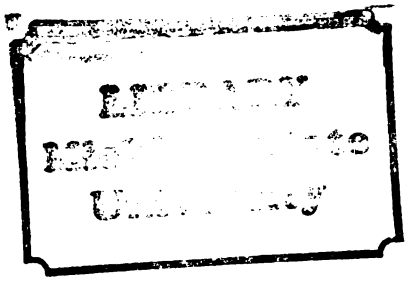
THESIS



This is to certify that the

dissertation entitled

BOUNDS ON THE EIGENVALUES FOR

CERTAIN CLASSES OF DYNAMIC SYSTEMS

presented by

MOHAMED ASHRAF ZEID

has been accepted towards fulfillment
of the requirements for

Doctor Of Philosophy degree in Mechanical Engineering

R C Rosenberg

Major professor

Date February 26, 1982

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BOUNDS ON THE EIGENVALUES FOR CERTAIN CLASSES OF DYNAMIC SYSTEMS

By

Mohamed Ashraf Zeid

A DISSERTATION

Submitted to
Michigan State University
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ABSTRACT

BOUNDS ON THE EIGENVALUES FOR
CERTAIN CLASSES OF DYNAMIC SYSTEMS

By

Mohamed Ashraf Zeid

Is it possible to estimate bounds on the eigenvalues of a system from a pictorial model? Classical methods first derive the state equations, then from the state matrix we obtain the bounds on the largest eigenvalue. For a class of systems, we obtain the bounds on the largest real part and the largest imaginary part of the eigenvalues by inspecting a graphical model of these systems. The graphical model used is a canonical form of the bond graph, namely the gyrobondgraph.

Bond graphs are graphical models of systems. They depict the intrinsic energy structure of the systems by lines called bonds. The nodes of the bond graphs are idealized energy handling devices depicted by a standard set of alpha-numeric characters. As a tool for estimating the eigenvalues of systems, bond graphs offer a suitable medium. They can model systems that handle different forms of energy, therefore, they cover a wide range of engineering problems. A canonical form of the bond graph, namely the gyrobondgraph, can be simplified to an oriented graph.

We first studied systems with uniform parameters. They are common in lumped models of continuous systems. The bounds obtained are used as a base to estimate the bounds for systems with general parameters.

For systems whose gyrobondgrpah is a simple full graph, we established a relation between the structure of the associated oriented graph and the largest real and largest imaginary parts of the eigenvalues. This relation is presented as an algorithm for computing the bounds and is based on the order of the system and the number of gyrators in the gyrobondgraph. The bounds given are comparable to bounds obtained using the standard methods while the number of computations required is substantially reduced. We also give a recursive formula for computing the characteristic polynomial of systems with uniform parameters. This formula was used to establish the bounds on the eigenvalues.

For a special case of systems whose gyrobondgraphs are a simple partial graph, we could obtain bounds on the largest real part and imaginary part of the eigenvalues. For the other classes, the standard techniques had to be used to obtain the bounds.

The yield of this work is the reduction in the computational effort required to obtain the bounds. This reduction becomes of major importance when the bounds are estimated for large scale systems. More study is required to extend this eigenvalue estimation procedure to cover other classes of systems.

DEDICATION

To my father and mother

ACKNOWLEDGMENTS

I wish to express my special appreciation to Dr. Ronald Rosenberg for his guidance, advice and support throughout the preparation of this thesis. His engineering insight has been an infinite source of inspiration, and working with him has been a professionally fulfilling and rewarding experience.

I also express my appreciation for the guidance and assistance of Dr. Albert Andry, Dr. Hassan Khalil, and Dr. Norman Hills. Their interest in my study and their ideas and critique have been most valuable.

A special word of appreciation is extended to my parents for their continuous and consistant support and assistance. Their encouragement has been of greal value. A special word of thanks to my wife Marilyn for her support, patience and for her help in typing the rough draft of this manuscript.

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CHAPTER I

INTRODUCTION

I.1 Motivation

Eigenvalues play a major role in the mathematical description of dynamic systems. The solution of many physical problems depend on the calculation, or at least the estimation, of some or all of the eigenvalues. In design, using interactive computers, the knowledge of an estimate of the largest eigenvalue of the designed system is useful for setting time limits of integration. For large scale systems, the use of techniques for estimating the largest eigenvalue from the state matrix proves to be expensive. A less expensive technique is to find the estimates from a graphical model of the system.

An estimate of the largest eigenvalue at the graphical model stage provides an early estimate since it is obtained without deriving the state equations. As we prove, in some cases estimates obtained from the graphical model are better than those obtained from the state matrix. The main advantage of using the graphical model is that it provides a relatively accurate and cheap estimate of the largest eigenvalue of the system.

Using the graphical model as a medium for estimating the largest eigenvalue will enable relating the structure of the system to its eigenvalues.

The goal of this work is to relate the eigenvalues of systems, through an estimation technique, to a graphical model of the system.

I.2 Computing the Eigenvalues

Current methods to compute the eigenvalues of a system start from the state matrix. As shown in Figure I.1, for each system a classical approach is used to determine the state matrix; then from the state matrix the eigenvalues are estimated or computed. Iterative techniques, for example the QR algorithm, are used to find all the eigenvalues of the system. A substantial presentation of these methods is found in Wilkinson (NA1). Computer software that uses iterative techniques is available in most mathematics software packages. Iterative techniques require extensive memory and a large number of computational operations, which is expensive in terms of computer time and memory. This becomes a basic issue when dealing with large scale systems. At early stages of design, a relatively accurate estimate of the largest eigenvalue is all that is required. Computing this estimate requires a relatively smaller amount of computations. Estimate of the largest eigenvalue are calculated from the state matrix. They are usually given in terms of inequalities that give an upper and lower bound on the largest eigenvalue. As an example, the well known Gerschgorin Theorem provides bounds on all the eigenvalues in terms of circles in the complex plane. The circles have their centers at the diagonal entries of the matrix and their radii are equal to sum of the rows entries. A complete survey of these inequalities is given in Marcus (NA10); more recent results can be found in Wolkowiz (NA11).

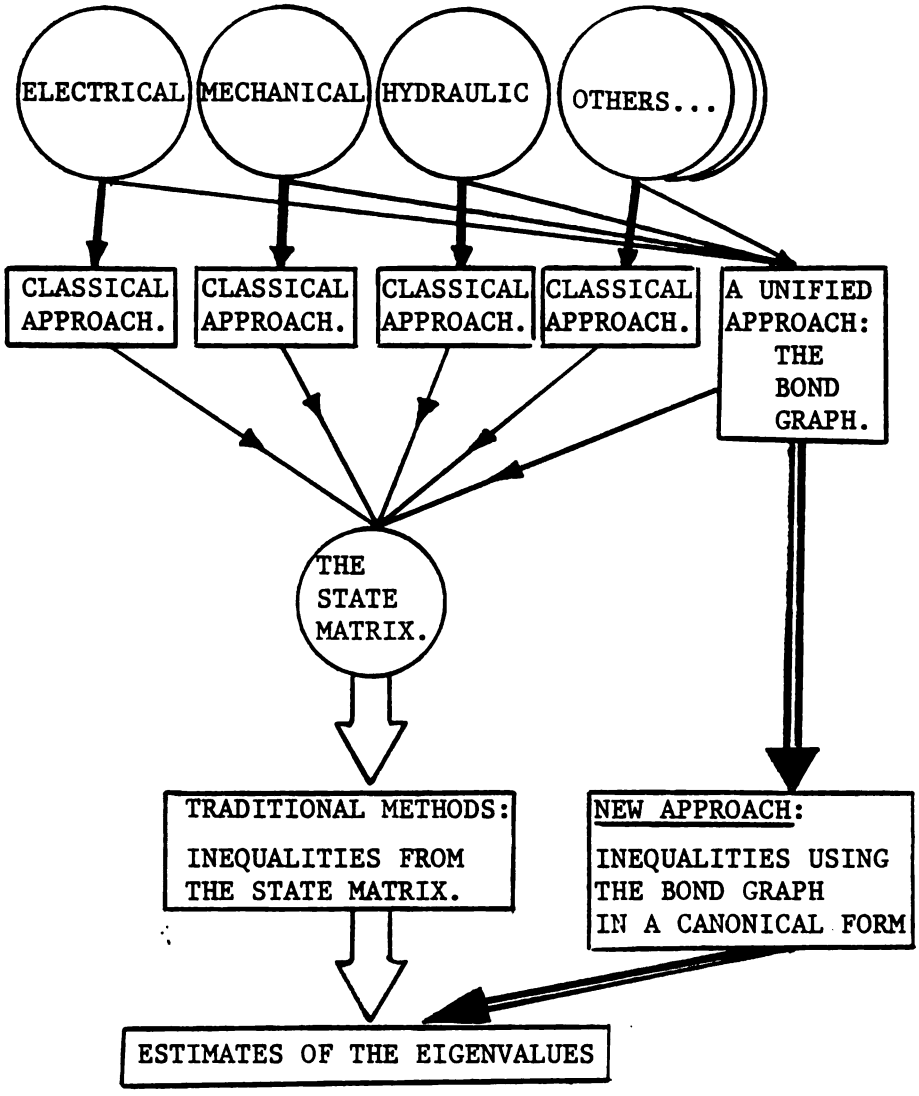


Figure I.1. A comparison of different approaches for estimating eigenvalues of systems.

I.3 Eigenvalues and Graphs

In this work we attempt to relate the eigenvalues of a system to a graphical model of that system, and develop a procedure that provides a relatively accurate estimate of the largest eigenvalue of the system, derived from that graphical model.

Eigenvalues are associated with graphs through the adjacency matrix of these graphs. In some cases, physical systems are represented by graphs. In electrical systems, signal flow graphs are represented by linear graphs, and their adjacency matrix was studied (GT9). In chemistry, the adjacency matrix represented the atom connectivity matrix (GT6). M. Kac used linear graphs and their adjacency matrix to study the vibration of a membrane (GT10). A complete account of linear graphs and their spectra is given in Cvetkovic (GT12). For the purpose of this study, we chose a graph that can be used to model a variety of lumped-parameter systems. These systems can represent different energy types: mechanical, electrical, hydraulic and others. The spectrum of the system is associated with the graph through the skew symmetric adjacency matrix (cf.II.3.2.). For a class of systems we succeeded in establishing bounds on their largest eigenvalue. These bounds are given as a function of the structure of the graphical model.

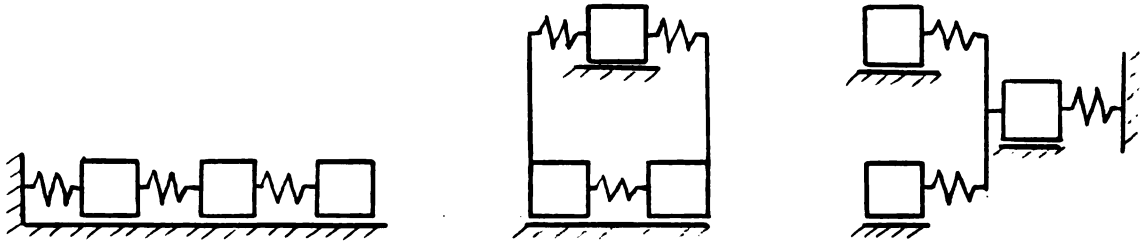
Bond graphs are a unified approach that provides graphical models for lumped-parameter systems that can belong to various energy domains. They use a standardized set of idealized elements, depicted by alphanumeric symbols, to represent the real elements of the system. The energy and power flow between these elements is depicted in the bond graph by lines called bonds. A complete description of bond graph modeling theory is found in (BG2) and a bibliography is found in (BG5). Bond

graphs offered a suitable medium for this study because they can be used to represent different types of systems; thus the results obtained cover a wide range of problems. Bond graphs can be reduced to a canonical form, namely the gyrobondgraph (BG1). This in turn is simplified to an oriented graph. Here we call the simplified graph: the point graph. Changing the lines of the point graph by adding or deleting them indicates a change in the structure of the associated system. Also the orderly fashion that the bond graph provides for deriving the state equations, enables relating the change in the structure of the point graph to the eigenvalues of the system. This relation was obtained here for a class of systems, namely those whose gyrobondgraphs are simple full graphs.

Figure I.3a,b and c represents the bond graph models for the systems sketched in Figure I.2a,b and c respectively. In Figure I.4a,b and c, we see the point graph resulting from simplifying the gyrobond graphs of the bond graphs in Figure I.3a,b and c.

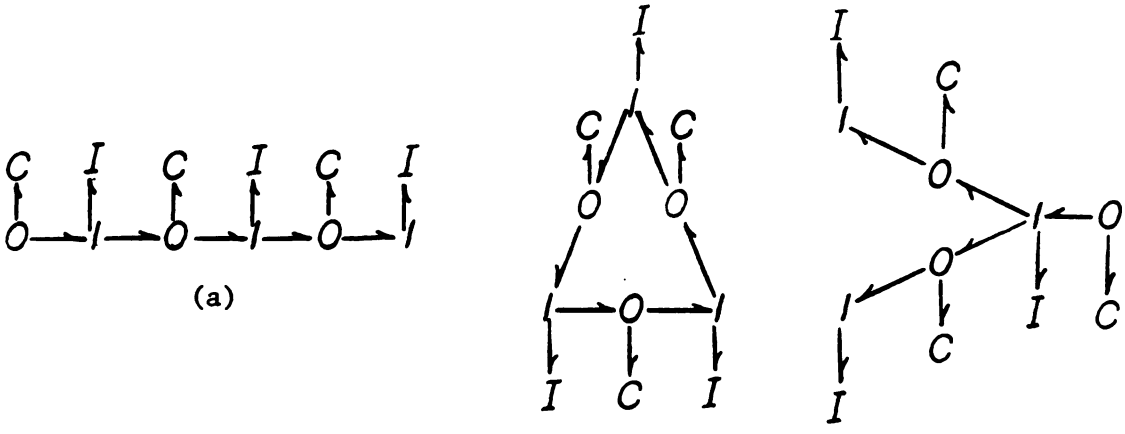
I.4 Summary of Results

For a class of systems, namely those whose gyrobondgraph is a simple full graph (BG1), we established bounds on the largest eigenvalue of the corresponding system. These bounds are given in terms of the structure of the point graph and the parameters of the elements of the system. A procedure to improve these bounds by means of further examination of the structure of the point graph is also given. For the same class of systems, we established a recursive formula for computing the characteristic equation of the system from the point graph. This formula provided us with an insight into the nature of the spectra of the system. It also offered interesting facts about the power orientation



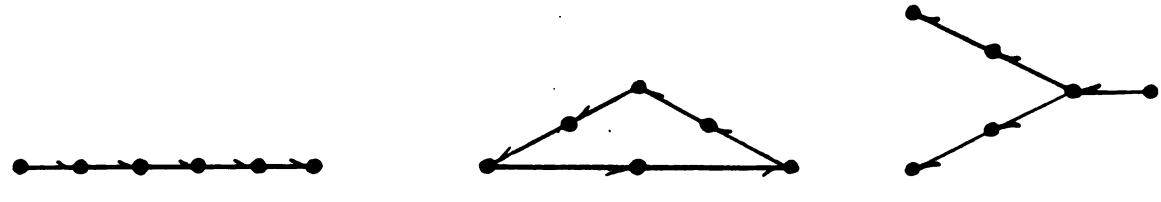
(a) (b) (c)

Figure I.2. Different mechanical transational systems.



(a) (b) (c)

Figure I.3. The bond graph models.



(a) (b) (c)

Figure I.4. The point graph.

of bond graph. For some special cases, namely those systems whose gyro-bondgraph is a simple partial graph, bounds for the eigenvalue could be established. For other cases, and the remaining classes of systems, we had to appeal to the traditional methods of using the state matrix.

A theoretical ground is now laid for further investigation of the subject. more study is needed to establish a complete understanding of this complex problem.

The results given in this work allow us to provide a certain description of the eigenvalues of systems. This description is obtained by examining the associated point graph. As an example, consider the point graphs given in Figure I.3. Let $\lambda = a + ib$, be an eigenvalue of the corresponding system. Let λ_1 be the eigenvalue with the largest modulus. If we assume unit parameters then from the point graphs we can conclude:

a) for all three systems in Figure I.1 we have:

$$2 \cos \frac{\pi}{6} \leq \lambda_1 \leq \sqrt{5}$$

b) λ_1 of the system in Figure I.1.b. is the smallest λ_1 in all three systems.

c) λ_1 of the system in Figure I.1.a. is smaller than λ_1 of the system in Figure I.1.c.

d) for the systems in Figure I.1.a. and Figure I.1.b. we know all their eigenvalues; they are:

$$[\pm i\, 2 \cos (\tfrac{1\pi}{7}), \pm i\, 2 \cos (\tfrac{2\pi}{7}), \pm i\, 2 \cos (\tfrac{3\pi}{7})]$$

for I.1.b. the eigenvalues are:

$$(\pm i\, 2 \cos (\tfrac{\pi}{6}), \pm i\, 2 \cos (\tfrac{2\pi}{6}), 0, 0)$$

e) for the system in Figure I.1.c. we can compute the characteristic equation which is given by:

$$(\lambda^2 + 1)(\lambda^4 + 4\lambda^2 + 1) = 0$$

The above equation can be solved to find the eigenvalues of the system.

We note that the results given above were obtained without having to find the state matrix. This, together with the small computational effort required for obtaining these bounds, offers an advantage over existing methods for estimating the eigenvalues for this class of problems.

I.5 Organization

Chapter II is a brief presentation of definitions and basic theorems used to prove the results presented in this work. This chapter is organized in four sections: Bond Graphs, Gyrobondgraphs, Graph Theory, and Linear Algebra. Some new definitions and concepts are introduced at the end of sections II.2. and II.3.

Chapters III and IV are the core results of this work. They deal with systems whose gyrobondgraph is a simple full graph. In Chapter III we establish upper and lower bounds on the largest eigenvalue of systems. The first part deals with systems that have elements with uniform parameters; they occur in some models of continuous systems. The results obtained were then used in the second part as a base to derive bounds for the more general case of systems with arbitrary parameters.

Chapter IV presents methods to improve the estimate found in Chapter III by examining the point graph and using a dissection procedure.

Also a procedure to find the coefficients of the characteristic polynomial is given, along with some results concerning the properties of the eigenvalues of a system in relation to its oriented graph. These results provide a theoretical basis for the extension of this work. The third part of this chapter presents some results that are used to estimate bounds in the case of general parameters.

In Chapter V we deal with simple partial graphs. We give some results concerning the largest eigenvalue for some special cases. Then for the remaining classes of systems some results concerning bounds on eigenvalues are given using the state matrix.

As discussed in the summary, this work does not answer the original question completely. It gives an answer for a certain class of systems, and lays the groundwork for answering the question more generally.

CHAPTER II

BACKGROUND AND DEFINITIONS

In this chapter we discuss briefly the background material used to derive the main results of this work, and we introduce some new definitions. We first present briefly the bond graph definitions and give two examples of bond graph models for a mechanical and an electrical system. We then discuss the gyrobondgraph, and present important properties of two classes of gyrobondgraphs. We proceed then to define the point graph and the gyroadjacency matrix associated with it. In section II.3. basic definitions and terminology of graph theory are given. In section II.4. matrices associated with graphs are discussed and the skew symmetric adjacency matrix is defined. For the sake of completeness, some basic definitions from linear algebra and matrix theory are given; also some useful theorems are presented.

The bond graph material is adapted from (BG7), (attached in appendix). The reader interested in detailed presentation of bond graphs is referred to Karnopp and Rosenberg (BG2). Graph theory terminology and definitions are adapted from Harary (GT1) and Cvetkovic (GT2).

II.1 The Bond Graph

Bond graphs are pictorial models of dynamic systems. Each real component of the system is modeled by one or more elements of an idealized set. The idealized elements form the nodes of the graph and are depicted by alphanumeric characters. They are connected by lines, called bonds, that are associated with the energy and power handling in

the real system.

For certain classes of dynamic systems, bond graphs offer some advantages over existing modeling procedures:

1) They offer a unified modeling approach. By the use of a standard set of graphical elements we can build a model for systems that handle different types of energy; for example, mechanical, electrical, hydraulic and others.
2) They are orderly and compact and they provide a systematic procedure for deriving the state equations.
3) Coupled with logical capabilities of digital computers, bond graphs are distinguished as a basis for a unified approach to computer aided modeling.

II.1.1 Bond Graph Definitions

Here we present a brief discussion of bond graph definitions. A complete presentation of bond graph language is given in the definition paper by Rosenberg and Karnopp (see appendix).

The basic idea in a bond graph model is the use of idealized energy and power handling elements to model real elements. This set of idealized elements is depicted by alphanumeric characters, and makes the nodes of the graph. The nodes are connected by lines called bonds, whenever power flows between idealized elements. The resulting picture is the bond graph. The standard set of elements of a bond graph are:

$$(S_e,\ S_f;\ I,\ C;\ R;\ TF,\ GY;\ 0,\ 1)$$

They are: source of effort, source of flow, inertia, capacitance, resistance; transformer, gyrator; common effort junction, and common flow junction.

In Table II.1. the elements of the standard set are described according to their idealized energy handling role in the model.

The ports are the places on the idealized elements where they are connected to other idealized elements.

The bond is a line defined together with two variables: the effort e(t), and the flow f(t). A bond connects two elements through their ports whenever power flows between them.

In Table II.2. examples of variables that can be modeled as effort or flow are given for various energy domains.

The power P(t), is defined as the scalar product of the effort and the flow:

$$P(t) = e(t) \cdot f(t)$$

The power direction is a half arrow on one end of the bond designating a positive power on that port.

The causality is a perpendicular stroke at one end of the bond. It denotes the input/output sense of the effort and flow variables.

In Table II.3. are possible causal assignments for bond graph elements. The sources and the juctions, by definition, have only one possible causal assignment; while the other elements each have two possible ways for causal assignment as shown. In the third column of the table, we used the convention that the variable on the left of the equality is the output.

The choice of one of two possible causality assignments on the storage elements (I,C) will result in either an integral or a differential relation for the element as shown. For the resistance, the relation in that sense is invariant to the causality choice. For the

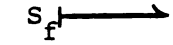
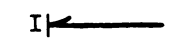
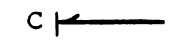
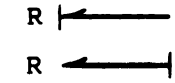
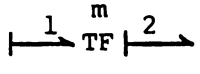
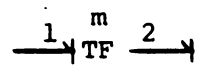
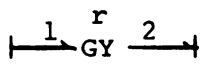
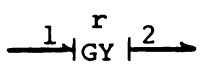
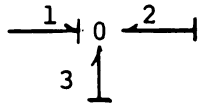
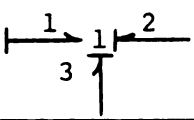
Table II.1. The Bond Graph Standard Set Element.

Basic Number of Ports	Type of Energy handling device	Name	Symbol	Definition Nonlinear	Definition Linear
Basic One Ports	Energy Sources	Effort Source	$S_e \longrightarrow$	e(t) given f(t) arbitrary	e(t) given f(t) arbitrary
		Flow Source	$S_f \longrightarrow$	f(t) given e(t) arbitrary	f(t) given e(t) arbitrary
	Energy Stores	Inertia	$I \xleftarrow[f]{e=\dot{p}}$	$f = \Phi_I^{-1}(p)$	$f = \frac{1}{I}p$
		Capacitor	$C \xleftarrow[f=\dot{q}]{c}$	$e = \Phi_C^{-1}(q)$	$e = \frac{1}{C}q$
	Energy Dissipator	Resistance	$R \xleftarrow[f]{e}$	$e = \Phi_R(f)$	$e = Rf$
Basic Two Ports	Transducers	Transformer	$\xleftarrow{e_1}\ m\ \xleftarrow{e_2}$ TF $f_1 \qquad f_2$	$e_2 = m(x)e_1$ $f_1 = m(x)e_1$	$e_2 = me_1$ $f_1 = mf_2$
		Gyrator	$\xleftarrow{e_1}\ r\ \xleftarrow{e_2}$ GY $f_1 \qquad f_2$	$e_2 = r(x)f_1$ $e_1 = r(x)f_2$	$e_2 = rf_1$ $f_2 = r^{-1}e_1$
Basic Three Ports	Junctions	Common Effort	$\xleftarrow{e_1}\ 0\ \xleftarrow{e_3}$ $f_1 \uparrow f_3$ $e_2 \mid f_2$	$e_1 = e_2 = e_3$ $f_1 + f_2 + f_3 = 0$	$e_1 = e_2 = e_3$ $f_1 + f_2 + f_3 = 0$
		Common Flow	$\xleftarrow{e_1}\ 1\ \xleftarrow{e_3}$ $f_1 \uparrow f_3$ $e_2 \mid f_2$	$e_1 + e_2 + e_3 = 0$ $f_1 = f_2 = f_3$	$e_1 + e_2 + e_3 = 0$ $f_1 = f_2 = f_3$

Table II.2. Variables that can be modeled as effort and flow for various energy domains.

Domain	Effort, e(t)	Flow, f(t)
Mechanical Translation	Force Component	Velocity Component
Mechanical Rotation	Torque Component	Angular Velocity Component
Hydraulic	Pressure	Volume Flow Rate
Electric	Voltage	Current

Table II.3. Possible causal assignments for bond graph elements.

Element	Possible Causal Form	Causal Relation
Effort Source		$e(t) = E(t)$
Flow Source		$f(t) = F(t)$
Inertia		$f = \Phi_I^{-1}(\int e\,dt)$...Integration causality $e = \frac{d}{dt}[\Phi_I(f)]$
Capacitor		$f = \frac{d}{dt}[\Phi_c(e)]$ $e = \Phi_c^{-1}(\int f\,dt)$...Inegration causality
Resistance		$f = \Phi_R^{-1}(e)$ $e = \Phi_R(f)$
Transformer	 	$e_1 = m\,e_2$ $f_2 = m\,f_1$ $f_1 = m^{-1}f_2$ $e_2 = m^{-1}e_1$
Gyrator	 	$e_1 = rf_2$ $e_2 = rf_1$ $f_1 = r^{-1}e_2$ $f_2 = r^{-1}e_1$
Common Effort Junction		$e_2 = e_1$, $e_3 = e_1$ $f_1 = -(f_2 + f_3)$
Common Effort Junction		$f_2 = f_1$, $f_3 = f_1$ $e_1 = -(e_2 + e_3)$

resistance, the relation in that sense is invariant to the causality choice. For the transformer and gyrator, the choice of causal assignment on one bond forces the causal assignment on the other bonds.

The common effort junction will have one input effort, forcing the causal assignment on the other bonds. The common flow junction will have one input flow, forcing the causal assignment on the other bonds to be output flow.

A systematic procedure that is given in (GB2) takes the sketch of a system to a bond graph model. From the bond graph, and following a systematic procedure (GB2), the state equations of the system are derived in the form:

$$\dot{x} = Ax + Bu$$

Now two simple examples for bond graph models of a mechanical and an electrical system are given.

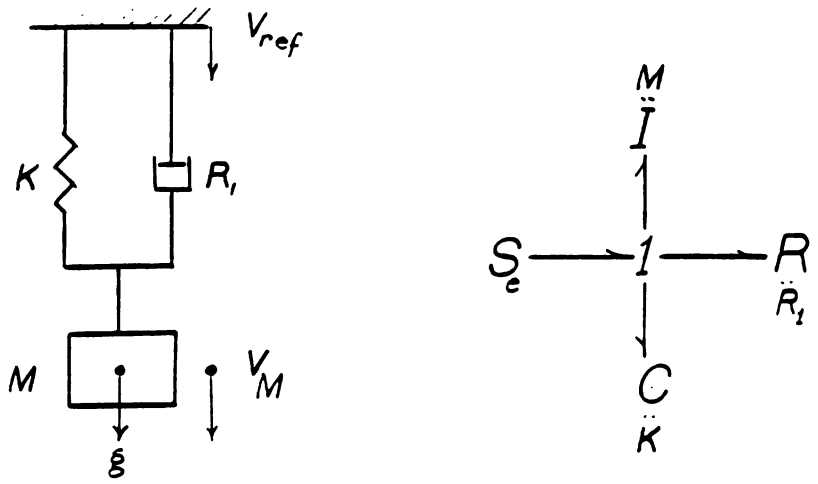
II.1.2 Examples of Bond Graph Models

A mechanical system

Consider the simple mechanical transational system given in Figure II.1.a. It consists of a Mass M, a spring K, and a hydraulic resistance R_1.

The bond graph model is shown in Figure II.1.b. The effect of gravitation is modeled as a source of effort, S_e; the spring is modeled as a capacitance C: the hydraulic resistance is modeled as a resistance R and the mass is modeled as an inertia I. In this case all elements are adjacent to a common velocity junction. The velocity on this 1-junction is the relative velocity:

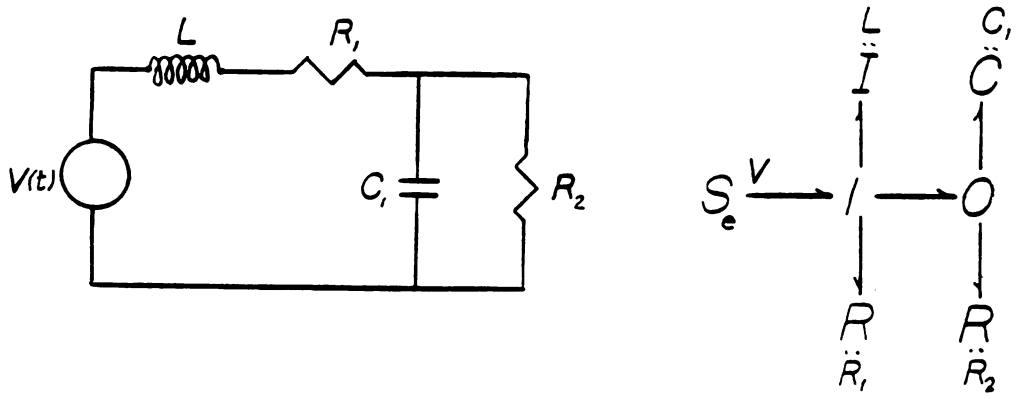
$$V = V_M - V_{ref}$$



(a) (b)

Figure II.1 (a) A sketch of a mechanical transational system

(b) The bond graph model



(a) (b)

Figure II.2 (a) A sketch of an electrical system

(b) The bond graph model

From the bond graph given in Figure II.1.b., and following a systematic procedure, the state equations are then derived.

An electrical system

In Figure II.2.a. is given an electrical system consisting of a voltage source, and inductance and a resistance in series, and a capacitance and a resistance in parallel.

In Figure II.2.b. is the bond graph model. Note that the voltage source is modeled as a source of effort; the inductance modeled as an inertia. The source of effort, the inertia and the resistance R_1 are adjacent to a common flow junction. The capacitance and the resistance R_2 are adjacent to a common effort junction.

From Figure II.2.b. the state equations of the system could be derived in the form:

$$\dot{x} = Ax + Bu$$

They are:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{C_1} \\ \frac{1}{L} & -\frac{1}{R_2 C_1} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V(t)$$

Where p is the flux linkage on the inductance and q is the charge on the capacitance.

II.2. The Gyrobondgraph

In the standard bond graph a set of 9 elements is used to model lumped-parameter systems. As stated before they are (S_e, S_f; I, C, R; TF, GY; 0, 1). Gyrobondgraphs are a canonical form of bond graph (BG1).

Gyrobondgraph model will have its elements from the set: (S_e, I, R, GY, 1). This is called a primitive set. A primitive set is any minimal set, that is irreducible, of which all lumped-parameters dynamical systems are composed (BG6). A primitive set must contain:

1) a gyrator
2) one type of source: effort source or flow source, (S_e or S_f).
3) one type of energy storage: inertia or capacitance, (I or C).
4) one type of ideal junction: common effort or common flow, (0 or 1).
5) a dissipation R.

According to the above conditions, there exist eight primitive sets whose elements are chosen from the standard set. The gyrobondgraph set is one of these sets.

A bond graph that has its elements from the set (S_e; I; R; GY; 1) is a gyrobondgraph if it satisfies the following adjacency conditions:

Adjacency Conditions

1) Each I is adjacent to a 1-junction at each I port.
2) Each R is adjacent to a 1-junction at each R port.
3) Each S_e is adjacent to a 1-junction.
4) Each GY is adjacent to two distinct 1-junctions.
5) Each 1-junction is adjacent to no more than one I, one R, and S_e, and to no other 1-junctions.
6) Pairs of 1-junctions have no more than one GY (or one multiport R) coupling between them.

