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SOME APPROXIMATIONS IN VIBRATIONS AND WAVE MOTION
OF ELASTIC MEDIA

presented by

MICHEL YVON RONDEAU

has been accepted towards fulfillment
of the requirements for

PH.D. degree in MECHANICAL ENG.

Matthew A. Medick
Major professor

Date May 4, 1979

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SOME APPROXIMATIONS IN VIBRATIONS AND WAVE MOTION
OF ELASTIC MEDIA

By

Michel Yvon Rondeau

A DISSERTATION

Submitted to
Michigan State University
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ABSTRACT

SOME APPROXIMATIONS IN VIBRATIONS AND WAVE MOTION OF ELASTIC MEDIA

By

Michel Yvon Rondeau

This paper is a study of wave propagation phenomena in a linear, elastic, isotropic, homogeneous layer. A layer is defined as an infinite plate bounded by a pair of parallel planes at $x_2 = \pm b$ as referred to a rectangular coordinate system x_i ($i = 1, 2, 3$). The motion is described by the integral method of Kirchhoff [11], derived through Hamilton's principle, which serves as the starting point for the various approximations to be developed.

In chapter one, a series expansion procedure coupled with a new truncation concept is used to construct two-dimensional partial differential equations for motions in the layer. Frequency spectra associated with traveling waves in a traction-free layer are studied in detail and compared with the spectra predictions of Medick [14], Nikodem [18], and that of the three-dimensional theory [1].

These two-dimensional layer equations also enter into the developments of chapters three and four. In chapter two, wave propagation problems are approached from a different viewpoint employing a symbolic technique. Applications are made to one, two and three-dimensional problems. This

technique will also be employed in chapter four where we discuss transient waves due to non-homogeneous boundary conditions.

In chapter three we subdivide a plate into (n) - subdomains. In each subdomain Hamilton's principle and the truncation procedure used in chapter one will be applied. This will give us two non-homogeneous P.D.E. coupling the uniform and linear amplitudes distributions. The non-homogeneous terms are stated at the interface of each subdomain. To further simplify the analysis, a new technique called Finite-point approximation is introduced. It gives a discrete solution in the direction of the plate thickness (i.e. x_2 -variable). The points are located at the top, bottom and interface of each subdomain. This technique will reduce the two non-homogeneous P.D.E. to a system of Differential-Difference equation. Finally, a solution in the form of a travelling wave with free traction at the top and bottom of the plate will generate an approximate Frequency Spectrum. The number of dispersion curves that we obtain through the spectrum depends on $\bar{N}$ the number of subdomains in the plate.

In chapter four, impact boundary conditions are applied at the top of the plate and free stresses are applied at the bottom of the plate. The Differential-Difference equations obtained in chapter three are reduced to Difference

equation using the Symbolic technique of chapter two. A solution for the Difference equation is developed for arbitrary $\bar{N}$ (the number of subdomains). Utilization of this solution in the impact boundary conditions mentioned above gives rise to a set of P.D.E. involving variables x_1, t .

These equations may readily be solved by transform techniques or numerical means.

A DEDICATION

This thesis is dedicated to my wife, Ah-ling, without whose encouragement and confidence this work would not have been completed.

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I wish to thank Professor Mathew A. Medick whose counsel and encouragement helped me establish a bridge between my earlier education in mathematics and the exciting and challenging problems abounding in physical phenomena. He introduced me to the creative use of intuition in problem solving. He first kindled my interest in elastic wave guide theories in his lectures, and subsequent encounters, he helped me to identify and shape the direction of my dissertation research. I am grateful for his guidance and encouragement, and most importantly, for his faith in me.

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SYMBOLS

CHAPTER ONE

u_i $\equiv$ displacements

ϵ_{ij} $\equiv$ strain tensor

σ_{ij} $\equiv$ stress tensor

$u_i^{(n)}$
(n)-order amplitudes of the cosine distributions

$\epsilon_{ij}^{(n)}$

$\bar{\epsilon}_{ij}^{(n)}$
(n)-order amplitudes of the sine distributions

$\bar{\sigma}_{ij}^{(n)}$

$\bar{T}_{ij}^{(n)}$
the (n)-order average thickness distributions

$T_{ij}^{(n)}$

$F_i^{(n)}$ $\equiv$ (n)-order face traction

$\bar{U}$ $\equiv$ the plate strain energy density

$\bar{K}$ $\equiv$ the plate Kinetic energy density

U_n $\equiv$ (n)-order strain energy density

K_n $\equiv$ (n)-order Kinetic energy density

$$e^{(n)} = u_{1,1}^{(n)} + u_{3,3}^{(n)} \equiv (n)\text{-order dillatation}$$

$$S_n = n\tau/2b$$

$$C_T = \frac{\mu}{\rho} \equiv \text{shear velocity}$$

$$C_L = (\lambda + \mu)/\rho \equiv \text{pressure velocity}$$

$$\nu = \text{Poisson ratio}$$

$$x^2 = C_L^2/C_T^2$$

$$x \equiv \text{frequency}$$

$$k \equiv \text{wavenumber}$$

$$\Omega \equiv \frac{2b}{\tau} \frac{\omega}{C_T}$$

$$z = \frac{2b}{\tau} k$$

$$c_1, c_2 \equiv \text{correction coefficients}$$

$$\nabla^2 = \partial_{x_1}^2 + \partial_2^2 + \partial_3^2$$

CHAPTER TWO

$$(S.O.D.E.) \equiv \text{symbolic O.D.E.}$$

$$(S.I.C.) \equiv \text{symbolic initial condition}$$

$$\partial_{xx} = \partial^2/\partial x^2$$

SECTION 2.2

$$\mathcal{L}^2 = C_L^{-2} d^2/dt^2$$

$$\tau = x/\lambda$$

transformation

$$T = \frac{c}{\lambda} t$$

$$m = \frac{c}{C_L} d/dT$$

$$\epsilon = b/\lambda$$

SECTION 2.3

$$R^2 = (C_T^{-2} \partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2)$$

$$\mathcal{L}^2 = (C_T^{-2} \partial_t^2 - \partial_{x_1}^2)$$

$$\eta = x_1/b$$

transformation

$$\epsilon = x_2/\lambda$$

$$\epsilon = b/\lambda$$

$$m^2 = \mathcal{L}^2 \ell^2 = \left[\left(\frac{c}{C_T} \right)^2 \partial_T^2 + \partial_\epsilon^2 \right]$$

SECTION 2.4

$$R_\alpha^2 = \nabla^2 - C_\alpha^{-2} \partial_t^2$$

$$C_\alpha = C_L \quad \text{and} \quad C_T, \quad \text{for} \quad \alpha = 1, 2$$

$$\bar{U}_1 = \bar{\nabla}_1$$

$$\bar{u}_2 = \bar{v} \times \bar{t}$$

$$\mathcal{L}_\alpha^2 = [C_\alpha^{-2} \partial_t^2 - \partial_{x_1}^2 - \partial_{x_3}^2], \quad \alpha = 1, 2$$

$$\bar{S}_\alpha = \sinh \mathcal{L}_\alpha b / \mathcal{L}_\alpha$$

$$\bar{C}_\alpha = \cosh \mathcal{L}_\alpha b, \quad \alpha = 1, 2$$

CHAPTER III and IV

$\bar{N} \equiv$ number of subdomain

$N \equiv$ the order of truncation

$u_n(x_1, x_2, t) =$ displacement on the (n)-subdomain

u_n (n)-displacement and (n)-stresses at the interface
 σ_n of each subdomain

$u^{(0)}$ and $v^{(0)}$ $\equiv$ uniform amplitude of displacements

$u^{(1)}$ and $v^{(1)}$ $\equiv$ linear amplitudes of displacements

$F^{(0)}, H^{(0)}$ $\equiv$ uniform amplitude of stresses

$F^{(1)}, H^{(1)}$ $\equiv$ linear amplitude of stresses

$$\bar{D}_{11} = 4b^2/\pi^2 \partial_{xx}^2$$

$$\bar{D}_1 = 2b/\pi \partial_x$$

$$\bar{D}_{tt} = 4b^2/\pi^2 C_T^2 \partial_{tt}^2$$

$$z = 2b/\pi k$$

$$\alpha = 2b/\pi C_T \quad (1)$$

$$k = \text{wavenumber}$$

$$\omega = \text{frequency}$$

$$x^2 = c_T^2/c^2$$

$$\mathcal{L}^2 = d^2/dt^2$$

$$L^2 = \mathcal{L}^2 + c^2$$

$$c = \pi/2b C_L$$

$$a_n = n\pi/2b$$

$$\Omega^2 = \omega^2 - c^2$$

SERIES EXPANSION-METHOD WITH TRUNCATION PROCEDURES FOR APPROXIMATION THEORY

1.1 Introduction

In this chapter we will use the general procedure for deducing approximate two-dimensional equations for elastic plates from the three-dimensional theory of elasticity which was introduced by Mindlin (16) based on the series expansion methods of Poisson (23) and Cauchy (3) and the integral method of Kirchhoff (11).

Legendre orthogonal polynomials were first used by Mindlin-Medick (18) as a basis in their series expansion. They constructed a 2nd-order extensional plate theory that for the first time exhibited and accounted for complex branches in the frequency spectra. Later Lee-Nikodem (18) repeated the development of Mindlin-Medick using the simple thickness modes as orthogonal basis.

In this chapter, representing the field variables by Fourier series, we briefly sketch the conversion of the three-dimensional field equations to two-dimensional partial differential equations. These results were previously obtained by Lee-Nikodem (14). A new formulation is introduced based upon a redefinition of the n th order stress-strain relations. A new truncation procedure is then used to generate appropriate plate theories.

Frequency spectra predictions of the new theories are obtained and compared with those of previous theories and with exact results where available.

1.2 Series Expansion of the Displacement

We first need to define the coordinates system and orthogonal basis to be used. Consider an infinite plate bounded by a pair of parallel planes at $x_2 = \pm b$ as referred to a rectangular coordinate system $x_i (i = 1, 2, 3)$. The thickness of the plate will be in the direction of x_2 and the displacement will be expanded using an orthogonal base $\phi_n(x_2)$. Thus, the finite domain $[-b, b]$ restricts our choice to Legendre or Fourier analysis. In this chapter we let $\phi_n(x_2) = \cos \frac{n\pi}{2} \left(1 - \frac{x_2}{b}\right)$. Hence, the displacement becomes

$$u_i(x_j, t) = \sum_{n=0}^{\infty} u_i^{(n)}(x_\alpha, t) \cos \frac{n\pi}{2} \left(1 - \frac{x_2}{b}\right), \quad \begin{matrix} i, j = 1, 2, 3 \\ \alpha = 1, 3 \end{matrix} \quad (1.1)$$

Using the orthogonality of the base we can write the amplitudes distributions as follows:

$$u_i^{(n)}(x_\alpha, t) = \frac{1}{b} \int_{-b}^b u_i(x_j, t) \cos \frac{2\pi}{2} \left(1 - \frac{x_2}{b}\right) dx_2. \quad (1.2)$$

The $u_i^{(n)}$ are the amplitudes of sinusoidal distributions of displacements across the thickness of the plates. For convenience, however, they will be referred to as n -order displacements. Since the plate is isotropic in this analysis, motion symmetric (extensional) and antisymmetric (flexural), with respect to the middle plane may be considered separately. In the case of extensional motion, only those

components of displacement $u_i^{(n)}$ are retained for which $i + n$ is odd. Similarly, $i + n$ even give rise to flexural motion.

In-plane amplitudes distributions are called compressional modes and the one normal to the plane are called face-shear modes. In the case of compressional modes we have $u_1^{(0)}$ and $u_2^{(0)}$ as extensional and transversal modes and $u_1^{(1)}$ and $u_2^{(1)}$ as thickness-shear and thickness-stretch modes. For face-shear modes, we have $u_3^{(0)}$ as shear mode and $u_3^{(1)}$ as face shear modes. Further explanations of their physical meanings can be found in Medick-Mindlin [18].

1.3 Derivation of the (n)-Order Equation of Motion

The motion is described by the integral method of Kirchhoff [11] which is derived through Hamilton's principle and becomes

$$\int_T \int_V (\sigma_{ij,j} - \rho u_{i,tt}) \delta u_i dv dt = 0 \quad \begin{matrix} ij = 1, 2, 3 \\ k = 1, 2, 3. \end{matrix} \quad (1.3)$$

Recall Equations (1.1)

$$u_i(x_j, t) = \sum_{n=0}^{\infty} u_i^{(n)}(x_\alpha, t) \cos \frac{n\pi}{2} \left(1 - \frac{x_2}{b}\right) \quad \begin{matrix} \alpha = 1, 3 \end{matrix} \quad (1.4)$$

which are the displacements equations. Substituting Equation (1.4) into Equation (1.3) yields

$$\sum_{n=0}^{\infty} \int_T \int_A \left[T_{ij,j}^{(n)} - s_n \bar{T}_{i2}^{(n)} + \frac{1}{b} F_i^{(n)} - \rho u_{i,tt}^{(n)} \right] \delta u_k dA dt = 0 \quad (1.5)$$

where

$$\int_{-b}^b \cos \frac{m\pi}{2} \left(1 - \frac{x_2}{b} \right) \cos \frac{n\pi}{2} \left(1 - \frac{x_2}{b} \right) dx_2 = \delta_{nm} b \quad (1.6)$$

$$\int_{-b}^b \sin \frac{m\pi}{2} \left(1 - \frac{x_2}{b} \right) \sin \frac{n\pi}{2} \left(1 - \frac{x_2}{b} \right) dx_2 = \delta_{nm} b$$

$$\int_{-b}^b \sin \frac{m\pi}{2} \left(1 - \frac{x_2}{b} \right) \cos \frac{n\pi}{2} \left(1 - \frac{x_2}{b} \right) dx_2 = 0$$

$$s_n = \frac{n\pi}{2b} \quad (1.7)$$

and

$$T_{ij}^{(n)} = \int_{-b}^b \sigma_{ij} \cos \frac{n\pi}{2} \left(1 - \frac{x_2}{b} \right) dx_2 \quad (1.8)$$

$$\bar{T}_{i2}^{(n)} = \int_{-b}^b \sigma_{i2} \sin \frac{n\pi}{2} \left(1 - \frac{x_2}{b} \right) dx_2$$

$$i = 1, 2, 3$$

$$F_i^{(n)} = \sigma_{i2}(b, x_\alpha, t) + (-1)^{n+1} \sigma_{i2}(-b, x_\alpha, t) \quad \alpha = 1, 3$$

which are the (n)-order average thickness stresses and (n)-order components of face-traction. Thus, (Vn) and $\delta u_k^{(n)} \neq 0$. Equation (1.17) gives us

$$T_{ij,j}^{(n)} - s_n \bar{T}_{i2}^{(n)} + \frac{1}{b} F_i^{(n)} - \rho u_{i,tt}^{(n)} = 0 \quad (1.9)$$

which represents the (n)-order stress equations of motion.

1.4 Derivation of the (n)-Order Strain-Displacements and Stress-Strains Relations

The three-dimensional strain-displacements relations are

$$\epsilon_{ij} = \frac{1}{2} [u_{i,j} + u_{j,i}] , \quad ij = 1,2,3 \quad (1.10)$$

Then if we substitute the displacements Equation (1.4), Equation (1.9) becomes

$$\epsilon_{ij}' = \sum_{n=0}^{\infty} \epsilon_{ij}^{(n)} \cos \frac{n\pi}{2} \left(1 - \frac{x_2}{b}\right) + \bar{\epsilon}_{ij}^{(n)} \sin \left(1 - \frac{x_2}{b}\right)$$

where

$$\epsilon_{ij}^{(n)} = \frac{1}{2} [u_{i,j}^{(n)} + u_{j,i}^{(n)}] \quad (1.11)$$

$$\bar{\epsilon}_{ij}^{(n)} = \frac{s_n}{2} \left(\delta_{2i} u_j^{(n)} + \delta_{2j} u_i^{(n)} \right) .$$

The above Equation (1.11) represents the (n)-order strain-displacement relations.

To obtain compatible n-order auxiliary stress-strain relations we start with the stress-strain relations for isotropic medium in the form:

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} . \quad (1.12)$$

where λ, μ are lamé constants.

Combining Equation (1.11) to above we obtain

$$\sigma_{ij} = \sum_{n=0}^{\infty} \sigma_{ij}^{(n)} \cos \frac{n\pi}{2} \left(1 - \frac{x_2}{b}\right) + \bar{\sigma}_{ij}^{(n)} \sin \frac{n\pi}{2} \left(1 - \frac{x_2}{b}\right) \quad (1.13)$$

where

$$\sigma_{ij}^{(n)} = \lambda \delta_{ij} \epsilon_{kk}^{(n)} + 2\mu \epsilon_{ij}^{(n)} \quad (1.14)$$

$$\bar{\sigma}_{ij}^{(n)} = \lambda \delta_{ij} \bar{\epsilon}_{kk}^{(n)} + 2\mu \bar{\epsilon}_{ij}^{(n)} .$$

This represents the (n)-order auxiliary stress-strain relations.

At this point, if we substitute Equation (1.13) in the first two equations of (1.8) we get

$$\frac{1}{b} T_{ij}^{(n)} = \sigma_{ij}^{(n)} + \sum_{m=0}^{\infty} A_{mn} \bar{\sigma}_{ij}^{(m)} \quad (1.15)$$

$$\frac{1}{b} \bar{T}_{i2}^{(n)} = \bar{\sigma}_{i2}^{(n)} + \sum_{m=0}^{\infty} A_{mn} \sigma_{i2}^{(m)} .$$

where $A_{nm} = 0$ for $n + m = \text{even}$, $A_{nm} = 4m/(m^2 - n^2)\pi$ for $n + m = \text{odd}$. Then the (n)-order stress equations of motion can be written in terms of the (n)-order auxiliary stress equations of motion.

$$\left[\sigma_{ij,j}^{(n)} - s_n \bar{\sigma}_{i2}^{(n)} \right] + \sum_{m=0}^{\infty} A_{nm} \left[\bar{\sigma}_{ij,j}^{(m)} - s_n \sigma_{2i}^{(m)} \right] + F_i^{(n)} = \rho b u_{i,tt}^{(n)} . \quad (1.16)$$

1.5 Derivation of the (n)-Order Strain Density and Kinetic Energy

The strain-density energy is defined as

$$\bar{U} = \frac{1}{2} \int_{-b}^b \sigma_{ij} \varepsilon_{ij} \, dx_2 \quad (1.17)$$

By substituting Equations (1.4) and (1.8) in the above equation it leads to

$$\bar{U} = \frac{b}{2} \sum_{n=0}^{\infty} U_n \quad (1.18)$$

where

$$U_n = \sigma_{ij}^{(n)} \varepsilon_{ij}^{(n)} + \bar{\sigma}_{ij}^{(n)} \bar{\varepsilon}_{ij}^{(n)} + \sum_{m=0}^{\infty} A_{nm} \left[\bar{\sigma}_{ij}^{(m)} \varepsilon_{ij}^{(n)} + \sigma_{ij}^{(m)} \bar{\varepsilon}_{ij}^{(n)} \right] \quad (1.19)$$

This represents the (n)-order strain-density energy equation.

The Kinetic energy is defined as

$$\bar{K} = \frac{\rho}{2} \int_{-b}^b u_{i,t} u_{i,t} \, dx_2 \quad (1.20)$$

then if we substitute Equation (1.1) in the above equation, we get

$$\bar{K} = \frac{\rho b}{2} \sum_{n=0}^{\infty} K_n \quad (1.21)$$

where

$$K_n = u_{i,t}^{(n)} u_{i,t}^{(n)} \quad (1.22)$$

This represents the (n)-order Kinetic energy equation. At this point we can use the (n)-order strain and Kinetic energy Equations (1.19) and (1.22) to derive Equations (1.15):

$$T_{ij}^{(n)} = \frac{1}{2} \frac{\partial U_n}{\partial \epsilon_{ij}^{(n)}} = \sigma_{ij}^{(n)} + \sum_{m=0}^{\infty} A_{nm} \bar{\sigma}_{ij}^{(n)} \quad (1.23)$$

$$\bar{T}_{i2}^{(n)} = \frac{1}{2} \frac{\partial U_n}{\partial \bar{\epsilon}_{ij}^{(n)}} = \bar{\sigma}_{i2}^{(n)} + \sum_{m=0}^{\infty} A_{nm} \sigma_{i2}^{(m)}$$

$$u_{i,tt}^{(n)} = \frac{1}{2} \frac{\partial}{\partial t} K_n .$$

1.6 Truncation Procedure

In this section we will first define our truncation procedure and correction coefficients necessary in the approximation theory. The coefficients will be added to the (n)-order strain density and Kinetic energy equations.

