2n) matrices, it is not recommended as a practical method. However, it does provide the basis for an improvement of the semi-iterative method.

Assume that $\underline{M}$ in Equation (76) is an nxn, Hermitian, convergent matrix with the special form

$$\underline{M} = \begin{bmatrix} \underline{0} & \underline{H} \\ \underline{H}^* & \underline{0} \end{bmatrix}$$

where the submatrices are square and $\underline{H}^*$ is the conjugate transpose of $\underline{H}$.

A matrix M which can be expressed in this form is said to be weakly cyclic of index 2. A weakly cyclic matrix $\underline{A}$ of index k is such that $\underline{A}^k$ has real non-negative, eigenvalues. For the assumed form of $\underline{M}$ the successive overrelaxation method for Equation (76) can be written (Varga (1962) p. 149)

$$\underline{U}_1^{(m+1)} = \omega \left[\underline{H}\, \underline{U}_2^{(m)} + \underline{F}_1 - \underline{U}_1^{(m)}\right] + \underline{U}_1^{(m)},$$
$$\underline{U}_2^{(m+1)} = \omega \left[\underline{H}^*\underline{U}_1^{(m+1)} + \underline{F}_2 - \underline{U}_2^{(m)}\right] + \underline{U}_2^{(m)}, \qquad m \geq 0$$

where $\underline{U}$ and $\underline{F}$ are partioned into $\underline{U}_1$, $\underline{U}_2$ and $\underline{F}_1$, $\underline{F}_2$ respectively.

The Cyclic Chebyshev Semi-iterative Method

The Chebyshev semi-iterative method corresponding to this special matrix $\underline{M}$ can be written (Varga (1962) p.150)

$$\underline{U}_1^{(m+1)} = \omega_{m+1} \left[\underline{H}\, \underline{U}_2^{(m)} + \underline{F}_1 - \underline{U}_1^{(m-1)}\right] + \underline{U}_1^{(m-1)},$$
$$\underline{U}_2^{(m+1)} = \omega_{m+1} \left[\underline{H}^*\underline{U}_1^{(m)} + \underline{F}_2 - \underline{U}_2^{(m-1)}\right] + \underline{U}_2^{(m-1)}, \qquad m \geq 1$$

Since the iteration parameter ω_{m+1} is a function of the number of iterations m, the cyclic characteristic of $\underline{M}$ permits the following form

$$\underline{U}_1^{(2m-1)} = \omega_{2m+1} \left[\underline{H}\, \underline{U}_2^{(2m)} + \underline{F}_1 - \underline{U}_1^{(2m-1)}\right] + \underline{U}_1^{(2m-1)}, \quad m \geq 1$$
$$\underline{U}_2^{(2m-2)} = \omega_{2m+2} \left[\underline{H}^*\underline{U}_1^{(2m+1)} + \underline{F}_2 - \underline{U}_2^{(2m)}\right] + \underline{U}_2^{(2m)} \quad m \geq 0$$

This is known as the cyclic Chebyshev semi-iterative method. Due to skipping half the vector iterates the rate of convergence of this method is twice that of the Chevyshev semi-iterative method. Basically though, this method is just a variation of the successive overrelaxation method. Varga (1962) shows that the average rate of convergence of the cyclic Chebyshev semi-iterative method is better than the average rate of convergence of the successive overrelaxation method. The cyclic Chebyshev semi-iterative method will be used to solve the model problem and the convergence rates of successive overrelaxation, the alternating direction implicit method and the cyclic Chebyshev semi-iterative methods will be compared.

Cyclic Chebyshev Semi-iterative Method for the Biharmonic Equation

Consider the solution of the biharmonic equation for a rectangular region with a grid of n interior mesh points in the x direction and p interior mesh points in the y direction.

The difference equation can be written in the form

$$\underline{A}\,\underline{U} = \underline{F} \qquad (84)$$

where $N = np$, $\underline{A}$ is an NxN coefficient matrix, $\underline{U}$ is the solution column vector and $\underline{F}$ is a column vector which accounts for the specified boundary conditions.

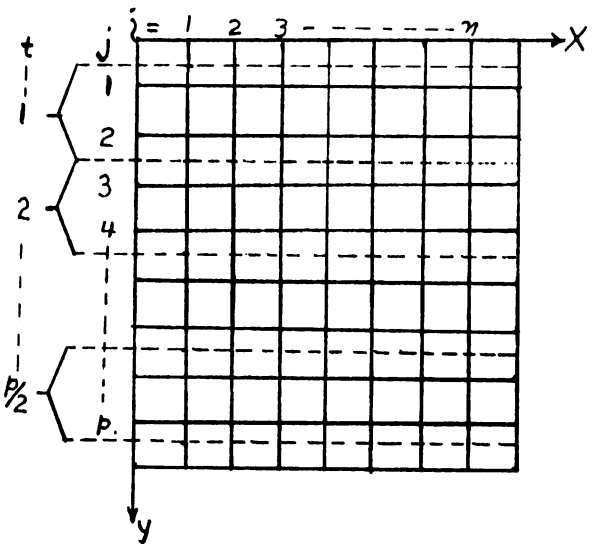


Figure 4.5 Two line blocks

The formulation of the model problem in terms of sub-matrices, Equation (43), can be extended for the general case

$$\begin{bmatrix} \underline{M} & \underline{B} & \underline{I} & \underline{0} & \underline{0} & \underline{0} & & \underline{0} \\ \underline{B} & \underline{M} & \underline{B} & \underline{I} & \underline{0} & \underline{0} & & \underline{0} \\ \underline{I} & \underline{B} & \underline{M} & \underline{B} & \underline{I} & \underline{0} & & \underline{0} \\ \underline{0} & \underline{I} & \underline{B} & \underline{M} & \underline{B} & \underline{I} & & \underline{0} \\ & & & & & & & \\ \underline{0} & \underline{0} & - & - & - & - & \underline{I} & \underline{B} & \underline{M} \end{bmatrix} \begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \\ \underline{U}_3 \\ \underline{U}_4 \\ \\ \underline{U}_p \end{bmatrix} = \begin{bmatrix} \underline{F}_1 \\ \underline{F}_2 \\ \underline{F}_3 \\ \underline{F}_4 \\ \\ \underline{F}_p \end{bmatrix}$$

The $\underline{M}$, $\underline{B}$, $\underline{I}$ are nxn matrices and $\underline{U}_k$ and $\underline{F}_k$ are n component column sub-matrices corresponding to the solution vector components along one row.

Griffin and Varga (1963) select a two row block of the components of $\underline{U}$ and show that the associated matrix $\underline{A}$ is of the form required for the cyclic Chebyshev scheme. The number of rows of interior mesh points, p, must be even. This permits the selection of $t = \frac{p}{2}$ two row column submatrices for the partitioning of $\underline{U}$.

$$\begin{bmatrix} \underline{L} & \underline{K} & \underline{0} & & \underline{0} \\ \underline{K}^T & \underline{L} & \underline{K} & & \underline{0} \\ \underline{0} & \underline{K}^T & \underline{L} & & \underline{0} \\ & & & & \\ \underline{0} & & & \underline{K}^T & \underline{L} \end{bmatrix} \begin{bmatrix} \underline{T}_1 \\ \underline{T}_2 \\ \underline{T}_3 \\ \\ \underline{T}_t \end{bmatrix} = \begin{bmatrix} \underline{F}'_1 \\ \underline{F}'_2 \\ \underline{F}'_3 \\ \\ \underline{F}'_t \end{bmatrix} \tag{85}$$

where

$$\underline{L} = \begin{bmatrix} \underline{M} & \underline{B} \\ \underline{B} & \underline{M} \end{bmatrix}, \quad \underline{K} = \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{B} & \underline{I} \end{bmatrix}, \quad \underline{K}^T = \begin{bmatrix} \underline{I} & \underline{B} \\ \underline{0} & \underline{I} \end{bmatrix}, \quad \underline{T}_q = \begin{bmatrix} \underline{U}_{q/2-1} \\ \underline{U}_{q/2} \end{bmatrix}$$

The difference equations (84) can be expressed in the form

$$\underline{L}\,\underline{T}_q = \underline{F}'_q - \underline{K}^T\,\underline{T}_{q-1} - \underline{K}\,\underline{T}_{q+1} \tag{86}$$

or since $\underline{L}$ is a 2nx2n symmetric positive-definite matrix, which assures the existence of the inverse $\underline{L}^{-1}$, it can be written:

$$\underline{T}_q = \underline{L}^{-1}\,\underline{F}'_q - \underline{L}^{-1}\,\underline{K}^T\,\underline{T}_{q-1} - \underline{L}^{-1}\,\underline{K}\,\underline{T}_{q+1}$$

which establishes the form required for use of the cyclic Chebyshev semi-iterative method.

Assuming the solution is known at all mesh points except those in the two-row block q, Equation (86) provides the basis for the determination of the remaining unknowns. The form of the matrix $\underline{L}$ is conveniently simplified if the components of $\underline{T}_q$ are selected alternately from the two rows.

Take $T_g = \{U_{1,j}, U_{1,j+1}, U_{2,j}, U_{2,j+1}, \dots\dots, U_{n,j}, U_{n,j+1}\}$;

then

$$\underline{L} = \begin{bmatrix}
20 & -8 & -8 & 2 & 1 & 0 & & & & \\
-8 & 20 & 2 & -8 & 0 & 1 & 0 & & & \\
-8 & 2 & 20 & -8 & -8 & 2 & 1 & 0 & & \\
2 & -8 & -8 & 20 & 2 & -8 & 0 & 1 & 0 & \\
1 & 0 & -8 & 2 & 20 & -8 & -8 & 2 & 1 & \\
0 & 1 & 2 & -8 & -8 & 20 & 2 & -8 & 0 & 1 \\
 & & & & & & & & & 2 \\
 & & & & & & & & & -8 \\
 & & & & & & & & & -8 \\
0 & 0 & & & 0 & 1 & 2 & -8 & -8 & 20
\end{bmatrix} \quad (2n \times 2n)$$

$$\underline{K} = \begin{bmatrix}
1 & 0 & 0 & & & & & & & \\
-8 & 1 & 2 & & & & & & & \\
0 & 0 & 1 & 0 & 0 & & & & & \\
2 & 0 & -8 & 1 & 2 & & & & & \\
0 & 0 & 0 & 0 & 1 & 0 & & & & \\
0 & 0 & 2 & 0 & -8 & 1 & 2 & & & \\
 & & & & & & & & & 0 \\
 & & & & & & & & & -8 \\
 & & & & & & & & & 2 \\
 & & & & 2 & 0 & -8 & 1 & 2 & 1
\end{bmatrix} \quad (2n \times 2n)$$

The matrix $\underline{L}$ is symmetric with non-zero elements appearing only in the main diagonal and the eight adjacent bands. Hence, the linear system of equations (86) can be solved by direct methods. The square-root method can be used to advantage. See p. 48.

After obtaining a method for the solution of (86) for one block, the full cyclic Chebyshev semi-iterative method is introduced

$$\underline{L}\,\underline{T}^{*(2m+1)}_{2k+1} = \underline{F}'_{2k+1} - \underline{K}^T\underline{T}^{(2m)}_{2k} - \underline{K}\,\underline{T}^{(2m)}_{2k+2}, \qquad 0 \le k \le \frac{t-1}{2} \tag{87}$$

$$\underline{T}^{(2m+1)}_{2k+1} = \underline{T}^{(2m-1)}_{2k+1} + \omega_{2m+1}\left[\underline{T}^{*(2m+1)}_{2k+1} - \underline{T}^{(2m-1)}_{2k+1}\right], \qquad m \ge 0$$

and

$$\underline{L}\,\underline{T}^{*(2m+2)}_{2k} = \underline{F}'_{2k} - \underline{K}^T\underline{T}^{(2m+1)}_{2k-1} - \underline{K}\,\underline{T}^{(2m+1)}_{2k+1}; \qquad m \ge 0, \tag{88}$$

$$\underline{T}^{(2m+2)}_{2k} = \underline{T}^{(2m)}_{2k} + \omega_{2m+2}\left[\underline{T}^{*(2m+2)}_{2k} - \underline{T}^{(2m)}_{2k}\right], \qquad 1 \le k \le t/2 .$$

The Chebyshev method consists of solving the difference equations (87) over the first two-row block, and all subsequent odd-numbered blocks, then difference equations (88) are solved over all even-numbered blocks.

The iteration parameters ω_{2m+1}, ω_{2m-2} are computed recursively using

$$\omega_1 = 1, \qquad \omega_2 = \frac{2}{2-\rho^2}$$

$$\omega_{s+1} = \frac{1}{\left[1 - \frac{\rho^2\omega_s}{4}\right]} \qquad \text{for } s \ge 2$$

where ρ is an approximation of the spectral radius of $\underline{A}$. If $\underline{A}$ is the coefficient matrix for a rectangular region, an approximation to the maximum eigenvalue, given by Griffin and Varga is

$$\rho \approx \frac{1}{\left[1 + \frac{\lambda^2 h^4}{2}\right]} \tag{89}$$

where

$$\lambda^2 = \left(\frac{\pi}{a}\right)^4\left[5.144\left(1 + \frac{a^4}{b^4}\right) + 3.115\,\frac{a^2}{b^2}\right]$$

h is the uniform mesh spacing in both the x and y directions and a and b are the dimensions of the plate. The relationship between the number of iterations and the value selected for ρ is shown in Fig. 4.6, from the model problem of Fig. 4.1, p. 20.

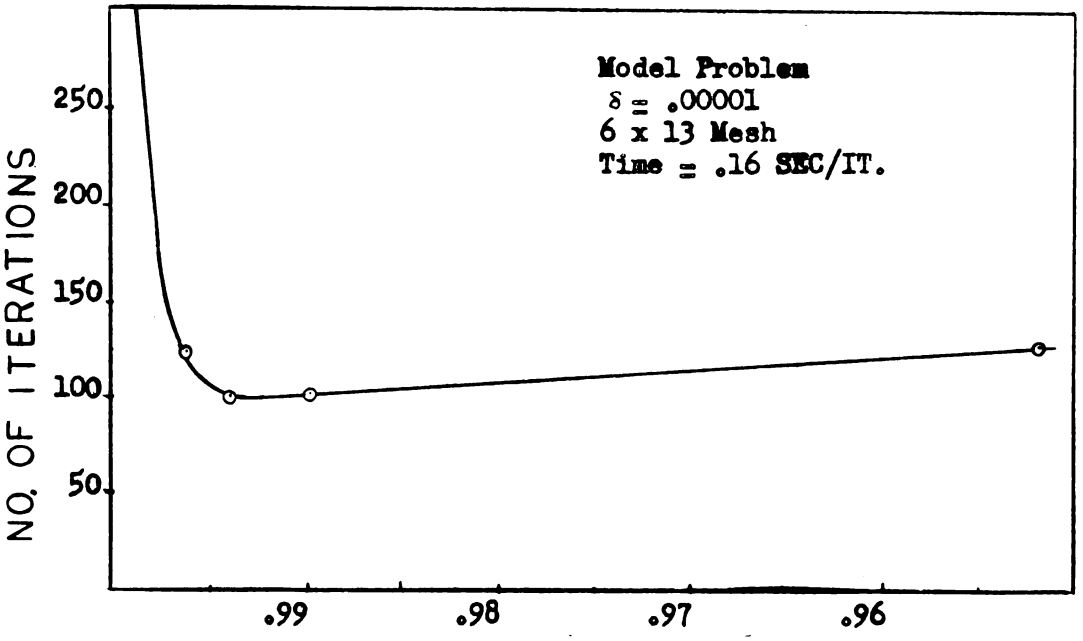


Figure 4.6 Cyclic Chebyshev semi-iterative method

Number of iterations vs. ρ

The Square Root Method

A direct-method algorithm for solving a system of linear equations, the square-root method, which is suggested by Faddeeva (1959) as one of the most efficient, is applicable only to symmetric systems. In Equation (86) which can be written

$$\underline{L}\,\underline{T}_q = \underline{P},$$

the matrix $\underline{L}$ is nine-diagonal symmetric and positive definite. Hence, $\underline{L}$ can be expressed as the product of two triangular matrices, one of which is the transpose of the other

$$\underline{L} = \underline{S}^T\,\underline{S},$$

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & 0 & \cdots & \\ 0 & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} & \cdots & \\ & 0 & & & & & & s_{2n-4,2n} \\ & & 0 & & & & & s_{2n-3,2n} \\ & & & 0 & & & & s_{2n-2,2n} \\ & & & & 0 & & & s_{2n-1,2n} \\ & & & & & & & s_{2n,2n} \end{bmatrix}$$

According to the rules for matrix multiplication the following relationships hold

$$\begin{aligned} l_{ij} &= s_{1i}\, s_{1j} + s_{2i}\, s_{2j} + \cdots + s_{ii}\, s_{ij}, \quad i<j \\ l_{ii} &= s_{1i}^2 + s_{2i}^2 \cdots + s_{ii}^2, \qquad i = j \end{aligned} \tag{90}$$

The s_{ij} are computed by recursive use of equations (90).

Since $s_{-3,j} = s_{-2,j} = s_{-1,j} = s_{0,j} = 0$,

$$s_{11} = \sqrt{l_{11}}, \qquad s_{1j} = \frac{l_{1j}}{s_{11}}, \quad 2 \le j \le 4$$

$$s_{ii} = \sqrt{l_{ii} - \sum_{z=1}^{i-1} s_{zi}^2}, \qquad i > 1,$$

$$s_{ij} = \frac{l_{ij} - \sum_{z=1}^{i-1} s_{zi}\, s_{zj}}{s_{ii}}, \qquad i+4 \ge j \ge i$$

$$s_{ij} = 0, \quad i>j$$

The solution of $\underline{L}\,\underline{T} = \underline{P}$ is obtained by solving two systems

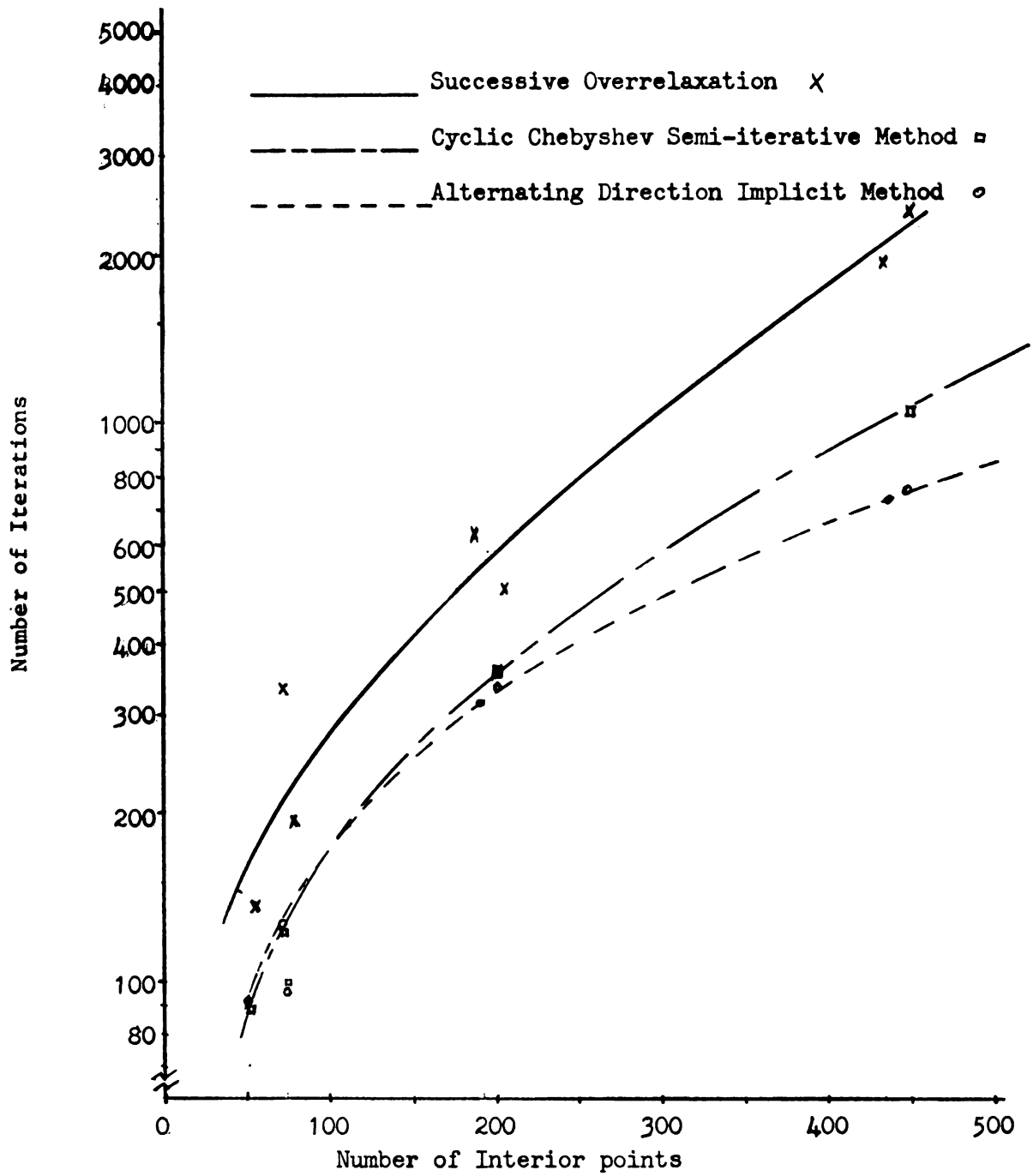
$$\underline{S}^T\,\underline{M} = \underline{P}, \qquad \underline{S}\,\underline{T} = \underline{M}$$

The components of the vector $\underline{M}$ are computed with the recurrence formulas

$$m_1 = \frac{p_1}{s_{11}}, \qquad m_i = \frac{p_i - \sum_{z=1}^{i-1} s_{zi}\, m_z}{s_{ii}}, \qquad i > 1$$

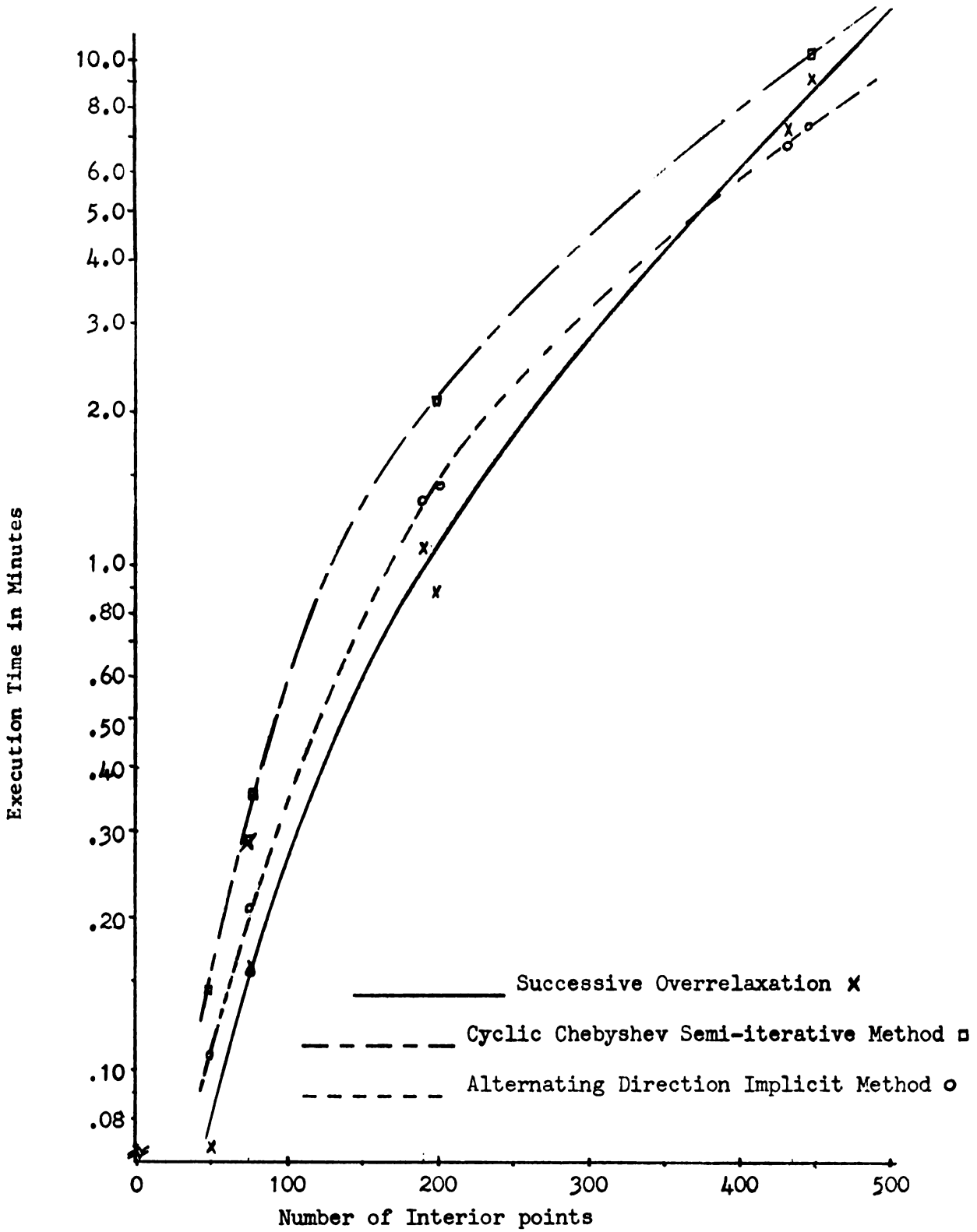
Similarly the components of $\underline{T}$ are determined with

$$T_{2n} = \frac{m_{2n}}{s_{2n,2n}}, \qquad T_i = \frac{m_i - \sum_{z=i+1}^{2n} s_{iz} T_z}{s_{ii}}, \qquad i < 2n$$



(a) Iterations vs. number of interior points

Figure 4.7 Comparison of Iterative methods



(b) Machine time vs. number of interior points

Figure 4.7 Comparison of Iterative methods

V. COMPARISON OF THREE ITERATIVE METHODS

A prime objective of this study is the identification of the relative merits of three iterative methods for the solution of the biharmonic difference equations. Commenting on the problem of selecting the best iterative method for solving engineering problems Varga (1962) p.245 makes several observations. (1) For the general case there are no theoretical arguments which rigorously establish the superiority of any one method. (2) The present evaluation of these methods has been based on the numerical experiments of many investigators. (3) The numerical results indicate that for each two-dimensional second-order partial differential equation boundary-value problem there is a critical mesh spacing h^* such that the two line cyclic Chebyshev semi-iterative method is superior for all mesh spacings $h \geq h^*$, while for $h < h^*$ a multiple-acceleration-parameter Peaceman-Rachford method is better.

Table 1. Comparison of iterative subroutine characteristics

Subroutine Name	Number of FORTRAN Statements	Storage of arrays for 450 points	Time (sec./iteration) 72 Mesh pts.	Time (sec./iteration) 450 Mesh pts.
SOR	32	646	.050	.223
ADI	89	1224	.098	.585
CHEB	162	1126	.167	.612

SOR - Successive overrelaxation method
ADI - Alternating direction implicit method
CHEB- Two line cyclic Chebyshev semi-iterative method

The evaluation of these iterative methods for biharmonic difference equations is more complicated than for Laplace's difference equations. For Laplace's equations, Golub and Varga (1961) indicate that the cyclic Chebyshev semi-iterative method requires effectively no more additional

arithmetic operations or vector storage than other iterative methods. This is not true for the biharmonic difference equations as shown by the comparison in Table 1 of the three iterative subroutines included in the ISOPEP program of Appendix B.

The overriding consideration for many is the actual machine time required for a solution which satisfies a specified convergence criterion.

Convergence criteria

The norm $\| \underline{E}^{(m)} \|_{II}$ of the error vector $\underline{E}^{(m)}$ is often used for terminating the iterations; the norm is defined by

$$\| \underline{E}^{(m)} \|_{II} = \sum_{i=1}^{k} | E_i^{(m)} |$$

A value is assigned to a parameter δ, and when

$$\| \underline{E}^{(m)} \|_{II} \leq \delta$$

the solution vector iterate $\underline{U}^{(m)}$ is accepted as the solution. Griffin (1963) indicates that for the two-line cyclic Chebyshev semi-iterative method, if the number of two-line blocks is large the average difference between corresponding values of U_i in successive iterations will be approximately $2\delta/k$, where k is the number of mesh points at which the stress function is unknown.

The criterion used in the ISOPEP program of Appendix B, which includes the subroutines SOR, ADI and CHEB, is

$$| E^{(m)} |_{max} \leq \delta$$

with $\delta = 10^{-5}$. For the model problem $| U_i |_{max} = 3.6$.

The three iterative methods were used to solve the model problem of Section VI - 1. Symmetry conditions were used for the stress function at the plate centerline. There is a summary of the results in Table 2 and comparison of the methods in terms of number of iterations and machine time required for different mesh intervals in Fig. 4.7.

Table 2. Comparison of machine time and number of iterations required for convergence

Mesh Space h	Mesh Size	Number of Interior Points	Number of Iterations			CDC 3600 Execution time in minutes		
			SOR	ADI	CCSI	SOR	ADI	CCSI
1/10	5x11	50	138	93	91	.070	.108	.145
1/12	6x13	72	194	95	99	.161	.156	.283 **
1/12	6x13	72	336	128	123	.280	.205	.342
1/20	10x20	190	636	317	- *	1.08	1.35	- *
1/20	10x21	200	500	334	357	.87	1.47	2.08
1/30	15x30	435	1984	712	- *	7.25	6.73	- *
1/30	15x31	450	2446	747	1053	9.10	7.29	10.75

* There must be an even number of rows for the CCSI subroutine.
** Optimum relaxation parameter used

Summary of numerical results

Numerical experimentation can provide insight for the appraisal of iterative methods and may provide a basis for theoretical investigations. The results obtained are based on solutions of the model problem only and conclusions should be qualified accordingly. A number of observations are presented for consideration.

For the model problem the cyclic Chebyshev semi-iterative method is iteratively faster than point successive overrelaxation for all mesh sizes considered. As shown in Fig. 4.7(a) it is also iteratively faster than the alternating direction implicit method for mesh spacing $h > 1/16$, or for less than 125 points. In terms of machine time required, successive overrelaxation is best for less than 350 mesh points or $h \geq 1/26$. If $h < 1/26$ the alternating direction implicit method is best.

