THE VAN HIELE THEORY THROUGH THE DISCURSIVE LENS: PROSPECTIVE TEACEHRS' GEOMETRIC DISCOURSES

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ABSTRACT

THE VAN HIELE THEORY THROUGH THE DISCURSIVE LENS:
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Over the past decade, there has been an increasing trend in the mathematics education research community to study students' reasoning in the teaching and learning of mathematics, and to examine issues emphasizing the use of vocabulary, terminology, and words in the mathematics classroom. In response, this study investigates changes in prospective elementary teachers' levels of geometric thinking, and the development of their geometric discourses in the classification of quadrilaterals.

In Sfard's (2008) *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing,* she introduces her commognitive framework, a systematic approach to analyzing the discursive features of mathematical thinking, including word use, visual mediators, routines, and endorsed narratives. To examine thinking about geometry, this study connects Sfard's analytic framework to another, namely the van Hiele theory (see van Hiele, 1959/1985). The van Hiele theory describes the development of students' five levels of thinking in geometry. Levels 1 to 5 are described as visual, descriptive, theoretical, formal logic and rigor, respectively. This study used the van Hiele Geometry Test from the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project (Usiskin, 1982) as the pretest and posttest to determine prospective elementary school teachers' van Hiele levels. This study also produces, on the basis of theoretical understandings and of empirical data, a detailed model, namely, *the Development of Geometric Discourse*. This model translates the

van Hiele levels into discursive stages of geometric discourses with respect to word use, visual mediators, routines, and endorsed narratives.

This study reveals discursive similarities and differences in participants' geometric discourses at the same van Hiele level, as well as changes in geometric discourse as a result of changes in levels of geometric thinking. The study also investigates the usefulness of a discursive framework in providing "rich descriptions" of participants' thinking processes.

To my grandparents, Changmao Wang and Zhongmin Xu for everything.

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CHAPTER ONE: INTRODUCTION

In a research report prepared for the U.S. Department of Education, Wilson, Floden and Ferrini-Mundy (2001) reported that research shows a positive connection between teachers' preparation in their subject matter and their performance and impact in the classroom, and found that "current results of subject matter preparation are disappointing" (p.35). Darken (2007) also pointed out that "the weak mathematical preparation of many elementary and middle school (K-8) teachers is one of the most serious problems afflicting American education" (p.20). These conclusions suggest that a teachers' preparation program needs to emphasize mathematics content knowledge for teaching. Knowing mathematics for teaching involves knowledge of mathematical ideas, mathematics reasoning skills, as well as communication skills, fluency with examples and terms, and thoughtfulness about the nature of mathematical proficiency.

Geometry, considered as a tool for understanding and interacting with the space in which we live, is perhaps the most intuitive, concrete and reality-linked part of mathematics (ICMI, 1998). It is in the language of geometry that the visual structure of our physical world is described and communicated between individuals, and the language of geometry helps students to reason deductively and to think interdependently. "It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth (Usiskin, 1996, p.231). Today, the language of geometry is used without its structure and grammar, and thus it is still a foreign language to many teachers (Pimm, 1987; Usiskin, 1996).

The National Council of Teachers of Mathematics (NCTM, 2000) *Principles and Standards for School Mathematics (PSSM)* recommended that students should "analyze characteristics and properties of two- and three dimensional geometric shapes and develop mathematical arguments about geometric relationships" (p.41). For instance, in the Geometry Standards for grades 3 to 5 students, it is recommended that all students should identify, compare and analyze polygons, and develop vocabulary to describe their attributes, as well as to classify polygons according to their properties, and to develop definitions of classes of shapes. Because students are expected to learn about geometrical concepts and attributes, as well as relationships between them, it is important for future teachers to know and be comfortable with the language of geometry.

In prior research and literature on students' learning of geometry, and in literature emphasizing the use of van Hiele theory to categorize students' levels of thinking, many studies address the complexity and difficulty of students' learning of geometry, as well as other educational and psychological concerns. Based on what others have studied about prospective teachers' learning in geometry and their geometric thinking, this study is guided by this overarching question: What do prospective teachers learn in geometry from their preparation for the work of teaching geometry?

The mathematics education community has always been interested in the teaching and learning of mathematics, and we became more aware of the importance of human interaction in the classrooms, and how it influences the effectiveness of teaching and learning. The notion of mathematics as discourse and students as being apprenticed into particular ways of *doing mathematics* in particular discursive contexts is now gaining prominence in mathematics education research. This phenomenon prompted the call for the study of teachers' knowledge in geometry and of their learning of geometry. This study is informed by prior research and literature on van Hiele theory (van Hiele, 1959/1985), a framework that describes students' levels of geometric thinking, and by studies using van Hiele theory in the context of methodology and teacher knowledge in geometry. Additionally, this study is informed by research in the past that investigates students' mathematics discourses in their discursive learning.

While previous work sheds light on prospective teachers' knowledge and thinking in geometry, it has not explored how examination of these teachers' geometric discourses could help in learning more about their levels of geometric thinking. This study, influenced by the discursive nature of van Hiele theory, and of discourse analysis in the form of the Commognition framework described in *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing* (Sfard, 2008), seeks to examine prospective teachers' knowledge in geometry, and to investigate as well their ways of communicating geometric thinking. The study revisits van Hiele levels with careful examination of key mathematical features at each level. These mathematical features include (1) use of mathematical words, (2) use of visual mediators in the form of geometric figures and their parts, and symbolic artifacts created for the purpose of communicating about geometry, (3) endorsed narratives such as mathematical propositions, axioms and definitions, and (4) mathematical routine procedures with which participants implement well-defined types of tasks. The discursive framework provides a new lens to investigate students' geometric thinking.

In Chapter 2, I position this study among studies addressing the teaching and learning of geometry in mathematics education in general, and studies that examine students' thinking in geometry using van Hiele theory, as well as studies emphasizing discursive learning. In addition, I describe Sfard's discursive framework in detail, including a description of each of its four key mathematical features and important phenomenon highlighted in this framework. Chapter 3 describes the methodology of the study, including descriptions of van Hiele Geometry Test instruments, interview tasks and an outline of the design of the study. Chapter 4 contains the results of the analyses conducted in this study along with interpretations of findings. These include the van Hiele Geometry pretest and posttest results and analyses of the whole group, and participants' in-depth interview results and analyses. Finally, Chapter 5 provides a discussion of the findings, and Chapter 6 summarizes the study's contributions to the field, its limitations, and suggestions for future research.

CHAPTER TWO: THEORETICAL BACKGROUND

Review of Relevant Literature

The upcoming sections detail the theoretical framework that will be used in the proposed study, and present reviews of relevant literature. The first section describes the van Hiele theory and then summarizes studies guided by the theory in the learning of geometry. In addition, this section summarizes research that addresses the knowledge of mathematics for teaching, specifically that related to geometry. The second section describes the commognitive framework related to discursive learning of mathematics. Included in this section are summaries of studies in discourse in the mathematics classroom, and the theoretical model of the development of geometric discourses that aligns the van Hiele theory through the discursive lens. The final section raises general research questions in discursive terms.

Regarding the teaching and learning of geometry, the van Hieles developed this influential theory of levels of geometric thinking. In discussing the profound impact of Pierre van Hiele's theory in mathematics education, Clements (2003) concludes, "van Hiele theory gave educators and researchers a model that promoted the understanding of important, conceptual based level of thinking… It is also a model of synergistic connections among theory, research, the practice of teaching, and students' thinking and learning"(p.151). To better describe the van Hiele Theory and how it has been used in the field of mathematics education, the following section provides the historical background and a general description of the theory.

The van Hiele Theory

The root of van Hiele theory emanated from the task of improving the teaching of geometry. A Dutch husband and wife, Pierre Marie van Hiele and Dina van Hiele-Geldof, developed "the van Hiele Theory" in their doctoral dissertations at the University of Utrecht, Netherlands, in 1957. Dina died shortly after completing her dissertation, and Pierre continued to develop and disseminate the theory (e.g., van Hiele, 1959/1985, 1986).

When Pierre and Dina worked at Montessori secondary schools as mathematics teachers, they were very disappointed with "students' low-level knowledge of geometry"(p.60, 1959/1985). On the other hand, they also realized that teachers and students often fail to communicate with each other because they "speak a very different language" (p.61). For example, one of Pierre and Dina's initial observations was that they seemed to speak about geometry in a different way than their students. When Pierre and Dina spoke about a square as a type of rectangle, students were confused because to them a square and a rectangle were quite different. This led Pierre and Dina to consider the existence of various levels of geometric thinking and the possibility that those students and teachers at different levels of thinking may have difficulty communicating with one another. Although Pierre and Dina developed the theory together, their views were quite different. As a result, Pierre's dissertation focused on identifying students' levels of thinking in learning geometry, while Dina's dissertation was more about a teaching experiment designed to investigate how students move from level to level.

The van Hiele theory includes five distinct levels that describe students' thought levels in the learning of geometry. However, P. M. van Hiele suggested that mathematics educators should focus on the first four van Hiele levels, because those are what teachers have to deal with in school most of the times (van Hiele, 1986). As P. M. van Hiele (1959/1985) described in "the Children's thought and geometry", the five van Hiele levels are as follows (p.62-63): Base Level, figures are judged by their appearance; First Level, figures are bearers of their properties, and they are recognized by their properties but not yet ordered; Second Level, properties are ordered, and they are deduced one from another; at this level, definitions of figure come into play but students did not understand the meaning of deduction; Third Level, thinking is concerned with the meaning of deduction, with the converse of a theorem, with axioms, with necessary and sufficient conditions; Fourth Level, thinking is concerned with a variety of axiomatic systems that are non-Euclidean. Geometry is seen in the abstract.

As described in the levels, students' levels of thinking attached to the learning of a particular geometric topic are inductive in nature. At level n-1 certain geometric objects are studied. Students are able to state some of the relationships explicitly about the objects. At level n the objects studied are now the statements that were explicitly made at level n-1 as well as explicit statements that were only implicit at level n-1. Therefore, the objects at level n consist of extensions of the objects at level n-1. One major purpose of distinguishing the levels is to recognize obstacles that are presented to students. For example, when a student who is thinking at level n-1 confronts a problem that requires vocabulary, concepts or thinking at level n, the student is unable to make progress on the problem, with expected consequences such as frustration, anxiety and even anger.

The van Hiele levels have several important properties: (1) The levels are discrete and sequential. *Discrete* indicates that the levels are qualitatively different from one another. *Sequential* refers to the fact that students pass through the levels in the same order, although varying at different rates, and it is not possible to skip levels. (2) That which was intrinsic at one level becomes extrinsic at the next level. For example, students operating at Level 1 are able to name geometric figures only by their appearance as a "whole" – the properties of a figure remain intrinsic. However, at Level 2, these properties become extrinsic and in fact are the new objects of study. (3) Each level has its own language and symbols. Van Hiele believed that "In general, the teacher and the student speak a very different language" (van Hiele, 1986, p. 62). Therefore, teachers and students often have difficulty communicating with one another about geometric concepts. This linguistic challenge can also extend to communicational difficulties between students in a classroom when they are functioning at different thought levels. (4) Instructional methods have a greater influence than either age or grade on a student's progress through the van Hiele levels. That is, a teacher's instructional activities can either foster or impede movement through the levels.

When assigning students to different van Hiele levels, P. M. van Hiele cautioned that it is possible to misjudge a student's level of thinking without careful analysis, because often students memorize or learn patterns in order to accomplish tasks, but do not really understand the underlying concepts. An example is when students recognize corresponding angles by finding the 'F' that is formed by parallel lines and the transversal. See Figure 2.1 below.

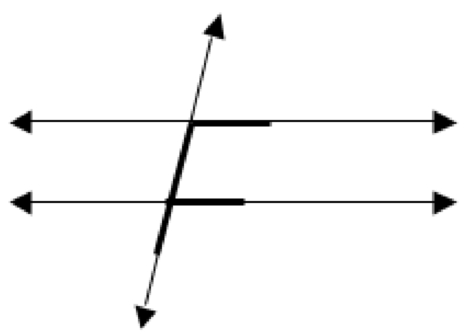


Figure 2.1 *Corresponding angles of parallel lines intersected by a transversal.*

This technique simplifies the relation between angles and lines. P. M. van Hiele claimed that it could be harmful to students if they only seek a quick result and avoid the 'crisis of thinking'. In saying 'crisis of thinking', P. M. van Hiele meant the difficulties that students need to transit from one level to a higher level. It is possible for students to derive the answer without recognizing the relationships between the angles in the figure (e.g., supplementary angles, angles at a point, interior angles). Van Hiele warned that these types of "tricks" might actually prevent students from moving to the subsequent level of reasoning (van Hiele, 1986, p.42).