Definition 1.1

The [N,q]-order truncation procedure is defined as follows:

$$a) \ u_j^{(n)} = 0 \quad n > q \quad (1.24)$$

$$b) \ T_{ij}^{(n)} = \bar{T}_{ij}^{(n)} = 0 \quad n > q$$

$$c) \ \partial_{x_i x_j}^2 \ (ij = 1, 2, 3), \ \partial_t^2 \text{ of } u_j^{(n)} = 0, \ N < n \leq q .$$

where q is the order of the series truncation and N is the order of higher elastics and inertia terms retained.

Condition a) and b) have the same physical meanings and explanations used by Poisson [2], Medick-Mindlin [18] and Lee-Nikodem [14]. The addition condition c) will reduce the system of P.D.E. in terms of the first q amplitudes distributions of both compressional and face-shear modes. Analytically it will reduce the wave-number k (in the x_1 -direction) and frequency ω relationships (which can be derived from the Partial Differential Equation by using straight-crested waves propagating in the x_1 -direction) to a polynomial of order q .

Definition 1.2

Correction coefficients c_1 and c_2 are going to be added to the (n) -order strain-density and kinetic energy Equations (1.19) and (1.22) as follows:

$$U_n = \delta_{1n}^{(2)} \sigma_{ij}^{(n)} \epsilon_{ij}^{(n)} + \bar{\sigma}_{ij}^{(n)} \bar{\epsilon}_{ij}^{(n)} + \sum_{m=0}^{\infty} A_{nm} \delta_{1m}^{(1)} \left[\sigma_{ij}^{(m)} \bar{\epsilon}_{ij}^{(n)} + \bar{\sigma}_{ij}^{(m)} \epsilon_{ij}^{(n)} \right] \quad (1.25)$$

$$K_n = \delta_{0n}^{(2)} u_{i,tt}^{(n)} u_{i,tt}^{(n)} \quad (1.26)$$

where

$$\delta_{1m}^{(1)} = \begin{cases} 1 & , \quad m \neq 1 \\ c_1 & , \quad m = 1 \end{cases} \quad (1.27)$$

$$\delta_{\alpha n}^{(2)} = \begin{cases} 1 & , \quad m \neq \alpha \\ c_2^p & , \quad m = \alpha \end{cases} , \quad p = \cos^2\left(\frac{i\pi}{2}\right) \quad \alpha = 0, 1$$

Now, using the above Equations (1.25) and (1.26), Equation (1.23) becomes

$$T_{ij}^{(n)} = \delta_{ln}^{(2)} \sigma_{ij}^{(n)} + \sum_{m=0}^{\infty} A_{nm} \delta_{lm}^{(1)} \bar{\sigma}_{ij}^{(m)} \quad (1.28)$$

$$\bar{T}_{i2}^{(n)} = \bar{\sigma}_{i2}^{(n)} + \sum_{m=0}^{\infty} A_{2m} \delta_{lm}^{(1)} \sigma_{i2}^{(m)}$$

$$u_{i,tt}^{(n)} = \delta_{on}^{(2)} u_{i,tt}^{(n)} .$$

Next, if we substitute Equation (1.28) in the (n)-order equation of motion Equation (1.16), we get

$$\left[\delta_{ln}^{(2)} \sigma_{ij,j}^{(n)} - s_n \bar{\sigma}_{i2}^{(n)} \right] + \sum_{m=0}^{\infty} A_{nm} \delta_{lm}^{(1)} \left[\bar{\sigma}_{ij,j}^{(m)} - s_n \sigma_{i2}^{(m)} \right] + F_i^{(n)} = \rho_b \delta_{on}^{(2)} u_{i,tt}^{(n)} \quad (1.29)$$

$$s_n = \frac{n\pi}{2b} ,$$

which represents the (n)-order auxiliary equation of motion with correction coefficients.

1.7 The Truncation Application for $[N, q] = [1, 1], [2, 1]$ and Arbitrary

In this section we will truncate for $N = 1, 2$ and $q = 1$ which will be needed for future discussions. An arbitrary truncation will also be generalized at the end of this section.

Case I ($N = 1, q = 1$)

Using definition 1.1 for $[1, 1]$ then Equation (1.29) gives us

$$\sigma_{ij,j}^{(0)} + A_{01} c_1 \sigma_{ij,j}^{(1)} + F_i^{(0)} = \rho b c_2^p u_{i,tt}^{(0)} \quad (1.30)$$

$$c_2^p \sigma_{ij,j}^{(1)} - s_1 \bar{\sigma}_{i2}^{(1)} + F_i^{(1)} = \rho b u_{i,tt}^{(1)}$$

From Equations (1.11) and (1.14), the (n)-order strain-displacements and stress-strain equations, Equation (1.15) becomes, in the case of the flexural motion ($i + n = \text{EVEN}$, $a = 1, 3$)

$$\mu \nabla^2 u_2^{(0)} + \frac{2\mu}{b} c_1 e^{(1)} + \frac{1}{b} F_2^{(0)} = \frac{\rho}{c_2} u_{2,tt}^{(0)} \quad (1.31)$$

$$\mu \nabla^2 u_a^{(1)} + (\lambda + \mu) e_a^{(1)} - \mu \left(\frac{\pi}{2b} \right)^2 u_a^{(1)} - \frac{2\mu}{b} c_1 u_{s,a}^{(0)} + \frac{1}{b} F_a^{(1)} = \rho u_{a,tt}^{(1)}$$

and for the extensional motion ($i + n = \text{ODD}$ 1 $a = 1, 3$)

$$\mu \nabla^2 u_a^{(0)} + (\lambda + \mu) e_a^{(0)} + \frac{2\lambda}{b} c_1 u_{2,a}^{(1)} + \frac{1}{b} F_a^{(0)} = \rho u_{a,tt}^{(0)} \quad (1.32)$$

$$c_2 \mu \nabla^2 u_2^{(1)} - (\lambda + 2\mu) \left(\frac{\pi}{2b} \right)^2 u_2^{(1)} - \frac{2\lambda}{b} c_1 e^{(0)} + \frac{1}{b} F_2^{(1)} = \rho u_{2,tt}^{(1)} .$$

where $e^{(n)} = u_{1,1}^{(n)} + u_{3,3}^{(n)}$.

These represent the zero and first orders displacement equations of motion.

Case II ($N = 2$, $q = 1$)

Using definition 1.1 for [2.1] then Equation (1.29) gives

us

$$\sigma_{ij,j}^{(0)} + A_{01} c_1 \bar{\sigma}_{ij,j}^{(1)} + F_i^{(0)} = \rho b c_2^p u_{i,tt}^{(0)} \quad (1.33)$$

$$c_2^p \sigma_{ij,j}^{(1)} - s_1 \bar{\sigma}_{i2}^{(1)} + A_{i2} \left[\bar{\sigma}_{ij,j}^{(2)} - s_1 \sigma_{i2}^{(2)} \right] + F_i^{(1)} = \rho b u_{i,tt}^{(1)}$$

$$\sigma_{ij,j}^{(2)} - s_2 \bar{\sigma}_{i2}^{(2)} + A_{2i} \left[\bar{\sigma}_{ij,j}^{(1)} - s_2 \sigma_{i2}^{(1)} \right] + F_i^{(2)} = 0 .$$

From Equations (1.11) and (1.14) the fluxural motion
($i + n = \text{EVEN}$, $a = 1, 3$) becomes

$$\mu \nabla^2 u_2^{(0)} + \frac{2\mu}{b} c_1 e^{(1)} + \frac{1}{b} F_2^{(0)} = \frac{\rho}{c_2} u_{2,tt}^{(0)} \quad (1.34)$$

$$\begin{aligned} \mu \nabla^2 u_a^{(1)} + (\lambda + \mu) e_a^{(1)} - \mu \left(\frac{\pi}{2b} \right)^2 u_a^{(1)} - \frac{2\mu}{b} c_1 u_{2,a}^{(0)} + \frac{2(4\lambda + \mu)}{3b} u_{2,a}^{(2)} \\ + \frac{1}{b} F_a^{(1)} = \rho u_{a,tt}^{(1)} \end{aligned}$$

$$+ (\lambda + 2\mu) \left(\frac{\pi}{b} \right)^2 u_2^{(2)} + \frac{2(4\lambda + \mu)}{3b} e^{(1)} - \frac{1}{b} F_2^{(2)} = 0$$

and for the extensional motion ($i + n = \text{ODD}$, $a = 1, 3$) we get

$$\mu \nabla^2 u_a^{(0)} + (\lambda + \mu) e_a^{(0)} + \frac{2\lambda}{b} c_1 u_{2,a}^{(1)} + \frac{1}{b} F_a^{(0)} = \rho u_{a,tt}^{(0)} \quad (1.35)$$

$$c_2 \mu \nabla^2 u_2^{(1)} - (\lambda + 2\mu) \left(\frac{\pi}{2b} \right)^2 u_2^{(1)} - \frac{2\lambda}{b} \alpha_1 e^{(0)} + \frac{2(\lambda + 4\mu)}{3b} e^{(2)}$$

$$+ \frac{1}{b} F_2^{(1)} = \rho u_{2,tt}^{(1)}$$

$$+ \mu \left(\frac{\pi}{b} \right)^2 u_a^{(2)} + \frac{2(\lambda + 4\mu)}{3b} u_{2,a}^{(1)} - \frac{1}{b} F_a^{(2)} = 0 .$$

These represent the zero, first and second orders displacement equations of motion.

Case III ($N \equiv \text{EVEN}$, $q = 1$)

Using definition 1.1 for [EVEN, 1] Equation (1.29) then gives us

$$\begin{aligned} \sigma_{ij,j}^{(0)} + \sum_{m=0}^{N-1} A_{O(2m+1)} \delta_{1(2m+1)}^{(1)} \bar{\sigma}_{ij,j}^{(2m+1)} + F_i^{(0)} &= \rho b c_2^p u_{i,tt}^{(0)} \\ c_2^p \sigma_{ij,j}^{(1)} - s_1 \bar{\sigma}_{i2}^{(1)} + \sum_{m=0}^N A_{1(2m)} \left[\bar{\sigma}_{ij,j}^{(2m)} - s_1 \sigma_{i2}^{(2m)} \right] + F_i^{(1)} \\ &= \rho b u_{i,tt}^{(1)}, \end{aligned} \quad (1.36)$$

and for $1 < n \leq N \equiv \text{EVEN}$

$$\begin{aligned} \sigma_{ij,j}^{(2n)} - s_{2n} \bar{\sigma}_{i2}^{(2n)} + \sum_{m=0}^{N-1} A_{(2m+1)}^{(2n)} \delta_{1(2m+1)}^{(1)} \left[\bar{\sigma}_{ij,j}^{(2m+1)} - s_{2n} \sigma_{i2}^{(2m+1)} \right] \\ + F_i^{(2n)} = 0 \\ \sigma_{ij,j}^{(2n+1)} - s_{2n+1} \bar{\sigma}_{i2}^{(2n+1)} + \sum_{m=0}^N A_{(2m)}^{(2n+1)} \left[\bar{\sigma}_{ij,j}^{(2m)} - s_{2n+1} \sigma_{i2}^{(2m)} \right] \\ + F_i^{(2n+1)} = 0. \end{aligned} \quad (1.37)$$

Similarly for $N \equiv \text{ODD}$, the displacements equation of motion can be derived using Equations (1.11) and (1.14) in the same fashion as the first and second orders.

In the above section we did not assign any numerical values for the correction coefficients c_1 and C_2 . According to Lee-Nikoden [14] it can be shown that in order to make

the slope of the lowest flexural branch in the first order theory coincide with that from the three-dimensional theory when both the frequency and wave number approach zero, the value of c_1 must be taken as $c_1 = \pi/4$. In order to make the phase velocities of the lowest extensional branches approach that of the Rayleigh surface waves [1] as both the values of the frequency and wave number get large, c_2 must be set equal to the real root of

$$c_2^3 - 8c_2^2 + 8\left(3 - 2/x^2\right)c_2 - 16\left(1 - 1/x^2\right) = 0 \quad (1.38)$$

for Rayleigh surface waves, where $x^2 = 2(1-\mu)/(1-2\mu)$. The values of c_2 for different values of Poisson's ratio are

Table 1

μ	0.00	0.100	0.200	0.250	0.300	0.350	0.400	0.500
c_2	0.764	0.798	0.830	0.845	0.860	0.874	0.888	0.963

1.8 Frequency Spectrum

In this section we are going to study the frequency spectrum using the truncation procedure discussed earlier. For $N = 2$ and $q = 1$ we will obtain a system of P.D.E. coupling the zero and first amplitudes distributions. The second amplitude can be computed by lower amplitudes. Similarly for $N = 3$ and $q = 2$ we have a system of P.D.E. coupling the zero, first and second terms. The third amplitude can be solved through the lower amplitudes also.

Finally, we will consider $N = 4$, $q = 2$ which will give us a system of P.D.E. coupling the zero, first and second amplitudes distributions. The third and fourth amplitudes can be solved through the lower amplitudes.

[1,2]-Order Truncation

Using Equations (1.29), (1.14) and (1.11) for $N = 2$ and $q = 2$ we obtain the displacement equations as follows:

$$\begin{bmatrix} B_{12} & A_{13} \\ A_{22} & A_{23} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(0)} \end{bmatrix} = 0 \quad \text{"extensional"} \quad (1.39)$$

$$\begin{bmatrix} A_{31} & A_{14} \\ A_{44} & B_{44} \end{bmatrix} \begin{bmatrix} u_1^{(0)} \\ u_2^{(1)} \end{bmatrix} = 0 \quad \text{"flexural"}$$

where $F_1^{(0)} = F_1^{(1)} = F_2^{(0)} = F_2^{(1)} = 0$ (since the B.C. are traction-free at $x_2 \pm b$) and

$$B_{12} = \left[x^2 - \frac{4}{9\pi^2} (4x^2 - 7) \left(4 - \frac{7}{x^2} \right) \right] \bar{D}_{11} + 1 - \bar{D}_{tt} \quad (1.40)$$

$$B_{44} = \left[1 - \frac{4}{9\pi^2} (4x^2 - 7)^2 \right] \bar{D}_{11} + x^2 - \bar{D}_{tt}$$

$$A_{13} = -c_1 \frac{4}{\pi} \bar{D}_1$$

$$A_{22} = c_1 \frac{4}{\pi} \bar{D}_1$$

$$A_{23} = \bar{D}_{11} - \frac{1}{c_2} \bar{D}_{tt}$$

$$A_{31} = x^2 \bar{D}_{11} - \bar{D}_{tt}$$

$$A_{34} = c_1 \frac{4}{\pi} (x^2 - 2) \bar{D}_1$$

$$A_{41} = -c_1 \frac{4}{\pi} (x^2 - 2) \bar{D}_1$$

$$A_{44} = c_2 \bar{D}_{11} + x^2 - \bar{D}_{tt}$$

and

$$\bar{D}_{11} = \frac{4b^2}{\pi^2} \partial_{xx} \quad (1.41)$$

$$\bar{D}_1 = \frac{2b}{\pi} \partial_x$$

$$\bar{D}_{tt} = \frac{4b^2}{\pi^2 C_T^2} \partial_{tt}$$

$$x^2 = C_L^2 / C_T^2 .$$

The quadratic amplitude becomes

$$u_2^{(2)} = - \frac{2}{3} \frac{b}{\pi^2} (4 - x^2) u_{1,1}^{(1)} \quad (1.42)$$

$$u_1^{(2)} = - \frac{2}{3} \frac{b}{\pi^2} (x^2 + 2) u_{2,1}^{(1)}$$

which can be solved later. The approximate frequency spectrum for flexural motion can be derived by assuming a solution as follows:

$$u_1^{(1)} = U_1^{(1)} e^{i[kx_1 - \omega t]} \quad (1.43)$$

$$u_2^{(0)} = U_2^{(0)} e^{i[kx_1 - \omega t]}.$$

Substituting the above in Equation (1.39) leads to

$$\begin{bmatrix} \bar{B}_{12} & A_{13} \\ \bar{A}_{22} & \bar{A}_{23} \end{bmatrix} \begin{bmatrix} U_1^{(1)} \\ U_2^{(0)} \end{bmatrix} = 0 \quad (1.44)$$

where

$$\bar{B}_{12} = - \left[x^2 - \frac{4}{9\pi^2} (4x^2 - 7) \left(4 - \frac{7}{x^2} \right) \right] z^2 + 1 + \Omega^2 \quad (1.45)$$

$$\bar{A}_{13} = - i c_1 \frac{4}{\pi} z$$

$$\bar{A}_{22} = i c_1 \frac{4}{\pi} z$$

$$\bar{A}_{23} = - z^2 + \frac{1}{c_2} \Omega^2$$

and

$$z = \frac{2b}{\pi} k \quad (1.46)$$

$$\Omega = \frac{2b}{\pi} \frac{\omega}{C_T}.$$

Thus, the determinant of Equation (1.44) becomes

$$\bar{B}_{12} \bar{A}_{23} - \bar{A}_{13} \bar{A}_{22} = 0. \quad (1.47)$$

It represents the approximate frequency spectrum for the flexural motion. The dispersion curves in Equation (1.47) are computed and compared with those of Lee-Nikodem and Rayleigh-Lamb [1] frequency equation as shown in Figures (1.1).

[2,3]-Order Truncation (First Approximation)

Using Equations (1.29), (1.14) and (1.11) for $N = 3$ and $q = 2$ the displacement equations for extensional motion become:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} U_1^{(0)} \\ U_2^{(1)} \\ U_1^{(2)} \end{bmatrix} = 0 \quad (1.48)$$

where

$$R_{11} = \left[-\frac{16}{9\pi^2} (x^2 - 2) \left(1 - \frac{2}{x^2}\right) + x^2 \right] \bar{D}_{11} - \bar{D}_{tt} \quad (1.49)$$

$$R_{12} = c_1 \frac{4}{\pi} (x^2 - 2) \bar{D}_1$$

$$R_{13} = \frac{8}{35\pi^2} (9x^2 - 14) \left(1 - \frac{2}{x^2}\right) \bar{D}_{11}$$

$$R_{21} = c_1 \frac{4}{\pi} (x^2 - 2) \bar{D}_1$$

$$R_{22} = x^2 + c_2 \bar{D}_{11} - \bar{D}_{tt}$$

$$R_{23} = -\frac{4}{3\pi} (x^2 + 2) \bar{D}_1$$

$$R_{31} = \frac{8}{35\pi^2} (9x^2 - 14) \left(1 - \frac{2}{x^2}\right) \bar{D}_{11}$$

$$R_{32} = -\frac{4}{3\pi} (x^2 + 2) \bar{D}_1$$

$$R_{33} = \left[-\frac{16}{225\pi^2} (9x^2 - 14) \left(9 - \frac{14}{x^2}\right) + x^2 \right] \bar{D}_{11} + 4 - \bar{D}_{tt}.$$

The extensional third amplitude becomes

$$u_2^{(3)} = -\frac{8b}{9\pi^2} \left[\left(1 - \frac{2}{x^2}\right) u_{1,1}^{(0)} + \frac{1}{5} \left(9 - \frac{14}{x^2}\right) u_{1,1}^{(2)} \right] \quad (1.50)$$

which can be solved later.