The relaxation factor or acceleration parameter was selected on the basis of the treatment given in the discussion of each of the three

methods in Section IV-B. For successive overrelaxation an initial value $\omega_b = 1.5$ was selected, and this was changed after every 10 iterations. As shown by the second and third problems in Table 2 this will not necessarily assure a good approximation of the optimum value of ω_b. When the cyclic change was used for the 6x13 mesh size problem approximately 70% more iterations were required than when the optimum relaxation factor was used. Another indication of the variation which can be expected when using this procedure is shown by the scatter of points in Figures 4.7(a) and (b) for the SOR subroutine. This contrasts with the curves fitted to the points for the other two methods. It should be noted that a random selection of values for ω will produce greater change in the number of iterations required for convergence for the successive overrelaxation method than for either of the other methods. Comparison of Figures 4.3, 4.4 and 4.5 shows that the alternating-direction implicit method and the cyclic Chebyshev semi-iterative method do not impose as severe a penalty on overestimation of the appropriate parameter as does point successive overrelaxation. For a considerable range of values above the optimum, the number of iterations increases only slightly above the minimum for these two methods, while the minimum in Fig. 4.3 for point successive overrelaxation is much sharper.

Another important consideration is the computer storage required by each of the three subroutines. The data in Table 1 clearly identifies the successive overrelaxation method as the one which requires least storage for both the arrays of data and the sequence of instructions. Because of the relative simplicity and minimal storage requirements the successive overrelaxation method was incorporated as the main iterative method in the ISOPEP program.

Since the number of iterations required for convergence appears to increase significantly with the number of mesh points, attention was directed toward the possibility of improving the rate of convergence by first solving the problem with a relatively coarse mesh, then interpolating between these values to generate a better initial solution vector for a finer mesh. A subroutine CHANG was written for this interpolation and some of the results are given in Table 3.

Table 3. Summary of problems solved with the ISOPEP program

Prob. No.	Converg. criterion	Mesh size	Number of Iterations	CDC 3600 Time		IBM 1620 Time (Hours)	Last Estimate of ρ
				Min	Sec		
101	10^{-5}	5x10	118		3.31		.9571
102	10^{-5}	10x20	521		51.9		.9804
103	10^{-5}	15x30	1548	5	38.3		.9942
201	10^{-5}	3x6	39			.85	.7194
202	10^{-5}	6x12	84			2.62	.9556
203	10^{-5}	12x24	86			6.31	.9906
205	10^{-5}	7x14	276		16.9		.9737
206	10^{-6}	14x28	1554	5	29.7		.9900
303	10^{-5}	6x12	225			2.33	.9506
306	10^{-6}	24x48	1038	4	49		.9793
401	10^{-5}	6x12	179			2.30	.9670
402	10^{-5}	12x24	126			5.67	.9648
408	10^{-6}	24x48	166	1	34.7		.8862
501	10^{-5}	6x12	135			1.80	.9476
502	10^{-5}	12x24	295			12.59	.9781
506	10^{-6}	24x48	1157	5	47.6		.9858
601	10^{-5}	15x15	176	1	20		.9802
604	10^{-7}	48x48	7072	57	14.2		.9998

Problems 101, 102 and 103 in Table 3 were solved using the subroutine CHANG. The final solution of Problem 103 required a total time of 6 minutes, 34 seconds and a total of 2187 iterations for the three problems. The same problem solved with initial values of the vector iterate set to .1, required 7 minutes 15 seconds and 1948 iterations as listed in Table 2.

The other sets of problems listed in Table 3, such as 401, 402, 408, were solved using the iterpolation procedure as the mesh was refined. For Problem 408, only 166 iterations were required after the last refinement of the mesh while for Problem 506, which has the same number of interior points, 1157 iterations were needed. Both of these problems have more than twice the number of mesh points used in Problem 103 yet the solution time is approximately the same for 506 and substantially less for 408.

Capabilities of the Computers used

Two computers have been used in this investigation. One, the Control Data Corporation, 3600 at Michigan State University Computing Center, is very fast and has a large main memory of 32,000 words of 12 decimal digits each. The floating-point multiplication of two numbers with 10-digit mantissas requires less than six microseconds. The other computer is the IBM 1620 at the University of Toledo Computation Center. The 1620 performs a floating-point multiplication of two numbers with eight-digit mantissas in approximately 12 milliseconds. The main memory provides for storage of 20,000 decimal digits which is about 1/19 of the 3600 memory. However, the main memory is supplemented with an IBM 1311 Disk File which provides 2,000,000 decimal digits of secondary storage. Several problems were solved on both computers and a comparison is provided in Table 4. The slight difference in the number of iterations

required for convergence is possibly a consequence of the difference in the mantissas of the floating point numbers used.

The difference in the speed of the two computers contrasts sharply. In fact for problems with a hundred or more mesh points the CDC 3600 is approximately 1200 times faster than the IBM 1620. At rates of $375.00 and $30.00 per hour for the 3600 and 1620 respectively, the cost of solving problem 1.102 would be $8.82 on the 3600 and $572.10 on the 1620. Two qualifications of the comparison should be noted. First, the time required for compiling a representative combination of the ISOPEP FORTRAN subroutines on the 3600 uses between 20 and 30 seconds. Compilation time on the 1620 ranges between 10 and 15 minutes. For this purpose the 3600 is only about 30 times faster than the 1620. A second consideration is the time saving possible on the 1620 through use of a machine language successive overrelaxation subroutine. The introduction of such a subroutine for the iterative solution of Laplace's equation resulted in a 50% reduction of machine time required.

Table 4. Comparison of the ISOPEP program for the IBM 1620 and CDC 3600 Computers

Prob. No.	Mesh Size	Number of Iterations		CDC 3600 Time				IBM 1620 Time	Method
				Total		Execution			
		3600	1620	Min.	Sec.	Min.	Sec.	Hours	
1.101	5x10	118	116		25		3.35	.97	SOR
2.101	5x10	90	90		29		6.0	1.78	ADI
1.102	10x20	636	622	1	25	1	5.0	19.07	SOR
2.102	10x20	317	344	2	2	1	21.	27.20	ADI
1.103	15x30	1984	1974	7	36	7	15.	158.00	SOR

VI. SOLUTIONS OF PLANE ELASTOSTATIC PROBLEMS

The second of three objectives of this dissertation is a general computer program for the iterative solution of plane elastostatic problems. The program ISOPEP includes six subroutines which contribute to this objective. The program is not completely general, however, since subroutines for the boundary conditions must be added for any specific problem. A flow diagram, listings of FORTRAN source decks, description of input preparation, and the output of a sample problem are provided in Appendix B. The analysis of physical phenomena with mathematical models, the analytical or numerical solutions of the models and the subsequent testing by comparison of the solutions with experimental observations, constitute major pursuits of scientists and engineers. The creation of the ISOPEP program contributes to the analysis of plane elastostatic problems and this computer program is considered a major part of the dissertation.

Demonstration of the utility of iterative methods for the solution of several problems is the third objective of this dissertation. Problems number two and six are included because analytical solutions are available for comparison with the numerical solutions. Problems three, four and five are representative of the practical problems for which analytical solutions are not available. The treatment of boundary conditions for irregular regions is included in the discussion of problem three.

1. The model problem

The square plate and edge loads as shown in Fig. 6.1 are symmetrical with respect to the x-axis. By proper choice of constants of integration the stress function will also be symmetrical with respect to the x-axis. Starting at the origin and proceeding counter-clockwise the boundary conditions are obtained from Equations (14) and the requirements for symmetry.

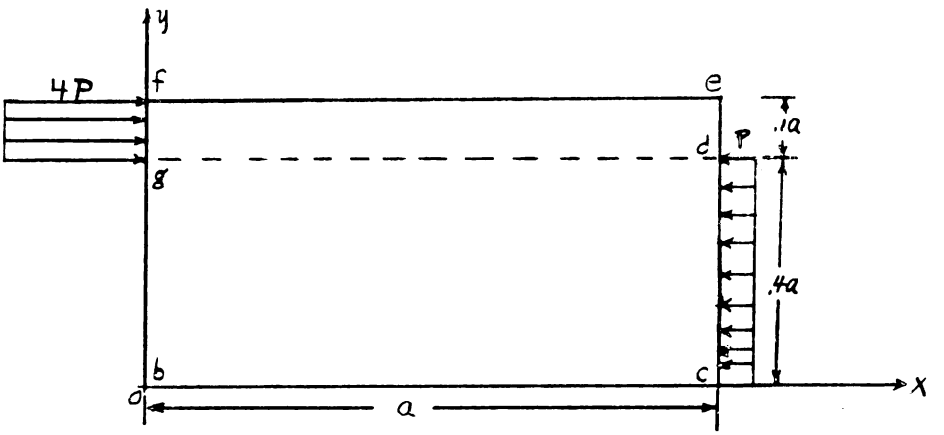


Figure 6.1 Boundaries of the model problem

The constants of integration A, B, C, D, E, and F in the following equations are determined so as to produce the desired symmetry of ϕ and consistent values of ϕ at each of the points b, c, d, e, f, g in the functional expressions for ϕ along neighboring line segments. Two of the constants are arbitrarily chosen to make $\phi = 0$ and $\frac{\partial \phi}{\partial x} = 0$ at b.

Along $\overline{bc}$

$$\frac{\partial^2 \phi}{\partial x \partial y} = 0 , \qquad \frac{\partial \phi}{\partial y} = 0 ;$$

along $\overline{cd}$

$$\frac{\partial \phi}{\partial y} = -py + \left(\frac{\partial \phi}{\partial y}\right)_c = -py , \qquad \frac{\partial \phi}{\partial x} = \left(\frac{\partial \phi}{\partial x}\right)_c = 0 ,$$

$$\phi = -py^2/2 + A = -py^2/2 + .1pa^2 ;$$

along $\overline{de}$

$$\frac{\partial \phi}{\partial y} = \left(\frac{\partial \phi}{\partial y}\right)_d = -.4ap , \qquad \frac{\partial \phi}{\partial x} = 0 ,$$

$$\phi = -.4apy + B = -.4apy + .18a^2p ;$$

along $\overline{ef}$

$$\frac{\partial \phi}{\partial y} = \left(\frac{\partial \phi}{\partial y}\right)_e = -.4ap , \qquad \frac{\partial \phi}{\partial x} = 0 ,$$

$$\phi = C = (\phi_e) = -0.02a^2p ;$$

along $\overline{gf}$

$$\frac{\partial \phi}{\partial y} = -4py + D \qquad \frac{\partial \phi}{\partial x} = 0,$$

$$\phi = -2py^2 + Dy + E = -2py^2 + 1.6apy - .32a^2p;$$

along $\overline{gb}$

$$\frac{\partial \phi}{\partial y} = \left(\frac{\partial \phi}{\partial y}\right)_g = 0, \qquad \frac{\partial \phi}{\partial x} = 0,$$

$$\phi = F = 0.$$

Values of the difference-equation approximation of the stress function U_{ij} and nodal point values of the normal stress component σ_{xij} are given in Fig. 6.2. The point values of the stress function are approximated

$$\frac{\phi_{ij}}{C} \approx U_{ij}$$

where $C = \frac{pa^2}{36}$. The constants p and a have been assigned values of unity.

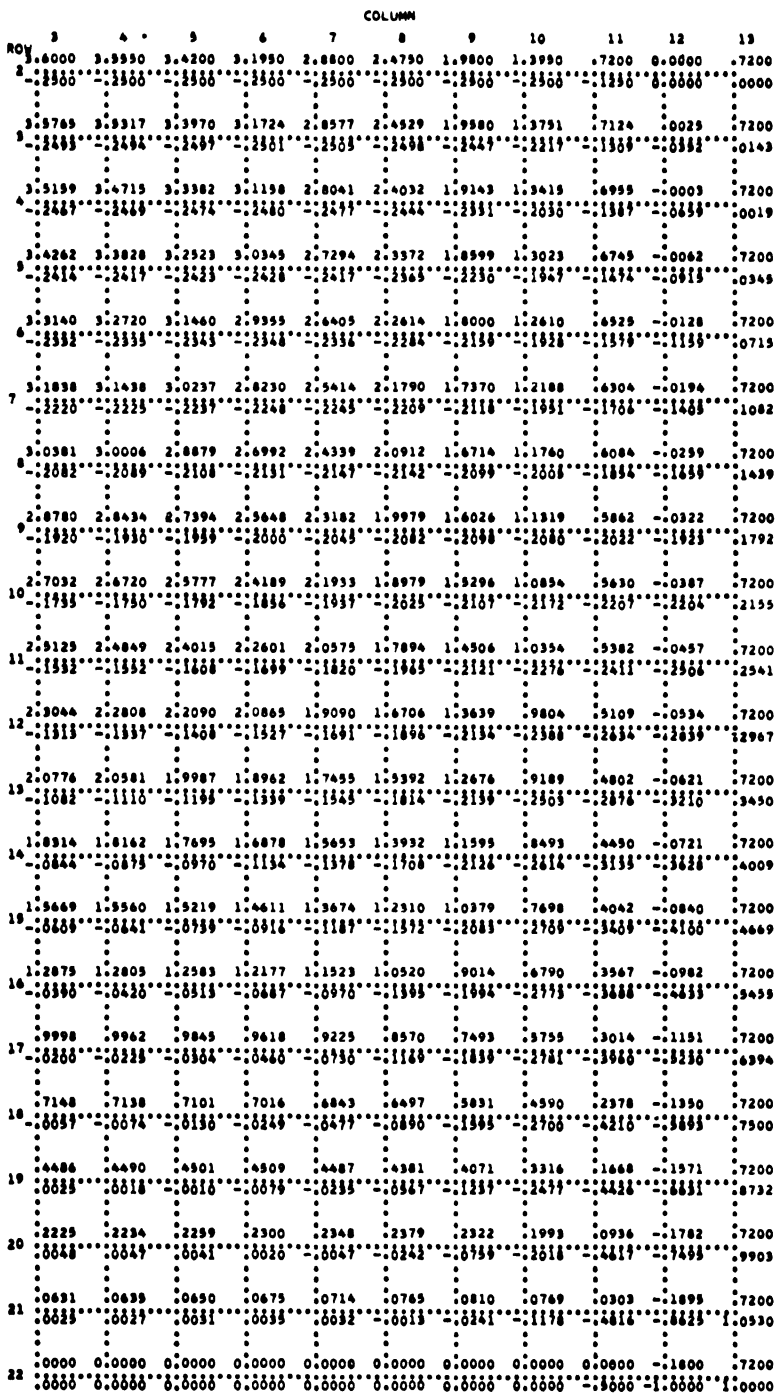
2. Semi-infinite plate with uniform load on the boundary

The exact solution of the problem of a semi-infinite plate with a uniformly distributed load of intensity P_o acting on an interval of the boundary $-c \leq x \leq c$, is given by Timoshenko and Goodier (1951). The stress function has the form

$$\phi = A\,(r^2\theta - r_1^2\theta_1)$$

Numerical solutions of this problem have been reported by Veyo and Hornbeck (1964), who used point successive overrelaxation of the biharmonic finite-difference equations, and Pisacane and Malvern (1963), who used a numerical mapping technique for application of the Muskhelishvili complex variable method. For comparison with the other numerical solutions a square region, 2 units by 2 units, of the semi-infinte plate will be considered. The intensity of the distributed load P_o is chosen as one and the value of c is $\frac{1}{4}$. For the assigned values of c and P_o the stress function can be written

FIGURE 6.2 Distributions of the Stress Function and σ_x for Problem 1



$\frac{U}{\sigma_x}$

$$\phi = \frac{1}{2\pi}\left[\{(X-\tfrac{1}{4})^2+y^2\}\tan^{-1}\left(\frac{X-1/4}{y}\right) - \{(X+\tfrac{1}{4})^2+y^2\}\tan^{-1}\left(\frac{X+1/4}{y}\right) - \frac{y}{2}\right] + \frac{1}{32}$$

Values of the stress function on the boundaries of the square region which are inside of the semi-infinite plate can be determined from the exact solution. The values of ϕ at points one mesh interval outside the square region can be similarly computed or obtained from the normal derivatives of the exact solution at the boundary. Due to symmetry it is possible to determine the solution by considering one half the square region. Along the y-axis which is the axis of symmetry, the normal derivative, $\frac{\partial \phi}{\partial X} = 0$, is used to determine the values of ϕ at points exterior to the solution domain.

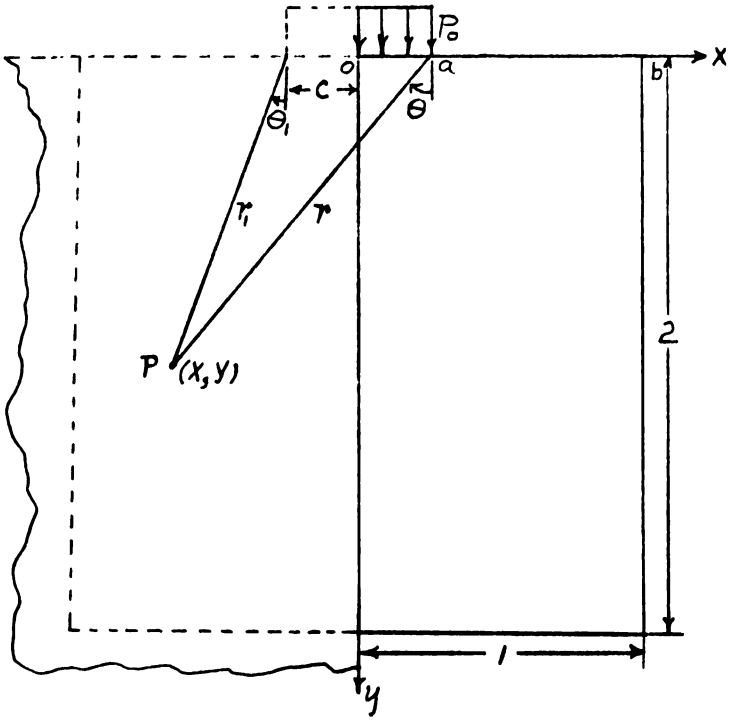


Figure 6.3
Problem 2 - Semi-infinite plate

Along $\overline{oa}$ we use Equations (14) and find

$$\frac{\partial^2 \phi}{\partial X^2} = -P_o = -1, \qquad \frac{\partial^2 \phi}{\partial X \partial y} = 0,$$

$$\frac{\partial \phi}{\partial X} = -X + B, \qquad \frac{\partial \phi}{\partial y} = A,$$

$$\phi = -\tfrac{1}{2}X^2 + BX + C.$$

Symmetry of ϕ with respect to the y-axis requires B=0. Two of the constants of integration may be arbitrarily selected. Let A = C = 0.

Then along $\overline{oa}$

$$\frac{\partial \phi}{\partial x} = -X, \qquad \frac{\partial \phi}{\partial y} = 0, \qquad \phi = -\frac{1}{2}X^2.$$

Along $\overline{ab}$

$$\frac{\partial \phi}{\partial x} = \left(\frac{\partial \phi}{\partial x}\right)_a = -\frac{1}{4}, \qquad \frac{\partial \phi}{\partial y} = \left(\frac{\partial \phi}{\partial y}\right)_a = 0,$$

$$\phi = -\frac{1}{4}X + D = -\frac{1}{4}X + \frac{1}{32}$$

The value of D = 1/32 is determined by the requirement $\phi_a = -1/32$ in the expressions for ϕ along $\overline{oa}$ and $\overline{ab}$. Note the last two terms of the stress function $-\frac{1}{4}x$ and 1/32 are necessary as a consequence of the arbitrary selection of A and C.

The values of the normal component of stress σ_y and the stress function are given in Fig. 6.5 for a 14x28 mesh. The convergence criterion used was 10^{-6}, and an initial estimate of the solution vector for the 14x28 mesh was obtained from interpolation between the solution vector elements for a 7x14 mesh. The total number of iterations required

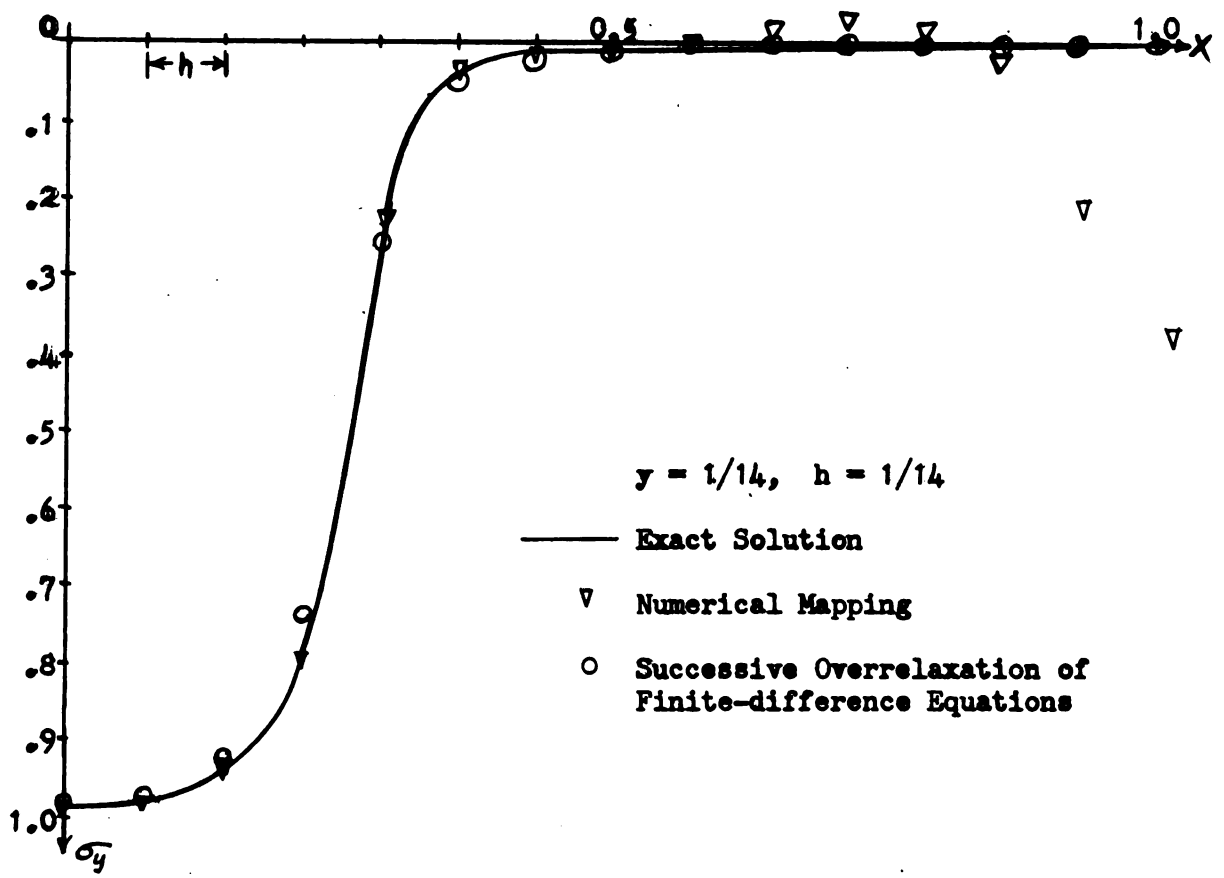


Figure 6.4 Comparison of numerical and exact solution for Problem 2

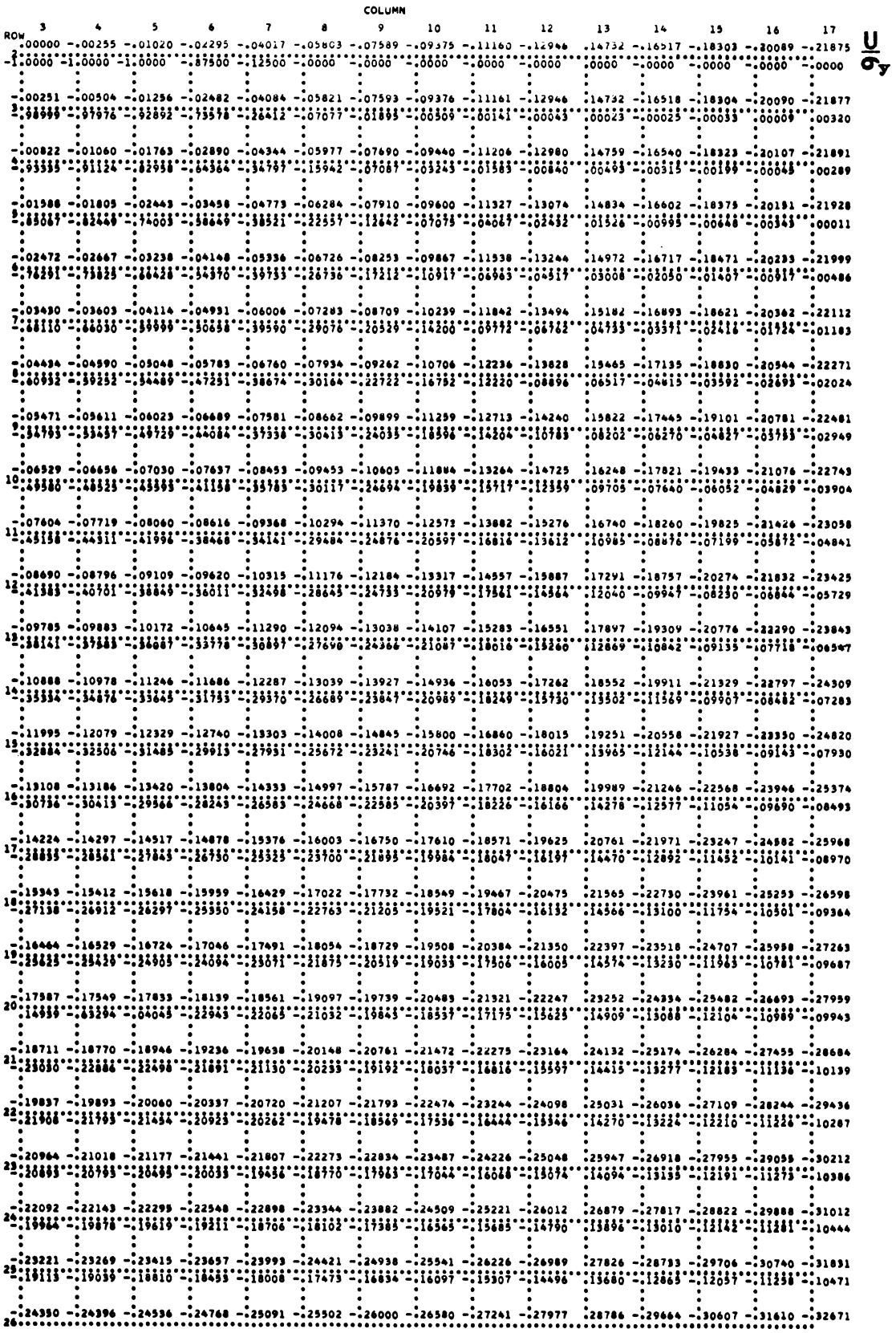
for both problems was 1830. A flat distribution of 0.1 at all interior points was used for the initial estimate for the 7x14 mesh. Veyo and Hornbeck reported approximately 3000 iterations were required for the 14x28 mesh with an initial value of -.0625 assigned at all interior points and a convergence criterion of $.5 \times 10^{-7}$. Though point values of the solution of the problem solved by Veyo and Hornbeck were not available for comparison, it was not possible to discern any significant difference in the results shown in Fig. 6.4 and similar graphical results in their report.

The deviation of the iterative solution of the finite-difference equations from the exact solution is greatest in those regions where the stress gradients are greatest. The stress gradients are large at the first row of interior points below the boundary subjected to the distributed load. This distribution of σ_y for the exact and finite-difference solutions at $y = 1/14$ is shown in Fig. 6.4. The numerical mapping solutions of Pisacane and Malvern are also shown. It is apparent that the numerical mapping technique provides only slightly better results in the region directly under the applied load where stress gradients are highest and much poorer results near the boundary $x = 1$. Both numerical methods provide better approximations to the exact solutions at rows of mesh points corresponding to $y \geq 2h$.

Problems 3, 4 and 5 - Notched tensile specimens

The problems of practical interest to the engineer frequently have irregular boundaries and attention is often centered on stress concentrations. An investigation of stress concentrations has been conducted by solving each notched tensile specimen problem for different mesh spacings. Three different types of notches have been considered:

FIGURE 6.5 Distributions of the Stress Function and σ_y for Problem 2



Problem 3 - Semicircular notches, Problem 4 - V-notches and Problem 5 - Rectangular notches. Sketches of the specimens are included in Figures 6.10, 6.11, 6.12.