The van Hiele theory recognizes the importance of language, which plays a significant role in communication. According to P. M. van Hiele, students' levels of thinking are important not in the sense of the way of their thinking, but in the results of thinking that are revealed in students' speaking and writing. For example, the meaning of a statement like, "This figure is a rhombus." depends on how one argues about it. For a student who is at Basic level, her answer could be, "This figure has a shape that looks like what I learned to call 'rhombus'." In contrast, if another student has already obtained the first van Hiele level or higher, her argument could be quite different. The figure that the student refers to is a collection of properties and those properties he/she has learned to call "rhombus" (van Hiele, 1986, p.109). By making the same statement, "This figure is a rhombus", one could use very different reasoning, and from a very different level of thinking. This example of students' responses to a rhombus illustrates how geometric language can vary among levels.

The van Hiele theory has been influential and extensively studied. In the next section, my review of the existing literatures focuses on how the van Hiele theory has been used in research in the years since the theory was developed.

<u>Research Guided by van Hiele Theory</u>

The van Hiele theory was introduced to the United States by the Russian mathematician Izaak Wirszup in a lecture entitled "Some Breakthroughs in the Psychology of Learning and Teaching Geometry" at the Closing General Sessions of the National Council of Teachers of Mathematics in 1974, following its incorporation into a new Soviet geometry curriculum (Wirszup, 1976; van Hiele, 1959/1985, 1986). After the van Hiele levels were translated into

English, they were widely used by many researchers in the United States. During the period of 1980-83, the National Science Foundation funded three major investigations of van Hiele levels in the United States: one directed by Burger and Shaughnessy at Oregon State University, another by Fuys, Geddes, and Tischler at Brooklyn College, and a third by Usiskin at the University of Chicago. Burger and Shaughnessy set out a study using clinical interviews to determine the usefulness of van Hiele levels for describing children's geometric thinking in elementary, middle, and high school grades. Fuys et al. focused their investigation on geometric thinking in adolescents using instructional models. Usiskin's project used a large-scale survey to test whether the van Hiele theory applied to the geometric reasoning of students enrolled in secondary geometry courses. These three intensive studies have been widely read, discussed, and cited. After these studies, dozens of other studies using the work of the van Hieles have been conducted in the United States (e.g., see Mayberry, 1983; Crowley, 1987; Senk, 1983, 1989). Internationally, Micheal de Villiers in the Netherlands and later in South Africa used van Hiele theory to develop geometry curricula (de Villiers, 1996), whereas Angel Gutierrez and Adele Jaime and their students in Spain, and John Pegg and his students in Australia used the theory to study students' learning in geometry (e.g., Gutierrez, 1996; Gutierrez, Jaime, & Fortuny, 1991; Gutierrez, Pegg, & Lawrie, 2000).

In the earlier writing of the van Hieles, the van Hiele levels of geometric thinking mainly refer to the classification of figures (van Hiele, 1959/1985). At that time, levels were descriptors and they were not labeled by single words (e.g., "Visual" for Level 1, etc.). Almost thirty years later, van Hiele (1986) referred to the five levels of thinking as visual, descriptive, theoretical, formal logic and rigor, and considered such classification to be suitable to a structure of mathematics (p.53).

Over the years, researchers not only used the levels to study students' levels of geometric thinking, but also expanded the area of research from classification of the quadrilaterals to the classification of similar figures, to reasoning and proof, to spatial geometry in three dimension measurement, etc. In these studies, researchers have proposed various descriptive labels for the van Hiele levels. As the first to name the van Hiele levels, Hoffer (1981) provided his descriptors, "levels of mental development in geometry" (p.13), which label Levels 1 through 5 as recognition, analysis, ordering, deduction, and rigor (p.13-14). Besides these five levels, Hoffer also suggested five basic skills that are expected at each level. These five skills are visual skills, verbal skills, drawing skills, logical skills, and applied skills. For instance, at Level 1 (recognition), the visual skills only focus on recognizing different figures from a picture, or on recognizing information labeled on a figure. At Level 2 (analysis), visual skills are developed to notice properties of a figure as well as to identify a figure as a part of a larger figure. At Level 3 (ordering), visual skills help to recognize interrelationships between different types of figures and common properties of different types of figures. At Level 4 (deduction), visual skills focus on using information about a figure to deduce more information. Finally, at Level 5 (rigor), visual skills are used to recognize unjustified assumptions made by using figures (p.15). Hoffer's descriptors suggested that various geometric skills might be expected of students at different levels of their development in geometry.

Other "level indicators", suggested by Burger and Shaughnessy (1986), describe the five levels as visualization, analysis, informal deduction, formal deduction, and rigor, for Levels 0 through 4, respectively. Using Burger and Shaughnessy's level indicators, Crowley (1987) provided additional examples of level-specific responses (except for Level 5), concerning how students would argue a given shape is a rectangle (p. l5).

Level 1	"It looks like one." or "Because it looks like a door."
Level 2	"Four sides, closed, two long sides, two shorter sides, opposite sides parallel, four right angles …"
Level 3	"It is a parallelogram with right angles."
Level 4	"This can be proved if I know this figure is a parallelogram and that one angle is a right angle."

Each response assigns to a level. The student at Level 1 gives answers based on a visual model and is identifying the rectangle by its overall appearance. At Level 2, the student is aware that the rectangle has properties; however, redundancies (i.e., properties that can be derived from other properties) are not noticed. A student operating at Level 3 will attempt to give a minimum number of properties (i.e., a definition), and finally, at Level 4, a student will seek to prove the fact deductively.

More recently, Battista (2009) elaborates and refines the van Hiele levels with regard to students' geometric reasoning. The descriptors of the levels he suggests are visual-holistic reasoning, descriptive-analytic reasoning, relational-inferential reasoning, formal deductive proof, and rigor (p.92-94), referring to Levels 1 through 5, respectively. For instance, at Level 1(visual-holistic reasoning), students argue that a square is not a rectangle because a rectangle is "long"; or claim that two figures have the "same shape" because they "look the same"(p.92). At this level, students' justifications of an argument are vague and holistic. At Level 2 (descriptive-analytic reasoning), students would assert that a square is a rectangle because "it has opposite sides equal and four right angles." At this level, students are able to explicitly specify shapes by their parts and spatial relationships among the parts; however they describe parts and properties informally and imprecisely using strictly informal language learned from everyday life. At Level

3 (relational-inferential reasoning), students start with empirical inference to reason that if a quadrilateral has four right angles (and this is a rectangle), its opposite sides have to be equal because by drawing a rectangle with a sequence of perpendiculars, they cannot make the opposite sides unequal; and then they use logical inference to recognize the classifications of shapes into a logical hierarchy (p.94).

These descriptors not only provide detailed information about how researchers identify students' levels of geometric thinking, but more importantly shed light on the geometric reasoning and language skills that students need to develop at each van Hiele level. When conducting studies using van Hiele theory, some researchers use clinical interviews, while others prefer open-ended survey tests. Among all the van Hiele studies, Usiskin's Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project, and Burger and Shuaghnessy's "Oregon Project" are two of the most frequently used and cited. In the following section I summarize the methods used in the van Hiele studies, as well as some important findings, beginning with these two projects that influence the teaching and learning of geometry.

Now let me move on to the research methods used in the van Hiele studies. The Usiskin CDASSG project (Usiskin, 1982; Senk, 1983) used a standard pretest and posttest, involving four tests, to assess 2699 students in full year geometry classes from 13 public high schools in five states. The four tests were: Entering Geometry Test, van Hiele Level Test, Comprehensive Assessment Program Geometry Test and Proof Test. The pretest (i.e., Entering Geometry Test and Van Hiele Level Test) was conducted during the first week of school, and the posttest (i.e., van Hiele Level Test, Comprehensive Assessment Program Geometry Test and Proof Test) was scheduled three to five weeks before the end of the school year.

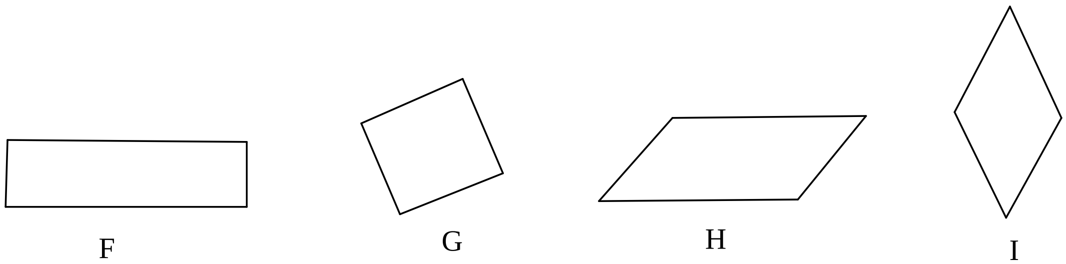
The van Hiele Geometry Test was designed to predict students' van Hiele levels at the beginning and the end of the school year. This test consists of 25 multiple-choice items, with 5 foils per item and 5 items per level, and was designed to capture the key thinking processes characteristic of each van Hiele level. In order to develop a rigorous test instrument that describes van Hiele levels in sufficient detail, researchers in the CDASSG project first reviewed nine original works of the van Hieles, including four originally written in English and five translated into English from Dutch, German or French. They compiled all the quotes from the van Hieles' writings (see appendix B) that describe behaviors of students at a given level. As an example of the quotes, the following is a selected list of Level 1 behaviors that Usiskin (1982) provided in the CDASSG project report:

Level 1 (their base level, level 0)

1. "Figures are judged according to their appearance."
2. "A child recognizes a rectangle by its form, shape"
3. "The rectangle seems different to him from a square."
4. "A child does not recognize a parallelogram in a rhombus."
5. "A student was able to produce these figures without error…"

The van Hiele Geometry Test (see Appendix B) instruments were based on the descriptions of students' behaviors at each given level. For example, Items 1-3 were derived from quote number one; Item 4 was derived from quote number eight and Item 5 was derived from quote number six. Figure 2.2 presents one van Hiele Level 1 item and its corresponding van Hieles' quotes.

Question 4: Which of these are squares?



(A) None of these are squares.
(B) G only
(C) F and G only
(D) G and I only
(E) All are squares.

Figure 2.2 An example of a Level 1 test item with its corresponding van Hiele quotes.

To grade students' responses to the van Hiele Geometry Test, the project used the 3 of 5 criterion (3 out of 5 correct) and 4 of 5 criterion (4 out of 5 correct), and compared the two criterions using the analyses of Type I and Type II error. The statistical analysis showed that, depending on whether one wishes to reduce Type I or Type II error, the 3 of 5 criterion minimizes the chance of missing a student and yields an optimistic picture of students' levels, whereas the 4 of 5 criterion minimizes the chance of a student being at a level by guessing (see Usiskin, 1982). Based on students' test responses, the students were assigned a weighted sum score according to the following:

1 point for meeting criterion on items 1-5 (Level 1)

2 points for meeting criterion on items 6-10 (Level 2)

4 points for meeting criterion on items 11-15 (Level 3)

8 points for meeting criterion on items 16-20 (Level 4)

16 points for meeting criterion on items 21-26 (Level 5) (Usiskin, 1982, p.22)

The points were added to give the weighted sum, and the weighted sums were calculated to allow a person to determine upon which levels the criterion has been reached from the weighted sum alone. For example, a score of 19 points indicates that the student has reached the criterion on Levels 1 (1 point), Level 2 (2 points) and Level 5 (16 points). The assigning of levels, however, was as follows: If a student met the criterion for passing each level up to and including level *n* and failed to meet the criterion for all levels above, then the student was assigned to level *n*; if the student could not be assigned to any level, then that student was not said to fit. Thus a student with a weighted sum of 1+2+16 =19 would satisfy the criterion at Level 1, Level 2 and Level 5 and was assigned to van Hiele Level 2 (p.25). The CDASSG project used Hoffer's (1981) descriptors, labeling the levels as recognition, analysis, ordering, deduction and rigor, from Levels 1 to 5. Additionally, the project reported results using both the classical theory (i.e., all five van Hiele levels are considered) and the modified theory (i.e., Level 5 is excluded from consideration) to classify students into van Hiele levels (see Usiskin, 1982).