The approximate frequency spectrum for extensional motion can be found by assuming a solution as follows:

$$u_1^{(0)} = U_1^{(0)} e^{i[kx_1 - \omega t]} \quad (1.51)$$

$$u_2^{(1)} = U_2^{(1)} e^{i[kx_1 - \omega t]}$$

$$u_1^{(2)} = U_1^{(2)} e^{i[kx_1 - \omega t]}.$$

Then Equation (1.48) becomes

$$\begin{bmatrix} \bar{R}_{11} & \bar{R}_{12} & \bar{R}_{13} \\ \bar{R}_{21} & \bar{R}_{22} & \bar{R}_{23} \\ \bar{R}_{31} & \bar{R}_{32} & \bar{R}_{33} \end{bmatrix} \begin{bmatrix} U_1^{(0)} \\ U_2^{(1)} \\ U_1^{(2)} \end{bmatrix} = 0. \quad (1.52)$$

The coefficients $\bar{R}_{ij}$ are obtained by substituting the followings in Equation (1.49):

$$\bar{D}_{11} = -z^2 \quad (1.53)$$

$$\bar{D}_1 = iz$$

$$\bar{D}_{tt} = -\Omega^2 .$$

Thus, the determinant of Equation (1.52) becomes

$$\bar{R}_{11}\bar{R}_{22}\bar{R}_{33} - \bar{R}_{11}\bar{R}_{23}\bar{R}_{32} - \bar{R}_{12}\bar{R}_{21}\bar{R}_{33} + \bar{R}_{12}\bar{R}_{23}\bar{R}_{31} \quad (1.54)$$

$$\bar{R}_{13}\bar{R}_{21}\bar{R}_{32} - \bar{R}_{13}\bar{R}_{22}\bar{R}_{31} = 0$$

and represents the approximate frequency spectrum for extensional motion. The dispersion curves of Equation (1.54) are computed and compared with those of Medick and Rayleigh-Lamb [1] frequency equation as shown in Figures (1.2). We can conclude that the first approximation technique improves the approximate frequency spectrum. We can also find the cubic amplitude distribution by using Equation (1.50) which will improve the solution for the displacements.

[2,4]-Order Truncation (Second Approximation)

Using Equation (1.29), (1.14) and (1.11) for $N = 4$ and $q = 2$ the displacement equations for extensional motion become:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} U_1^{(0)} \\ U_2^{(1)} \\ U_1^{(2)} \end{bmatrix} = 0 \quad (1.55)$$

where

$$R_{11} = [x^2 - \frac{16}{9\pi} 2 (x^2 - 2) (1 - \frac{2}{x}) \bar{D}_{11} - \bar{D}_{tt}] \quad (1.56)$$

$$R_{12} = \frac{16}{\pi} c_1 (x^2 - s) \bar{D}_1$$

$$R_{13} = - [\frac{16}{45\pi} 2 (x^2 - x) (9 - \frac{14}{x^2}) \bar{D}_{11}]$$

$$R_{21} = - \frac{4}{\pi} c_1 (x^2 - 2) \bar{D}_1$$

$$R_{22} = [c_2 - \frac{4}{900\pi} 2 (x^2 + 14)^2 \bar{D}_{11} - \bar{D}_{tt} - x^2]$$

$$R_{23} = \frac{4}{3\pi} (x^2 + 2) \bar{D}_1$$

$$R_{31} = [\frac{16}{45\pi} 2 (x^2 - 2) (9 - \frac{14}{x^2}) \bar{D}_{11}]$$

$$R_{32} = -\frac{4}{3\pi} (x^2 + 2) \bar{D}_1$$

$$R_{33} = [(x^2 - \frac{16}{255\pi} 2 (4x^2 - 14) (9 - \frac{14}{x^2})) \bar{D}_{11} - \bar{D}_{tt} - 2]$$

The extensional third and fourth amplitudes are

$$U_2^{(3)} = \frac{-4}{9\pi} 2 \left[(1 - \frac{2}{x^2}) U_1^{(0)}, 1 + \frac{1}{5} (9 - \frac{14}{x^2}) U_1^{(2)}, 1 \right] \quad (1.57)$$

$$U_1^{(4)} = - \frac{1}{60\pi} (x^2 + 14) U_2^{(1)}, 1$$

which can be solved later.

The approximate frequency spectrum for extensional motion can be found by assuming a solution as follows:

$$U_1^{(0)} = U_1^{(0)} e^{i[kx_1 - \omega t]} \quad (1.58)$$

$$U_2^{(1)} = U_2^{(1)} e^{i[kx_1 - \omega t]}$$

$$U_1^{(2)} = U_1^{(2)} e^{i[kx_1 - \omega t]}$$

Then Equation (1.55) becomes

$$\begin{bmatrix} \bar{R}_{11} & \bar{R}_{12} & \bar{R}_{13} \\ \bar{R}_{21} & \bar{R}_{22} & \bar{R}_{23} \\ \bar{R}_{31} & \bar{R}_{32} & \bar{R}_{33} \end{bmatrix} \begin{bmatrix} U_1^{(0)} \\ U_2^{(1)} \\ U_1^{(2)} \end{bmatrix} = 0 \quad (1.59)$$

The coefficients $\bar{R}_{ij}$ are obtained by substituting the following in Equation (1.55):

$$\bar{D}_{11} = -z^2 \quad (1.60)$$

$$\bar{D}_1 = iz$$

$$\bar{D}_{tt} = -\Omega^2$$

Thus, the determinant of Equation (1.39) becomes

$$\bar{R}_{11}\bar{R}_{22}\bar{R}_{33} - \bar{R}_{11}\bar{R}_{23}\bar{R}_{32} - \bar{R}_{12}\bar{R}_{21}\bar{R}_{33} + \bar{R}_{13}\bar{R}_{23}\bar{R}_{31} \quad (1.61)$$

$$\bar{R}_{13}\bar{R}_{21}\bar{R}_{32} - \bar{R}_{13}\bar{R}_{22}\bar{R}_{31} = 0$$

and represents the approximate frequency spectrum for extensional motion. The dispersion curves of Equation (1.61) are computed and compared with those of Medick, of the first and second approximation and Rayleigh-Lamb [1] frequency equation as shown in Figure (1.3). We can conclude that the second approximation improves the approximate frequency spectrum.

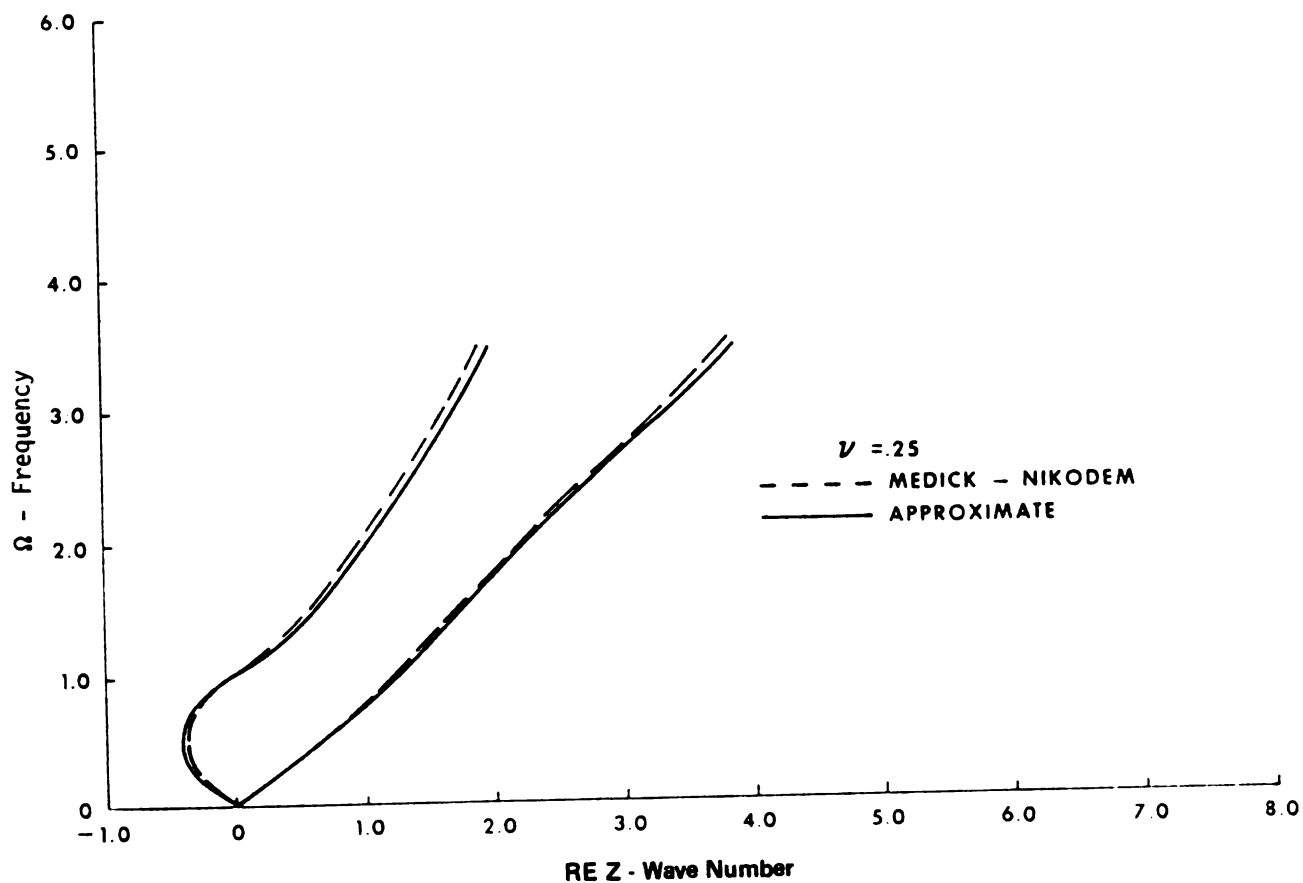


Figure 1.1

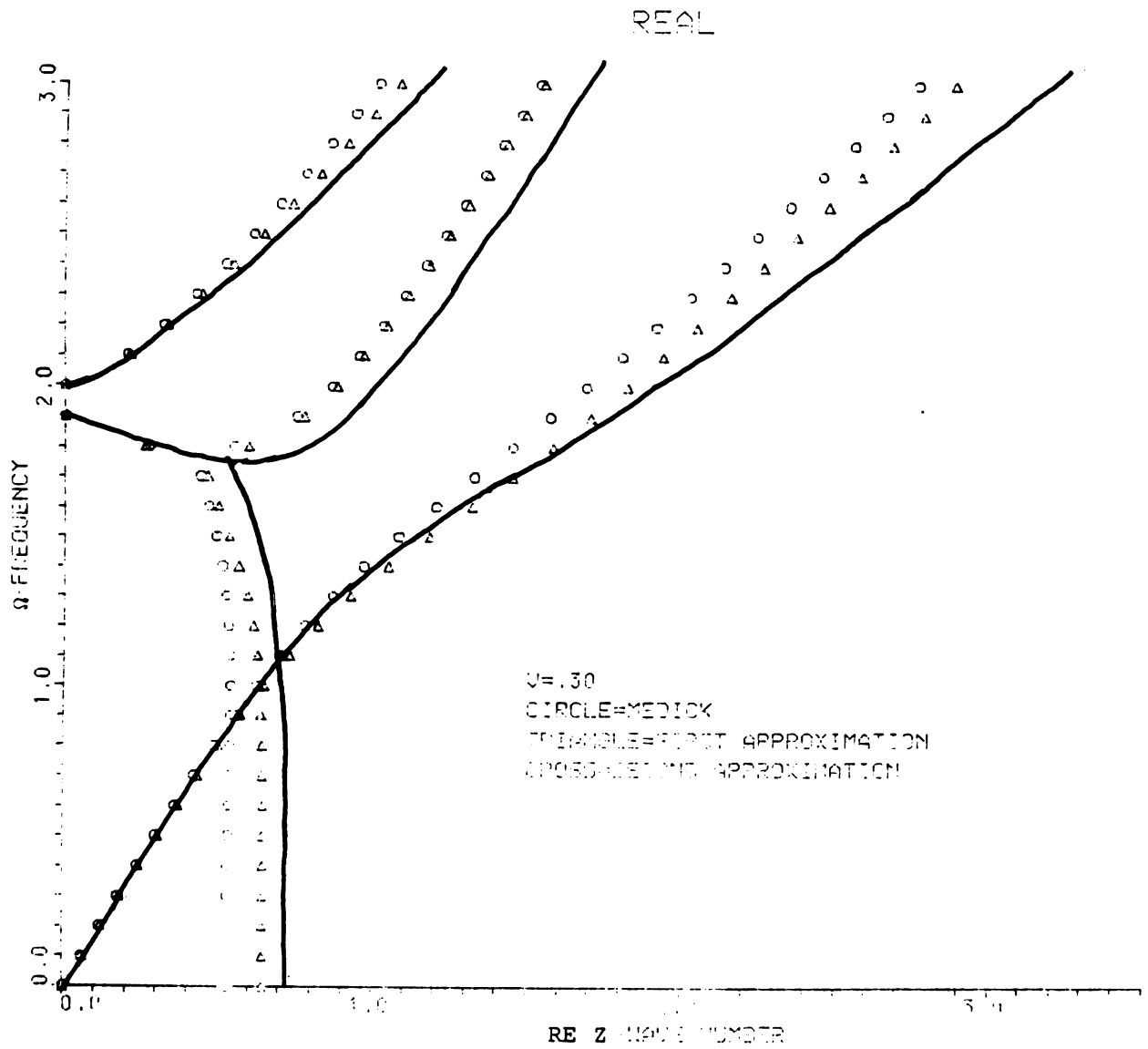


FIGURE (1.2)

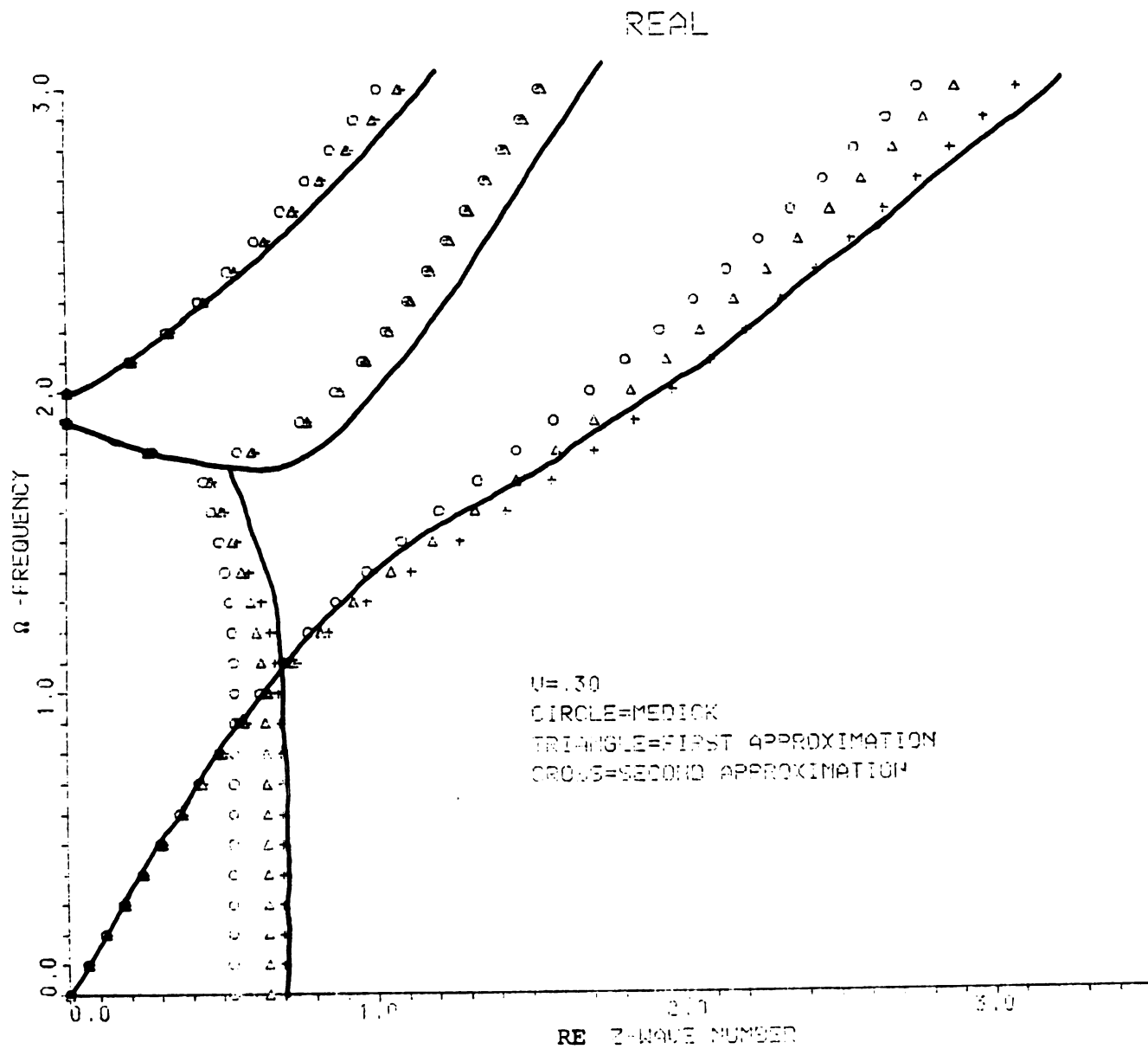


FIGURE (1.3)

CHAPTER II

SYMBOLIC METHOD IN WAVE PROPAGATION PROBLEMS

2.1 Introduction

In this chapter we are going to generate an approximation theory using Symbolic technique [13]. The P.D.E. and B.C. are reduced to an infinite differential equation. This is done by letting all partial derivatives behave as constants, except for one of them. Thus, this will reduce the P.D.E. to a symbolic ordinary differential equation (S.O.D.E.) with symbolic initial condition (S.I.C.). Then, to generate the infinite differential equation, we substitute the symbolic solution (S.S.) into the B.C.

First we will examine the string equation because the simplicity of the operator permits us to solve this problem exactly. Then we non-dimensionalize the equation and use the parameter to find an approximation solution. Similarly a two-dimensional problem will be presented. At the end of this chapter the three-dimensional elastic plates problem with free stresses at the top and bottom of plates will be considered. The approximation will be compared to other techniques used in the preceding chapter.