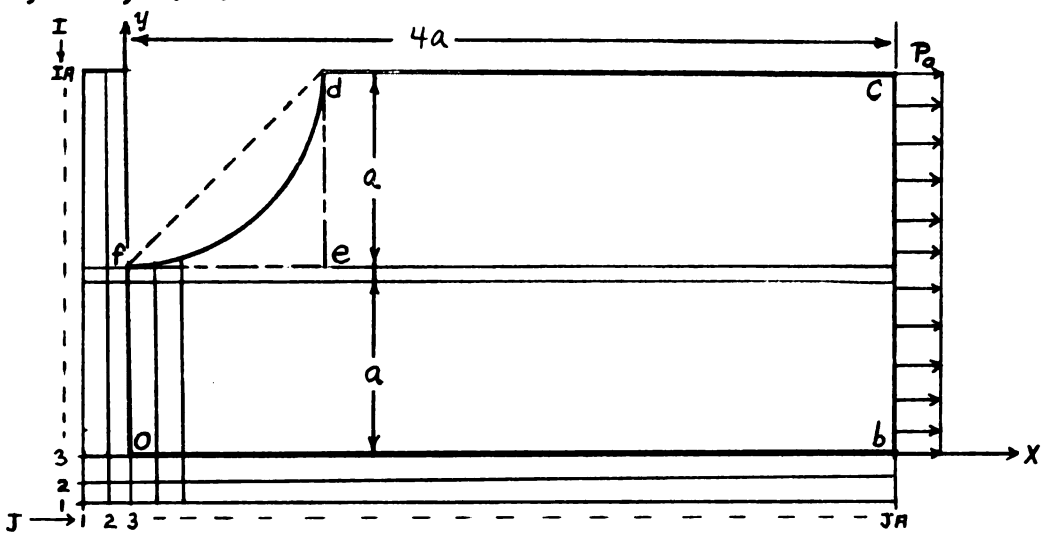


Figure 6.6 Solution domain for Problems 3, 4 and 5

Boundary conditions for the three problems are identical. Introducing a Cartesian coordinate system as shown in Fig. 6.6 integration constants will be chosen so the stress function is symmetrical with respect to the x and y axes and it is necessary to solve each problem for only one quarter of the whole plate. At the end $\overline{bc}$ a uniformly distributed tensile load of P_o units intensity is applied. Using Equations (14) we find

$$\frac{\partial \phi}{\partial x} = A , \qquad \frac{\partial \phi}{\partial y} = P_o y + \left(\frac{\partial \phi}{\partial y}\right)_b .$$

But $\left(\frac{\partial \phi}{\partial y}\right)_b = 0$, if ϕ is symmetric with respect to the x-axis; hence

$$\phi = P_o y^2/2 + B = P_o y^2/2 .$$

The constants A and B are arbitrarily selected as zero. Then along $\overline{cd}$

$$\frac{\partial \phi}{\partial x} = 0 , \qquad \frac{\partial \phi}{\partial y} = 2a P_o ,$$

$$\phi = \phi_c = 2a^2 P_o .$$

Along the two axes of symmetry the normal derivative conditions are used to compute values of the stress function in two mesh rows exterior to

FIGURE 6.7 Distributions of the Stress Function and σ_x for Problem 3

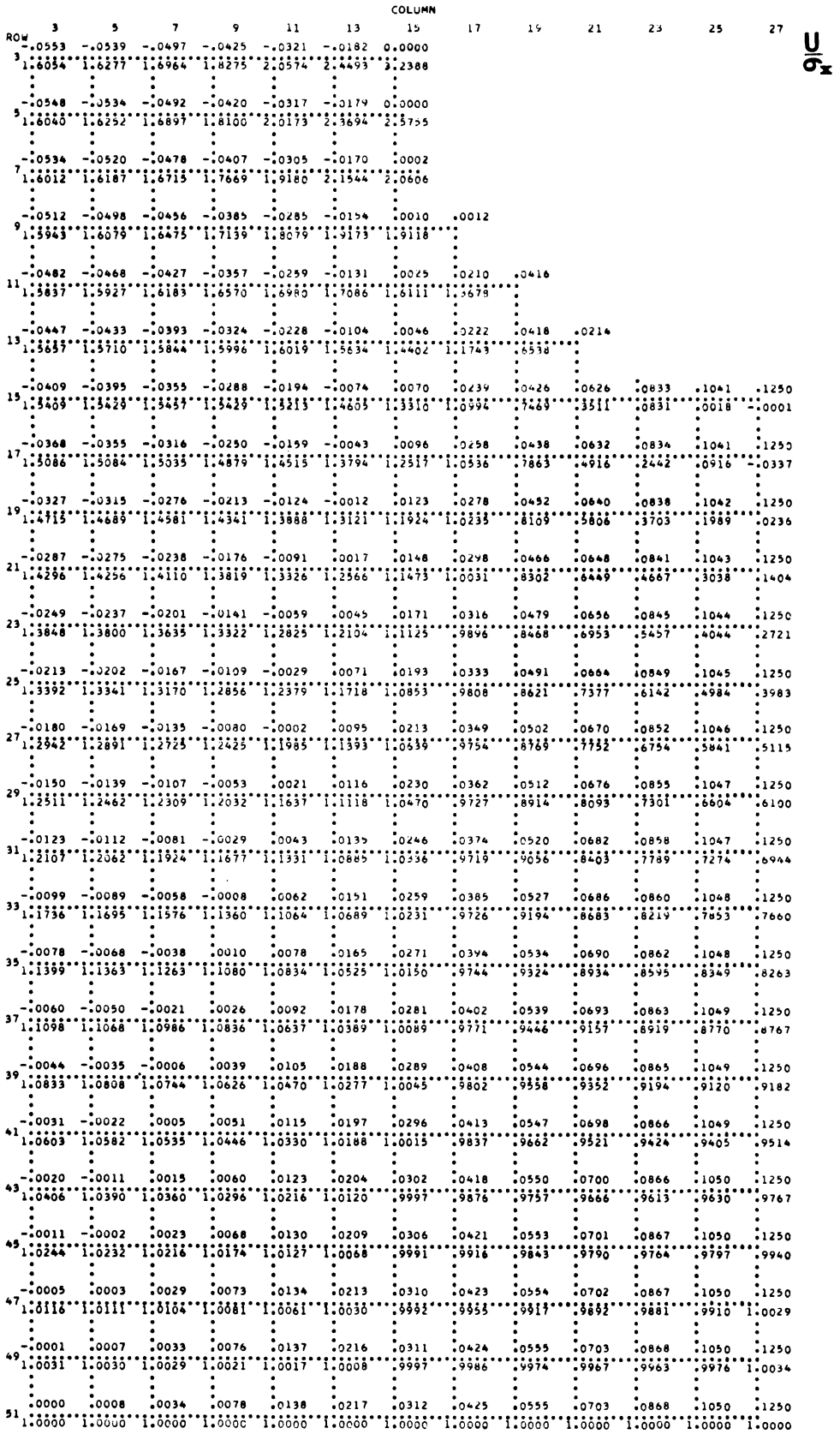


FIGURE 6.8 Distributions of the Stress Function and σ_x for Problem 4

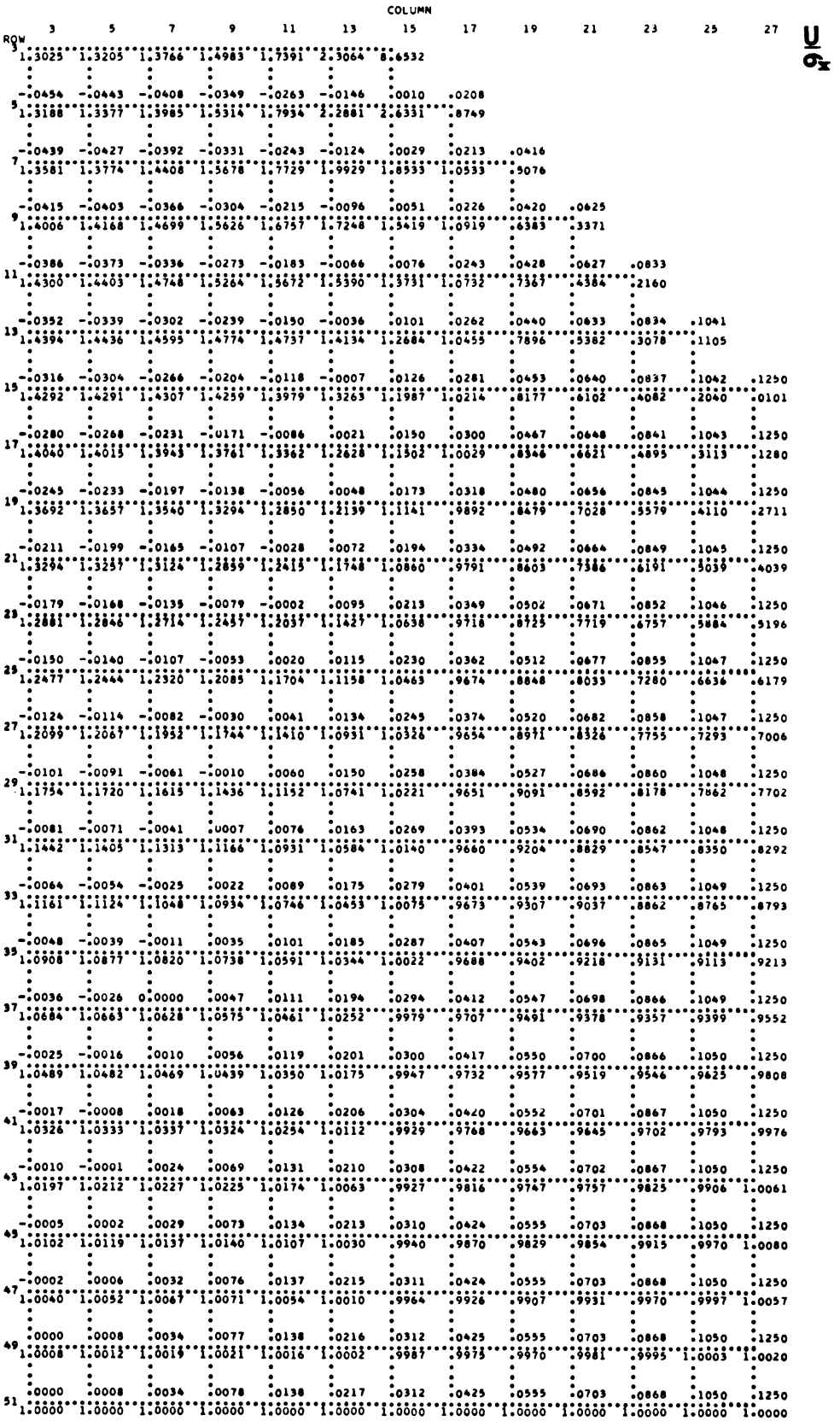
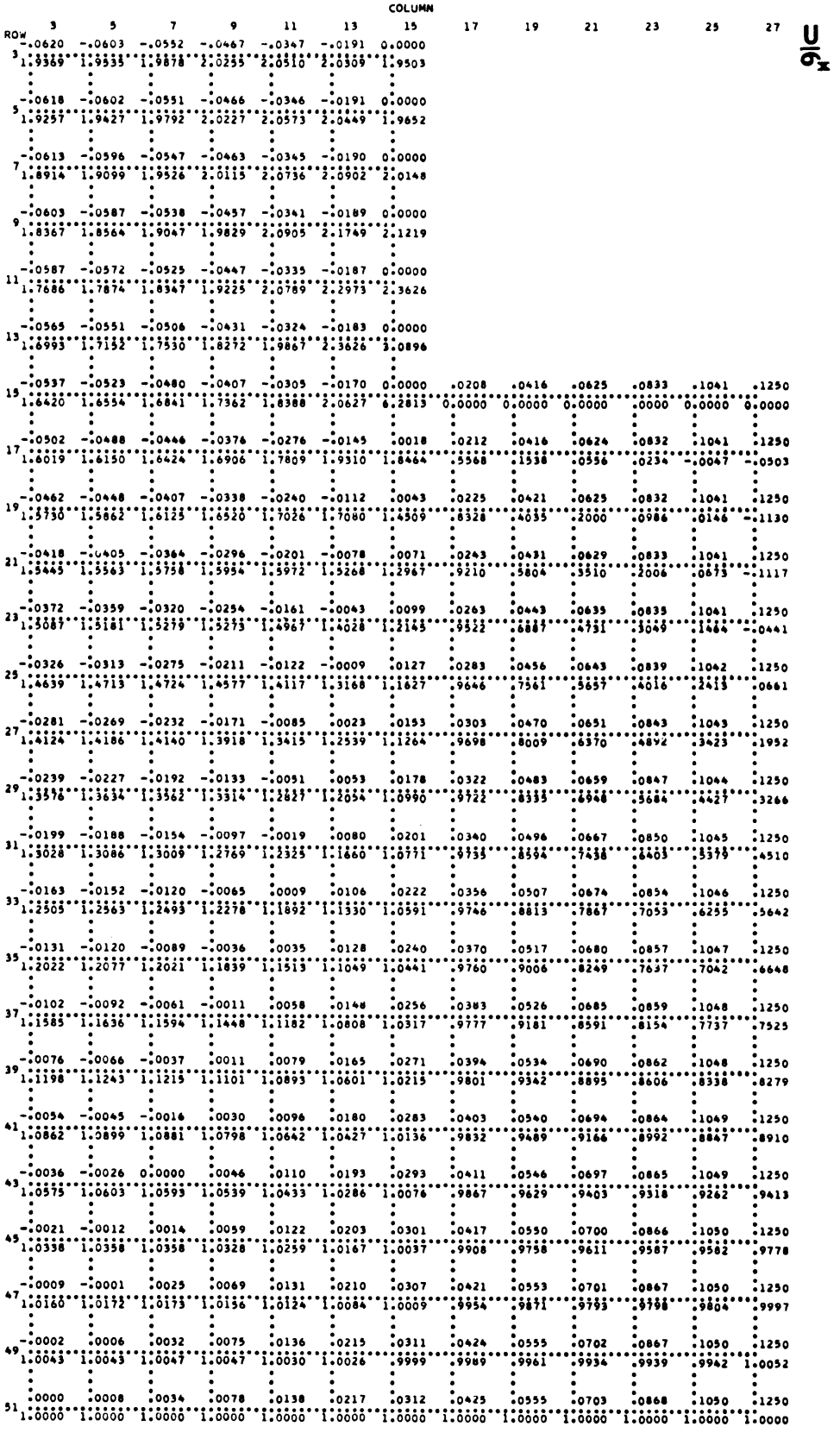


FIGURE 6.9 Distributions of the Stress Function and σ_x for Problem 5



the solution domain. Along the x-axis, $\frac{\partial \phi}{\partial y} = 0$, while along the y-axis $\frac{\partial \phi}{\partial x} = 0$. Along the notch edge $\overline{df}$ or $\overline{de}$ and $\overline{ef}$

$$\frac{\partial \phi}{\partial x} = 0, \qquad \frac{\partial \phi}{\partial y} = \left(\frac{\partial \phi}{\partial y}\right)_d = 2a P_o$$

$$\phi = 2a P_o y + C = 2a P_o (y - a)$$

The value of C is determined by the value of ϕ_d in the equation used along $\overline{cd}$. The last expression for ϕ is valid for all 3 problems but for Problem 5 it simplifies to $\phi = 0$ along $\overline{ef}$.

The solutions given in Figures 6.7, 6.8 and 6.9 were obtained with boundary conditions for which $P_o = 1$ and $a = \frac{1}{4}$. Though a 12x24 mesh is used for listing the solution in all three figures, the results are taken from the solution obtained for a 24x48 mesh.

Stress concentrations occur at the base of the notch at $y = a$. For the V and semi-circular notches the concentration occurs along the y-axis. For the rectangular notch the concentration occurs at $x = a$, $y = a$. The comparisons of stress concentrations for different choices of mesh size are given in Figures 6.10, 6.11, 6.12. R. E. Peterson (1953) gives an exact stress concentration of 3.08 at the base of the semi-circular notches in a tensile specimen infinitely long. The stress concentration would be slightly higher in a specimen of finite length. Southwell (1956) determined a stress concentration factor of 3.0 using a "relaxation" solution for a mesh of 116 points. Solutions obtained with the ISOPEP program produce stress concentrations of 2.92 and 3.24 for meshes of 65 and 1038 points respectively.

At the base of the V-notch or the corner of the rectangular notch even slight strains induce stresses of high magnitude and the stresses determined from a solution of the biharmonic difference equations would hardly represent an actual physical state of stress. However, stress

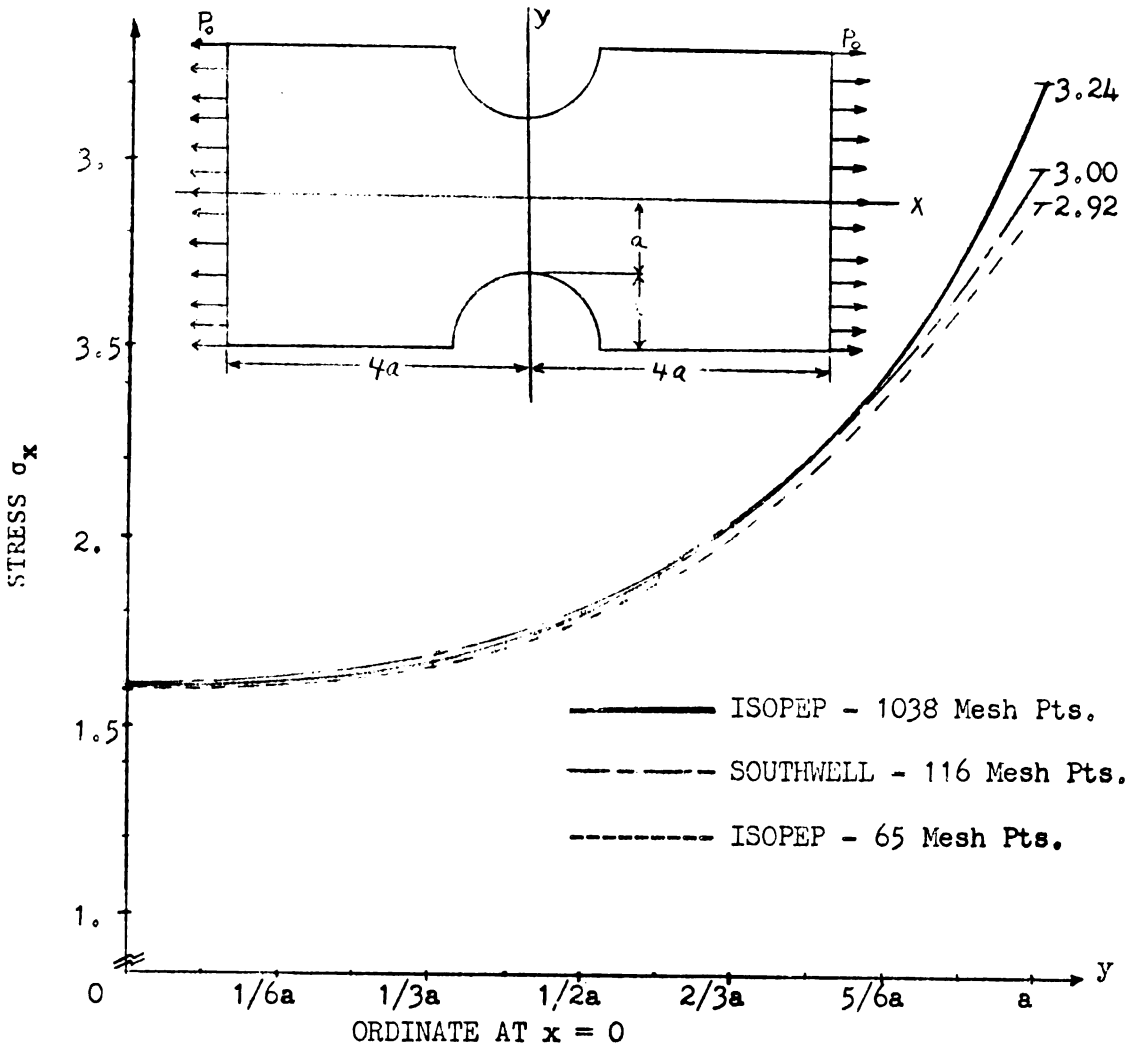


Figure 6.10 Stress Concentration in a semi-circular notch

distributions obtained under the assumption of an ideally elastic body are useful in the analysis of the plasticity problem. Exact solutions of these problems would provide infinite stress concentrations for these two problems. The numerical solution provides a set of discrete values of the stress function. The value at each mesh point is an average value associated with an area surrounding the point. The area in general is proportional to h^2. As the uniform mesh space h is decreased, the stress function solution at a point of high stress concentration would provide a better approximation of the high stress. Figures 6.11 and 6.12 illustrate this phenomenon for the V and rectangular notches.

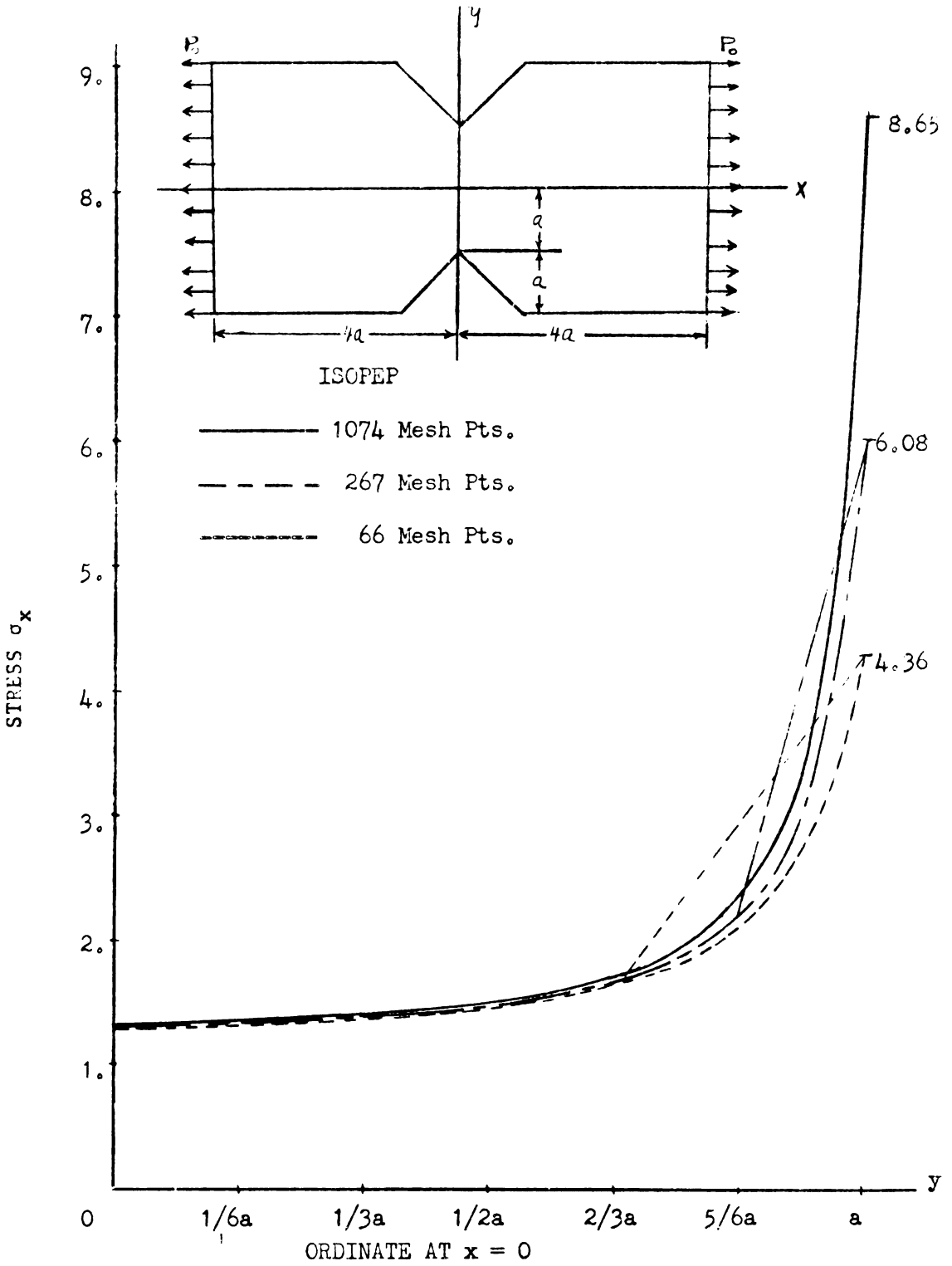


Figure 6.11 Stress concentration in the V-notch

TABLE 5 STRESS FUNCTION AT POINTS ADJACENT TO THE SEMI-CIRCULAR NOTCH

	COLUMN						
	12	13	14	15	16	17	18
ROW							
3	-.02569	-.01823	-.00971	0.00000			
4	-.02560	-.01816	-.00966	0.00000			
5	-.02533	-.01795	-.00953	.00002			
6	-.02488	-.01757	-.00928	.00009			
7	-.02423	-.01703	-.00890	.00025			
8	-.02340	-.01631	-.00833	.00056	.01042		
9	-.02239	-.01541	-.00759	.00107	.01057		
10	-.02123	-.01435	-.00670	.00173	.01093	.02085	
11	-.01992	-.01316	-.00566	.00256	.01149	.02107	.03125
12	-.01851	-.01186	-.00450	.00353	.01223	.02155	.03140
13	-.01700	-.01046	-.00324	.00463	.01313	.02221	.03180
14	-.01541	-.00898	-.00189	.00582	.01415	.02302	.03239
15	-.01378	-.00745	-.00049	.00708	.01523	.02392	.03309
16	-.01212	-.00590	.00094	.00837	.01637	.02488	.03386
17	-.01045	-.00433	.00239	.00969	.01753	.02588	.03468

	COLUMN						
	19	20	21	22	23	24	25
ROW							
12	.04166						
13	.04180	.05209					
14	.04216	.05224	.06253	.07291			
15	.04266	.05255	.06269	.07297	.08334	.09375	.10416
16	.04324	.05296	.06294	.07310	.08339	.09376	.10416
17	.04389	.05344	.06326	.07330	.08349	.09379	.10416

TABLE 6 X-STRESS COMPONENT AT POINTS ADJACENT TO THE SEMI-CIRCULAR NOTCH

	COLUMN									
	13	14	15	16	17	18	19	20	21	22
ROW										
3	2.449	2.745	3.238							
4	2.431	2.701	2.952							
5	2.369	2.629	2.576							
6	2.275	2.506	2.111							
7	2.154	2.345	2.061							
8	2.031	2.130	2.193							
9	1.917	1.955	1.911	1.386						
10	1.805	1.801	1.745	1.664	1.417					
11	1.708	1.678	1.611	1.506	1.367	1.106				
12	1.628	1.586	1.514	1.404	1.243	.955	.695			
13	1.563	1.514	1.440	1.330	1.174	.944	.653	.439		
14	1.508	1.455	1.380	1.274	1.131	.940	.711	.460	.238	
15	1.460	1.405	1.331	1.229	1.099	.936	.746	.543	.351	.612
16	1.417	1.362	1.288	1.193	1.074	.931	.769	.598	.431	.286
17	1.379	1.323	1.251	1.162	1.053	.927	.786	.638	.491	.358

TABLE 7 Y-STRESS COMPONENT AT POINTS ADJACENT TO THE SEMI-CIRCULAR NOTCH

	COLUMN									
	13	14	15	16	17	18	19	20	21	22
ROW										
3	.329	.209	0.000							
4	.328	.199	.044							
5	.361	.263	.114							
6	.395	.326	.220							
7	.425	.412	.345							
8	.400	.402	.444							
9	.350	.343	.357	.487						
10	.312	.326	.372	.451						
11	.271	.295	.348	.438	.584					
12	.222	.244	.286	.350	.443	.590	.323			
13	.169	.185	.214	.257	.318	.405	.500			
14	.116	.126	.145	.175	.216	.268	.321	.352		
15	.068	.074	.088	.108	.136	.170	.205	.226	.219	.180
16	.025	.030	.040	.055	.076	.102	.127	.147	.152	.135

TABLE 8 XY-SHEAR STRESS AT POINTS ADJACENT TO THE SEMI-CIRCULAR NOTCH

	COLUMN									
	13	14	15	16	17	18	19	20	21	22
ROW										
3	0.000	0.000								
4	-.104	-.153								
5	-.198	-.285								
6	-.266	-.391								
7	-.300	-.455								
8	-.307	-.464								
9	-.312	-.451	-.645							
10	-.312	-.434	-.578							
11	-.302	-.399	-.511	-.633						
12	-.287	-.365	-.448	-.535	-.628	-.578				
13	-.275	-.337	-.404	-.470	-.530	-.562				
14	-.266	-.319	-.374	-.426	-.470	-.491	-.469			
15	-.261	-.307	-.353	-.396	-.431	-.447	-.435	-.385	-.303	
16	-.258	-.298	-.338	-.375	-.403	-.417	-.409	-.376	-.317	-.242

TABLE 9 STRESS FUNCTION AT POINTS ADJACENT TO THE THE V-NOTCH

	COLUMN						
	12	13	14	15	16	17	18
ROW							
3	-.02187	-.01570	-.00853	0.00000			
4	-.02162	-.01540	-.00816	.00035	.01041		
5	-.02095	-.01466	-.00737	.00104	.01060	.02083	
6	-.01997	-.01362	-.00634	.00191	.01111	.02097	.03125
7	-.01876	-.01240	-.00518	.00291	.01181	.02136	.03136
8	-.01740	-.01106	-.00392	.00401	.01266	.02193	.03167
9	-.01595	-.00965	-.00260	.00517	.01361	.02264	.03214
10	-.01442	-.00818	-.00123	.00638	.01464	.02345	.03273
11	-.01286	-.00669	.00015	.00763	.01571	.02433	.03341
12	-.01128	-.00518	.00155	.00890	.01682	.02526	.03415
13	-.00969	-.00367	.00295	.01017	.01794	.02621	.03494
14	-.00811	-.00218	.00434	.01143	.01906	.02718	.03575
15	-.00656	-.00071	.00570	.01268	.02017	.02815	.03657
16	-.00503	.00072	.00704	.01390	.02127	.02911	.03739
17	-.00353	.00213	.00834	.01509	.02233	.03004	.03819

	COLUMN						
	19	20	21	22	23	24	25
ROW							
7	.04166						
8	.04175	.05208					
9	.04201	.05215	.06250				
10	.04239	.05236	.06255	.07291			
11	.04288	.05268	.06272	.07296	.08333		
12	.04345	.05308	.06298	.07309	.08336	.09375	
13	.04407	.05354	.06330	.07329	.08346	.09377	.10416
14	.04472	.05404	.06366	.07353	.08361	.09383	.10417
15	.04539	.05457	.06406	.07381	.08378	.09393	.10421
16	.04607	.05511	.06447	.07410	.08397	.09404	.10426
17	.04674	.05565	.06488	.07440	.08417	.09416	.10431

TABLE 10 X-STRESS COMPONENT AT POINTS ADJACENT TO THE V-NOTCH

	COLUMN									
	13	14	15	16	17	18	19	20	21	22
ROW										
3	2.306	3.158	8.653							
4	2.350	2.952	3.557	1.621						
5	2.288	2.601	2.633	1.534	.874					
6	2.149	2.269	2.139	1.540	.958	.642				
7	1.992	2.011	1.953	1.473	1.053	.708	.507			
8	1.848	1.818	1.670	1.400	1.086	.798	.559	.412		
9	1.725	1.672	1.542	1.335	1.091	.852	.638	.456	.337	
10	1.622	1.562	1.446	1.280	1.085	.884	.694	.525	.375	.273
11	1.539	1.475	1.372	1.235	1.072	.902	.736	.581	.438	.308
12	1.470	1.407	1.315	1.196	1.059	.913	.767	.625	.492	.367
13	1.413	1.352	1.268	1.164	1.045	.919	.789	.661	.538	.420
14	1.366	1.306	1.230	1.137	1.032	.921	.805	.690	.577	.467
15	1.326	1.269	1.198	1.115	1.021	.921	.817	.713	.610	.509
16	1.292	1.238	1.172	1.096	1.011	.921	.826	.732	.637	.545
17	1.263	1.211	1.150	1.079	1.003	.920	.834	.748	.661	.576

TABLE 11 Y-STRESS COMPONENT AT POINTS ADJACENT TO THE V-NOTCH

	COLUMN									
	13	14	15	16	17	18	19	20	21	22
ROW										
3	1.376	1.705	1.622							
4	1.033	.983	.787	.874						
5	.665	.529	.412	.725	.642					
6	.421	.318	.289	.468	.573	.507				
7	.273	.212	.217	.325	.426	.464	.412			
8	.179	.147	.163	.233	.313	.365	.380	.337		
9	.115	.098	.117	.167	.229	.278	.305	.311	.273	
10	.066	.059	.076	.115	.164	.206	.236	.250	.251	.216
11	.028	.026	.041	.072	.110	.146	.175	.192	.199	.196
12	-.002	-.003	.010	.035	.066	.097	.123	.141	.149	.150
13	-.029	-.027	-.016	.003	.028	.055	.078	.094	.104	.107
14	-.051	-.049	-.039	-.023	-.002	.019	.040	.056	.067	.071
15	-.069	-.066	-.058	-.044	-.027	-.008	.008	.024	.035	.041
16	-.084	-.080	-.073	-.061	-.047	-.031	-.015	-.001	.009	.017

TABLE 12 XY-SHEAR STRESS COMPONENT AT POINTS ADJACENT TO THE V-NOTCH

	COLUMN									
ROW	13	14	15	16	17	18	19	20	21	22
3	0.000	0.000	0.000							
4	.143	0.000	-.672							
5	.096	-.124	-.650	-.902						
6	0.000	-.221	-.563	-.773	-.698					
7	-.080	-.268	-.498	-.650	-.653	-.554				
8	-.134	-.286	-.448	-.561	-.586	-.540	-.450			
9	-.169	-.289	-.409	-.495	-.525	-.504	-.449	-.369		
10	-.191	-.286	-.378	-.445	-.475	-.467	-.430	-.374	-.302	
11	-.204	-.281	-.352	-.406	-.434	-.433	-.407	-.364	-.310	-.243
12	-.211	-.274	-.331	-.375	-.399	-.402	-.385	-.352	-.307	-.254
13	-.216	-.267	-.314	-.350	-.371	-.376	-.364	-.338	-.301	-.255
14	-.219	-.261	-.299	-.329	-.347	-.353	-.345	-.325	-.294	-.254
15	-.220	-.255	-.287	-.312	-.328	-.333	-.328	-.312	-.286	-.252
16	-.220	-.250	-.277	-.298	-.311	-.316	-.312	-.299	-.277	-.247

The finest mesh used for the notched tensile specimen problems was 24X48. The corresponding mesh interval h is 1/48. The ISPOEP program includes a subroutine STRESS for the calculation of any or all of the stress components σ_x, σ_y and τ_{xy}. Values of all three stress components and the stress function in a region surrounding the point of stress concentration are provided in Tables 5 to 12 for the semi-circular and V-notches. See Appendix B for the finite-difference equations used. These equations are valid at interior points only. Calculation of stresses at points on an irregular boundary are not included in the subroutine STRESS.