Gyrobondgraphs can be obtained from bond graphs using the following two step procedure:

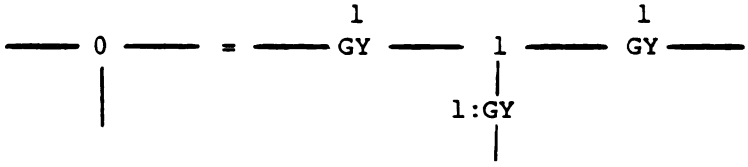
(a) Replacements

Replace each S_f by an S_e and a GY.

Replace each C by any equivalent I and a GY.

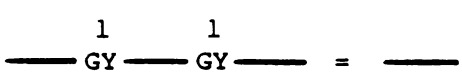
Replace each TF by two GY's in cascade, one of which has unity modulus.

Replace each 0-junction by a 1-junction and a GY at each port. The moduli are unity.



(b) Simplifications

1. Simplify each two unit gyrators in cascade by a single bond.

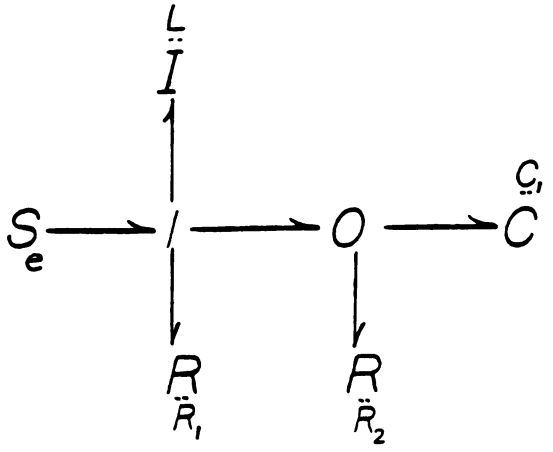


2. Combine all 1-junctions that are directly bonded.

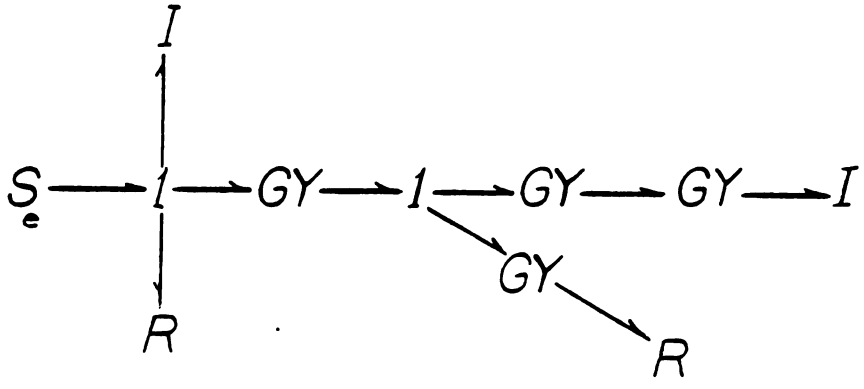
Example

In Figure II.3.a. is the bond graph obtained in Figure II.2.b. for the system sketched in Figure II.2.a. Applying the replacement steps, we obtain the graph in Figure II.3.b. Applying the simplication

(a)



(b)



(c)

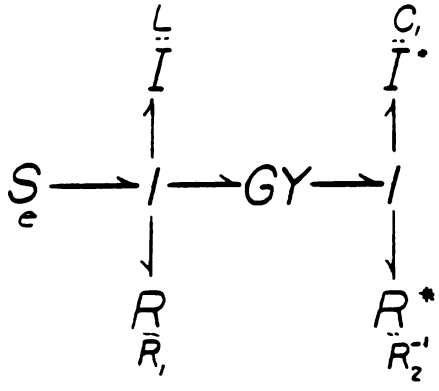


Figure II.3. Transforming the bondgraph to a gyrobondgraph.

steps we obtain the gyrobondgraph in Figure II.3.c. The * on I and R denotes transformed elements. The parameters for the transformed elements are

$$I^* = C_1$$

$$R^* = \frac{1}{R_2}$$

The state equations in terms of the transformed elements are

$$\begin{bmatrix} \dot{p} \\ \dot{p}^* \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{C_1} \\ \frac{1}{L} & -\frac{R_2^{-1}}{C_1} \end{bmatrix} \begin{bmatrix} p \\ p^* \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V(t)$$

Classification of the gyrobondgraphs

Gyrobondgraphs are classified as simple or complex as follows:

"A gyrobondgraph is simple if every I element and every R element is a one port. Otherwise the gyrobondgraph is complex."

They are classified as full I or partial I as follows:

"If the number of I-field ports is equal to the number of 1-junctions, the gyrobondgraph is full. Otherwise it is partial."

For the purpose of estimating eigenvalues, a classification defining the fullness of the R field is useful. Following the line of the above definitions, we classify a full R or partial R gyrobondgraph as follows:

"If the number of R-field ports is equal to the number of 1-junctions, the gyrobondgraph is full R. Otherwise it is partial R."

Note: (1) While "Full I, full R" characterizes all the 1-junctions in the gyrobondgraph as being adjacent to one I and one R port, the partial I, partial R does not provide such characterization.

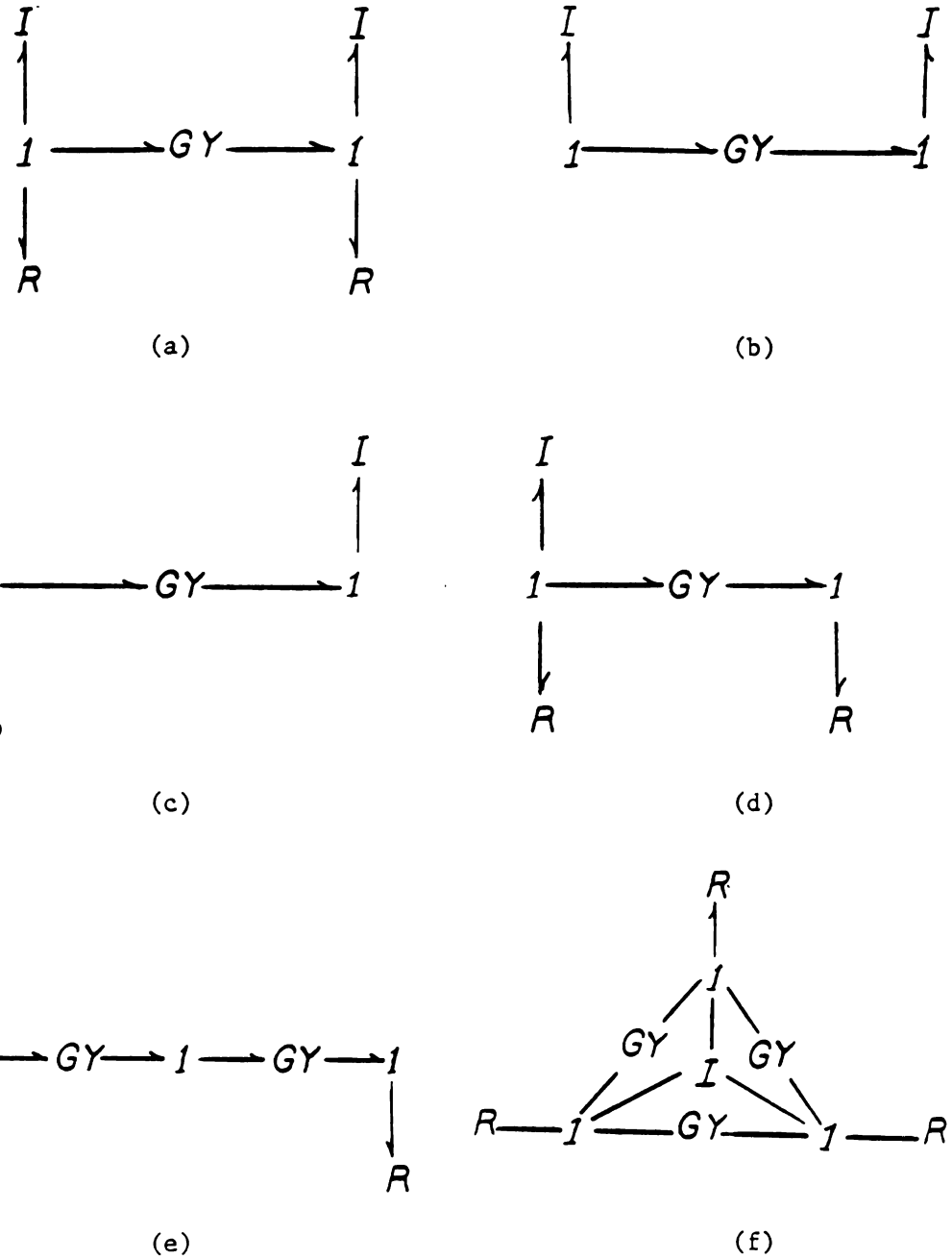


Figure II.4. (a) and (b) simple full graphs; (c) simple full I, partial R graph; (d) simple partial I, full R graph; (e) simple partial I, partial R graph; (f) complex full graph.

CLASSIFICATION OF GYROBONDGRAPHS

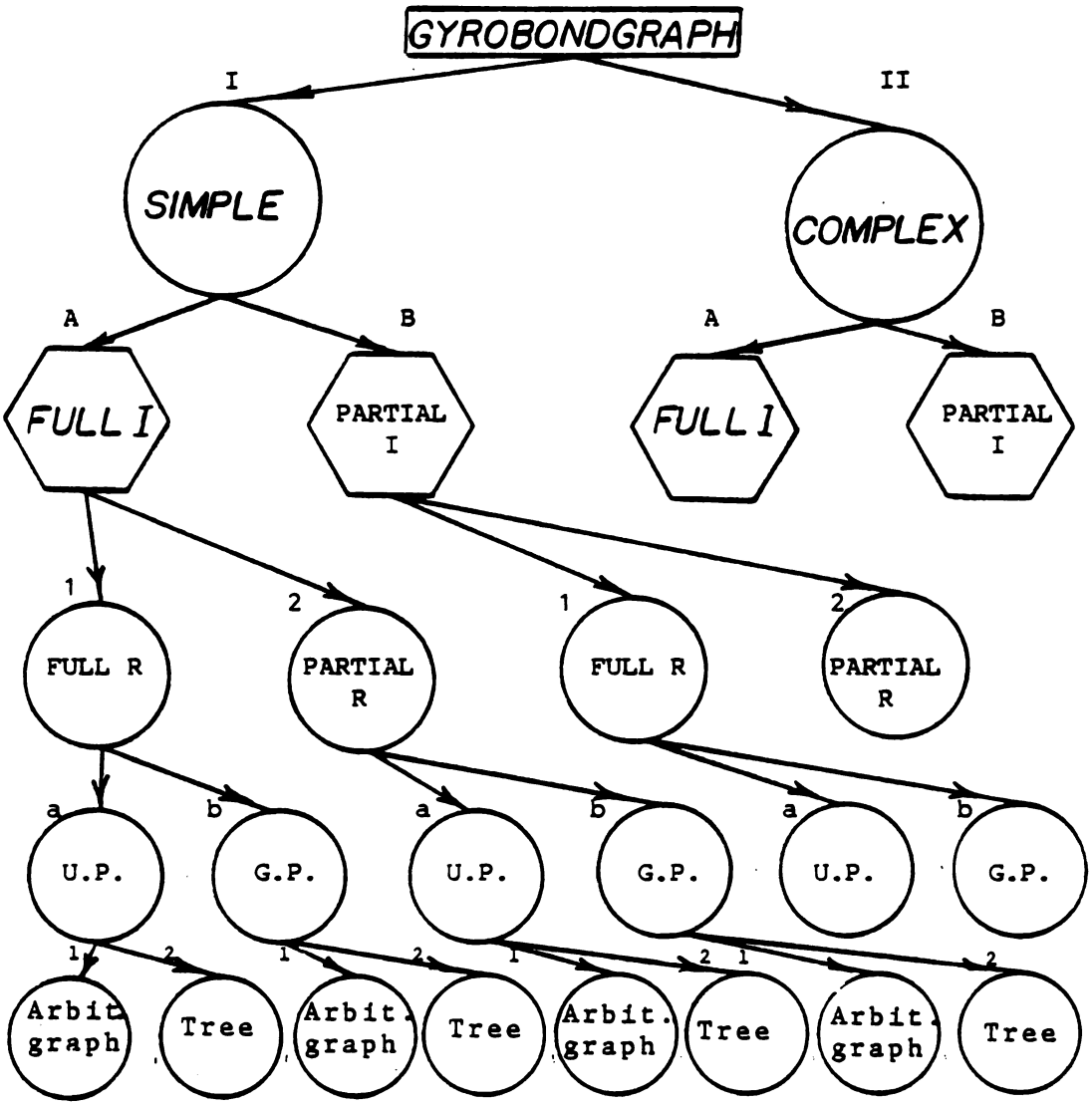


Figure II.5.
- I) Simple: All R,I elements are one port.
- II) Complex: At least one R or I element is a multiport.
- A) Full I: Each 1-junction has one I element adjacent.
- B) Partial I: At least one 1-junction does not have an I element.
- 1) Full R: Each 1-junction has one R element adjacent.
- 2) Partial R: At least one 1-junction does not have an R element.
- a) U.P.: All parameters of the system are equal 1.
- b) G.P.: The parameters of the system may assume arbitrary values.
- $_1$) Arbit. graph: The oriented graph has at least one cycle.
- $_2$) Tree: The oriented graph has no cycles.

(2) From this arises a need to define the 1-junctions according to their field adjacencies.

(3) "Full graph" will be used to denote a full I graph where the R field can be either full R or partial R.

(4) "Partial graph" will be used to denote a partial I graph where the R field can be either full R or partial R.

In Figure II.4. are examples of different classes of gyrobondgraphs. In Figure II.5. is the classification of gyrobondgraphs.

II.2.1 The Point Graph

The point graph is a simplified representation of a gyrobondgraph. The reasons for the simplification are:

(1) The gyrobondgraph becomes an abstract conventional form namely the linear graph. The properties of this linear graph can then be studied and related to the eigenvalues of the associated system.

(2) Compactness is achieved, especially when plotting large scale systems on CRT. This plotting could be achieved in a hierarchical fashion (LA6).

Before proceeding with the simplification we classify the 1-junctions according to their adjacencies as follows:

(1) A full point: is a 1-junction that has an adjacent I element. It may or may not have an R element. The graphical symbol of a full point is a darkened circle (a point) as in Figure II.6.a.

(2) A partial R point: is a 1-junction that has only an R element adjacent. The parameter of the R is non-zero.

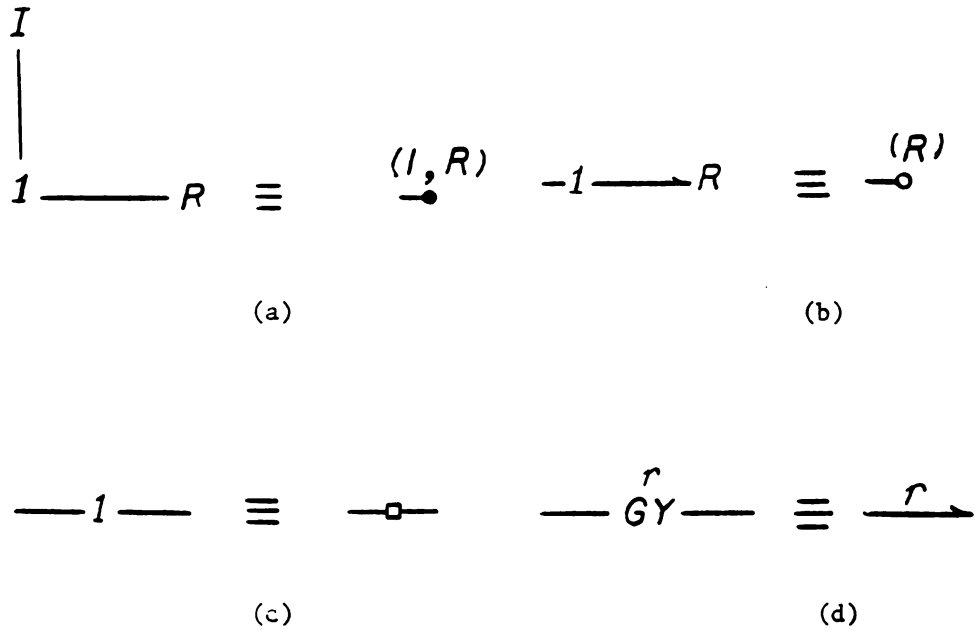


Figure II.6. Simplification of the gyrobondgraph to a point graph.

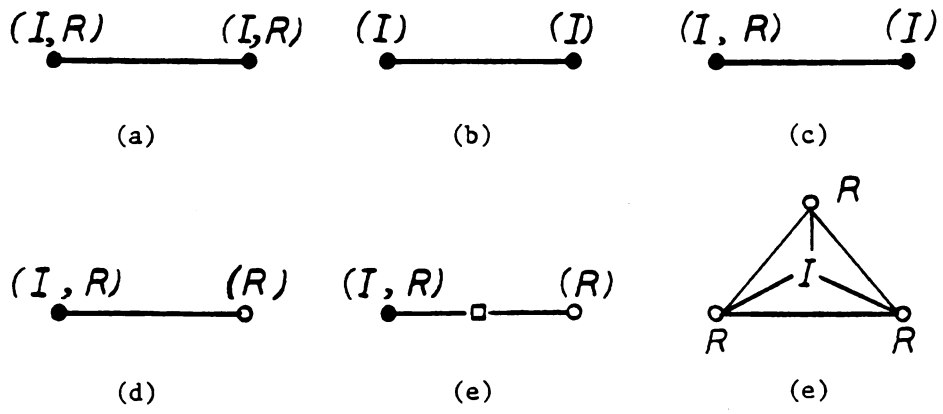


Figure II.7. The point graphs corresponding to the gyrobondgraphs in Figure II.4.

The graphical symbol of a partial point is a small circle as in Figure II.6.b.

(3) A partial point: is a 1-junction that has neither I or R elements adjacent. Its graphical symbol is a small square as in Figure II.6.c.

Simplification of the gyrobondgraph

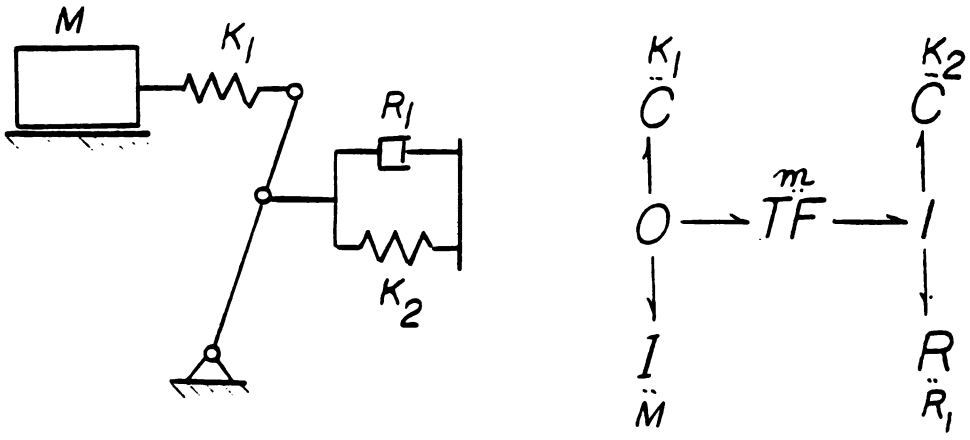
To simplify a gyrobondgraph to a point graph the following two step procedure is used.

(1) Replace each 1-junction by the corresponding point as stated above.

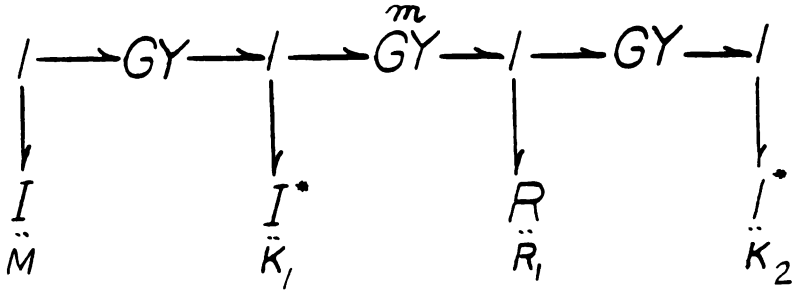
(2) Replace each gyrator together with its bonds by a directed line joining the two adjacent points.

The linear oriented graph obtained will be called the point graph. The gyrator modulus (if different than one) can be written beside the directed line. The value of the parameters of the inertia and the resistance can be written beside the points. No confusion will occur here between the directed line and a bond since the latter will be adjacent to a multi-port element. The compactness of the point graph can be seen when comparing Figure II.4. and II.7. In Figure II.7. are the point graphs corresponding to the gyrobondgraphs in Figure II.4. The point graph could be considered as a more abstract form of bond graph. It offered however a suitable medium for examining the eigenvalues of the associated system. An example of a system and the associated point graph is given in Figure II.8.

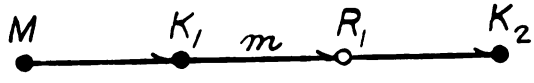
Throughout this work we will examine the point graph. The points of the point graph will be labeled by numbers 1,2,...N, where N is the number of total points in the point graph.



(a) (b)



(c)



(d)

Figure II.8. (a) A mechanical system; (b) the bond graph model; (c) the gyrobondgraph; (d) the point graph.

The objective of this work will be to relate the structure of the point graph to the eigenvalues of the associated system. We will develop a procedure to estimate bounds on the largest eigenvalue of a system if its gyrobondgraph is full, by using the structure of the point graph.

II.2.2 A Class of Systems

In this section we present the class of systems studied through this work. They are the dynamic systems whose gyrobondgraph is a simple partial graph, with the condition that each 1-junction has at least one I or one R element adjacent.

In terms of the point graph they are systems whose point graph has full points and partial R points only.

It could be easily proven that for such a class of systems, all storage elements in the gyrobondgraph have integration causality.

A basic matrix in studying this class is the gyroadjacency matrix. It is defined in relation to the point graph as follows:

For a point graph having f full points and p partial R points, the total number of points in the point graph is: $N = f + p$; we associate the gyroadjacency matrix $S_{NxN} = [s_{ij}]$ such that:

$s_{ij} = r_{ij}$ if there is a directed line from point i to point j with gyrator modulus equal r_{ij}; and

$s_{ij} = -r_{ij}$ if there is a directed line from point j to point i with gyrator modulus equal r_{ij}; and

$s_{ij} = 0$ otherwise.

The state matrix

For the class of systems defined above, we give the state matrix in terms of the gyroadjacency matrix and diagonal matrices of the resistances and the inertias defined below.

Let R_F ... be the diagonal matrix whose entry R_{Fii} is the resistance of the full point i,

$$R_F = \begin{bmatrix} R_1 & & & 0 \\ & R_2 & & \\ & & \ddots & \\ 0 & & & R_f \end{bmatrix}$$

and let R_P . . . be the diagonal matrix whose entry R_{Pii} is the resistance of the partial R point i,

$$R_P = \begin{bmatrix} R_{f+1} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & R_N \end{bmatrix}$$

and let I^{-1}. . . be the diagonal matrix whose i^{th} entry is the inverse of the inertia on the full point i,

$$I^{-1} = \begin{bmatrix} \frac{1}{I_1} & & & 0 \\ & \frac{1}{I_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{I_f} \end{bmatrix}$$

(The points are not necessarily labeled in order in the graph, but they are ordered in the above matrices: first the full points, then the partial points.)

If we write $$S = \begin{bmatrix} S_F & S_{12} \\ -S^T_{12} & S_P \end{bmatrix}$$

then $$A = [(S_F - R_F) + S_{12}\,(S_p - R_p)^{-1}\, S_{12}^T]\;\; [I^{-1}] \qquad \ldots(1)$$

provided $(S_p - R_p)^{-1}$ exists.

where S_F...is the gyroadjacency matrix of the subgraph having all its points full,

and S_p...is the gyroadjacency matrix of the subgraph having all its points partial.

The proof for the above theorem is given in Appendix 8.

The simple full graph

A simple full graph has a point graph with all its points being full points. Thus

$$S_p = 0$$
$$R_p = 0$$

and the state matrix is given by

$$A = [S_F - R_F]\;\; (I^{-1})$$

This result could also be obtained from (BG1).

Example

Consider the system in Figure II.3. The gyrobondgraph given in Figure II.3.c. is simple full, hence

the gyroadjacency matrix is

$$S_F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

the resistance matrix is

$$(R_F) = \begin{bmatrix} R_1 & 0 \\ 0 & R_2^{-1} \end{bmatrix},$$

and the inertia matrix is

$$(I^{-1}) = \begin{bmatrix} 1/L & 0 \\ 0 & 1/C_1 \end{bmatrix}.$$

Applying the expression for the state matrix of a simple full graph, we obtain

$$A = \left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} R_1 & 0 \\ 0 & R_2^{-1} \end{bmatrix}\right]\begin{bmatrix} 1/L & 0 \\ 0 & 1/C_1 \end{bmatrix} = \begin{bmatrix} -R_{1/L} & 1/C_1 \\ -1/L & -R_{2/C_1}^{-1} \end{bmatrix}$$

which is the same result obtained earlier.

II.3 Graph Theory

Next we introduce some basic definitions of graph theory with some illustrating examples. A new matrix associated with the oriented graph is introduced; we call it skew symmetric adjacency matrix*. Also here we state some theorems from the theory of graph spectra. These theorems were used in some of the proofs in this work.

II.3.1 Graphs

Here we introduce the mathematical definition of graphs and the terminology related to them.

A Graph: denoted G or G(p,q) consists of the following: a finite nonempty set V=V(G) of p points** together with a prescribed set x of q unordered pairs of distinct points of V.

---

*The skew symmetric matrix introduced here has been introduced as an adjacency matrix of a special class of tournaments (LA7).

**Sometimes called nodes, vertices, junctions.

A Line: each pair x=(u,v) of points of X is a line***. u and v are called adjacent; x and u are called incident to each other.

Adjacent Lines: if two lines are incident to a common point they are called adjacent lines.

In Figure II.9. G is a graph, number of lines q=5 and number of points p=4, line x is incident to point u, line y is incident to point u; and thus lines x and y are adjacent.

Directed Line: is an ordered pair of distinct points. Line z is a directed line. Because the direction of the line is the power direction in a point graph, the term power direction will be used throughout this work.

Oriented Graph: A graph with all its lines being directed lines. No multiple lines between same points and no directed line from one point to itself. Figure II.10 is an Oriented graph.

A Walk: of a graph G is an alternating sequence of points and lines beginning with a point and ending with a point. In this alternating sequence a line is incident with the point preceding it in the sequence and with the point following it. The sequence (w,y,u,x,v,x,u) in Figure II.9. is a walk.

A Closed Walk: is a walk beginning and ending with the same point.

A Path: is a walk with all its points and lines distinct. In Figure II.9 (w,y,u,x,v) is a path. A graph that is a path and that consists of p points will be denoted by: P_p. In Figure II.11.a. is a path with four points: P_4.

***Sometimes called edge, arc, branch.

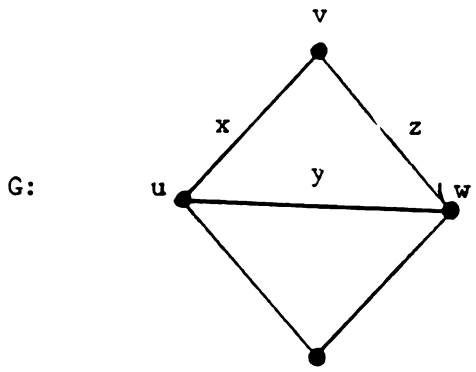


Figure II.9. A graph with four points.

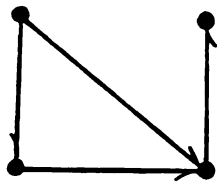


Figure II.10. An oriented graph.

Figure II.11. (a) A path with four points: P_4; (b) a cycle with three points: C_3.

A Cycle: is a walk with distinct points that is closed. In Figure II.9. (w,y,u,x,v,z,w) is a cycle. A cycle with p points will be denoted by C_p. Figure II.11.b. is a cycle with three points.

In oriented graphs we classify cycles according to their number of points and their power orientation as follows:

Odd (even) numbered cycles are cycles having odd (even) number of points.

We define power clockwise (counter clockwise) of a cycle as the number of directed lines in that cycle that are directed clockwise (counter clockwise).

Odd powered cycles are cycles having their power clockwise or counter clockwise equal an odd number; otherwise they are even powered.

In Figure II.12.a. the graph is a cycle. It is odd numbered, and is 3 clockwise powered, zero counter clockwise powered and thus odd powered.

Note: this definition implies all odd numbered cycles are odd powered.

In Figure II.12.b. the graph is a cycle. It is even numbered, and is 4 clockwise powered, zero counter clockwise powered and thus even powered.

In Figure II.12.c. the graph is a cycle. It is even numbered, and is 3 clockwise powered, 1 counter clockwise powered and thus odd powered. Thus even numbered cycles can be odd or even powered.

Labeling a Graph: is assigning to the N points of the graph numbers from 1 to N. There exists more than one way to label a graph (or an oriented graph). Figure II.13.a. is an oriented graph; in b and c are two different labelings for the same oriented graph.

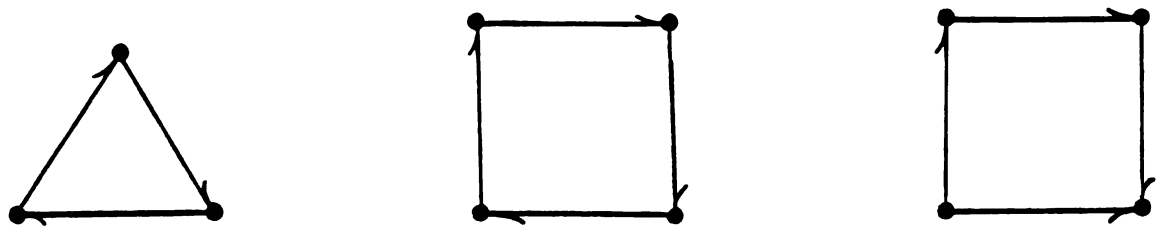


Figure II.12. (a) An odd length (numbered) cycle; (b) An even length (numbered) cycle with even power; (c) An even length (numbered) cycle with odd power.

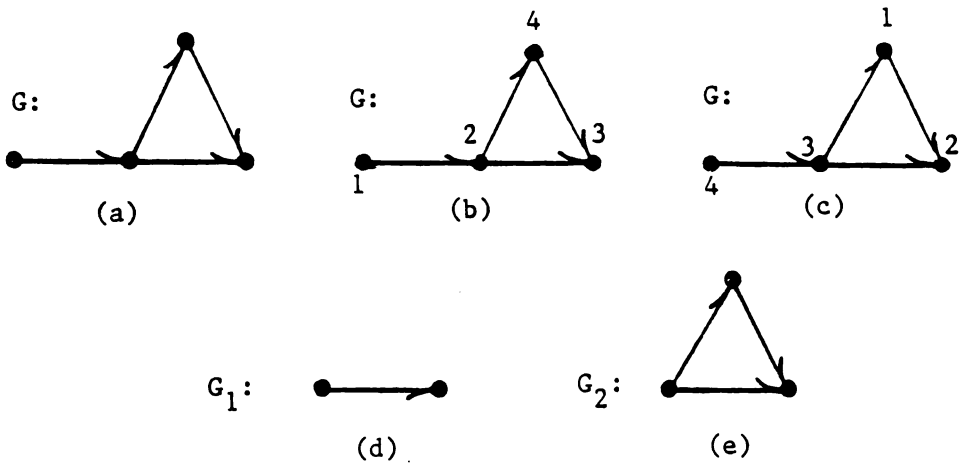


Figure II.13. (a) A directed unlabeled graph; (b) and (c) two different labelings of G; (d) and (e) two subgraphs of G.

Subgraphs: a subgraph of the graph G is a graph having all its points and lines in G. G_1 and G_2 in Figure II.13.d. and Figure II.13.e. are subgraphs of G.

Removal of a Point: the subgraph obtained from a graph G by removing a point v, and denoted (G-v), consists of all points of G except v, and all lines of G except those adjacent to v. In Figure 14.b. the subgraph G_1 is obtained from the graph G in Figure II.14.a. after removing the point v.

Removal of a Line: the subgraph obtained from a graph G by removal of a line x, and denoted (G-x), is the graph having all lines of G except x. In Figure II.14.c. the graph G_2 is obtained from the graph G by removing the line x.