2.2 Symbolic Method to Solve the String Equation

The partial differential equation for the string problem is

$$(2.1) \quad u_{xx} - c_2^{-2} u_{tt} = 0 \quad (\text{P.D.E.})$$

with B.C.

$$(2.2a) \quad u_x(\pm b, t) = 0 \quad (\text{B.C.})$$

and I.C.

$$(2.2b) \quad \begin{aligned} u(x, 0) &= 0 \\ u_t(x, 0) &= f(x) \end{aligned} \quad (\text{I.C.})$$

Let ∂_t^2 behaves as a constant then equation (2.1) can be written as follows:

$$(2.3) \quad u'' - \mathcal{L}^2 u = 0 \quad (\text{S.O.D.E}) \quad \text{where} \quad \mathcal{L}^2 = \frac{d^2}{c_2^2 dt^2}$$

$$(2.4) \quad \begin{aligned} u(0, t) &= H(t) \\ u'(0, t) &= \mathcal{L} F(t) \end{aligned} \quad (\text{S.I.C.})$$

The solution for the (S.O.D.E) is

$$u(x, t) = a(t) \cosh \mathcal{L} x + b(t) \sinh \mathcal{L} x$$

and by using the (S.I.C.), the above equation becomes

$$(2.5) \quad u(x, t) = H(t) \cosh \mathcal{L} x + F(t) \sinh \mathcal{L} x .$$

Next, we substitute the above in the B.C. equation (2.2a) which will give us an infinite O.D.E.

$$(2.6) \quad H(t) \sinh \mathcal{L}b \pm F(t) \cosh \mathcal{L}b = 0$$

or

$$(2.7) \quad \begin{aligned} [\sinh \mathcal{L}b] H(t) &= 0 \\ [\cosh \mathcal{L}b] F(t) &= 0. \end{aligned}$$

The above infinite operator on $H(t)$ and $F(t)$ can be solved exactly by assuming a solution in the form of an expotential such as

$$(2.8) \quad \begin{aligned} H(t) &= A e^{\omega t} \\ F(t) &= i B e^{\omega t}. \end{aligned}$$

Expanding $\cosh \mathcal{L}b$ and $\sinh \mathcal{L}b$ in terms of power series and operating equation (2.8) term by term yields

$$(2.9) \quad \begin{aligned} [\sinh \frac{\omega b}{C_L}] H(t) &= 0 \\ [\cosh \frac{\omega b}{C_L}] F(t) &= 0. \end{aligned}$$

If A and B are arbitrarities, then the above equations are satisfied for

$$(2.10) \quad \begin{aligned} \omega &= i \frac{n\pi}{2b} C_L, \quad n \text{ EVEN} \\ \omega &= i \frac{n\pi}{2b} C_L, \quad n \text{ ODD.} \end{aligned}$$

Therefore, using the above result, the solution (2.8) becomes

$$H(t) = \sum_{n=\text{EVEN}}^{\infty} A_n \cos \frac{n\pi}{2b} C_L t$$

(2.11)

$$F(t) = \sum_{n=\text{ODD}}^{\infty} B_n \sin \frac{n\pi}{2b} C_L t.$$

Next, if we substitute equation (2.11) in equation (2.5), operate and rearrange terms, we obtain a solution for our problem as follows:

$$(2.12) \quad u(x,t) = \sum_{n=\text{EVEN}}^{\infty} A_n \cos \frac{n\pi}{2b} C_L t \cos \frac{n\pi}{2b} x + \sum_{n=\text{ODD}}^{\infty} B_n \sin \frac{n\pi}{2b} C_L t \sin \frac{n\pi}{2b} x$$

with the time variable I.C. equation (2.2b).

$$u(x,0) = 0$$

(2.13)

$$u_t(x,0) = f(x).$$

Note that equation (2.12) is the exact solution of the P.D.E. with boundaries. We are now going to non-dimensionalize equation (2.14) given below.

$$[\sinh \mathcal{L}b] H(t) = 0$$

(2.14)

$$[\cosh \mathcal{L}b] F(t) = 0.$$

Using these transformations given below

$$\eta = \frac{x}{l}, \quad l = \text{max-wavelength}$$

(2.15)

$$T = \frac{c}{l} t, \quad c \text{ is the velocity}$$

then,

$$(2.16) \quad \mathcal{L} = \frac{m}{l} \quad \text{where} \quad m = \frac{c}{C_L} \frac{d}{dT}, \quad \epsilon = \frac{b}{l}$$

and equation (2.14) becomes

$$(2.17) \quad \begin{aligned} [\sinh m\epsilon] h(T) &= 0 & H\left(\frac{l}{c} T\right) &= h(T) \\ [\cosh m\epsilon] f(T) &= 0 & F\left(\frac{l}{c} T\right) &= f(T) \end{aligned} \quad \text{where}$$

and equation (2.5) becomes

$$(2.18) \quad \bar{u}(\eta, T) = h(T) \cosh \epsilon m \eta + f(T) \sinh \epsilon m \eta \quad \text{where} \\ u(l\eta, \frac{l}{c} T) = \bar{u}(\eta, T).$$

We can use the parameter ϵ as a way to truncate equation (2.17). The approximation will be of order ϵ^{2N} ($N = 0, 1, 2, \dots$). Hence, we will proceed to consider two cases ($N = 0, 1$).

Case I ($N = 0, \epsilon^0$)

This reduces equation (2.17) to

$$(2.19) \quad \begin{aligned} h(T) &= 0 \\ m f(T) &= 0, \end{aligned}$$

then, equation (2.18) gives us $u'(x, t) = 0$ (or)

$$(2.20) \quad u(x, t) = \text{constant}.$$

Case II ($N = 1, \epsilon^2$)

This reduces equation (2.17) to

$$(2.21) \quad \begin{aligned} \left(1 + \frac{m^2 \epsilon^2}{2}\right) h(T) &= 0 & g(T) &= m f(T) \\ \left(1 + \frac{m^2 \epsilon^2}{6}\right) g(T) &= 0 & G(t) &= \mathcal{L} F(t) \end{aligned} \quad \text{where}$$

and equation (2.18) gives us

$$(2.22) \quad \bar{u}(\eta, T) = h(T) \left[1 + \frac{m^2 \eta^2 \epsilon^2}{2} \right] + g(T) \left[1 + \frac{\epsilon^2 m^2 \eta^2}{6} \right] \in \eta.$$

Using equation (2.15), equations (2.21) and (2.22) becomes

$$(2.23) \quad \begin{aligned} \left(1 + \frac{b^2 \ell^2}{2} \right) H(t) &= 0 \\ \left(1 + \frac{b^2 \ell^2}{6} \right) G(t) &= 0 \end{aligned}$$

$$u(x, t) = H(t) + G(t)bx + \ell^2 H(t) \frac{b^2 x^2}{2} + \ell^2 G(t) b^3 \frac{x^3}{6}.$$

Note the above solution is cubic in x (spatial) and the operator is a second order in t (time). Now assume a solution in the form of

$$(2.24) \quad H(t) = A \cos \omega t$$

$$G(t) = i B \sin \omega t,$$

then equation (2.23) gives us two quadratic equations in ω as follows

$$(2.25) \quad \begin{aligned} (2 - b^2 \ell^2) &= 0 \quad \text{ROOTS} \equiv \omega_1, \omega_2 \\ (6 - b^2 \ell^2) &= 0 \quad \text{ROOTS} \equiv \Omega_1, \Omega_2. \end{aligned}$$

Therefore, equation (2.23) becomes

$$(2.26) \quad \begin{aligned} H(t) &= A_1 \cos \omega_1 t + A_2 \cos \omega_2 t \\ G(t) &= B_1 \sin \Omega_1 t + B_2 \cos \Omega_2 t \end{aligned}$$

Hence equations (2.26) and (2.23) are the approximation of order $(N = 1, \epsilon^2)$.

2.3 Symbolic Method to Solve the S.H. in an Elastic Layer

The equations of motion (without body force) are

$$(2.27) \quad \sigma_{ij,j} - \rho u_{i,tt} = 0 \quad i, j = 1, 2, 3.$$

Now if the B.C. are

$$(2.28) \quad \sigma_{23}(+b, x_\alpha, t) = 0 \quad \text{"FREE VIBRATION"} ,$$

then the stresses are independent of x_3 . Therefore, we consider a two-dimensional problem which decouples equation (2.27) into antiplane shear and inplane motions. The antiplane shear motion is

$$(2.29) \quad \begin{aligned} \sigma_{3\beta,\beta} - \rho u_{3,tt}(x_1, x_2, t) &= 0, \quad \beta = 1, 2 \\ \sigma_{23}(\pm b, x_1, t) &= 0. \end{aligned}$$

The stress-displacement equations are

$$(2.30) \quad \sigma_{3\beta} = \mu u_{3,\beta} \quad \beta = 1, 2.$$

It follows that equation (2.29) becomes

$$(2.31) \quad \begin{aligned} (\partial_{x_1}^2 + \partial_{x_2}^2 - c_T^2 \partial_t^2) u_3(x_1, x_2, t) &= 0 \\ \partial_{x_2} u(x_1, \pm b, t) &= 0 \end{aligned}$$

or

$$(2.32) \quad \begin{aligned} \nabla^2 u &= 0 \\ \partial_{x_2} u(x_1 \pm b, t) &= 0 \end{aligned}$$

where

$$(2.33) \quad \mathcal{R}^2 = (c_T^{-2} \partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2)$$

$$u_3 = u(x_1, x_2, t) \quad .$$

Let ∂_t^2 and $\partial_{x_1}^2$ behave as constants then equation (2.32) can be written as follows:

$$(2.34) \quad u' - \mathcal{L}^2 u = 0 \quad (\text{S.O.D.E.})$$

$$(2.35) \quad \begin{aligned} u(x_1, 0, t) &= H(t) \\ u'(x_1, 0, t) &= \mathcal{L} F(t) \end{aligned} \quad (\text{S.I.C.})$$

where

$$(2.36) \quad \mathcal{L}^2 = (c_T^{-2} \partial_t^2 - \partial_{x_1}^2) \quad .$$

The solution for the (S.O.D.E) is

$$(2.37) \quad u(x_1, x_2, t) = a(x_1, t) \cosh \mathcal{L} x_2 + b(x_1, t) \sinh \mathcal{L} x_2$$

and by using the (S.I.C) equation (2.35) the above equation becomes

$$(2.38) \quad u(x_1, x_2, t) = H(x_1, t) \cosh \mathcal{L} x_2 + F(x_1, t) \sinh \mathcal{L} x_2 \quad .$$

Next, we substitute the above in the B.C. equation (2.32) which gives us an infinite P.D.E.,

$$(2.39) \quad H(x_1, t) \sinh \mathcal{L} b \pm F(x_1, t) \cosh \mathcal{L} b = 0$$

or

$$\begin{aligned}
 \sinh \mathcal{L}b \quad H(x_1, t) &= 0 \\
 \text{where } \mathcal{L}^2 &= (c_T^{-2} \partial_t^2 - \partial_{x_1}^2) \\
 (2.40) \quad \cosh \mathcal{L}b \quad F(x_1, t) &= 0.
 \end{aligned}$$

These infinite operators can be solved exactly by assuming an exponential solution

$$\begin{aligned}
 H(x_1, t) &= A e^{i[kx_1 - \omega t]} \\
 (2.41) \quad F(x_1, t) &= i B e^{i[kx_1 - \omega t]}.
 \end{aligned}$$

If we substitute (2.41) in (2.40), then for A and B arbitrariables, we are going to get

$$\begin{aligned}
 \frac{\omega^2}{c_T^2} - k^2 &= \frac{n\pi}{2b}, \quad n \text{ EVEN} \\
 (2.42) \quad \frac{\omega^2}{c_T^2} - k^2 &= \frac{n\pi}{2b}, \quad n \text{ ODD}.
 \end{aligned}$$

Equation (2.42) are the SH-symmetrical and SH-antisymmetrical modes of the Frequency Spectrum, which are derived in Achenback [11]. Hence, if we substitute equations (7.42) and (2.41) in equation (2.38) we get the exact solution as follows:

$$(2.43) \quad u(x_1, x_2, t) = \begin{Bmatrix} z_I(x_2) \\ z_{II}(x_2) \end{Bmatrix} \exp i(kx_1 - \omega t)$$

where

$$\begin{aligned}
 (2.44) \quad z_I(x_2) &= A \cos \frac{n\pi}{2b} x_2, \quad n \equiv \text{EVEN}, \quad \left(\frac{n\pi}{2b}\right)^2 = \frac{w^2}{c_T^2} - k^2 \\
 z_{II}(x_2) &= B \sin \frac{n\pi}{2b} x_2, \quad n \equiv \text{ODD}, \quad \left(\frac{n\pi}{2b}\right)^2 = \frac{w^2}{c_T^2} - k^2.
 \end{aligned}$$

We are now going to non-dimensionalize equation (2.40) which is

$$\begin{aligned}
 (2.45) \quad \sinh \mathcal{L}b \, H(x_1, t) &= 0 \\
 \cosh \mathcal{L}b \, F(x_1, t) &= 0.
 \end{aligned}$$

Using these transformations given below

$$\begin{aligned}
 (2.46) \quad \eta &= \frac{x_1}{b} & \epsilon &= \frac{b}{\ell} \\
 \epsilon &= \frac{x_2}{\ell} & \text{then} \quad \partial_t^2 &= \left(\frac{c}{\ell}\right)^2 \partial_T^2 \\
 T &= \frac{c}{\ell} t & \partial_{x_2}^2 &= \left(\frac{1}{\ell}\right)^2 \partial_\epsilon^2
 \end{aligned}$$

then

$$(2.47) \quad m^2 = \mathcal{L}^2 \ell^2 = \left[\left(\frac{c}{c_T}\right)^2 \partial_T^2 - \partial_\epsilon^2 \right]$$

and equation (2.45) becomes

$$\begin{aligned}
 (2.48) \quad \sinh \epsilon m \, h(\eta, T) &= 0 & H(h\eta, \frac{\ell}{c} T) &= h(\eta, T) \\
 \cosh \epsilon m \, f(\eta, T) &= 0 & F(h\eta, \frac{\ell}{c} T) &= f(\eta, T)
 \end{aligned}$$

where

and equation (2.38) becomes

$$(2.49) \quad \bar{u}(\eta, \epsilon, T) = h(\eta, T) \cosh \epsilon m \epsilon + f(\eta, T) \sinh \epsilon m \epsilon,$$

where

$$(2.50) \quad \bar{u}(\eta, \varepsilon, T) = u(b\eta, l\varepsilon, \frac{l}{c} T) .$$

The parameter ε can be used as a means to truncate equation (2.48). The approximation will be of order ε^{2N} ($N = 0, 1, 2, \dots$). Now, let us proceed by considering two cases ($N = 0, 1$).

Case I ($N = 0, \varepsilon^0 \equiv 1$)

This reduces equation (2.48) to

$$(2.51) \quad h(\eta, T) = 0$$

$$m g(\eta, T) = 0 \quad \text{where} \quad g(\eta, T) = m f(\eta, T).$$

And equation (2.49) leads to

$$(2.52) \quad \frac{\partial}{\partial \varepsilon} \bar{u}(\eta, \varepsilon, T) = 0$$

or

$$(2.53) \quad u(x_1, x_2, t) = \text{constant}.$$

Case II ($N = 1, \varepsilon^2$)

This reduces equation (2.48) to

$$(2.54) \quad \begin{aligned} (2 + m^2 \varepsilon^2) h(\eta, T) &= 0 \\ (6 + m^2 \varepsilon^2) g(\eta, T) &= 0 \end{aligned} \quad \begin{aligned} \text{where } g(\eta, t) &= m f(T) \\ G(x_1, t) &= \mathcal{L} F(x_1, t) \end{aligned}$$

and equation (2.49) gives us

$$(2.55) \quad \bar{u}(\eta, T) = h(\eta, T) \left(1 + \frac{m^2 \varepsilon^2 \varepsilon^2}{2}\right) + g(\eta, T) \left(1 + \frac{\varepsilon^2 m^2 \varepsilon^2}{6}\right) t \varepsilon .$$

By using equation (2.46), equation (2.54) and (2.55) become

$$(2.56) \quad \begin{aligned} (2 + b^2 \mathcal{L}^2) H(x_1, t) &= 0 \\ (6 + b^2 \mathcal{L}^2) G(x_1, t) &= 0 \end{aligned} \quad \text{where } \mathcal{L}^2 = c_T^2 = c_T^{-2} \partial_t^2 - \partial_x^2$$

$$(2.57) \quad u(x_1, x_2, t) = H(x_1, t) + G(x_1, t) b x_2 + \mathcal{L}^2 H(x_1, t) \frac{b^2 x_2^2}{2} + \mathcal{L}^2 G(x_1, t) \frac{b^3 x_2^3}{6}.$$

Thus, the above equation gives us the approximation of order $(N=1, \epsilon^2)$.

2.4 Symbolic Method Approximation to Solve Three-Dimensional Wave Problem

Consider the free stresses condition at $x_2 = \pm b$, then we have Navier equation and B.C. as follows:

$$(2.58) \quad c_L^2 \bar{\nabla} (\bar{\nabla} \cdot \bar{u}) - c_T^2 \bar{\nabla} \times (\bar{\nabla} \times \bar{u}) - \frac{\rho}{u} = 0$$

$$(2.58a) \quad \sigma_{2j}(x_1, \pm b, x_3, t) = 0$$

and for solutions, Fredholm vector decomposition is used in

$$(2.59) \quad \bar{u} = \bar{\nabla} \phi + \bar{\nabla} \times \bar{\psi}, \quad \bar{\nabla} \cdot \bar{\psi} = 0$$

where $\bar{u}(x_j, t)$, $j = 1, 2, 3$. From equation (2.59) the divergence and curl of the displacement vector are

$$(2.60) \quad \bar{\nabla} \cdot \bar{u} = \nabla^2 \phi, \quad \text{scalar potential}$$

$$\bar{\nabla} \times \bar{u} = -\nabla^2 \bar{\psi}, \quad \text{vector potential}$$

and by substituting the above equation in (2.58) we get

$$(2.61) \quad \begin{aligned} R_1^2 \bar{U}_1 &= 0 \quad \text{and} \quad \bar{\nabla} \times \bar{U}_1 = 0 \\ R_2^2 \bar{U}_2 &= 0 \quad \text{and} \quad \bar{\nabla} \cdot \bar{U}_2 = 0 \end{aligned}$$

where

$$(2.62) \quad \begin{aligned} R_\alpha^2 &= \nabla^2 - \frac{1}{C_\alpha^2} \frac{\partial^2}{\partial t^2}, \quad C_\alpha \equiv C_L \text{ and } c_T \text{ for } \alpha = 1, 2 \\ \nabla^2 &= \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 \end{aligned}$$

$$\bar{U}_1 = \bar{\nabla} \psi$$

$$U_2 = \bar{\nabla} \times \bar{\psi}.$$

The displacement vector (2.59) becomes

$$(2.63) \quad \bar{u}(x_j, t) = \bar{U}_1(x_j, t) + \bar{U}_2(x_j, t), \quad j = 1, 2, 3.$$

Let $\partial_t^2, \partial_{x_1}^2, \partial_{x_3}^2$ behave as constants, then equation (2.61) can be written as follows:

$$(2.65) \quad \begin{aligned} \bar{U}_\alpha^{11} - \mathcal{L}_\alpha^2 \bar{U}_\alpha &= 0 \quad (\text{S.O.D.E.}), \quad \mathcal{L}_\alpha^2 = \left[\frac{1}{c_\alpha^2} \partial_t^2 - \partial_{x_1}^2 - \partial_{x_3}^2 \right] \\ \bar{U}_\alpha(x_1, 0, x_3, t) &= \bar{F}_\alpha(x_1, x_3, t) \\ \bar{U}_\alpha^1(x_1, 0, x_3, t) &= \mathcal{L}_\alpha \bar{G}_\alpha(x_1, x_3, t) \end{aligned} \quad (\text{S.I.C.}), \quad \alpha = 1, 2$$

The solution for the (S.O.D.E.) with (S.I.C.) is

$$(2.66) \quad U_{i(\alpha)} = F_{i(\alpha)} \cosh \mathcal{L}_\alpha x_2 + G_{2(\alpha)} \sin \mathcal{L}_\alpha x_2, \quad \alpha = 1, 2$$

where $\bar{U}_{(\alpha)} = (U_{1(\alpha)}, U_{2(\alpha)}, U_{3(\alpha)})$ are the components of vector $\bar{U}_{(\alpha)}$, $\alpha = 1, 2$.