This large-scale research study showed that nearly 40% of students in the United States finish high school functioning below van Hiele Level 2 (Analysis). Students entering high school geometry courses with higher van Hiele levels, such as Level 2 or Level 3 (Ordering), were more likely to succeed in writing proofs by the end of the school year (Senk, 1983, 1989). Of those studied, students who entered geometry courses functioning at van Hiele Level 1 had a 30% chance of success in proof writing. Entering geometry at Level 2 provided students with a 56% chance of success at proof writing, and all students entering at Level 3 experienced success at proof writing by the end of the school year. These results show that high school students' achievements in writing proofs are positively related to van Hiele levels of geometric thinking and to achievement on standard non-proof geometry content (p.318). The study also concluded, “In the form given by the van Hieles, Level 5 either does not exist or is not testable. All other levels are testable”(Usiskin, 1982, p.79).

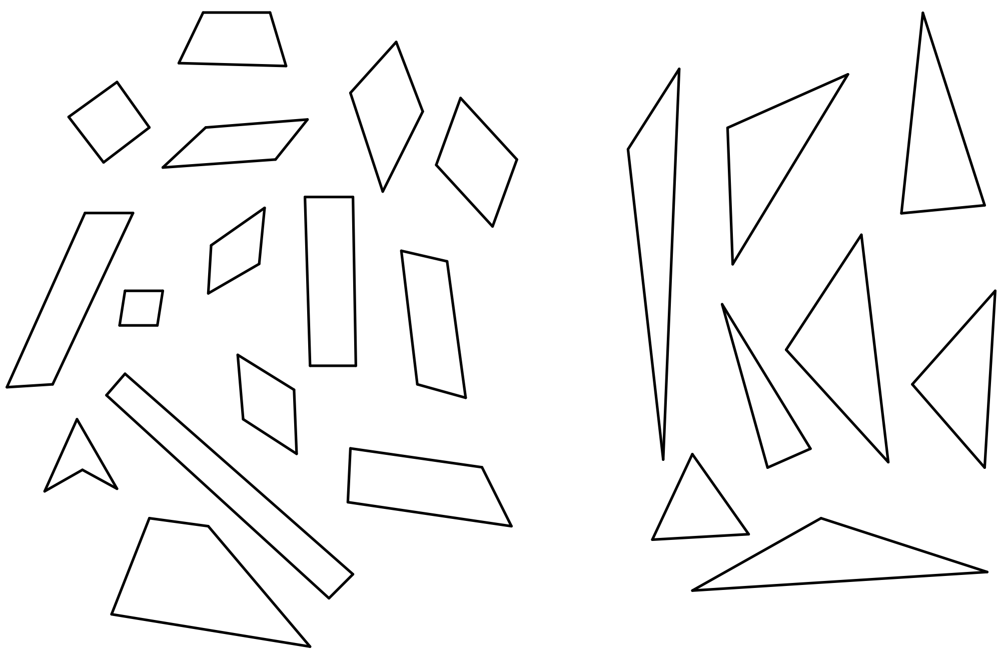
It is helpful to use these data to determine the initial status of students’ geometric backgrounds and to assess their progress. However, there have been questions and doubts about the feasibility of measuring reasoning by means of items, and about the internal consistency of the items (Crawley, 1990; Wilson, 1990). Also, one might question what information might be missed in a paper-pencil test, and how the details of students’ thinking processes might be better detected. Nevertheless, the main advantage of this method is that it can be administered to many indiviudals, and it is easy and quick to distinguish between the thought levels of students.

In contrast, Burger and Shaughnessy’s Oregon project (Burger & Shaugnessy, 1986) used clinical interviews to determine students’ van Hiele levels. They interviewed 45 students from 5 school districts in 3 states, ranging from elementary to middle to high school. The interviews consisted of eight tasks focusing on geometric shapes, and those tasks were designed to reflect the descriptions of the van Hiele levels.

The design of the interview tasks involved drawing shapes, identifying and defining shapes, sorting shapes (e.g., triangles and quadrilaterals); and the interview protocols were designed to engage participants in both informal and formal reasoning about geometric shapes. Six of the eight tasks, focusing on drawing, identifying, and sorting, were expected to elicit the characterizations of van Hiele Levels 0-2 from the protocols. To give an example of the design, Figure 2.3 shows two tasks that were used, *Identifying and defining* (2.3a) and *Sorting* (2.3b).

<u>Identifying and Defining</u>

Students were given a sheet of quadrilaterals (Figure 2.3a), and they were asked to write an S on each square and R on each rectangle, and if the student was familiar with the terms, a P on each parallelogram and a B on each rhombus. During the interviews, students were asked to justify their marking. In the defining part of the activity, the student was asked, "What would you tell someone to look for in order to pick out all the rectangles on a sheet of figures?" Or, an equivalent question was asked, "Could you make a shorter list? Is No. 2 a rectangle? Is No. 9 a parallelogram?"(p.34).



2.3a. Quadrilateral to be identified　　　　2.3b. Triangles to be sorted

Figure 2.3　　Two experimental tasks from the Oregon Project

Sorting

A set of cut out triangles was spread out on the table (Figure 2.3b). The student was asked, " Can you put some of these together that are alike in some way? How are they alike? Can you put some together that are alike in a different way? How are they alike?" (p.34) This line of questioning was continued as long as the student could come up with new sorting strategies. These activities sought to explore the student's definitions and class inclusions.

The project collected and analyzed students' original written works during the interviews, and the dialogs between interviewers and the students were analyzed and documented as well. For example, on the Drawing Triangles task, interviewees were asked to draw "different" triangles. Based on the interviewees' drawings during the interviews, Burger and Shaughnessy found that for Bud, a 5th grade student, "different triangles" meant triangles in different orientations or positions only. In contrast, for Amy, an 8th grade student, "different triangles" meant having different angle measures and sizes, and for Don, a 10th grade student, "different triangles" meant different types of triangles. Figure 2.4 shows the drawings from Bud, Amy and Don.

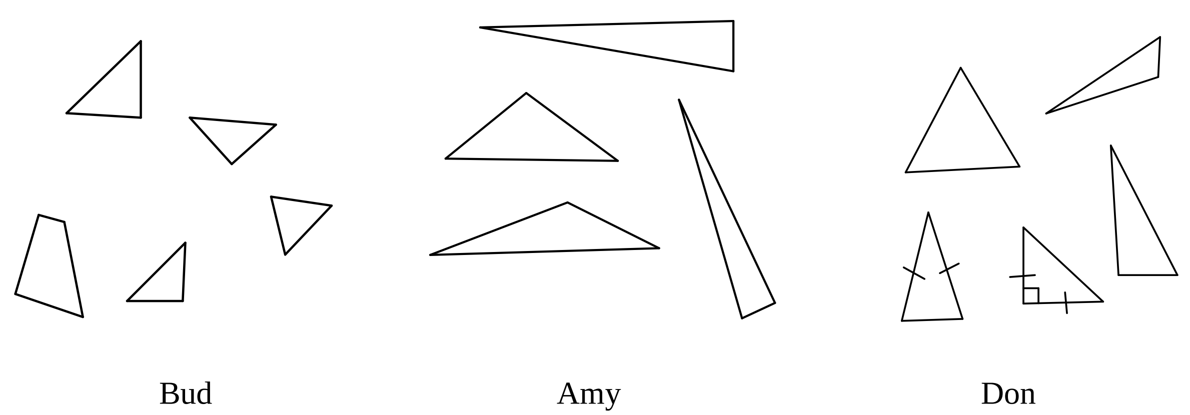


Figure 2.4 Bud, Amy and Don's drawings of different triangles.

Recall that the "level indicators" developed by Burger and Shaughnessy (1986) describe Levels 0 through 4, respectively, as visualization, analysis, informal deduction, formal deduction, and rigor. Pursuant to students' responses during the interviews, it turned out that, even though all three students were to reason about what is meant by "different triangles," and could provide drawings, all three students were later assigned to three different van Hiele levels: Bud (Level 0), Amy (Level 1) and Don (Level 2). This example illustrates use of the “same language” but very different reasoning. Burger and Shaughnessy also documented the original scripts in which questions were asked for interviewees to complete the task. For example, for the activity “Drawing Triangles”, interviewees were asked to draw a triangle (called No.1), and then another triangle (called No.2) that is different in some way from the first one. After the interviewee had done so, he/she was asked to draw a third triangle that is different from the first two triangles, and so on. Later, the interviewees were asked questions such as “How is #2 different from #1?” and “How would they be all different from each other?”(p.37).

Burger and Shaughnessy’s project confirmed the hierarchy nature of the levels. They also found that age is not significantly related to the levels. However, the reviewers of the project had disagreements and experienced some difficulties of assigning a level to students who appeared to be in the transition between Levels 0 and 1(p.42).

The interviewees’ written works on one hand, and their verbal responses to the questions on the other hand, combined to increase the reliability of the data and provide strong evidence on how the data were analyzed and interpreted by researchers. The great advantage of clinical interviews is that the information obtained from the interviews results in a deeper knowledge of the ways students reason. However, this study is clinical with a small sample of students representing a very broad range of ages (Kindergarten to College). This kind of research is time-consuming and is unsuitable for assessing many people.

The review of the methods used in van Hiele studies influenced the design of the proposed study. The van Hiele Geometry Test, used to distinguish students’ van Hiele levels, is effective in getting initial information about students’ levels of thinking, and the CDSSAG project showed that it is a well-tested and designed test instrument. The Oregon project, on the other hand, gives an example of how clinical interviews could well detect students' thinking processes when engaged in informal and formal reasoning about basic geometric shapes.

There are many other studies using the van Hiele theory that pertain to the learning and teaching of geometry. One such study by Fuys et al. (1988) focused on clinical interviews with sequences of instructions known as "Instructional Module Activities" (p.11). In this project, all subjects were interviewed individually in six to eight 45-minutes sessions as they worked with an interviewer on the Instructional Modules. The subjects were selected to reflect the diversity of sixth-grade students from New York City public schools. To categorize the subjects' levels of thinking, the interviews focused on their progress (or lack of it) within the levels or to higher levels, and on learning difficulties as well (p.78). This project was designed to investigate whether or not instructional modules would help subjects move through the levels. Fuys et al. (1988) also documented the dialogues between interviewers and subjects. For example, in the assessment of subjects' understanding of the exterior angle of a triangle, subjects were given an open question of finding a possible relationship among the three angles indicated in Figure 2.5.

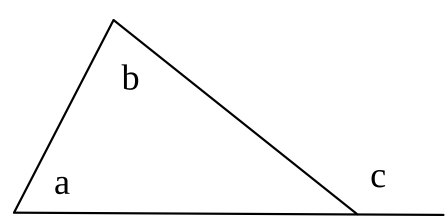


Figure 2.5 An exterior angle of a triangle.

During the interviews, the interviewer gave several prompts to the subjects such as: " Is any part of angle *c* related to angle *a* or angle *b*?" With the help of the interviewer, the subjects would sometimes successfully complete the argument of a relationship between two interior angles (angles *a* and *b*) and their exterior angle (angle *c*) of the triangle.

In addition to the clinical interviews, Fuys et al. (1988) also analyzed geometric content of three widely used K-8 textbook series with regard to the van Hiele levels. They reported that no more than 2% of the lessons contained content that require geometric thinking at Level 3 (formal deduction), and all of Level 3 lessons appeared at Grades 7 and 8. The remaining 98% represented van Hiele Levels 0, 1, and 2. Analyzing their findings, Fuys et al. concluded that " average students do not need to think above Level 0 (visual) for almost all of their geometry experience through grade 8" (p.169). Not surprisingly, the overall results for the van Hiele levels of students in the United States were discouraging. In their study of the geometric reasoning of sixth and ninth grade students, Fuy et al. (1988) found the following: 19% of sixth graders performed consistently at Level 0 (visual), 31% performed sometimes at Level 1 (analysis) and sometimes at Level 2 (informal deduction), and the remaining 50% performed sometimes at Level 2 and sometimes at Level 3 (formal deduction). The ninth graders' corresponding percentages were 12%, 44%, and 44%.