Connected Graph: a graph is connected if every pair of points are joined by a path.

The Degree of a Point: is the number of lines incident on this point.

Minimum Degree: of a graph G: denoted d(G) is the smallest degree of all points of G.

Maximum Degree: of a graph G; denoted $\Delta(G)$ is the largest degree of all points of G.

End Point: is a point of degree 1.

The Complete Graph: denoted K_p, is a graph having every pair of its p points adjacent. Examples of oriented K_3 and K_4 are given in Figure II.15.a,b.c. Note that for oriented graphs with four points, K_4 is not unique because of possible different power orientations; for example: Figure II.15.b. and Figure II.15.c. are both K_4.

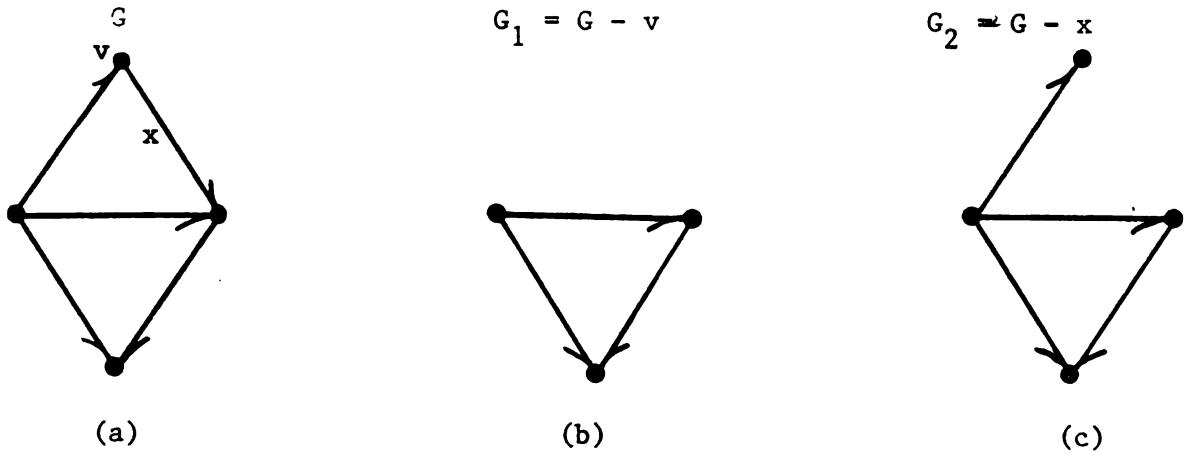


Figure II.14. (a) A graph G, with point v and line x; (b) A graph G_1 obtained from G by removing the point v; (c) A graph G_2 obtained from G by removing the line x.

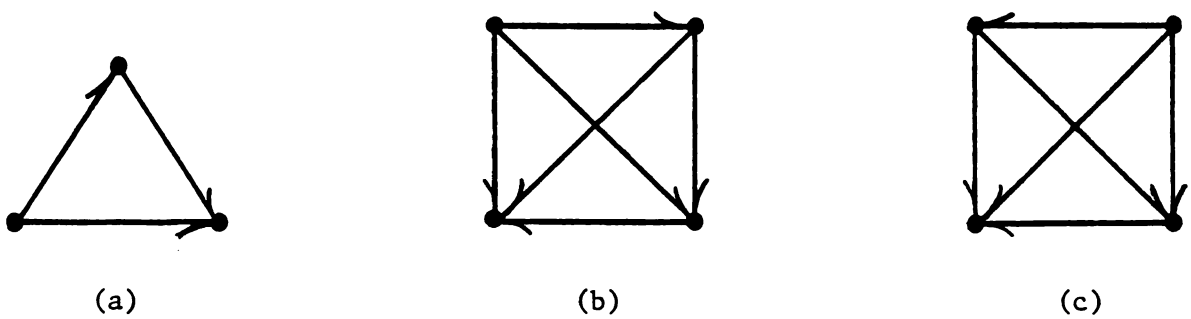


Figure II.15. (a) A complete graph with three points: K_3; (b) and (c) two possible power orientations on a complete graph K_4.

Figure II.16. (a) A complete bigraph: $K_{2,3}$; (b) A complete bigraph $K_{1,3}$ (a star).

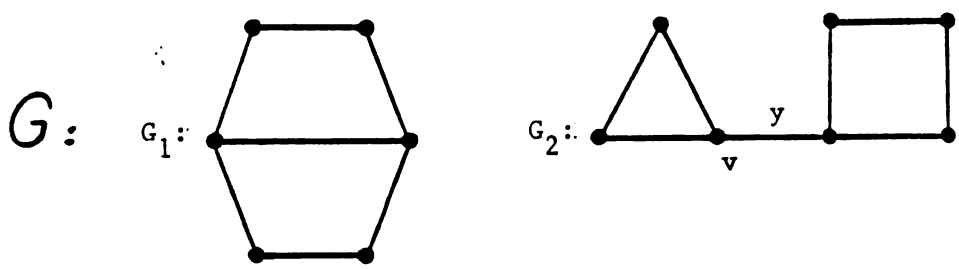


Figure II.17. A graph G that has two components: G_1 and G_2. The subgraph G_2 has a bridge y and a cut point v.

The Tree: is a graph with no cycles.

Bigraph: a graph whose set of points V can be partioned into subsets V_1 and V_2 such that every line of G joins V_1 and V_2.

Complete Bigraph: denoted by $K_{m,n}$. If V_1 and V_2 have m and n points and G contains every line joining V_1 and V_2. Example in Figure II.16.a. the graph is $K_{2,3}$.

A Component: of a graph G is a maximal connected subgraph of G.

A Cutpoint: of a graph is a point whose removal increases the number of components.

A Bridge: of a graph is line whose removal increases the number of components.

In Figure II.17 is a graph G having two components G_1, G_2. Removal of point v increases number of components, and removal of line y will increase number of components. v is a cut point, y is a bridge.

A Star: is a complete bigraph with V_1 having only one point. A star with p points is denoted $K_{1,p-1}$. $K_{1,3}$ is a star with 4 points and is shown in Figure II.16.b.

The Direct Sum: of two oriented graphs $G_1(u_i,x_i)$ and $G_2(v_i,y_i)$, denoted by $G_1 + G_2$, is a graph $G = G_1 + G_2$. The points of G are the cartesian product of the points of G_1 and G_2; and there exists a directed line in G from the point $w_i = (u_i,v_i)$ to the point $w_j = (u_j,v_j)$ whenever the following is true: $u_i = u_j$ and there is a directed line in G_2 from v_i to v_j; or $v_i = v_j$ and there is a directed line in G_1 from u_i to u_j.

In Figure II.18.b., the point a and the point b have the same first coordinate: (1); and in G_2 there is a directed line from point 3

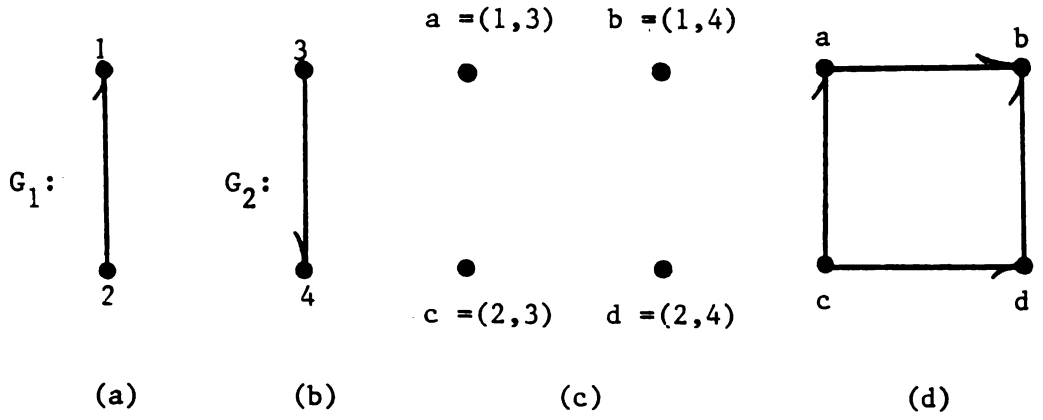


Figure II.18. The direct sum of two labeled orineted graphs. (a) and (b) are two labeled oriented graphs; (c) the cartesian product of their points; (d) the direct sum graph.

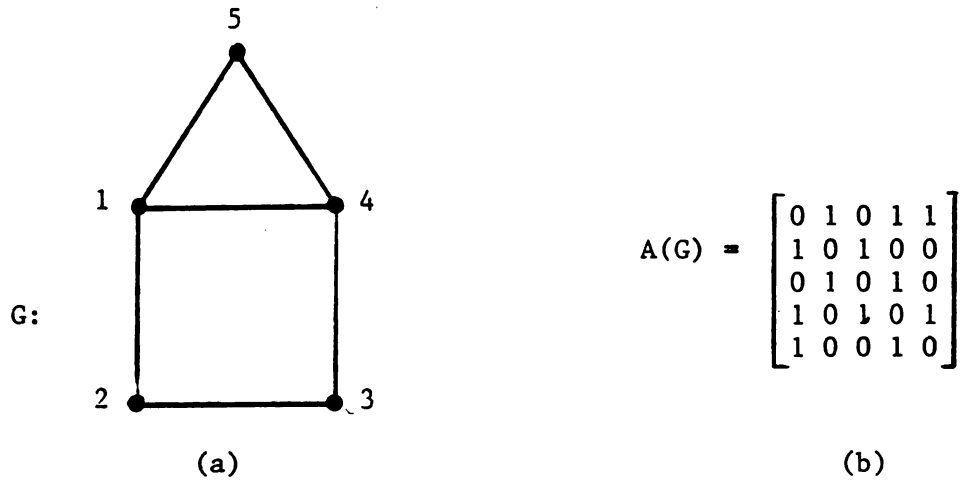


Figure II.19. (a) a (unoriented) graph G; (b) the adjacency matrix of the graph G.

to point 4. They are the second coordinates in point a and point b. So we join point a and b by a directed line as shown in Figure II.18.d. On the other hand, point a and point d do not have any coordinates that are identical so no line joins a and d in Figure II.18.d. Point b and point d have the second coordiantes identical and there is a directed line in G_1 joining point 2 to point 1, so we insert a directed line from d and b as in Figure II.18.d.

II.3.2. Matrices Associated with Graphs

A graph can be completely determined by its adjacencies and incidence. This information can be stated in matrix form. Here we introduce two of those matrices, namely the adjacency and incidence matrices. They have been studied in detail in graph theory and spectral graph theory. We also introduce a new matrix, the skew symmetric adjacency matrix. It is associated with oriented graph, and is basic to the results proven in this work.

The adjacency matrix of an unoriented graph

Let G be an unoriented graph with p points. Then the adjacency matrix $A(G) = [a_{ij}]$ is a pxp matrix with its entries a_{ij} as follows:

a_{ij} = 1 if point i is adjacent to point i,

a_{ij} = 0 otherwise.

In Figure II.19.a. is a labeled unoriented graph and in Figure II.19.b. is its adjacency matrix. Note that the adjacency matrix of an unoriented graph is symmetric.

The adjacency matrix of an oriented graph

Let G be an oriented graph with p points, then the adjacency matrix $A(G) = [a_{ij}]$ is a pxp matrix with its entries a_{ij} as follows:

$a_{ij} = 1$ if there is a directed line form point i to point j

$a_{ij} = 0$ otherwise.

Note that this matrix can be symmetric only in case of two directed lines between two points. This matrix is mentioned here for the purpose of not confusing it with the skew symmetric adjacency matrix defined below.

Incidence matrix

Associated with a graph G(p,q) a matrix B pxq with entries $b_{ij} = 1$ if v_i and x_j are incident, $b_{ij} = 0$ otherwise.

The skew symmetric adjacency matrix

The skew symmetric adjacency matrix is the gyroadjacency matrix, defined in section II.2., if all gyrators have unit modulus. For an oriented graph G(p,q) define a pxp matrix $\tilde{A} = [\tilde{a}_{ij}]$ such that;

$\tilde{a}_{ij} = +1$ if there is a directed line from j to i,

$\tilde{a}_{ij} = -1$ if there is a directed line from i to j, and

$a_{ij} = 0$ otherwise.

In Figure II.20.a. is an oriented graph G. The skew symmetric adjacency matrix associated with G is:

$$\tilde{A}(G) = \begin{bmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$$

(a) (b)

Figure II.20 (a) an oriented graph with 4 points

(b) the unoriented graph obtained from G by relaxing the orientation.

If we relax the orientation on G, the obtained unoriented graph G shown in Figure II.20.b. has an adjacency matrix $A = [a_{ij}]$:

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Note that a_{ij} is the absolute value of the entries of the skew symmetric adjacency matrix $\tilde{A} = [\tilde{a}_{ij}]$:

$$a_{ij} = |\tilde{a}_{ij}|$$

We conclude that the operaiton of relaxing the orientation on the oriented graph, will result in an equivalent operation on the adjacency matrices. This operation will produce a symmetric adjacency matrix. The above definition could be also extended to a weighted oriented graph.

II.3.3 Spectrum of a Graph

Throughout this work we will be dealing with connected graphs, no multiple lines between two points, and no loops (a line between a point and itself). Thus we will be dealing with two different types of adjacency matrices. The skew symmetric adjacency matrix will be associated with oriented graphs and the symmetric adjacency matrix with unoriented graphs.

We mean by spectrum of a graph the roots of the characteristic polynomials of the adjacency matrix, A, given by $\det(A-\lambda U)=0$. 'A' here denotes the symmetric adjacency matrix if we are dealing with graphs that are not oriented, and the skew symmetric adjacency matrix as defined in II.3.2. if we are talking about oriented graphs.

Labeling: the spectrum of a grpah is invariant under different labeling. This is true for symmetric spectra and skew symmetric spectra

as well. If A is the symmetric or the skew symmetric adjacency matrix, and A^* is the new adjacency matrix obtained by the relabeling operation, then $A^* = P^{-1} AP$, where P are orthogonal permutation matrices.

Inverting Powers: on one point of an oriented graph. The skew symmetric spectrum of an oriented graph obtained from its skew symmetric adjacency matrix is invariant under the operation of switching all powers on one point. This operation is shown in Figure II.21. This operation is equivalent to the similarity transformation of multiplying the matrix A by the matrix E and E^{-1} where E is an identity matrix except for some entry e_{ii} of E; and $e_{ii} = -1$,

$$A = E^{-1}AE.$$

Next we introduce some theorems from theory of graph spectra. These theorems are applicable to the symmetric adjacency matrix. An asterisk will denote that the theorem applies also to the spectra of skew symmetric adjacency matrices.

Theorem 1 (GT2)

(a) If G is disconnected; then the spectrum of the graph G is equal to the union of the spectra of the components of G. (*)

(b) The sum of the eigenvalues of a graph is equal to zero. (*)

(c) If G has N points; and if C is the number of closed walks of length k in G; then

$$C = \text{tr } A^k = \sum_{i=1}^{N} \lambda_j^k$$

Theorem 2 (GT5)

Let G be a graph with N points, and let λ_i be an eigenvalue of G such that:

$$|\lambda_1| \leq \ldots \leq |\lambda_N|$$

then:

(a) $2 \cos \frac{\pi}{N+1} \leq |\lambda_1| \leq N-1$

(b) If G' is a subgraph of G, and λ_i' are the eigenvalues of the subgraph; then

$$|\lambda_1'| \leq |\lambda_1|$$

For the skew symmetric spectra, the bounds in theorem 2 (a) apply; however for this case, closer bounds exist. We hive these bounds in Chapter III.

Theorem 2 (b) applies in case of the skew symmetric spectra only in case G' is obtained from G by deleting at least one point. The case of a subgraph obtained by deleting a line does not apply here since the Frobenius theorem used for the proof of theorem 2 is applicable only for non-negative matrices.

II.4 Linear Algebra and Matrix Theory

In the following section we present some concepts and notations from Linear Algebra and Matrix theory. We also give some theorems relevant to graph spectra.

Eigenvalues: if A is an NxN matrix, a vector $\emptyset = x \in C^N$ is an eigenvector of A if there exists a scalar λ such that:

$$Ax = \lambda x.$$

λ is an eigenvalue of A.

The set of all eigenvalues of A are called the spectra of A. The polynomial

$$\det (A - \lambda U) = 0, \text{ where}$$

U is the identity matrix, is the characteristic polynomial. The roots of the characteristic polynomial are the eigenvalues of the matrix A.

We arrange the eigenvalues as follows:

$$|\lambda_N| \leq \ldots\ldots \leq |\lambda_1| \text{ , and}$$

we arrange their real and imaginary parts as follows: If $\lambda = a + ib$, and $i=\sqrt{-1}$, then:

$$|b_N| \leq \ldots\ldots \leq |b_1| \text{ , and}$$

$$|a_N| \leq \ldots\ldots \leq |a_1| \text{ .}$$

Note that λ_1 is not necessary equal to a_1+ib_1.

The following theorems from matrix theory are useful in studying the spectra of graphs:

1. For symmetric or hermitian matrices all eigenvalues are real.
2. For skew symmetric matrices all eigenvalues are pure imaginary.
3. The sum of the diagonal elements of a matrix, called the trace, is equal to the sum of the eigenvalues of that matrix.
4. If A is a skew symmetric then iA is Hermiltian, $i=\sqrt{-1}$.

Norms

A norm of a matrix is a real valued function defined on the space of matrices and satisfying the following relations:

For an arbitrary matrix A and B, and an arbitrary scalar c,

a) $N(A) \geq 0$; and $N(A) = 0$ if and only if $A=0$,

b) $N(cA) = |c|\, N(A)$

c) $N(A+B) \leq N(A) + N(B)$

d) $N(AB) \leq N(A).N(B)$

We define the following norms:

$N_2(A) = \max_{x \neq 0} \left[\frac{x^* A^* x}{x^* x} \right]^{\frac{1}{2}}$ called spectral norm;

$N_E(A) = \left[\Sigma |a_{ij}|^2 \right]^{\frac{1}{2}}$ called Euclidean norm.

Where the superscript * denotes the transpose of the matrix.

For any square matrix of dimension n:

1. $N_E(A) = N_E(|A|)$, where the entries of A are the moduli of the entries of A.
2. $N_2(|A|) \leq N_2(A)$
3. $N_2(A) \leq N_E(A) \leq n^{\frac{1}{2}} N_2(A)$
4. $N_2(A) \leq n^{\frac{1}{2}} N_2(|A|)$

Theorems for locating eigenvalues:

The following are theorems frequently used to local the eigenvalues of matrices:

1. Gerschgorin Theorem: for a matrix $A_{nxn} = [a_{ij}]$ the eigenvalues lay in the union of discs with center at a_{ij} and radius r_i given by:

$$r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}$$

2. Interlacing Theorem: for relating the eigenvalues of a matrix to the eigenvalues of its principal submatrices: Let A_{nxn} be a hermitian matrix with eigenvalues λ_i such that

$$|\lambda_n| \leq \ldots \leq |\lambda_1| \text{ , then}$$

$$\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$$

where μ_i are the eigenvalues of the mxm principal submatrix of A. We say that the eigenvalues of the principal submatrices of a matrix <u>interlace</u> with its eigenvalues.

3. <u>Average row sum theorem</u>: (Haemers) (GT3)

Let A be partitioned as follows:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix}$$

and A_{ii} is a square matrix for i = 1, ... , m;

let $B_{mxm} = [b_{ij}]$ where b_{ij} = the average row sum of A_{ij}; then the eigenvalues of B interlace with the eigenvalues of A in the sense defined in theorem 2.

4. <u>Interlacing theorem for the sum of two hermitian matrices</u>: if A, B and C are hermitian matrices with eigenvalues λ_i, μ_i and ν_i respectively, and if:

$$C = A + B$$

then

$$\nu_{r+s-1} \leq \lambda_r + \mu_s$$

The above theorems will be used in proving some of the results given in this work.

5. <u>Properties of skew symmetric matrices</u>: there are important properties to know since the gyroadjacency matrix S is skew symmetric.

1) If S is NxN and N is odd, then there exists at least one zero eigenvalue.

2) If we denote by Spec (S), or spectrum of S, the the set of all the eigenvalues of S:

$$\text{Spec } S = [+ i \lambda_1, + i \lambda_2, \ldots, -i\lambda_2, -i \lambda_1]$$

then iS, $i = \sqrt{-1}$, is a hermitian matrix and

$$\text{Spec } (iS) = [+\lambda_1, + \lambda_2, \ldots, -\lambda_2, -\lambda_1]$$

This is an important fact since all theorems applicable to the spectrum of hermitian matrices will be applicable to the imaginary part of the spectrum of the skew symmetric matrices.

This will allow us to speak about the interlacing in the eigenvalues of skew symmetric matrices, meaning the interlacing in the imaginary parts of the eigenvalues.

CHAPTER III

BOUNDS FOR SIMPLE FULL GRAPHS

III.0 Introduction

Simple full graphs as defined in section II.2. are gyrobond-graphs (BG1) that have every I and every R element a one port, and the number of I field ports is equal to the number of 1-junctions.

The point graph of a simple full graph has full points only. (cf. II.2.1.).

A large class of systems can be represented by simple full graphs. Examples of these systems are given throughout this chapter. The bounds given here are used in some other classes of systems where simple full graphs are subgraphs of the point graph (cf. Chapter V). Also for large scale systems an estimate of the largest eigenvalue obtained from the point graph will save a large computational effort. In the following chapter we provide these bounds as a function of the number of 1-junctions in the simple full graph, or the number of points N in the point graph, and the structure of the graph.

We present bounds on the largest eigenvalue of systems. The information required to obtain these bounds is:

1) Number of points of the point graph of the system N.
2) The type of parameters the system has:
   a. Uniform parameters (defined below). (Sec. III.1.)
   b. General parameters. (Sec. III.2.)

3) The structure of the point graph:

a. a tree

b. a general graph

In III.1. we give the results for systems with uniform parameters. In section III.1.1. are three examples of physical systems that have their point graph a tree, a cycle, and a general graph. In section III.1.2. we give the bounds for systems whose point graph is a tree, then for systems whose point graph is a general graph. In section III.1.3. we present important properties of cycles. Section III.1.4. the case of simple full graph with partial R field is discussed and an example demonstrating application of the results is given. In III.2. bounds for systems with general parameters are given. In section III.2.1. two examples of physical systems that have general parameters are presented along with the bounds obtained on their largest eigenvalues. In section III.2.2. the procedure for obtaining the bounds for systems with general parameters is given. Next we present definitions relevant to this chapter.

For a system represented by a simple full graph, the <u>state matrix</u> is given by:

$$A = (S_F - R_F)\ (I^{-1})$$

where (cf. II.2.2)

S_F...is the gyroadjacency matrix,

R_F...is the diagonal matrix of resistances, and

I^{-1}...is the diagonal matrix of the inverse of inertias.

A similarity transformation

Two systems are similar if they have the same spectrum. For a given system with general parameters, a similarity transformation using the point graph and a simple procedure will produce a similar system. The new system will have the same point graph as the original system with all the parameters of its inertias equal 1.

Tranformation procedure:

1) On each line replace each gyrator modulus r_{ij} by:

$$r^*_{ij} = \frac{r_{ij}}{\sqrt{I_i\ I_j}}$$

where I_i, I_j are the inertias on points i and j adjacent to r_{ij}.

2) On each point replace each resistance R_i by:

$$R^*_i = \frac{R_i}{I_i}$$

where I_i is the adjacent inertia.

In Figure III.1. is an example of the reduction procedure. The proof could be easily found from the fact that the matrices A^* and A^{**} are similar if:

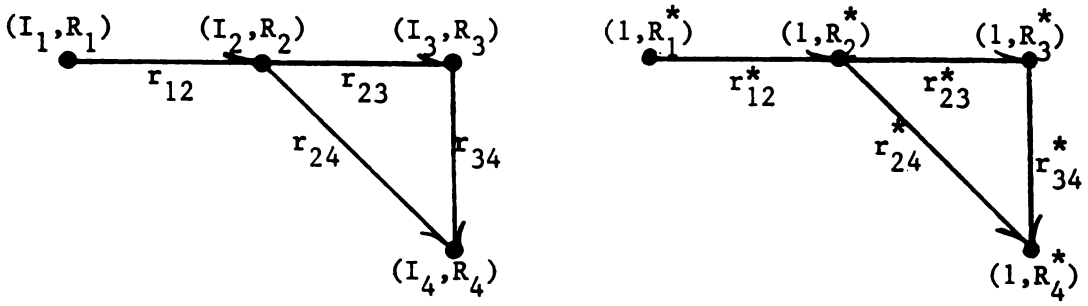
$$A^* = AD \quad \text{and} \quad A^{**} = D^{\frac{1}{2}}\ AD^{\frac{1}{2}}$$

and D is diagonal.

Systems with uniform parameters

Systems with uniform parameters will have:

$$R^*_i = \alpha \text{ and } r^*_{ij} = \beta$$



(a) (b)

$$R_i^* = \frac{R_i}{I_i} \qquad \text{for } i = 1, 2, 3, 4$$

$$r_{ij}^* = \frac{r_{ij}}{\sqrt{I_i I_j}} \qquad \text{for } i,j \text{ as above.}$$

Figure III.1. (a) and (b) two point graphs for two identical systems except for the parameters of their energy storage elements.

where α, β are constants and R_i^* and r_{ij}^* are defined as above. A possible case of uniform parameters will occur when $\alpha = \beta = 1$, in this case we say the system has unit parameters. Systems with uniform parameters are easily detected from their transformed point graph. This can be seen in Figure III.12.

Ordering the eigenvalues

If a system has an NxN state matrix A, and if

$$\lambda_j = a_k + ib_\ell \ , \ i = \sqrt{-1}$$

is an eigenvalue of the matrix A, we arrange the eigenvalues as follows:

$$|\lambda_N| \leq \ldots \leq |\lambda_1| \ , \text{ and}$$

we arrange the real and imaginary parts of the eigenvalues as follows:

$$|a_N| \leq \ldots \leq |a_1| \ , \text{ and}$$

$$|b_N| \leq \ldots \leq |b_1| \ .$$

Upper bounds on the largest eigenvalue, denoted by λ_u, is any number such that:

$$|\lambda_1| \leq \lambda_u$$

In the same sense the subscript u on either a or b, will denote a number that is an upper bound on either the real or the imaginary part of the eigenvalue:

$$|a_1| \leq |a_u|$$

$$|b_1| \leq |b_u|$$

Note that the upper bound on the largest eigenvalue is an upper bound on all the eigenvalues. A lower bound on the largest eigenvalue will not necessarily be a bound on any other eigenvalue.

Gerschgorin theorem

Applying the Gerschgorin theorm to the skew symmetric adjacency matrix or the gyroadjacency matrix of the transformed point graph gives:

$$b_1 \leq \Delta \quad \text{or} \quad b_1 \leq \max_i \sum_j r^*_{ij}$$

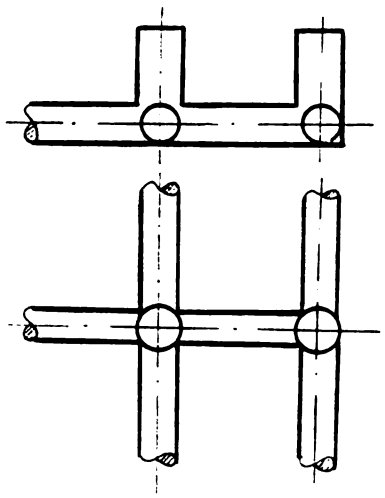
where Δ is the largest degree of the point graph associated with the system. This bound becomes efficient to use in case the number of points in the point graph is much larger than Δ. This is demonstrated in the example given in III.1.1 for a system with a general point graph.

III.1 Uniform Parameters

The bounds on the largest eigenvalue of systems with uniform parameters are studied for two major reasons:

1) Some physical systems are modeled as systems that have unit parameters. See, for example, lumped models of single power line continuous systems (cf. example 1 section III.1.1).
2) The bounds obtained for systems with unit parameters are used as a base to obtain bounds for the more general case of systems with general parameters.

Systems with uniform parameters could be recognized from the transformed point graph. This can be easily done by scanning the parameters on the r^*_{ij} and R^*_i on the point graph. Examples of these systems are given next in III.1.1. In section III.1.2. we give bounds first for tree point graphs, then for the general point graph. In III.1.3. properties of cycles are discussed; and in III.1.4. partial R field graph is discussed.



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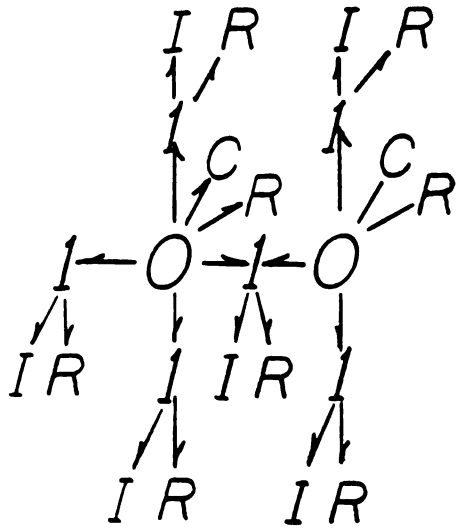

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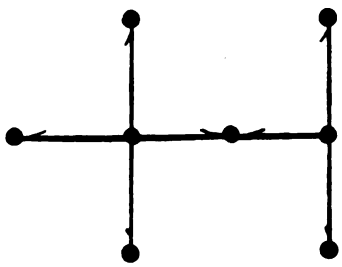


(c)

Estimates		Computed values
Using the bond graph	Using Gerschgorin	Using iterative methods
$1.87 \leq b_1 \leq 2.64$	$b_1 \leq 4$	$b_1 = 2.14$

Figure III.2. (a) A hydraulic system; (b) the bond graph model; (c) the point graph.

III.1.1 Examples of the Bounds

A tree

In this example we present a hydraulic system whose point graph is a tree. We give the bounds on the largest eigenvalue and compare it with the values obtained using iterative algorithms and also compare it with bounds obtained using the Gerschgorin Theorem.

In Figure III.2.a. we have a hydraulic system as shown. It consists of six pipes connected to two gravity tanks. In III.2.b. is the bond graph model and in III.2.c. is the point graph model. Here the number of points N= 8. If we assume all parameters equal unity: C = I = R = 1, then from results presented in this chapter the following bounds are given:

$$a = -1$$

$$1.87 \leq b_1 \leq 2.64$$

A bound computed using Gerschgorin Theorem would give an upper bound.

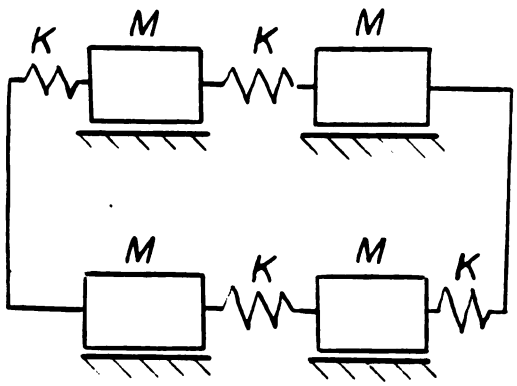
$$b_1 \leq 4$$

The iterative methods provide accurate values; using the EISPACK routines we get for b_1,

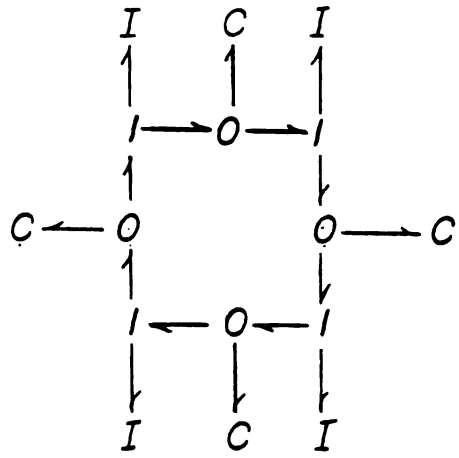
$$b_1 = 2.14$$

A cycle

Consider the system of Figure III.3.a. Here we have four masses of mass M and four massless springs of spring constant K. In Figure III.3.b. is the bond graph and in Figure III.3.c. is the point graph. Here the number of points is N = 8. This is a case of uniform parameters. From results presented in this chapter and for this case we



(a)



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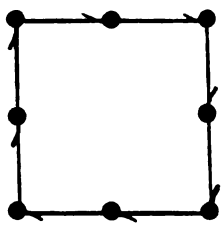


(c)

Figure III.3. (a) A mechanical vibrating system; (b) the bond graph model; (c) the point graph, with spectra given by:

$$\text{Spec}(G) = [\pm i\, 2\sqrt{\frac{K}{M}},\ \pm i\sqrt{\frac{2K}{M}},\ \pm i\sqrt{\frac{2K}{M}},\ 0,0]$$

know all the eigenvalues exactly. They are:

$$\text{Spec } (G) = (\pm i2\sqrt{\frac{K}{M}},\ \pm i\sqrt{\frac{2K}{M}},\ \pm i\sqrt{\frac{2K}{M}},\ 0,0)$$

The above results coincide with the results given by J.N. Boyd (BG8), except here we used massless springs. The example was set as masses arranged around a circle with all motion confined to the circle. It demonstrated the use of projection operators in obtaining natural frequencies of a one-dimensional crystal.