We also have two additional conditions which are the divergence of $\bar{U}_2$ and the curl of $\bar{U}_1$ to be satisfied. Hence, we let

$$\begin{aligned}
 (2.67) \quad \bar{\nabla} \cdot \bar{U}_2 &= U_{1(2),1} + U_{2(2),2} + U_{3(2),3} \\
 &= [F_{1(2),1} + F_{3(2),3} + \mathcal{L}_2 G_{2(2)}] \cosh \mathcal{L}_2 x_2 \\
 &\quad + [G_{1(2),1} + G_{3(2),3} + \mathcal{L}_2 F_{2(2)}] \sinh \mathcal{L}_2 x_2 \\
 &= 0.
 \end{aligned}$$

Thus, $(\forall x_2)$ the above equation gives us

$$\begin{aligned}
 (2.68) \quad F_{1(2),1} + F_{3(2),3} + \mathcal{L}_2 G_{2(2)} &= 0 \\
 G_{1(2),1} + G_{3(2),3} + \mathcal{L}_2 F_{2(2)} &= 0
 \end{aligned}$$

Similarly, the curl of $\bar{U}_1$ is

$$(2.69) \quad \bar{\nabla} \times \bar{U}_1 = [e_{ijk} U_{j(1),k}] \hat{e}_i = 0.$$

So for $(\forall i)$ we must have

$$(2.70) \quad e_{ijk} U_{j(1),k} = 0$$

or

$$\begin{aligned}
 (2.71) \quad F_{2(1),\beta} &= \mathcal{L}_1 G_{\beta(1)} \\
 G_{2(1),\beta} &= \mathcal{L}_1 F_{\beta(1)} \\
 F_{3(1),1} &= F_{1(1),3} \\
 G_{3(1),1} &= G_{1(1),3}
 \end{aligned} \quad , \quad \beta = 1 \text{ or } 3$$

Next let us consider the stresses equation at the plane
($x_2 = \pm b$)

$$(2.72) \quad \sigma_{2j} = \delta_{2j} \lambda [u_{1,1} + u_{2,2} + u_{3,3}] + 2\mu [u_{2,j} + u_{j,2}]$$

or

$$\sigma_{21} = \mu [u_{2,1} + u_{1,2}]$$

$$(2.73) \quad \sigma_{22} = \lambda [u_{1,1} + u_{3,3}] + (\lambda + 2\mu) u_{2,2}$$

$$\sigma_{23} = \mu [u_{3,2} + u_{2,3}]$$

and substitute equations (2.63) and (2.66) in the above equation and use the B.C. (2.58a) (at $x_2 = \pm b$) we then have

$$(2.74) \quad \sigma_{21} = \sum_{\alpha=1}^2 [\mu (F_{2(\alpha),1} + H_{1(\alpha)}) \bar{C}_{\alpha} + \mu (H_{2(\alpha),1} + \ell_{\alpha}^2 F_{1(\alpha)}) \pm \bar{S}_{\alpha}] = 0$$

$$(2.75) \quad \sigma_{23} = \sum_{\alpha=1}^2 [\mu (F_{2(\alpha),3} + H_{3(\alpha)}) \bar{C}_{\alpha} + \mu (H_{2(\alpha),3} + \ell_{\alpha}^2 F_{3(\alpha)}) \pm \bar{S}_{\alpha}] = 0$$

$$(2.76) \quad \sigma_{22} = \sum_{\alpha=1}^2 \left[[\lambda (F_{1(\alpha),1} + F_{3(\alpha),3}) + (\lambda + 2\mu) H_{2(\alpha)}] \bar{C}_{\alpha} + [\lambda (H_{1(\alpha),1} + H_{3(\alpha),3}) + (\lambda + 2\mu) \ell_{\alpha}^2 F_{2(\alpha)}] \pm \bar{S}_{\alpha} \right] = 0$$

where

$$(2.77) \quad \bar{S}_\alpha = \frac{\sinh \mathcal{L}_\alpha b}{\mathcal{L}_\alpha} \quad \alpha = 1, 2$$

$$\bar{C}_\alpha = \cosh \mathcal{L}_\alpha b$$

$$H_i(\alpha) = \mathcal{L}_\alpha G_i(\alpha) \quad .$$

At this point the extensional motion and the flexural motion are going to be considered separately in two cases.

Case I Extensional Motion

In the case of extensional motion, the following conditions must hold:

- (2.78) a) u_1, u_3 EVEN WITH x_2 -VARIABLE
 b) u_2 ODD WITH x_2 -VARIABLE
 c) $G_\alpha(1) = G_\alpha(2) \equiv 0$, $\alpha = 1, 3$
 d) $F_2(1) = F_2(2) \equiv 0$.

Then the displacement (2.66) becomes

$$(2.79) \quad u_\alpha(x_i, t) = F_{\alpha(1)} \cosh \mathcal{L}_1 x_2 + F_{\alpha(2)} \cosh \mathcal{L}_2 x_2 \quad , \quad \alpha = 1, 3$$

$$u_2'(x_i, t) = H_{2(1)} \cosh \mathcal{L}_1 x_2 + H_{2(2)} \cosh \mathcal{L}_2 x_2$$

and equations (2.68) and (2.71) are reduced to

$$(2.79a) \quad F_{1(2),1} + F_{3(2),3} + H_{2(2)} = 0$$

$$H_{2(1),\alpha} = \mathcal{L}_1^2 F_{\alpha(1)}$$

$$F_{3(1),1} = F_{1(1),3} \quad .$$

The B.C. (2.74-76) takes the following forms:

$$\begin{aligned}
(2.80) \quad \sigma_{21} &= \sum_{\alpha=1}^2 [H_2(\alpha),_1 + \mathcal{L}_\alpha^2 F_1(\alpha)] \bar{S}_\alpha = 0 \\
\sigma_{23} &= \sum_{\alpha=1}^2 [H_2(\alpha),_3 + \mathcal{L}_\alpha^2 F_3(\alpha)] \bar{S}_\alpha = 0 \\
\sigma_{22} &= \sum_{\alpha=1}^2 [\lambda(F_1(\alpha),_1 + F_3(\alpha),_3) + (\lambda + 2\mu)H_2(\alpha)] \bar{C}_\alpha = 0.
\end{aligned}$$

From equation (2.62) we have

$$\begin{aligned}
(2.81) \quad \bar{U}_1 &= \bar{\nabla} \psi, \quad \text{Pressure waves} \\
\bar{U}_2 &= \bar{\nabla} \times \bar{\psi}, \quad \text{Shear waves.}
\end{aligned}$$

Therefore, if we let $\bar{U}_1 \equiv 0$ (i.e. $F_{\alpha(1)} = G_{2(1)} = 0$), equation (2.79) can be written as

$$\begin{aligned}
(2.82) \quad u_\alpha(x_i, t) &= F_\alpha(2) \cosh \mathcal{L}_2 x_2, \quad \alpha = 1, 3 \\
u'_2(x_i, t) &= H_2(2) \sinh \mathcal{L}_2 x_2, \quad i = 1, 2, 3
\end{aligned}$$

and the infinite P.D.E. (2.80) are reduced to

$$\begin{aligned}
(2.83) \quad \sigma_{21} &= [H_2(2),_1 + \mathcal{L}_2^2 F_1(2)] \bar{S}_2 = 0 \\
\sigma_{23} &= [H_2(2),_3 + \mathcal{L}_2^2 F_3(2)] \bar{S}_2 = 0 \\
\sigma_{22} &= [\lambda(F_1(2),_1 + F_3(2),_3) + (\lambda + 2\mu)H_2(2)] \bar{C}_2 = 0
\end{aligned}$$

or

$$(2.84) \quad \left[\partial_{tt}^2 - \frac{4\mu(\lambda + 2\mu)}{\rho(\lambda + 2\mu)} (\partial_{x_1}^2 + \partial_{x_3}^2) \right] \bar{S}_1 \bar{C}_1 u_\alpha(x_1, x_3, t) = 0$$

$\alpha = 1, 3.$

For the theory of order 0 ($\bar{S}_1 = \bar{C}_1 = 1$) equation (2.84) is identical to the results of Poisson-Cauchy [3] and Hegemier-Bache [6].

Case II Flexural motion

In the case of flexural motion, the following conditions must hold:

- (2.85) a) u_1, u_3 ODD WITH x_2 -VARIABLE
 b) u_2 EVEN WITH x_2 -VARIABLE
 c) $F_{\alpha(1)} = F_{\alpha(2)} = 0$, $\alpha = 1, 3$
 d) $G_{2(1)} = G_{2(2)} = 0$.

Then the displacement (2.66) becomes

$$(2.86) \quad u_2(x_i, t) = F_{2(1)} \cosh \mathcal{L}_1 x_2 + F_{2(2)} \cosh \mathcal{L}_2 x_2$$

$$u_{\alpha}(x_i, t) = G_{\alpha(1)} \sinh \mathcal{L}_1 x_2 + G_{\alpha(2)} \sinh \mathcal{L}_2 x_2$$

$$\alpha = 1, 3$$

and

$$(2.87) \quad u'_{\alpha}(x_i, t) = H_{\alpha(1)} \cosh \mathcal{L}_1 x_2 + H_{\alpha(2)} \cosh \mathcal{L}_2 x_2$$

$$\alpha = 1, 3.$$

If we let $\bar{u}_1 = 0$, then $F_{2(1)} = G_{\alpha(1)} = 0$ ($\alpha = 1, 3$) and equation (2.86) becomes

$$(2.88) \quad u'_{\alpha}(x_i, t) = H_{\alpha(2)} \cosh \mathcal{L}_2 x_2, \quad \alpha = 1, 3$$

$$u_2(x_i, t) = F_{2(2)} \cosh \mathcal{L}_2 x_2$$

and the stresses (2.74-76) take the forms

(2.89)

$$\sigma_{21} = [F_{2(2),1} + H_1(2)] \bar{C}_2 = 0 \quad .$$

$$\sigma_{23} = [F_{2(2),3} + H_3(2)] \bar{C}_2 = 0$$

$$\sigma_{22} = [\lambda(F_{1(2),1} + F_{3(2),3}) + (\lambda + 2\mu)H_{2(2)}] \bar{C}_2 = 0.$$

The above equation can be reduced to equation (2.84).

CHAPTER III
REDUCTION OF SERIES-EXPANSION TO A
FINITE-POINTS APPROXIMATION

3.1 Introduction

In this chapter an elastic plate with thickness $2b\bar{N}$ is being analyzed. $\bar{N}$ -subdomain thickness $2b$ with local coordinates x_j ($j = 1, 2, 3$) will be related to a global coordinates $\bar{x}_j$ ($j = 1, 2, 3$) by a transformation. Hamilton's principle and the truncation procedure used in Chapter one will be applied in each subdomain. Continuity of the stresses and displacements is required at each interface dividing subdomains. Hamilton's principle gives us two non-homogeneous P.D.E. coupling the zero and first amplitudes distributions. The non-homogeneous terms are stated at the interface of each subdomain. The finite points approximation gives a discrete solution in the direction of the plate thickness (i.e. x_2 -variable). The points are located at the top, bottom and interface of each subdomain. This technique will reduce the two non-homogeneous P.D.E. to a system of Differential-Difference equation. Dispersion Relationships of Harmonic Wave will be studied for various values of $\bar{N}$, the number of subdomain.

3.2 Transformation and Partition of a Plate in (n)-Subdomain

Recall Hamilton's principle and displacements obtained in the first chapter:

$$(3.1) \quad \int_T \int_V (\sigma_{ij,j} - \rho u_{i,tt}) \delta u_k dv dt = 0, \quad i, j, k = 1, 2, 3.$$

$$(3.2) \quad u_i(\bar{x}_j, t) = \sum_{m=0}^{\infty} u_i^{(m)}(\bar{x}_\alpha, t) \cos \frac{m\pi}{2} \left(1 - \frac{\bar{x}_2}{b}\right), \quad \alpha = 1, 3.$$

We define a transformation

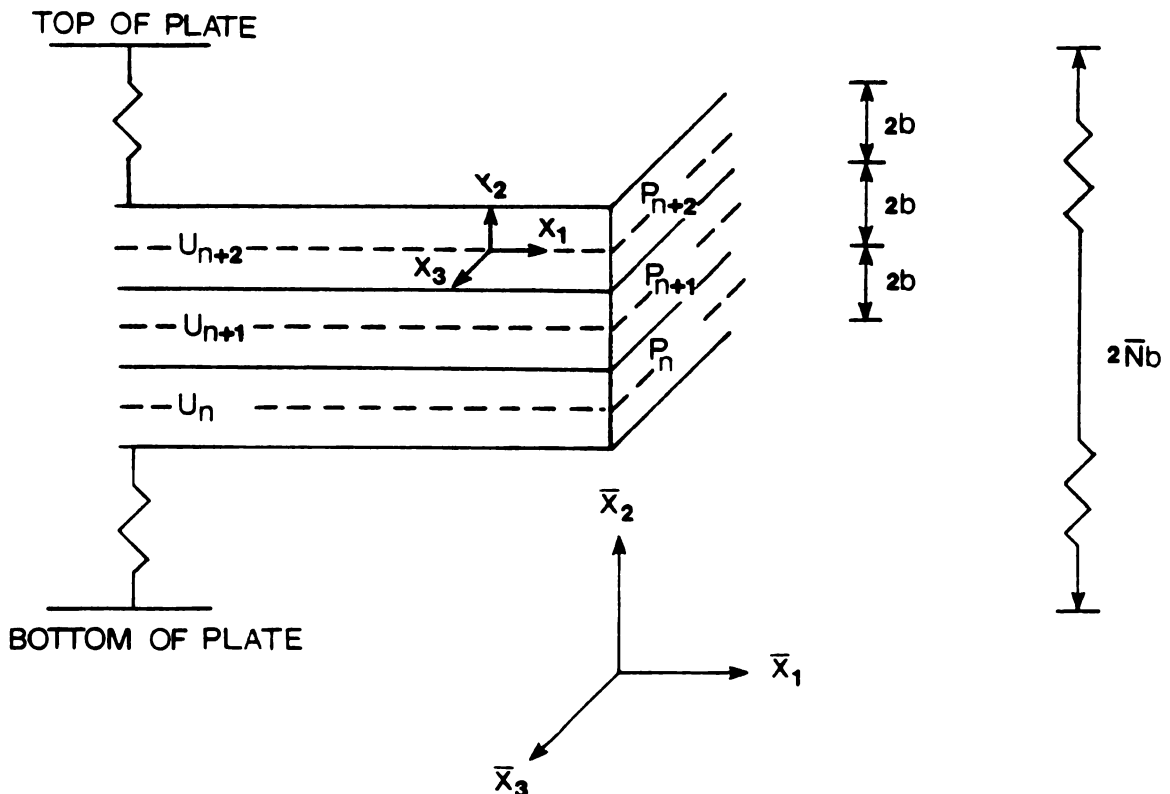
$$(3.3) \quad \bar{x}_1 = x_1$$

$$\bar{x}_2 = x_2 + P_n, \quad \text{where } P_n = 2nb$$

$$\bar{x}_3 = x_3$$

which relate the global coordinate to the local description.

The figure below will show the plate is partitioned in (n)-subdomain.



We shall proceed to analyze a plate of thickness $2b\bar{N}$. In each subdomain we have:

- a. Hamilton's principle (3.1) describing the motion.
- b. The truncation procedure of Chapter I applied to Hamilton's principle.
- c. The truncation displacements $u^{(m)} = 0, m > 1$.

Therefore equation (3.2) becomes

$$(3.4) \quad u_i(x_j, t, n) = u^{(0)}(x_\alpha, t) + \cos n\pi u^{(1)}(x_\alpha, t) \sin \frac{\pi}{2b} x_2$$

where the transformation (3.3) is used and n describes the displacements in the n -subdomain.

- d. The displacements at the interface which are continuous (i.e.)

$$(3.5) \quad u_i(x_\alpha, b, t, n) = u_i(x_\alpha, -b, t, n+1), \quad i \leq 1, 2, 3$$

$$(3.6) \quad u_i(x_\alpha, b, t, n+1) = u_i(x_\alpha, -b, t, n+2), \quad \alpha = 1, 3.$$

Similar results can be obtained for the stresses.

3.3 Truncation and Finite-Point Method Application

From definition 1.1 in Chapter I, the general truncation procedure for the $[N, 1]$ -order is

$$(3.7) \quad a) \quad u_j^{(n)} = 0 \quad n > N$$

$$b) \quad \sigma_{ij}^{(n)} = \bar{\sigma}_{ij}^{(n)} = 0 \quad n > N$$

$$c) \quad \partial_{x_i x_j}^2 (i, j = 1, 2, 3), \quad \partial_t^2 \text{ of } u_j^{(n)} \equiv 0 \\ 1 \leq n \leq N.$$

Plain Strain will also be added to further reducing the complexity of our problem, (i.e.) $u_3 \equiv 0$ and $\partial_{x_3}(\quad) \equiv 0$. A new notation is now introduced,

$$\begin{aligned}
 (3.8) \quad u^{(0)} &= u_1^{(0)}, & F^{(0)} &= F_1^{(0)} \\
 u^{(1)} &= u_1^{(1)}, & F^{(1)} &= F_1^{(1)} \\
 v^{(0)} &= u_2^{(0)}, & H^{(0)} &= F_2^{(0)} \\
 v^{(1)} &= u_2^{(1)}, & H^{(1)} &= F_2^{(1)}
 \end{aligned}$$

for the displacements and stress components. Hence, equation (3.4) becomes

$$\begin{aligned}
 (3.9) \quad u_n(x_1, x_2, t) &= u^{(0)}(x_1, t) + u^{(1)}(x_1, t) \cos n\pi \sin \frac{\pi}{2b} x_2 \\
 v_n(x_1, x_2, t) &= v^{(0)}(x_1, t) + v^{(1)}(x_1, t) \cos n\pi \sin \frac{\pi}{2b} x_2
 \end{aligned}$$

and

$$\begin{aligned}
 (3.10) \quad u_{n+1}(x_1, x_2, t) &= u^{(0)}(x_1, t) - u^{(1)}(x_1, t) \cos n\pi \sin \frac{\pi}{2b} x_2 \\
 v_{n+1}(x_1, x_2, t) &= v^{(0)}(x_1, t) - v^{(1)}(x_1, t) \cos n\pi \sin \frac{\pi}{2b} x_2.
 \end{aligned}$$

The Finite-point method at $(x_2 = b)$ reduces the above equation to

$$\begin{aligned}
 (3.11) \quad u_n &= u^{(0)} + u^{(1)} \\
 u_{n+1} &= u^{(0)} + u^{(1)} \\
 v_n &= v^{(0)} + v^{(1)} \\
 v_{n+1} &= v^{(0)} + v^{(1)}
 \end{aligned}$$

which represents points in the plates. The approximation depends on the number of subdomains. If $\bar{N}$ increases, then better results can be obtained from the above theory. Next, the inversion of equation (3.11) yields

$$\begin{aligned}
 u^{(0)} &= \frac{1}{2} [u_{n+1} + u_n] \\
 u^{(1)} &= \frac{1}{2} [u_{n+1} + u_n] \\
 v^{(0)} &= \frac{1}{2} [v_{n+1} + v_n] \\
 v^{(1)} &= \frac{1}{2} [v_{n+1} + v_n]
 \end{aligned}
 \tag{3.12}$$

which are the uniform and linear terms of the series expansion. Similarly, the stresses can be obtained as follows:

$$\begin{aligned}
 F^{(0)} &= \sigma_{n+1} - \sigma_n \\
 F^{(1)} &= \sigma_{n+1} + \sigma_n \\
 H^{(0)} &= \tau_{n+1} - \tau_n \\
 H^{(1)} &= \tau_{n+1} + \tau_n
 \end{aligned}
 \tag{3.13}$$

Next, we are going to consider a truncation for $N = 1, 2$ and $q = 1$ (which was derived in section 1.7, Chapter I) for each subdomain then the finite-point method defined above will give rise to a set of Differential-Difference equation: Case 1 ($N=1, q+1$)/the zero and first order displacement equations of motion for $[1,1]$ -order truncation are given in equations 1.31 and 1.32. Thus, if we substitute equation 3.12 in these equations, we are generating a Differential-Difference operation as follows:

$$(3.14) \quad \begin{bmatrix} 0 & A_{12} & A_{13} & 0 \\ 0 & A_{22} & A_{23} & 0 \\ A_{31} & 0 & 0 & A_{34} \\ A_{41} & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} (u_{n+1} + u_n) \\ (u_{n+1} - u_n) \\ (v_{n+1} + v_n) \\ (v_{n+1} - v_n) \end{bmatrix} +$$

$$\begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \begin{bmatrix} (\sigma_{n+1} + \sigma_n) \\ (\tau_{n+1} - \tau_n) \\ (\sigma_{n+1} - \sigma_n) \\ (\tau_{n+1} + \tau_n) \end{bmatrix} = 0$$

where

$$(3.15) \quad \begin{aligned} A_{12} &= x^2 \bar{D}_{11} + 1 - \bar{D}_{tt} \\ A_{13} &= \frac{-4}{\pi} C_1 \bar{D}_1 \\ A_{22} &= \frac{4}{\pi} C_1 \bar{D}_1 \\ A_{23} &= \bar{D}_{11} - C_2^{-1} \bar{D}_{tt} \\ A_{31} &= x^2 \bar{D}_{11} - \bar{D}_{tt} \\ A_{34} &= \frac{4}{\pi} (x^2 - 2) C_1 \bar{D}_1 \\ A_{41} &= \frac{-4}{\pi} (x^2 - 2) C_1 \bar{D}_1 \\ A_{44} &= C_2 \bar{D}_{11} + x^2 - \bar{D}_{tt} \end{aligned}$$

and

$$\begin{aligned}
 p_i &= \frac{8b}{\mu\pi^2} \quad , \quad i = 1, 2, 3, 4 \\
 \bar{D}_{11} &= \frac{4b^2}{\pi^2} \partial_{xx}^2 \\
 \bar{D}_1 &= \frac{2b}{\pi} \partial_x \\
 \bar{D}_{tt} &= \frac{4b^2}{\pi^2 C_T^2} \partial_{tt}^2 .
 \end{aligned}
 \tag{3.16}$$