More recently, Newton (2010) used van Hiele levels to analyze K-8 geometry state standards. Of the 5,710 Grade Level Expectations (GLEs) contained in the K-8 Geometry and Measurement strands of 42 states, 1,667 GLEs (approximately 29 %) were labeled as descriptive geometry. The analysis of the descriptive geometry GLEs indicated that approximately 47% of the GLEs are Level 1(visualization), 49% are Level 2 (analysis), and 4% are Level 3 (informal deduction). According to Newton, the absence of Level 3 GLEs in more than 40% of the states and the near absence in the remaining 60% represent the most compelling result of the analyses, since formal deduction (Level 4) is generally expected in high school geometry courses.

The van Hiele theory has informed and shaped the improvement of the geometry curriculum (Wirszup, 1976; de Villers, 1999). For example, de Villiers cautioned "no amount of effort and fancy teaching methods at the secondary school will be successful, unless we embark on a major revision of the primary school geometry curriculum along van Hiele lines". In 1999, Clements et al. encouraged the van Hiele level's use in guiding curriculum development, and suggested that "helping children move through these levels may be taken as a critical educational goal" (p. 193). The following year, in *Principles and Standards in School Mathematics*, the NCTM cited van Hiele and others who have studied his theory to develop the importance of "… building understanding in geometry across the grades, from informal to more formal thinking" (2000, p. 40).

Knowledge of Geometry for Teaching

"Mathematical knowledge for teaching means the mathematical knowledge used to carry out the work of teaching mathematics" (Hill, Rowan & Ball, 2005, p.373). When suggesting what it means to know mathematics for teaching, Ball, Hill and Bass (2005) argue that teaching involves knowledge of mathematical ideas, mathematics reasoning skills, and communication, fluency with examples and terms, and thoughtfulness about the nature of mathematical proficiency. For instance, additional mathematical insight and understanding are required to explain, listen, and examine students' work, and more mathematical analysis is required when correcting students' errors. In addition to mathematical knowledge for teaching, Ball et al address the need for teachers to have a specialized fluency with mathematical language, and to know what counts as a mathematical explanation.

In this section, I summarize research that emphasizes prospective teachers' knowledge of geometry. Mayberry (1983) investigated nineteen undergraduate prospective teachers' geometric understanding when encountering seven geometry concepts: squares, right triangles, isosceles triangles, circles, parallel lines, similarity and congruence, all common topics to elementary school textbooks. The study found that two students had difficulty in recognizing a square with a nonstandard orientation (Basic level), while the properties of figures were often not perceived (Level I). For example, when asked, “does a right triangle have a longest side?” (p.60), twelve students responded that they did not think that such a triangle had to have a longest side. With regard to Level II, the study concluded that class inclusions, relationships, and implications were not perceived by many of the students, because they answer the questions for particular figures rather than generalized ones. Responses to questions about congruence (Level III) show that fifteen out of the nineteen prospective teachers believed that two right triangles with ten-centimeter hypotenuses are always congruent. Also, ten of nineteen assert that two circles with ten-centimeter chords are always congruent (Table 2.1).

Table 2.1 Responses to Relation Questions about Congruency

Figure. “ Are these Congruent?”	Always	Sometimes	Never	Don’t know
A square and a triangle	0	1	17	1
Two squares with 10-cm sides	16	2	0	1
Two right triangles with 10-cm hypotenuses	15	3	0	1
Two circles with 10-cm chords	10	8	0	1
Two similar triangles	3	11	3	2

Findings suggested that the prospective teachers in the study did not yet perceive some of the properties of basic geometric shapes, and they did not perceive a proof as a logical chain leading from the “given” to the “conclusion.”

Hershkowitz, Bruckheimer and Vinner conducted in Israel a study of 518 students (grades 5-8), 142 prospective teachers, and 25 in-service teachers. Findings showed that “teachers have patterns of misconceptions similar to those of students in grades 5-8” (1987, p.222). More specifically, they assessed 142 prospective elementary teachers’ understanding of geometry concepts in the context of angles, altitudes of triangles, and diagonals of polygons. For example, one of the tasks was to assess the understanding of the angle concept by recognizing the drawing of an angle on a sheet of paper. Results suggested that sixty-eight percent of the prospective teachers had a proper understanding of the concept of angle (p.223). After assessing prospective teachers’ understanding of the diagonals of polygons, Hershkowitz, Bruckheimer and Vinner (1987) found that most of the prospective teachers only drew diagonals that were inside the polygons and rejected the possibility of an exterior diagonal (see Figure 2.6). This result suggests that most prospective teachers did not use definitions as their primary tool when working with these tasks. They tended to use their own individual image of a given concept (e.g, concept of diagonals of a polygon). These prospective teachers’ individual concept images were misleading in the case of the diagonals of concave polygons.

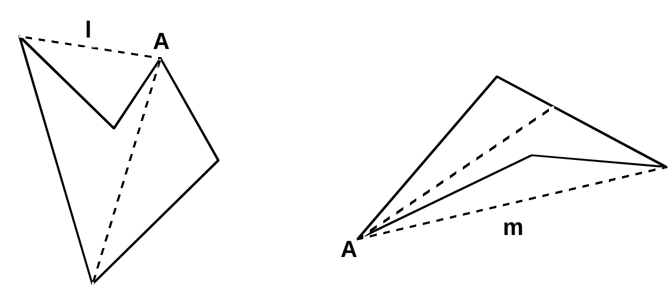


Figure 2.6 The “diagonals” of concave polygons

Gutierrez, Jaime, and Fornruny (1991) conducted a study to evaluate 32 prospective teachers’ spatial reasoning in three-dimensional geometry. Nine items of the Spatial Geometry Test were grouped into five activities. These activities were designed to elicit prospective teachers’ conceptual understanding of three-dimensional figures, by paying attention either to visual qualities or to the properties of basic geometric shapes such as squares and parallelograms. Activities that asked prospective teachers to select solids, based on given properties, focused on how prospective teachers use definitions and properties to identify the polyhedron from a given set of solids. With regard to the question involving writing the differences and similarities between a cube and the Solid I (see Figure 2.7), this activity focused on observation and manipulation of the polyhedran. In response to this question, for instance, one prospective teacher argued, “In both solids [a cube and Solid I] the faces are parallelograms and both have six faces. And the differences were, the angles in solid I are not right” (p.246). Another prospective teacher argued, “Solid I and a cube were alike because both solids have parallel faces and all edges are the same, and they differed because they don’t have the same shape”(p. 246).

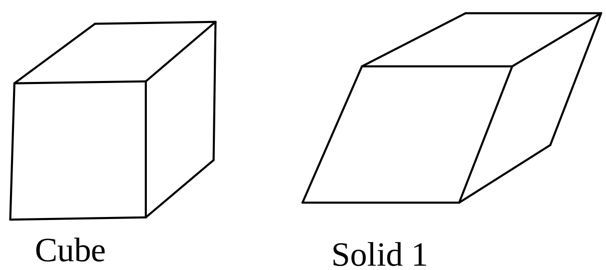


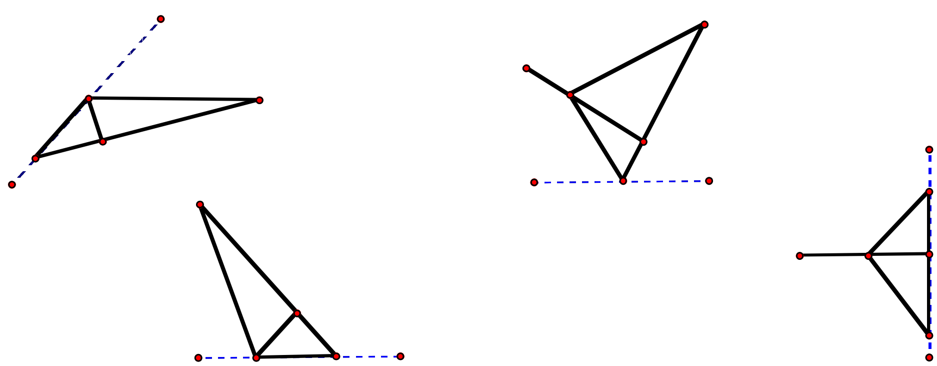
Figure 2.7 A cube (left) and Solid I (right)

These two responses both reasoned about the differences and similarities between a cube and a solid, but the arguments provided by the two prospective teachers were quite different. The first response focused on the properties of the geometric figures; whereas the latter response mainly paid attention to the visual qualities of figures.

Based on Vinner and Hershkowitz’s (1987) notion of concept image, Gutierrez and Jaime (1999) conducted a study of prospective elementary teachers’ understanding of the concept of the altitude of a triangle. They analyzed 190 prospective teachers’ written tests, focusing on concept images, difficulties, and errors related to the concept of the altitude of a triangle. Analysis showed that there were more correct responses in the test containing the definition of the altitude than in the test without the definition. This observation suggested that the definition seemed to provide information helping these prospective teachers to improve their understanding of the concept of altitude.

From the responses that prospective teachers provided in the study, we learned that there was confusion and misunderstanding between the concepts of altitudes and medians of a triangle, and the concepts of altitudes and perpendicular bisectors of the sides of a triangle. These misunderstandings could be reasons why prospective teachers responded incorrectly to a partial image that excludes external altitude (Figure 2.8a), and to a partial image that does not take into account the length of the altitude (Figure 2.8b).

2.8a. 2.8b.



"For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of the dissertation"

Figure 2.8 Responses regarding the altitude of a triangle

Another issue worth mentioning is the influence of the position of a triangle, as a consequence of a different rotation of the figure. The easier item appeared to be the prototypical position with a vertical altitude, which suggests that prospective teachers' concept images of altitudes of triangles were limited to only certain types of triangles, and with certain orientations of how triangles are positioned.

In the previous section I have presented examples from existing literatures regarding prospective teachers' understanding in the context of Euclidean geometry. The review included teachers from Israel, Spain, and the United States. Some investigations were done through large-scale studies; others were done through clinical interviews. These prospective teachers who were tested all shared some common difficulties and weakness in their learning of geometry.

Theoretical Framework

According to the van Hieles, a learning process is also a process of learning a new language, because "each level has its own linguistic symbols" (van Hiele, 1959/1985, p.4). The van Hiele levels reveal the importance of language use, and emphasize that language is a critical factor in the movement through the levels. The word "language" is not clearly defined in the broad use of it (see van Hiele, 1986). Moschkovich (2007) argued that the language of mathematics does not mean a list of vocabulary words or grammar rules, but rather the communicative competence necessary and sufficient for competent participation in mathematical discourse.

> A real concept is an image of an objective thing in its complexity. Only when we recognize the thing in all its connections and relations, only when this diversity is synthesized in a word, in an integral image through the multitude of determinations, do we develop a concept. – Vygotsky

In this section, I summarize issues of language and discourse in mathematical learning from existing literature. Sfard (2008) uses a discursive approach inspired by Vygotsky to make a distinction between language and discourse - language is a tool, whereas discourse is an activity in which the tool is used or mediates. This perspective provides an understanding of mathematics as a social and discursive accomplishment in which talk, gesture, diagrams, representations, and objects play an important role; consequently, mathematics learning requires several modes of communication (Sfard, 2002).

Researchers such as Ball (1993) elaborated the relationship between discourse, content and community in their research to illustrate how these elements help students to develop what Schoenfeld (1992) calls a “mathematics point view.” From this perspective, classroom mathematical discourse is essentially a progress of establishing true claims about mathematics. In her work, Lampert (1998) advances the case for classroom-based research to consider the impact of language and discourse on mathematical learning. For example, Lampert illustrates aspects of “mathematical talk” that includes position taking, question asking, proof and justification, expanding ideas, use of evidence, conjectures, symbolic reference, and so on.

Kerslake’s (1991) focus on language in mathematics classrooms is an example of work that attempts to examine the specific language of the content area, and helps to identify how the use of language becomes a resource for understanding students’ misconceptions. For example, based on the interviews with students, Kerslake found that students fail to conceive fractions as numbers because they perceive them as “broken numbers” instead. Moreover, students tend to rely on the everyday language of “sharing” to describe division, and Kerslake surmises that this happens because sharing is likely to have been the students’ first experiences of dividing. Kerslake suggested a closer look at how students think of and talk about fractions in the course of learning.

Moschkovich (2010) also acknowledges the significant role of discourse in learning. She demonstrated this view, through an analysis of a third grade bilingual mathematics classroom, to illustrate two features of mathematical discourse: situated meaning of words (utterances), and focus of attention. She suggests that learning mathematics is a discursive activity that involves participating in a community of practice using multiple materials, linguistic, and social resources.