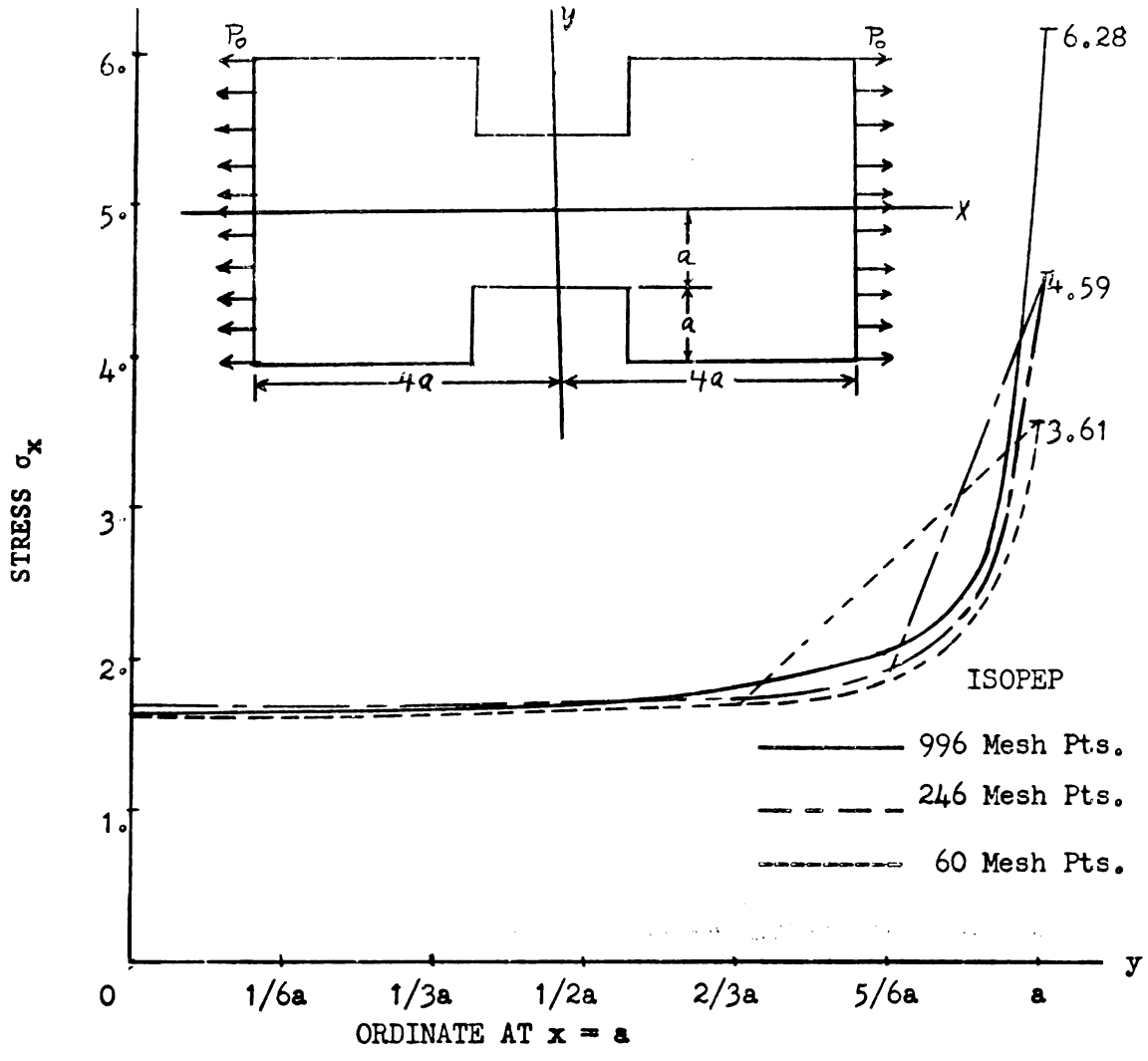


Figure 6.12 Stress concentration in a rectangular notch

Though a solution obtained with a coarse mesh does not provide a good measure of the stress at a point of stress concentration, the results shown in Figures 6.10, 6.11 and 6.12 indicate a decrease in mesh interval h produces only slight changes in the stresses computed at all other points.

Use of an average value of the stress function over an area surrounding a mesh point depends more on the magnitude of the area than on the geometry of the region. For example at the base of the V-notch the area surrounding the mesh point is taken as about half the area for an interior mesh point. Does this not imply that the solution obtained at this point is a better approximation for a tensile specimen with a

rounded rather than a sharp corner at the base of the notch? Some evidence for this conjecture is provided by comparison of the stress concentration factors found from the numerical solutions of Problems 401, 402 and 408 with the stress concentration factors provided by Peterson (1953) for notched flat bars in tension. Peterson provides values of the stress concentration factor K_t for tensile specimens with deep notches with parallel sides and a semicircular base as shown in Figure 6.13. The stress concentration is defined

$$K_t = \frac{\sigma_{max}}{\sigma_{norm}}$$

where σ_{norm} is the average stress across the minimum cross section of the specimen. The curve plotted in Figure 6.13 shows the relationship of K_t to r/D where r is the notch radius and D is the width of the bar. This curve is based on the Neuber-theory solutions tabulated by Peterson (pp 26-27) for d/D = .5 where d is the minimum distance across the bar at the base of the notch. The stress concentration factors for three different numerical solutions of Problem 4 are listed in Table 13.

Table 13 Stress concentrations factors for the V-notch tensile specimen

Problem	Mesh Size	h	σ_{max}	σ_{norm}	K_t	r/D
401	6x12	1/12	4.362	2	2.181	1/12
402	12x24	1/24	6.083	2	3.042	1/24
408	24x48	1/48	8.653	2	4.326	1/48

Assuming the numerical solution approximates the solution for a bar with circular notch such that r = h, the stress concentration factors K_t for problems 401, 402, 408 are plotted as three points labeled in Figure 6.13. There is good agreement between the numerical solutions and the Neuber theory solutions. Peterson reports that for notches with inclined sides having an included angle α and a circular arc at

the base, the notch angle has very little effect if $0^{\circ} < \alpha \leq 90^{\circ}$.

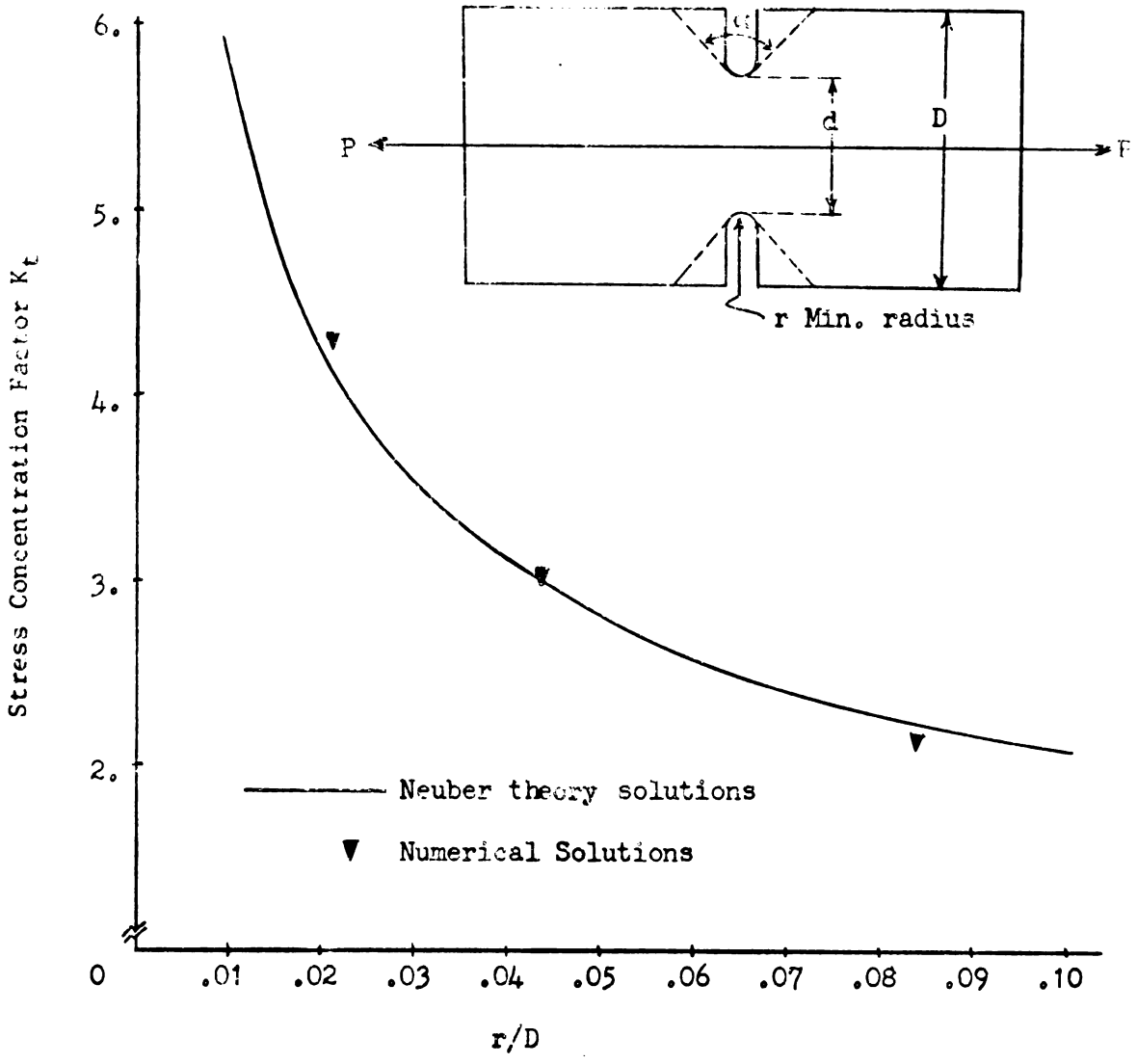


Figure 6.13 Stress Concentration factor K_t for a notched flat bar

Treatment of irregular boundaries

The semi-circular notch is typical of the irregular boundaries often encountered in practical problems. Griffin and Varga (1962) show how different mesh intervals can be used so that each horizontal grid line crossing the irregular boundary intersects a vertical grid line on the boundary. There are several advantages to this approach, but it does increase the number of arithmetic operations needed for the calculation of each vector iterate. For this reason a uniform mesh

interval is used in ISOPEP, and the irregular boundary is approximated with straight line segments, introduced arbitrarily as shown in Fig. 6.14, connecting nodes of the primary mesh. (The dual mesh is formed by lines halfway between lines of the primary mesh.) The points labeled B are on the boundary and values of U_{Bij} are computed using Equation (28) of Chapter III. The only complication is the evaluation of $\nabla^2 U_i$ for points on the boundary. A few examples will illustrate the technique.

$$\nabla^2 U_o = \frac{1}{A_o} \oint \frac{\partial U_o}{\partial n} ds$$

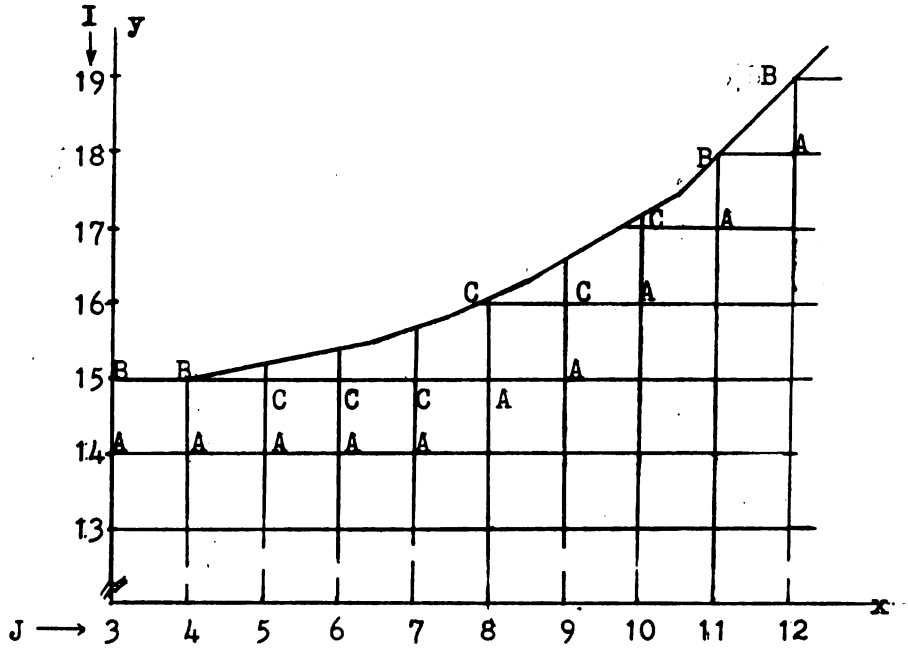


Figure 6.14 Boundary points for the semi-circular notch

At an interior point where $A_o = h^2$ the line integral is approximated (see Fig. 3.3 and Equation (27))

$$\nabla^2 U_o = \frac{1}{A_o}\left[(U_1 - U_o)\frac{S_{o1}}{h_{o1}} + (U_2 - U_o)\frac{S_{o2}}{h_{o2}} + (U_3 - U_o)\frac{S_{o3}}{h_{o3}} + (U_4 - U_o)\frac{S_{o4}}{h_{o4}}\right]$$

For the notch boundary

$$\frac{\partial \phi}{\partial y} = 1/2 , \qquad \frac{\partial \phi}{\partial x} = 0.$$

At a boundary point such as $P_{15,3}$ (I = 15, J = 3), Figures 6.14 and 6.15(a),

$$\int_a^b \frac{\partial\phi}{\partial\eta}\,ds \approx (U_4 - U_0), \qquad \int_b^c \frac{\partial\phi}{\partial\eta}\,ds \approx \tfrac{1}{2}(U_1 - U_0),$$

$$\int_c^d \frac{\partial\phi}{\partial\eta}\,ds \approx \left(\frac{\partial\phi}{\partial y}\right)_0 h = \frac{h}{2}, \qquad \int_d^a \frac{\partial\phi}{\partial\eta}\,ds \approx \tfrac{1}{2}(U_3 - U_0),$$

$$\nabla^2 U_0 = \frac{2}{h^2}\left[U_4 + \tfrac{1}{2}(U_1+U_3) - 2U_0 + \frac{h}{2}\right].$$

Thus the approximation to the line integrals along $\overline{bc}$ and $\overline{da}$ is made in terms of central difference quotients at the ends of the intervals instead of at an interior point. This illustrates an additional complication of the irregular boundary, which appears at most of the boundary points.

At point $P_{18,11}$, Figures 6.14 and 6.15(b),

$$\int_a^b \frac{\partial\phi}{\partial\eta}\,ds \approx (U_4 - U_0), \qquad \int_b^c \frac{\partial\phi}{\partial\eta}\,ds \approx (U_1 - U_0),$$

$$\int_c^a \frac{\partial\phi}{\partial\eta}\,ds \approx \left[\left(\frac{\partial\phi}{\partial x}\right)_0 \cos\theta + \left(\frac{\partial\theta}{\partial y}\right)_0 \sin\theta\right]\frac{h}{\sin\theta} = \frac{h}{2}, \quad \theta = 135^\circ$$

$$\nabla^2 U_0 = \frac{2}{h^2}\left[U_1 + U_4 - 2U_0 + \frac{h}{2}\right].$$

At the point $P_{15,6}$, Figures 6.14 and 6.15(c),

$$\int_a^b \frac{\partial\phi}{\partial\eta}\,ds \approx (U_4 - U_0), \qquad \int_b^c \frac{\partial\phi}{\partial\eta}\,ds \approx (U_1 - U_0)$$

$$\int_c^d \frac{\partial\phi}{\partial\eta}\,ds \approx \left[\left(\frac{\partial\phi}{\partial x}\right)_0 \cos\theta + \left(\frac{\partial\phi}{\partial y}\right)_0 \sin\theta\right]\frac{h}{\sin\theta} = \frac{h}{2},$$

$$\int_d^a \frac{\partial\phi}{\partial\eta}\,ds \approx \left[\tfrac{1}{2} + \frac{d_2}{h}\right](U_3 - U_0)$$

$$\nabla^2 U_0 = \left[\frac{U_4 + U_1 - 2U_0 + \frac{h}{2} + (\frac{1}{2} + \frac{d_2}{h})(U_3 - U_0)}{(\frac{1}{2} + d_1/h)\,h^2}\right]$$

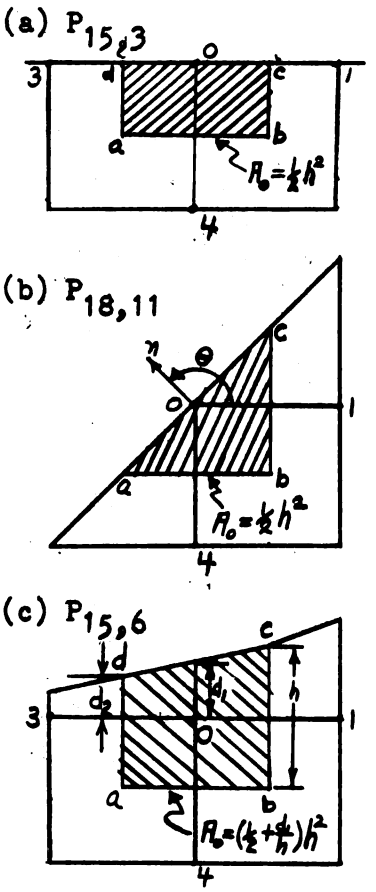


Figure 6.15 Treatment of boundary points

The evaluation of $\nabla^2 U_{(15,6)}$ is used in Equation (28) for the finite-difference equations at the point $P_{14,6}$. It is not possible to write finite-difference equations for point $P_{15,6}$ using Equation (28). Hence the evaluation of $U_{(15,6)}$ and the discrete values of the stress function at other points labeled with a C in Fig. 6.14 are obtained by interpolation. Fox (1950) treats this problem with the Gregory-Newton forward interpolation formula. A point 0 exterior to the boundary is introduced as shown in Fig. 6.16. With the value of U_B and $U_B' = (\frac{\partial U}{\partial X})_B$ specified on the boundary, it is possible to eliminate U_0 and determine U_1 at the first interior point in terms of the boundary conditions and one or more interior points as follows

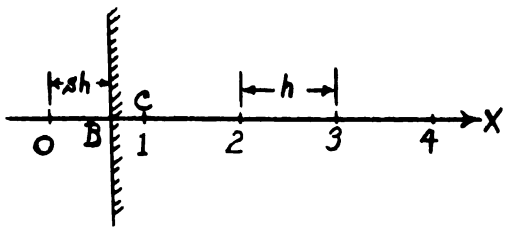


Figure 6.16 Interior points near a boundary

$$U = U_o + \Delta U_o \frac{(X-X_o)}{h} + \frac{\Delta^2 U_o}{2!}\frac{(X-X_o)(X-X_1)}{h^2} + \frac{\Delta^3 U_o}{3!}\frac{(X-X_o)(X-X_1)(X-X_2)}{h^3} + \frac{\Delta^4 U_o}{4!}\frac{(X-X_o)(X-X_1)(X-X_2)(X-X_3)}{h^4} + \cdots\cdots .$$

Let $s = (x_B - x_o)/h$ then

$$U_B = U_o + (U_1 - U_o)s + (U_2 - 2U_1 + U_o)\frac{s(s-1)}{2!} + (U_3 - 3U_2 + 3U_1 - U_o)\frac{s(s-1)(s-2)}{3!} + (U_4 - 4U_3 + 6U_2 - 4U_1 + U_o)\frac{s(s-1)(s-2)(s-3)}{4!} + \cdots\cdots ,$$

$$U_B' = (U_1 - U_o) + (U_2 - 2U_1 + U_o)(s - \tfrac{1}{2}) + (U_3 - 3U_2 + 3U_1 - U_o)\frac{3s^2 - 6s + 2}{3!} + (U_4 - 4U_3 + 6U_2 - 4U_1 + U_o)\frac{4s^3 - 18s^2 - 22s - 6}{4!} + \cdots\cdots ,$$

or

$$U_B = D_1 U_o + D_2 U_1 + D_3 U_2 + D_4 U_3 + D_5 U_4 + H_1$$

$$hU_B' = E_1 U_o + E_2 U_1 + E_3 U_2 + E_4 U_3 + E_5 U_4 + H_2$$

where the D_1 and E_1 are fourth degree polynomials in s obtained by retaining differences up to fourth order while H_1 and H_2 involve fifth

and higher differences of U as well as higher degree terms in s. Neglecting H_1 and H_2 the external point value can be eliminated and the expression of U_1 is

$$U_1 = \left[\frac{U_B}{D_1} - \frac{hU'_B}{E_1} + \left(\frac{E_3}{E_1} - \frac{D_3}{D_1}\right)U_2 + \left(\frac{E_4}{E_1} - \frac{D_4}{D_1}\right)U_3 + \left(\frac{E_5}{E_1} - \frac{D_5}{D_1}\right)U_4\right] \Big/ \left(\frac{D_2}{D_1} - \frac{E_2}{E_1}\right).$$

Though it is preferable to retain differences of at least the fourth order, similar formulas can be written for higher or lower order differences.

Problem 6 - A rectangular hole in an infinite plate

The solution for an infinite plate subjected to uniform tensile stress P_0 at $x = \infty$ and $x = -\infty$ is provided by Savin (1961) for several different approximations of the boundaries of a rectangular hole. The stresses on the boundary of a square hole obtained from one of Savin's solutions will be used for comparison with the stresses obtained from a numerical solution. The problem solved numerically is for a 2 unit by 2 unit region surrounding the hole. The hole dimensions are 2/3 by 2/3. Values of the stress function and normal derivatives on the outer boundary of the selected region are determined from Savin's solution. The numerical solution of the biharmonic equation provides a basis for the determination of the stresses on the boundary of the hole, and these are compared with stresses from Savin's solution for a curvilinear

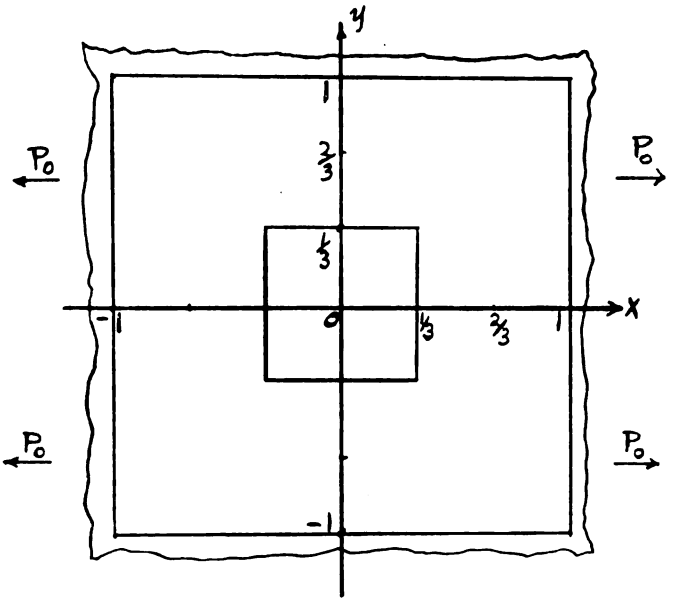


Figure 6.17 Problem 6
Infinite plate with a square hole

approximation to the square hole.

Savin's solution is obtained using Muskhelivshili's method. It is known that the stress function ϕ (x,y) can be written

$$\phi(x,y) = \mathrm{Re}\left[\bar{z}\,\alpha(z) + \beta(z)\right]$$

where Re means the real part of the complex expression, $\alpha(z)$ and $\beta(z)$ are analytic functions of the complex variable $z = x + iy$. $\bar{z} = x - iy$.

Savin's solution is obtained by conformal mapping of the region exterior to the hole onto the interior of the unit circle in the complex ζ-plane by a mapping function $z = \omega(\zeta)$. The mapping function, approximated by three terms of an infinite series, and the complex stress functions $\alpha(\zeta) \equiv \alpha[\omega(\zeta)]$ and the derivative of $\beta(\zeta) \equiv \beta[\omega(\zeta)]$ are given by Savin (1961) pp 51-53 as

$$\omega(\zeta) = R\left[\frac{1}{\zeta} - \frac{1}{6}\zeta^3 + \frac{1}{56}\zeta^7\right],$$

$$\alpha(\zeta) = P_0 R\left[\frac{1}{4\zeta} + 0.426\zeta + 0.046\zeta^3 + 0.008\zeta^5 + 0.004\zeta^7\right],$$

$$\psi(\zeta) = \frac{d\beta(\zeta)}{d\zeta} = -P_0 R\left[\frac{1}{2\zeta} + \frac{0.548\zeta - 0.457\zeta^3 - 0.026\zeta^5 - 0.029\zeta^7}{1 + 0.5\zeta^4 - 0.125\zeta^8}\right],$$

where $\zeta = \xi + i\eta = \rho e^{i\theta}$, (ξ, η), (ρ, θ) are rectangular and polar coordinates respectively in the complex ζ plane. Points on the circumference of the circle correspond to points on the boundary of the square hole, but when the series for $\omega(\zeta)$ is truncated, the correspondence is not exact. The Cartesian coordinates of any point in the plate in terms of the polar coordinates in the complex plane are given by

$$x = R\left(\frac{1}{\rho}\cos\theta - \frac{\rho^3}{6}\cos 3\theta + \frac{\rho^7}{56}\cos 7\theta\right),$$

$$y = -R\left(\frac{1}{\rho}\sin\theta + \frac{\rho^3}{6}\sin 3\theta - \frac{\rho^7}{56}\sin 7\theta\right).$$

The boundary of the hole defined by these equations when $\rho = 1$ is not an exact square. The edges are slightly bowed and the corners are approximated with circular arcs of radius $r = 0.0245a$, where a is the distance along the x-axis from one edge of the curvilinear square to

the other. R is the length of one side of the square hole. The value of R = .39321508 is used in the problem solved numerically. This assures the transformation of the hole boundary at a distance of 1/3 from the origin onto the boundary of the circle $\rho = 1$ in the complex plane. Thus $a = 2/3$, and the corner radius of curvature for Savin's solution is $r = 0.0163$.

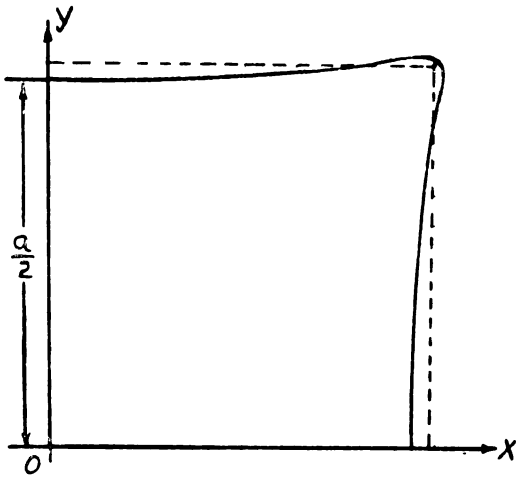


Figure 6.18 Approximation of the square hole

The equations given by Savin do not permit the direct evaluation of the stress function along the boundaries $x = 1$ and $y = 1$. Derivatives can be found from

$$\frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y} = \alpha(\varsigma) + \frac{\omega(\varsigma)}{\overline{\omega'(\varsigma)}}\overline{\alpha'(\varsigma)} + \psi(\varsigma)$$

See, for example, Savin (1961) p. 6 or Muskhelishvili (1953) p. 183. Values of ρ which correspond to points on $x = 1$ and $y = 1$, are determined for $0^\circ \leq \theta \leq 90^\circ$ at intervals $\Delta\theta = 3^\circ$. For these values of and θ, $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ are calculated. The availability of a basic set of complex variable subroutines and statements in 3600 FORTRAN simplify these calculations. The values of the derivatives of ϕ thus determined relate to points unequally spaced along $x = 1$ and $y = 1$. A five point Lagrange interpolation formula is used to find values of the derivatives at equally spaced points. Along the x-axis, quadrature of $\frac{\partial \phi}{\partial x}$ provides the point values of ϕ up to an additive constant. The $\frac{\partial \phi}{\partial y}$ values specify a normal derivative. A similar procedure is used to obtain values of ϕ and normal derivatives along the boundary $y = 1$.