CASE II (N=2, q=1)

Similarly, the zero and first order displacement equations for $N=2$ were given in equations (1.34) and (1.35). Thus, if we substitute equation (3.12) in the first two equations, we get

$$\begin{aligned}
 (3.17) \quad & \begin{bmatrix} 0 & B_{12} & A_{13} & 0 \\ 0 & A_{22} & A_{23} & 0 \\ A_{31} & 0 & 0 & A_{14} \\ A_{41} & 0 & 0 & B_{44} \end{bmatrix} \begin{bmatrix} (u_{n+1} + u_n) \\ (u_{n+1} - u_n) \\ (v_{n+1} + v_n) \\ (v_{n+1} - v_n) \end{bmatrix} + \\
 & \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & \bar{p}_4 \end{bmatrix} \begin{bmatrix} (\sigma_{n+1} + \sigma_n) \\ (\tau_{n+1} - \tau_n) \\ (\sigma_{n+1} - \sigma_n) \\ (\sigma_{n+1} + \tau_n) \end{bmatrix} = 0
 \end{aligned}$$

where

$$\begin{aligned}
 B_{12} &= \left[x^2 - \frac{4}{9\pi^2} (4x^2 - 7) \left(4 - \frac{7}{x^2} \right) \right] \bar{D}_{11} + 1 - \bar{D}_{tt} \\
 B_{44} &= \left[1 - \frac{4}{9\pi^2} (4x^2 - 7)^2 \right] \bar{D}_{11} + x^2 - \bar{D}_{tt} .
 \end{aligned}
 \tag{3.18}$$

The A_{ij} can be computed from equation (3.15) and

$$\begin{aligned} \bar{p}_1 &= \frac{8b}{\mu\pi^2} \left[1 + \frac{1}{3} \left(4 - \frac{7}{x^2} \right) \bar{D}_1 \right] \\ \bar{p}_4 &= \frac{8b}{\mu\pi^2} \left[1 + \frac{1}{3} (4x^2 - 7) \bar{D}_1 \right] \end{aligned} \quad (3.19)$$

The last equation of each of the equations (1.34) and (1.35) are used in the decoupling of higher modes contained in the first two equations of each of the equations (1.34) and (1.35). These modes can be written as

$$\begin{aligned} u_2^{(2)} &= \left(\frac{b}{\pi} \right)^2 \frac{1}{(\lambda + 2\mu)} \left[- \frac{2(4\lambda + \mu)}{3b} u_{1,1}^{(1)} + \frac{1}{b} F_2^{(0)} \right] \\ u_1^{(2)} &= \left(\frac{b}{\pi} \right)^2 \frac{1}{\mu} \left[- \frac{2(\lambda + 4\mu)}{3b} u_{2,1}^{(1)} + \frac{1}{b} F_1^{(0)} \right] \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} F_2^{(2)} &= F_2^{(0)} \\ F_1^{(2)} &= F_1^{(0)} . \end{aligned} \quad (3.21)$$

3.4 Dispersion Relationships of Harmonic Waves

Now, we investigate the dispersion relations of harmonic waves in a plate divided in $\bar{N}$ subdomains and governed by the approximate equation of motion in last section. Equations (3.14) and (3.17) correspond to a truncation [1,1] and [2,1] respectively and can be written in the form:

$$\begin{aligned}
 (3.22) \quad & \begin{bmatrix} -A_{12} & A_{12} & A_{13} & A_{13} \\ -A_{22} & A_{22} & A_{23} & A_{23} \\ A_{31} & A_{31} & -A_{34} & A_{34} \\ A_{41} & A_{41} & -A_{44} & A_{44} \end{bmatrix} \begin{bmatrix} u_n \\ u_{n+1} \\ v_n \\ v_{n+1} \end{bmatrix} \\
 & + \begin{bmatrix} p_1 & p_1 & 0 & 0 \\ 0 & 0 & -p_2 & p_2 \\ -p_3 & p_3 & 0 & 0 \\ 0 & 0 & p_4 & p_4 \end{bmatrix} \begin{bmatrix} \sigma_n \\ \sigma_{n+1} \\ \tau_n \\ \tau_{n+1} \end{bmatrix} = 0
 \end{aligned}$$

and

$$\begin{aligned}
 (3.23) \quad & \begin{bmatrix} -B_{12} & B_{12} & A_{13} & A_{13} \\ -A_{22} & A_{22} & A_{23} & A_{23} \\ A_{31} & A_{31} & -A_{34} & A_{34} \\ A_{41} & A_{41} & -B_{44} & B_{44} \end{bmatrix} \begin{bmatrix} u_n \\ u_{n+1} \\ v_n \\ v_{n+1} \end{bmatrix} \\
 & + \begin{bmatrix} \bar{p}_1 & \bar{p}_1 & 0 & 0 \\ 0 & 0 & -p_2 & p_2 \\ -p_3 & p_3 & 0 & 0 \\ 0 & 0 & \bar{p}_4 & \bar{p}_4 \end{bmatrix} \begin{bmatrix} \sigma_n \\ \sigma_{n+1} \\ \tau_n \\ \tau_{n+1} \end{bmatrix} = 0
 \end{aligned}$$

First, let us consider the case where $\bar{N} = 1$ (one domain). Then for harmonic waves propagating along the x_1 axis, we assume

$$(3.24) \quad (u_n, v_n, \tau_n, T_n) = (U_n, V_n, \Sigma_n, T_n) e^{i[kx_1 - \omega t]}.$$

Substituting this into the approximate equations of motions (3.22) we obtain a set of algebraic difference equations with constant coefficients in terms of U_n, V_n, Σ_n , and T_n where n takes the values between zero and $\bar{N}$ for a plate made of $\bar{N}$ subdomain. Boundary conditions for the dispersion relationships require both surfaces of the plate be traction free. For the case of $\bar{N} = 1$

$$(3.25) \quad \begin{aligned} \Sigma_0 &= T_0 = 0 \\ \Sigma_1 &= T_1 = 0 \end{aligned}$$

and the Differential-Difference equation (3.22) becomes

$$(3.26) \quad \begin{bmatrix} -\bar{\bar{A}}_{12}' & \bar{\bar{A}}_{12}' & \bar{\bar{A}}_{13}' & \bar{\bar{A}}_{13}' \\ -\bar{\bar{A}}_{22}' & \bar{\bar{A}}_{22}' & \bar{\bar{A}}_{23}' & \bar{\bar{A}}_{23}' \\ \bar{\bar{A}}_{31}' & \bar{\bar{A}}_{31}' & -\bar{\bar{A}}_{34}' & \bar{\bar{A}}_{34}' \\ \bar{\bar{A}}_{41}' & \bar{\bar{A}}_{41}' & -\bar{\bar{A}}_{44}' & \bar{\bar{A}}_{44}' \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ V_0 \\ V_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant represents the approximate frequency spectrum for $\bar{N} = 1$ and $[1,1]$ (the general truncation procedure); the coefficients in the above matrix are functions of $\bar{\bar{A}}_{ij}(k, \omega)$, the wavenumber and frequency. Now let us consider the cut-off frequency modes which can be shown as

$$\lim_{k \rightarrow 0} \bar{\bar{A}}_{ij}(k, \omega).$$

It reduces the coefficients of equation (3.15) to

$$\begin{aligned}\bar{\bar{A}}_{13} &= \bar{\bar{A}}_{22} = \bar{\bar{A}}_{34} = \bar{\bar{A}}_{41} = 0 \\ \bar{\bar{A}}_{23} &= \bar{\bar{A}}_{31} = \Omega^2 \\ \bar{\bar{A}}_{12} &= \Omega^2 - 1 \\ A_{44} &= \Omega^2 - x^2.\end{aligned}$$

Therefore, we can write equation (3.26) in the form:

$$(3.27) \quad \begin{bmatrix} \Omega^2 - 1 & 0 & 0 & 0 \\ 0 & 0 & \Omega^2 & 2\Omega^2 \\ \Omega^2 & 2\Omega^2 & 0 & 0 \\ 0 & 0 & \Omega^2 - x^2 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ v_0 \\ v_1 \end{bmatrix} = 0.$$

The determinant for the above matrix leads to

$$\Omega^4 [\Omega^2 - 1] [\Omega^2 - x^2] = 0$$

which is an equation of order eight. Thus the roots will correspond to the lowest cut-off modes of uniform and linear amplitudes distributions. These are

$$(3.28) \quad \begin{aligned} \Omega_{1,2} &= 0 & \equiv u^{(0)} & \quad \text{SYM} \equiv \text{EXTENSIONAL} \\ \Omega_{3,4} &= 0 & \equiv v^{(0)} & \quad \text{AN-SYM} \equiv \text{TRANSVERSAL} \\ \Omega_{5,6} &= \pm 1 & \equiv u^{(1)} & \quad \text{AN-SYM} \equiv \text{THICK-SHEAR} \\ \Omega_{7,8} &= \pm x & \equiv v^{(1)} & \quad \text{SYM} \equiv \text{THICK-STRETCH.} \end{aligned}$$

Next using equation (3.23) as the approximate equation of motion, then equation (3.24) for $\bar{N} = 1$ leads to

$$(3.29) \quad \begin{bmatrix} -\bar{B}_{12}', & \bar{B}_{12}', & \bar{A}_{13}', & \bar{A}_{13}' \\ -\bar{A}_{22}', & \bar{A}_{22}', & \bar{A}_{23}', & \bar{A}_{23}' \\ \bar{A}_{31}', & \bar{A}_{31}', & -\bar{A}_{34}', & \bar{A}_{34}' \\ \bar{A}_{41}', & \bar{A}_{41}', & -\bar{B}_{44}', & \bar{B}_{44}' \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ V_1 \\ V_2 \end{bmatrix} = 0$$

and represents the approximate Frequency Spectrum for $\bar{N} = 1$ and [2,1] (the general truncation procedure). The cut-off frequency modes (wavenumber $\rightarrow 0$) which can be shown as

$$(3.30) \quad \lim_{k \rightarrow 0} \bar{A}_{ij} \quad \text{and} \quad \lim_{k \rightarrow 0} \bar{B}_{12} \quad \text{and} \quad \bar{B}_{44}.$$

But we have

$$(3.31) \quad \begin{aligned} \lim_{k \rightarrow 0} [\bar{B}_{12} &= \bar{A}_{12}] \\ \lim_{k \rightarrow 0} [\bar{B}_{44} &= \bar{A}_{44}], \end{aligned}$$

therefore equations (3.26) and (3.29) become identical and the same cut-off modes can be generated.

Consequently, for a general truncation of order [N,1] and $\bar{N} = 1$, the exact cut-off frequency can also be derived. If $\bar{N}$ (the number of subdomain) is increased, then higher modes will be generated by this process.

Next, we consider $\bar{N} = 2$ (two subdomain), then equation (3.23) becomes

$$(3.32) \quad \begin{bmatrix} -B_{12} & B_{12} & 0 & A_{13} & A_{13} & 0 & \bar{p}_1 & 0 \\ -A_{22} & A_{22} & 0 & A_{23} & A_{23} & 0 & 0 & p_2 \\ A_{31} & A_{31} & 0 & -A_{34} & A_{34} & 0 & p_3 & 0 \\ A_{41} & A_{41} & 0 & -B_{44} & B_{44} & 0 & 0 & \bar{p}_4 \\ 0 & -B_{12} & B_{12} & 0 & A_{13} & A_{13} & \bar{p}_1 & 0 \\ 0 & -A_{22} & A_{22} & 0 & A_{23} & A_{23} & 0 & -p_2 \\ 0 & A_{31} & A_{31} & 0 & -A_{34} & A_{34} & -p_3 & 0 \\ 0 & A_{41} & A_{41} & 0 & -B_{44} & B_{44} & 0 & \bar{p}_4 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ v_0 \\ v_1 \\ v_2 \\ \sigma_1 \\ T_1 \end{bmatrix} = 0 .$$

Using a harmonic wave propagating along the x_1 axis (equation 3.24) and considering the cut-off frequency modes ($k \rightarrow 0$), then the determinant of equation (3.32) can be written in the form

$$(3.33) \quad \begin{bmatrix} \Omega^2 - 1 & 0 & 0 & 0 & 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & \Omega^2 & 2\Omega^2 & 0 & 0 & p_1 \\ \Omega^2 & 2\Omega^2 & 0 & 0 & 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & \Omega^2 - x^2 & 0 & 0 & 0 & p_1 \\ 0 & \Omega^2 - 1 & 0 & 0 & 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & 0 & \Omega^2 & 2\Omega^2 & 0 & -p_1 \\ 0 & \Omega^2 & 2\Omega^2 & 0 & 0 & 0 & -p_1 & 0 \\ 0 & 0 & 0 & 0 & \Omega^2 - x^2 & 0 & 0 & p_1 \end{bmatrix} = 0 .$$

This generates six cut-off frequency modes. Hence, a plate divided into two subdomains will give rise to two additional modes corresponding to extensional and flexural motions respectively.

Finally, we are going to study the Frequency Spectrum derived by the technique of this chapter. The approximate Frequency Spectrum for $\bar{N} = 1$ is obtained by taking the determinant of equation (3.26) which is

$$(3.34) \quad \det. \equiv \bar{A}_{41} \bar{A}_{12} \bar{A}_{23} \bar{A}_{34} - \bar{A}_{41} \bar{A}_{13} \bar{A}_{22} \bar{A}_{34} + \bar{A}_{44} \bar{A}_{31} \bar{A}_{12} \bar{A}_{23} - \bar{A}_{44} \bar{A}_{31} \bar{A}_{13} \bar{A}_{22} = 0.$$

The above coefficients are defined in equation (3.15).

The dispersion curves of equation (3.34) are computed and shown in Figures (3.1). Hence, for $\bar{N} = 1$, we generate the first and second dispersion curves of extensional and flexural motion.

3.5 Longitudinal Strain Problem

Due to the symmetry in the x_1, x_3 coordinates, the displacements $u_1 = u_3 = 0$ and $u_2 = u_2(x_2, t)$. Thus, the stresses become

$$\sigma_{22} = (\lambda + 2\mu) u_{2,2}$$

$$\sigma_{11} = \sigma_{33} = \lambda u_{2,2}$$

and Hamilton's principle in section (3.2) becomes

$$(3.35) \quad \int_{-b}^b \int_T^T (\sigma_{22,2} - \rho u_{2,tt}) \delta u_2 dl dt = 0$$

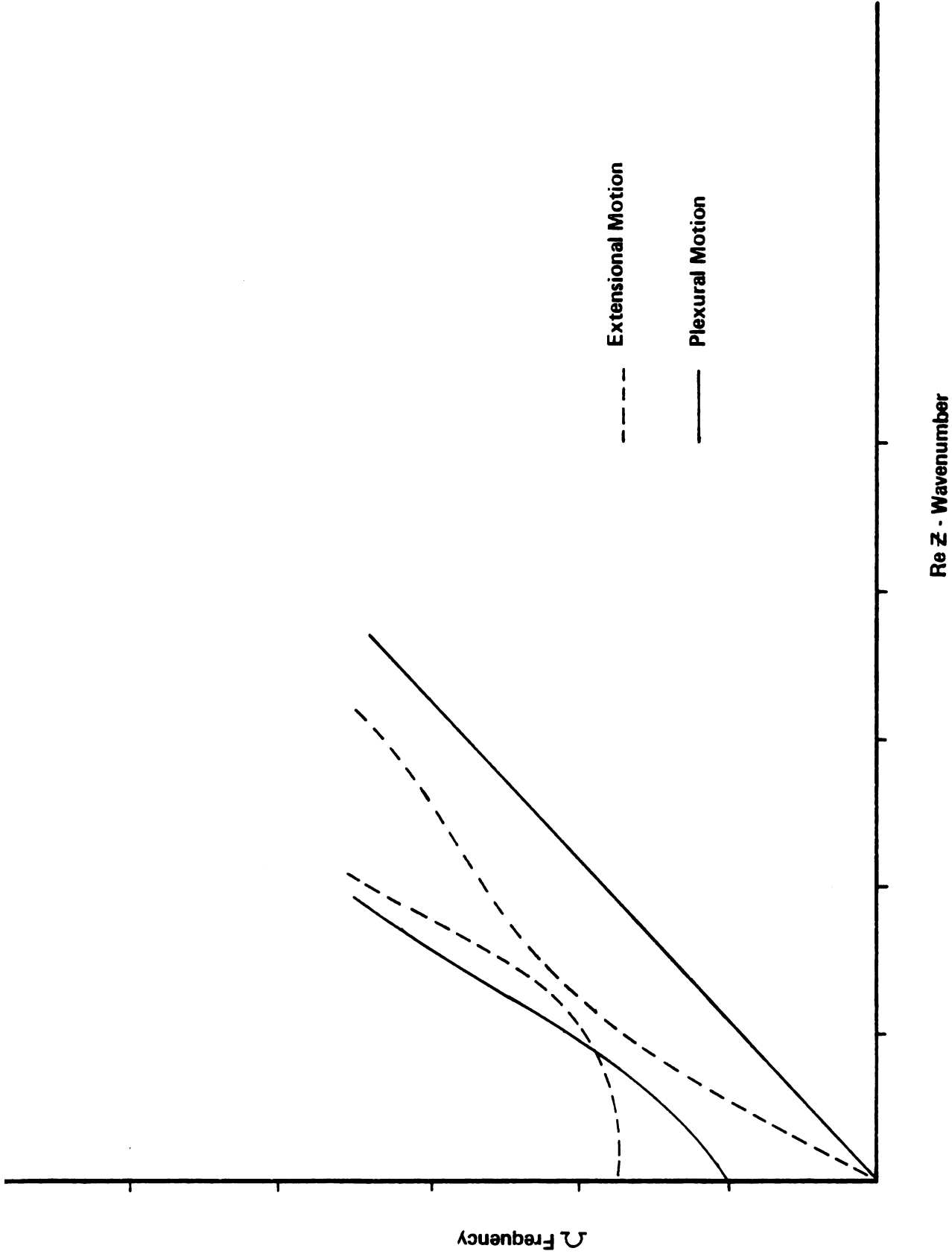
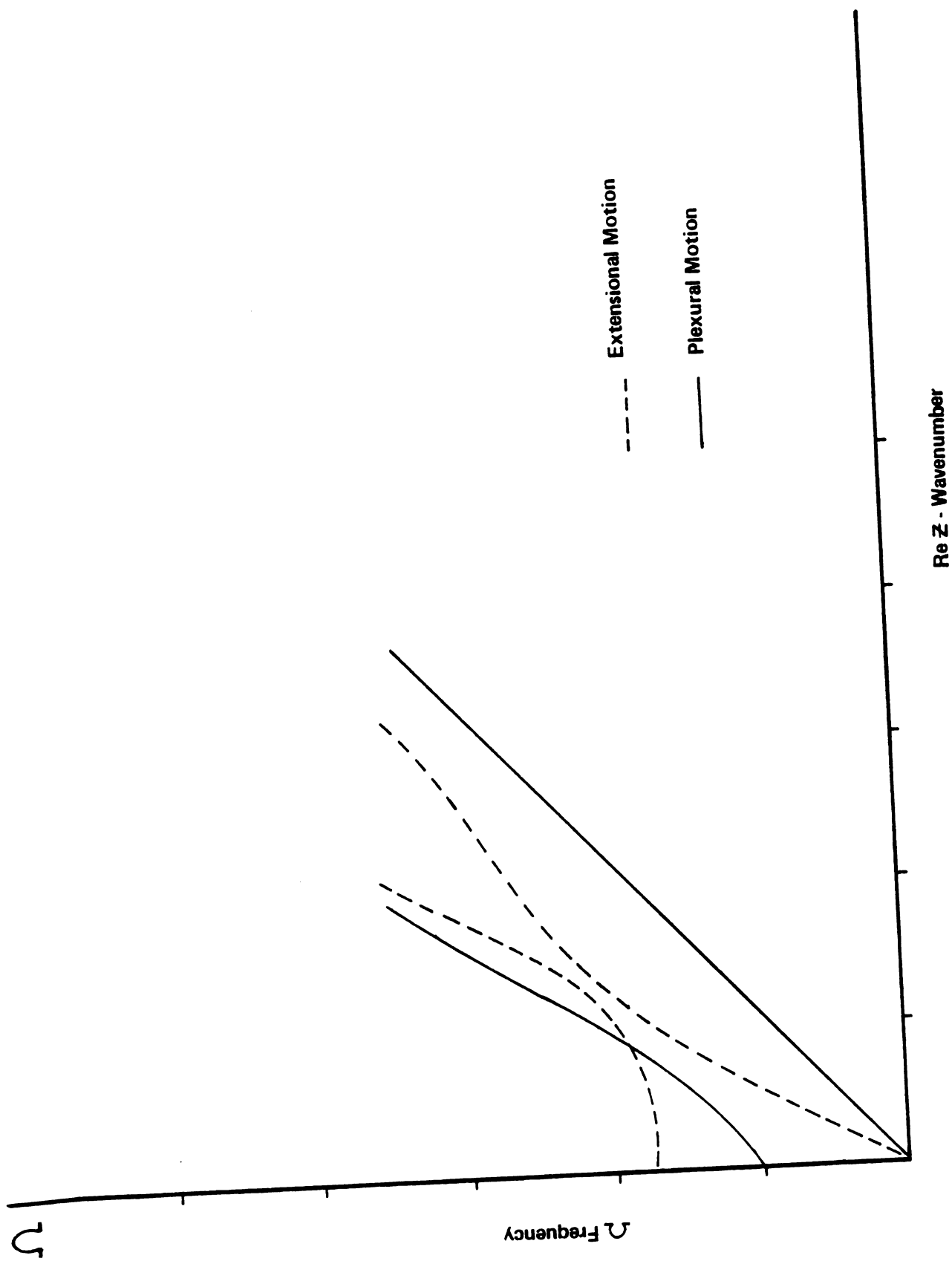