Many researchers have attempted to develop frameworks to examine students' discourses in learning mathematics. As an example, Sfard's (2008) communicational approach to mathematical learning provides a notion of mathematical discourse that distinguishes her framework from others in several ways. In particular, Sfard (2002) argues that the knowing of mathematics is synonymous with the ability to participate in mathematics discourse. From this perspective, conceptualizing mathematical learning as the development of a discourse and investigating learning means getting to know the ways in which children modify their discursive actions in these three respects: "its vocabulary, the visual means with which the communication is mediated, and the meta discursive rules that navigate the flow of communication and tacitly tell the participants what kind of discursive moves would count as suitable for this particular discourse, and which would be deemed inappropriate." (Sfard, 2002, p.4) Therefore, Sfard's (2008) commognitive approach is grounded in the assumption that thinking is a form of communication and that learning mathematics is learning to modify and extend one's discourse.

<u>Commognition</u>

In Sfard's (2008) *Thinking as communicating: Human development, the growth of discourses, and mathematizing*, she defines discourse as "any act of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions" (Sfard, 2008, p. 297). A discourse is considered to be mathematical when it features mathematical vocabulary that relates to numbers and shapes. Geometric discourse, a subcategory of mathematical discourse, features mathematical vocabulary specifically relating to geometric shapes, definitions and proofs, etc (p.245). Mathematical discourses are distinguishable by their vocabularies, visual mediators, routines, and endorsed narratives.

From the commognitve point of view the development of discourse involves modifying colloquial mathematical discourse into a more precise mathematical discourse, one that follows meta-discursive rules. For example, in mathematics, geometric shapes are analytically classified by their properties, not just by how they appear to us holistically. Thus, a stretched out triangle is still a triangle even if it looks distorted. As long as it has three line segments joined at vertices, it is a triangle; and because we count those segments and vertices, we engage in a linguistic act (see Sfard, 2008). In Sfard's terms, the mathematical discourse develops from colloquial mathematical discourse; the colloquial mathematical discourse is an important starting point, and to develop mathematical discourse requires a fundamental change in the discourse practices.

To identify whether a discourse is "mathematical", four characteristics can be considered as critical: word use, routines, visual mediators and endorsed narratives. Following is a very brief description of each of these.

In a mathematical discourse, numbers words and comparison words (e.g., bigger, smaller) will appear in the conversations discussing numbers and shapes. In this proposed study, the geometric discourse deals with triangles and quadrilaterals, and their properties; therefore words will appear such as "angles", "sides", "parallel", "diagonals", "congruent", "same", etc. and sometimes these words will have multiple meanings depending on how the person uses them. *Word use* is an all-important matter because, being tantamount to what others call word meaning, it is responsible for how the user sees the world, and it is one of the distinctive characteristics of discourses (Sfard, 2005, p.245). In particular, a students' word use reflects his/her levels of mathematical thinking, which are crucial in this proposed study.

Visual mediators are objects that are operated upon as a part of the process of communication. In colloquial discourses, visual mediators are images of material things existing independently of the discourse; whereas in mathematical discourses, visual mediators are often involved with symbolic artifacts, created specially for the sake of this particular form of communication. Communication-related operations on visual mediators often become automated and embodied. For example, "the procedures of scanning the mediator with one's eyes… With some experience, this procedure would be remembered, activated, and implemented in the direct response to certain discursive prompts, as opposed to implementation that requires deliberate decisions and the explicit recall of a verbal prescription for these operations" (Sfard, 2008, p.134).

Narrative is defined as "a set of utterances, spoken or written, that is framed as a description of objects, of relations between objects or processes with or by objects, and which is subject to *endorsement* or rejection, that is, to being labeled as *true* or *false*" (Sfard, 2008, p.300). *Endorsed narratives* are sets of propositions that are accepted and labeled as *true* by the given community. In the case of geometric discourse, endorsed narratives are known as mathematical theories, including definitions, proofs, axioms, and theorems. The statement "a parallelogram is a quadrilateral with two pairs of parallel sides" is an endorsed narrative of parallelogram, defining what a parallelogram is mathematically. Mathematical discourse is conceived as one that should be impervious to any considerations other than purely deductive relations between narratives. In this proposed study, the narratives will be those utterances produced by prospective teachers when identifying and classifying basic geometric shapes, whereas the endorsed narratives will be the definitions of different quadrilaterals from textbooks that prospective teachers encounter in their geometry class, and narratives that will be constructed or endorsed by prospective teachers during the interview.

Routines are repetitive patterns characteristic of the given discourse. Specifically, mathematical regularities can be noticed whether one is watching the use of mathematical words and mediators, or follows the process of creating and substantiating narratives about numbers or geometrical shapes. In fact, such repetitive patterns can be seen in almost any aspect of mathematical discourses: in mathematical forms of categorizing, in mathematical modes of attending to the environment, in the ways of viewing situations as “the same” or different, which is crucial for the interlocutors’ ability to apply mathematical discourse whenever appropriate – and the list is still long.

When someone is doing mathematics, or to be more specific, is engaging in a mathematics task in geometry, patterns such as how one is carefully using mathematical words, or how one is following certain steps when substantiating narratives about geometrical shapes, can be observed. In fact, those repetitive patterns can be seen in almost any aspect of mathematical discourses (Sfard, 2005; Sfard 2008). In this proposed study, when prospective teachers identify and classify basic geometric figures, mathematical regularities will be noticed through the conversations, to determine whether these prospective teachers are paying attention to the use of mathematical words and following the process of creating and substantiating narratives about geometrical shapes.

A *Routine* is defined as a set of meta-rules defining a discursive pattern that repeats itself in certain types of situations, and the set can be divided into two subsets – the *how* of a routine, and the *when* of a routine (Sfard, 2008, p.208): The *how* of a routine: a set of meta-rules that determine the course of the patterned discursive performance (the course of routine, or procedure); the *when* of a routine: a collection of meta-rules that determine those situations in which the discussant would deem this performance as appropriate.

In this proposed study, "the how of a routine" will likely be observed (through interviews), whereas "the when of a routine" will not be discovered because it requires observations over a period of time (consistent observations over weeks, months or even years).

The Levels of Geometric Discourses

Many researchers have confirmed the usefulness of van Hiele theory when describing the development of students' geometry thinking. However the same researchers often find the van Hiele theory lacking in depth with respect to the broad description of the levels, and they would like a more detailed description of students' levels of thinking. Recall that Hoffer's (1981) "Sample Skills and Problems" (p.11) provided a framework that connects the levels of development with five basic skills (e.g., visual skills, verbal skills, drawing skills, etc) that are expected at each van Hiele level. This work inspired me to consider the possibility of connecting the van Hiele theory with a discursive framework, and to translate the van Hiele levels into discursive terms. I claim that when a student's geometric thinking develops to a higher level, simultaneously there is a development of the student's geometric discourse in discursive terms. That is, the levels of geometric thinking can be viewed also as the levels of geometric discourses. If so, the question is: "what additional information do the analyses of geometric discourse provide about the student's level of thinking?" To investigate the usefulness of discursive framework, the study produced a model, on the basis of the theoretical understandings, that describes each van Hiele level as a geometric discourse with respect to *word use*, *routines*, *visual mediators* and *endorsed narratives*.

By “the theoretical understandings”, I mean the initial translation of the van Hiele levels to discursive terms based on the quotes from the van Hieles’ writings, the same quotes used to design the van Hiele Geometry Test in the CDASSG project. Taking the van Hieles’ descriptions of students’ behaviors at each level, I analyzed them into the four characteristics described in the Sfard framework; such translation illustrates student geometric discourse at each van Hiele level. The translation for five van Hiele levels is shown in Tables 2.2 (Levels 1-3) and 2.3 (Levels 4-5).

Table 2.2 Discursive Translation of van Hiele Levels 1-3

Key terms	Characteristic Geometric Discourse
Van Hieles’ description of Level 1. Figures are judged by their appearance.	
Word use	Naming a figure is matching the figure with its name.
Routines	Direct recognition, a perceptual experience that is self-evident.
Endorsed narratives	Descriptions of how one perceives. “This one (square) looks different from this one (a rectangle)”.
Visual mediators	2-D geometric shapes, the openness of angles, positions of the lines or physical orientations of a figure are parts of the process of direct recognition.
Van Hieles’ description of Level 2. Figures are bearers of their properties.	
Word use	Naming a figure is associated with its properties.
Routines	Direct recognition. Object level routines include checking, measuring and comparing partial properties of figures.
Endorsed narratives	Descriptions of visual properties. “Squares are not rectangles because they have all sides equal, but rectangles are not”.
Visual mediators	2-D geometric shapes, the openness of angles, positions of segments or physical orientations of a figure are parts of the process of direct recognition and identification of visual properties.
Van Hieles’ description of Level 3. Properties are ordered and are deduced one from another.	

Table 2.2 (cont'd)

Word use	Naming a figure signifies the realization of the figure regarding its endorsed narratives.
Routines	Including routines at Level 2. Object level routines producing endorsed narratives.
Endorsed narratives	Descriptions of a definition of a figure and actions on a figure. "Rectangle is a parallelogram having four right angles".
Visual mediators	Figures, lines and angles are parts of the process of identifying necessary and sufficient condition of a definition.

Table 2.3 Discursive Translation of van Hiele Levels 4-5

Key terms	Characteristic Geometric Discourse
Van Hieles' description of Level 4. Thinking is concerned with the meaning of deduction.	
Word use	Naming a figure signifies the realization of the figure regarding its endorsed narratives and its connections with other figures.
Routines	Using abstract symbols. Abstract level of routines producing endorsed narratives and making connections among them
Endorsed narratives	Descriptions of abstract relations. Constructions of narratives using deductive reasoning.
Visual mediators	All level 3 visual mediators, plus abstract symbols, mathematical diagrams.
Van Hieles' description of Level 5. Figures are bearers of their properties.	
Word use	Naming a figure signifies the realization with its endorsed narratives and connections with other figures in both Euclidean and non-Euclidean geometry.
Routines	Routines are connected with creativity.
Endorsed narratives	Descriptions of abstract relations in both Euclidean and non-Euclidean geometry.
Visual mediators	Mathematical symbols and artifacts used in both Euclidean and non-Euclidean geometry.

During this process of translating, the van Hieles' quotes at each level were reviewed and analyzed into possible characteristics of a mathematical discourse. For example, the quote No. 2, "A child recognizes a rectangle by its form, shape", provides information about how a child identifies a figure, what it calls a "rectangle", based on its physical appearance. When this quote is translated into discursive terms, the word, "rectangle" signifies a geometric shape (i.e., a shape that we call a "rectangle"), thus the *word use* here is a name or a label of *the* figure. The phrase, "recognizes… by its form, shape" suggests that the direct recognition triggers the decision making, and therefore the *routine procedure* is a perceptual experience and is self-evident (e.g., [it is] a rectangle [because I see it] by its form, shape). *Narratives* are utterances, verbal or written, that describe objects, and/or relations between objects. The narrative statement is "what is said" about the object during the interview or observation; a student with behavior described in the quote is very likely to say, "it is a rectangle because it looks like one". The *visual mediator* in this situation could include a drawing or picture of a four-sided figure looking like a rectangle.

The translation of van Hiele levels into discursive terms provides additional information about what a student might say (*word use* and *narratives*) and do (*routines*), as well as what visible objects (visual mediators) are operated upon as a part of the process of communication through the geometric discourses. Moreover, the translation provides more details about the development of the levels through the development of geometric discourses. The van Hiele levels provide a useful framework to distinguish students in different levels, whereas the geometric discourses at the van Hiele levels provide in-depth analyses of students' levels of geometric thinking.

<u>The General Research Question in Commognitive Terms</u>

In the previous section I described Sfard's (2008) commognitive framework, a systematic approach to analyzing the discursive features of mathematical thinking. To examine thinking about geometry, the study connected Sfard's framework to the van Hiele theory (1959/1985), and produced a detailed model, namely *the Development of Geometric Discourse*. This model translates the five van Hiele levels into five discursive stages of geometric discourses. I used empirical data of the study to enrich and refine the model with these questions:

1. At what van Hiele levels do prospective teachers operate?
2. What additional information does the model, *the Development of Geometric Discourse*, provide regarding prospective teachers' levels of geometric thinking?

In view of my interest in investigating prospective teachers' knowledge in geometry after taking a university geometry course, I also asked this question:

3. How do prospective elementary teachers' competencies in geometry change as a result of their participation in a university geometry course?