The general problem of a multiply-connected region is discussed by Griffin and Varga (1962). The constants of integration can be arbitrarily chosen on an exterior closed boundary. However, for each interior closed boundary it is necessary to determine ϕ_k, $\frac{\partial \phi}{\partial x_k}$, $\frac{\partial \phi}{\partial y_k}$ for one point P_k on the boundary in such a way that the components of displacement u,v and the rotation $\omega_z = \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$ will be single-valued. The three additional unknowns for each interior boundary are related to the point values of the stress function in the region surrounding the hole in a manner which permits use of iterative methods.

Since the purpose of the present numerical solution was only to test the efficacy of the finite-difference method in reproducing details of the rapidly-varying stress near a stress-concentration point by comparison with Savin's solution, the procedure of Griffin and Varga for a multiply-connected region was not followed. Instead the results of Savin's solution were used to choose the integration constants on the boundary x = 1 in such a way that $\phi = 0$ at the point (1/3,0), and the assumption that ϕ is symmetric with respect to the x and y axes will satisfy the required conditions of single-valuedness. The symmetry conditions then took care of the other constants on the hole boundary, and it was possible to solve the problem with one quarter of the plate as a simply-connected region, see Fig. 6.19. The x-axis is a line of symmetry. Hence $\frac{\partial \phi}{\partial y} = 0$ along the x-axis. The

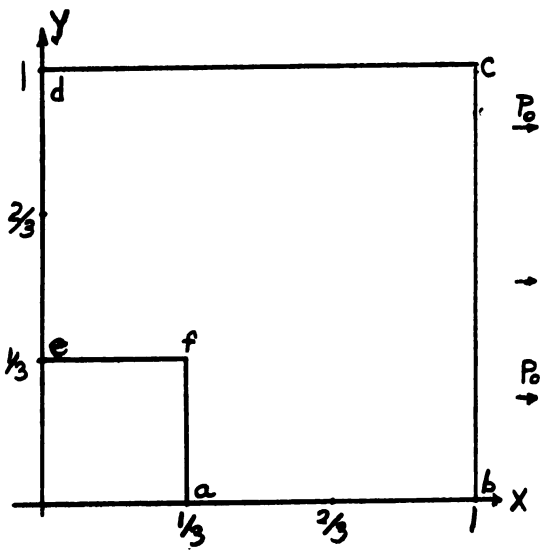


Figure 6.19 Domain of solution for Problem 6

y-axis is also a line of symmetry. Hence $\frac{\partial \phi}{\partial x} = 0$ along the y-axis.

Along $\overline{af}$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0,$$

$$\frac{\partial \phi}{\partial y} = \left(\frac{\partial \phi}{\partial y}\right)_a = 0, \qquad \frac{\partial \phi}{\partial x} = \left(\frac{\partial \phi}{\partial x}\right)_a = A$$

$$\phi = B = 0$$

The value of B is zero since it has been assumed that $\phi_a = 0$.

Along $\overline{ef}$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0, \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0,$$

$$\frac{\partial \phi}{\partial x} = \left(\frac{\partial \phi}{\partial x}\right)_e = 0, \qquad \frac{\partial \phi}{\partial y} = \left(\frac{\partial \phi}{\partial y}\right)_f = 0,$$

$$\phi = C = \phi_f = 0$$

Hence $A = 0$.

Along $\overline{bc}$ and $\overline{cd}$ boundary values of the stress function and its normal derivatives are obtained from Savin's solution.

Iterative solutions were obtained for mesh intervals of 1/15, 1/30, and 1/48. Values of the stress function and σ_x at the intersection of every fourth grid row and grid column of the 48x48 mesh are given in Fig. 6.20.

A comparison of the stress concentrations found in the numerical solution and those given on p. 53 of Savin (1961) is provided in Table 14. Values of σ_θ given by Savin are the values of σ_y in a rectangular coordinate system which has the origin at the point under investigation and the y-axis tangent at the given point to the curve $\rho = 1$. See Savin p. 8. The table includes comparable stresses, σ_A, for the iterative solution with a convergence criterion of 10^{-6} and a second set σ_B for the solution obtained with a convergence criterion of 10^{-7}. Both are included because this appears to be an exception to the general case in which stress concentrations increase as the convergence

FIGURE 6.20 Stress Function and σ_x Distributions for Problem 6

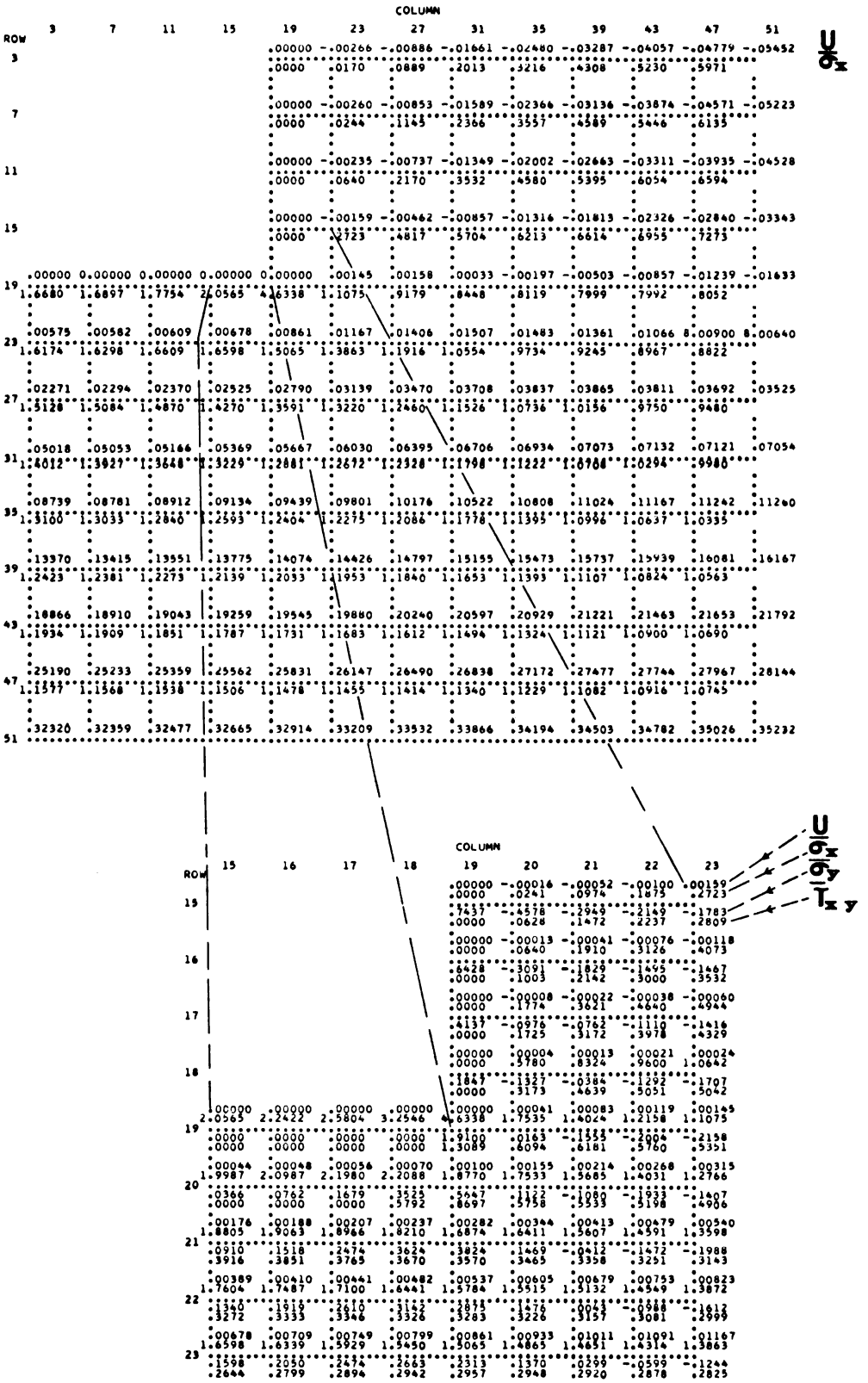


Table 14. Comparison of boundary values of σ_θ obtained from the iterative solution with those from Savin's solution.

Savin's Solution				ISOPEP Solutions			
θ	x	y	σ_θ	σ_A	σ_B	x	y
0°	.3347	0	-.936	-1.005	-.902	.3333	0
				-.991	-.895	.3333	.0833
				-.973	-.883	.3333	.1250
				-.944	-.864	.3333	.1667
				-.897	-.828	.3333	.2083
				-.801	-.744	.3333	.2500
				-.452	-.414	.3333	.2917
35°	.3361	.2952	-.544				
				.170	.185	.3333	.3125
40°	.3352	.3164	.605				
45°	.3294	.3294	4.368	4.78	4.52	.3333	.3333
50°	.3164	.3352	4.460				.3333
				3.402	3.281	.3125	.3333
55°	.2952	.3361	2.888				
				2.710	2.645	.2915	.3333
				2.174	2.057	.2500	.3333
				1.979	1.869	.2083	.3333
				1.880	1.775	.1667	.3333
				1.819	1.722	.1250	.3333
				1.780	1.690	.0833	.3333
90°	0	.3347	1.760	1.748	1.668	0	.3333

The stresses σ_A and σ_B are comparable to σ_θ for iterative solutions with the convergence criteron of 10^{-6} and 10^{-7} respectively.

criterion is decreased. The values of σ_A and σ_B along $x = .3333$ for $y < .3333$ are values of σ_y computed from the point values of the stress function. Along $y = .3333$ for $x < .3333$ the values of σ_A and σ_B are the values of σ_x computed from the stress function solution. At $x = .3333$, $y = .3333$ which corresponds to $\theta = 45^\circ$, the values of σ_θ given by Savin would be along a line which makes an angle of 45° with the x-axis. Values of σ_x, σ_y and τ_{xy} computed at the corner of the square were used to compute values of σ_A and σ_B in the same direction as σ_θ. The values, $\sigma_\theta = 4.368$, $\sigma_A = 4.78$ and $\sigma_B = 4.52$ are the largest stress concentrations occuring at corresponding points in the solutions considered. The fact that slightly different values are found for the numerical solutions might be expected from the different representation of the boundary at the corner in the numerical solution and Savin's solution.

VII SUMMARY AND CONCLUSIONS

The objective of this study has been three-fold: (1) The identification of efficient iterative methods for the solution of plane elastostatic problems; (2) The preparation of a system of computer programs for solving this class of problems; and (3) A demonstration of the use of iterative methods.

An investigation of the numerical solution of elliptic differential equations resulted in the selection of three matrix iterative methods as the alternatives which should be considered for inclusion in a digital computer program for the solution of plane elastostatic problems. Computer programs have been written for the point successive overrelaxation method, the alternating-direction implicit method and the cyclic Chebyshev semi-iterative method. Solutions of a model problem for various mesh intervals and convergence parameters are used for comparing the methods. The model problem is a square plate with uniformly distributed loads on portions of two edges. See Section VI-1.

The superiority of one iterative method over another may be judged by comparing the number of iterations required to satisfy a given convergence criterion. The results given in Fig. 4.7 (a) show the cyclic Chebyshev semi-iterative method is iteratively faster than the other methods when the number of mesh points is less than approximately 125. This number of mesh points corresponds to a mesh interval $h = 1/16$. For $h < 1/16$ the alternating-direction implicit method is iteratively faster. Whether one method is iteratively faster than another may not be an adequate basis for selecting the best method for a computer program.

Cost, which is directly related to the computer time required, may be a better measure of the superiority of a particular method. The results given in Fig. 4.7 (b) show the point successive overrelaxation method is better than the other methods for a mesh interval $h > 1/26$. For a finer mesh, say the number of interior mesh points is more then 350, the alternating-direction implicit method is the best for the problems examined.

The results given in Fig. 4.7 were obtained using various methods for approximating the optimum relaxation factors. How do these three schemes compare when the optimum parameters for accelerating convergence are used? A series of problems were run in an attempt to answer this question. A plot of the number of iterations against the associated parameter for accelerating convergence is given for each method. See Figures 4.3, 4.4, 4.6. In Table 2 the data for the second and third problems provide a comparison of the number of iterations and machine time for a problem with mesh interval $h = 1/12$. Optimum parameters were used in the second problem and the standard approximations given in Section IV were used in problem 3. Use of the optimum parameters produced substantial improvement in the machine time required; 42% for successive overrelaxation, 24% for the alternating-direction implicit method and 18% for the cyclic Chebyshev semi-iterative method. These results show the magnitude of the improvement which could be made in iterative methods if better estimates of the optimum parameters could be found. This is a problem which warrants further investigation.

Another important consideration in the selection of an iterative method for the solution of plane elastostatic problems is the storage requirements of instruction and arrays. A summary of the storage

requirements for each method is provided in Table 1. The successive overrelaxation method requires less than 50% of the storage needed for either of the other methods. Using successive overrelaxation, problems with up to 435 mesh points have been solved on an IBM 1620 with a core memory of 20,000 decimal digits. Comparisons of the times required to solve problems on the CDC 3600 and the IBM 1620 are given in Table 4.

A system of FORTRAN computer programs for the solution of plane elastostatic problems has been written and tested. Flow diagrams, listing of FORTRAN source decks, specification of input and the output for a sample problem are provided in Appendix B. A subroutine for each of the iterative methods is included. In addition there are subroutines for: calculation of stress components; calculation of initial values when the mesh is refined; input and output. The main program ISOPEP provides for the linkage of these subroutines and an additional pair of subroutines which account for the boundary conditions of a particular problem. The boundary condition subroutines for the six problems discussed in Section VI are also included in Appendix B.

A set of six problems has been solved using the ISOPEP program. The first problem is a square plate with edge load on two sides as shown in Fig. 4.1. This is a model problem used to compare the selected iterative methods. Distributions of the stress function and σ_x for Problem 1 are given in Fig. 6.2.

The second and sixth problems were selected because they provide a basis for comparing analytic and numerical solutions. Problem 2 is a semi-infinite plate with a uniformly distributed load along a segment of the edge. The stress function and σ_y distributions from the numerical solution are given in Fig. 6.5. A comparison of the distributions of

σ_y at one mesh interval from the plate edge is provided in Fig. 6.4; values from an exact solution, from the ISOPEP solution, and from a numerical mapping solution are compared. Problem 6 is an infinite plate with a square hole. Comparison of the analytical solution of G.N. Savin with the iterative solution is provided in Table 14. Stresses on the boundary of the square hole are used for this comparison. The numerical solution shows a slightly higher stress concentration at the corner than Savin's solution, but the agreement is fairly good, considering that the effective rounding of the corner implicit in the finite-difference solution approximates the boundary in a way different from Savin's truncated series mapping.

The other three problems are notched tensile specimens. Problem 3 has semi-circular notches, Problem 4 has V-notches, and Problem 5 rectangular notches. These problems were selected for an investigation of the numerical determination of stress concentrations. Stress function and σ_x distributions are given in Figures 6.7, 6.8 and 6.9 for Problems 3, 4 and 5 respectively. Additional details of solution values at the base of the semi-circular and V-notches are given in Tables 5 through 13. Values of the shear stress τ_{xy} and σ_y are included.

Comparisons are made of the stress concentrations computed with different choices of mesh intervals h. See Figures 6.10, 6.11, 6.12. In addition, for the V-notch the stress concentrations for the ISOPEP solutions are compared with the Neuber theory solutions provided by R. E. Peterson (1953). Under the assumption that the numerical solution at the base point of the V-notch represents a solution for a specimen with a semi-circular notch of radius equal to the mesh interval, the two solutions show good agreement. See Fig. 6.13. This agreement may

be somewhat fortuitous, but it does lend support to the usefulness of numerical analysis in the vicinity of a singularity. In any actual body the notch would be somewhat rounded, and the numerical analysis for a sharp notch approximates the actual state of stress in a rounded notch with radius equal to the mesh interval of the analysis.

The numerical results for the six problems solved indicate that high speed digital computers provide an effective means for the analysis of plane elastostatic problems. The stress distributions obtained from numerical solutions compare very well with explicit solutions. Though it is necessary to use a fine mesh in the neighborhood of a singularity, the ISOPEP program for the CDC 3600 permits the use of over 3600 mesh points. This should permit the analysis of quite complicated problems. ISOPEP solutions on the CDC 3600 are reasonable in cost. At an hourly rate of $375.00 per hour for computer time, the cost of solving a problem with 200 mesh points would be $5.41. The cost for the solution of a problem with approximately 1100 mesh points would be $36.23.

The possibility of augmenting the ISOPEP program has been considered. Since ISOPEP is a set of linked subprograms, additions could be easily made. The additions which have been considered include: (1) A better treatment of boundaries formed with circular arcs, perhaps, through use of polar coordinates; (2) Use of a refined mesh in a subregion of the solution domain; (3) Use of a different formulation of the differential equations. The variational method for deriving difference equations as used in Section III provides a finite-difference form of the Navier equations, see Eq. (37). A study of the iterative solution of these would be of value.

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APPENDIX A

CONVERGENCE OF ITERATIVE METHODS

The convergence characteristics of an iterative solution of a system of linear equations of the form

$$\underline{A}\,\underline{U} = \underline{F} \qquad (A\text{-}1)$$

can be judged by examination of the matrix $\underline{A}$. The system of linear equations associated with the biharmonic difference equations can be written so $\underline{A}$ is positive definite and of the form

$$\underline{A} = \underline{D} + \underline{G} + \underline{H} \qquad (A\text{-}2)$$

where $\underline{D}$ is a nonsingular block diagonal matrix and the form of $\underline{G} + \underline{H}$ associated with a particular iterative method establishes the convergence of the method. The basic conditions for convergence are given in Theorem A.5 of Subsection 3 below, while estimates of the relative rates of convergence of different methods are cited in Subsection 4. Theorems A.8, A.9 and A.10 give some alternative convergence conditions for the Richardson's, Gauss-Seidel and successive overrelaxation methods. Subsection 5 is concerned with the selection of optimum relaxation factors.

For fuller discussion, with proofs and bibliography, see Varga (1962) and Faddeeva (1959). Specific page references in these readily available sources are cited. No attempt has been made to give a historical account here with credits to the originator of each result. The purpose of this appendix is a summary review of basic matrix properties and the convergence of matrix iterative methods.

1. Matrix and Vector Norms, the Spectral Norm of a Matrix

Definition: If $\underline{A}$ is an n x n matrix, then $\underline{A}$ is convergent if the sequence of matrices $\underline{A}$, $\underline{A}^2$, $\underline{A}^3$ ..... converges to the n x n null matrix, $\underline{O}$. Otherwise $\underline{A}$ is divergent.

In the investigation of iterative schemes it is important to judge not only the convergence of the solution vector $\underline{U}$, error vector $\underline{E}$, and associated matrices $\underline{B}$, but also the rates of convergence. Norms of vectors and matrices are of importance in the discussion of rates of convergence. The norms have been defined in many different ways. Some of these norms are discussed below. Except in this section $||X||$ will always denote the Euclidean vector norm defined as $||X||_{III}$ below and the matrix norm $||A||$ will mean the spectal norm $||A||_{III}$. We will also use the term spectral radius, denoted by $\rho(\underline{A})$ for the magnitude of the largest eigenvalue of the matrix, defined in the next subsection.

The norm of a vector $\underline{X}$ is an associated non-negative number designated $||\underline{X}||$ which satisfies

(a) $||\underline{X}|| > 0$ for $\underline{X} \neq \underline{O}$, and $||\underline{O}|| = 0$,

(b) $||C\,\underline{X}|| = |C|\,||\underline{X}||$ where C is any scalar,

(c) $||\underline{X} + \underline{Y}|| \leq ||\underline{X}|| + ||\underline{Y}||$, the triangular inequality.

There are three norms of interest. For the vector $\underline{X} = \{X_1, X_2, X_3 \text{ ------ } X_n\}$, we define

I. $||\underline{X}||_I = \max_i |X_i|$,

II. $||\underline{X}||_{II} = |X_1| + |X_2| + \text{----} + |X_n|$,

III. $||\underline{X}||_{III} = \sqrt{|X_1|^2 + \text{----} + |X_n|^2}$, Euclidean norm

Since norms are often used as bounds it should be noted that

$$||\underline{X}||_I \leq ||\underline{X}||_{II} \leq n\,||\underline{X}||_I$$

$$||\underline{X}||_{II} \leq ||\underline{X}||_{III} \leq \sqrt{n}\,||\underline{X}||_I$$

Fadeeva (1959) states that a necessary and sufficient condition for a sequence $\underline{X}^{(m)}$ of vectors to converge to a vector $\underline{X}$ is that

$$|| \underline{X}^{(m)} - \underline{X} || \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

for any norm satisfying conditions (a), (b) and (c); he gives a proof for the three norms I, II, and III.

The norm of an n x n matrix $\underline{A}$ is a non-negative number $|| \underline{A} ||$ which satisfies

(a) $|| A || > 0$ if $\underline{A} \neq \underline{0}$, and $|| \underline{0} || = 0$,

(b) $|| C \underline{A} || = |C| \, || \underline{A} ||$

(c) $|| \underline{A} + \underline{B} || \leq || \underline{A} || + || \underline{B} ||$

(d) $|| \underline{A} \, \underline{B} || \leq || \underline{A} || \, || \underline{B} ||$

The necessary and sufficient condition for convergence of a sequence of n x n matrices $\underline{A}^{(m)}$ is that $|| \underline{A}^{(m)} - \underline{A} || \rightarrow 0$ as $m \rightarrow \infty$. Any norm which satisfies (a), (b), (c) and (d) can be used to establish convergence. If $|| \underline{A}^{(m)} - \underline{A} || \rightarrow 0$ then $|| \underline{A}^{(m)} || \rightarrow || \underline{A} ||$.

As in the case of vectors it is possible to introduce matrix norms in a variety of ways. But for the purpose at hand it is convenient to introduce a matrix norm which satisfies some special requirements. One of these is the requirement for the matrix norm to be <u>compatible</u> with a given vector norm. This condition holds if

$|| \underline{A} \, \underline{X} || \leq || \underline{A} || \, || \underline{X} ||$. <u>Compatibility condition</u>

It is possible to determine a matrix norm which satisfies (a), (b), (c), (d) and which is compatible with a specified vector norm. For example, as a matrix norm compatible with the Euclidean vector norm $|| \underline{X} ||_{III}$, we define the <u>spectral norm</u> $|| A ||_{III}$ of a matrix $\underline{A}$ as the least upper bound of the norm of the vector $\underline{A} \, \underline{X}$ as $\underline{X}$ runs over all vectors of norm unity:

$$\|\underline{A}\|_{III} = \max \|\underline{A}\,\underline{X}\|_{III} \text{ with } \|\underline{X}\|_{III} = 1$$

2. Eigenvectors and eigenvalues

An eigenvector (also called proper vector, characteristic vector or latent vector) of a linear transformation $\underline{A}$ is any non-zero vector $\underline{X}$ such that

$$\underline{A}\,\underline{X} = \lambda \underline{X}$$

where λ may be a complex number and is known as an eigenvalue (or proper number, latent root or characteristic number).

If a vector $\underline{X}$ is an eigenvector it must satisify

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots a_{1n}x_n &= \lambda x_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots\cdots a_{2n}x_n &= \lambda x_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots a_{nn}x_n &= \lambda x_n
\end{aligned}$$

a system of homogeneous equations, which can have a non-trivial solution only if

$$\begin{vmatrix}
a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\
\vdots & & & \\
a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda
\end{vmatrix} = 0$$

The determinant is equivalent to an n^{th} degree polynomial in λ which is called the characteristic polyomial of the matrix $\underline{A}$. For a real symmetric matrix all the eigenvalues are real.

Definition: If $\underline{A}$ is an n x n complex matrix with eigenvalues λ_i, $1 \leq i \leq n$, then

$$\rho(\underline{A}) = \max |\lambda_i|$$

is the spectral radius of $\underline{A}$.

3. Properties of Matrices

Theorem A.1: If $\underline{A}$ is any arbitrary matrix then

$$\|\underline{A}\| \geq \rho(\underline{A}), \qquad \text{[Varga (1962) p.10]}$$

Theorem A.2: If $\underline{A}$ is an n x n matrix then

$$\|\underline{A}\| = \sqrt{\rho(\underline{A}^*\underline{A})}$$

where $\underline{A}^*$ is the conjugate transpose of $\underline{A}$. [Varga (1962) p.11]

Definition: An arbitrary n x n complex nonsingular matrix $\underline{A}$ and an n-dimensional vector $\underline{X}$ can be combined as $\underline{X}^T \underline{A} \underline{X}$ to form a homogeneous second degree expression in terms of the components of $\underline{X}$ which is called a quadratic form. If the value of the quadratic form is positive for all non-trivial $\underline{X}$ then the form and the matrix $\underline{A}$ are said to be positive definite.

Definition: If $\underline{A}$ is a n x n complex matrix such that any two elements situated symmetrically with respect to the principal diagonal are complex conjugates, $a_{ji} = \bar{a}_{ij}$, then $\underline{A}$ is called a Hermitian matrix. Notable characteristics are:

1. Diagonal elements must be real.

2. Any symmetric matrix with real elements is a Hermitian matrix.

3. The eigenvalues of a Hermitian matrix are real.

4. The eigenvalues of a positive-definite Hermitian matrix are positive.

Definition: A square matrix $\underline{A}$ is diagonally dominant if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

If the strict inequality holds for all i, $\underline{A}$ is said to be strictly diagonally dominant.

Theorem A.3: If $\underline{A}$ is a Hermitian n x n strictly diagonally dominant matrix with positive real diagonal entries, then $\underline{A}$ is positive definite.

[Varga (1962) p. 23]

Theorem A.4: If $\underline{A}$ is an n x n Hermitian matrix, then

$$|| \underline{A} || = \rho(\underline{A}) \qquad \text{[Varga (1962) p.11]}$$

Theorem A.5: If $\underline{A}$ is an n x n complex matrix, then $\underline{A}$ is convergent if and only if

$$\rho(\underline{A}) < 1 \qquad \text{[Varga (1962) p. 13]}$$

Thus, we see that if the matrix $\underline{A}$ in Eq. (A-1) is symmetric and positive definite, then $\underline{A}$ is convergent if the spectral radius of $\underline{A}$ is less than 1. In the next subsection the convergence rates of several iterative methods will be considered.

4. Convergence Rates of Iterative Methods

The system of linear equations (A-1) can be written in the form

$$(\underline{I} - \underline{M})\ \underline{U} = \underline{F} \qquad \text{(A-3)}$$

where $\underline{M}$ is an n x n complex matrix and $\underline{I}$ is the identity matrix. If $(\underline{I} - \underline{M})$ is non-singular a unique solution vector $\underline{U}$ exists. Consider an iterative method for the solution of Eq. (A-3)

$$\underline{U}^{(m+1)} = \underline{M}\ U^{(m)} + \underline{F} \qquad m = 1, 2, 3 \text{ ----- } \qquad \text{(A-4)}$$

For any vector iterate $\underline{U}^{(m)}$ the difference $\underline{U}^{(m)} - \underline{U} = \underline{E}^{(m)}$ is a measure of the deviation of the vector iterate from the solution vector. The error vector $\underline{E}^{(m)}$ can be written in terms of the preceding error vector

$$\underline{E}^{(m)} = \underline{M}\ \underline{E}^{(m-1)}$$

This is obtained by subtracting (A-3) from (A-4) and it can be readily shown

$$\underline{E}^{(m)} = \underline{M}^{m}\ \underline{E}^{(o)}$$

From Theorem A.5 we conclude that the error vectors $\underline{E}^{(m)}$ will tend to zero for an arbitrary $\underline{E}^{(o)}$ if and only if $\underline{M}$ has a spectral radius $\rho(\underline{M}) < 1$.

Theorem A.6: If $\underline{A}$ and $\underline{B}$ are n x n matrices, then

(1) $|| \underline{A} || > 0$ or $\underline{A} = \underline{0}$;

(2) if k is a scalar, $\|k\,\underline{A}\| = |k|\,\|\underline{A}\|$;

(3) $\|\underline{A}+\underline{B}\| \leq \|\underline{A}\| + \|\underline{B}\|$;

(4) $\|\underline{A}\,\underline{B}\| \leq \|\underline{A}\|\,\|\underline{B}\|$;

(5) $\|\underline{A}\,\underline{U}\| \leq \|\underline{A}\|\,\|\underline{U}\|$ for all vectors $\underline{U}$ of n components.

Also, there is one nonzero vector $\underline{V}$ in the n-dimensional vector space such that

$$\|\underline{A}\,\underline{V}\| = \|\underline{A}\|\,\|\underline{V}\| \qquad \text{[Varga (1962) p.9]}$$

Since the solution vector iterates $\underline{U}^{(m)}$ and the associated error vector iterates $\underline{E}^{(m)}$ are n-dimensional, Theorem A.6 can be used as justification for

$$\|\underline{E}^{(m)}\| = \|\underline{M}^m\,\underline{E}^{(o)}\| \leq \|\underline{M}^m\|\;\|\underline{E}^{(o)}\|$$

$$\|\underline{M}^m\| \geq \frac{\|\underline{E}^{(m)}\|}{\|\underline{E}^{(o)}\|}$$

Hence, $\|\underline{M}^m\|$ as an upper bound estimate of the ratio of $\|\underline{E}^{(m)}\|$ to $\|\underline{E}^{(o)}\|$ provides a basis for the comparison of different iterative methods. If $\underline{M}$ is a Hermitian matrix then

$$\|\underline{M}^m\| = [\rho(\underline{M})]^m$$

Definition: Given two n x n complex matrices $\underline{M}$ and $\underline{B}$. If for some $m > 0$, $\|\underline{M}^m\| < 1$ then

$$R(\underline{M}^m) = -\ln\left[\|\underline{M}^m\|^{1/m}\right] = \frac{-\ln\|\underline{M}^m\|}{m} \qquad \text{(A-5)}$$

is defined as the *average rate of convergence for m iterations* of the matrix $\underline{M}$. [Varga (1962) p.62.] The convergence of $\underline{B}$ is said to be iteratively faster than the convergence of $\underline{M}$ when $R(\underline{B}^m) > R(\underline{M}^m)$.