Figure 3.1



Re Z - Wavenumber

Figure 3.1

and the displacements of equation (3.2) are reduced to

$$(3.36) \quad u_2(\bar{x}_2, t) = \sum_{n=0}^{\infty} u_2^{(n)}(t) \cos \frac{n\pi}{2} \left(1 - \frac{\bar{x}_2}{b}\right).$$

Next, the substitution of equation (3.30) in equation

(3.35) gives $(\forall n)$

$$(3.37) \quad \ddot{v}^{(n)}(t) + \alpha_n^2 c_2^2 v^{(n)}(t) + F^{(n)}(t) = 0, \quad \alpha_n = \frac{n\pi}{2b}$$

$$, \quad c_2 = \frac{\lambda + 2\mu}{\rho}$$

where

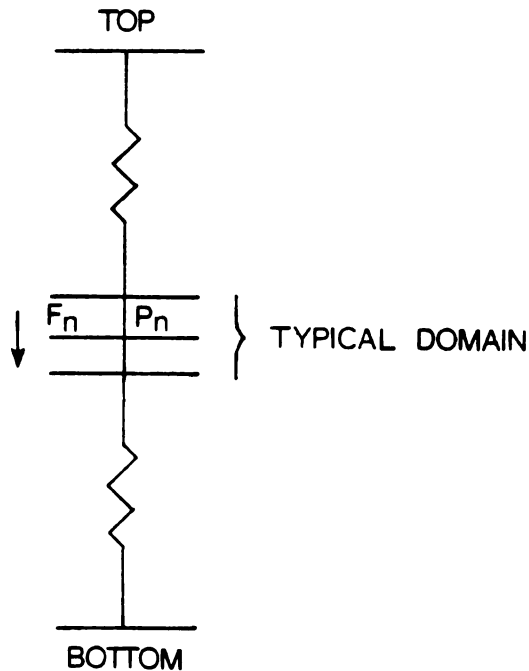
$$(3.38) \quad F^{(n)}(t) = \frac{1}{b} [\sigma_x(b, t) + (-1)^{n+1} \sigma_x(-b, t)]$$

which represents the (n) -order displacements equation of motion.

The transformation of coordinates (3.3) takes the form

$$(3.39) \quad \bar{x}_2 = x_2 + P_n, \quad P_n = 2nb.$$

GRAPH



Again, we shall proceed to analyze a plate of thickness $2b\bar{N}$ with spacial variable x_2 and time. In each subdomain we have

- a) Hamilton's principle (3.35) describing the motion.
- b) The truncation procedure of Chapter I as

$$v^{(n)}(t) = 0, \quad n > 1$$

where
$$v_2^{(n)}(t) = v^{(n)}(t).$$

Setting the condition $v^{(n)} = 0 \quad n > 1$ in equation (3.37) and equation (3.38), we get

$$(3.40) \quad \ddot{v}^{(0)}(t) + F^{(0)}(t) = 0$$

$$\ddot{v}^{(1)}(t) + c_{21}^2 v^{(1)}(t) + F^{(1)}(t) = 0$$

where

$$F^{(0)}(t) = \frac{1}{b} [\sigma_x(b, t) - \sigma_x(-b, t)]$$

$$F^{(1)}(t) = \frac{1}{b} [\sigma_x(b, t) + \sigma_x(-b, t)].$$

Similarly, equation (3.36) becomes

$$(3.41) \quad v_n(x, t) = v^{(0)}(t) + (-1)^n v^{(1)}(t) \sin \frac{\pi}{2b} x$$

where the transformation in equation (3.39) is used and n describes the displacements in the (n) -subdomain.

The finite-point method at $x_2 = b$ reduces equation (3.41) to

$$(3.42) \quad \begin{aligned} v_n &= v^{(0)} + v^{(1)} \\ v_{n+1} &= v^{(0)} + v^{(1)} \end{aligned}$$

and the inversion of above equation is

$$(3.43) \quad \begin{aligned} v^{(0)} &= \frac{1}{2} [v_{n+1} + v_n] \\ v^{(1)} &= \frac{1}{2} [v_{n+1} - v_n]. \end{aligned}$$

Similarly, the stress can be derived as follows:

$$(3.44) \quad \begin{aligned} F^{(1)}(t) &= \frac{1}{b} (\sigma_{n+1} + \sigma_n) \\ F^{(0)}(t) &= \frac{1}{b} (\sigma_{n+1} - \sigma_n) \end{aligned}$$

Substituting equations (3.43) and (3.44) in zero and first order displacement equations of motion (3.40) yields a set of differential-difference equations:

$$(3.45) \quad \begin{aligned} \mathcal{L}^2 [v_{n+1} + v_n] + \frac{2}{b} [\sigma_{n+1} - \sigma_n] &= 0 \\ L^2 [v_{n+1} - v_n] + \frac{2}{b} [\sigma_{n+1} - \sigma_n] &= 0 \end{aligned}$$

where

$$(3.46) \quad \begin{aligned} \mathcal{L}^2 &= \frac{d^2}{dt^2}, \quad c = \frac{\pi}{2b} c_L \\ L^2 &= \mathcal{L}^2 + c^2. \end{aligned}$$

Next, we investigate this dispersion relationship of harmonic waves governed by the Differential-Difference equation (3.45) by assuming a harmonic wave in time.

$$(3.47) \quad [v_n, \tau_n] = [V_n, T_n] e^{i\omega t},$$

then (3.45) becomes a Difference equation as follows:

$$(3.48) \quad x^2 [v_{n+1} + v_n] + (-1)^n \frac{2}{b} [T_{n+1} - T_n] = 0$$

$$\Omega^2 [v_{n+1} - v_n] + (-1)^n \frac{2}{b} [T_{n+1} - T_n] = 0$$

where $\Omega^2 = (x^2 - c^2)$.

The Boundary conditions for the dispersion relationship require that both surfaces of the plate be traction free, i.e.,

$$(3.49) \quad T_0 = 0$$

$$T_{\bar{N}} = 0.$$

Three cases are going to be considered, a) $\bar{N} = 1$ which corresponds to one region. b) $\bar{N} = 2$ and c) $\bar{N} = 3$.

Case 1 ($\bar{N} = 1$)

The Boundary condition (3.49) is reduced to

$$(3.50) \quad T_0 = 0$$

$$T_1 = 0 \quad \text{"FREE VIBRATION"}$$

and substituting the above in the Difference equation

(3.46) yields

$$\begin{aligned}
 (3.51) \quad & \omega^2 [v_1 + v_0] = 0 \\
 & \Omega^2 [v_1 - v_0] = 0, \quad \Omega^2 = (\omega^2 - c^2), \\
 & c = \frac{\pi}{2b} c_L
 \end{aligned}$$

or

$$(3.52) \quad \begin{bmatrix} \omega^2 & \omega^2 \\ -\Omega^2 & \Omega^2 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The determinant of the above matrix is

$$(3.53) \quad \det. = \omega^2 (\omega^2 - c^2) = 0.$$

The roots will correspond to the first and second modes of vibrations. These are

$$\begin{aligned}
 (3.54) \quad & \omega_{1,2} = 0 \\
 & \omega_{3,4} = \pm c \quad \text{where} \quad c = \frac{\pi}{2b} c_L.
 \end{aligned}$$

Case II ($\bar{N} = 2$)

The Boundary condition (3.49) is reduced to

$$\begin{aligned}
 (3.55) \quad & T_0 = 0 \\
 & T_2 = 0
 \end{aligned}$$

and substituting the equation (3.47) in the Difference equation (3.45) yields

$$\begin{aligned}
 (3.56) \quad & b\omega^2[v_1 + v_0] - 2 T_1 = 0 \\
 & b\omega^2[v_2 + v_1] + 2 T_1 = 0 \\
 & b\Omega^2[v_1 - v_0] - 2 T_1 = 0 \\
 & b\Omega^2[v_2 - v_1] - 2 T_1 = 0
 \end{aligned}$$

or

$$(3.57) \quad \begin{bmatrix} \omega^2 & \omega^2 & 0 & -1 \\ 0 & \omega^2 & \omega^2 & 1 \\ -\Omega^2 & \Omega^2 & 0 & -1 \\ 0 & -\Omega^2 & \Omega^2 & -1 \end{bmatrix} \begin{bmatrix} b v_0 \\ b v_1 \\ b v_2 \\ 2 T_1 \end{bmatrix} = 0 .$$

The determinant of the above matrix is

$$\det. = \omega^2(\omega^2 - c^2)(2\omega^2 - c^2) = 0.$$

The roots:

$$\begin{aligned}
 (3.58) \quad & \omega_{1,2} = 0 \\
 & \omega_{3,4} = \pm c
 \end{aligned}$$

correspond to the first and second modes of vibrations
and

$$(3.59) \quad \omega_{5,6} = \pm \frac{c}{2}$$

is a damping frequency due to the non-homogeneous term of
equation (3.48).

Case III ($\bar{N} = 3$)

The B.C. (3.49) is reduced to

$$(3.60) \quad T_0 = 0$$

$$T_3 = 0$$

and substituting equation (3.56) in equation (3.46) gives

$$(3.61) \quad \begin{bmatrix} x^2 & x^2 & 0 & 0 & -1 & 0 \\ 0 & x^2 & x^2 & 0 & 1 & -1 \\ 0 & 0 & x^2 & x^2 & 0 & -1 \\ -\Omega^2 & \Omega^2 & 0 & 0 & -1 & 0 \\ 0 & -\Omega^2 & \Omega^2 & 0 & -1 & -1 \\ 0 & 0 & -\Omega^2 & \Omega^2 & 0 & -1 \end{bmatrix} \begin{bmatrix} b v_0 \\ b v_1 \\ b v_2 \\ b v_3 \\ 2 T_1 \\ 2 T_2 \end{bmatrix} = 0.$$

The determinant of the above is a polynomial of order eight and its roots represent the naturals and damping frequency of our system. Hence, the determinant is

$$\det. \equiv x^2 (4x^2 - c^2) (4x^2 - 3c^2) (x^2 - c^2) = 0.$$

The lowest and highest roots are

$$(3.61.1) \quad \omega_{12} = 0$$

$$\omega_{34} = \pm c, \quad c = \frac{\pi}{2b}$$

and correspond to the first and second modes of vibrations.

The remaining roots are

$$(3.61.2) \quad \begin{aligned} x_{56} &= \pm c/2 \\ u_{78} &= \pm \sqrt{3}c/2 . \end{aligned}$$

They are the damping frequency modes.

CHAPTER IV

TRANSIENT WAVES DUE TO IMPACT

In this chapter impact boundary conditions are applied at the top of a plate. Free stresses are applied at the bottom of the plate. Similar conditions had been used by Kim-Moon [10] but in composite plates. The Differential-Difference equations obtained in Chapter III are reduced to difference equation using symbolic technique. A solution for this difference equation for various $\bar{N}$ (the number of subdomain) will be developed under plain strain and longitudinal strain conditions. Then, application of this solution to the impact boundary conditions mentioned above all give rise to a set of P.D.E. involving the x_1 and t variables. Various numerical techniques can be employed to solve these P.D.E. such as transform techniques and computer application.

4.2 Plane Strain Transient Wave Due to Impact

In section 1.7 of Chapter 1, we found the six P.D.E. needed for [1,1] and [2,1]. In Chapter III we reduced these six P.D.E. to a Differential-Difference equation. These two cases will be studied under impact conditions. In the case of [1,1] the Differential-Difference equations (3.14) are:

$$(4.1) \quad \begin{bmatrix} 0 & A_{12} & A_{13} & 0 \\ 0 & A_{22} & A_{23} & 0 \\ A_{31} & 0 & 0 & A_{34} \\ A_{41} & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} b(u_{n+1} + u_n) \\ b(u_{n+1} - u_n) \\ b(v_{n+1} + v_n) \\ b(v_{n+1} - v_n) \end{bmatrix} + \begin{bmatrix} 2(\sigma_{n+1} + \sigma_n) \\ 2(\tau_{n+1} - \tau_n) \\ 2(\sigma_{n+1} - \sigma_n) \\ 2(\tau_{n+1} - \tau_n) \end{bmatrix} = 0$$

where $D_{11} = \partial_x^2$, $D_{tt} = \partial_{tt}^2$ and

$$(4.2) \quad \begin{aligned} A_{12} &= (\lambda + 2\mu) D_{11} - \mu \left(\frac{\pi}{2b}\right)^2 - \rho D_{tt} \\ A_{13} &= \frac{-2\mu}{b} c_1 D_1 \\ A_{22} &= \frac{2\mu}{b} c_1 D_{13} \\ A_{23} &= \mu D_{11} - \rho c_2^{-1} D_{tt} \\ A_{31} &= (\lambda + 2\mu) D_{11} - \rho D_{tt} \\ A_{34} &= \frac{2\lambda}{b} c_1 D_1 \\ A_{41} &= \frac{-2\lambda}{b} c_1 D_1 \\ A_{44} &= \mu D_{11} - (\lambda + 2\mu) \left(\frac{\pi}{2b}\right)^2 - \rho c_2^{-1} D_{tt} \end{aligned}$$

and

$$(4.3) \quad \begin{aligned} F^{(0)} &= \sigma_{n+1} - \sigma_n \\ F^{(1)} &= \sigma_{n+1} + \sigma_n \\ H^{(0)} &= \tau_{n+1} - \tau_n \\ H^{(1)} &= \tau_{n+1} + \tau_n \end{aligned}$$

Let ∂_t and ∂_x behave as constants (Symbolic technique) then the Differential-Difference equation (4.1) becomes a Difference equation. Assume a solution in the form

$$(4.4) \quad [u_n, v_n, \sigma_n, T_n] = [A, B, C, D] e^{in\theta}.$$

Then equation (4.1) becomes

$$(4.5) \quad \begin{bmatrix} 0 & A_{12} & A_{13} & 0 \\ 0 & A_{22} & A_{23} & 0 \\ A_{31} & 0 & 0 & A_{34} \\ A_{41} & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} b(1+e^{i2\theta}) A \\ b(1-e^{i2\theta}) A \\ b(1+e^{i2\theta}) B \\ b(1-e^{i2\theta}) B \end{bmatrix} + \begin{bmatrix} 2(1+e^{i2\theta}) C \\ 2(1-e^{i2\theta}) D \\ 2(1-e^{i2\theta}) C \\ 2(1+e^{i2\theta}) D \end{bmatrix} = 0$$

or

$$(4.6) \quad \begin{bmatrix} (a_2 + b_2 z), & (a_3 + b_3 z), & 0, & 0 \\ (a_4 + b_4 z), & (a_1 + b_1 z), & 0, & 0 \\ 0, & 0, & -(1-z), & 0 \\ 0, & 0, & 0, & (1+z) \end{bmatrix} \begin{bmatrix} b A \\ b B \\ 2 C \\ 2 D \end{bmatrix} = 0$$

where

$$(4.7) \quad z = e^{i2\theta}$$

and

$$\begin{aligned}
 (4.8) \quad a_1 &= A_{41} - A_{44} = -[c_2 \mu D_{11} - (\lambda + 2\mu) \left(\frac{\pi}{2b}\right)^2 + \frac{2\lambda}{b} c_1 D_1 - \rho D_{tt}] \\
 a_2 &= A_{12} - A_{13} = [(\lambda + 2\mu) D_{11} - \mu \left(\frac{\pi}{2b}\right)^2 + \frac{2\mu}{b} c_1 D_1 - \rho D_{tt}] \\
 a_3 &= A_{31} - A_{34} = [(x + 2\mu) D_{11} - \frac{2\lambda}{b} c_1 D_1 - \rho D_{tt}] \\
 a_4 &= A_{22} - A_{23} = -[\mu D_{11} - \frac{2\mu}{b} c_1 D_1 - \rho c_2^{-1} D_{tt}]
 \end{aligned}$$

$$\begin{aligned}
 (4.9) \quad b_1 &= A_{41} + A_{44} = [\mu D_{11} - (\lambda + 2\mu) \left(\frac{\pi}{2b}\right)^2 - \frac{2\lambda}{b} c_1 D_1 - \rho D_{tt}] \\
 b_2 &= A_{12} + A_{13} = [(x + 2\mu) D_{11} - \mu \left(\frac{\pi}{2b}\right)^2 - \frac{2\mu}{b} c_1 D_1 - \rho D_{tt}] \\
 b_3 &= A_{31} + A_{34} = [(\lambda + 2\mu) D_{11} + \frac{2\lambda}{b} c_1 D_1 - \rho D_{tt}] \\
 b_4 &= A_{22} + A_{23} = [\mu D_{11} + \frac{2\mu}{b} c_1 D_1 - \rho c_2^{-1} D_{tt}].
 \end{aligned}$$

The determinant of equation (4.6) becomes

$$(4.10) \quad \det. \equiv (1 - z^2)(az^2 + dz + c) = 0$$

where

$$\begin{aligned}
 (4.11) \quad a &= [b_1 b_2 - b_3 b_4] \\
 d &= [a_1 b_2 + a_2 b_1 - a_3 b_4 - a_4 b_3] \\
 c &= [a_1 a_2 - a_3 a_4]
 \end{aligned}$$

The roots of equation (4.10) give us

$$\begin{aligned}
 (4.12) \quad z_1 &= -1 \\
 z_2 &= 1 \\
 z_3 &= \text{Re}^{i\psi} \\
 z_4 &= \text{Re}^{-i\psi}
 \end{aligned}$$

where

$$(4.13) \quad R = \sqrt{\alpha^2 + \beta^2} \quad \text{and} \quad \alpha = -d$$

$$\tan \psi = \frac{\beta}{\alpha} \quad \beta = (4ac - d^2)^{1/2}.$$

Recall equation (4.7) which is $z = e^{i2\theta}$, then equation (4.17) leads to

$$(4.14) \quad e^{i2\theta} = \pm 1$$

$$e^{i2\theta} = \text{Re}^{\pm i\psi}$$

or

$$(4.15) \quad \begin{aligned} \theta_1 &= 0 & \text{where} \\ \theta_2 &= \pi/2 & \alpha^2 + \beta^2 = 1 \\ \theta_3 &= \psi/2 & \tan \psi = \frac{\beta}{\alpha} \\ \theta_4 &= -\psi/2 & \alpha = \cos \psi. \end{aligned}$$

At this point we can observe that $\theta_i (i=1,2,3,4)$ are the four independent roots necessary for solution (4.4). Hence, we can write this solution in the form

$$(4.16) \quad [u_n, v_n, \sigma_n, \tau_n] = [A_1, B_1, C_1, D_1] + [A_2 B_2 C_2 D_2] e^{in\pi}$$

$$+ [A_3 B_3 C_3 D_3] e^{in\psi} + [A_4 B_4 C_4 D_4] e^{-in\psi}$$

where $\alpha = \cos \psi$ is an infinite operator resulting from the Symbolic technique.