A general graph

Consider the lumped-parameter bond graph model of a two power distributed system. The bond graph model of the Timoshenko model for transverse vibration of a prismatic bar with shear corrections is adapted from Bonderson (BG9) and given for 3 microelements in Figure III.4.a. In Figure III.4.b. is the point graph; it has N = 12 points. Assume that all parameters are equal to 1. Then from results presented in the next two sections, the following bounds could be estimated for b_1:

$$1.941 \leq b_1 \leq 7.59$$

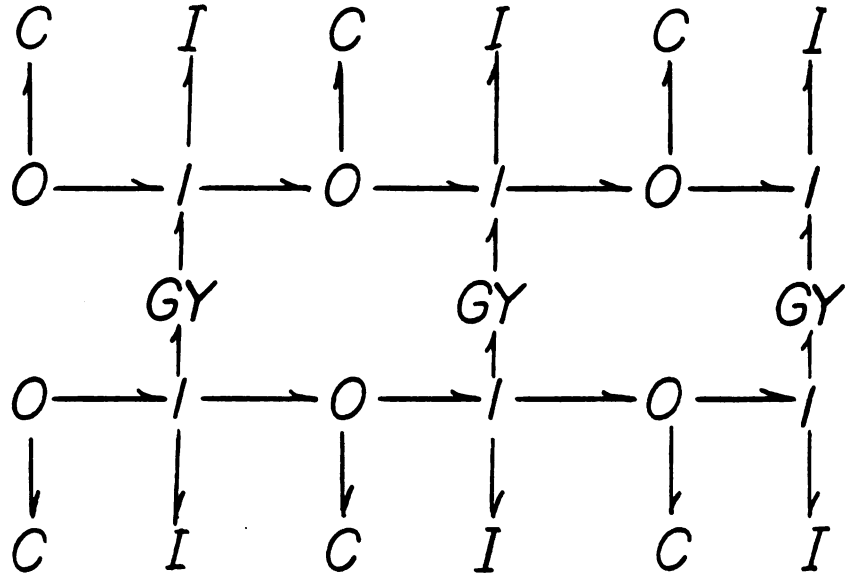
Using Gerschgorin Theorem gives a better upper bound on b_1.

$$b_1 \leq 3$$

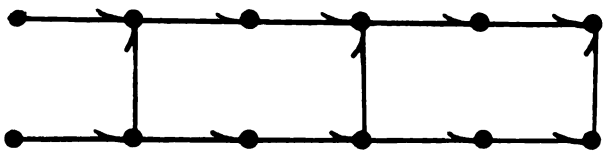
The computed value using interative techniques gives:

$$b_1 = 2.370$$

We note that the difference between the upper bound and the real value is too large. In this case the use of Gerschgorin Theorem would give closer bounds. (In Chapter IV we present techniques that would reduce the upper bound to $b_1 \leq 2.802$.)



(a)



(b)

Estimates		Computed value
Using the point graph	Using Gerschgorin	Using iterative methods
$1.941 \leq b_1 \leq 7.59$	$b_1 \leq 3$	$b_1 = 2.370$

Figure III.4. a) A bond graph of a lumped parameter model of a two power distributed system: transverse vibration in a prismatic beam with shear corrections. The model is adapted from L.S. Bonderson (BG9). Here we used three elements. b) Is the point graph model.

III.1.2 Bounds on Largest Eigenvalue

In this section we give the bounds on the largest eigenvalue of systems with uniform parameters. A plot that depicts the bounds for trees is given in Figure III.5. A plot of the bounds for general graphs is in Figure III.6. In Figure III.7. is a plot showing the lower bounds on arbitrary graphs.

Trees

Let G be a point graph associated with a system with uniform parameters. Let G be such that:

(a) G has no cycles

(b) G has N points; and all the points are full.

Let $\lambda = a + ib$, be an eigenvalue of the system,

(a) If G is a star, then

$$\text{Spec } (G) = [\pm i\beta\sqrt{N-1},\ 0,\ \ldots\ 0] \quad \text{and}$$

$$b_u = \beta\sqrt{N-1} \equiv b_{uS}$$

(b) If G is a path, then

$$\text{Spec } (G) = [\pm\, i\ 2\beta \cos \frac{\pi}{N+1}, \ldots, i\ 2\beta \cos \frac{k\pi}{N+1}],\ k=1,\ldots,N$$

and $$b_u = 2\beta \cos \frac{\pi}{N+1} \equiv b_{uP}$$

(c) For any G that is a tree with N points

$$2\beta \cos \frac{\pi}{N+1} = b_{uP} \leq b_l \leq b_{uS} = \beta\sqrt{N-1}$$

(d) The spectrum of the symmetric adjacency matrix of the oriented graph obtained by relaxing the orientation on G, and the spectrum of the skew symmetric adjacency matrix are identical. (Apart from $i = \sqrt{-1}$.)

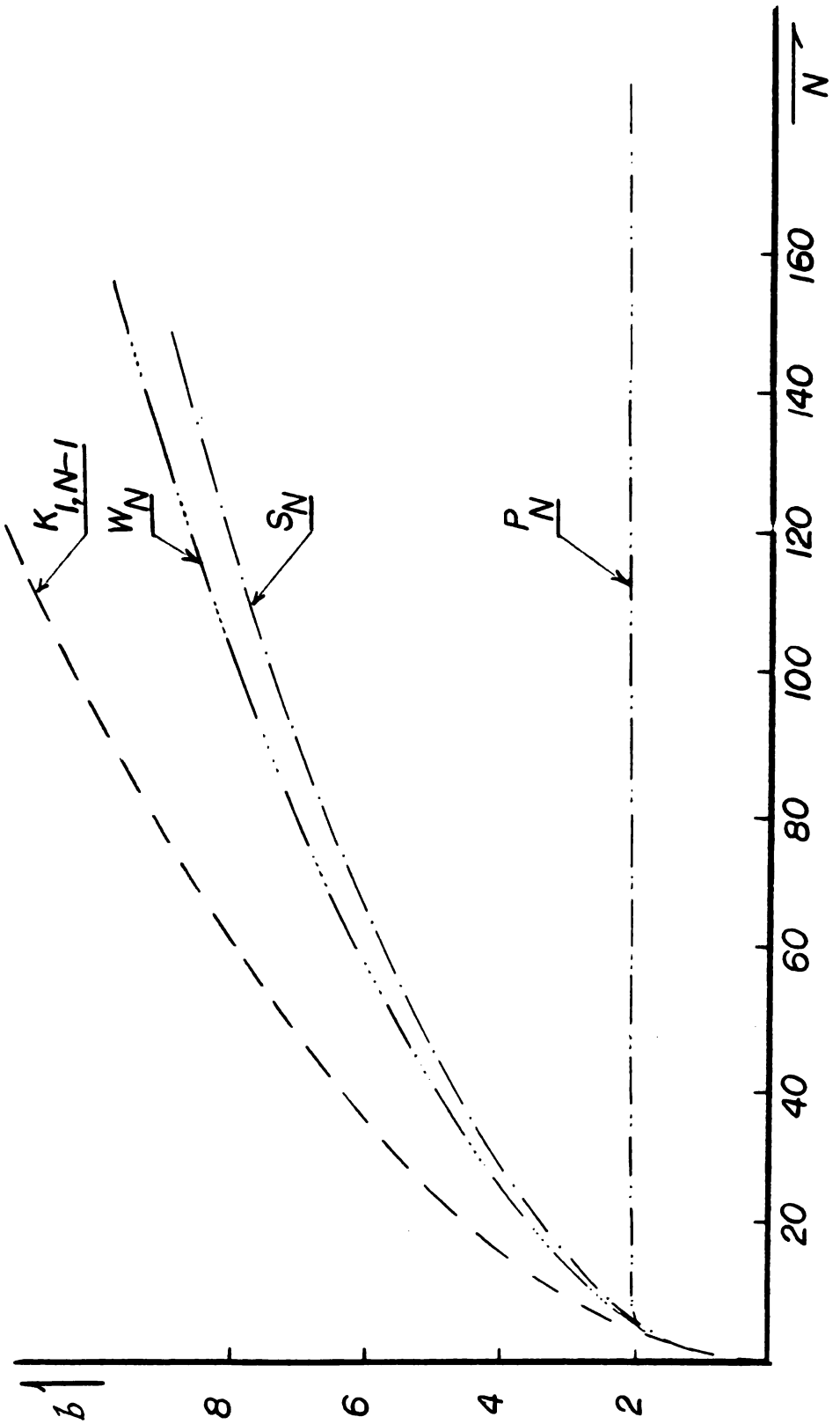


Figure III.5. The largest imaginary part of the eigenvalue of some trees: b; versus their number of points N.

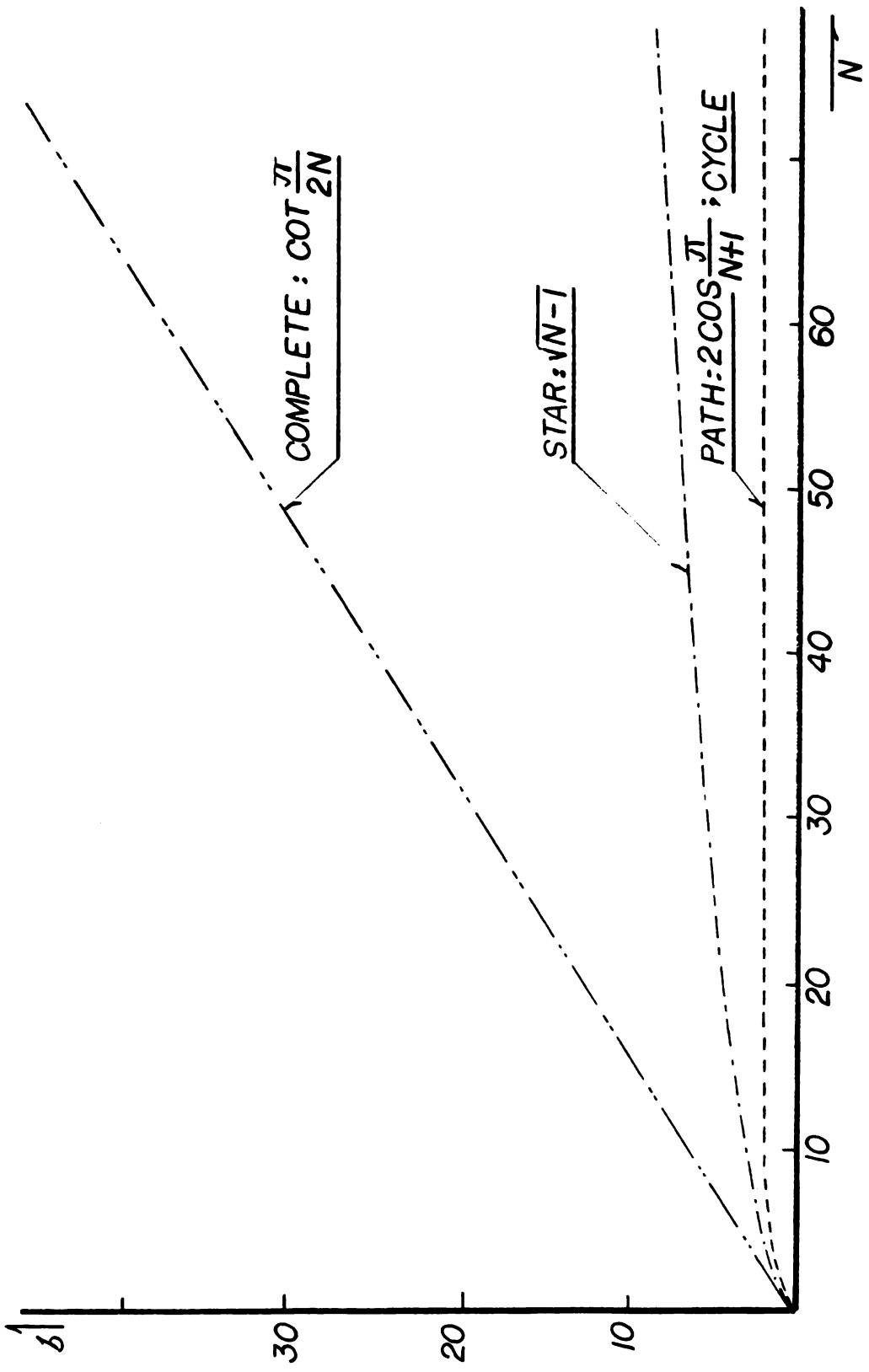
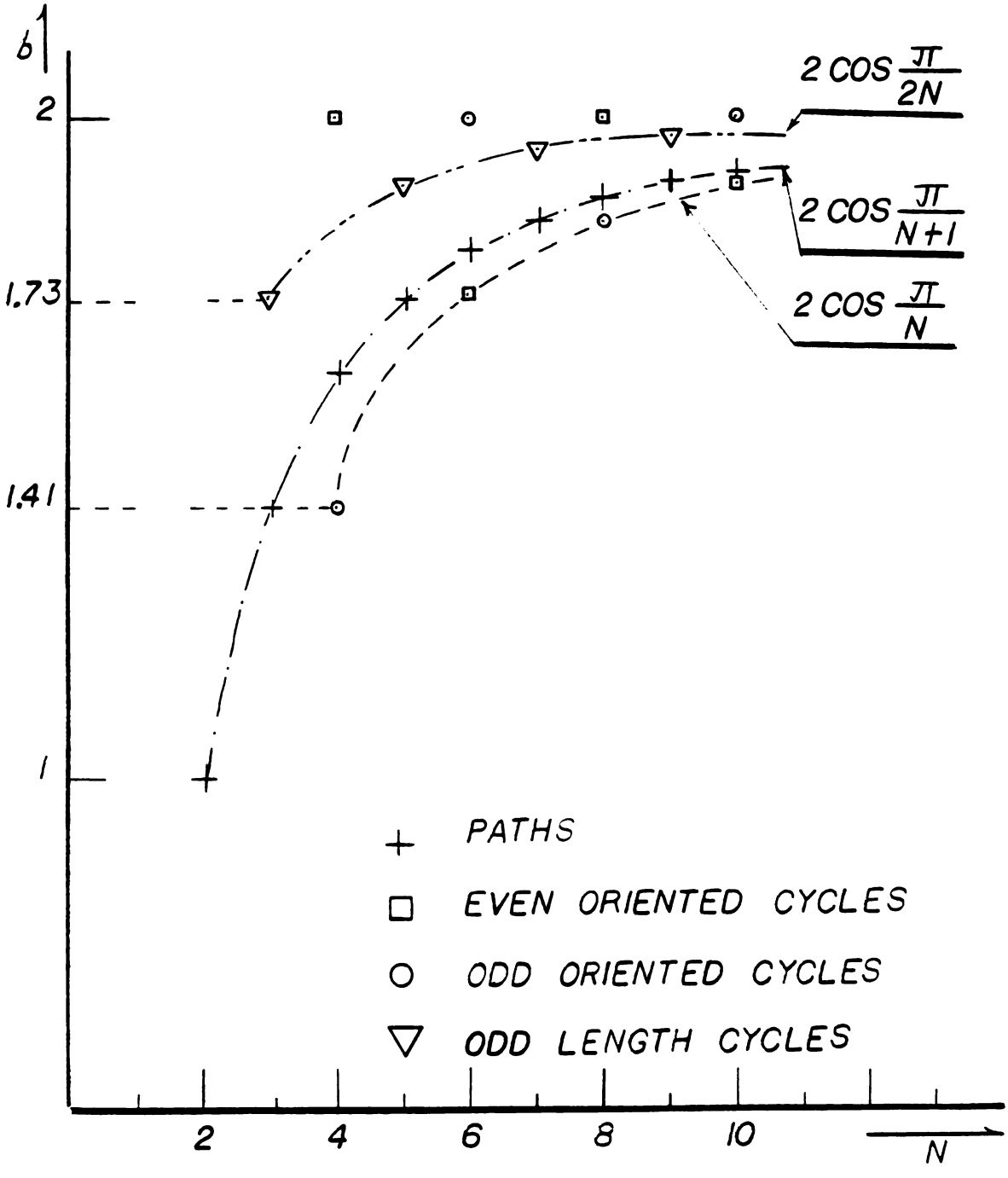
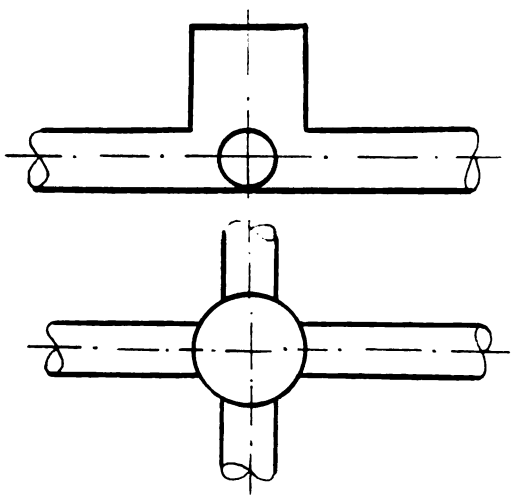


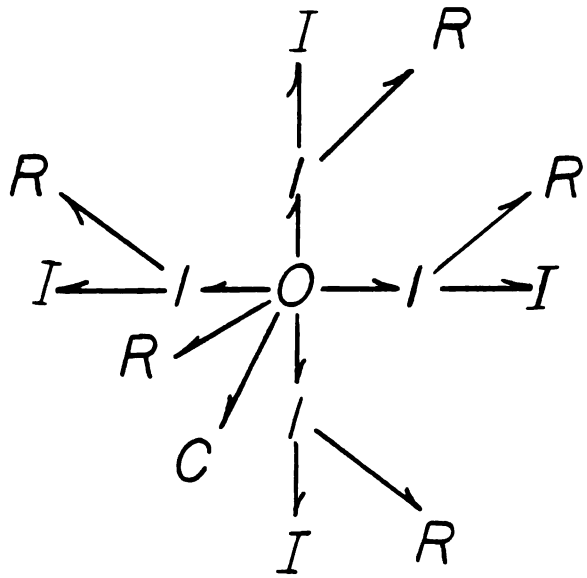
Figure III.6. Bounds on the largest imaginary part of the eigenvalue: b versus number of points N for bound graphs: K_N, $K_{1,N-1}$, P_N, C_N.



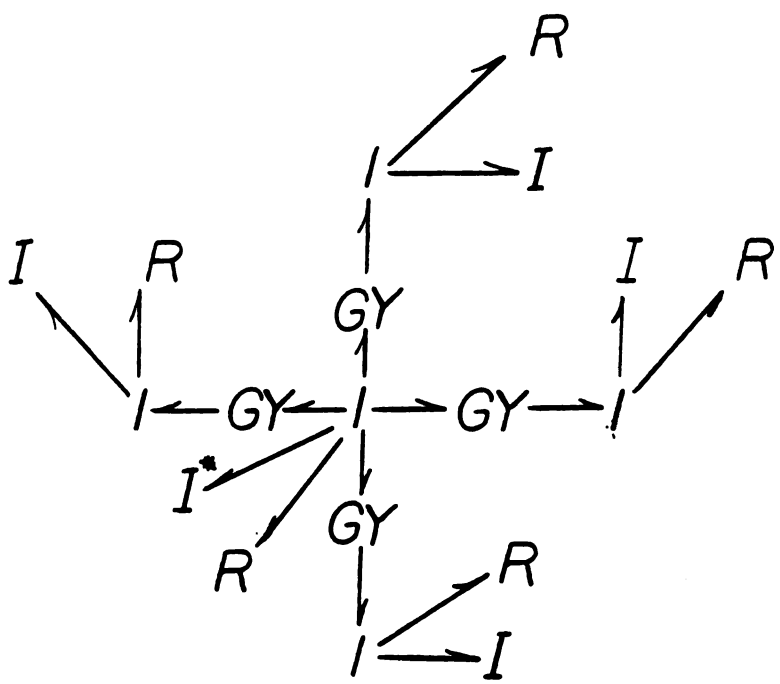
× Figure III.7. The largest imaginary part (b) of the eigenvalues of paths and cycles versus the number of points N.



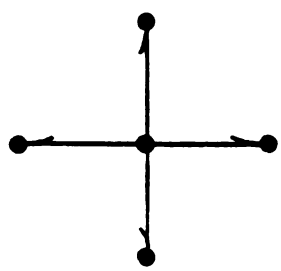
(a)



(b)



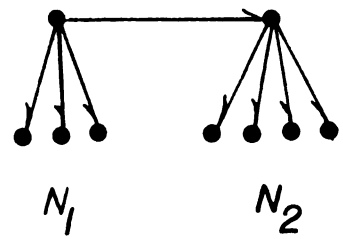
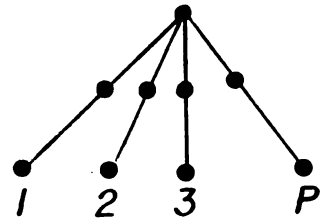
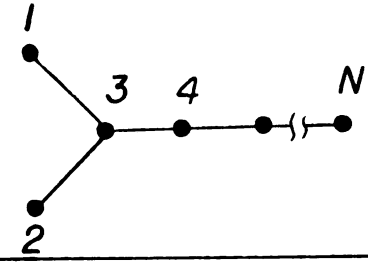
(c)



(d)

Figure III.8. a) A hydraulic system composed of a water tank and four tubes; b) the bond graph model; c) the gyrobondgrahp; d) the oriented graph with 5 points and spectrum: (2i, -2i, 0,0).

Table III.1 The Spectra of some trees.

Name of tree and structure	Spectrum of corresponding system
W_N 	$\frac{\pm i\sqrt{+(N-1)-\sqrt{(N-1)^2-4N_1 \cdot N_2}}}{\sqrt{2}}$, $\frac{\pm i\sqrt{+(N-1)+\sqrt{(N-1)^2-4N_1 \cdot N_2}}}{\sqrt{2}}$, 0, 0... Note: $b_u = \frac{\sqrt{+(N-1)+\sqrt{(N-1)^2-4N_1 \cdot N_2}}}{\sqrt{2}}$, and we have: (N_1+N_2-2) zero roots.
S_N 	$\pm i\sqrt{p+1}, i, i, \ldots, 0, \ldots, -i, -i, -i\sqrt{p+1}$ Note: $b_u = \sqrt{p+1}$, and one zero root. and (N-3) root with modulus equal one.
Z_N 	$2\cos\frac{\pi}{2(N-1)}$, $2\cos\frac{3\pi}{2(N-1)}$, ..., $2\cos\frac{(2k+1)\pi}{2(N-1)}$, ... Note: $b_u = 2\cos\frac{\pi}{2(N-1)}$

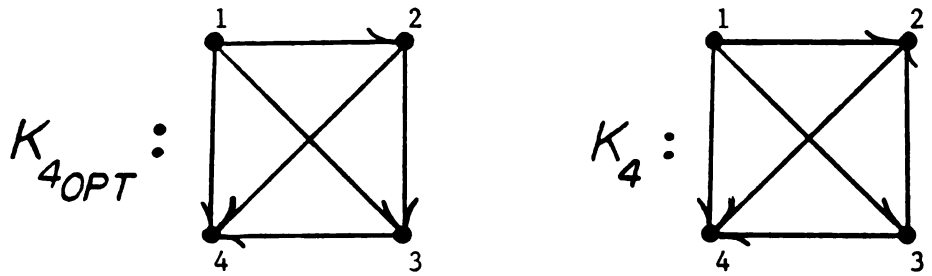


Figure III.9. (a) Is an optimal graph for the graph in (b).

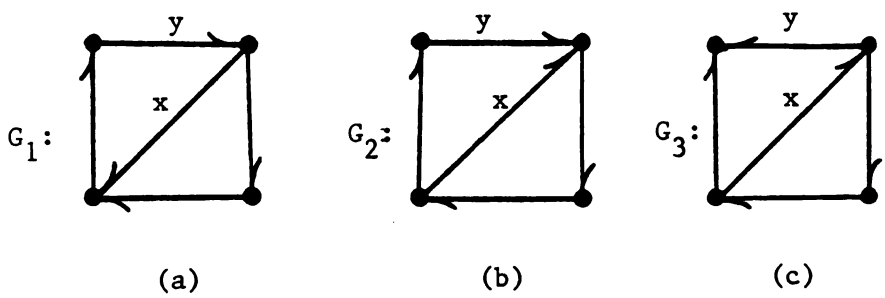


Figure III.10. In (a), (b) and (c) Are three oriented graphs with four points and five lines. The orientation is different on the lines. The spectra of the first two are identical, the spectrum of the third is different.

(e) Spec (G) is invariant under power orientation.

(f) The real parts of the system's eigenvalues: $a = -\alpha$

An example of a system that has a star point graph is given in Figure III.8. The constant α, β are as defined in uniform parameter systems.

Some special cases of trees

In Table III.1. are the spectra of some trees. These spectra could be found easily by applying the recursive formula in Chapter IV. The result (d) will allow us to use the results of the well developed spectral graph theory to obtain insight into the spectra of systems whose point graph is simple full and has no cycles. Symmetric spectra of all trees with $N \leq 10$ are given in Cvetkovic (CT2).

General graph

Here we present bounds on systems whose gyrobondgraph is a simple full graph. The condition of no cycle is relaxed. The bounds are given for a point graph that may have cycles.

Define the optimal graph (G_{opt}) of a point graph G with N points and q lines as a graph with N points and q_1 lines such that

$$q \leq q_1$$

and the power orientation on the lines of G_{opt} is such that if $b_1(G)$ is the largest eigenvalue in Spec (G), then

$$b_1\ (G) \leq b_1\ (G_{opt})$$

The optimal graph (K_{Nopt}) of a complete point graph with N points (K_N) is given in Figure III.9.a. for N = 4.

Table III.2. Summary of the spectra of cycles. K = 0, 1, . . ., N-1

N = number of points \ Power		For Even Power b =	For Odd Power b =
N odd		$2\cos\frac{(2K+1)\pi}{2N}$ $b_u = 2\cos\frac{\pi}{2N}$	
N even	N/2 even	$2\cos\frac{2K\pi}{N}$ ** $b_u = 2$	$2\cos\frac{(2K+1)\pi}{N}$ * $b_u = 2\cos\frac{\pi}{N} = b_{uC}o$
	N/2 odd	$2\cos\frac{(2K+1)\pi}{N}$ ** $b_u = 2\cos\frac{\pi}{N} = b_{uC}e$	$2\cos\frac{2K\pi}{N}$ * $b_u = 2$

*For odd power even points, the spectra does not have a zero.
**For evenpower even point, the spectra have two zero eigenvalues.

For a system with uniform parameters, if the associated point graph G has N points all of them are full points. If $\lambda = a + ib$ is an eigenvalue of the system, then

If G is K_N then

$$\text{Spec } (K_{N\text{opt}}) = [i\beta \cot \frac{\pi}{2N} , i\beta \cot \frac{3\pi}{2N} , \ldots , i\beta \cot \frac{(2k+1)\pi}{2N}]$$

$$k = 0,1,\ldots, N-1$$

and $b_u = \beta \cot \frac{\pi}{2N} = b_{uKopt}$

If G is a cycle, the spectrum is given in Table III.2.2.

If G is a general graph and if N is odd, then

$$2\beta \cos \frac{\pi}{N+1} = b_{uP} \leq b_1 \leq b_{uKopt} = \beta \cot \frac{\pi}{2N} , \text{ and}$$

If N/2 is even

$$2\beta \cos \frac{\pi}{N} = b_{uC}{}^{o} \leq b_1 \leq b_{uKopt} = \beta \cot \frac{\pi}{2N} , \text{ and}$$

If N/2 is odd, then

$$2\beta \cos \frac{\pi}{N} = b_{uC}{}^{e} \leq b_1 \leq b_{uKopt} = \beta \cot \frac{\pi}{2N} .$$

The proof of the upper bound could be derived easily by finding the graph associated with the matrix H given in Pick's Theorem (see Bodewig (NA2)). The matrix H has the largest eigenvalue over all skew symmetric matrices of the same order.

The proof for lower bounds is given in appendix 7 .

Power Orientation (on lines of general graphs)

If the power orientation on a line that is not part of any cycle or that is part of odd numbered cycles only, is changed, the spectrum of the point graph does not change. This power orientation rule is a

corollary of results in Chapter IV concerning the characteristic polynomial of the point graph. For more information about the effect of power orientation see Chapter IV. An example of the power orientation rule is given in Figure III.10.a. and Figure III.10.b. We see two point graphs with the same spectrum since the power orientation is changed on the line x that belongs only to odd numbered cycles. The point graphs in Figure 10.c. will have different spectra since the power orientation is changed on line y that belongs to even numbered cycles.

III.1.3. Properties of Cycles

The spectrum of a cycle with large number of points can be equivalent to the union of spectra of graphs of smaller numbers of points. This can be used to substantially decrease the number of computations, when computing the eigenvalues of a cycle that has a large number of points.

Equivalence of cycles

Let C_N^e denote a cycle with N points and even power, and

let C_N^o denote a cycle with N points and odd power, and

let Z_N, P_N denote the tree in Table III.1, and the paths with N points, then

(a) For N = odd number

$$\text{Spec } (C_N) = \text{Spec } (Z_{N+1}) - (0)$$

(b) For N/2 = odd number (N = 6,10,...)

$$\text{Spec } (C_N^e = \text{Spec } (C_{N/2}) \cup \text{Spec } (C_{N/2})), \text{ and}$$

$$\text{Spec } (C_N^o) = \text{Spec } (P_{(N-2)/2}) \cup \text{Spec } (P_{(N-2)/2}) \cup (2, -2)$$

(c) For N/2 = even number (N = 8,12,...)

$$\text{Spec } (C_N^e) = \text{Spec } (C_{N/2}^o) \text{ U Spec } (C_{N/2}^o), \text{ and}$$

$$\text{Spec } (C_N^o) = \text{Spec } (Z_{(N+2)/2}) \text{ U Spec } (Z_{(N+2)/2}) - (0,0)$$

The above properties are due to the fact that for a cycle, the characteristic polynomial has a cosine function as follows:

If N is odd, then

$$\det (S_F - \lambda U) = \cos N\theta = 0, \text{ and}$$

If N is even and the cycle has even power, then

$$\det (S_F - \lambda U) = \cos N\theta - i^N = 0, \text{ and}$$

If N is even and the cycle has odd power, then

$$\det (S_F - \lambda U) = \cos N\theta + i^N = 0.$$

The determinants described above could be obtained from the recursive formula given in Chapter IV. The solution of the characteristic equations given above, gives the properties that are given in Table III.2. The equivalence properties could be deduced from Table III.2. From the equivalence identities we can conclude that a cycle, on which power change will produce a change in the associated spectrum is reducible to a graph whose spectrum is invariant under power change.

The possible reduction in the number of computations is given in the following examples.

Suppose we need to find the spectrum of C_{48}^e. From the equivalence identities we have

$$\begin{aligned}\text{Spec } (C_{48}^e) &= \text{Spec } (C_{24}^e) \text{ U Spec } (C_{24}^0) \\ &= \text{Spec } (C_{12}^e) \text{ U Spec } (C_{12}^0) \text{ U 2 * Spec } (Z_{13}) - (0,0) \\ &= \text{Spec } (C_6^e) \text{ U Spec } (C_6^0) \text{ U 2 * Spec } (Z_7) \text{ U 2 *} \\ &\quad \text{Spec } (Z_{13}) - (0,0,0,0)\end{aligned}$$

$$= 2 * \text{Spec } (C_3) \text{ U } 2 * \text{Spec } P_2 \text{ U } 2 * \text{Spec } (Z_7)$$

$$\text{U } 2 * \text{Spec } (Z_{13}) - (0,0,0,0) + (2,2)$$

where for shorthand we used

2 * Spec (.) for:Spec (.) U Spec (.)