According to Sfard (2008), mathematical knowledge in geometry involves two levels of discursive process: the object level and the abstract level. For example, the geometrical narrative on geometric shapes, "the sum of the angles in a quadrilateral is 360º", is considered at the object level because it expresses a property of quadrilaterals; whereas the abstract level involves a patterned activity of formulation and substantiation of these object-level narratives. This study investigates two components when examining prospective teachers' knowledge in geometry. The first component is students' knowledge of the names, properties and classification of geometric shapes (object level); the second is competence in reasoning. The new knowledge about geometrical constructs comes from a deductive process (abstract level). The question "How do prospective elementary teachers' competencies in geometry change as a result of their participation in a university geometry course?" pertains to prospective teachers' familiarities with basic geometric shapes and their properties, abilities to formulate conjectures, and abilities to construct geometric proofs.

In discursive terms, an analysis of “familiarity with basic geometric shapes, their properties and classification” implies an examination of students’ narratives about geometric figures utilizing geometric names, where narratives speak about the properties of figures and relations between them. The analysis of “ability to formulate conjectures about figures, their properties and relations, and abilities to construct geometric proofs” suggests an examination of students’ formulation and substantiation of these object level rules about geometric figures.

In the next section, I will discuss the methodology to be used in this study.

CHAPTER THREE: METHODOLOGY

The purpose of this study is to explore a discursive framework as an analytic tool to describe students' geometric thinking through the analysis of their geometric discourses. To this end, I used three primary data sources, (1) written responses to the van Hiele Geometry Test (from a Pretest and Posttest), (2) transcripts (from two in-depth interviews, the first interview conducted right after the pretest, and the second right after the posttest), (3) other written artifacts (participants' written statements, answer sheets to the three tasks during the interviews). I describe these sources in greater detail here followed by the methods used to analyze these data sources.

Participants

All participants in the study were prospective elementary school teachers. In a certain mid-west university teacher education program, prospective elementary teachers were required to complete a sequence of two mathematics content courses designed for elementary school teachers. The first of these courses deals with on numbers and operations, and the second with measurement and geometry. The participants of the study were prospective elementary teachers enrolled in the measurement and geometry course; most of them were juniors and sophomores, and a few were seniors. All seventy-four students enrolled in the course in the fall of 2010 participated in the pretest, and sixty-three of these participated in the posttest, both tests being given as class assignments. Twenty-one participants voluntarily participated in the interview part of the study soon after the pretest, and twenty of these participated in the second part of the interview soon after the posttest.

All participants enrolled in the geometry and measurement course for teachers used Parker and Baldridge's (2008) textbook, *Elementary Geometry for Teachers*. Ten chapters are included in this textbook, all discussing mathematical topics related to geometry and measurement for prospective elementary school teachers. Most participants had studied geometry in K-12, therefore the contents of this study related to triangles, quadrilaterals and proof introduced in Chapter 2 (*Geometric Figures*) and Chapter 4 (*Deductive Geometry*) were partly review to them. For example, in Chapter 2 students were introduced to triangles and parallelograms. The discussion includes the introduction of angles, perpendicular and parallel lines, as well as the classification of quadrilaterals. In the classification of quadrilaterals, students studied parallelograms, rectangles, rhombuses, squares, trapezoids, and kites. In Chapter 4 students learned how to derive new geometric facts from previously known facts using logical arguments. For instance, in the beginning of Chapter 4, where a problem of finding an unknown angle in a quadrilateral leads to an unknown angle proof, students learned from a natural computation to deduce a general fact about a quadrilateral. Later in the chapter, students learned to construct proofs for congruent triangles, and to use congruent triangles to verify properties of quadrilaterals. Thus, these participants were introduced to the topics in this study by the textbook for the course.

The van Hiele Geometry Test

As discussed in the earlier chapter, many mathematics educators have used van Hiele theory to determine students' levels of mathematical thinking. In order to identify suitable survey instruments for the study, literature on the van Hiele levels was reviewed. The van Hiele Geometry Test (see Appendix C), used in the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project, was chosen because it was carefully designed and tested by the researchers of the project (Usiskin, 1982, p.19). This test was used as the instrument for pretest and posttest to determine the van Hiele level of the students.

The van Hiele Geometry Test contains 25 multiple-choice items, distributed into five van Hiele levels: Items 1-5 (Level 1), Items 6-10 (Level 2), Items 11-15 (Level 3), Items 16-20 (Level 4) and Items 21-25 (Level 5). These items are designed to identify students' geometric thinking at five van Hiele levels. For example, Items 1 to 5 of are designed to identify students' thinking related to van Hiele Level 1, at which figures are judged according to their appearance. Items 5 to10 identify students' behaviors related to van Hiele Level 2, at which figures are the supports of their properties. The van Hiele Geometry Test was given to provide initial information on students' levels of geometric thinking at the two end points: beginning (pretest) and the end of the semester (posttest). The analyses of the pretest and posttest help to determine students' changes in geometric thinking resulting from participating the measurement and geometry class.

The goals of the interviews were (1) to gather information about students' knowledge of with triangles and quadrilaterals, as well as the parts of the triangles and quadrilaterals (e.g., angles, sides, etc), (2) students' abilities in verifying their claims and deriving mathematical proofs, and (3) to probe further into students' geometric discourses as revealed through these mathematical activities.

Interview Tasks

Three tasks (see Appendix D) and corresponding interview protocols were designed for the interviews. All three tasks were printed individually on four standard 8″× 11″ sheets of white paper. Task One presents eighteen geometric shapes, labeled with alphabet capital letters shown in Figure 3.1. Among these eighteen polygons, thirteen are quadrilaterals, four are triangles, and one is a hexagon. Sixteen of these polygons were chosen from the van Hiele Geometry Test Items 1-5. Two more polygons were added, consisting of Fig. Q, a quadrilateral and Fig. S, a triangle with no pair of sides equal. These shapes also commonly appear in elementary and middle school textbooks used by many researchers over the past two decades to categorize students' geometric thinking with respect to the van Hiele levels (e.g., Mayberry, 1983; Burger & Shaughessy, 1986; Gutierrez, Jaime, & Fortuny, 1991).

<u>Task One: Sorting Geometric Figures</u>

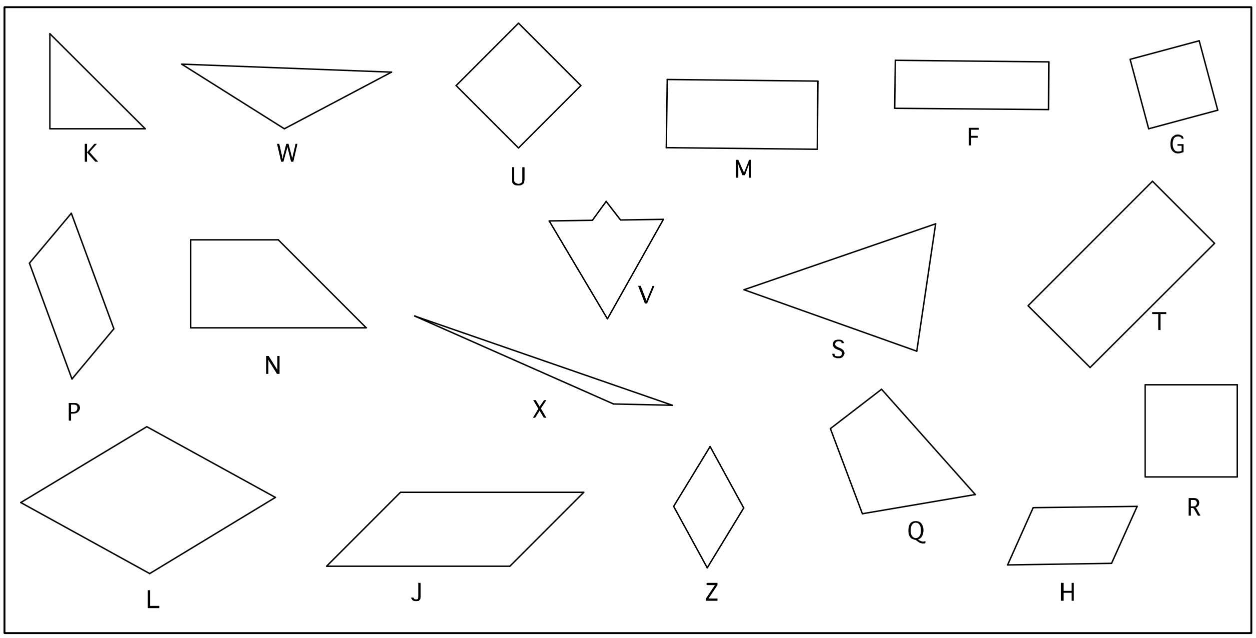


Figure 3.1 Task One: Sorting Geometric Figures

Task One was presented to participants at the beginning of the interviews, and each was asked to sort the eighteen polygons into groups. After the first round of sorting, each participant was asked to regroup or/ and subgroup the polygons. For example, some participants sorted the polygons into a group of rectangles (see U, M, F, T, R, G in Figure 3.1), and a group of triangles (see X, K, W, S in Figure 3.1). The questions "Can you describe each group to me?" and " Can you find another way to sort these shapes into groups?" allowed participants to produce narratives about triangles and quadrilaterals based on their knowledge of polygons. Analysis of the act of grouping gave information on how participants classify triangles and quadrilaterals.

At the end of Task One, I asked each participant to write the definitions of rectangle, square, parallelogram, rhombus, trapezoid and isosceles triangle, and collected their written narratives. This information revealed how participants defined these mathematical terms, and how they made connections between a name and a recognized parallelogram, as well as how quadrilaterals are related to one another.

Task Two: Investigating Properties of Parallelograms

Task Two of the interview had two components. The first component, divided into Part A and Part B, was designed to collect participants' drawings of the parallelograms (see Table 3.1), and to gather more information on their knowledge of parallelograms.

Table 3.1 Investigating the Properties of Parallelograms

Draw a parallelogram in the space below.
o What can you say about the angles of this parallelogram? o What can you say about the sides of this parallelogram? o What can you say about the diagonals of this parallelogram?
Draw a new parallelogram that is different from the one you drew previously.
o What can you say about the angles of this parallelogram? o What can you say about the sides of this parallelogram? o What can you say about the diagonals of this parallelogram?

Part A begins with *"Draw a parallelogram in the space below"*, and next asks participants to describe the angles, sides and diagonals of the parallelogram. For instance, the question, “What can you say about the angles of this parallelogram?” was to find out about participants’ familiarity with the angles of parallelograms. Part B starts with *“In the space below, draw a new parallelogram that is* <u>*different*</u> *from the one you drew previously”*, and asks participants to describe the angles, sides and diagonals of the new parallelogram. This part of the task investigated how participants define parallelograms, and their thinking of *different* parallelograms.

After participants completed Parts A and B, they were presented pictures of parallelograms not included in their drawings. Four pictures of parallelograms were prepared for the interviews, consisting of a parallelogram, a rhombus, a rectangle and a square, each drawn on a 3″× 5″ white index card. Figure 3.2 shows the four parallelograms.

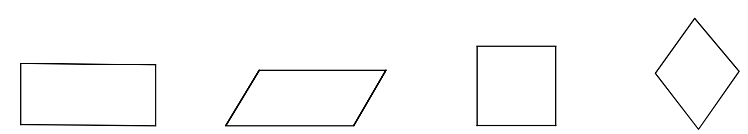


Figure 3.2 Pictures of four parallelograms

The purpose of these pictures was to encourage discussion of different parallelograms and their parts. They helped me to explore out why a participant included some parallelograms but excluded others. For example, after a participant drew a picture of a parallelogram in Part A and drew a rectangle as a different parallelogram in Part B of Task Two, I presented a picture of a square, and asked whether it was also parallelogram and why. Thereby, I gathered more information about participants’ understanding of parallelograms, and was able to gain insights missed in Task One regarding to participants’ ways of identifying and defining parallelograms. A set of interview scripts was designed to further aid in analyzing participants’ understanding of parallelograms (see Appendix E). These scripts were written to help participants make claims about the angles, sides and diagonals of parallelograms. Task Two shed additional light on participants' knowledge of parallelograms, and familiarity with their angles, sides and diagonals.