Now, by substituting equation (4.16) in equation (4.4) we will have a relationship between the arbitrary constants as follows:

(4.17)

$$\begin{aligned}
& \begin{bmatrix} [a_2 + b_2 z_1], [a_3 + b_3 z_1], & 0 & , & 0 \\ [a_4 + b_4 z_1], [a_1 + b_1 z_1], & 0 & , & 0 \\ 0 & , & 0 & , & -(1-z_1), & 0 \\ 0 & , & 0 & , & 0 & , & +(1+z_1) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix} + \\
& \begin{bmatrix} [a_2 + b_2 z_2], [a_3 + b_3 z_2], & 0 & , & 0 \\ [a_4 + b_4 z_2], [a_1 + b_1 z_2], & 0 & , & 0 \\ 0 & , & 0 & , & -(1-z_2), & 0 \\ 0 & , & 0 & , & 0 & , & (1+z_2) \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix} e^{in\pi} + \\
& \begin{bmatrix} [a_2 + b_2 z_3], [a_3 + b_3 z_3], & 0 & , & 0 \\ [a_4 + b_4 z_3], [a_1 + b_1 z_3], & 0 & , & 0 \\ 0 & , & 0 & , & -(1-z_3), & 0 \\ 0 & , & 0 & , & 0 & , & (1+z_3) \end{bmatrix} \begin{bmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{bmatrix} e^{in\psi} + \\
& \begin{bmatrix} [a_2 + b_2 z_4], [a_3 + b_3 z_4], & 0 & , & 0 \\ [a_4 + b_4 z_4], [a_1 + b_1 z_4], & 0 & , & 0 \\ 0 & , & 0 & , & -(1-z_4), & 0 \\ 0 & , & 0 & , & 0 & , & (1+z_4) \end{bmatrix} \begin{bmatrix} A_4 \\ B_4 \\ C_4 \\ D_4 \end{bmatrix} e^{-in\psi} = 0
\end{aligned}$$

where

$$z_1 = 1$$

$$z_2 = -1$$

(4.18)

$$z_3 = e^{i\psi}$$

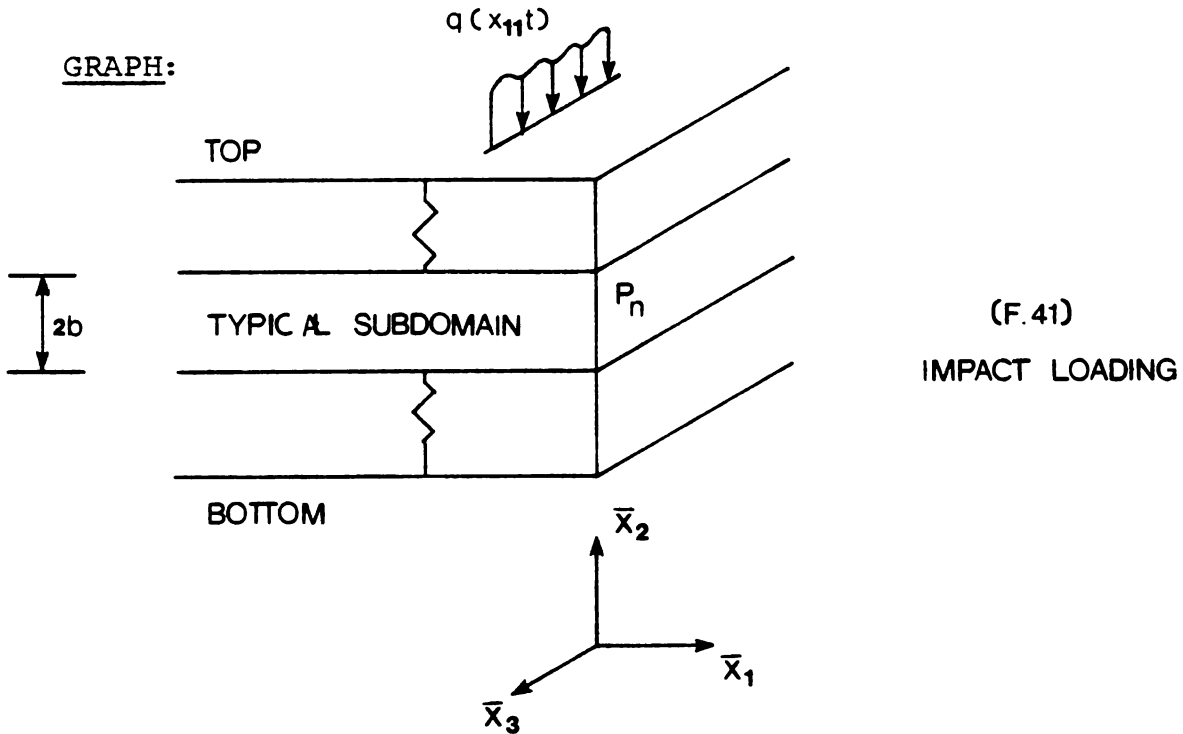
$$z_4 = e^{-i\psi}.$$

Next, we will examine the impact problem where a load is going to be applied at the top of the plate in the following manner:

$$(4.19) \quad \begin{aligned} \sigma_0(x_1, t) &= q(x_1, t) ; & |x_1| < x_0, & \quad 0 < t < t_0 \\ \sigma_0 &= 0 & ; & \quad |x_1| > x_0 \quad \text{or} \quad t > t_0 \end{aligned}$$

which is a transversal loading. The other stresses are

$$(4.20) \quad \sigma_{\bar{N}+1} = \tau_0 = \tau_{\bar{N}+1} = 0.$$



Using solution (4.16) which is

$$(4.21) \quad \begin{aligned} [u_n, v_n, \sigma_n, \tau_n] &= [A_1 B_1 C_1 D_1] + [A_2 B_2 C_2 D_2] e^{in\pi} \\ &+ [A_3 B_3 C_3 D_3] e^{in\psi} + [A_4 B_4 C_4 D_4] e^{-in\psi} \end{aligned}$$

and apply the B.C. equation (4.19) to it, we get

$$\begin{aligned}
C_1 + C_2 + C_3 + C_4 &= q(x_1, t) \\
D_1 + D_2 + D_3 + D_4 &= 0 \\
(4.22) \quad C_1 + C_2 e^{i[\bar{N}+1]\pi} + C_3 e^{i[\bar{N}+1]\psi} + C_4 e^{-i[\bar{N}+1]\psi} &= 0 \\
D_1 + D_2 e^{i[\bar{N}+1]\pi} + D_3 e^{i[\bar{N}+1]\psi} + D_4 e^{-i[\bar{N}+1]\psi} &= 0.
\end{aligned}$$

Therefore, from equations (4.17) and (4.22) we have eight P.D.E. of infinite order with 8 unknown $C_i(x_1, t)$, $D_i(x_1, t)$ where $i=1,2,3,4$ which can be solved numerically. The coefficients of the displacements $A_i(x_1, t), B_i(x_1, t)$ where $i=1,2,3,4$ can be found by using equation (4.17).

Next, in the case of $N=2$, the Differential Difference equations (3.17) are

$$\begin{aligned}
(4.23) \quad & \begin{bmatrix} 0 & B_{12} & A_{13} & 0 \\ 0 & A_{22} & A_{23} & 0 \\ A_{31} & 0 & 0 & A_{14} \\ A_{41} & 0 & 0 & B_{44} \end{bmatrix} \begin{bmatrix} (u_{n+1} + u_n) \\ (u_{n+1} - u_n) \\ (v_{n+1} + v_n) \\ (v_{n+1} - v_n) \end{bmatrix} \\
& + \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & 1/b & 0 & 0 \\ 0 & 0 & 1/b & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \begin{bmatrix} F^{(1)} \\ H^{(0)} \\ F^{(0)} \\ H^{(1)} \end{bmatrix} = 0
\end{aligned}$$

where

$$\begin{aligned}
C_1 + C_2 + C_3 + C_4 &= \alpha(x_1, t) \\
D_1 + D_2 + D_3 + D_4 &= 0 \\
(4.22) \quad C_1 + C_2 e^{i[\bar{N}+1]\pi} + C_3 e^{i[\bar{N}+1]\psi} + C_4 e^{-i[\bar{N}+1]\psi} &= 0 \\
D_1 + D_2 e^{i[\bar{N}+1]\pi} + D_3 e^{i[\bar{N}+1]\psi} + D_4 e^{-i[\bar{N}+1]\psi} &= 0.
\end{aligned}$$

Therefore, from equations (4.17) and (4.22) we have eight P.D.E. of infinite order with 8 unknown $C_i(x_1, t)$, $D_i(x_1, t)$ where $i=1,2,3,4$ which can be solved numerically. The coefficients of the displacements $A_i(x_1, t), B_i(x_1, t)$ where $i=1,2,3,4$ can be found by using equation (4.17).

Next, in the case of $N=2$, the Differential Difference equations (3.17) are

$$\begin{aligned}
(4.23) \quad & \begin{bmatrix} 0 & B_{12} & A_{13} & 0 \\ 0 & A_{22} & A_{23} & 0 \\ A_{31} & 0 & 0 & A_{14} \\ A_{41} & 0 & 0 & B_{44} \end{bmatrix} \begin{bmatrix} (u_{n+1} + u_n) \\ (u_{n+1} - u_n) \\ (v_{n+1} + v_n) \\ (v_{n+1} - v_n) \end{bmatrix} \\
& + \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & 1/b & 0 & 0 \\ 0 & 0 & 1/b & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \begin{bmatrix} F^{(1)} \\ H^{(0)} \\ F^{(0)} \\ H^{(1)} \end{bmatrix} = 0
\end{aligned}$$

where

$$\begin{aligned}
 B_{12} &= \left[\frac{9\pi(\lambda + 2\mu)^2 - 4(4\lambda + \mu)^2}{9(\lambda + 2\mu)\pi^2} \right] D_{11} + \mu \left(\frac{\pi}{2b} \right)^2 - \rho D_{tt} \\
 (4.24) \quad B_{44} &= \left[\frac{\rho\pi c_2 \mu^2 - 4(4\lambda + \mu)^2}{9\mu\pi^2} \right] D_{11} - (\lambda + 2\mu) \left(\frac{\pi}{2b} \right)^2 - \rho D_{tt}
 \end{aligned}$$

and the A_i can be computed from

$$\begin{aligned}
 p_1 &= 1/b + \frac{2}{3} \frac{(4\lambda + \mu)}{(\lambda + 2\mu)\pi} D_1 \\
 (4.25) \quad p_4 &= 1/b + \frac{2}{3} \frac{(4\lambda + \mu)}{\mu\pi} D_1
 \end{aligned}$$

If we replace A_{44} and A_{12} by B_{44} and B_{12} in equations (4.8) and (4.9), the solution (4.16) can be applied to the above problem. Therefore, Case I ($N=1$) and Case II ($N=2$) are identical except for the coefficients B_{44} and B_{12} .

4.3 Longitudinal Strain Transient Wave due to Impact

In Chapter 3, section 3.5 we set the symmetrical conditions with variables x_1, x_3 which reduced the three-dimensional problem to one spacial variable x_2 and t , the time variable. The Differential-Difference equations (3.46) are

$$\begin{aligned}
 \mathcal{L}^2[u_{n+1} - u_n] + \frac{2}{b} [\sigma_{n+1} - \sigma_n] &= 0 \\
 (4.26) \quad L^2[u_{n+1} - u_n] + \frac{2}{b} [\sigma_{n+1} - \sigma_n] &= 0
 \end{aligned}$$

where

$$(4.27) \quad \begin{aligned} \mathcal{L}^2 &= \frac{d^2}{dt^2} \\ L^2 &= \mathcal{L}^2 + C^2, \quad C = \frac{\pi}{2b} C_L. \end{aligned}$$

Let ∂_t behaves as a constant (Symbolic technique), then equation (4.26) becomes a Difference equation. We assume a solution in the form of

$$(4.28) \quad [u_n, \sigma_n] = [A, B] e^{i2n\theta},$$

then equation (4.26) becomes

$$(4.29) \quad \begin{bmatrix} \mathcal{L}^2 (1 + e^{i2\theta}) & , & -(1 - e^{i2\theta}) \\ -L^2 (1 - e^{i2\theta}) & , & (1 + e^{i2\theta}) \end{bmatrix} \begin{bmatrix} bA \\ 2B \end{bmatrix} = 0.$$

For A and B arbitraries the determinant must be set to zero, which yields

$$(4.30) \quad \det \equiv (e^{i2\theta} - e^{i\beta})(e^{i2\theta} - e^{-i\beta}) = 0$$

where

$$(4.31) \quad \cos \beta = \left(\frac{1}{c^2} \frac{d^2}{dt^2} + 1 \right).$$

The roots of equation (4.30) are

$$\theta_1 = \beta/2$$

$$\theta_2 = -\beta/2.$$

Therefore, equation (4.28) has two independent solutions and by superposition, we get

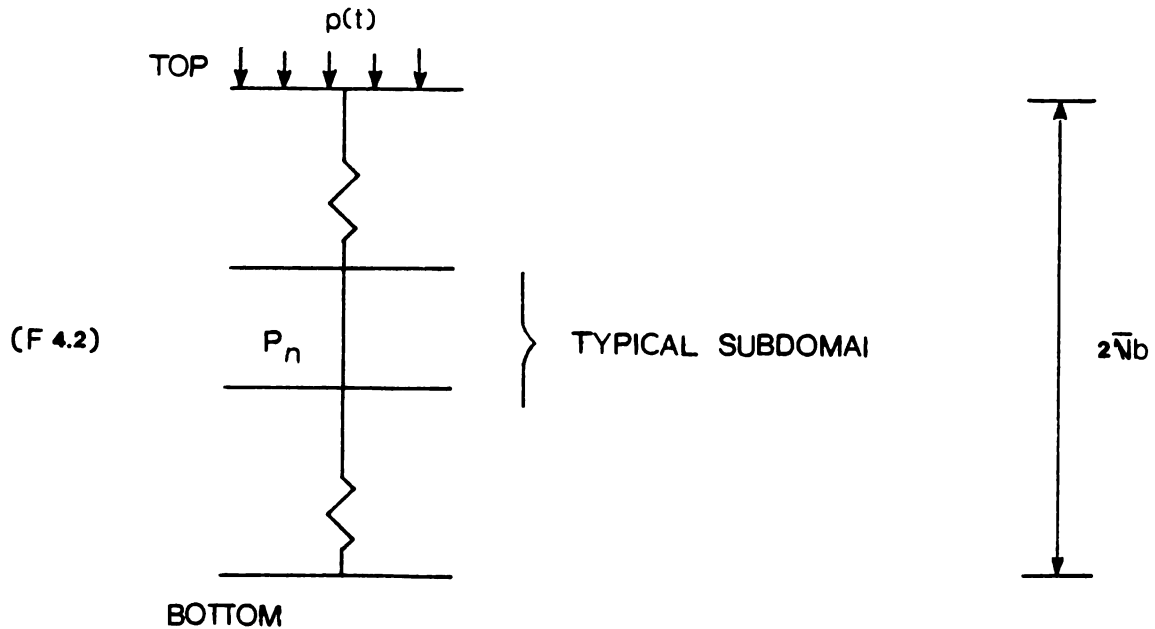
$$(4.32) \quad [u_n, \sigma_n] = [A_1, \beta_1] e^{in\beta} + [A_2, \beta_2] e^{-in\beta}.$$

Now, we let the Boundary conditions be

$$\begin{aligned}
 \sigma_0 &= -p(t) & t \geq 0 \\
 (4.33) \quad \sigma_0 &= 0 & t < 0 \\
 \sigma_{\bar{N}} &= 0, & .
 \end{aligned}$$

The figure below will have a better representation of the B.C. which give rise to a transient wave.

FIGURE:



Recall solution (4.32) which is

$$(4.34) \quad [u_n, \sigma_n] = [A_1, B_1]e^{in\beta} + [A_2, B_2]e^{-in\beta}$$

where $\cos \beta = \left(\frac{1}{c^2} \frac{d^2}{dt^2} + 1\right)$. Thus, if we apply the B.C.

(4.33) to the above equation yields

$$\begin{aligned}
 (4.35) \quad & B_1 + B_2 = -p(t) \\
 & B_1 e^{i[\bar{N}] \beta} + B_2 e^{-i[\bar{N}] \beta} = 0
 \end{aligned}$$

or

$$\begin{aligned}
 (4.36) \quad & B_1 = -[B_2 + p(t)] \\
 & [1 - \cos 2(\bar{N}) \beta] B_2(t) = -p(t) .
 \end{aligned}$$

The above equation gives us two infinite O.D.E. depending on $\bar{N}$, the number of subdomain. Next, we will consider two cases $\bar{N} = 1, 2$.

Case I ($\bar{N} = 1$)

Using this condition, equation (4.36) becomes

$$\begin{aligned}
 (4.37) \quad & B_1 = -[B_2 + p(t)] \\
 & (1 - \cos 2\beta) B_2(t) = -p(t) .
 \end{aligned}$$

Thus, substituting equation (4.31) in the above result, yields

$$\begin{aligned}
 (4.38) \quad & B_1(t) = -[B_2(t) + p(t)] \\
 & \frac{d^2}{dt^2} \left(1 + \frac{1}{c^2} \frac{d^2}{dt^2}\right) B_2(t) = \frac{1}{2} p(t)
 \end{aligned}$$

which is the O.D.E. needed to solve $B_2(t)$. From $B_2(t)$ we get $B_1(t)$. Hence, the stress becomes

$$(4.39) \quad \sigma_n = B_1(t) e^{in\beta} + B_2(t) e^{-in\beta}$$

and the displacements can be obtained using equation (4.29) .

Case II ($\bar{N} = 1$)

Using this condition, equation (4.36) becomes

$$\begin{aligned} B_1 &= -[B_2 + p(t)] \\ (4.40) \quad (1 - 2 \cos^2 \beta) B_2(t) &= -\frac{1}{2} p(t) \end{aligned}$$

and by substituting equation (4.31) in the above result, yields

$$\begin{aligned} [2c^{-4} \frac{d^4}{dt^4} + 4c^{-2} \frac{d^2}{dt^2} + 1] B_2(t) &= p(t) \\ (4.41) \quad \beta_1 &= -[B_2 + p(t)] \end{aligned}$$

which is the O.D.E. needed to solve $B_2(t)$. From $B_2(t)$ we get $B_1(t)$. Hence, the stress and displacement can be found using equations (4.29) and (4.36).

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