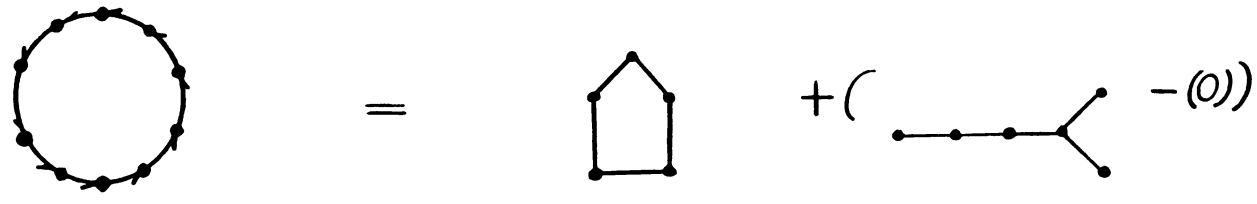
In this case the reduction of computation is obvious. Instead of computing the spectrum of a graph with 48 points, all we need is to find the spectrum of a tree with at most 13 points. The spectrum of Z_N is given in Table III.1.

In Table III.3. are more examples for the spectra of cycles and the equivalent graphs.

A question that is not yet answered is: could the above equivalence be generalized to some more general graphs?

The zeros in the spectrum of a cycle: a double zero in the spectrum of a mechanical transational system, denotes the existance of a rigid body mode of motion in the system. The only cycles that have a double zero in their spectra are cycles with even number of points and even power. This fact can be deduced from the general term for the eigenvalues of the cycles given in Table III.II. Examples of systems whose point graphs are cycles with even number of points and even powers are given in Figure I.2.b. in Chapter I, and in Figure III.3. Notice that in these systems a rigid body motion mode is possible. Cycles with even number of points and odd power do not have zeros in their spectra. An example of a system whose point graph is a cycle with even number of points and odd power is given in Figure III.11. We note that this system does not have a possible rigid body motion.

Table III.3. Example of equivalence of the spectra of cycles.

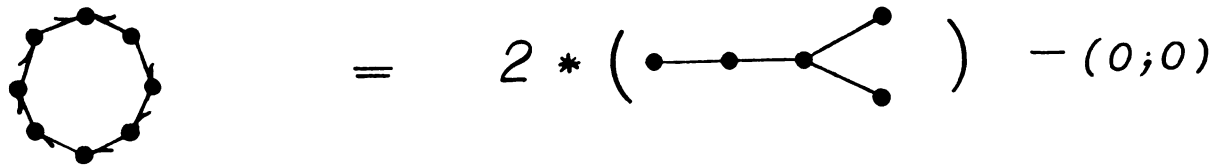


$Spec(C_{10}^{e}) = Spec(C_5) + [Spec(Z_6) - 0\ ]$

$Spec(C_{10}^{e})$ = (1.902,1.176,0.0,-1.902,-1.176,-1.902,1.176,0.0,-1.902,-1.176)

Spec (C_5) = (1.902,1.176,0.0,-1.902,-1.176)

Spec (Z_6) = (1.902,1.176,0.0,0.0, -1.902,-1.176)

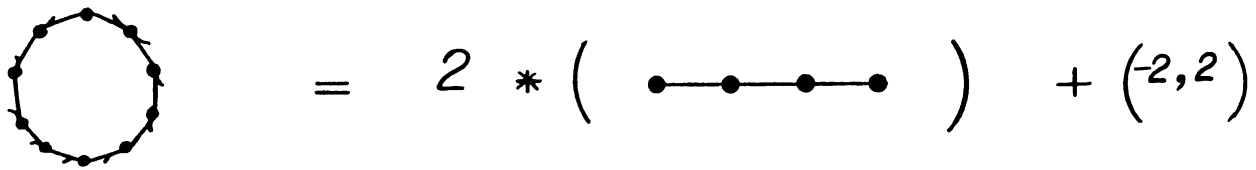


$Spec(C_8^{o}) = Spec(Z_5) + Spec(Z_5) - (0,0)$

$Spec(C_8^{o})$ = (1.84,0.765,-1.84,-0.765,1.84,0.765,-1.84,-0.765)

$Spec(Z_5)$ = (1.84,0.765,-1.84,-0.765,0.0)

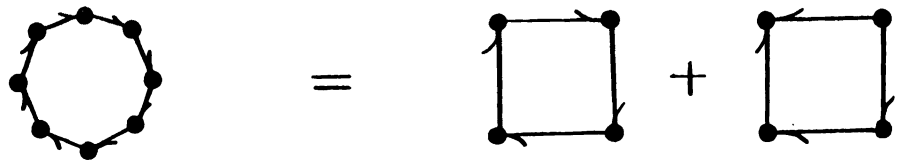
Table III.3. (continued)



$Spec(C_{10}^{o})$ = 2 * Spec (P_4) + (-2,2)

$Spec(C_{10}^{o})$ = (2,-2,1.618,0.618,-1.618,-0.618,1.618,0.618,1.618,-0.618)

$Spec(P_4)$ = (1.618,0.618,-1.618,-0.618)



$Spec(C_8^e)$ = $Spec(C_4^e)$ + $Spec(C_4^o)$

$Spec(C_8^e)$ = (2,-2,1.414,-1.414,1.414,-1.414,0.0,0.0)

$Spec(C_4^e)$ = (2,-2,0,0)

$Spec(C_4^o)$ = (1.414,-1.414,1.414,-1.414)

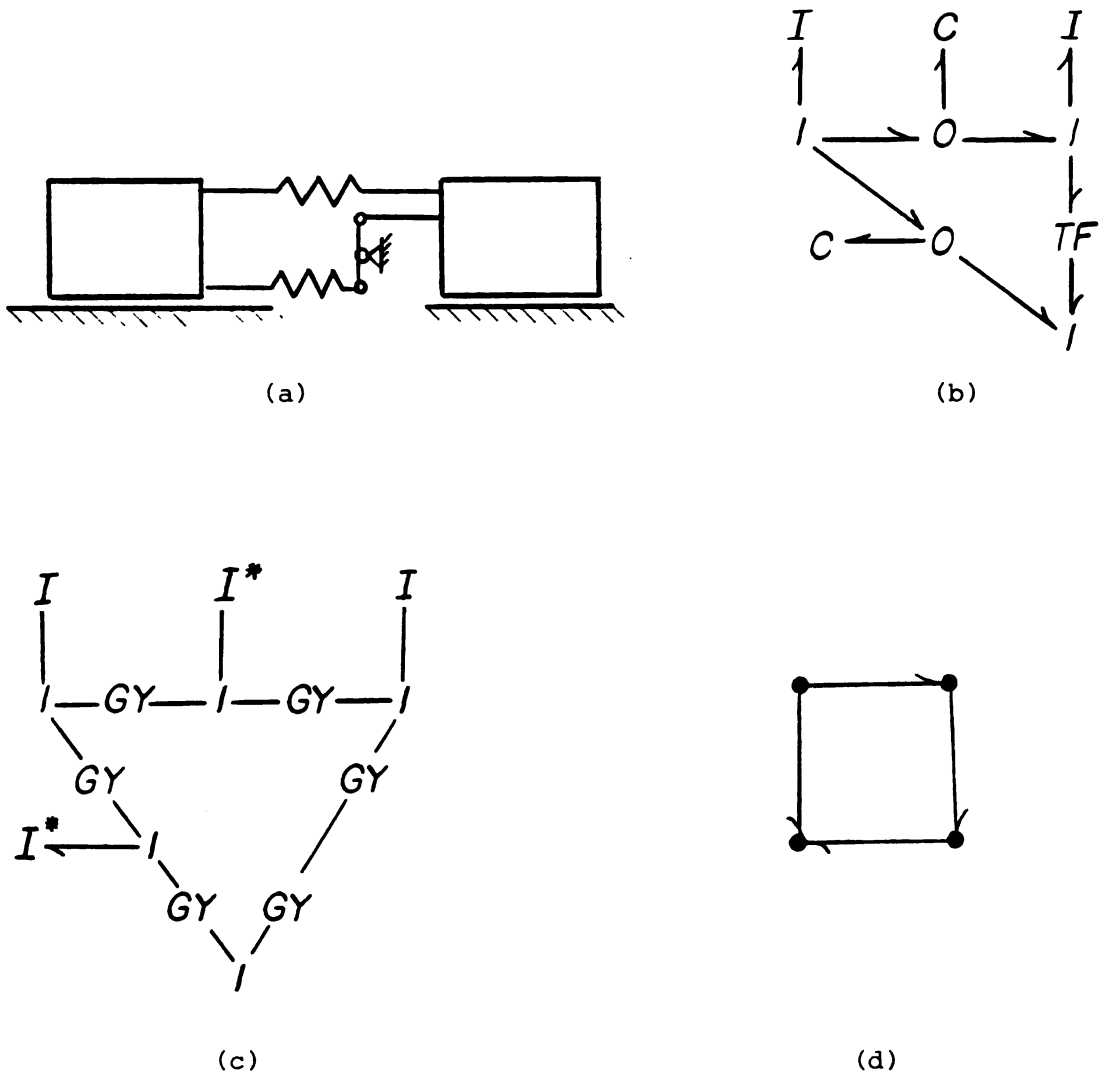


Figure III.11. a) A mechanical transational system consisting of two masses, two springs, a lever with transformation ratio equal 1. b) The bond graph model; c) the gyrobondgraph reduction; d) the corresponding oriented graph with four points and spectrum:

$$\text{Spec}(G)=[i\sqrt{2},\ i\sqrt{2},\ -i\sqrt{2},\ -i\sqrt{2}]$$

III.1.4 Partial R Field

If a system has uniform parameters, as defined in III.1., and if the associated point graph has N full I points and m full R points then:

$$-\alpha \leq a_1 \leq -\alpha \cdot \frac{m}{N}$$

$$-\alpha \cdot \frac{m}{N} \leq a \leq 0$$

$$b_1(A) \leq b_1(S)$$

where α, β are as defined in the introduction of Chapter III and $b_1(A)$ denotes the imaginary part of the largest eigenvalue of the state matrix of the system A, and $b_1(S)$ is the largest eigenvalue of the skew symmetric adjacency matrix S of the point graph.

The above result is proved using a theorem given by Amir-Moéz (NA12) and by finding the symmetric and the skew symmetric components of the state matrix.

An example of this class of systems is the beam in Figure III. 12.a. Consider the single power lumped-parameter model of the beam in Figure III.12.b. Note that here we have two resistances on the second and the fifth masses. The bond graph model is given in Figure III.12.d. In the point graph we have: N = 10, and m = 2.

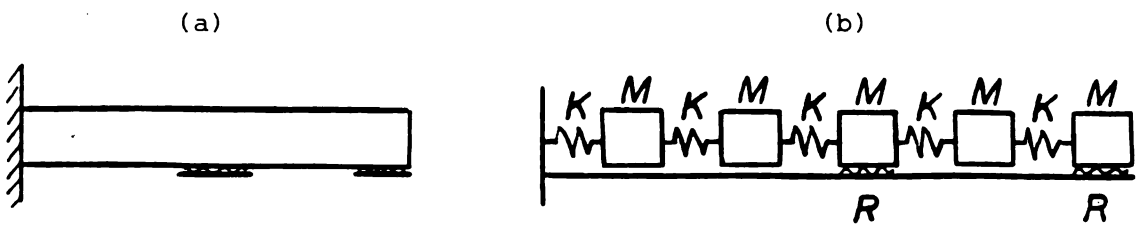
In Figure III.12.e. we have the reduced point graph after reducing the inertias to unit parameters. We find α, β :

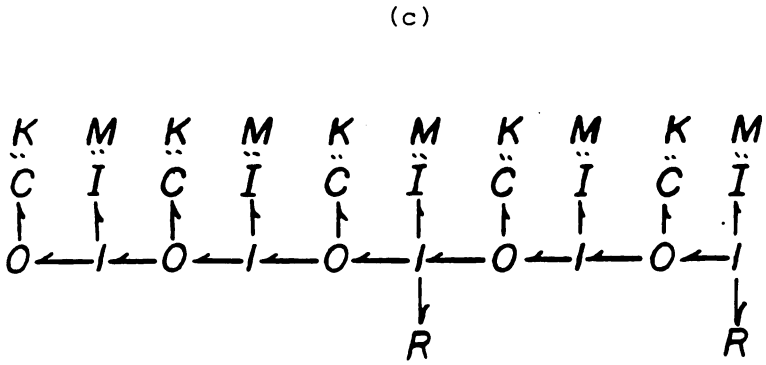
$$\alpha = \frac{R}{M}$$

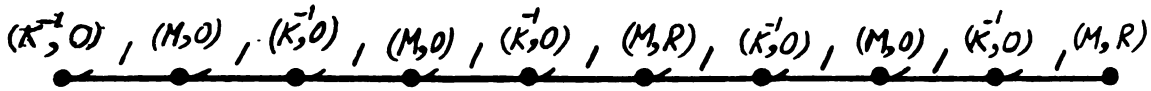
$$\beta = \sqrt{\frac{K}{M}}$$

The point graph here is a path and its spectrum is given by:

$$\text{Spec } (G) = i\ 2 \cos \frac{k\pi}{11},\ k=1,\ldots,N$$







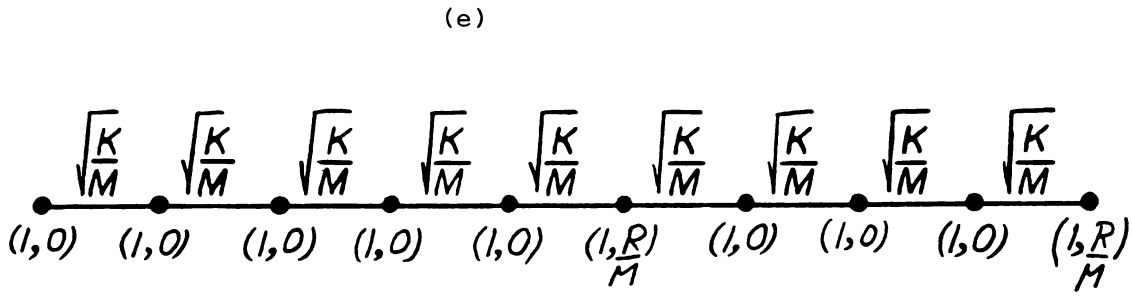


Figure III.12. The point graph of a lumped parameter model of a beam.

This spectrum represents the set of all the eigenvalues of the undamped system. Since here we have two full R points: m = 2 we conclude the following bounds:

$$-\frac{R}{M} \leq a_1 \leq -\frac{R}{M}\cdot\frac{2}{10}$$

$$-\frac{R}{M}\cdot\frac{2}{10} \leq a_N \leq 0$$

$$b_1 \leq 2\cdot\sqrt{\frac{K}{M}}\cdot\cos\frac{\pi}{11}$$

III.2 General Parameters

Here we relax the condition of uniform parameters set in III.1. The bounds in this section are given for system whose gyrobondgraph is simple full and whose parameters can assume any arbitrary values. In III.2.1. we give two examples of such systems and compare the bounds that we obtained to the computed eigenvalues. In III.2.2. we give the procedure used to obtain the bounds.

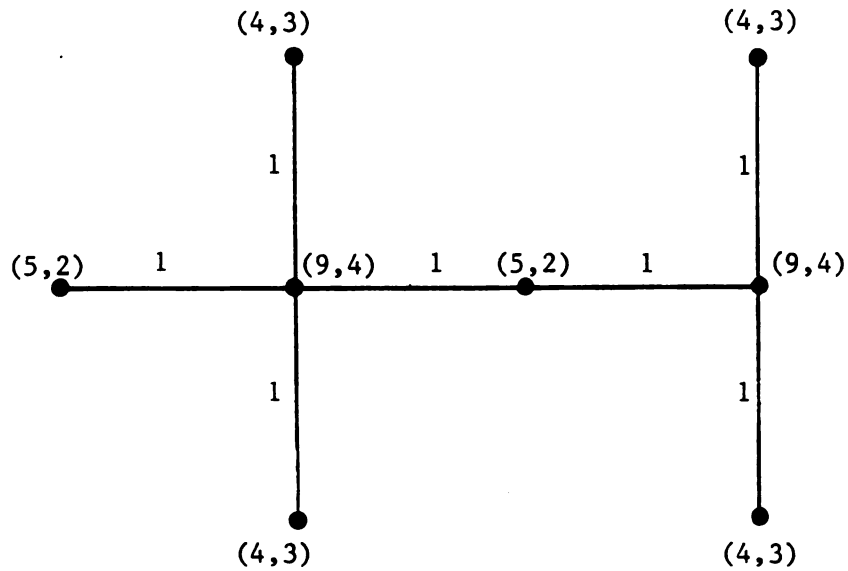
III.2.1 Examples: A Tree

Consider the example of a hydraulic system in Figure III.2. If we assume the system has general parameters, then the point graph associated with the system will be as shown in Figure III.13.a. We then use the reduced unit inertia graph in III.13.b. and the upper bound given in section III.2. to obtain:

$$b_1 \leq .439 \text{ , and}$$

$$-.75 \leq a_1 \leq -.4$$

(a)



(b)

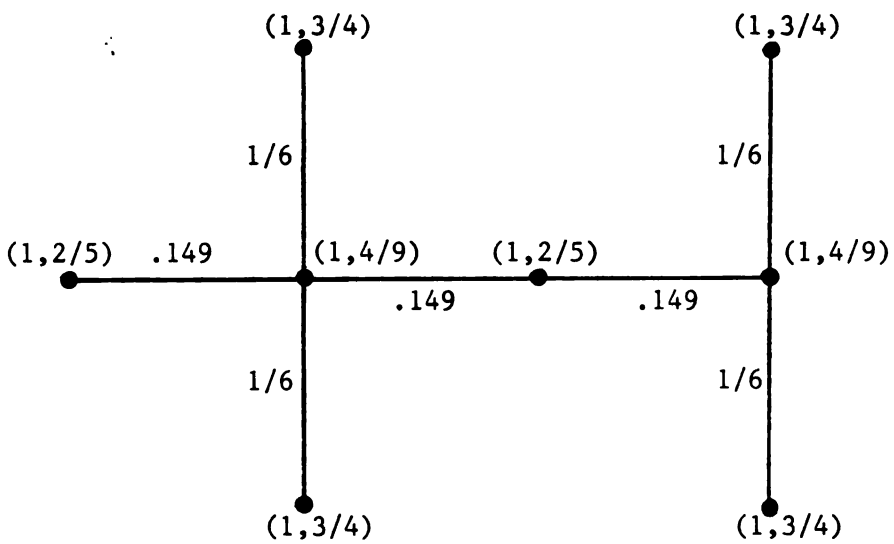


Figure III.13. Reducing the point graph of a system to an equivalent point graph with unit inertias.

Using the iterative techniques, the interated values are:

$$b_1 = .289$$

$$a_1 = .750$$

A general graph

Consider the example of the lumped model of the two power distributed system discussed in section III.1. Here we will use the parameters as shown in Figure III.14.a. From the reduced unit inertia graph in Figure III.14.b. we can estimate the following bounds:

$$b_1 \leq 1.253$$

The computed value of b_1 using iterative methods gives:

$$b_1 = .715$$

III.2.2 Bounds on the Largest Eigenvalue

We next give the bounds on the largest imaginary part and the largest real part for systems with arbitrary parameters.

For systems with arbitrary parameters, reduce the point graph to a similar point graph that has all inertias equal to 1. This procedure is described in III.0.

From the obtained point graph, find:

$$\beta_1 = \max r^*_{ij} = \max \frac{r_{ij}}{\sqrt{I_i I_j}} \quad , \text{ and}$$

$$\alpha_1 = \max R^*_i = \max \frac{R_i}{I_i} \quad , \text{ and}$$

$$\alpha_N = \min R^*_i = \min \frac{R_i}{I_i} \quad , \text{ and}$$

$$\alpha_m = \frac{1}{N} \cdot \sum_{i=1}^{N} \alpha_i \quad .$$

This procedure is shown in Figure III.13.b. and III.14.b. After finding the values of α_1, α_m, α_N and β_N we proceed to estimate the bounds as follows:

A. If the Point Graph G is a Tree:

1) Find b_u (G)

2) The bounds are: $b_1 \leq \beta_1 \times b_u$ (G)

$$-\alpha_1 \leq a_1 \leq -\alpha_m$$

$$-\alpha_m \leq a_N \leq -\alpha_N$$

In the first step we find the upper bound on the largest eigenvalue of the skew symmetric adjacency matrix. This is done using the bounds given in III.1. or the improved bounds that will be given in IV.1.

In the example given in III.13.b. we can easily find that:

$\beta_1 = .25$, and

$\alpha_1 = .75$, and

$\alpha_8 = .4$, and

$\alpha_m = .58$

From the example of a tree given in III.1.1. we have:

$b_u = 2.64$

Hence the bounds given in the example of a hydraulic system in Figure III.2.a. with general parameters as in Figure III.13 are: $b_1 \leq \frac{1}{6} \times 2.64 = .439$, and

$-.75 \leq a_1 \leq -.58$, and

$-.58 \leq a_N \leq -.4$

B. If the Point Graph G is a General Graph:

1) Find the optimal graph G_{opt} of G.

2) Find $b_u\ (G_{opt})$.

3) The bounds than are:

$$b_1 \leq \beta_1 \times b_1\ (G_{opt})$$

$$-\alpha_1 \leq a_1 \leq -\alpha_m$$

$$-\alpha_m \leq a_N \leq -\alpha_N$$

The optimal graph is defined in III.1.2. In Chapter IV we give a procedure to find G_{opt} for a class of point graphs. More work however is needed at this point to establish a more general procedure for finding G_{opt}. If for G we do not know any G_{opt}, we use K_{opt}. This will give bound on b_1 similar to the bound given in Pick's Theorem (see NA2). In the example in Figure III.14.a. using K_{12} as the optimal graph will give:

$$b_1\ (K_{12}) = 7.59$$

However using information in IV.3.2. about the sum of two graphs, we can deduce that the graph shown in Figure III.14.c. is an optimal graph of G and that:

$$b_1\ (G_{opt}) = 2.802$$

Then from the graph with unit inertia, in Figure III.14.b. we find:

$$\beta_1 = .447$$

and thus: $b_1 \leq .447 \times 2.802 = 1.252$

Obviously the first bound is better. If we compare the bounds obtained here to the computed value of b_1:

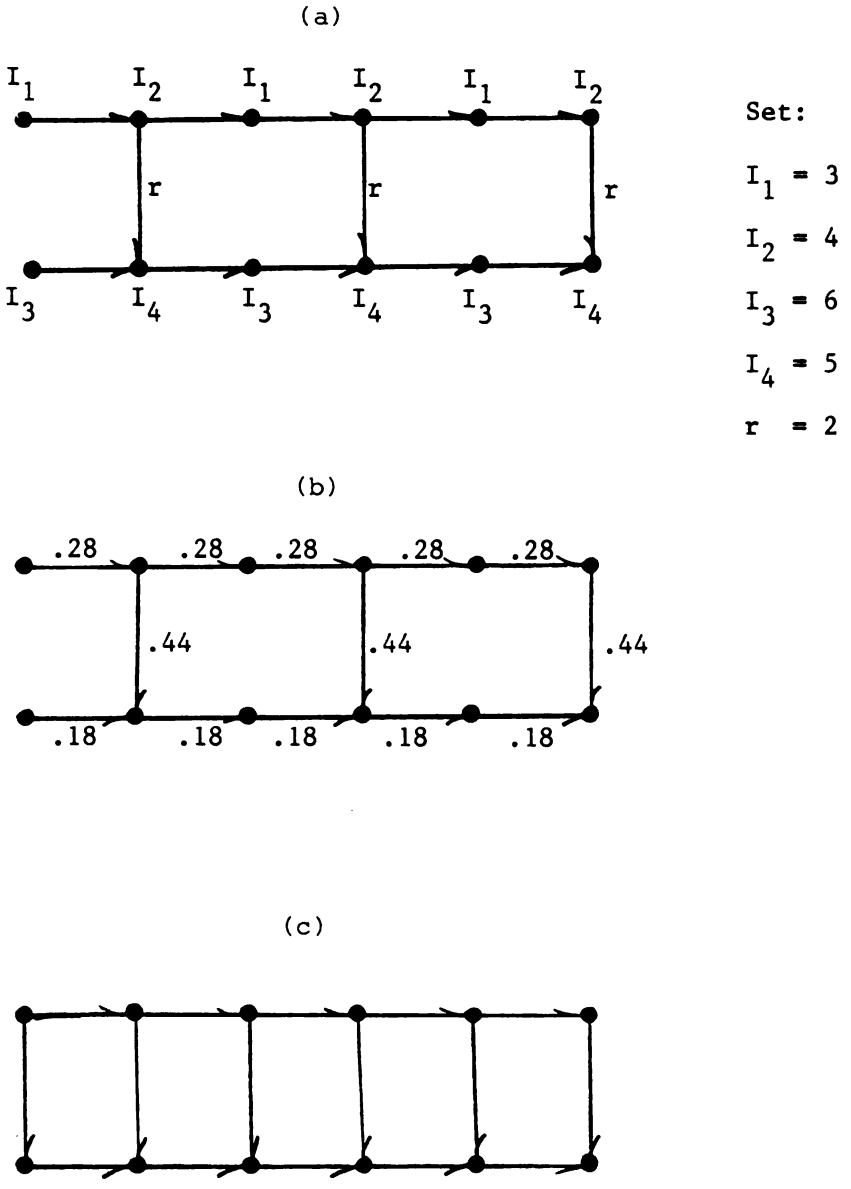


Figure III.14. A two power model of a beam.

$I_1 = \rho A \Delta x$, $I_2 = \Delta x / K^2 \Gamma A$, $I_3 = \Delta x / Ej$, $I_4 = \rho j \Delta x$, $r = 1/\Delta x$.

A is the beam cross-section area, ρ is the mass density, j is cross-section area moment, E is Young's modulus, K^2 is Timoshenko shear coefficient, Γ is the shear modulus.(adapted from BG9).

Computed $b_1 = .715$

Upper bound using G_{opt}: $b_u = 1.252$

Upper bound using K_{12}: $b_u = 3.392$

Upper bound using Gerschgorin theorem: $b_u = 1.000$

Obviously in this case, using Gerschgorin theorem applied to the reduced point graph in Figure III.14.b. gives a better bound.

CHAPTER IV

IMPROVING THE BOUNDS FOR SIMPLE FULL GRAPHS

IV.0. Introduction

To refine the bounds obtained earlier, more information needs to be extracted from the point graph. Here we give a procedure based on inspecting the point graph structure with the objective of improving the bounds on the largest eigenvalue of the associated system. We also give a formula for computing the characteristic polynomial of a special case of uniform parameters. This formula was basically used in proving the results obtained in Chapter III. It can also be used as a tool to calculate the eigenvalues of the systems by using available techniques for obtaining the roots of a polynomial.

Section IV.1. deals with improving the bounds obtained in Chapter III. We first give two examples, (sec.IV.1.), to show the possible improvement in the bounds. Section IV.2. presents a procedure aimed at improving the bounds through a dissection technique. In IV.3. an important property of the eigenvalues of the direct sum of two graphs is given. We then related the graph obtained by the direct sum to the class of optimal graphs. Section IV.2. provides results concerning the characteristic polynomial of a point graph. IV.2.1. introduces a formula for the coefficients, and in IV.2.2 a recursive formula to obtain the characteristic polynomial of a point graph is given.

Two definitions relevant to this chapter are presented as follows:

A dissection of a point graph G will mean partitioning G into two or more subgraphs. None of the subgraphs are empty. The lines common to two subgraphs will be called boundary lines.

A boundary degree, denoted by δ_i^b of the boundary point i, is the number of boundary lines incident with i. The largest boundary degree will be denoted by Δ_i^b. In Figure IV.1. the point graph G is dissected into the subgraphs G_1, G_2, and G_3. Point 2 is a boundary point and its boundary degree is $\delta_2^b = 1$.

IV.1 Improving the Bounds

In Chapter III we derived bounds on the largest eigenvalue of a system with uniform parameters. The bounds were based on the following information:

(a) The number of points N in the point graph.

(b) The existence of cycles in the point graph.

The bounds deduced from this information are the best possible. This is due to the fact that they correspond to the largest eigenvalue of existing graphs. For example, the upper bound for trees corresponds to the largest eigenvalue of a star tree. Thus in order to improve these bounds, more information needs to be known about the point graph. In section IV.1.2. we give a procedure that improves the bounds by applying the rules given in Chapter III to chosen subgraphs of the point graph. Two examples of systems and their improved bounds are given in IV.1.1.

Optimal graphs are necessary for obtaining bounds on the largest eigenvalue of systems with general parameters. In V.1.3. graphs that can be optimal are presented together with a procedure to obtain them for a given general graph.

IV.1.1 Examples

A tree

Consider a system with unit parameters whose point graph is as shown in Figure IV.2. The number of points N = 10, and the graph has no cycles.

From results given in Chapter III, the bounds on the eigenvalues are

$$a_1 = 1,$$

$$1.918 = 2\cos\frac{\pi}{11} \leq b_1 \leq \sqrt{10-1} = 3.$$

Using the Gerschgorin Theorem applied to the point graph we obtain

$$a_1 = -1$$

$$b_1 \leq 7$$

In this chapter we will give the following bounds:

$$a_1 = -1$$

$$2.67 \leq b_1 \leq 3.44$$

Iterative methods will give the following value

$$a_1 = -1$$

$$b_1 = 2.682$$

In this example we see that the lower bound is improved while the upper bound is not. The upper bound given in IV.2. is better than the upper bound given in Chapter III if

$$\frac{N}{\Delta} \geq 1.5$$

Where N is the number of points and Δ is the largest degree in the tree.

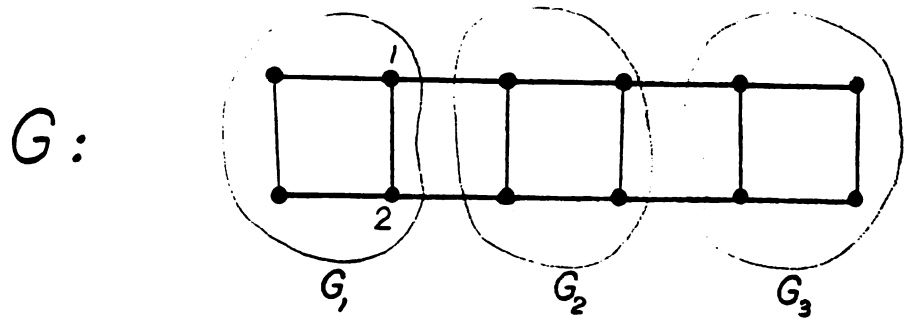


Figure IV.1. Dissecting a graph G into subgraphs G_1, G_2 and G_3.

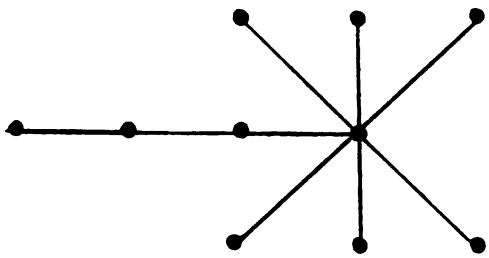


Figure IV.2. A point graph with ten points and largest degree $\Delta = 7$.

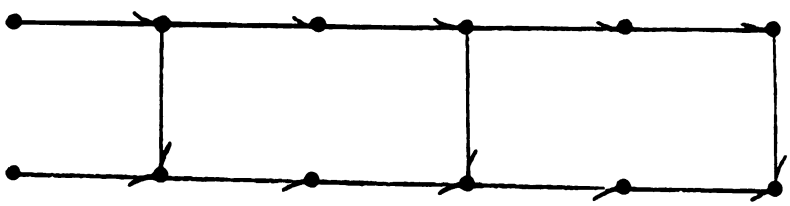


Figure IV.3. The point graph of a Timoshenko beam with three microelements.

A general graph

Consider the point graph in Figure IV.3. The associated system is a lumped-parameter, two-power distributed system which is discussed in Chapter III.1. The bounds on the largest eigenvalues obtained earlier as compared to the improved bounds given in this chapter are

The bounds of Chapter III

$$1.941 \leq b_1 \leq 7.59$$

Using Gersgorin Theorem

$$b_1 \leq 3$$

Using the dissection procedure given in section IV.II

$$2 \leq b_1 \leq 2.732$$

Using the optimal graph as a bound as given in section IV.III

$$b_1 \leq 2.802$$

The computed value using the EISPACK subroutines

$$b_1 = 2.370$$

We see that in this case, and as compared to the previous example, the upper and the lower bounds are improved. The bound given using the optimal graph appears weaker than the bound given by the dissection procedure. However, the bound given by the optimal graph is essential for obtaining bounds for the general parameter case.