The significance of $R(\underline{M}^m)$ as a measure of the average rate of convergence may be seen by consideration of the average error reduction factor per iteration σ.

$$\sigma = \left[\frac{\|\underline{E}^{(m)}\|}{\|\underline{E}^{(o)}\|}\right]^{1/m}$$

For the case when $\|\underline{M}^m\| < 1$

$$\sigma \leq \|\underline{M}^m\|^{\frac{1}{m}} = e^{-R(\underline{M}^m)} \qquad (A\text{-}6)$$

where e is the base of the natural logarithms. Thus $R(\underline{M}^m)$ is the exponential decay rate for the upper bound of the average error reduction σ, per iteration in an m-step iterative process.

Let $N_m = 1/R(\underline{M}^m)$. Substituting into (A-6) we find

$$\sigma^{N_m} \leq \frac{1}{e}$$

from which we conclude N_m is a measure of the number of iterations required to reduce the norm of $\underline{E}^{(o)}$ by a factor e.

Theorem A.7: $R(\underline{A}^m)$, the average rate of convergence for m iterations of an n x n convergent matrix $\underline{A}$ has a limiting value of $-\ln \rho(\underline{A})$ as m increases without bound. [Varga (1962) p.67]

Definition: The asymptotic rate of convergence $R_\infty(\underline{A})$ is

$$R_\infty(A) = -\ln \rho(\underline{A})$$

Corollary: Let $\underline{A}$ be a convergent n x n matrix, then

$$R_\infty(\underline{A}) \geq R(\underline{A}^m)$$

for any positive integer m.

For Hermitian matrices $\underline{A}$ and $\underline{B}$, the spectral radii may be used for comparison of rates of convergence, since by Theorem A.4,

$\rho(\underline{A}) < \rho(\underline{B}) < 1$ implies $\|\underline{A}^m\| < \|\underline{B}^m\| < 1$ and hence $R(\underline{A}^m) > R(\underline{B}^m)$, so that of two convergent Hermitian matrices, the one with the smaller spectral radius will have a faster average rate of convergence for any m.

It should be noted that though $\|\underline{M}^m\| \to 0$ as $m \to \infty$ for two iterative schemes with matrices $\underline{A}$ and $\underline{B}$, it is possible in general that for a selected value m_1, matrix $\underline{A}$ may be iteratively faster than $\underline{B}$ but for a second value m_2, $\underline{B}$ may be iteratively faster than $\underline{A}$.

Identification of the convergent iterative methods for biharmonic difference equations is aided by consideration of the following three theorems. The form of matrix $\underline{A}$ in Eq. (A-1) is the determining factor.

Theorem A.8: If $\underline{A}$ is a strictly diagonally dominant n x n complex matrix, then the associated Richardson's and Gauss-Seidel matrices are convergent and the corresponding methods are convergent for an arbitrary initial vector $\underline{U}^{(0)}$. [Varga (1962) p. 73.]

The matrix $\underline{A}$ associated with the biharmonic difference equations is not strictly diagonally dominant and Windsor (1957) has shown that Richardson's method is not convergent for the biharmonic equation.

Theorem A.9: If $\underline{A} = \underline{D} + \underline{G} + \underline{G}^*$ is an n x n Hermitian matrix where $\underline{D}$ is Hermitian and positive definite and $(\underline{D} + \underline{G})$ is nonsingular, then the Gauss-Seidel iterative method is convergent if and only if $\underline{A}$ is positive definite. ($\underline{G}^*$ is the conjugate transpose of $\underline{G}$). [Varga p.78.]

Theorem A.10: If $\underline{A} = \underline{D} + \underline{G} + G^*$ is an n x n Hermitian matrix and $\underline{D}$ is Hermitian and positive definite, then the successive overrelaxation method is convergent for any arbitrary $\underline{U}^{(o)}$ if and only if $\underline{A}$ is positive definite and $(\underline{D} + \omega \underline{G})$ is nonsingular for $0 < \omega < 2$. [Varga (1962) p. 80.]

5. Optimum relaxation factors

Definition: An n x n matrix $\underline{P}$ which has zeroes and ones for elements and only one non-zero element in each row and each column is called a permutation matrix.

Definition: Given $\underline{A}$ an n x n complex matrix with $n > 1$, if there exists an n x n permutation matrix $\underline{P}$ such that

$$\underline{P}\,\underline{A}\,\underline{P}^{T} = \begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{0} & \underline{A}_{22} \end{bmatrix}$$

where $\underline{A}_{11}$ and $\underline{A}_{22}$ are square matrices of order k and (n-k) respectively, then $\underline{A}$ is called reducible. Otherwise $\underline{A}$ is called irreducible.

Theorem A.11: If $\underline{A}$ is an irreducible n x n matrix, then: (1) $\underline{A}$ has a positive real simple eigenvalue equal to $\rho(\underline{A})$ the spectral radius; (2) Increasing the value of any element of $\underline{A}$ will increase $\rho(\underline{A})$; (3) Corresponding to the eigenvalue $\rho(\underline{A})$ there is an eigenvector with all its elements positive. [Theorem of Perron and Frobenius, Varga (1962) p.30]

Definition: If $\underline{A}$ is an n x n irreducible matrix with non-negative elements which has a single eigenvalue of modulus $\rho(\underline{A})$, it is said to be primitive. If $\underline{A}$ has k eigenvalues with modulus of $\rho(\underline{A})$ then $\underline{A}$ is cyclic of index k, $k \geq 2$. Each eigenvalue of modulus $\rho(\underline{A})$ is a simple eigenvalue. [Varga (1962) p.35]

Definition: A square matrix $\underline{A}$ of order n is said to be weakly cyclic of index k ($k > 1$) if an n x n permutation matrix $\underline{P}$ exists such that

$$\underline{P}\,\underline{A}\,\underline{P}^{T} = \begin{bmatrix} \underline{0} & \underline{0} & & \underline{0} & \underline{A}_{1,k} \\ \underline{A}_{2,1} & \underline{0} & & \underline{0} & \underline{0} \\ \underline{0} & \underline{A}_{3,1} & \underline{0} & \underline{0} & \underline{0} \\ \vdots & & \ddots & & \vdots \\ \underline{0} & \underline{0} & & \underline{A}_{k,k-1} & \underline{0} \end{bmatrix}$$

where the null diagonal submatrices are square. $\underline{A}$ may or may not be reducible. A matrix can be simultaneously weakly cyclic of different indices. [Varga (1962) p. 39]

Again consider the matrix equation

$$\underline{A}\,\underline{U} = \underline{F}$$

where $\underline{A}$ is an n x n complex matrix which can be partitioned

$$\underline{A} = \begin{bmatrix} \underline{A}_{1,1} & \underline{A}_{1,2} & - & - & - & \underline{A}_{1,N} \\ \underline{A}_{2,1} & \underline{A}_{2,2} & - & - & - & \underline{A}_{2,N} \\ - & & & & & - \\ - & & & & & - \\ - & & & & & - \\ \underline{A}_{N,1} & \underline{A}_{N,2} & - & - & - & \underline{A}_{N,N} \end{bmatrix} \tag{A-7}$$

where the square diagonal submatrices are nonsingular. Let $\underline{D}$ be formed of the submatrices $\underline{A}_{i,i}$,

$$\underline{D} = \begin{bmatrix} \underline{A}_{1,1} & \underline{0} & & \underline{0} \\ \underline{0} & \underline{A}_{2,2} & & \underline{0} \\ & & & \\ & & & \\ & & & \\ \underline{0} & \underline{0} & & \underline{A}_{N,N} \end{bmatrix}$$

then $\underline{D}$ is also nonsingular, and the matrix equation can be written

$$(\underline{A} - \underline{D})\,\underline{U} + \underline{D}\,\underline{U} = \underline{F}$$

or

$$\underline{U} = \underline{B}\,\underline{U} + \underline{D}^{-1}\,\underline{F} \tag{A-8}$$

where $\underline{B} = \underline{I} - \underline{D}^{-1}\,\underline{A}$ is the iteration matrix for the block Jacobi iterative method

$$\underline{U}^{(m+1)} = \underline{B}\,\underline{U}^{(m)} + \underline{D}^{-1}\,\underline{F}$$

The matrix $\underline{B}$ is called the block Jacobi matrix of $\underline{A}$.

Definition: If the block Jacobi matrix $\underline{B}$ is weakly cyclic of index p, then $\underline{A}$ is p-cyclic in the partitioned form (A-7). [Varga (1962) p.99]

Definition: The p-cyclic matrix $\underline{A}$ is consistently ordered if all the eigenvalues of the matrix

$$\underline{B}(k) = k\underline{H} + k^{-(p-1)}\,\underline{G}$$

are independent of k for $k \neq 0$, where $\underline{B}$ is the block Jacobi matrix with zero diagonal elements, $\underline{H}$ and $\underline{G}$ are respectively strictly lower and upper triangular matrices such that $\underline{B} = \underline{H} + \underline{G}$. [Varga (1962) p. 101]

If the partitioned form of $\underline{A}$ is block tri-diagonal, it is consistently ordered and 2-cyclic. The optimum relaxation factor ω_b which maximizes the asymptotic rate of convergence of the block successive overrelaxation matrix for p = 2 is given by

$$\omega_b = \frac{2}{1 + \sqrt{1 - \rho^2(\underline{B})}}$$

where $\underline{B}$ is the block Jacobi matrix. [Varga (1962) p. 110]

For the cyclic Chebyshev semi-iterative method the acceleration parameters are given

$$\omega_m = \frac{2\,C_{m-1}(1/\rho(\underline{B}))}{\rho(\underline{B})\,C_m(1/\rho(\underline{B}))}$$

where $\underline{B}$ is the block Jacobi matrix. [Varga (1962) p. 138]

The alternating-direction implicit method for the case of a fixed acceleration parameter r is a slight variation of Eq. (67) p.33

$$\underline{U}^{(m+1)} = \underline{R}_r\,\underline{U}^{(m)} + \underline{S}_r\,\underline{F}$$

If there is a block Jacobi matrix $\underline{B}_R$ associated with $\underline{R}_r$, the asymptotic rate of convergence will be a function of $\rho^2(\underline{B}_R)$.

Thus we find that the optimum relaxation factors for successive overrelaxation and the cyclic Chebyshev semi-iterative method as well as the acceleration parameter for the alternating direction implicit method are functions of the spectral norm of an associated block Jacobi matrix. In fact Varga (1962) shows that the three iterative methods considered have the same asymptotic rates of convergence for Laplace's equation solved for a square.

There appears to be a dearth of good approximations of ρ for irregularly shaped regions. It is common practice to use approximations based on the numerical results obtained in the iterative solution.

Consider point successive overrelaxation. Forsythe and Wasow (1960) p. 250 show that the eigenvalues λ_i of the matrix of the simplest iterative scheme, the point Jacobi method, are related to the eigenvalues η_i of the matrix of the successive overrelaxation method by $\eta_i = \lambda_i^2$. This is applicable to the eigenvalue λ_1 of maximum modulus and hence to the spectral radius of the successive overrelaxation matrix. The optimum relaxation factor as derived by Forsythe and Wasow p. 253 is

$$\omega_b = \frac{2}{1+\sqrt{1-\lambda_1^2}} = \frac{2}{1+\sqrt{1-\eta_1}} \qquad \text{(A-9)}$$

The error vectors from one iteration to the next are related

$$\|\underline{E}^{(m)}\| \leq \|\underline{M}\| \, \|\underline{E}^{(m-1)}\|$$

where $\underline{M}$ is the appropriate matrix for the iterative scheme being considered. The dominant eigenvalue η_1 is the limit of $\|\underline{E}^{(m)}\| / \|\underline{E}^{(m-1)}\|$ as m increases without bound. Any vector norm of $\underline{E}$ may be used. One computational procedure for estimating ω_b consists of starting the problem solution with $\omega = 1$, then after a number of iterations approximate η_1

$$\eta_1 \approx \|\underline{E}^{(m)}\| \, / \, \|\underline{E}^{(m-1)}\| \qquad \text{(A-10)}$$

and solve (A-9) for an approximate value of ω_b.

Another approach to approximating ω_b consists of selecting various values of ω, running through several iterations for each and then by comparison of results, select the best.

The procedure used in the ISOPEP code consists of computing a set

of values for ω obtained by applying Equations (A-9) and (A-10) every 10 or 20 iterations. This has the advantage of being an automatic procedure, and reasonably good results have been obtained for a number of problems. Approximating ω_b numerically as the solution proceeds is more computing art than science. Better methods for determining optimum relaxation factors would contribute much to the usefulness of iterative methods.

APPENDIX B

ISOPEP-A FORTRAN PROGRAM FOR THE ITERATIVE SOLUTION OF PLANE ELASTOSTATIC PROBLEMS

The system of computer programs named ISOPEP was written as a general system for the analysis of plane elastostatic problems. It includes a set of FORTRAN-II subroutine subprograms and a main program which provides linkage of a selected subset of these six subroutines and additional boundary-value subroutines which must be provided for each problem.

This appendix includes a brief description of the general subroutines, six examples of boundary-value subroutines, instructions for preparing input data for the program, a sample of the output, a flow diagram and listings of the FORTRAN Source decks.

ISOPEP

This name is used for the system of programs and also for the main program which provides linkage between the subroutines. The system was designed for use on an IBM 1620 with a 20 K main memory and a CDC 3600. The number of subroutines linked by ISOPEP may be reduced to increase the storage available for arrays. Only the Call statements in ISOPEP and all Dimension statements need to be changed. Only one of the three iterative method subroutines SORLX, ADI or CHEB is normally used.

OUTIN

OUTIN is the input and output subroutine. It provides for the initialization of arrays. When problems using a relatively coarse mesh

spacing have converged, the point values are stored for later use in calculating an initial stress-function distribution for a finer mesh.

CHANG

The initial stress function distribution for the second or subsequent mesh refinements of a problem can be computed by interpolating between the values obtained from an earlier coarse-mesh solution. The problems may be solved consecutively, with the earlier solution saved in memory, or the preceding solution may be read from punched cards. Execution of this option is controlled by the input of an appropriate value of the control number MESH.

STRESS

The stress components σ_x, σ_y and τ_{xy} will be computed and punched out on cards if specified by one of the options determined by the input of a control number NSTRS. The difference equations used to calculate the stress components at each interior point are

$$\sigma_{x_{ij}} \approx [U_{i,j+1} - 2U_{i,j} + U_{i,j-1}]/h^2 ,$$

$$\sigma_{y_{ij}} \approx [U_{i+1,j} - 2U_{i,j} + U_{i-1,j}]/h^2 ,$$

$$\tau_{xy_{ij}} \approx -[U_{i+1,j+1} - U_{i+1,j-1} - U_{i-1,j+1} + U_{i-1,j-1}]/4h^2 .$$

SORLX

SORLX is the subroutine for the point successive overrelaxation iterative method. This has been adopted for general problems with irregular boundaries. As written, it is limited to simply-connected regions for which all mesh lines, parallel to one of the Cartesian coordinate axes, are continuous segments connecting two boundary points.

ADI

The alternating-direction implicit method has been written as two

subroutines ADI1 and ADI2. The second is executed immediately after the first. The specification of two subroutines makes more storage available for arrays when the programs are run on the disk-oriented IBM 1620. The solution domain is limited to rectangular regions.

CHEB

The cyclic-Chebyshev semi-iterative method subroutine has been run only on the CDC 3600. It is written in FORTRAN II and could be divided into two or more subroutines for a computer with a limited main memory. Only rectangular regions can be treated with this program. It is necessary to specify an even number of interior mesh rows for this subroutine.

PB(N)BD

The values of the stress function at points on the boundary must be computed only at the beginning of the problem solution. This is part of the initialization of the solution array. A subroutine of this type must be provided for each problem. Examples are included for $1 \leq N \leq 6$.

PB(N)EX

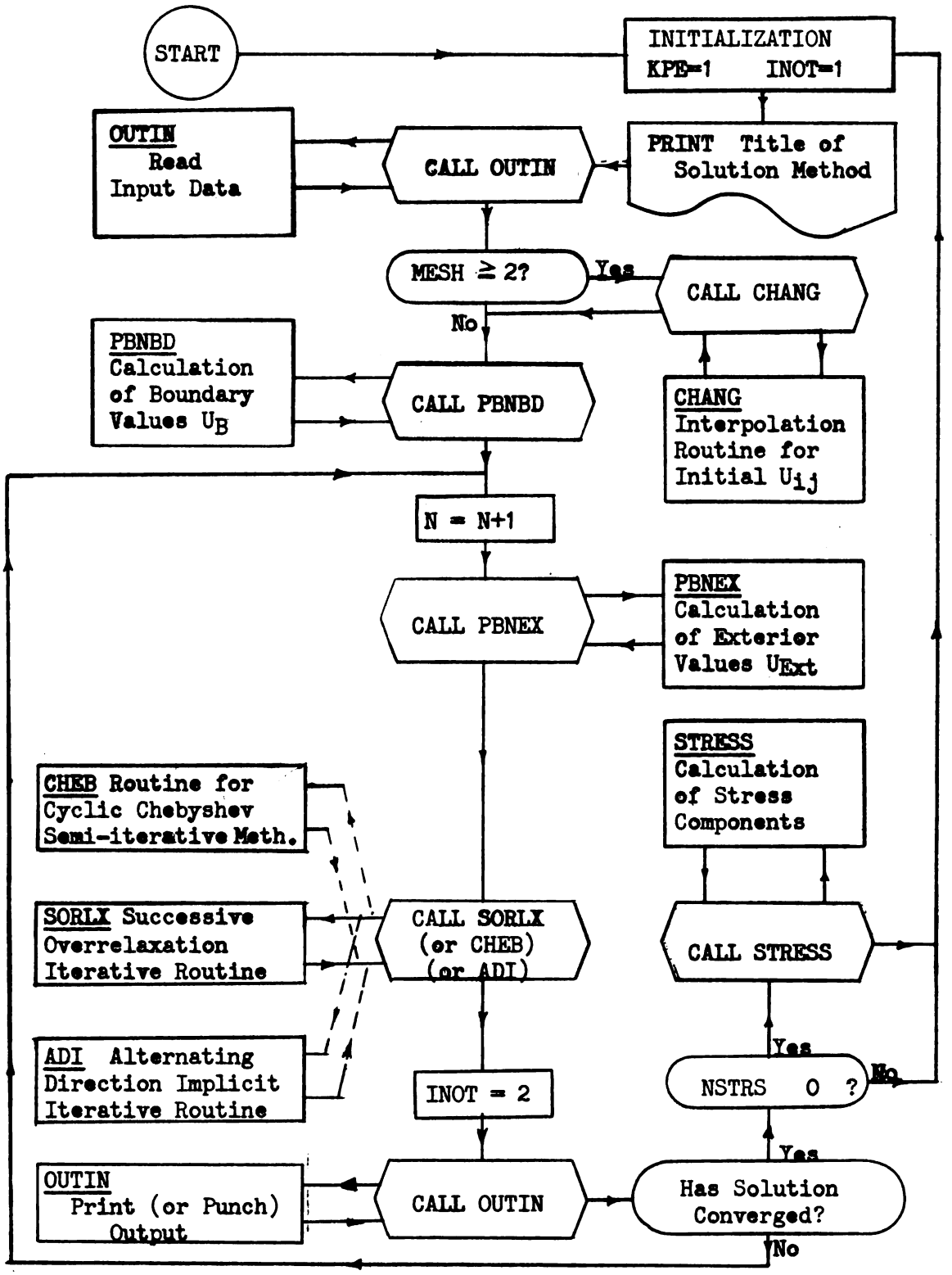
Derivative boundary conditions must also be provided for each problem. These may be treated by the introduction of exterior points or with special finite-difference equations for each point on the boundary. Both approaches are illustrated in the examples included for $1 \leq N \leq 6$. This subroutine must be executed during each iteration.

Input Preparation

The input to the problem can be provided on a single 80 column card unless point values of the stress function U_{ij} are provided. The deck of stress function values would be placed immediately after the

ITERATIVE SOLUTION OF PLANE ELASTOSTATIC PROBLEMS

FLOW DIAGRAM OF ISOPEP



control card which includes identification, dimensions and control numbers for input and output options. The stress function values must be provided in agreement with the FORMAT (I5,7F10.7/(5X,7F10.7)). Output is punched in this form in order to simplify restart procedures.

The FORMAT specification (F10.3,F10.8,2F10.4,10I4), is used for the first card. The following list gives the use and symbolic name of each of the 14 numeric entries on the card.

Column Numbers	Symbolic Name	Function
1-10	PRNO	Problem number for users identification.
11-20	DE	Convergence criterion, usually in the range 10^{-5}-10^{-7}.
21-30	RFA	Relaxation factor
31-40	SPYIJ	$\|\|\underline{E}\|\|_{II}$ Norm of the error vector in the preceding iteration when restarting. Set to 1.0 initially.
41-44	MX	Number of mesh intervals along the x-axis.
45-48	MY	Number of mesh intervals along the y-axis.
49-52	IF	(2) If y = 0 is an exterior boundary (3) If y = 0 is a line of symmetry
53-56	JF	(2) If x = 0 is an exterior boundary (3) If x = 0 is a line of symmetry
		Note: Two mesh lines exterior to the domain are reserved for derivative conditions if the boundary is a line of symmetry otherwise only one line is reserved.
57-60	N	N=0: Only one card of input is required. The initial value of the stress function at all interior points is set to 0.1. N>0: Count of number of iterations completed. A deck of

Column Numbers	Symbolic Names	Function
		point values of the stress function must be provided immediately following the first data card.
		N < 0: This is the control number which terminates the processing of a sequence of data sets. A final data card should always be provided with this entry.
61-64	NT	Maximum number of iterations permitted. This is the choice of the user and provides for punched output for restarting.
65-68	NOUT	Output will be printed after NOUT iterations and every subsequent set of NOUT iterations.
69-72	ND	ND $\geq$ NT: The initial value of RFA will be used for all iterations.
		ND < NT: The relaxation factor will be computed and changed at the end of every ND iterations.
73-76	MESH	(1) No mesh refinement calculations
		(2) Read in values of the stress function from a prior problem and interpolate to find an initial stress function distribution for the current problem. Another control card must follow the first card.
		(3) Use the stress function stored in memory from the preceding problem as the basis for interpolating to find an initial stress function distribution for the current problem. The user must be sure there is at least one problem specified on input cards preceding this one.

Column Numbers	Symbolic Names	Function
77-80	NSTRS	< 0 Punch specified stress components
		|NSTRS| = 1 Print (and punch) σ_y
		|NSTRS| = 2 Print (and punch) σ_y and σ_x
		|NSTRS| = 3 Print (and punch) τ_{xy}, σ_y and σ_x

The second control card is used only if MESH = 2 on the first card. The four integers read in with a (4I5) FORMAT specification provide the dimensions of the problem solution provided on the cards following this control card and the number of mesh spaces along the y-axis of the problem to be solved.

Column Number	Symbolic Name	Function
1-5	IP	Number of mesh columns of input data
6-10	JP	Number of mesh rows of input data
11-15	M1	MY for the input solution
16-20	M2	MY of the problem to be solved.

Example of Input Data

First half of the first card

Card columns

```
0         1         2         3         4
12345678901234567890123456789012345678901234567890
```

1.190 .00001 1.5 1.0

Second half of the first card

Card columns

```
         5         6         7         8
12345678901234567890123456789012345678901234567890
```

5 10 3 2 1 200 200 10 1 -3

IBM 1620 OUTPUT OF SAMPLE PROBLEM

```
‡‡JOB 5                                ISOPEP      C L DAVIS
‡‡FORX53 1
00784 CORES USED                         15207 NEXT COMMON

1  SUCCESSIVE OVERRELAXATION

0  SOLUTION OF THE BIHARMONIC EQUATION          C.L. DAVIS

  PROBLEM  CONVERG.     MESH SPACES    MESH
  NUMBER   CRITERION    MX  MY  IF  JF     STRESS

   1.190   .0000100     5  10   3   2   1  -3

  RELAX.     SUM OF    ITERATIONS     POINTS NOT
  FACTOR     ERRORS   COUNT  MAX.NO.  CONVERGED

    1.5000    1.0000     0      200       50
    1.6352    1.7275    10      200       45
    1.6324    1.4140    20      200       45
    1.5618     .7068    30      200       45
    1.4707     .1865    40      200       45
    1.4761     .0372    50      200       41
    1.5587     .0136    60      200       44
    1.5988     .0082    70      200       43
    1.5832     .0045    80      200       40
    1.5579     .0019    90      200       30
    1.5258     .0006   100      200       24
    1.4878     .0002   110      200        9

   1.190 .00001000    1.4878      .0001   5  10   3   2 116 200
200  10  1  -3

  STRESS FUNCTION

   2 3.6000000 3.4200000 2.8800000 1.9800000  .7200000 -.7200000
   3 3.5072542 3.3295735 2.7961993 1.9099653  .6964044 -.7200000
   4 3.3027094 3.1353795 2.6327760 1.7980644  .6555203 -.7200000
   5 3.0285702 2.8790977 2.4277819 1.6699096  .6106627 -.7200000
   6 2.6981876 2.5725967 2.1885523 1.5266087  .5626013 -.7200000
   7 2.3065274 2.2096326 1.9063169 1.3587928  .5068047 -.7200000
   8 1.8437695 1.7788512 1.5669582 1.1528863  .4365079 -.7200000
   9 1.3121754 1.2792279 1.1621542  .8963012  .3443551 -.7200000
  10  .7482642  .7410047  .7057714  .5860380  .2259561 -.7200000
  11  .2501268  .2542441  .2625820  .2490407  .0918619 -.7200000
  12 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000 -.7200000
```

STRESS DISTRIBUTION

PROBLEM NUMBER 1.190 DIMENSION = 5 X 10

(THE X COMPONENTS OF STRESS)

2	-36.0000	-36.0000	-36.0000	-36.0000	-18.0000	0.0000
3	-35.5362	-35.5693	-35.2859	-32.7326	-20.2843	-4.7191
4	-33.4664	-33.5273	-33.2108	-30.7832	-23.2976	-12.8959
5	-29.8952	-30.1843	-30.6556	-30.1374	-27.1415	-21.8675
6	-25.1189	-25.8453	-27.7899	-30.2063	-31.8593	-31.4798
7	-19.3796	-20.6420	-24.4208	-30.4464	-37.4816	-42.6391
8	-12.9842	-14.6974	-20.2178	-30.2306	-44.0129	-56.6984
9	-6.5898	-8.4126	-14.8779	-28.6093	-51.2408	-75.1290
10	-1.4520	-2.7973	-8.4500	-24.0348	-58.5874	-98.8088
11	.8234	.4220	-2.1879	-14.3637	-65.4683	-125.6276
12	0.0000	0.0000	0.0000	0.0000	-72.0000	-144.0000

(THE Y COMPONENTS OF STRESS)

2	-18.5493	-18.0854	-16.7602	-14.0069	-4.7191	0.0000
3	-11.1799	-10.3767	-7.9622	-4.1866	-1.7288	0.0000
4	-6.9594	-6.2087	-4.1570	-1.6253	-.3973	0.0000
5	-5.6243	-5.0219	-3.4235	-1.5146	-.3203	0.0000
6	-6.1277	-5.6463	-4.3005	-2.4515	-.7735	0.0000
7	-7.1097	-6.7817	-5.7123	-3.8090	-1.4500	0.0000
8	-6.8836	-6.8841	-6.5445	-5.0678	-2.1856	0.0000
9	-3.2317	-3.8599	-5.1578	-5.3678	-2.6246	0.0000
10	6.5773	5.1462	1.3193	-2.6734	-1.5695	0.0000
11	24.8010	23.2516	18.0607	8.7956	4.2232	0.0000
12	50.0252	50.8487	52.5163	49.8081	18.3723	0.0000

(THE XY SHEAR STRESS COMPONENTS)

2	0.0000	0.0000	0.0000	0.0000	0.0000	-57.0898
3	.0001	1.2516	2.5671	4.5686	4.5483	0.0000
4	.0001	2.7566	5.2605	7.0668	6.0013	0.0000
5	0.0000	4.0074	7.2831	8.7826	6.7863	0.0000
6	0.0000	5.0144	8.9587	10.4401	7.7779	0.0000
7	0.0000	5.8206	10.5005	12.3875	9.3430	0.0000
8	0.0000	6.2547	11.6978	14.5428	11.5622	0.0000
9	0.0000	5.8579	11.7749	16.2658	14.1712	0.0000
10	0.0000	4.0619	9.4430	16.1769	16.1815	0.0000
11	0.0000	1.0623	3.8741	11.9953	14.6509	0.0000
12	0.0000	0.0000	0.0000	0.0000	0.0000	94.7034