In completing Task Two, participants were engaged in verifying their claims regarding the properties of parallelograms, constructing informal and/or formal proofs. For example, when a participant made a statement that the diagonals of a rectangle were equal, she was asked to justify the claim. In order to convince me, the participant had to engage in a reasoning process. Such requests of "how do you" or "why" were designed to learn how participants' verify mathematical arguments.

Task Three: Prove the Equivalence of Two Definitions

The participants in my study are future teachers, and need to be aware that, although only one definition of parallelogram dominates books (e.g., a parallelogram is a quadrilateral with two pairs of parallel sides), other equivalent definitions could be given (e.g., a trapezoid with two pairs of parallel sides)(Usiskin & Griffin, 2008). The requirement for another definition to be equivalent to the standard definition is that the defining conditions yield the same figures, and only such figures. Although Task Three focused on deriving mathematical propositions from previously known propositions, the choice was made to ask participants to construct mathematical proofs in order to learn about their understanding of mathematical proof.

Two definitions of parallelogram were presented in Task Three:

1. *"A quadrilateral is a parallelogram if and only if both pairs of opposite sides have the same length"*
2. *"A quadrilateral is a parallelogram if and only if both pairs of opposite angles have the same measure"*

Definitions 1 and 2 describe properties of sides and angles, respectively. Before introducing this task, I first asked what it meant to prove two definitions equivalent mathematically. I then explained as follows: "To show that two definitions are equivalent, you must verify that each set of defining conditions implies the other". That is, to show that the two definitions of parallelogram are equivalent, one must prove the following implications: (1) "if a quadrilateral has two pairs of opposite sides of the same length, then the quadrilateral also has two pairs of opposite angles of the same measure"; <u>and</u> (2) "if a quadrilateral has two pairs of opposite angles of the same measure, then the quadrilateral also has two pair of opposite sides of the same length."

I anticipated that participants would respond to Task Three differently. The task was designed to learn about their skill in constructing proofs, about their familiarity with "If P, then Q" statements, and about their use of mathematical symbols. The task also helped measure abilities to derive a geometric proposition from other ones, more precisely in the context of quadrilaterals.

Data Collection

Data collection for this study took place in four phases, as summarizes in Figure 3.3.

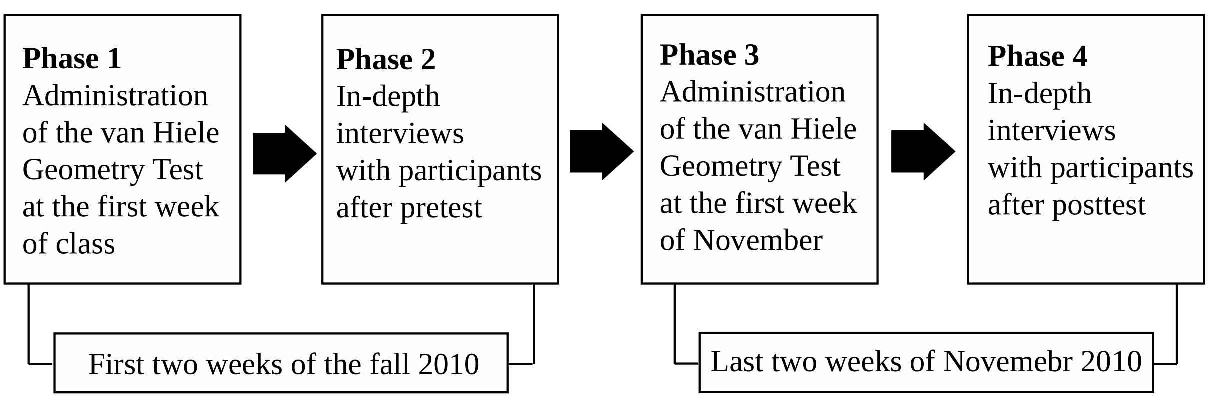


Figure 3.3 Summary of Data Collection Phases

The first phase of the data collection was the pretest, a 35-minutes van Hiele Geometry Test. All students (n=74) enrolled in the measurement and geometry classes took the van Hiele Geometry Test during the first class of the fall semester of 2010. All the tests were collected and analyzed, in order to determine participants' van Hiele levels at the beginning of the semester.

In the second phase, twenty-one students voluntarily participated in a 90-minutes in-depth interview with the same researcher a week after the pretest was given. All interviews were video and audio recorded, and transcribed to serve as the main data recourse for analyzing interviewees' geometric discourses relating to triangles and quadrilaterals. All interviews were completed before the students were introduced to geometric figures in their mathematics content course.

The third phase of the data collection was the posttest, consisting again of the van Hiele Geometry Test. Among the seventy-four participants, sixty-three repeated test ten weeks later during their class time. Again the test responses were collected and analyzed, in order to determine changes in van Hiele levels between the two tests.

The last phase of the data collection consisted interviews with students who had participated in the interviews at the beginning of the semester. Among the twenty-one original interviewees, twenty were interviewed individually for 90-minutes a week after the posttest. Again the interviews were video and audio recorded, and transcribed and analyzed in order to observe changes in interviewees' geometric discourse. All interviews were conducted after students had finished the chapter introducing deductive reasoning in their mathematics content course.

Data Analysis

In my data analysis I applied Sfard's discursive framework to analyze interviews. I used this analysis to investigate participants' levels of geometric thinking through a discursive lens, and to gain some perspective on the usefulness of the framework to describe levels of thinking. I briefly outline the data analyses for this study. First, I compared written responses from the van Hiele Geometry pretest and posttest, obtaining from the test scores information about the changes in these prospective elementary school teachers' van Hiele levels as a whole group. To determine students' van Hiele levels, I followed the test grading method used in the CDASSG project from the University of Chicago in 1982 (Usiskin, 1982). Following the CDASSG project's grading method, I used the 4 of 5 criterion[1] to determine if a student had reached a given level. I chose the 4 out of 5 correction criterion because it minimized the chance of a participant being at that level by guessing (Usiskin, 1982). When assigning a student to a level, I used the classical van Hiele levels 0-5 introduced by CDASSSG.

The assigning of levels required that the student at level n satisfy the criterion not only at that level but also at all proceeding levels. For example, if a participant scored 4 of 5 correct for Levels 1, 2 and 3, 2 of 5 correct for Level 4, and 1of 5 correct for Level 5, this participant was assigned to van Hiele Level 3 because she not only satisfied the criterion at Level 3, but at *all* preceding levels as well. However, in this study there were participants assigned as *nofit* because their van Hiele levels could not be determined from the van Hiele Geometry Test. For example, a

[1]According to the CDASSG project, the 3 of 5 criterion minimizes the chance of missing a student and yields an optimistic picture of students' levels, whereas the 4 of 5 criterion minimizes the chance of a student being at a level by guessing. I decided to use the 3 of 5 criterion.

participant was assigned *nofit* because her test results satisfied criterions at the levels 1, 3 and 5, but not at *all* preceding levels (Usiskin, 1982). I also counted and analyzed students' overall performances for each item in the pretest and posttest, looking for changes in answering single questions as a group (see the results of both analyses in Chapter 4).

My main object of attention in this study is geometric discourse. I try to use the voices of my participants in describing interview's whatever possible, so that readers can draw their own conclusions. In analyzing participants' geometric discourse, I identified the mathematical features in interview transcripts using four categories in the framework: (1) Mathematical words, (2) Visual mediators, (3) Endorsed narratives and (4) Mathematical routines. *Mathematical words* and *visual mediators* utilized mathematical objects of mathematical discourse, whereas *mathematical routines* aimed to produce narratives in given situations. To investigate *changes* in participants' geometric discourse, I analyzed (1) participants' words use regarding to the names of triangles and quadrilaterals (e.g., rectangle, rhombus, etc), and the hierarchy of classifications of quadrilaterals, comparing results of the analyses from both interviews, (2) participants' routine procedures of verifying claims about properties of parallelograms regarding angles, sides and diagonals, comparing results of the analyses from both interviews, and (3) participants' routine procedures of deriving geometric propositions from other geometric propositions, and in constructing mathematical proofs (see the descriptions of interviews and results of these analyses in Chapter 4).

I also investigated the usefulness of a discursive framework as an analytical tool to describe participants' behaviors at each van Hiele level in greater detail and depth. The study produced a theoretical model, the *Development of Geometric Discourse*, describing participants' geometric discourse at each van Hiele level. The model includes the descriptions of (1)

Geometric Objects, (2) Routines, (3) Endorsed Narratives, and (4) Visual Mediators, at van Hiele levels 1 to 4. The model provides a new perspective to present levels of geometric thinking as geometric discourse (see the descriptions of the model in Chapter 4).

CHAPTER FOUR: FINDINGS

Changes in van Hiele Levels of Geometric Thinking

During the fall of 2010, the van Hiele Geometry Tests were conducted at the beginning of the semester (pretest) and ten weeks later (posttest). Sixty-three prospective teachers participated in both tests; among these sixty-three participants, twenty of them voluntarily participated in the interview part of the study. In this section I present results of these sixty-three participants' van Hiele Geometric Tests as a whole group, as well as the results of the twenty interviewees' van Hiele Geometric Tests, in order to give some background information on their changes in van Hiele levels in the paper-pencil test.

Changes in van Hiele Geometry Test as a Whole Group

The van Hiele Geometry Test contains 25 multiple-choice items, distributed into five van Hiele levels: Items 1-5 (Level 1), Items 6-10 (Level 2), Items 11-15 (Level 3), Items 16-20 (Level 4) and Items 21-25 (Level 5). These items are designed to identify participants' geometric thinking at five van Hiele levels. For example, Items 1 to 5 of are designed to identify students' thinking related to van Hiele Level 1, at which figures are judged according to their appearance. Items 5 to10 are designed to identify participants' thinking related to van Hiele Level 2, at which figures are the supports of their properties.

Following the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project's grading method, I had a choice of either the 3 out of 5 correction criterion, or the 4 out of 5 correction criterion, to determine whether a participant has reached a given van Hiele level. I chose the 4 out of 5 criterion for this study because it minimized the chance of a participant being at that level by guessing (Usiskin, 1982). When assigning a student to a level, this study used the classic van Hiele levels introduced by CDASSSG, which included Levels from 0 to 5. The assigning of levels required that the participants at level n satisfied the criterion not only at that level but also at *all* proceeding levels. For example, if a participant scored 5 of 5 correct for Levels 1 and 2, 4 of 5 correct for Level 3, 2 of 5 correct for Level 4, and 1of 5 correct for Level 5, this participant was assigned at van Hiele Level 3 because she not only satisfied the criterion at Level 3, but *all* preceding levels as well. However, in this study there were participants assigned as *nofit* because their van Hiele levels could not be determined from the van Hiele Geometric Test. For example, if a participant scored 5 of 5 correct for Level 1, 3 of 5 correct for Level 2, 4 of 5 correct for Level 3, 2 of 5 correct for Level 4, and 4 of 5 correct for Level 5, she was assigned *nofit* because her test results satisfied criterions at the levels 1, 3 and 5, but not *all* preceding levels (Usiskin, 1982).

To give an overall idea of participants' van Hiele levels as a whole group, Table 4.1 presents the distributions of levels in both number and percentage from the pretest and the posttest.

Table 4.1 Distributions of participants' van Hiele levels at both tests

Level	VHB		VHE	
	N	%	N	%
0	6	9.52	4	6.35
1	8	12.70	9	14.29
2	9	14.29	8	12.70
3	17	26.99	30	47.62
4	2	3.17	2	3.17
5	0	0	1	1.59
Total fitting	42	66.67	54	85.71
Nofit	21	33.33	9	14.29
Totals	63	100	63	100

Note: VHB indicates interviewees' van Hiele levels at the beginning of the fall semester, whereas VHE indicates their van Hiele levels ten weeks later.

Table 4.1 shows about forty-eight percent of participants at Level 3 (n=30) at the posttest, and that number almost doubled over the pretest. One participant moved to Level 5 at the posttest. There was a slight reduction at Level 0 and Level 2, and a slight increase at Level 1 from the pretest to the posttest. There were twenty-one participants assigned to *nofit* at the pretest, but only nine such participants at the posttest. These numbers show that a little more than half (52.4%) of participants were able to reach Level 3 or above ten weeks later after their first day of class in the fall semester 2010. The reduction of the *nofit* students suggests that participants' responses were more consistent at the posttest than the pretest.

Participants' responses for each item were counted and analyzed. Table 4.2 presents the frequencies of each item regarding responses in both the pretest and posttest. The bolded numbers represent the number of participants having the correct answer for that item.