IV.1.2 Procedures for Inspecting the Point Graph

Let G be a point graph with all its points full

Let G_1, G_2... be the subgraphs of G obtained by a certain dissection

Let $b_1(G_k)$ denote the largest eigenvalue of the skew symmetric adjacency matrix of the graph G_k,

Let the ordering of the subgraphs $G_1, G_2, \ldots$ be such that

$$b_1(G_k) \leq b_1(G_1) \quad ; \quad k = 2,3,\ldots$$

Let Δ^b be the largest boundary degree in all subgraphs,

Let A_B be the NxN block diagonal matrix whose diagonal block matrices are the skew symmetric matrices of the subgraphs $G_1, G_2, \ldots$

Let the matrix C by such that

$$C = A - A_B$$

then $$b_1(G) \leq b_1(G_1) + b_1(C) \quad \ldots (1)$$

or $$b_1(G) \leq b_1(G_1) + \Delta^b \quad \ldots (2)$$

and $$b_1(G_1) \leq b_1(G) \quad \ldots (3)$$

(for proof see appendix 10)

To improve the bounds obtained earlier we need to minimize the left hand side of the inequality (1) or (2); and to maximize the left hand side of inequality (3). We next give two procedures aimed at achieving this optimization. We first give the procedure for trees.

Trees

The procedure to improve the estimates on the bounds for the largest eigenvalue is based on the following inspection of the point graph G.

1. To Improve the Lower Bound

   a) If the largest degree Δ in the point graph is $\Delta \leq 3$, no improvement is obtained by the steps b or c.

b) Find the star S_M that is a subgraph of G and has the largest degree Δ. A first improvement then is $\Delta \leq b_1(G)$.

c) Find the tree W_K or S_K, described in Table III.1, such that $K > M$, an improvement of the bound given in (b), then is

$$b_1(W_K) \leq b_1(G)$$

or

$$b_1(S_K) \leq b_1(G).$$

If both W_K and S_K are present, choose the one with the largest number of points. If both of them have the same number of points, choose W_K.

2. To Improve the Upper Bound

a) If the ratio $\frac{N}{\Delta} \leq 1.5$, no improvement in the bounds will be obtained by step b.

b) Choose a dissection of the graph G that minimizes the largest degree of the boundary points and at the same time, maximizes the largest degree of G_1; then an improved upper bound is

$$b_1(G) \leq b_1(G_1) + \Delta^b.$$

For cases when the graph G has the ratio of its number of points N to its largest degree Δ: $\frac{N}{\Delta} \leq 1.5$, no significant improvement in the bounds obtained in Chapter III will be obtained by the step (b). This could be concluded by comparing both bounds. Generally speaking, if $\frac{N}{\Delta} \tilde{=} 1$, the tree is approximate to the structure of the star, therefore, the bound given by the largest eigenvalue of the star is close enough.

From the above procedure we see the need to extend the properties of the spectra of trees given in Table III.1. This will improve the comparing capabilities and improve the lower bounds.

General graph

For a general graph the presence of cycles requires a different procedure. The following is the procedure.

1) To Improve the Lower Bound

Find a dissection that produces a subgraph G_1 that is a complete graph K_p or a star S_n (or a tree W_n). If both K_p and S_n exists and if $\sqrt{n} > p$, take $G_1 = S_n$. Otherwise take $G_1 = K_p$. The lower bound is then given as

$$b_1 (G_1) \leq b_1 (G)$$

2) To Improve an Upper Bound

- If $\frac{N}{\Delta} \cong 1$, use bounds given in Chapter III. Otherwise
- Dissect the graph into subgraphs of known b_1, and minimizing Δ^b. Then

$$b_1 (G) \leq b_1 (G_1) + \Delta^b.$$

Examples

A tree

An example of applying this procedure to a point graph that is a tree, consider the point graph in Figure IV.4.

Procedure: 1.(a). The star with the largest degree is the subgraph G_1 and has $\Delta = 7$, thus

$$2.64 = \sqrt{7} \leq b_1 (G).$$

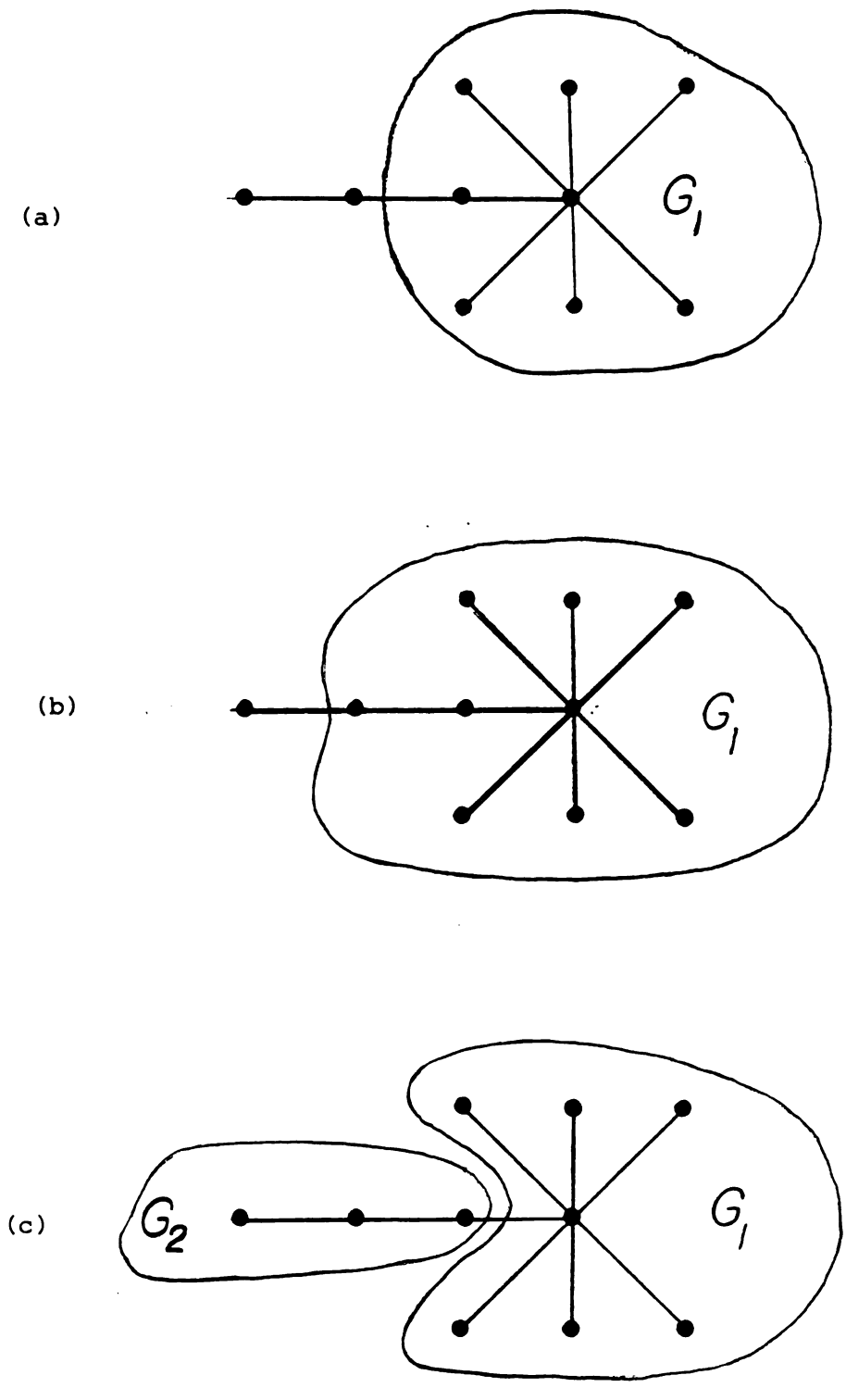


Figure IV.4. Three possible dissection of a tree into subgraph.

Procedure: 1.(b). We recognize the tree W_N (cf. Table III.1.) with N=9. It is the subgraph G_1 in Figure IV.4.b. From Table III.1.

$$b_1 (W_q) = \frac{\sqrt{8 + \sqrt{64 - 4.1.6}}}{2} = 2.67$$

and $$2.67 \leq b_1 (G).$$

Procedure: 2. A dissection that minimizes Δ^b is shown in Figure IV.4.c. Here $\Delta^b = 1$. G_1 is shown as a star with N=7

and $$b_1 (G) \leq \sqrt{7-1} + 1$$

$$\leq 3.449$$

Note that the ratio is $\frac{N}{\Delta} = \frac{10}{7} = 1.4$. Therefore, the upper bound given in Chapter III is stronger and is not improved by this procedure. The calculated value of b_1 using EISPAK subroutines is

$$b_1 = 2.682$$

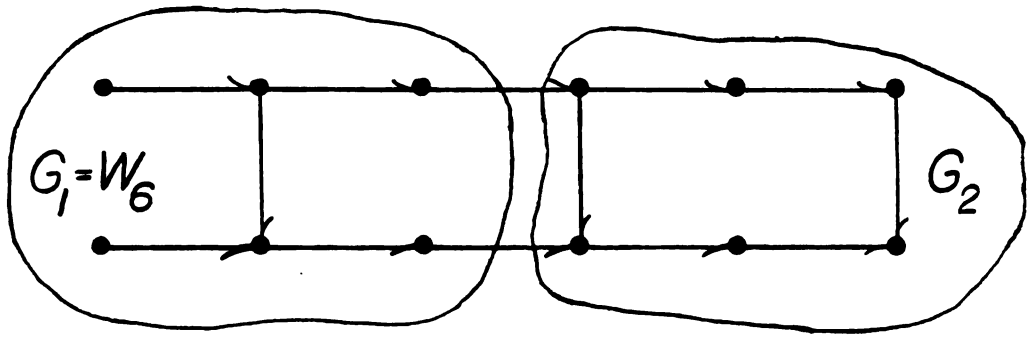
A general graph

An example of applying the procedure for a general graph is given in Figure IV.5.a. The point graph was studied in Chapter III and is the point graph of a three microelement model of a Timoshenko beam (BG9).

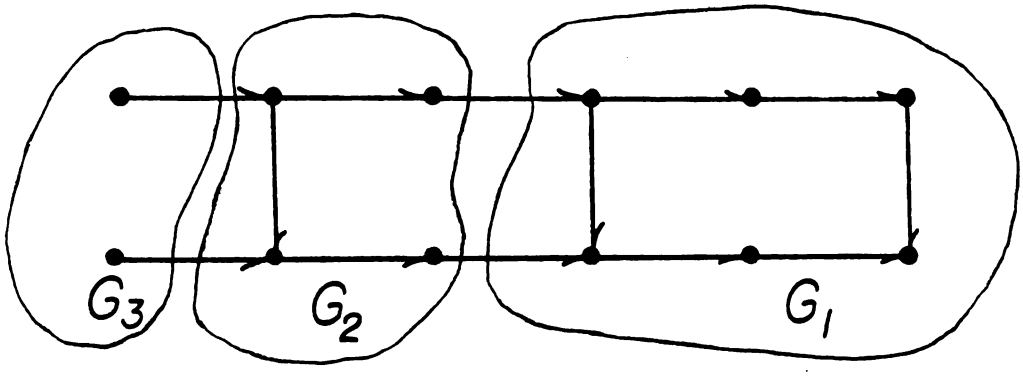
By inspecting the graph, we see that there exists an uncomplete graph as a subgraph. We also see that a subgraph W_6 exists. From Table III.1. we have

$$b_1 (W_6) = \frac{\sqrt{5 + \sqrt{25 - 4.2.2}}}{2} = 2$$

and the lower bound is then $2 \leq b_1 (G)$.



(a)



(b)

Figure IV.5. The point graph of a lumped model of a Timoshenko beam. (a) a dissection that gives a lower bound on b_1; (b) a dissection that gives the upper bound on b_1.

To obtain an upper bound, the dissection shown in Figure IV.5.b. will provide a minimal boundary degree.

$$\Delta^b = 1$$

and the subgraph G_1 is the cycle C_6, thus

$$b_1 (G_1) = 2\cos \frac{\pi}{6} = 1.732$$

and the upper bound on $b_1 (G)$ is

$$b_1 (G) \leq 1.732+1$$

The actual computed value of $b_1 (G)$ is

$$b_1 (G) = 2.370$$

IV.1.3 The Optimal Graph

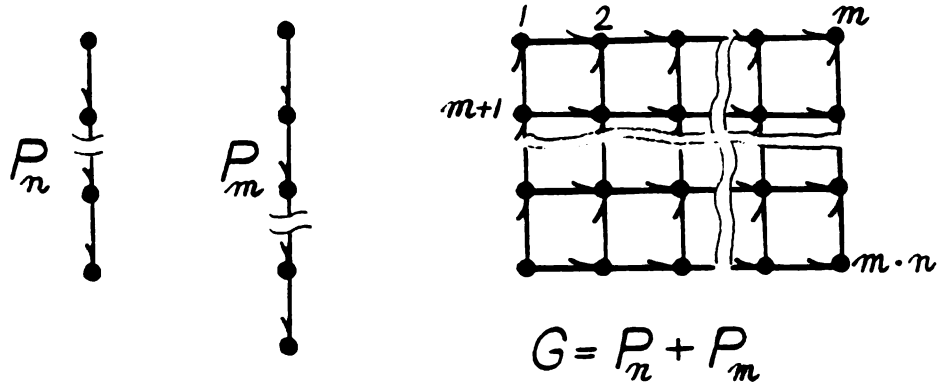
Optimal graphs are defined in III.1.2. They are necessary to obtain an effective upper bound on the largest eigenvalue of systems with general parameters when their associated point graph is a general graph. Optimal graphs also provide upper bounds for systems with uniform parameters when their point graph is a general graph. A major property of optimal graphs is defined below (for the proof see appendix 8).

If G_1 and G_2 are two optimal graphs, then the graph G obtained from their direct sum is an optimal graph, and

$$b_1 (G) = b_1 (G) + b_1 (G_2)$$

A point graph that is the direct sum of the path P_n and the path P_m, is a mxn grid as shown in Figure IV.6. The largest eigenvalue of the grid is

$$b_1 (G) = 2\cos \frac{\pi}{m+1} + 2\cos \frac{\pi}{n+1}$$



(a)

Figure IV.6. (a) and (b) are two directed paths with m and n points. In (c) is a grid that is the direct sum of P_m and P_n.

(b) (c)

Figure IV.7. (a) a path P_2; (b) an even cycle C_4^e and in (c) is the direct sum of P_2 and C_4^e .

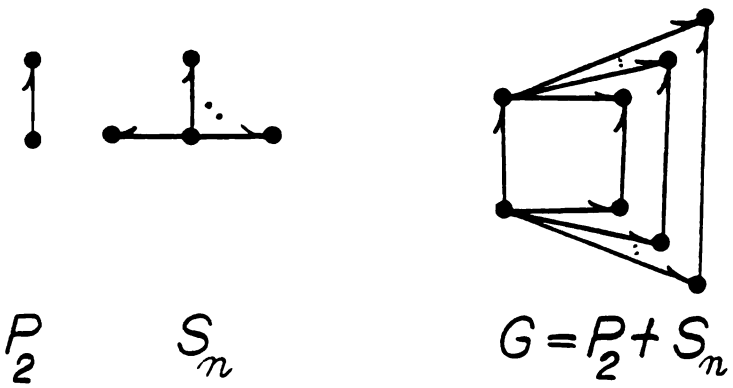


Figure IV.8. The direct sum of a path P_2 and a star S_n.

Since the path is an optimal graph (by definition any tree is an optimal graph), then the grid shown in IV.6.c. is an optimal graph. The graph G (Figure IV.7.) is the direct sum of the path P_2 and the cycle C_4^e. The largest eigenvalue of G is

$$b_1(G) = b_1(P_2) + (C_4)$$

$$= 1 + 2 = 3$$

The graph G is an optimal graph because the cycle C_4^e is an optimal graph, and the path P_2 is an optimal graph.

In Figure IV.8. the graph G is the direct sum of the path P_2 and a star S_n. The largest eigenvalue of G is

$$b_1(G) = b_1(P_2) + b_1(S_n)$$

$$= 1 + \sqrt{n-1}$$

Here also G is an optimal graph.

To obtain an optimal graph for a given general graph, add as many lines as required until the obtained graph becomes an optimal graph. For graphs with an odd number of points, this procedure does not render an optimal graph and other procedures have to be used.

IV.2 The Characteristic Polynomial

The formula we give here is for the characteristic polynomial of a system whose point graph is simple full and whose parameters are general except for the resistances; all resistances are assumed equal to zero. This formula, in other words, computes the characteristic polynomial of the gyroadjacency matrix defined in II.2.2. In case of a system with unit parameters, the formula then gives the characteristic polynomial of the skew symmetric adjacency matrix as defined in II.3.2.

The formula given here was mainly used to prove the bounds given in Chapter III, and to obtain the spectra of the trees in table III.1. It can provide an alternative in calculating all the imaginary eigenvalues of systems with zero resistances by using standard polynomial solving techniques.

In IV.2.1. we give the coefficients of the characteristic polynomial from the point graph. Section IV.2.2. presents a recursive formula for finding the characteristic polynomial of a point graph in terms of its subgraphs. For proofs of the formulae given here, see appendix 1,2,3. A similar formula for the characteristic polynomial of the symmetric adjacency matrix of an unoriented graph was derived by A. Schwenk (1973), and is found in (GT1). Another similar formula computed for the symmetric adjacency matrix for a Sigraph was found by M.K. Gill and B.D. Acharya (1980), (GT7).

IV.2.1. The Coefficients of the Characteristic Polynomial

Let G be a point graph, and let P(G) be the characteristic polynomial of the gyroadjacency matrix S of G such that

$$P(G) = \det (S - \lambda I) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_o = 0$$

Define: An elementary figure is either

a) The graph K_2 (the path with two points), or

b) Every graph C_q, $q \geq 3$ (the cycle with q points).

A basic figure U is every graph all of whose components are elementary figures.

Let $\mathcal{U}_i$ be the set of all basic figures contained in G and having i points.

Let $\Pi(U)$ be the product of the gyrators moduli on the lines belonging to the cycles C in U, and the square of the gyrators moduli on K_2 belonging to U.

Let c(U) be the total number of cycles with even length in U and $c^e(U)$ be the total number of cycles with even power in U. The coefficients of P(G) are given by

$$a_i = 0 \quad \text{if } i \text{ is odd}, \quad \text{otherwise}$$

$$a_i = \sum_{U \in \mathcal{U}_i} (-1)^{c^e(U)} \; 2^{c(U)} \; \Pi(U) \qquad \ldots (1)$$

in the case of <u>unit parameters</u>:

$$\Pi(U) = 1$$

Thus the coefficients of the characteristic polynomial of the skew symmetric adjacency matrix are

$$a_i = 0 \text{ if } i \text{ is odd}, \quad \text{otherwise}$$

$$a_i = \sum_{U \in \mathcal{U}_i} (-1)^{c^e(U)} \; 2^{c(U)} \qquad \ldots (2)$$

<u>Example</u>

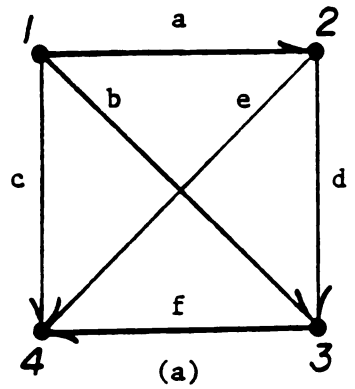
Consider the point graph given in Figure IV.9.a. In Figure IV.9.b. is the set of all basic figures with two points: $\mathcal{U}_2$. By definition of point graph, no cycles exist in $\mathcal{U}_2$. Thus the powers of (-1) and 2 in equation (1) equal zero.

$$c(U) = 0$$

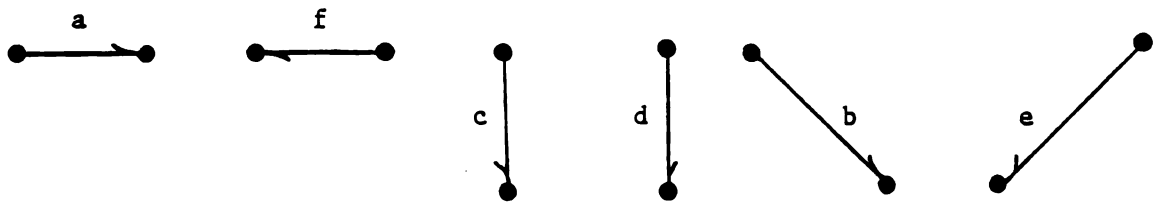
$$c^e(U) = 0$$

and

$$a_2 = a^2 + b^2 + c^2 + d^2 + e^2 + f^2$$



(a)



(b)

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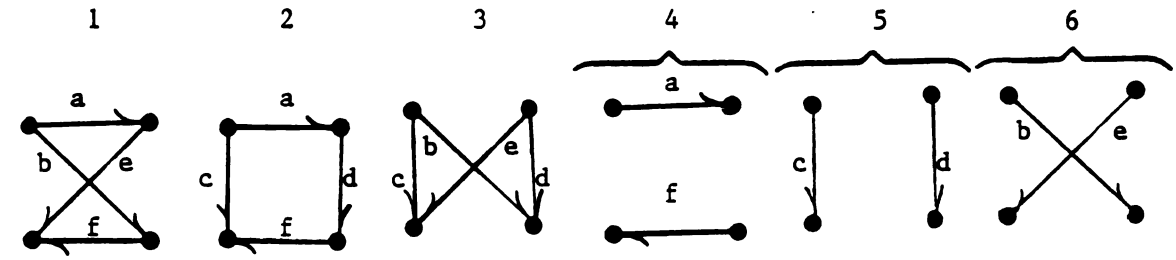
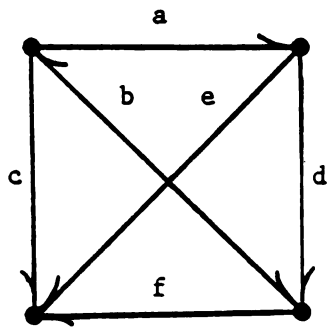


(c)

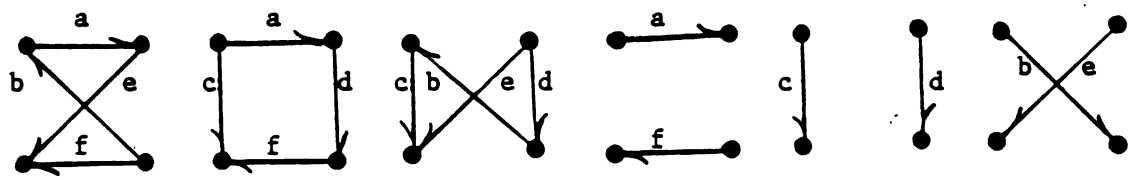
Figure IV.9. a) The point graph with gyrator constants: a,b,c,d,e,f. b) The six basic figures with two points; c) The six basic figures with four points.



(a)

Basic figure number:

1 2 3 4 5 6



(b)

Figure IV.10. a) An oriented graph similar to the point graph in Figure IV.9. except for the orientation on line b. b) The basic figures with four points.

The set of all basic figures with four point $\mathcal{U}_4$ are given in IV.9.c. The first three basic figures consist of one cycle each so the power of (-1) in all three is

$$c(U) = 1$$

The first and the third basic figure are both even power cycles thus the power of (2) in both is

$$c^e(U) = 1$$

The second basic figure is an odd cycle thus the power of (2) is

$$c^e(U) = 0$$

The last three basic figures consists of two paths K_2 each and no cycles, so the power of (-1) and 2 in all three of them are equal zero. Thus the coefficients of λ^0 is

$$a_4 = (-1)^1 . 2^1 . (abef) + (-1)^0 . 2^1 . (acfd) + (-1)^1 . 2^1 . (cbed) + a^2 . f^2 + c^2 . d^2 + b^2 . e^2$$

Now consider the point graph in Figure IV.10.a. It is basically the same as the point graph in Figure IV.9.a. except for the power orientation on line b. This power change does not effect the coefficients of λ^2, since the basic figuer with 2 points does not have cycles. The power change on b changes the power direction on the first and third basic figures with four points. They both become odd powered as seen in Figure IV.10.b. and thus the coefficient a_4 becomes

$$a_4 = (-1)^0\ 2^1\ (abef) + (-1)^0\ 2^1\ (acfd) + (-1)^0\ 2^1\ (abed) + a^2\ f^2\ + c^2\ d^2 + b^2\ e^2$$

If the parameters a = b = c = d = e = f = 1, then for Figure IV.9.a. the characteristic polynomial is thus

$$\lambda^4 + 6\lambda^2 + 1 = 0 \text{ , and}$$

for Figure IV.10.a., the characteristic polynomial is

$$\lambda^4 + 6\lambda^2 + 9 = 0$$

These results could be easily checked by finding the determinant of the skew symmetric adjacency matrix for both point graphs.

Corollary

1. For Trees:

   a) The spectrum of the gyroadjacency matrix of a tree is invariant under power orientation. This can be proved form equation (1) since the term with negative sign is only dependent on cycles.

   b) The spectra of the skew symmetric adjacency matrix and the symmetric adjacency matrix are identical, (apart from the factor $i = \sqrt{-1}$). This can be proved by substituting λ by $i\lambda$ in the characteristic polynomial of the symmetric adjacency matrix. The coefficients given in Cvetkovic (GT2) then becomes

$$a_i = 0 \qquad \text{if } i \text{ is odd, otherwise}$$

$$a_j = \sum_{U \in \mathcal{U}_j} (-1)^{p(U)} \, (-1)^j$$

   where p(U) is the number of components in U, for a tree p(U) = j and thus

$$a_j = \sum_{U \in \mathcal{U}} (-1)^{2j} = \sum_{U \in \mathcal{U}} 1$$

which is identical to equation (1) reduced to unit parameters and zero cycles.

2. Cycles with Odd number of Points

The spectrum of the gyroadjacency matrix is invariant under the change of power orientation on any single line that belongs only to odd number cycles. This can be easily proven from equation (1) since odd number cycles do not effect the sign in the summation. Only even numbered cycles with even powers contribute to the power of (-1). Note also that cycles with odd number of points do not contribute to the coefficients of the characteristic equation.

IV.2.2 A Recursive Formula

This recursive formula serves to obtain the characteristic polynomial of a point graph if we know the characteristic polynomial of one of its subgraphs.

Let r_{uv} denote the gyrator modulus on the line between the point u and the point v,

Let $C^e(v)$ denote a cycle with even power that passes through the point v.

Let $C^o(v)$ denote a cycle with odd power that passes through the point v.

Let $\Pi C(v)$ denote the product of all gyrator moduli of the lines of the cycle C that passes through the point v.

Let P(G-C(v)) denote the characteristic polynomial of the point graph obtained from G by deleting all points and lines belonging to the cycle C(v).

Let P(G-v) denote the characteristic polynomial of the point graph obtained from G by deleting the point v and all lines incident with it.

Then

$$P(G) = \lambda \cdot P(G - v) + \sum_{*} r_{uv}^2 \cdot P(G - v - u) +$$

$$2. \sum_{**} \Pi C^0(v) \cdot P(G - C^0(v)) - \Pi C^e(v) \cdot P(G - C^e(v)) \quad \ldots (3)$$

where $\sum_{*}$ is the summation over all points u that are adjacent to the point v, and $\sum_{**}$ is the summation over all the odd and even power cycles that pass through the point v.

Note that cycles with odd number of points do not contribute to the summation.

The following notions, although of an abstract nature, are useful in applying the above formula.

1. If G is a point, then $P(G) = \lambda$
2. If G is an empty graph, then $P(G) = 1$
3. If G has two disconnected components, G_1 and G_2, then

$$P(G) = P(G_1) \cdot P(G_2)$$

An example of the use of this formula is given below. Consider again the point graph given in Figure IV.9. Applying equation (3):

$P(G) = \lambda P(G - 1) + a^2 \cdot P(G - 1 - 2) + b^2 \cdot P(G - 1 - 3) + c^2 \cdot P(G - 1 - 4) + 2[adfc \cdot P(G - adfc) - abef \cdot P(G - abef) - cbed \cdot P(G - cbed)]$

$P(G - 1) = \lambda^3 + (d^2 + e^2 + f^2)\lambda$

$P(G - 1 - 2) = \lambda^2 + f^2$

$P(G - 1 - 3) = \lambda^2 + e^2$

$P(G - 1 - 4) = \lambda^2 + d^2$

$P(G - adfc) = 1$ (since it is empty)

$P(G - adef) = 1$

$P(G - abed) = 1$

Substituting, we get

$$P(G) = \lambda[\lambda^3 + (d^2 + e^2 + f^2)\lambda] + a^2(\lambda^2 + f) + b^2(\lambda^2 + e^2) + c^2(\lambda^2 + d^2) + 2(adfc - abef - cbed)$$

$$= \lambda^4 + (a^2 + b^2 + c^2 + d^2 + e^2 + f^2)\lambda^2 + 2[(adfc - abef - cbed)] + [a^2 f^2 + b^2 e^2 + c^2 d^2]$$

If we substitute for $a = b = c = d = e = f = 1$, we obtain

$$p(G) = \lambda^4 + 6\lambda^2 + 1$$

which verifies the results obtained in the proceeding section.

Corollary

Let G be a point graph having two subgraphs G_1 and G_2 connected by a bridge. Let the bridge be incident to point u in G_1 and point v in G_2. Let r_{uv} be the gyrator modulus on the bridge uv. Then

$$P(G) = P(G_1) \cdot P(G_2) + r_{uv}^2 P(G_1 - u) \cdot P(G_2 - v) \quad \ldots (4)$$

The use of the above equation to obtain the characteristic polynomial of a system is shown in the following example. Consider the point G shown in Figure IV.11. It can be dissected as two subgraphs G_1 and G_2 with the bridge uv connecting them. If we assume unit parameters and if the characteristic polynomial of a path with N points is written as P_N, and the characteristic polynomial of a star with N points is written as S_N, then

$$P(G) = P(G_1) \cdot P(G_2) + P(G_1 - u) \cdot P(G - v)$$

$$= S_4 \cdot P_{N-4} + S_3 \cdot P_{N-5}$$

We know that $S_N = \lambda^{N-2} (\lambda^2 + (N-2))$ then

$$P(G) = \lambda^2 (\lambda^2 + 3) \cdot P_{N-4} + \lambda(\lambda^2 + 2) \cdot P_{N-5}.$$

After some algebraic manipulations and using the following properties of P_N

$$\lambda P_N = P_{N+1} - P_{N-1}$$

and $$i^N \cdot P_N = \frac{\sin (n+1)\theta}{\sin} , \theta = \cos^{-1} \left(\frac{i\lambda}{2}\right)$$

We obtain $$P(G) = \lambda \cdot \cos (N-1)\theta = 0$$

thus $$\theta = \frac{(sk + 1)\pi}{2(N-1)}$$

and the roots of the characteristic polynomial are

$$\lambda = 0, \lambda = 2i \cos \frac{(2k + 1)\pi}{2(N-1)} , k = 0 ,.., N - 1; i = \sqrt{-1}.$$

The above example is an illustration of the method used to obtain the spectra of trees given in Table III.1. It gives an example of a possible technique for extending these results.

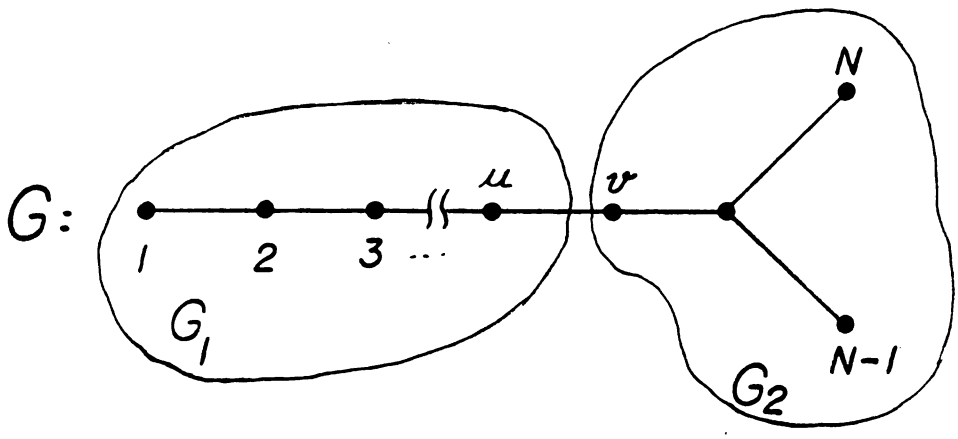


Figure IV.11. A point graph with a bridge uv, the characteristic polynomial is:

$$P(G) = P_{N-4} \cdot S_4 + P_{N-3} \cdot S_3$$

The number of walks

The concept of the number of walks is related to the sum of the eigenvalues of the skew symmetric adjacency matrix. Let $N_k(i,j)$ be the sum of signed walks of length k from point i to point j. Each line in the walk is signed negative (-) if the walk is in the same direction of the power orientation, and is signed positive (+) otherwise. The sign of the walk from i to j is the product of all signs on the lines of the walk. If $S^k = (s_{ij}^{(k)})$ is the kth power of the skew symmetric adjacency matrix then

$$s_{ij}^k = N_k(i,j) \qquad \text{and}$$

$$\sum_{i=1}^{N} \lambda_i^k = \sum_{i=1}^{N} N_k(i,i)$$

The above result is expected to be used as a basis to investigate the spectrum of the complements of the point graph with a goal of obtaining more insight into the effect of the change of power orientation on the spectrum of a point graph.