NOTE TITLES ENCLOSED IN PARENTHESIS ARE NOT PART OF OUTPUT

C LISTING OF ISOPEP FORTRAN SOURCE DECKS

```
C                    ISOPEP FOR THE IBM 1620
C      THE DECKS LISTED HAVE BEEN RUN UNDER THE MONITOR I
C      SYSTEM ON AN IBM 1620 WITH 20K MEMORY AND A 1311 DISK FILE.
C      DECKS ARE LISTED IN APPROPIATE ORDER WITH MAIN DECK LAST.

340003200701360003200702490240251196361130010 2
‡‡JOB 5                                OUTIN 10/17/65
‡‡FOR 53
*LDISK
      SUBROUTINE OUTIN
C       INPUT AND OUTPUT SUBROUTINE
      DIMENSION U(14,24),IK(24),IL(24),D(5),P(13,7 ),U1(13)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
   10 FORMAT (F10.3,F10.8,2F10.4,1014)
   11 FORMAT (F 8.3,F11.7,I5,5I4/)
   12 FORMAT (   50H0  SOLUTION OF THE BIHARMONIC EQUATION
     119H           C.L.DAVIS/)
   14 FORMAT (45H  RELAX.     SUM OF    ITERATIONS     POINTS NOT/
     12X,        42HFACTOR     ERRORS  COUNT  MAX.NO. CONVERGED/)
   15 FORMAT (41H  PROBLEM  CONVERG.     MESH SPACES    MESH/
     12X,        45HNUMBER  CRITERION    MX  MY  IF  JF      STRESS/)
   16 FORMAT(2F10.4,I5,2I8)
   17 FORMAT (17H  STRESS FUNCTION/)
   20 FORMAT(I5,7F10.7/(5X,7F10.7))
   22 FORMAT(I5,7F10.7/(5X,7F10.7))
      IF(1-INOT)133,80,80
   80 READ  10,PRNO,DE,RFA,SPYIJ,MX,MY,IF,JF,N,NT,NOUT,ND,MESH,NSTRS
      PRINT 12
      PRINT 15
      PRINT 11,PRNO,DE,MX,MY,IF,JF,MESH,NSTRS
      PRINT 14
      K=MX*MY
      PRINT 16,RFA,SPYIJ,N,NT,K
      IA=MX+IF
      JA=MY+JF
      MI=IA+1
      MJ=JA+1
      NDL=ND
      NINC=NOUT
      IF(MESH-3) 91,94,94
   91 DO 92 J=1,13
      DO 92 I=1,7
   92 P(J,I)=0.0
   94 DO 101 J=1,MJ
      DO 100 I=1,MI
  100 U(I,J)=0.1
      IL(J)=3
  101 IK(J)=IA
      IF(N) 200,103,102
  102 DO 104 J=JF,JA
  104 READ                20, K,(U(I,J),I=IF,IA)
      NDL=N+ND
  103 RETURN
```

```
  133 IF(  KPE   ) 90,90,119
  119 IF (SENSE SWITCH 1)  90,121
  121 IF(N-NOUT) 123,126,127
  123 IF(N-NT) 115,89,89
  127 NOUT=NOUT+NINC
      GO TO 121
   89 INOT=1
      GO TO 90
  126 INOT=3
   90 PRINT 10,PRNO,DE,RFA,SPYIJ,MX,MY,IF,JF,N,NT,NOUT,ND,MESH,NSTRS
      PRINT 17
      DO 105 J=JF,JA
  105 PRINT                20,J,(U(I,J),I=IF,IA)
      IF(SENSE SWITCH 1) 112,111
  111 IF(INOT-2) 112,110,115
  110 IF(KPE) 115,116,115
  200 STOP
C   MX MUST  BE LESS THAN I IN THE DIMENSION OF P(J,I) IF  THE
C       SOLUTION IS TO BE SAVED FOR THE NEXT PROB.
  116 IF(MX-6 ) 117,117,114
  117 DO 118 J=JF,JA
      K=J-JF+1
      DO 118 I=IF,IA
      L=I-IF+1
  118 P(K,L)=U(I,J)
      IP=MX+1
      JP=MY+1
      AM1=1./D1
  114 IF(NSTRS) 112,115,115
  112 PUNCH 10,PRNO,DE,RFA,SPYIJ,MX,MY,IF,JF,N,NT,NOUT,ND,MESH,NSTRS
      PUNCH 17
      DO 113 J=JF,JA
  113 PUNCH 22,J,(U(I,J),I=IF,IA)
      IF(SENSE SWITCH 1) 200,115
  115 RETURN
      END

‡‡JOB 5                          STRESS 10/17/65                  CLD
‡‡FOR 53
*LDISK
      SUBROUTINE STRESS
C      STRESS CALCULATION SUBROUTINE
      DIMENSION U(14,24),IK(24),IL(24),D(5),P(13,7 ),U1(13)
      DIMENSION S(14),SX(14),TXY(14)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
   14 FORMAT (18H    PROBLEM NUMBER,F6.3,5X,12H DIMENSION =,I3,2H X,I3/)
   27 FORMAT (//)
   30 FORMAT (21HO STRESS DISTRIBUTION/)
   40 FORMAT (I5, 7F10.4/(5X, 7F10.4))
   42 FORMAT (I5, 7F10.4/(5X, 7F10.4))
  805 AI=1./(D1*D1)
      PRINT 30
```

```
      PRINT 14,PRNO,MX,MY
      DO 814 J=JF,JA
  820 K1=IK(J  )
      K2=IL(J)
      IF(J-IL(3)) 822,821,822
  821 K2=3
  822 DO 812 I=IF,K1
      IF (I-K2) 811,810,810
  811 S(I)=0.0
      GO TO 812
  810 S(I)=(U(I+1,J)+U(I-1,J)-2.*U(I,J))*AI
  812 CONTINUE
  813 K=K1
      IF(NSTRS) 815,814,814
  815 PUNCH 42,J,(S(L) ,L=IF,K)
  814 PRINT                40, J,(S(L),L=IF,K)
      PRINT 27
      IF( ABSF(NSTRS)-2) 832,833,833
  833 DO 844 J=JF,JA
      K1=IK(J)
      K2=IL(J)
      IF(J-IL(3)) 835,834,835
  834 K2=3
  835 DO 841 I=IF,K1
      IF(I-K2) 839,840,840
  839 SX(I)=0.0
      GO TO  841
  840 SX(I)=(U(I,J+1)+U(I,J-1)-2.*U(I,J))*AI
  841 CONTINUE
      K=K1
      IF(NSTRS) 842,844,844
  842 PUNCH 42,J,(SX(L),L=IF,K)
  844 PRINT 40,J,(SX(L),L=IF,K)
      PRINT 27
      IF(ABSF(NSTRS)-3) 832,845,845
  845 DO 854 J=JF,JA
      K1=IK(J)
      K2=IL(J)
      IF(J-IL(3)) 847,846,847
  846 K2=3
  847 DO 850 I=IF,K1
      IF(I-K2) 848,849,849
  848 TXY(I)=0.0
      GO TO 850
  849 TXY(I)=(U(I+1,J+1)-U(I-1,J+1)-U(I+1,J-1)+U(I-1,J-1))*AI/4.
  850 CONTINUE
      K=K1
      IF(NSTRS) 852,854,854
  852 PUNCH 42,J,(TXY(L),L=IF,K)
  854 PRINT 40,J,(TXY(L),L=IF,K)
  832 RETURN
      END
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```
‡‡JOB 5                          PB1BD  10/17/65                CLD
‡‡FOR 53
*LDISK
      SUBROUTINE PB1BD
C   CALCULATION OF BOUNDARY VALUES      PROBLEM 1
      DIMENSION U(14,24),IK(24),IL(24),D(5),P(13,7 ),U1(13)
      DIMENSION PH(14,24)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
      EQUIVALENCE (U,PH)
      A=MY
      D1=1./A
      X=0.0
      DO 103 I=3,IA
      PH(I,JA)=0.0
      IF(X-.4) 103,103,104
  103 X=X+D1
  104 K=I
      DO 105 I=K,IA
      PH(I,JA)=(2.*X*X-1.6*X+.32)*(-36.)
  105 X=X+D1
  106 DO 108 J=2,JA
  108 PH(IA,J)= -.72
      X=0.0
      DO 110 I=3,IA
      PH(I,2 )=(.5*X*X-.1)*(-36.)
      IF(X-.4) 110,110,114
  110 X=X+D1
  114 K=I
      DO 115 I=K,IA
      PH(I,2 )=(.4*X-.18)*(-36.)
  115 X=X+D1
      RETURN
      END

‡‡JOB 5                          PB1EX  10/17/65                CLD
‡‡FOR 53
*LDISK
      SUBROUTINE PB1EX
C     CALCULATION OF EXTERIOR VALUES        PROBLEM 1
      DIMENSION U(14,24),IK(24),IL(24),D(5),P(13,7 ),U1(13)
      DIMENSION PH(14,24)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
      EQUIVALENCE (U,PH)
      DO 117 J=2,JA
      PH(2,J)=PH(4,J)
      PH(1,J)=PH(5,J)
  117 PH(MI,J)=PH(IA-1,J)-0.8*D1*36.
      DO 116 I=1,IA
      PH(I,1)=PH(I,3)
  116 PH(I,MJ)=PH(I,JA-1)
      RETURN
      END
```

```
‡‡JOB 5                                CHANG 10/17/65                    CLD
‡‡FOR 53
*LDISK
      SUBROUTINE CHANG
C     CHANGE MESH SPACING
      DIMENSION U(14,24),IK(24),IL(24),D(5),P(13,7 ),U1(13)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KFE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
  100 FORMAT(4I5)
  101 FORMAT(I5,7F10.8)
  102 FORMAT(I5,7F10.7/(5X,7F10.7))
  103 FORMAT(I5,7F10.7/(5X,7F10.7))
      IF(MESH-2) 22,20,22
   20 READ                  100,IP,JP,M1,M2
      AM1=M1
      AM2=M2
      DO 18 J=1,JP
      READ                  102, K,(P(J,I),I=1,IP)
   18 PRINT                   103,J,(P(J,I),I=1,IP)
      GO TO 24
   22 M2=MY
      M1=AM1+.000001
      AM2=M2
   24 D2=1./AM2
      D3=1./AM1
      JA2=M2+1
      J=1
      L=0
      Y2=0.
    6 I=1
      X1=0.
      X2=-D2
      I2=0
      DY=Y2
      JY=L+3
    5 I2=I2+1
      IX=I2+2
      IF(I2-JP )7,7,10
    7 X2=X2+D2
      DX=X2-X1
      IF(ABSF(DX-D3)-.000001) 9,9,11
   11 IF(DX-D3) 8,9,9
    8 U1(I2)=P(J,I)+DX*((P(J,I+1)-P(J,I))/D3)+DY*((P(J+1,I)-P(J,I))/D3)
      U(IX,JY)=U1(I2)
      IF(I2-JP) 5,5,10
    9 I=I+1
      X1=X1+D3
      DX=X2-X1
      IF(I-IP) 8,8,10
   10 L=L+1
      PRINT                  103,L,(U1(I2),I2=1,JP)
      IF(L-JA2) 15,14,14
   15 Y2=Y2+D2
      IF(ABSF(Y2-D3)-.000001) 12,12,16
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```
   16 IF(Y2-D3) 6,12,12
   12 J=J+1
      Y2=Y2-D3
      IF(J -M1-1)  6,6,14
   14 RETURN
      END

‡‡JOB 5                              SORLX 10/17/65                    CLD
‡‡FOR 53
*LDISK
      SUBROUTINE SORLX
C     POINT SUCCESSIVE OVERRELAXATION                SORLX
      DIMENSION U(14,24),IK(24),IL(24),D(5),P(13,7 ),U1(13)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
   10 FORMAT(2F10.4,I5,218)
      SYIJ=0.0
      KPE=0
      KT=0
      JB=JA-1
  130 DO 140 J= 3,JB
      IB=IK(J)-1
      IC=IL(J)
      DO 133 I=IC,IB
      IM=I-1
      IMM=I-2
  131 YIJ    =.05*RFA*(8.*( U(I+1,J)+ U(IM,J)+ U(I,J-1)+ U(I,J+1))
     1 -2.*( U(I+1,J+1)+ U(IM,J+1)+ U(I+1,J-1)+ U(IM,J-1))
     2 - U(I+2,J)- U(IMM,J)- U(I,J+2)- U(I,J-2))-RFA* U(I,J)
      Y2=ABSF(YIJ)
      SYIJ=SYIJ+Y2
      U (I,J)= U(I,J)+YIJ
      IF (Y2-DE) 133,132,132
  132 KPE=KPE+1
  133 CONTINUE
C    ADD CARDS FOR SOR3,SOR4 OR SOR5 BETWEEN STATEMENTS 133 AND 140
  140 CONTINUE
      RF1=SYIJ/SPYIJ
      SPYIJ=SYIJ
  134 IF (N-NDL) 137,136,136
  136 NDL=NDL+ND
      RFA=2./(1.+SQRTF(ABSF(1.-RF1)))
      PRINT                10,RFA,SPYIJ,N,NT,KPE
  137 RETURN
      END

‡‡JOB 5                              ISOPEP      C L DAVIS             10/17/6
‡‡FORX53 1
C     PROGRAM ISOPEP
C     ITERATIVE SOLUTION OF PLANE ELASTOSTATIC PROBLEMS
      DIMENSION U(14,24),IK(24),IL(24),D(5),P(13,7 ),U1(13)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
    5 FORMAT (28H1  SUCCESSIVE OVERRELAXATION/)
```

```
   50 INOT=1
      RF1=1.
      PRINT 5
      KPE=1
      CALL OUTIN
      IF(MESH-2) 55,52,52
   52 CALL CHANG
C       NEXT INSTRUCTION ASSURES COMPATIBILITY OF SUBROUTINES ON IBM 1620
   55 U(MI,MJ)=ABSF(SQRTF(1.0))
      CALL PB1BD
   60 N=N+1
      CALL PB1EX
      CALL SORLX
      INOT = 2
      CALL OUTIN
      IF(INOT-2) 50,68,60
   68 IF(KPE) 80,80,60
   80 IF(ABSF(NSTRS)-1) 50,82,82
   82 CALL STRESS
      GO TO 50
      END
*LOCAL,OUTIN,SORLX,CHANG,STRESS,PB1BD,PB1EX
     1.190    .00001        1.5        1.0   5  10   3   2   0 200 200  10
     1.198    .00001        1.5        1.0  10  20   3   2   0 200 200  10
     1.190    .00001      1.678      .0001   5  10   3   2  -1 200 200  10
C     LAST THREE CARDS ARE EXAMPLES OF INPUT DATA

C                    ISOPEP FOR THE CDC 3600
C     THE DECKS LISTED HAVE BEEN RUN UNDER THE SCOPE SYSTEM
C     THE MAIN DECK IS PLACED FIRST. THIS IS FOLLOWED BY THE
C     SUBROUTINE SUBPROGRAMS OUTIN, CHANGE, STRESS, SORLX,
C     THE DATA.THE SAME SUBROUTINES AS LISTED FOR THE IBM 1620 ARE
C     PBNBD, PBNEX AND THEN AFTER NECESSARY SCOPE CONTROL CARDS
C     USED EXCEPT ARRAY DIMENSIONS ARE CHANGED AND THE FIRST THREE
C     CARDS(‡‡JOB 5), (‡‡FOR 5), (*LDISK) ARE OMITTED.  THE
C     SUBROUTINES PB6BD AND PB6EX ARE LISTED. NOTE PB6BD USES
C     COMPLEX TYPE VARIABLES WHICH ARE NOT AVAILABLE IN
C     FORTRAN II BUT ARE AVAILABLE IN 3600 FORTRAN.

9JOB,031547,ISOPEP6, 20.DAVIS,C,L.  9/21/65  GROUP C
9FTN,X,*
      PROGRAM ISOPEP6
C     ITERATIVE SOLUTION OF PLANE ELASTOSTATIC PROBLEMS
      DIMENSION U(64,64),IK(64),IL(64),D(5),P(31,31),U1(64)
      DIMENSION UXD(64),UYD(64)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
      COMMON UXD,UYD
    5 FORMAT (28H1  SUCCESSIVE OVERRELAXATION)
   50 INOT=1
      RF1=1.
      PRINT 5
      KPE=1
      CALL OUTIN
```

```
      IF(MESH-2) 55,52,52
   52 CALL CHANG
   55 CALL PB6BD
   60 N=N+1
      CALL PB6EX
      CALL SORLX
      INOT = 2
      CALL OUTIN
      IF(INOT-2) 50,68,60
   68 IF(KPE) 80,80,60
   80 IF(ABSF(NSTRS)-1) 50,82,82
   82 CALL STRESS
      GO TO 50
      END

C     INSERT SUBROUTINE OUTIN - CHECK ARRAY DIMENSIONS
C     INSERT SUBROUTINE CHANGE - CHECK ARRAY DIMENSIONS
C     INSERT SUBROUTINE STRESS - CHECK ARRAY DIMENSIONS
C     INSERT SUBROUTINE SORLX - CHECK ARRAY DIMENSIONS

      SUBROUTINE PB6BD
C    BOUNDARY VALUES FOR REGION OF INFINITE PLATE WITH A SQUARE HOLE
      TYPE COMPLEX PHI,PSI,PDU,OMEG,DOMG,DOMGB,DPHI,DPHIB,PSIB,ZET
      DIMENSION U(64,64),IK(64),IL(64),D(5),P(31,31),U1(64)
      DIMENSION UXD(64),UYD(64)
      DIMENSION DUX(62),DUY(62),X(62),Y(62),UBX(31),UBY(31)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
      COMMON UXD,UYD
C      LAGRANGE INTERPOLATION FORMULA FOR UNEQUAL INTERVALS
      GRANF (XA)                                    =UA*(
     1(XA-X2)*(XA-X3)*(XA-X4)*(XA-X5))/((X1-X2)*(X1-X3)*(X1-X4)*(X1-X5))
     2+UB*(
     3(XA-X1)*(XA-X3)*(XA-X4)*(XA-X5))/((X2-X1)*(X2-X3)*(X2-X4)*(X2-X5))
     4+UC*(
     5(XA-X1)*(XA-X2)*(XA-X4)*(XA-X5))/((X3-X1)*(X3-X2)*(X3-X4)*(X3-X5))
     6+UD*(
     7(XA-X1)*(XA-X2)*(XA-X3)*(XA-X5))/((X4-X1)*(X4-X2)*(X4-X3)*(X4-X5))
     8+UE*(
     9(XA-X1)*(XA-X2)*(XA-X3)*(XA-X4))/((X5-X1)*(X5-X2)*(X5-X3)*(X5-X4))
      CORXF(RH,TH)=(COSF(TH)/RH-RH**3*COSF(3.*TH)/6.+RH**7*COSF(7.*TH)/
     156.)*R
      CORYF(RH,TH)=(SINF(TH)/RH+RH**3*SINF(3.*TH)/6.-RH**7*SINF(7.*TH)/
     156.)*(-R)
   22 FORMAT (20I4)
   50 FORMAT (48HO  INFINITE PLATE WITH A SQUARE HOLE   C L DAVIS//)
   60 FORMAT (11HO  THETA = ,F10.5,6HRHO = ,F8.2,15H NO BOUNDRY PT./)
      PRINT 50
      A=MX
      D1=1./A
      R=1.17964523/3.
      PI=3.14159265
      THETA=2.*PI
      LIM=37
```

```
      DANG=LIM-1
      DANG=-PI/(2.*DANG)
      XMAX=1.0
      YMAX=1.0
      DLRO=-.05
      DO 640 I=1,LIM
      RHO=1.0
      IF(THETA-7.*PI/4.) 620,610,610
  610 RIDX=XMAX-ABSF(CORXF(RHO,THETA))
      IF(RIDX) 616,630,612
  612 RHO=RHO+DLRO
      IF(.10-RHO) 610,610,614
  614 PRINT 60,THETA,RHO
      GO TO 630
  616 DO 618 K=1,50
      DRO=((CORXF(RHO,THETA)/XMAX-1.)*RHO)/(R*(RHO**7*COSF(7.*THETA)/7.-
     12.*RHO**3*COSF(3.*THETA)/3.)/XMAX-1.)
      IF(ABSF(DRO)-.0000001)  630,630,618
  618 RHO=RHO-DRO
      GO TO 614
  620 RIDY=YMAX-ABSF(CORYF(RHO,THETA))
      IF(RIDY) 624,630,622
  622 RHO =RHO+DLRO
      IF(0.1-RHO) 620,620,614
  624 DO 626 K=1,50
      DRO=((CORYF(RHO,THETA)/YMAX-1.)*RHO)/(R*(RHO**7*SINF(7.*THETA)/7.-
     12.*RHO**3*SINF(3.*THETA)/3.)/YMAX-1.)
      IF(ABSF(DRO)-.0000001)  630,630,626
  626 RHO=RHO-DRO
      GO TO 614
  630 X(I)=CORXF(RHO,THETA)
      Y(I)=CORYF(RHO,THETA)
      ZET=CMPLX(RHO*COSF(THETA),RHO*SINF(THETA))
      OMEG=R*(1./ZET-ZET**3/6.+ZET**7/56.)
      DOMG=R*(.125*ZET**6-.5*ZET**2-1./ZET**2)
      DOMGB= CMPLX(REAL(DOMG), -AIMAG(DOMG))
      PHI=R*(.25/ZET+.426*ZET+.046*ZET**3+.008*ZET**5+.004*ZET**7)
      DPHI=R*(.028*ZET**6+.04*ZET**4+.138*ZET**2+.426-.25/ZET**2)
      DPHIB= CMPLX(REAL(DPHI), -AIMAG( DPHI))
      PSI=-R*(.5/ZET+(.548*ZET-.457*ZET**3-.026*ZET**5-.029*ZET**7)/
     1 (1.+.5*ZET**4-.125*ZET**8))
      PSIB=  CMPLX(REAL( PSI), -AIMAG(PSI))
      PDU=PHI+(OMEG*DPHIB)/DOMGB+PSIB
      DUX(I)=REAL(PDU)
      DUY(I)=AIMAG(PDU)
  640 THETA =THETA + DANG
C     BOUNDARY POINTS  REGION OF AN INFINITE  PLATE
      READ 22,(IL(J),J=3,JA)
      UBY(1)=-.0545259
      UBX(1)=.3232002
      LUM=LIM/2
      DO 650 I=1,LUM
      K=LIM-I
      UBX(I+1)=UBX(I)-.5*(DUX(K)+DUX(K+1))*(X(K+1)-X(K))
```

```
  650 UBY(I+1)=UBY(I)+.5*(DUY(I)+DUY(I+1))*(Y(I+1)-Y(I))
C     BOUNDARY POINTS ON THE SQUARE
      K=IL(3)
      DO 652 I=3,K
      U(K,I)=0.0
  652 U(I,K)=0.0
      YVAR=0.0
      K=3
      DO 662 J=JF,JA
  654 IF(YVAR-Y(K)) 660,660,656
  656 IF(K-LUM+2)  658,658,660
  658 K=K+1
      GO TO 654
  660 X1=Y(K-2)
      X2=Y(K-1)
      X3=Y(K  )
      X4=Y(K+1)
      X5=Y(K+2)
      UA=UBY(K-2)
      UB=UBY(K-1)
      UC=UBY(K  )
      UD=UBY(K+1)
      UE=UBY(K+2)
      U(IA,J)=GRANF(YVAR)
      UA=DUX(K-2)
      UB=DUX(K-1)
      UC=DUX(K  )
      UD=DUX(K+1)
      UE=DUX(K+2)
      UXD(J)=GRANF(YVAR)
  662 YVAR=YVAR+D1
      XVAR=0.0
      K=14
      DO  672 I=IF,IA
  664 KB=LUM+K
      KC=LUM+2-K
      IF(XVAR-X(KB)) 670,670,666
  666 IF(4-K    ) 668,668,670
  668 K=K-1
      GO TO  664
  670 X1=X(KB+2)
      X2=X(KB+1)
      X3=X(KB  )
      X4=X(KB-1)
      X5=X(KB-2)
      UA=UBX(KC-2)
      UB=UBX(KC-1)
      UC=UBX(KC )
      UD=UBX(KC+1)
      UE=UBX(KC+2)
      U(I,JA) = GRANF(XVAR)
      UA=DUY(KB+2)
      UB=DUY(KB+1)
      UC=DUY(KB )
```

```
      UD=DUY(KB-1)
      UE=DUY(KB-2)
      UYD(I)   = GRANF(XVAR)
  672 XVAR=XVAR+D1
      RETURN
      END

      SUBROUTINE PB6EX
C     VALUES OF STRESS FUNCTION AT EXTERIOR POINTS WHICH SATISFY
C     BOUNDARY DERIVATIVE  CONDITIONS
      DIMENSION U(64,64),IK(64),IL(64),D(5),P(31,31),U1(64)
      DIMENSION UXD(64),UYD(64)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
      COMMON UXD,UYD
      K=IL(3)-1
      DO 685 J=JF,JA
      IF(J-K) 682,683,684
  683 U(2,J)=U(K+2,J)
      GO TO 685
  682 U(K,J)=U(K+2,J)
      GO TO 685
  684 U(1,J)=U(5,J)
      U(2,J)=U(4,J)
  685 U(MI,J)=U(IA-1,J)+UXD(J)*2.*D1
      DO 690 I=IF,IA
      IF(I-K)  686,686,688
  686 U(I  ,K)=U(I,K+2)
      GO TO 690
  688 U(I,1)=U(I,5)
      U(I,2)=U(I,4)
  690 U(I,MJ)=U(I,JA-1)+UYD(I)*2.*D1
      RETURN
      END
         SCOPE
9LOAD
9RUN,20,3000
     1.704    .00001        1.5        1.0  48  48   3   3   025002500  1
  19  19  19  19  19  19  19  19  19  19  19  19  19  19  19  19  19
   3   3   3   3   3   3   3   3   3   3   3   3   3   3   3   3   3
   3   3   3   3   3   3   3   3   3   3   3   3   3   3   3   3   3
     1.701    .00001        1.5        1.0  15  15   3   3  -1 200 200  1
C     LAST FIVE CARDS ARE EXAMPLES OD INPUT DATA

C     TO USE ADI, REPLACE THE CALL SORLX CARD IN THE ISOPEP
C     MAIN DECK WITH TWO CARDS, CALL ADI1, CALL ADI2
C     CHECK FOR AGREEMENT OF DIMENSION AND COMMON STATEMENTS
      SUBROUTINE ADI1
C   ALTERNATING DIRECTION IMPLICIT METHOD ROW SOLUTION
      DIMENSION U(28,52),UH(24,48),F(50),H(50),G(50),B(50)
      DIMENSION IK(52),IL(52),D(5)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
      COMMON UH,F,G,B,I,J,X,K
```

```
      EQUIVALENCE (F,H),(F(1),RNR),(B(1),CJ ),(G(1),RFA)
      JB=JA-1
      DO 166 J=3,JB
      IB=IA-1
      SYIJ=0.0
      K=IB-1
      L=J-2
      DO 160 I=3,IB
      F(I  )=8.*(U(I,J+1)+U(I,J-1))+4.*(U(I+1,J)+U(I-1,J))-U(I,J+2)-
     1 2.*(U(I+1,J+1)+U(I-1,J+1)+U(I-1,J-1)+U(I+1,J-1))-U(I,J-2)+
     2 (RNR-14.)*U(I,J)
      IF(I-4) 150,151,152
  150 G(I)=1.0/CJ
      B(I)=-4.*G(I)
      H(I)=(F(I)-U(I-2,J)+4.*U(I-1,J))*G(I)
      GO TO 160
  151 F(I)=F(I)-U(I-2,J)
  152 DELN=4.+B(I-2)
      WJ=CJ-G(I-2)+DELN*B(I-1)
      G(I)=1.0/WJ
      B(I)=(-4.+DELN*G(I-1))*G(I)
      IF(K-I) 153,154,155
  153 F(I)=F(I)+4.*U(I+1,J)
  154 F(I)=F(I)-U(I+2,J)
      G(I)=0.0
  155 H(I)=(F(I)-H(I-2)+DELN*H(I-1))/WJ
  160 CONTINUE
      IB=IB-2
      DO 166 I2=1,IB
      I=IA-I2
      K=I-2
      YIJ=H(I)
      IF(I2-2) 166,162,161
  161 YIJ=YIJ-G(I)*UH(K+2,L)
  162 YIJ=YIJ-B(I)*UH(K+1,L)
  166 UH(K,L)=YIJ
      RETURN
      END

      SUBROUTINE AD12
C   ALTERNATING DIRECTION IMPLICIT METHOD COLUMN SOLUTION
      DIMENSION U(28,52),UH(24,48),F(50),H(50),G(50),B(50)
      DIMENSION IK(52),IL(52),D(5)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
      COMMON UH,F,G,B,I,J,X,K
      EQUIVALENCE (F,H),(F(1),RNR),(B(1),CJ ),(G(1),RFA)
      SYIJ=0.0
      KPE=0
      IB=IA-1
      DO 181 I=3,IB
      K=I-2
      JB=JA-1
      DO 176 J=3,JB
```

```
      L=J-2
      F(J)=RNR*UH(K,L)+6.*U(I,J)-4.*(U(I,J+1)+U(I,J-1))+U(I,J+2)+U(I,L)
      IF(J-4) 170,171,172
  170 G(J)=1.0/CJ
      B(J)=-4.*G(J)
      H(J)=(F(J)-U(I,J-2)+4.*U(I,J-1))*G(J)
      GO TO 176
  171 F(J)=F(J)-U(I,J-2)
  172 DELN=4.+B(J-2)
      WJ=CJ-G(J-2)+DELN*B(J-1)
      G(J)=1.0/WJ
      B(J)=(-4.+DELN *G(J-1))/WJ
      IF (JB-1-J) 173,174,175
  173 F(J) =F(J)+4.*U(I,J+1)
  174 F(J)=F(J)-U(I,J+2)
      G(J)=0.0
  175 H(J)=(F(J)-H(J-2)+DELN*H(J-1))/WJ
  176 CONTINUE
      JB=JB-2
      DO 181 J2=1,JB
      J=JA-J2
      YIJ=H(J)
      IF(J2-2) 179,178,177
  177 YIJ=YIJ-G(J)*U(I,J+2)
  178 YIJ=YIJ-B(J)*U(I,J+1)
  179 Y2=ABSF(YIJ-U(I,J))
      SYIJ=SYIJ+Y2
      IF(Y2-DE) 181,180,180
  180 KPE=KPE+1
  181 U(I,J)=YIJ
      SPYIJ=SYIJ
      RETURN
      END

C     TO USE CHEB INSERT THE FOLLOWING CARD AFTER THE
C     FIRST CALL OUTIN STATEMENT IN THE MAIN ISOPEP DECK
      NCHB = 1
C     REPLACE THE CALL SORLX CARD IN THE MAIN ISOPEP DECK WITH
C     THE FOLLOWING TWO CARDS
      CALL CHEB(NCHB)
      NCHB = 2
C     CHECK DIMENSION AND COMMON STATEMENTS

      SUBROUTINE CHEB(NCHB)
C     CYCLIC CHEBYSHEV SEMI-ITERATIVE METHOD
      DIMENSION U(28,52),IK(52),D(5),AK(30),AL(30,30),SA(30,30),U2(30)
      DIMENSION IL(52)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
      COMMON AK,SA,AL,U2,RHO
  300 IF(NCHB-1) 302,302,345
  302 NQ=(JA-3)/2
      L=(JA-2)/2
      IF(L-NQ) 310,310,304
```

```
304 PRINT 305
305 FORMAT (26H  ERROR ODD NUMBER OF ROWS)
    STOP 1010
310 NI=IA-3
    NII=NI+NI
    DO 312 K=1,NII
    U2(K)=0.0
    AK(K)=0.0
    DO 312 L=1,NII
    SA(K,L)=0.0
312 AL(K,L)= 0.0
    DO 314 K=1,NII
314 AL(K,K)=20.0
    AL(1,2)=-8.0
    AL(1,3)=-8.0
    AL(2,4)=-8.0
    AL(3,4)=-8.0
    AL(3,5)=-8.0
    AL(1,4)=2.0
    AL(2,3)=2.0
    AL(4,5)=2.0
    AL(1,5)=1.0
    AL(2,5)=0.0
    DO 315 K=5,NII
    AL(K-3,K+1)=AL(K-4,K  )
    AL(K-2,K+1)=AL(K-4,K-1)
    AL(K-1,K+1)=AL(K-4,K-2)
315 AL(K  ,K+1)=AL(K-4,K-3)
    DO 316 L=1,NII
    DO 316 K=L,NII
316 AL(K,L)=AL(L,K)
    SA(1,1)=SQRTF(AL(1,1))
    DO 320 L=2,5
320 SA(1,L)=AL(1,L)/SA(1,1)
    K=1
324 DO 340 K=2,NII
    K1=K+1
    KLR=K-4
    IF(KLR) 325,325,326
325 KLR=1
326 KUP=K-1
    TEMP=0.0
    DO 328 L=KLR,KUP
328 TEMP=TEMP+SA(L,K)*SA(L,K)
    SA(K,K)=SQRTF(AL(K,K)-TEMP)
    IF(K-NII) 329,340,340
329 LUP=K+4
    IF(LUP-NII) 332,332,330
330 LUP=NII
332 DO 336 L=K1,LUP
    TEMP=0.0
    DO 334 L1=KLR,KUP
334 TEMP=TEMP + SA(L1,K)*SA(L1,L)
336 SA(K,L)=(AL(K,L)-TEMP)/SA(K,K)
```

```
340 CONTINUE
    IF (RFA) 282,282,284
282 A=MY
    A=A*D1
    B=MX
    B=B*D1
    B=(A/B)**2
    PI=3.14159265
    RLAM=(5.144*(1.0+B*B)+3.155*B)*(PI/A)**4
    RHO=1.0/(1.0+RLAM*D1**4/2.0)
    GO TO 285
284 RHO=SQRTF( 2.-2./RFA)
285 RF2=2.0/(2.0-RHO*RHO)
    RF1=1.0
345 JC=JA-1
    KPE=0
    SYIJ=0.0
    RFA=RF1
    JB=3
347 DO 390 J=JB,JC,2
    I=3
    DO 360 K=1,NII,2
    P1=8.*U(I,J-1)-2.*(U(I-1,J-1)+U(I+1,J-1))-U(I,J-2)-U(I,J+2)
    P2=8.*U(I,J+2)-2.*(U(I+1,J+2)+U(I-1,J+2))-U(I,J-1)-U(I,J+3)
    IF(K-4) 348,354,354
348 IF(K-2) 350,350,352
350 P1=P1+8.*U(I-1,J)-2.*U(I-1,J+1)-U(I-2,J)
    P2=P2+8.*U(I-1,J+1)-2.*U(I-1,J)-U(I-2,J+1)
    AK(1)=P1/SA(1,1)
    AK(2)=(P2-AK(1)*SA(1,2))/SA(2,2)
    GO TO 360
352 P1=P1-U(I-2,J)
    P2=P2-U(I-2,J+1)
    AK(3)=(P1-AK(1)*SA(1,3)-AK(2)*SA(2,3))/SA(3,3)
    AK(4)=(P2-AK(1)*SA(1,4)-AK(2)*SA(2,4)-AK(3)*SA(3,4))/SA(4,4)
    GO TO 360
354 IF(K+3-NII) 357,355,356
355 P1=P1-U(I+2,J)
    P2=P2-U(I+2,J+1)
    GO TO 357
356 P1=P1+8.*U(I+1,J)-2.*U(I+1,J+1)-U(I+2,J)
    P2=P2+8.*U(I+1,J+1)-2.*U(I+1,J)-U(I+2,J+1)
357 KUP=K-1
    KLR=K-4
    TEMP=0.0
    DO 358 K1=KLR,KUP
358 TEMP=TEMP +AK(K1)*SA(K1,K)
    AK(K )=(P1-TEMP)/SA(K,K)
    KLR=KLR+1
    TEMP=0.0
    DO 359 K1=KLR,K
359 TEMP=TEMP+AK(K1)*SA(K1,K+1)
    AK(K+1)=(P2-TEMP)/SA(K+1,K+1)
360 I=I+1
```

```
      K=NII
      U2(K)  =AK(K)/SA(K,K)
      U2(K-1)=(AK(K-1) -SA(K-1,K)*U2(K))/SA(K-1,K-1)
      K1=NII-1
      DO 380 L=2,K1,2
      K=NII-L
      KUP=K+4
      IF (KUP-NII) 368,368,367
  367 KUP=NII
  368 TEMP=0.0
      KLR=K+1
      DO 370 L1=KLR,KUP
  370 TEMP=TEMP +SA(K,L1)*U2(L1)
      U2(K)=(AK(K)-TEMP)/SA(K,K)
      KLR=KLR-1
      IF(KUP-NII) 372,374,374
  372 KUP=KUP-1
  374 TEMP =0.0
      DO 376 L1=KLR,KUP
  376 TEMP=TEMP+SA(K-1,L1)*U2(L1)
      U2(K-1)=(AK(K-1)-TEMP)/SA(K-1,K-1)
  380 CONTINUE
      I=3
      DO 390 K=1,NII,2
      YIJ1=RFA*(U2(K)-U(I,J))
      YIJ2=RFA*(U2(K+1)-U(I,J+1))
      Y1=ABSF(YIJ1)
      Y2=ABSF(YIJ2)
      SYIJ=SYIJ+Y1+Y2
      U(I,J  )=U(I,J  )+YIJ1
      U(I,J+1)=U(I,J+1)+YIJ2
      IF(Y1-DE) 384,382,382
  382 KPE=KPE+1
  384 IF(Y2-DE) 390,386,386
  386 KPE=KPE+1
  390 I=I+1
      IF (JB-3) 392,392,394
  392 JB=5
      RFA=RF2
      GO TO 347
  394 RF1=1.0/(1.0-RHO*RHO*RFA/4.0)
      RF2=1.0/(1.0-RHO*RHO*RF1/4.0)
      SPYIJ=SYIJ
      RETURN
      END

C     TO RUN PROBLEM 2 REPLACE THE CORRESPONDING CARDS IN THE
C     MAIN ISOPEP DECK WITH THE FOLLOWING TWO CARDS
      CALL PB2EX
      CALL PB2BD
C     CHECK DIMENSION AND COMMON STATEMENTS

      SUBROUTINE PB2BD
```

```
C     CALCULATION OF BOUNDARY VALUES OF PROBLEM 2
      DIMENSION U(28,52),IK(52),IL(52),D(5),P(25,13),U1(25)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
  100 RTPI=.5/3.1415927
      A=MX
      D1=1./A
C                                                  ALONG Y=0
      X=0.0
      DO 105 I=3,IA
      U(I,2)= -.5*X*X
      IF (X-.25) 105,105,106
  105 X=X+D1
  106 K=I
      DO 107 I=K,IA
      U(I,2)=0.03125-.25*X
  107 X=X+D1
C                                                        ALONG X=1
      Y=1.E-10
      DO 108 J=2,JA
      U(IA,J)=((9./16.+Y*Y)*ATANF(.75/Y)-(25./16.+Y*Y)*ATANF(1.25/Y)
     1 -.5*Y)*RTPI+.03125
  108 Y=Y+D1
C                                                        ALONG Y=2
      X=0.0
      DO 109 I=3,IA
      U(I,JA)=RTPI* ((4.+(X-.25)**2)*ATANF((X-.25)/2.)-((X+.25)**2+4.)*
     1 ATANF((X+.25 )/2.)-1.)+.03125
  109 X=X+D1
      RETURN
      END

      SUBROUTINE PB2EX
      DIMENSION U(28,52),IK(52),IL(52),D(5),P(25,13),U1(25)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
C     CALCULATION OF EXTERIOR VALUES
C       POINTS OUSIDE OF RANGE  X=0 OR GREATER AND X=1 OR LESS
      RTPI=.5/3.1415927
  110 Y=1.E-10
      DO 112 J=2,JA
      U(2,J)=U(4,J)
      U(1,J)=U(5,J)
      U(MI,J)=U(IA-1,J)+2.*D1*RTPI*(1.5*ATANF( .75/Y)-2.5*ATANF(1.25/Y))
  112 Y=Y+D1
C             POINTS OUTSIDE OF RANGE Y=0 OR GREATER AND Y=2 OR LESS
      X=0.0
      DO 114 I=3,IA
      U(I,1)=U(I,3)
      U(I,MJ)=U(I,JA-1)+2.*D1*RTPI*(    4.*(ATANF((X-.25)/2.)-
     1 ATANF((X+.25)/2.)))
  114 X=X+D1
      RETURN
      END
```

```
      SUBROUTINE FB3SD
C     BOUNDARY CONDITIONS FOR THE NOTCHED PLATE   PROB. NO. 3
      DIMENSION U(28,52),IK(52),D(5),IL(52)
      DIMENSION AR(7)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KFE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
      EQUIVALENCE (AR(1),U(129 )), (DS1,U(157 )),(DS2,U(158 ))
      EQUIVALENCE(DS3,U( 159)),(DS4,U( 160)),(DS5,U( 161)),(DS6,U( 162))
      EQUIVALENCE (DH1,U( 163))
   22 FORMAT (2014)
      A=MX
      D1=1./A
      IB=IA-1
      READ 22, (IK(J),J=3,JA)
      PRINT 22,(IK(J),J=3,JA)
      JB=IK(3)+1
      DO 210 J=JB,JA
  210 U(IA,J)=.125
      Y=0.0
      DO 211 I=3,IB
      U(I,JA)=.5*Y*Y
      U(I,2)=Y
C     THE VALUE OF U  ON THE CIRCULAR ARC
      U(I,3)=.5*(Y-.25)
  211 Y=Y+D1
      U(22,14)=U(22,3 )
      U(26,15)=U(26,3 )
      X=0.0
      JB=JB-1
      DO 216 J=3,JB
C     THE VALUE OF Y  ON THE CIRCULAR ARC
      U(MI,J)=.5-SQRTF(.0625-X*X)
C     THE VALUE OF U  ON THE CIRCULAR ARC
      U(IA,J)=.5*(U(MI,J)-.25)
C     THE VALUE OF DELY/H FROM THE MESH POINT TO THE CIRCULAR ARC
      IF(J-3) 216,212,212
  212 DEL=(U(MI,J)-.25)/D1
  213 IF(DEL-1.) 215,214,214
  214 DEL=DEL-1.
      GO TO 213
  215 U(IA-1,J)=DEL
  216 X=X+D1
      U(15,4 )=U(27,4 )
      U(16,8 )=U(27,8 )
      U(18,11)=U(18,3 )
      U(19,12)=U(19,3 )
      DO 217 K=2,11
      KK=IA-K
  217 U(KK  ,4)=U(IA-1,K+3)
      DS1=.5*U(IA-1,5)
      DS2=   U(IA-1,5)+DS1
      DS3=2.*U(IA-1,7)-1.
      DS4=.5*U(IA-1,9)
      DS5=1.5*U(IA-1,9)-.5
```

```
      DS6=.5+DS4
      DH1=5.*DS5/3.
      AR(1)=.5+DS1
      AR(2)=.5+U(IA-1,5)
      AR(3)=.5+U(IA-1,6)
      AR(4)=.25*(DS3+DS6+1.0)
      AR(5)=.5*DS5*DH1
      AR(6)=.5+U(IA-1,9)-AR(5)
      AR(7)=.5+U(IA-1,10)
      RETURN
      END

      SUBROUTINE PB3EX
C   EXTERIOR POINTS FOR NOTCHED PLATE
      DIMENSION U(28,52),IK(52),D(5),IL(52)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
      IB=IA-1
  220 DO 223 I=3,IB
      U(I,MJ)=U(I,JA-1)
      IF(I-IK(3)-1) 222,222,223
  222 U(I,1)=U(I,5)
      U(I,2)=U(I,4)
  223 CONTINUE
C     INTERIOR POINT ADJACENT TO CIRCULAR ARC ON HORIZONTAL LINE
      I=17
      J=10
      GO TO 224
  218 I=20
      J=13
      GO TO 224
  219 J=14
  224 CA=1.-U(I,4)
      CB=CA*(CA-1.)/2.
      CC=CB*(CA-2.)/3.
      CD=CC*(CA-3.)/4.
      BB=CA-.5
      BC=((3.*CA-6.)*CA+2.)/6.
      BD=(((4.*CA-18.)*CA+22.)*CA-6.)/24.
      DA1=1.-CA+CB-CC+CD
      DA2=CA-2.*CB+3.*CC-4.*CD
      DA3=CB-3.*CC+6.*CD
      DA4=CC-4.*CD
      DB1=BB+BD-BC-1.
      DB2=1.-2.*BB+3.*BC-4.*BD
      DB3=BB-3.*BC+6.*BD
      DB4=BC-4.*BD
      BE=1./(DA2/DA1-DB2/DB1)
      U(I,J)=BE*(U(I,3)/DA1+(DB3/DB1-DA3/DA1)* U(I,J+1)
     1+(DB4/DB1-DA4/DA1)*U(I,J+2)+(BD/DB1-CD/DA1)*U(I,J+3))
      I=I+1
      IF (J-13) 218,219,221
  221 IF(I-24) 225,224,226
  225 I=23
```

```
      J=15
      GO TO 224
  226 IF(I-25) 224,224,227
  227 DO 229 J=1,JA
      U(1,J)=U(5,J)
      U(2,J)=U(4,J)
      IF(J-IK(3)) 229,229,228
  228 U(MI,J)=U(IA-1,J)+D1
  229 CONTINUE
C     INTERIOR POINT ADJACENT TO CIRCULAR ARC ON VERTICAL LINE
      DUB=.5
      I=15
      J=5
  230 CA=1.-U(IA-1,J)
      CB=CA*(CA-1.)/2.
      CC=CB*(CA-2.)/3.
      CD=CC*(CA-3.)/4.
      BB=CA-.5
      BC=((3.*CA-6.)*CA+2.)/6.
      BD=(((4.*CA-18.)*CA+22.)*CA-6.)/24.
      DA1=1.-CA+CB-CC+CD
      DA2=CA-2.*CB+3.*CC-4.*CD
      DA3=CB-3.*CC+6.*CD
      DA4=CC-4.*CD
      DB1=BB+BD-BC-1.
      DB2=1.-2.*BB+3.*BC-4.*BD
      DB3=BB-3.*BC+6.*BD
      DB4=BC-4.*BD
      BE=1./(DA2/DA1-DB2/DB1)
      U(I,J)=BE*(U(IA,J)/DA1+DUB*D1/DB1+(DB3/DB1-DA3/DA1)*U(I-1,J)
     1+(DB4/DB1-DA4/DA1)*U(I-2,J)+(BD/DB1-CD/DA1)*U(I-3,J))
      J=J+1
      IF(J-7) 230,230,232
  232 I=16
      IF(J-9) 230,230,233
  233 U(15,1)=U(15,5)
      U(15,2)=U(15,4)
  244 RETURN
      END

      SUBROUTINE PB4BD
C     BOUNDARY CONDITIONS FOR THE V-NOTCHED PLATE        PROB. NO. 4
      DIMENSION U(28,52),IK(52),D(5),P(25,13),U1(25),IL(52)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
   22 FORMAT (2014)
      A=MX
      D1=1./A
      IB=IA-1
      READ                22,(IK(J),J=3,JA)
      PRINT               22,(IK(J),J=3,JA)
      JB=IK(3)+1
      DO 210 J=JB,JA
  210 U(IA,J)=.125
```

```
      Y=0.0
      DO 215 I=3,IB
      U(I,JA)=.5*Y*Y
      IF(I-IK(3)) 215,215,212
C     BOUNDARY VALUES ON NOTCH EDGE
  212 K=I-M/4
      U(I,K)=.5*(Y-.25)
  215 Y=Y+D1
      RETURN
      END

      SUBROUTINE PB4EX
C     EXTERIOR POINTS FOR THE V-NOTCHED PLATE   PROB. NO. 4
      DIMENSION U(28,52),IK(52),D(5),P(25,13),U1(25),IL(52)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
  220 DO 224 I=3,IA
      U(I,MJ)=U(I,JA-1)
      IF(I-IK(3)-1) 222,222,224
  222 U(I,1)=U(I,5)
      U(I,2)=U(I,4)
  224 CONTINUE
      DO 228 J=1,JA
      U(1,J)=U(5,J)
      U(2,J)=U(4,J)
      IF(J-IK(3)) 228,228,226
  226 U(MI,J)=U(IA-1,J)+D1
  228 CONTINUE
      RETURN
      END

      SUBROUTINE PB5BD
C     BOUNDARY CONDITIONS FOR  A PLATE WITH A SQUARE NOTCH  PROB. NO. 5
      DIMENSION U(28,52),IK(52),D(5),IL(52)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
   22 FORMAT (20I4)
      IB=IA-1
      A=MX
      D1=1./A
      READ                22, (IK(J),J=3,JA)
      PRINT               22, (IK(J),J=3,JA)
      JB=IK(3)+1
      DO 205 J=3,JB
  205 U(JB,J)=0.0
      DO 210 J=JB,JA
  210 U(IA,J)=.125
      Y=0.0
      DO 215 I=3,IB
      U(I,JA)=.5*Y*Y
      IF(I-JB) 215,215,212
  212 U(I,JB)=.5*(Y-.25)
  215 Y=Y+D1
      RETURN
      END
```

```
      SUBROUTINE PB5EX
C   EXTERIOR POINTS FOR A PLATE WITH A SQUARE NOTCH
      DIMENSION U(28,52),IK(52),D(5),IL(52)
      COMMON U,IK,D,MX,IF,IA,MI,MY,JF,JA,MJ,N,NOUT,ND,NT,NDL,RFA,RF1,
     1 SPYIJ,PRNO,DE,D1,KPE,INOT,MESH,NSTRS,NINC,IP,JP,AM1,IL,P,U1
      JB=IK(3)+1
      IB=IA-1
  220 DO 225 I=3,IB
      U(I,MJ)=U(I,JA-1)
      IF(I-IK(3)-1)222,222,223
  222 U(I,1)=U(I,5)
      U(I,2)=U(I,4)
      GO TO 225
  223 U(I,JB-1)=U(I,JB+1)
  225 CONTINUE
      DO 228 J=1,JA
      U(1,J)=U(5,J)
      U(2,J)=U(4,J)
      IF(J -JB) 228,226,226
  226 U(MI,J)=U(IA-1,J)+D1
  228 CONTINUE
      RETURN
      END

C     BOUNDRY CONDITIONS CAN BE INTRODUCED USING EQUATION (28)
C     AS INDICATED IN SECTION 6 - TREATMENT OF IRREGULAR
C     BOUNDARIES. THE ADDITIONAL INSTRUCTIONS REQUIRED
C     FOR SORLX FOR PROBLEMS 3, 4 AND 5 ARE LISTED UNDER
C     THE HEADINGS SOR3, SOR4 AND SOR5 RESPECTIVELY.
C     THE CARDS ARE INSERTED BETWEEN THE SUBROUTINE SORLX
C     STATEMENT NUMBERS 133 AND 140.

C                              SOR3
C      ADDITION TO SORLX FOR SEMI-CIRCULAR NOTCH
      I=IB+1
      IF(J-IK(3)-2) 230,230,140
  230 DO 231 L1=1,5
      K=I-(L1/3-1)*(-1)**L1
      L=J-((L1-1)/3)*(-1)**(L1-1)
  231 D(L1)=(U(K+1,L)+U(K-1,L)+U(K,L+1)+U(K,L-1)-4.*U(K,L))
      K=J-2
      GO TO (233,235,236,237,238,239,240,248,250,250,250,252,256,260),K
  233 TEMP=0.0
      ANV=2.0
  234 D(2)=(TEMP+U(I,J)+.5*(U(I+1,J+1)+U(I+1,J-1)+D1)-2.*U(I+1,J))*ANV
      GO TO 242
  235 ANV=1./AR(1)
      TEMP=DS1*(U(I+1,J+1)-U(I+1,J))
      GO TO 234
  236 ANV=1./AR(2)
      TEMP1=DS2*(U(I+1,J+1)-U(I+1,J))
      TEMP=-TEMP+TEMP1
      GO TO 234
  237 ANV=1./AR(3)
```

```
      TEMP=-TEMP1+.5*(U(I+1,J+1)-U(I+1,J))
      GO TO 234
  238 D(2)= U(I,J)+U(I+1,J-1)+U(I+1,J+1)-3.*U(I+1,J)+(U(IA,J)-U(I+1,J))
     1*1.5
      D5=D(2)
      GO TO 242
  239 D(2)=(U(I,J)+.8*U(I+1,J+1)-1.8*U(I+1,J)+.5*D1)/AR(4)
      D(5)=D5
      D5=D(2)
      GO TO 242
  240 D(2)=U(I,J)+U(I+1,J+1)+U(I+1,J-1)-3.*U(I+1,J)+(U(IA,J)-U(I+1,J))
     1 /U(IA-1,J)
      D5=D(2)
      GO TO 242
  248 D(5)=D5
      D(2)=(U(I+1,J+1)+U(I,J)-2.*U(I+1,J)+D1/2.)/AR(7)
      D5=D(2)
      GO TO 242
  250 D(5)=D5
      D(2)=2.*(U(I+1,J+1)+U(I,J)-2.*U(I+1,J))+D1
      D5=D(2)
      IF(J-13)242,251,251
  251 D(2)=.5*D5/AR(7)
      D5=D(2)
      GO TO 242
  252 D(2)=(U(I+1,J+1)+U(I,J)+DS6*U(I+2,J)-(2.+DS6)*U(I+1,J)+
     1DH1*(U(I+1,3)-U(I+1,J))/U(I+1,4)+(1.-DS6)*D1*.5)/AR(6)
      D(5)=D5
      GO TO 242
  256 IF(I-21) 257,257,258
  257 D(5)=(U(I,J)+(U(I-1,J-1)+U(I+1,J-1)-2.*U(I,J-1))*DS6+(U(I,3)-
     1 U(I,J-1))/U(I,4)-U(I,J-1))/DS6
      KT=1
      GO TO 242
  258 D(5)=(U(I,J)+DS6*U(I-1,J-1)-(1.+DS6)*U(I,J-1)+.5*D1*DS6)/AR(4)
      D(2)=U(I,J)+U(I+1,J+1)+U(I+2,J)-3.*U(I+1,J)+( U(I+1,3)-U(I+1,J))
     1/U(I+1,4)
      D5=D(2)-U(I,J)
      KT=0
      GO TO 242
  260 IF(I-24) 262,263,264
  262 D(5)=D5+U(I-1,J-1)
      KT=1
      GO TO 242
  263 TEMP1=(U(I+1,J-1)-U(I,J-1))*DS2
      TEMP=TEMP1+(U(I-1,J-1)-U(I,J-1))*.5+.5*(.5-DS2)*D1
      ANV=1./AR(3)
      GO TO268
  264 IF(I-26) 265,266,242
  265 TEMP=TEMP1
      TEMP1=(U(I+1,J-1)-U(I,J-1))*DS1
      TEMP=-TEMP+TEMP1+.5*(DS2-DS1)*D1
      ANV=1./AR(2)
      GO TO268
```

```
  266 KT=0
      TEMP=-TEMP1+.5*DS1*D1
      ANV=1./AR(1)
  268 D(5)=(TEMP+U(I,J)-2.*U(I,J-1)+.5*(U(I-1,J-1)+U(I+1,J-1)))*ANV
  242 YIJ=(4.*D(3)-D(1)-D(2)-D(4)-D(5))*.05*RFA
      Y2=ABSF(YIJ)
      SYIJ=SYIJ+Y2
      U(I,J)=U(I,J)+YIJ
      IF(Y2-DE) 243,244,244
  244 KPE=KPE+1
  243 IF (KT)  140,140,245
  245 I=I+1
      GO TO 230

C         SOR4********(SEE COMENTS ON PAGE 146)
C     ADDITION TO SORLX FOR A V NOTCH
      I=IB+1
      IF(J-IK(3)-1) 230,230,140
  230 DO 231 L1=1,5
      K=I-(L1/3-1)*(-1)**L1
      L=J-((L1-1)/3)*(-1)**(L1-1)
  231 D(L1)=(U(K+1,L)+U(K-1,L)+U(K,L+1)+U(K,L-1)-4.*U(K,L))
      IF(J-4) 232,234,234
  232 D(2)=4.*(U(I+1,J+1)+U(I,J)+U(I+1,J-1)-3.*U(I+1,J)+D1*.5)/3.
      D5=D(2)
      GO TO 242
  234 D(5)=D5
      IF(J-IK(3)-1) 235,236,236
  235 D(2)=2.*(U(I+1,J+1)+U(I,J)-2.*U(I+1,J))+D1
      D5=D(2)
      GO TO 242
  236 D(2)=8.*(1.5*(U(I,J)-U(I+1,J))+.75*D1)/5.
  242 YIJ=(4.*D(3)-D(1)-D(2)-D(4)-D(5))*.05*RFA
      Y2=ABSF(YIJ)
      SYIJ=SYIJ+Y2
      U(I,J)=U(I,J)+YIJ
      IF(Y2-DE) 140,244,244
  244 KPE=KPE+1

C         SOR5********(SEE COMENTS ON PAGE 146)
C     ADDITION TO SORLX FOR A SQUARE NOTCH
      I=IB+1
      IF(J-IK(3)-1) 230,230,140
  230 DO 231 L1=1,5
      K=I-(L1/3-1)*(-1)**L1
      L=J-((L1-1)/3)*(-1)**(L1-1)
  231 D(L1)=(U(K+1,L)+U(K-1,L)+U(K,L+1)+U(K,L-1)-4.*U(K,L))
      IF(J-IK(3)-1) 232,242,242
  232 D(2)=(U(I,J)*2.+U(I+1,J+1)+U(I+1,J-1)-4.*U(I+1,J))+D1
  242 YIJ=(4.*D(3)-D(1)-D(2)-D(4)-D(5))*.05*RFA
      Y2=ABSF(YIJ)
      SYIJ=SYIJ+Y2
      IF(Y2-DE) 140,244,244
      U(I,J)=U(I,J)+YIJ
  244 KPE=KPE+1
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TABLE OF SYMBOLS

PRNO, DE, RFA, SPYIJ, MX, MY, IF, JF, N, NT, NOUT, ND, MESH, NSTRS, IP, JP, M4, M2. SEE INPUT PREPARATION FOR DEFINITION.

U(I,J) - DISCRETE VALUES OF STRESS FUNCTIONS.

I - ROW INDEX.

J - COLUMN INDEX.

MI = MAXIMUM I = MX+IF+1

IA = MI-1 = LAST BOUNDARY ROW INDEX

MJ = MAXIMUM J = MY+JF+1

JA = MJ-1 = LAST BOUNDARY COLUMN INDEX

P(J,I) - STRESS FUNCTION DISTRIBUTION SAVED FOR GENERATION OF INITIAL ESTIMATE OF U(I,J) IN NEXT PROBLEM (SEE CHANG P. 129).

U1(J) - A COLUMN ARRAY USED IN CHANG FOR TEMPORARY STORAGE.

D1 - MESH INTERVAL H COMPUTED IN PBNBD, D1 DEPENDS ON PHYSICAL DIMENSIONS OF THE SOLUTION DOMAIN.

KPE - COUNT OF POINTS AT WHICH CONVERGENCE CRITERION WAS NOT SATISFIED IN THE LAST ITERATION.

INOT - CONTROL SWITCH FOR SUBROUTINE OUTIN. INOT = 1 SIGNALS INPUT OF NEXT DATA SET. INOT = 2 SIGNALS USE OF THE OUTPUT PORTION OF THE SUBROUTINE.

IK(J) = LAST INTERIOR POINT ROW INDEX IN COLUMN J.

IL(J) = FIRST INTERIOR POINT ROW INDEX IN COLUMN J.

IK(J) AND IL(J) ARE INTRODUCED IN PBNBD FOR SPECIFICATION OF IRREGULAR BOUNDARIES.

SPECIFICATION OF COLUMN TERMINAL POINTS FOR SORLX

A RECTANGULAR MESH WITH THE SPECIFIED DIMENSIONS OF THE ARRAY U ENCOMPASSES THE PROBLEM SOLUTION DOMAIN. FOR A RECTANGULAR REGION THE ITERATIVE METHOD SUBROUTINE SORLX SWEEPS THE MESH BY COLUMNS STARTING WITH J = 3, USING $3 \leq I < IA$ AND CONTINUING UNTIL J = JA-1. IF SYMMETRY CONDITIONS ARE NOT USED THE BOUNDARIES CORRESPOND TO THE ROWS I = 2 AND I = IA AND THE COLUMNS J = 2 AND J = JA. THE ROWS I = 1, I = MI AND THE COLUMNS J = 1 AND J = MJ ARE EXTERIOR TO THE SOLUTION DOMAIN AND CAN BE USED BY THE WRITER OF THE PBNEX SUBROUTINE TO SATISIFY BOUNDARY DERIVATIVE CONDITIONS. IF I = 3 (OR J = 3) IS A LINE OF SYMMETRY THE VALUES IN THE MESH LINE I = 2 (OR J = 2) ALSO MUST BE PROVIDED IN PBNEX USING DERIVATIVE CONDITIONS ALONG A LINE OF SYMMETRY.

A PROBLEM WITH IRREGULAR BOUNDARIES MAY SPECIFY A DIFFERENT FIRST INTERIOR POINT, IL(J), AND LAST INTERIOR POINT, IK(J), FOR EACH COLUMN. THESE VALUES MAY BE READ FROM CARDS OR GENERATED IN THE SUBROUTINE PBNBD. IF THIS IS NOT DONE ALL IL(J) ARE SET TO 3 AND ALL IK(J) TO IA-1 BEFORE TRANSFER TO PBNBD AND THE DOMAIN IS TREATED AS A RECTANGLE. THOUGH IL(J) AND IK(J) LIMIT THE SWEEP ALONG THE COLUMNS WITHIN SORLX THEIR SPECIFICATION DEPENDS ON THE CHOICE OF THE WRITER OF THE BOUNDARY CONDITION SUBROUTINES. IF HE ELECTS TO COMPUTE SOME INTERIOR POINTS BY INTERPOLATION IN PBNEX THEN HE CAN SET THE INTERIOR POINT ROW INDEX RANGES ACCORDINGLY. (SEE PB3EX P. 143)