CHAPTER V

SIMPLE PARTIAL GRAPHS

Simple partial graphs are the most general form of simple graphs and thus obtaining an estimate for the largest eigenvalue is a more complex problem. For a special case, we can give estimates; however for the majority of this class we still have to refer to the traditional procedure of obtaining the estimate through the state matrix.

If the simple partial point graph has only full points and partial R points and if no two partial R points are adjacent, then the bounds on the imaginary part of the eigenvalue is given in V.1. At this point more work is required to extend these results.

In V.2., and for the purpose of completeness, we present a numerical technique given by H. Wolkowiz (NA5) for estimating the eigenvalues of non-normal matrices. The choice of method is due to the fact that remaining classes do not have a distinct reduction to normal matrices as for systems represented by simple full graphs.

Similarity Transformation

The procedure described here for a system whose point graph is simple full-R partial-I will give a new system that has the same spectrum and the same point graph as the original system. The advantage is that the new system has all inertia parameters equal 1. This reduces

the state matrix given in II.22. equation 2 from:

$$A = [(S_F - R_F) + S_{12}\,(S_P - R_P)^{-1}\,S_{12}^T]\;\;[I^{-1}] \text{ to}$$

$$\tilde{A} = (\tilde{S}_F - \tilde{R}_F) + \tilde{S}_{12}(S_P - R_P)^{-1}\,\tilde{S}_{12}^T \qquad \ldots (1)$$

and A is similar to $\tilde{A}$.

The procedure is

1. Replace the gyrator modulus r_{ij} between point i and point j by r_{ij}^* where

   $r_{ij}^* = \dfrac{r_{ij}}{\sqrt{I_i I_j}}$ if points i and j are both full points

   $r_{ij}^* = \dfrac{r_{ij}}{\sqrt{I_i}}$ if the point j is a partial R point

2. Replace the resistance on the full point i by R_i^* where

   $R_i^* = \dfrac{R_i}{I_i}$

   This procedure gives a simpler system whose eigenvalues can be studied and then related to the original system.

Notations

If G is a point graph that has N full points then

G_F ... is the subgraph of G that contains all the N full points and will be called the simple full subgraph, and

G_P ... is the subgraph of G that contains all the partial R points and will be called the simple partial R subgraph.

b_1 (G) will denote the largest imaginary part of the eigenvalue of the system associated with the point graph G.

Next we present some results for the case when all points of the partial R graph are not connected with each other.

V.1 A Non-Connected Partial R Subgraph

If every partial R point of the point graph G is not adjacent to any other partial R point, we have a non-connected partial R subgraph. In this case the bounds on the largest real and imaginary part of the eigenvalue of the system associated with G are

$$b_N(G_F) \leq b_1(G) \leq b_1(G_F), \qquad \text{and}$$

if we let $B = -(\tilde{R}_F + \tilde{S}_{12} R_P^{-1} \tilde{S}_{12}^T)$; then

$$a_N(B) \leq a_1(G) \leq a_1(B).$$

The value of $b_1(G_F)$ can be found for the simple full subgraph by applying the procedures given in Chapter III and IV on the subgraph G_F.

A special case

Sparse partial R points: If no two partial R point are adjacent to the same full point, we say that the partial R points are sparse. The bounds on the imaginary part of the eigenvalue are as above, and the bounds on the real parts become

$$\min_i \left(R^*_{Fi} + \sum_j r^{*\,2}_{ij}\, R_{Pi}^{-1}\right) \leq a_1(G) \leq \max_i \left(R^*_{Fi} + \sum_j r^{*2}_{ij}\, R^{-1}_{Pi}\right)$$

Where

r^*_{ij} . . . is the gyrator modulus between the full point i and the partial point j,

R^*_{Fi} . . . is the resistance on the full point i, and

R_{Pi} . . . is the resistance on the partial point j.

If we assume unit parameters

$$R_{Pi} = R^{*}_{Fi} = r^{*}_{ij} = 1 \quad \text{for all } i = 1 \ldots$$

then the bounds on $a_1(G)$ reduce to

$$-(1 + \Delta^{b}_{j}) \leq a_1 \leq -1$$

where

Δ^{b}_{j} is the largest boundary degree of all partial R points.

The above bounds could be easily proven using the results given in Amir-Moez (NA12) on the bounds of the real and imaginary part of the eigenvalues; and noting that when the partial I subgraph is not connected then

$$S_p = 0$$

At this point more investigation is needed in the following directions:

1. In the case of non-connected partial R subgraph, the matrix $\tilde{S}_{12}$ can be considered as a signed adjacency matrix of the subgraph whose points are the boundary points in the partial R subgraph and the boundary points in the full subgraph. The eigenvalues of such graphs are studied in (GT7) and more development will lead to relating the bound of the real part to the structure of the above mentioned subgraph.
2. The effect of different structures of G_p on $a_1(G)$ could be found for some special cases of boundary points.
3. In the most general case; we have two possibilities:
   a) Either relating the bounds to some function of the structure of the subgraphs G_F, G_p and others or

b) starting from the point graph and constructing a related oriented and non-oriented linear graph that can be used to estimate bounds on the largest real and imaginary part of the eigenvalues.

For the purpose of completness, we present next a numerical method that gives bounds on the eigenvalues and that uses the state matrix. These bounds are given in Wolkowitz and Styan (NA5). They give bounds on the eigenvalues, their real and imaginary part using the state matrix.

V.2 Estimates From the State Matrix

For the remaining classes of systems, the state matrix is in general a non-normal matrix. This justifies the choice of the estimator given in this chapter.

Let: A be an nxn matrix, and N(A) be the euclidean norm of A, define

$$D = AA^* - A^*A$$

$$B = \frac{1}{2}(A + A^*)$$

$$C = \frac{1}{2}(A - A^*)$$

$$K_A^u = [N^4(A) - \frac{1}{2}N^2(D)]^{\frac{1}{2}}$$

$$K_A^\ell = [N^2(A) - (\frac{n^3 - n}{12})^{\frac{1}{2}} N(D)]$$

$$K_B^u = \begin{cases} [N^4(B) - \frac{1}{8}N^2(D)]^{\frac{1}{2}} & \text{if } N(B) \geq N(C) \\ [N^2(B) - \frac{1}{12}\frac{N^2(D)}{N^2(A)}] & \text{otherwise,} \end{cases}$$

$$K_B^\ell = [N^2(B) - (\frac{n^3 - n}{48})^{\frac{1}{2}} N(D)]$$

$$K_C^u = \begin{cases} [N^2(C) - \frac{1}{8}N^2(D)]^{\frac{1}{2}} & \text{if } N(C) \geq N(B) \\ [N^2(C) - \frac{1}{12}\frac{N^2(D)}{N^2(A)}] & \text{otherwise,} \end{cases}$$

$$K_C^{\ell} = [\, N^2(C) - (\frac{n^3 - n}{48})^{1/2} N(D) \,]$$

$$(s_T^u)^2 = \frac{K_T^u - |Tr(T)|^2/n}{n} \qquad \text{for } T = A, B \text{ or } C$$

$$(s_T^{\ell}) = \max\,(0, \frac{K_T^{\ell} - |Tr(T)|^2/n}{n})$$

$$m_A^u = (\frac{K_A^u}{n})^{1/2} \quad , \quad m_A^{\ell} = \frac{|Tr(A)|}{n} \quad ,$$

$$m_B^u = m_B^{\ell} = \frac{Tr(B)}{n} \quad , \quad m_C^u = m_C^{\ell} = \frac{Tr(C)}{n}$$

Theorem

Let λ_i^T be the i^{th} eigenvalue of the matrix $T \equiv A$, B or C; and let the matrix A be a real nxn matrix, then for $1 \le j \le k \le n$,

$$m_T^{\ell} - s_T^u \left(\frac{j-1}{n-j+1}\right)^{1/2} \le \frac{1}{k-j+1} \sum_{i=j}^{k} \lambda_i^T \le m_T^u + s_T^u \left(\frac{n-k}{k}\right)^{1/2}$$

$$\frac{1}{k-j+1} \sum_{i=j}^{k} \lambda_i^C \le \left(\frac{K_C^u}{2k}\right)^{1/2}$$

This theorem gives bounds that are closer than gerschgorin discs. It requires more computations, but it offers more information about the eigenvalues of the matrix.

This theorem can be used together with equation (1) to obtain bounds on the eigenvalues for some special structures of simple full R graphs. The theorem given here can be programed as a subroutine to obtain estimates in cases when the methods of Chapter III are not applicable.

VI. CONCLUSION

Summary of Results

Bounds on the largest eigenvalue of a class of systems are obtained by inspecting the bond graph model. A summary of the procedure is given in the chart in Figure VI.1.and Table VI.1. The given procedure is obtained from the results given in Chapter III and Chapter V, and can be coded to be used in digital computers. To be able to program the results given in Chapter IV, some graph theory subroutine will be needed to detect various structures of graphs. These are not needed when coding the results of Chapter III because we need only to discirminate between trees and general graphs. This discrimination could be obtained as follows: If the number of gyrators is equal to or greater than the number of 1-junctions, then we have a general graph (cf.GT1). We next compare the results obtained in this work to some classical results.

Comparison To Other Methods

We compare the bounds given here to the bounds given by three other methods. They are:

Pick's Theorem (1919) for obtaining an upper bound on the largest real and imaginary part of the eigenvalue of a matrix, (NA10).

Gersgorin Theorem (1931).

The Euclidian Norm: $N_E(A)$, (cf. section II.4).

The reason for this choice is that the information, the number of operations required, and the accuracy for the four methods are comparable.

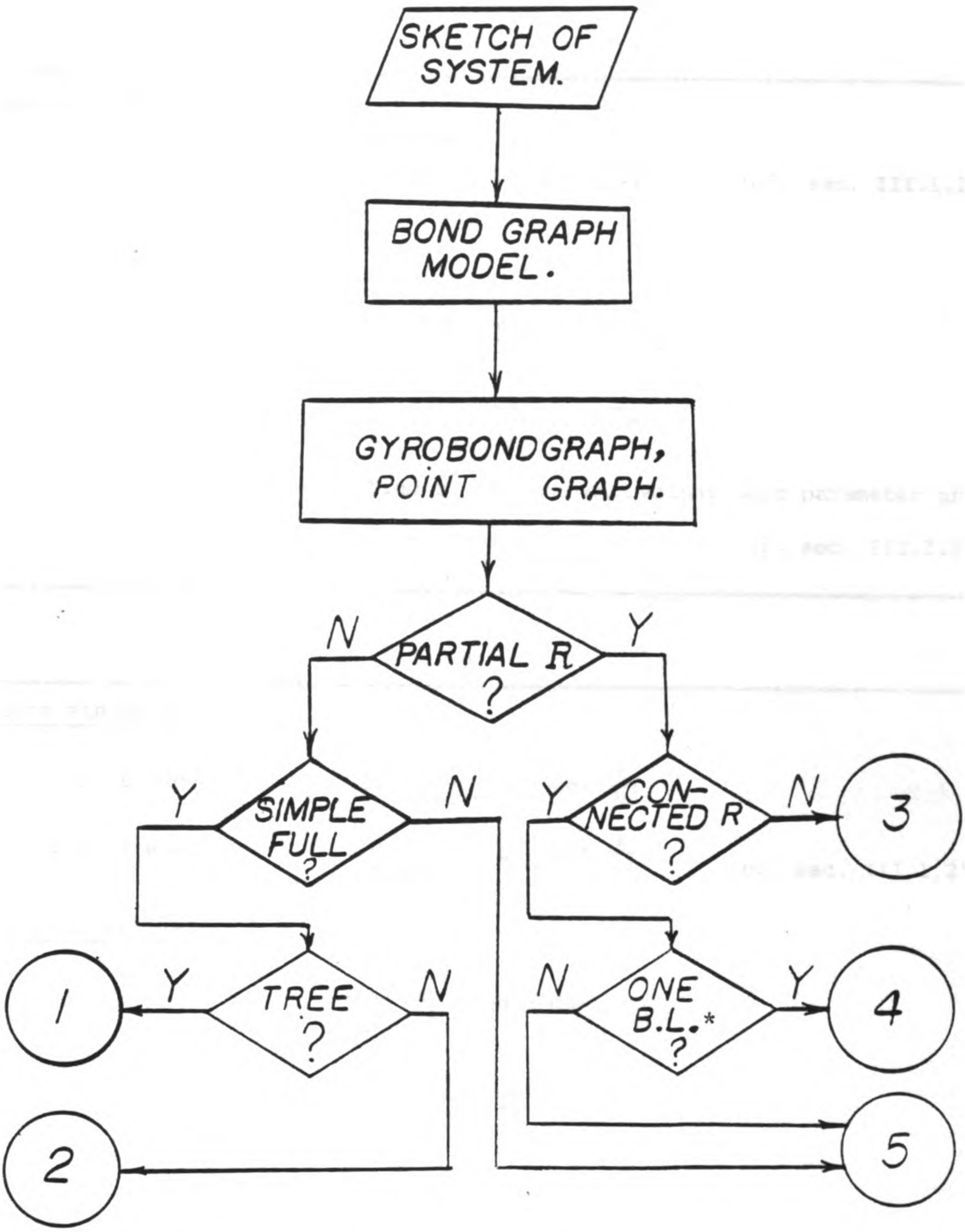


Figure VI.1. A flow chart summarizing the results. B.L. are boundary lines. The number in the circles refer to a block in Table VI.1.

Table VI.1. Summary of Results.

(1)

UNIT PARAMETERS:

$$2\cos\frac{\pi}{N+1} \leq b \leq \sqrt{N-1} \qquad a = -1 \qquad \text{(cf. sec. III.1.2)}$$

GENERAL PARAMETERS:

$$\min_i (R_i) \leq |a| \leq \max_i (R_i)$$

$$\text{Find } \max_{i,j} \frac{r_{ij}}{\sqrt{I_i I_j}} = r_m$$

Find b_u for the equivalent unit parameter graph

$$b \leq b_u x\ r_m \qquad \text{(cf. sec. III.2.2)}$$

(2)

UNIT PARAMETERS:

If N is odd: $2\cos\frac{\pi}{N+1} \leq b \leq \cot\frac{\pi}{2N}$ $\quad a = -1$

If N is even: $2\cos\frac{\pi}{N} \leq b \leq \cot\frac{\pi}{2N}$ $\quad$ (cf. sec. III.1.2)

GENERAL PARAMETERS:

$$\min_i (R_i) \leq |a| \leq \max_i (R_i)$$

$$\text{Find } \max_{i,j} \frac{r_{ij}}{\sqrt{I_i I_j}} = r_m$$

Find the optimal graph

Find b_u for the optimal graph

$$b \leq b_u x\ r_m \qquad \text{(cf. sec. III.2.2)}$$

continued next page

Table VI.1 (continued)

(cf. sec. V.1)

(3)

UNIT PARAMETERS AND GENERAL PARAMETERS:

For bounds on b: apply (1) or (2) to the simple full subgraph.

For bounds on a: apply Gerschgorin theorem to the matrix

$$b = -R_F - S_{12}R_P^{-1}S_{12}^T$$

(4)

UNIT PARAMETERS:

For bounds on b: apply (1) or (2) to the simple full subgraph.

For bounds on a:

$$-a \leq a \leq -1$$

GENERAL PARAMETERS:

For bounds on b: apply (1) or (2) to the simple full subgraph.

For bounds on a: apply Gerschgorin theorem to the matrix

$$B = -R_F - S_{12}R_P^{-1}S_{12}^T$$

(5)

Find the state matrix and then use Gerschgorin theorem (or see V.2).

Bond Graph Method

Limitations:

Applies currently to only systems whose gyrobondgraph is a simple full graph or a special form of simple partial graph.

Information Needed:

The simplified gyrobondgraph (the point graph).

The largest parameter on the similar gyrobondgraphs (that has all inertias parameters equal 1).

Number of Computations:

For a system of order N:

If the system has general parameters then the number of required computations n_C is

$$N + 2 \leq n_C \leq N^2 + 4$$

depending on the structure of the gyrobondgraph.

If the systems has uniform parameters then the number of required computations $n_C \tilde{=} 5$ depending on the structure of the gyrobondgraphs.

Accuracy:

For systems with uniform parameters, the bounds are the best in comparison to the three other theorems.

For systems with general parameters, the upper bound is better only to the bound given by Pick's theorem.

Pick's Theorem

Limitations:

For any real matrix A and if $G = \frac{A+A^*}{2}$ and $T = \frac{A-A^*}{2}$ and $\lambda = a + ib$ is an eigenvalue of A_{NxN} then

$$|a_1| \leq g.N \quad , \quad \text{and}$$

$$b \leq t.\cot \frac{\pi}{2N} \quad ,$$

where g is the largest entry of G and t is the largest entry of T.

Information Needed:

The order of the matrix and the largest entry of the real and imaginary components of the state matrix.

Number of Computations:

For a matrix of order N, we need $3N+5$ operations for each of the real and imaginary part, in addition to the computations required to find the state matrix.

Accuracy:

The upper bounds given here are larger than any upper bound obtained by the three other methods.

Gerschgorin Theorem

Limitations:

For any complex matrix, if the union of the discs is the disc with the largest radius, then we cannot obtain a lower bound.

Information Needed: The state matrix.

Number of Computations:

For a real matrix, to obtain bounds on the largest real and largest imaginary part of the eigenvalues, then we need $3N + N^2$ operations for each.

Accuracy:

For a system with uniform parameters the bounds obtained using the bond graph are better. In case of general parameters, the bounds obtained here are better than any of the methods discussed.

The Eucledian Norm

Limitations:

Gives only upper bound.

Number of Computations:

For a real matrix, to obtain bounds on the largest imaginary part of the eigenvalue, we need $(N + 1)^2$ operation for each.

Accuracy:

For the general case comes after the Gerschgorin method.

Further Directions of Research

From the above comparison we note a substantial decrease in the number of computations required to obtain the bounds. This justifies an extension of this work aimed at broadening the range of problems to which the estimation procedure given here is applicable. The following points are suggested directions for further work.

1. For systems with general parameters we obtained an upper bound on the largest imaginary part of the eigenvalues. The results need to be extended to provide a lower bound.
2. For systems whose point graph is simple partial, can we use the results developed in this work to provide bounds on the imaginary part of the eigenvalues, and use the spectral graph theory to obtain bounds on the real parts?
3. For systems whose gyrobondgraph is complex can we construct a linear graph that can be used to estimate the eigenvalues of the system?

4. Given a system with general parameters and with the constraint that $\frac{R_i}{I_i}$ = constant for all i, we know the effect of changing the power direction on one line on the characteristic equation. This result needs to be generalized as to the constraint given above. We also need to investigate the effect of the power change on the largest eigenvalue.

APPENDICES

APPENDIX 1

The Coefficients of the Characteristic Equation:

The expression for the coefficients of the characteristic equation of the skew symmetric adjacency matrix with arbitrary entries can be obtained from a theorem by A. Spialter (GT5). The theorem is also described in (GT5) by Cvetokovic. In their work they give the coefficients of the symmetric adjacency matrix. This could be extended as shown below to the case of a skew symmetric adjacency matrix associated with a directed graph. We first present the theorem for the symmetric case.

Theorem 1. [GT2], [GT6]

Let $\tilde{G}$ be a graph with weighted edges, and let

$$P(\tilde{G}) = \det(\lambda I - A(\tilde{G})) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_o = 0$$

be the characteristic polynomial of $\tilde{G}$. Define:

Elementary figure E: The graph K_2 or
The graph C_q $(q > 2)$,

Basic figure U: Every graph all of its components are elementary figures

$p(U)$: the number of components in the basic figure U,

$c_t(U)$: the total number of cycles in the basic figure U,

$\mathcal{U}_i$: the set of all basic figures contained in G and having exactly i points,

$e(E)$: the set of edges belonging to an elementary figure E

$w(u)$: the weight on the edge u,

$\zeta(u,U)$ = 1 if u is contained in a cycle in U, and

= 2 if u belongs to K_2; $K_2 \varepsilon U$.

Then:

$$a_i = \sum_{U \varepsilon \mathcal{U}_i} (-1)^{p(U)} \cdot 2^{c_t(U)} \cdot \Pi(U) \qquad \ldots (1)$$

where

$$\Pi(U) = \prod_{**} \prod_{*} (w(u)^{\zeta(u,U)}) \qquad \ldots (2)$$

* is the product over all e(E).

** is the product over all $E \varepsilon U$.

Next theorem 1 is used to derive a formula for the coefficients of the characteristic polynomial of the skew symmetric adjacency matrix. We note that from the definition of S(G) the sign of the weight on any direct edge of G changes according to the direction of traversal of the edge. This produces the need for classifying cycles according to their orientation.

Define:

Odd (even) cycle: as a cycle with an odd (even) length.

Odd oriented cycle (even oriented cycle): as an even cycle in which the number of edges that have a clockwise direction is odd (even).

We note that the definition for the orientation of cycles does not apply to odd cycles.

Theorem 2

Let c(U) be the number of even cycles in the basic figure U,

$c^e(U)$ be the number of even oriented cycles, and let

$$P(G) = \det(\lambda I - S(G)) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_o = 0$$

be the characteristic polynomial of the oriented graph G.

If i is odd, then

$$a_i = 0 \quad , \text{ and} \qquad \ldots (3.a)$$

if i is even, then

$$a_i = \sum_{U \in \mathcal{U}_i} (-1)^{c^e(U)} \, 2^{c(U)} \, \Pi(U) \qquad \ldots (3.b)$$

where $\Pi(U)$, U, and $\mathcal{U}_i$ are as defined in Theorem 1.

Proof:

In equation 1, Theorem 1, the factor of 2 is due to the two directions of traversal of each cycle in every elementary figure [GT6]. In the case of the coefficients of $P(\tilde{G})$, the weights on the edges do not change signs for different directions of traversal. However in the case of P(G) the weights change signs depending on the direction of traversal of the cycle.

We have three possible cases for cycles $C_q = E$; $E \in U$.

1. q is odd: In this case the product $\Pi(w(u)^{\zeta(u,U)})$ in equation 4 will have a positive sign in one direction of traversal and a negative sign in the other direction. So if $E = C_q$, q is odd then $\Pi(U) = 0$ whenever $C_q \in U$, and the contribution of the basic figure in the summation 3.b is zero.

   Now whenever i is odd then any $U \in \mathcal{U}_i$ will have an elementary figure E that is a cycle C_q, q is odd. Thus every term in summation (3) will be equal zero, and 3.a follows.

2. If q is even and C_q is odd oriented: In this case the product $\Pi(w(u)^{\zeta(u,U)})$ will have the same sign in both directions of traversal, and the sign of the product is negative. Thus the power of (-1) in equation 1 will be decreased by the number of even cycles that have odd orientation and that belongs to U. The power of 2 in equation 1 will contain the number of C_q with odd orientation, q even.

3. If q is even and C_q is even oriented: Here the product $\Pi(w(u)^{\zeta(u,U)})$ will also have the same sign for both directions of traversal and the sign of the product is positive. We conclude that the power of (-1) in equation 3 will contain the number of cycles, with even orientation and even length, that belongs to U.

   The power of 2 in equation 3 will also contain the number of cycles, with even orientation and even length, that belongs to U.

If $E = K_2$, the product of the weight on K_2 is negative, and the power of (-1) in equation 3 is decreased by the number of $K_2 \in U$.

Now in equation 1 if we subtract from the powers of 2 the number of odd cycles in U, and subtract from the power of (-1) the number of K_2 and the number of even length cycles with odd orientation, we obtain equations 3.b.

Corollary 1

Changing the direction on any edge that does not belong to an even cycle does not change the spectrum of the oriented graph G.

This fact can be easily proved by observing that odd cycles do not contribute to any coefficient of P(G). Also the orientation on any edge that does not belong to a cycle does not effect the terms of equation 3.b.

Corollary 2

If G is an oriented tree, and $\tilde{G}$ is obtained from G by relaxing all the directions on its edges, and if the spectrum of S(G) is:

$$\text{Spec } (G) = [\pm\ j\ \lambda_1, \pm\ j\ \lambda_2, \ldots], \quad j = \sqrt{-1}$$

Then the spectrum of A(G) is

$$\text{Spec } (\tilde{G}) = [\pm\ \lambda_1, \pm\ \lambda_2, \ldots]$$

Proof:

If $\tilde{G}$ is a tree then Theorem 1 reduced to:

$$a_i = \sum_{U \in \mathcal{U}_i} (-1)^{p(U)} \quad \Pi(U) \qquad \ldots (4)$$

Now in (2) if every w(u) is replaced by jw(u), $j = \sqrt{-1}$, then equation 5.b, for G is a tree, becomes identical to equation 4 as follows:

$$a_i = \sum \prod_{**} \prod_{*} (jw(u)^2) = (-1)^{p(U)} \Pi(U).$$

From equation 4, and for trees if i is odd:

$$\mathcal{U}_i = \emptyset \qquad \text{and}$$

$$a_i = 0$$

APPENDIX 2

Recurrence Formula for Computing the Characteristic Polynomial:

Starting from the formula for the coefficients of the characteristic equation developed above; we use a proof similar to the proof used by A. Shwenk to develop a recursive formula for the characteristic equation of the symmetric adjacency matrix. Here we prove a general parameters formula for a skew symmetric case.

Theorem

Let G-v be the oriented graph obtained from G by deleting the node v and all edges that have v as their initial or terminal node,

$C^e(v)$ be the even cycle with even orientation that contains the node v, and

$C^o(v)$ be the even cycle with odd orientation that contains the node v, then

the characteristic polynomial of the skew symmetric adjacency matrix of the oriented graph G is given by:

$$P(G) = \lambda.P(G-v) + \sum_{*} r_{uv}^2 \; . \; P(G - v - u) + 2. \sum_{**} (\Pi C^o(v)) \; . \; P(G - C^o(v)) - 2. \sum_{***} (\Pi C^e(v)) \; . \; P(G - C^e(v)) \quad ...(5)$$

Where:

v is any node of the oriented graph G of order n,

r_{uv} is the weight on the directed edge (uv),

* is the summation over all the nodes u adjacent to v,

** is the summation over all even cycles with odd orientation that contains the node v, and

*** is the summation over all even cycles with even orientation that ocntains the node v.

Proof

From equation 3.b we have

$$a_i = \sum_{U \in \mathcal{U}_i}{}' (-1)^{c^e(U)} \cdot 2^{c(U)} \cdot \Pi(U)$$

We prove (5) by establishing a one to one correspondence between the basic figures contributing to the coefficient a_i on the left hand side of equation 5, and the basic figures contributing to one term on the right hand side.

Let $U \in \mathcal{U}_i$ be a basic figure of G and let m be its contribution to the coefficient a_i. We ahve two possibilities:

(1) $v \notin U$ take U' = U and U' is viewed as a subgraph of (G-v).

(2) $v \in E \in U$ take U' = U - E and U' is viewed as a subgraph of (G - E).

In case (1) since U' = U - E, the U' will contribute m to the coefficient of λ^{n-i-1} in P(G - v) and thus supplies m towards the coefficient of λ^{n-i} in $\lambda P(G - v)$.

In case (2), since U' = U - E, then U' will contribute:

$$(-1)^{c^e(U')} \cdot 2^{c(U')} \cdot \Pi(U') \qquad \ldots (6)$$

to the coefficient of $\lambda^{n-\ell-(i-\ell)} = \lambda^{n-i}$ in P(G - E) and ℓ is the number of vertices of E.

Now if $E = K_2$, then expression 6 becomes:

$$(-1)^{c^e(U)} \cdot 2^{c(U)} \cdot \frac{\Pi(U)}{r^2_{uv}} = \frac{m}{r^2_{uv}} \quad .$$

Hence U' supplies m to the coefficient of λ^{n-i} in:

$$r^2_{uv} \cdot P(G - u - v) \quad .$$

If $E = C^e(v)$, then expression 6 becomes:

$$(-1)^{c^e(U)} \cdot (-1)^{-1} \cdot 2^{c(U)} \cdot 2^{-1} \cdot \frac{\Pi(U)}{\Pi C^e(v)} = \frac{-m}{2.(\Pi C^e(v))}$$

Hence U' supplies m to the coefficient of λ^{n-i} in:

$$-2.(\Pi C^e(v)).P(G - C^e(v)) \quad .$$

If $E = C^o(v)$, then expression 6 becomes:

$$(-1)^{c^e(U)} \cdot 2^{c(U)} \cdot 2^{-1} \cdot \frac{\Pi(U)}{\Pi C^o(v)} = \frac{m}{2.(\Pi C^o(v)} \quad .$$

Hence U' supplies m to the coefficient of λ^{n-i} in:

$$2.(\Pi C^o(v)).P(G - C^o(v)) \quad .$$

Now if $E = C_q(v)$, and q is odd, then the contribution of U to the coefficient a_i is equal to zero. In this case let $m = 0.k$, where the zero is due to the presence of C_q, and k is the contribution of all other elementary figures in U. Hence U' supplies k to the coefficients of λ^{n-i} in the term:

$$0.P(G - C_q(v)) \quad .$$

We conclude that the term $P(G - C_q(v))$ does not exist on the left hand side of equation 5.

Thus the contribution of each basic figure U to the left hand side is matched by a corresponding contribution to the right hand side by a basic figure U', which completes the proof.

APPENDIX 3

The Characteristic Polynomial of an Oriented Graph in Relation to Two Subgraphs Connected by a Bridge

Let G be an oriented graph with characteristic polynomial P(G) let G_1 and G_2 be two subgraphs connected by a bridge as shown in Figure 1. Then

$$P(G) = P(G_1) \cdot P(G_2) + r_{uw}^2 \cdot P(G_2 - w) \cdot P(G_1 - u) \quad \ldots (6)$$

Proof:

Applying the recursive formula 3 successively we get:

$$P(G) = \lambda P(G - u) + \sum_{*} r_{wu}^2 P(G - u - w) +$$

$$2 \cdot \sum_{**} \Pi C^o(u) \cdot P(G - C^o(u)) - \Pi C^e(u) \cdot P(G - C^e(u))$$

$$= \lambda P(G_2) \cdot P(G_1 - u) + r_{wu}^2 P(G_1 - u) \cdot P(G_2 - w) +$$

$$P(G_2) \cdot \sum_{*} r_{wu}^2 P(G_1 - u - w) +$$

$$P(G_2) \cdot (2 \cdot \sum_{**} \Pi C^o(u) P(G - C^o(u)) - \Pi C^e(u) P(G - C^e(u))$$

$$= \ldots (6)$$

where * is the summation over all w adjacent to u,

and ** is the summation over all $C^o(u)$ and $C^e(u)$.

The above formula is usefull computing the characteristic polynomial of sparse systems.

Using equation 5 we can find a general formula for the path as follows.

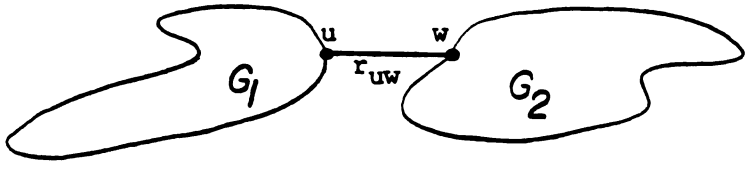


Figure 1. Example of two subgraphs joined by a bridge.

APPENDIX 4

The Characteristic Polynomial of a Path

For a path with N point let

$$\det (S_F - \lambda U) = P_N = 0 \quad , \text{ U is the identity matrix.}$$

then:

$$\lambda P_N = P_{N-1} - P_{N+1}$$

From equation 5, and starting with a path with N points; construct a path with N+1 points, then

$$P_{N+1} = \lambda P_N + P_{N-1}$$

This is the recursion formula for Chebichev's polynomial U_j of the second kind as follows:

$$(i)^{j+1} \cdot P_j = U_{j=1}\left(\frac{i\lambda}{2}\right) \quad , i = \sqrt{-1}$$

and

$$U_j = \frac{\sin\left((j+1)\cos^{-1}(i\lambda/2)\right)}{\sin\cos^{-1}(i\lambda/2)}$$

The above results could also be obtained from the characteristic polynomial of the symmetric adjacency matrix, since for trees the absolute values of the spectrum of the symmetric and skew symmetric are the same.

APPENDIX 5

The Characteristic Polynomial of a Star

For a star with N points let

$$\det (S_F - \lambda U) = S_N = 0$$

then $$S_N = \lambda^{N-2} (\lambda^2 = (N-1)$$

Proof:

Applying equation 5

$$S_N = \lambda P(G-u) + (N-1) P(G-u-w)$$

$$P(G-u) = \lambda^{N-1}$$

$$P(G-u-w) = \lambda^{N-2}$$

So $$S_N = \lambda\lambda^{N-1} + (N-1) \lambda^{N-2}$$

$$= \lambda^{N-2} (\lambda^2 + (N-1))$$

which gives $$b_u = \sqrt{N-1} = b_{us}$$

where b_{us} is the largest eigenvalue of the star tree.

APPENDIX 6

Bounds for Largest Eigenvalues of Trees

$$b_u < b_{us} = \sqrt{N-1}$$

Since all trees with N points have the same number of lines: N-1; then the number of entries of the matrix S_F is = 2(N-1)

for a tree $\quad ||S_F||_E^2 = 2(N-1)$

but $\quad ||-||_E^2 = \Sigma\lambda_i^2$

so $\quad \sum_{i=1}^{N} \lambda_i^2 = 2(N-1)$

and since λ_i occurs in complex conjugates then:

$$2 \sum_{i=1}^{N/2} \lambda_i^2 = 2(N-1)$$

$$\lambda_1^2 \leq (N-1)$$

$$\lambda_1 \leq \sqrt{N-1} = b_{us}$$

APPENDIX 7

General Parameters Problem

Given a skew symmetric matrix $S_F = [s_{ij}]$ define a matrix $H = [h_{ij}]$ such that $h_{ij} = \text{sgn}(s_{ij})$

let: $s = \max_{i,j} s_{ij}$,

let G be an oriented graph associated with the matrix H, such that H is the skew symmetric adjacency matrix of the oriented graph G, if ib(.) is an eigenvalue of a skew symmetric matrix such that

$$b_n(.) \leq \ldots\ldots \leq b_1(.)$$

then:

(A) If G is a tree or having only cycles with odd number of points then:

$$b_1(S_F) \leq b_1(H) * s$$

(B) If G has cycles with even number of points then:

$$b_1(S_F) \leq b_1(H_{opt}) * s$$

where H_{opt} is the adjacency matrix associated with an optimal graph G_{opt} of G.

i.e. $b_1(H) \leq b_1(H_{opt})$

Proof:

Let $Ax = ib_1 x$

$\frac{1}{i} Ax = b_1 x$

$$\frac{1}{i}Ax = b_1 x$$

$$x^* x = 1$$

$$b_1 = x^* \frac{1}{1} \sum_{r<s} a_{rs} (x_r^* x_s - x_r x_x^*) \quad \ldots (1)$$

Proceed as follows:

(A) Whenever x_r has a negative real part, change the sign of x_r so we have the real part of all x's positive. At the same time change the sign of the corresponding a_{rs} such that equation (1) remains unchanged. This is equivalent to the similarity transformation of multiplying row r and column r by -1. In the graph G this corresponds to inverting all powers on point r. After this operation we will have x_r having their arguments Φ_r such that:

$$-\pi/2 < \Phi \leq \pi/2$$

Rename the new vector y and equation 1 now becomes:

$$b_1 = \frac{1}{i} \sum_{r<s} c_{rs} (y_r^* y_s - y_r y_s^*) \quad \ldots (2)$$

Note that:

$$|c_{rs}| = |b_{rs}|$$

(B) Reorder the vectors y in equation (2) by changing the indeces such that we always have:

$$\Phi_r \geq \Phi_{r+1}$$

At the same time change the signs of b_{rs} whenever needed such that equation (2) remains unchanged.

This operation is equivalent to a similarity transformation by permuting row k and column k. In the oriented graph G this is equivalent to a relabeling operation. Now rename the new vectors z and equation (2) becomes:

$$b_1 = \frac{1}{i} \sum_{r<s} d_{rs} \, (z_r^* z_s - z_r z_s^*)$$

Note that here

$$|d_{rs}| = |c_{rs}|$$

(C) We have for $r \not< s$

$$\frac{1}{i} (z_r^* z_s - z_r z_s^*) = 2.r_r r_s \sin (\Phi_r - \Phi_s) > 0$$

(since in step 2 we made all $\Phi_r > \Phi_s$, and in step 1 we made all $-\pi/2 < \Phi_r \leq \pi/2$)

r_r, r_s are the modulus of the vectors z_r, z_s.

(D) Equation (3) now becomes

$$b_1 = \sum_{r<s} 2.d_{rs}.r_r r_s \sin(\Phi_r - \Phi_s)$$

$$\leq \sum_{r<s} 2. \, d \, .r_r r_s \sin (\Phi_r - \Phi_s) \leq \frac{d}{i} \sum (z_r^* z_s - z_r z_s^*)$$

where $$d = \max_{r,s} |d_{rs}|$$

so $$b_1 \leq \frac{d}{i} \cdot z^* H z = d \, z^* \cdot \frac{1}{i}.H. \, z \leq d.b_1(H)$$

H is a skew symmetric matrix with the same number of zeros as the matrix S_F and has all the upper diagonal enteries positive.

(E) In the case of a tree or all odd numbered cycle graph; changing the power direction on the individual lines does not change the spectra of the graph. Therefore H can be taken as an adjacency matrix of the oriented graph $\tilde{G}$ that is obtained from G by operations in (a), (b) and then by changing some powers on individual lines in step (d).

(F) If G has even numbered cycles, H can be taken as the skew symmetric adjacency matrix of the oriented graph $\tilde{G}$ that has the same numberof lines as G but with different power orientation. Since $\tilde{G}$ is spectrally different from G, and $\tilde{G}$ is unknown, choose $\tilde{\tilde{G}}$ as the optimal graph such as:

$$b_1\ (H(\tilde{G})) \leq b_1\ (H(\tilde{\tilde{G}}))$$

APPENDIX 8

The State Matrix of a Class of Simple Partial Graphs

Let G be a point graph that has full points and partial I points only. The state matrix of the system associated with such point graph is given by:

$$A = [(S_F - R_F) + S_{12} (S_p - R_p) S_{12}^T] [I^{-1}]$$

where:

S_F... is the gyroadjacency matrix of the simple full subgraph, and

S_p... is the gyroadjacency matrix of the simple partial subgraph, and

$S = \begin{bmatrix} S_F & S_2 \\ -S_{12}^T & S_P \end{bmatrix}$ is the gyroadjacency matrix of the point graph, and

R_F... is the diagonal matrix of resistances adjacent to full points

R_p... is the diagonal matrix of resistances adjacent to partial R points

I^{-1}... is the diagonal matrix of the inverse of the inertias adjacent to the full points.

Proof:

In a gyrobondgraph we have only effort sources. Assigning causalities to sources and integration causalities on storage elements does not complete the augmentation of the bondgraph. In this case a proper but not unique (see BG2 pp 150), choice is to assign conductance causalities to resistances adjacent to partial points.

Let e_f, f_f be the effort and the flow on bonds adjacent to a storage element.

Let e_p, f_p be the effort and the flow on bonds adjacent to a resistance that is adjacent to a partial point.

Let _ denote a column vector; then for the simple partial graph with causality as discussed we can write:

$$\underline{e} = \begin{bmatrix} \underline{e}_f \\ \underline{e}_p \end{bmatrix} \left[\begin{bmatrix} S_F & S_{12} \\ -S_{12}^T & S_P \end{bmatrix} - \begin{bmatrix} R_F & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} \underline{f}_f \\ \underline{f}_p \end{bmatrix} \quad \ldots (1)$$

Also we have for the storage elements:

$$\underline{f}_f = I^{-1} \underline{p} \quad \ldots (2)$$

And for resistances adjacent to partial points:

$$\underline{e}_p = R_p \underline{f}_p \quad \ldots (3)$$

By substituting (3) into the equation of $\underline{e}_p$ and then into $\underline{e}_f$ and knowing that $\dot{\underline{p}} = \underline{e}_f$, we obtain the state matrix A:

$$\dot{\underline{p}} = A\underline{p}$$

and $$A = [(S_F - R_F) + S_{12} (S_P - R_P)^{-1} S_{12}^T] [I^{-1}]$$

APPENDIX 9

Lower Bounds on Largest Eigenvalues for General Graphs

Let G be a general graph with N points.

Let S (G) be its skew symmetric adjacency matrix.

Let $b_1(S(G))$ be its largest eigenvalue.

Let $b_{uC_N^0}$ be the largest eigenvalue of an odd power cycle having N points.

Let $b_{uC_N^e}$ be the largest eigenvalue of an even power cycle having N points.

Then: If N/2 is odd

$$2\cos\frac{\pi}{N} = b_{uC_N^e} \leq b_1 \ (G)$$

and If N/2 is even

$$2\cos\frac{\pi}{N} = b_{uC_N^0} \leq b_1 \ (G)$$

and If N is odd

$$S\cos\frac{\pi}{N+1} = b_{uP} \leq b_1 \ (G)$$

Proof:

Arrange all graphs into two classes:

1) All trees with N points, and all optimal graphs with N points.

2) Graphs that are not optimal.

Now delete a line from a graph belonging to the first class. The obtained graph will have smaller b_1. Classify the obtained graph according to whether it is optimal or not. If it is optimal we can

proceed with deleting a line. The minimal connected graph resulting from repeating this operation is a tree, and from results in appendix 1 and Cvetkovic (BG2, p73), the graph with smallest b_1 is the path. In the second class, deleting a line will either produce a graph of the first class with b_1 larger than the original (because of continuity property of the eigenvalues as function of parameters on the line of the graph), or will produce a graph of the second class with smaller b_1. The first case was discussed above. The minimal connected graph obtained if the second case continues to happen is the cycle with $b_1 = 2 \cos \frac{\pi}{N}$. From properties of cycles presented in Chapter III, in Table III.1, we conclude the hypothesis.

APPENDIX 10

Bounds on Largest Eigenvalue Using Subgraphs

Let G be a point graph with all its points being full;

let A be its skew symmetric adjacency matrix,

let G_1, G_2 . . . be subgraphs of G obtained by a certain dissection,

let Δ^b be the largest boundary degree in the subgraphs otained,

let A_B be the NxN block diagonal matrix with its diagonal blocks being the skew symmetric adjacency matrices of the subgraphs G_1, G_2, . . .

let the matrix C be such that

$$C = A - A_B,$$

and let $b_1(.)$ denote the largest eigenvalue of skew symmetric adjacency matrices, then

$$b_1(G) \leq b_1(G_1) + b_1(C) \quad \ldots (1)$$

or

$$b_1(G) \leq b_1(G_1) + \Delta^b \quad \ldots (2)$$

and

$$b_1(G_1) \leq b_1(G) \quad \ldots (3)$$

Proof:

The Lower Bound (inequality 3)

From the definition of dissection, G_1 has at least one point less than G. Thus, the skew symmetric adjacency matrix of G_1 is a principle submatrix of the skew symmetric adjacency matrix of G. Then by the use of the theorem of interlacing of the eigenvalues of principal

submatrices, (Theorem 2, Chapter II.4), and using property (2) for skew symmetric matrices, (Chapter II.4), we obtain the lower bound.

The Upper Bound

We have

$$A = A_B + C$$

and the eigenvalues of A interlace with the sum of the eigenvalues of A^B and C, and

$$b_1(A) \leq b_1(A_B) + b_1(C)$$

Using Gerschgorin theorem on matrix C and knowing that the largest row sum in C is Δ^b, we get

$$b_1(C) \leq \Delta^b$$

and $$b_1(A^B) = b_1(G_1) \quad \text{by definition}$$

and thus we conclude

$$\therefore \quad b_1(A) \leq b_1(G_1) + \Delta^b$$

Note: 1. The lower bound given in the inequality 3 is weaker than the equivalent lower bound for the symmetric adjacency matrix. This is due to the fact that the latter was based on Frobenius-Perron theorem for non-negative matrices (GT2) which apply to the symmetric adjacency matrix but not the skew symmetric case (cf.II.3.2.).

2. An application of a result given by Slepian and Landaw in (NA7) would give the bound

$$b_1(G) \leq \frac{(b_1(G') + \sqrt{(b_1(G'))^2 + 4\delta})}{2}$$

where G' is the subgraph obtained from G by deleting only one point with degree δ.

APPENDIX 11

Optimal Graphs

Let G_1 and G_2 be two optimal point graphs, then the point graph obtained from their direct sum, $G = G_1 + G_2$, is an optimal graph and

$$b_1(G) = b_1(G_1) + b_2(G_2) \qquad \ldots (1)$$

Proof:

If A(G) is the skew symmetric adjacency matrix of the point graph G, then using the theorem given in (LA5), and used in (GT2) for giving the same result for symmetric adjacency, we obtain

$$A(G) = A(G_1) \otimes I_2 + I_1 \otimes A(G_2) \qquad \ldots (2)$$

where I_1, I_2 are identity matrices of the same order as $A(G_1)$ and $A(G_2)$ and $\otimes$ is the Kronkner product, then

$$b_i(G) = b_i(G_1) + b_i(G_2) \qquad \text{(LA5)} \ldots (3)$$

Now the matrix A(G) can be written as

$$A(G) = A_{B2} + C$$

where A_{B2} is a block diagonal matrix with all its block diagonal matrices are $A(G_2)$, and C is as defined in the previous proof.

Let $$A(G') = A_{B2} + C'$$

where C' is different from C by changing in G the power direction on one or more lines or deleting one or more lines.

Now from the theorem of interlacing of the eigenvalues of the sum of matrices (cf.II.4, Theorem 4), we have

$$b_1(G') \leq b_1(A_{B2}) + b_1(C').$$

Now the matrix $C = A(G_1) \otimes I_2$, is similar to a diagonal matrix whose diagonals blocks are the skew symmetric adjacency matrix $A(G_1)$ and C' will also be a block diagonal with its diagonal blocks the same as for C except for the changes. If G_1 is an optimal graph, then

$$b_1(C') \leq b(C)$$

and

$$b_1(G') \leq b_1(G)$$

The same proof is repeated for G_2 as an optimal graph.

APPENDIX 12

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A Definition of the Bond Graph Language

Introduction

THE purpose of this paper is to present the basic definitions of the bond graph language in a compact but general form. The language presented herein is a formal mathematical system of definitions and symbolism. The descriptive names are stated in terms related to energy and power, because that is the historical basis of the multiport concept.

It is important that the fundamental definitions of the language be standardized because an increasing number of people around the world are using and developing the bond graph language as a modeling tool in relation to multiport systems. A common set of reference definitions will be an aid to all in promoting ease of communication.

Some care has been taken from the start to construct definitions and notation which are helpful in communicating with digital computers through special programs, such as ENPORT [5].[1] It is hoped that any subsequent modifications and extensions to the language will give due consideration to this goal.

Principal sources of extended descriptions of the language and physical applications and interpretations will be found in Paynter [1], Karnopp and Rosenberg [2, 3], and Takahashi, et al. [4]. This paper is the most highly codified version of language definition, drawing as it does upon all previous efforts.

Basic Definitions

Multiport Elements, Ports, and Bonds. *Multiport elements* are the nodes of the graph, and are designated by alpha-numeric characters. They are referred to as elements, for convenience. For example, in Fig. 1(*a*) two multiport elements, 1 and *R*, are shown. *Ports* of a multiport element are designated by line segments incident on the element at one end. Ports are places where the element can interact with its environment.

For example, in Fig. 1(*b*) the 1 element has three ports and the *R* element has one port. We say that the 1 element is a 3-port, and the *R* element is a 1-port.

Bonds are formed when pairs of ports are joined. Thus bonds are connections between pairs of multiport elements.

For example, in Fig. 1(*c*) two ports have been joined, forming a bond between the 1 and the *R*.

Bond Graphs. A *bond graph* is a collection of multiport elements bonded together. In the general sense it is a linear graph whose nodes are multiport elements and whose branches are bonds.

A bond graph may have one part or several parts, may have no loops or several loops, and in general has the characteristics of any linear graph.

An example of a bond graph is given in Fig. 2. In part (*a*) a bond graph with seven elements and six bonds is shown. In part (*b*) the same graph has had its powers directed and bonds labeled.

A *bond graph fragment* is a bond graph not all of whose ports have been paired as bonds.

An example of a bond graph fragment is given in Fig. 1(*c*), which has one bond and two open, or unconnected, ports.

Port Variables. Associated with a given port are three direct and three integral quantities.

Effort, $e(t)$, and *flow*, $f(t)$, are directly associated with a given port, and are called the port power variables. They are assumed to be scalar functions of an independent variable (t).

Power, $P(t)$, is found directly from the scalar product of effort and flow, as

$$P(t) = e(t) \cdot f(t).$$

The direction of positive power is indicated by a half-arrow on the bond.

Momentum, $p(t)$, and *displacement*, $q(t)$, are related to the effort and flow at a port by integral relations. That is,

[1] Numbers in brackets designate References at end of paper.

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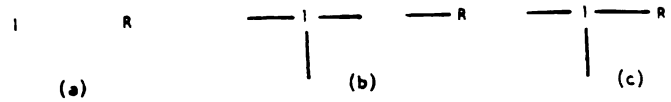


Fig. 1 Multiport elements, ports, and bonds: (a) two multiport elements; (b) the elements and their ports; (c) formation of a bond

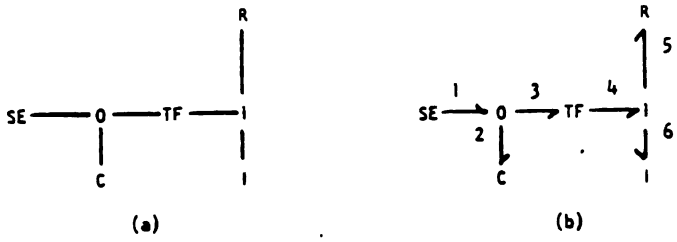


Fig. 2 An example of a bond graph: (a) a bond graph; (b) the bond graph with powers directed and bonds labeled

$$p(t) = p(t_0) + \int_{t_0}^{t} e(\lambda)d\lambda$$

and $q(t) = q(t_0) + \int_{t_0}^{t} f(\lambda)d\lambda$, respectively.

Momentum and displacement are sometimes referred to as energy variables.

Energy, $E(t)$, is related to the power at a port by

$$E(t) = E(t_0) + \int_{t_0}^{t} P(\lambda)d\lambda.$$

The quantity $E(t) - E(t_0)$ represents the net energy transferred through the port in the direction of the half-arrow (i.e., positive power) over the interval (t_0, t).

In common bond graph usage the effort and the flow are often shown explicitly next to the port (or bond). The power, displacement, momentum, and energy quantities are all implied.

Basic Multiport Elements. There are nine basic multiport elements, grouped into four categories according to their energy characteristics. These elements and their definitions are summarized in Fig. 3.

Sources.

Source of effort, written $SE\ e$, is defined by $e = e(t)$.

Source of flow, written $SF\ f$, is defined by $f = f(t)$.

Storages.

Capacitance, written $\frac{e}{f}\ C$, is defined by

$$e = \Phi(q) \text{ and } q(t) = q(t_0) + \int_{t_0}^{t} f(\lambda)d\lambda.$$

That is, the effort is a static function of the displacement and the displacement is the time integral of the flow.

Inertance, written $\frac{e}{f}\ I$, is defined by

$$f = \Phi(p) \text{ and } p(t) = p(t_0) + \int_{t_0}^{t} e(\lambda)d\lambda.$$

That is, the flow is a static function of the momentum and the momentum is the time integral of the effort.

Dissipation.

Resistance, written $\frac{e}{f}\ R$, is defined by

$$\Phi(e, f) = 0.$$

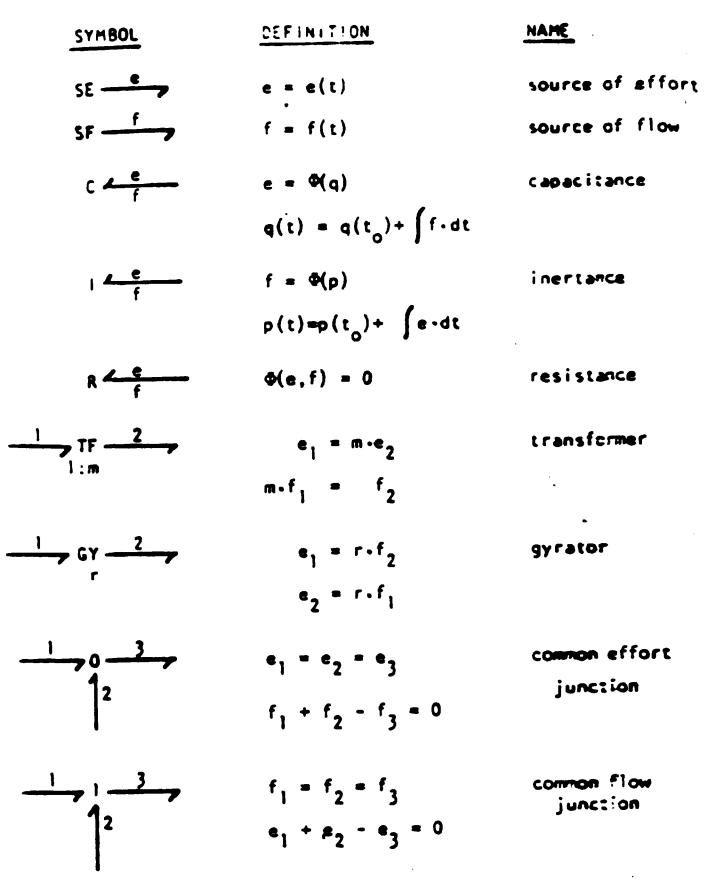


Fig. 3 Definitions of the basic multiport elements

That is, a static relation exists between the effort and flow at the port.

Junctions: 2-Port.

Transformer, written $\frac{e_1}{f_1}\ TF\ \frac{e_2}{f_2}$, is a linear 2-port element defined by

$$e_1 = m \cdot e_2$$

and

$$m \cdot f_1 = f_2,$$

where m is the modulus.

Gyrator, written $\frac{e_1}{f_1}\ GY\ \frac{e_2}{f_2}$, is a linear 2-port element defined by

$$e_1 = r \cdot f_2$$

and

$$e_2 = r \cdot f_1,$$

where r is the modulus.

Both the transformer and gyrator preserve power (i.e., $P_1 = P_2$ in each case shown), and they must each have two ports, so they are called essential 2-port junctions.

Junctions: 3-Port.

Common effort junction, written $\xrightarrow{1}\ 0\ \xrightarrow{3}$ with $\uparrow 2$

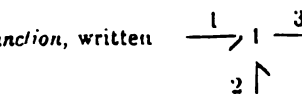
is a linear 3-port element defined by

$$e_1 = e_2 = e_3 \qquad \text{(common effort)}$$

and

$$f_1 + f_2 - f_3 = 0. \qquad \text{(flow summation)}$$

Other names for this element are the *flow* junction and the

zero junction. *Common flow junction*, written  , is a linear 3-port element defined by

$$f_1 = f_2 = f_3 \qquad \text{(common flow)}$$

and

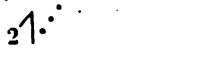
$$e_1 + e_2 - e_3 = 0. \qquad \text{(effort summation)}$$

Other names for this element are the *effort* junction and the *one* junction.

Both the common effort junction and the common flow junction preserve power (i.e., the *net* power in is zero at all times), so they are called junctions. If the reference power directions are changed the signs on the summation relation must change accordingly.

Extended Definitions

Multiport Fields.

Storage Fields. *Multiport capacitances*, or *C-fields*, are written  , and characterized by

$$e_i = \Phi_i(q_1, q_2, \ldots q_n),\ i = 1 \text{ to } n,$$

and $q_i(t) = q_i(t_0) + \int_{t_0}^{t} f_i(\lambda)d\lambda,\ i = 1 \text{ to } n.$

Multiport inertances, or *I-fields*, are written  ,

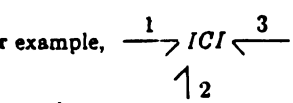
and characterized by

$$f_i = \Phi_i(p_1, p_2, \ldots p_n),\ i = 1 \text{ to } n,$$

and $p_i(t) = p_i(t_0) + \int_{t_0}^{t} e_i(\lambda)d\lambda,\ i = 1 \text{ to } n.$

If a *C*-field or *I*-field is to have an associated "energy" state function then certain integrability conditions must be met by the Φ_i functions. In multiport terms the relations given in the foregoing are sufficient to define a *C*-field and *I*-field, respectively.

Mixed multiport storage fields can arise when both *C* and *I*-type storage effects are present simultaneously. The symbol for such an element consists of a set of *C*'s and *I*'s with appropriate ports indicated.

For example,  indicates the existence of a set of relations

$$f_1 = \Phi_1(p_1, q_2, p_3),$$

$$e_2 = \Phi_2(p_1, q_2, p_3),$$

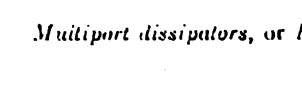
$$f_3 = \Phi_3(p_1, q_2, p_3),$$

and

$$p_1(t) = p_1(t_0) + \int_{t_0}^{t} e_1(\lambda)d\lambda,$$

$$q_2(t) = q_2(t_0) + \int_{t_0}^{t} f_2(\lambda)d\lambda,$$

$$p_3(t) = p_3(t_0) + \int_{t_0}^{t} e_3(\lambda)d\lambda.$$

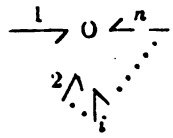
Multiport dissipators, or *R*-fields, are written 

and are characterized by

$$\Phi_i(e_1, f_1, e_2, f_2, \ldots e_n, f_n) = 0,\ i = 1 \text{ to } n.$$

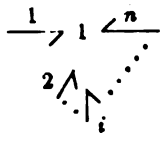
If the *R*-field is to represent pure dissipation, then the power function associated with the *R*-field must be positive definite.

Multiport junctions include 0 junctions and 1 - junctions with *n* ports, $n \geq 2$. The general case for each junction is given in the following.



$$e_1 = e_2 = \ldots = e_n$$

$$\sum_{i=1}^{n} f_i = 0$$



$$f_1 = f_2 = \ldots = f_n$$

$$\sum_{i=1}^{n} e_i = 0$$

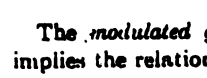
Modulated 2-Port Junctions. The *modulated transformer*, or *MTF* written implies the realtions

$$e_1 = m(\mathbf{x})\cdot e_2$$

and

$$m(\mathbf{x})\cdot f_1 = f_2,$$

where $m(\mathbf{x})$ is a function of a set of variables, $\mathbf{x}$. The modulated transformer preserves power; i.e., $P_1(t) = P_2(t)$.

The *modulated gyrator*, or *MGY*, written  implies the relations

$$e_1 = r(\mathbf{x})\cdot f_2$$

and

$$e_2 = r(\mathbf{x})\cdot f_1,$$

where $r(\mathbf{x})$ is a function of set of variables, $\mathbf{x}$. The modulated gyrator preserves power; i.e., $P_1(t) = P_2(t)$.

Junction Structure. The *junction structure* of a bond graph is the set of all 0, 1, *GY*, and *TF* elements and their bonds and ports. The junction structure is an *n*-port that preserves power (i.e., the *net* power in is zero). The junction structure may be modulated (if it contains any *MGY*'s or *MTF*'s) or unmodulated.

For example, the junction structure of the graph in Fig. 2(*b*) is a 4-port element with ports 1, 2, 5, and 6 and bonds 3 and 4. It contains the elements 0, *TF*, and 1.

Physical Interpretations

The physical interpretations given in this section are very succinctly stated. References [1], [2], and [3] contain extensive descriptions of physical applications and the interested reader is encouraged to consult them.

Mechanical Translation. To represent mechanical translational phenomena we may make the following variable associations:

1 *effort*, *e*, is interpreted as *force;*
2 *flow*, *f*, is interpreted as *velocity;*
3 momentum, *p*, is interpreted as impulse-momentum;
4 displacement, *q*, is interpreted as mechanical displacement.

Then the basic bond graph elements have the following interpretations:

1 source of effort, *SE*, is a force source;
2 source of flow, *SF*, is a velocity source (or may be thought of as a geometric constraint);

3 resistance, R, represents friction and other mechanical loss mechanisms;

4 capacitance, C, represents potential or elastic energy storage effects (or spring-like behavior);

5 inertance, I, represents kinetic energy storage (or mass effects);

6 transformer, TF, represents linear lever or linkage action (motion restricted to small angles);

7 gyrator, GY, represents gryational coupling or interaction between two ports;

8 0-junction represents a common force coupling among the several incident ports (or among the ports of the system bonded to the 0-junction); and

9 1-junction represents a common velocity constraint among the several incident ports (or among the ports of the system bonded to the 1-junction).

The extension of the interpretation to rotational mechanics is a natural one. It is based on the following associations:

1 effort, e, is associated with torque; and

2 flow, f, is associated with angular velocity.

Because the development is so similar to the one for translational mechanics it will not be repeated here.

Electrical Networks. In electrical networks the key step is to interpret a port as a terminal-pair. Then variable associations may be made as follows:

1 *effort*, e, is interpreted as *voltage*;

2 *flow*, f, is interpreted as *current*;

3 momentum, p, is interpreted as flux linkage;

4 displacement, q, is interpreted as charge.

The basic bond graph elements have the following interpretations:

1 source of effort, SE, is a voltage source;

2 source of flow, SF, is a current source;

3 resistance, R, represents electrical resistance;

4 capacitance, C, represents capacitance effect (stored electric energy);

5 inertance, I, represents inductance (stored magnetic energy);

6 transformer, TF, represents ideal transformer coupling;

7 gyrator, GY, represents gyrational coupling;

8 0-junction represents a parallel connection of ports (common voltage across the terminal pairs); and

9 1-junction represents a series connection of ports (common current through the terminal pairs).

Hydraulic Circuits. For fluid systems in which the significant fluid power is given as the product of pressure times volume flow, the following variable associations are useful:

1 *effort*, e, is interpreted as *pressure*;

2 *flow*, f, is interpreted as *volume flow*.

3 momentum, p, is interpreted as pressure-momentum;

4 displacement, q, is interpreted as volume.

The basic bond graph elements have the following interpretations:

1 source of effort, SE, is a pressure source;

2 source of flow, SF, is a volume flow source;

3 resistance, R, represents loss effects (e.g., due to leakage, valves, orifices, etc.);

4 capacitance, C, represents accumulation or tank-like effects (head storage);

5 inertance, I, represents slug-flow inertia effects;

6 0-junction represents a set of ports having a common pressure (e.g., a pipe tee);

7 1-junction represents a set of ports having a common volume flow (i.e., series).

Other Interpretations. This brief listing of physical interpretations of bond graph elements is restricted to the simplest, most direct, applications. Such applications came first by virtue of historical development, and they are a natural point of departure for most classically trained scientists and engineers. As references [1–4] and the special issue collection in the JOURNAL OF DYNAMIC SYSTEMS, MEASUREMENT, AND CONTROL, TRANS. ASME, Sept. 1972, indicate, bond graph elements can be used to describe an amazingly rich variety of complex dynamic systems. The limits of applicability are not bound by energy and power in the sense of physics; they include any areas in which there exist useful analogous quantities to energy.

Concluding Remarks

In this brief definition of the bond graph language two important concepts have been omitted. The first is the concept of *bond activation*, in which one of the two power variables is suppressed, producing a pure signal coupling in place of the bond. This is very useful modeling device in active systems. Further discussion of activation will be found in reference [3], section 2.4, as well as in references [1] and [2].

Another concept omitted from discussion in this definitional paper is that of *operational causality*. It is by means of causality operations applied to bond graphs that the algebraic and differential relations implied by the graph and its elements may be organized and reduced to state-space form in a systematic manner. Extensive discussion of causality will be found in reference [3], section 3.4 and chapter 5. Systematic formulation of relations is presented in reference [6].

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