Table 4.2 Van Hiele Geometry Test: Item Analysis for Pretest (B) and Posttest (E)

Level	Choice	Item	1B	1E	2B	2E	3B	3E	4B	4E	5B	5E
1	A		0	0	0	0	0	0	0	1	0	0
	B		**53**	**59**	0	0	0	0	**41**	**50**	0	0
	C		0	0	4	0	**63**	**63**	6	5	15	4
	D		10	4	**59**	**63**	0	0	1	5	0	1
	E		0	0	0	0	0	0	6	2	**48**	**58**
2		Item	6B	6E	7B	7E	8B	8E	9B	9E	10B	10E
	A		5	5	2	1	**39**	**46**	3	1	4	5
	B		**43**	**46**	1	0	3	0	2	0	4	0
	C		9	7	2	1	8	3	**54**	**60**	4	3
	D		6	5	3	0	2	4	1	0	**40**	**44**
	E		0	0	**55**	**61**	11	10	3	2	1	11
3		Item	11B	11E	12B	12E	13B	13E	14B	14E	15B	15E
	A		4	3	6	4	**45**	**56**	**31**	**46**	2	6
	B		3	0	**47**	**51**	0	0	14	7	**29**	**33**
	C		**54**	**58**	5	4	0	0	10	3	4	1
	D		1	0	0	0	0	0	4	2	7	5
	E		1	1	5	4	18	7	4	5	21	18

Table 4.2 (cont'd)												
4		Item	<u>16B</u>	<u>16E</u>	<u>17B</u>	<u>17E</u>	<u>18B</u>	<u>18E</u>	<u>19B</u>	<u>19E</u>	<u>20B</u>	<u>20E</u>
	A		9	10	14	12	14	10	27	37	**19**	**30**
	B		8	6	9	11	14	15	10	12	2	2
	C		**31**	**30**	**23**	**29**	3	2	14	7	5	1
	D		10	13	9	4	**27**	**28**	**10**	**5**	32	29
	E		5	4	8	7	4	8	2	2	5	1
5		Item	<u>21B</u>	<u>21E</u>	<u>22B</u>	<u>22E</u>	<u>23B</u>	<u>23E</u>	<u>24B</u>	<u>24E</u>	<u>25B</u>	<u>25E</u>
	A		33	28	11	12	20	26	2	5	1	2
	B		**16**	**16**	9	14	8	4	3	5	16	14
	C		4	7	6	6	2	5	11	13	2	1
	D		1	1	15	19	**25**	**26**	16	22	**33**	**37**
	E		9	10	**22**	**12**	8	2	**30**	**18**	10	9

Note the number of students (n=63) who participated in both tests.

Table 4.2 shows an increase in correct answers for Items 1 to 15 from the pretest to the posttest, indicating that participants did better at the posttest in items related to van Hiele Levels 1, 2 and 3. There was an increase in correct answers at Level 4 and Level 5, but not for every item individually. Figure 4.1 compares participants' performance based on the number of correct answers for each item at the pretest and the posttest.

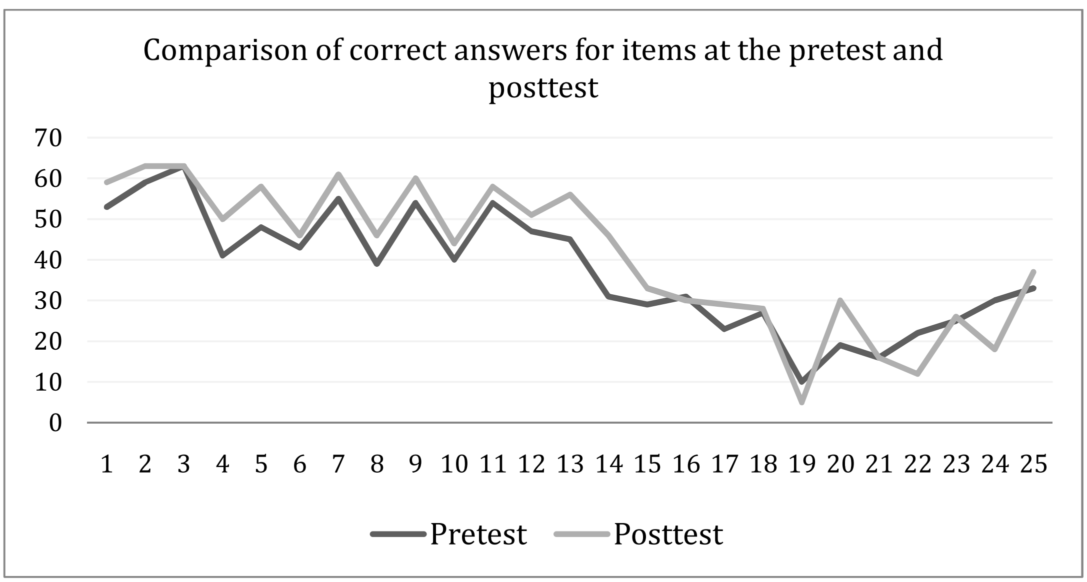


Figure 4.1 Comparison of correct answers for items at both tests

Figure 4.1 shows an increase in correct answers for Items 1 to 15, indicating that the group of participants (n=63) had a better performance on these items at the posttest than at the pretest. Thus, participants were better in answering questions relating to van Hiele levels 1 to 3 at the posttest than at the pretest. However, there was a reduction in correct answers starting at Item 16, and that continued intermittently to Item 25. Items 16 to 25 are designed to identify participants' Level 4 and Level 5 thinking, and this inconsistency was no surprise because the course was not designed to train students at theses levels. In looking at these results, it is clear that participants were getting more familiar with triangles and quadrilaterals, as well as with properties related to these polygons. However, it is not clear that participants' improved in doing proofs and thinking at an abstract level, mathematical activities related to van Hiele Level 4 and Level 5.

Twenty students voluntarily participated in the interviews conducted after the pretest and the posttest. In the next section, I briefly discuss the results of these twenty interviewees' van Hiele Geometry Tests before they entered the interview part of the study.

Changes in van Hiele Geometry Test among Interviewees

Twenty-one students voluntarily participated in the interviews shortly after the pretest, and twenty of these participated in the second interviews ten weeks later after the posttest. Table 4.3 presents distributions of interviewees' van Hiele levels at the pretest and the posttest.

Table 4.3 Distributions of interviewees' van Hiele levels at both tests

Level	VHB		VHE	
	N	%	N	%
0	2	10	1	5
1	1	5	0	0
2	2	10	4	20
3	6	30	10	50
4	1	5	2	10
5	0	0	0	0

Table 4.3 (cont'd)				
Total fitting	12	60	17	85
Not fit	8	40	3	15
Totals	20	100	20	100

The table shows that the percentages at each van Hiele level of the twenty interviewees matched closely with the corresponding percentages of the whole group (n=63). For example, thirty percent of interviewees were at Level 3 in the pretest, whereas it increased to fifty percent at the posttest. These numbers are close to those for the whole group, twenty eight percent and forty eight percent, respectively (see Table 4.2). Thus it appears that the twenty interviewees are a good sample size for the study. Table 4.4 gives, for these interviewees, frequencies of changes from one van Hiele level to another between the two tests.

Table 4.4 Interviewees' changes in van Hiele levels between tests

	Van Hiele levels at the posttest (n=20)							
	Levels	0	1	2	3	4	5	Nofit
Van Hiele levels at the pretest (n=20)	0	-	-	-	**2**	-	-	-
	1	-	-	-	**1**	-	-	-
	2	**1**	-	**1**	-	-	-	-
	3	-	-	-	**4**	**1**	-	1
	4	-	-	-	-	-	-	1
	5	-	-	-	-	-	-	-
	Nofit	-	-	3	3	1	-	1

For example, among these interviewees (n=20), one stayed at Level 2, and four stayed at Level 3 from the pretest to the posttest. There were two interviewees moving three van Hiele levels from Level 0 to Level 3, while one interviewee moved two van Hiele levels from Level 1 to Level 3, and one interviewee changed from Level 3 to Level 4. A total of seven interviewees changed from *nofit* to Level 2 (n=3), Level 3(n=3) or Level 4 (n=1). Note that among the ten interviewees assigned to van Hiele levels at both the pretest and the posttest, five showed no change, and four showed changes from a lower to a higher van Hiele level.

The van Hiele Geometry Test provided initial information about participants' van Hiele levels at the time of the study, but it did not provide rich descriptions about the changes in participants' levels of thinking. For a deeper analysis of participants' thinking, twenty participants were interviewed soon after the pretest and posttest. In the next section I describe my interviews with these participants, as well as my findings from these interviews.

Changes in Geometric Discourse

In this section I describe findings about the interviewees who participated in the interview parts of the study. To narrow the scope, this section focuses on the analyses of five interviewees' geometric discourses as examples of various scenarios I encountered during the interviews. My analyses are organized with regard to the results of their van Hiele levels from the van Hiele Geometry Tests. These five interviewees have been assigned the names ATL, ANI, ALY, AYA and ARI.

To analyze interviewees' geometric discourses in the context of quadrilaterals and triangles, I devoted my attention to interviewees' familiarity with polygons in regard to their *word use*, including use of the names of polygons (e.g., parallelogram, rectangle, etc), and the names of the parts of polygons (e.g., angle, side, etc). Also, I analyzed interviewees' various routines while engaging in solving geometric tasks during the interviews; these routines included *routines* of *sorting*, *identifying, defining, conjecturing* and *substantiating.*

Recall that a *routine* is a set of meta-rules that describes a repetitive discursive action. As described in an earlier section, this set of rules is divided into the *how* of the routine and the *when* of the routine. The *when* of the routine, was influenced by my direct requests during the interviews, as I asked interviewees directly for implementation of given tasks (e.g., sorting the polygons differently from the way they did before). So for a better analysis of the *when* of routines, I needed to observe interviewees' spontaneously using procedures as a part of more complex tasks. Therefore, in this study I mainly analyze the *how* of routines, the routine procedures that determine the course of patterned discursive performance (Sfard, 2008).

In this study different routines are involved given the nature of the tasks: the *routine of sorting* is a set of routine procedures that describes repetitive actions in classifying polygons (e.g, by their family appearances, by visual properties, etc); the *routine of identifying* is a set of routine procedures that describes repetitive actions in identifying polygons (e.g., by visual recognition, by partial properties check); whereas the *routine of defining* is a set of repetitive actions related to how polygons are described or defined (e.g, by visual properties, by mathematical definition, etc). In endorsed narratives such as mathematical definitions or axioms, the *routine of recalling,* a subcategory of the *routine of defining,* is a set of repetitive actions using previously endorsed narratives (e.g, I remember this definition because I learned it), and "it can indicate a lot not just about how the narratives were memorized, but also about how they were constructed and substantiated originally" (Sfard, 2008, p.236).

With regard to performing mathematical tasks, "guessing and checking" are seen as common activities. The *routine of conjecturing* is a set of repetitive actions that describe a process of how conjecture is formed; and the *routine of substantiating* is a set of patterns describing a process of using endorsed narratives to produce new narratives that are true (e.g., an informal or formal proof using a triangle congruence criterion).

To better understand how learning takes place, and how mathematical concepts are developed, it is helpful to conceptualize mathematical learning as the development of a discourse, or a change in discourse. Among the twenty interviewees, three showed a change in their van Hiele levels from lower to higher according to the van Hiele Geometric Test conducted at the beginning of the semester (the pretest) and ten weeks later (posttest). The three interviewees include ATL, who moved two van Hiele levels from Level 1 to Level 3; ANI, who moved three van Hiele levels from Level 0 to Level 3; and ABU who moved one van Hiele level from Level 3 to Level 4. In the following subsections I describe each interviewee's geometric discourse with regard to their *routines* and *word use*. From my observations I present evidence to point out changes in each interviewee's geometric discourse. I will refer to the interview conducted at the beginning of the semester as the Pre-Interview, and the interview conducted ten weeks later as the Post-Interview.

Case 1: Changes in ATL's Geometric Discourse

ATL was a sophomore at the time of the interviews. ATL took her last geometry class five years prior to the geometry and measurement class. The van Hiele Geometry Test showed that she was at Level 1 at the pretest, and ten weeks later she moved two van Hiele levels, to Level 3 at the posttest. I interviewed ATL after both tests, and analyzed and compared ATL's geometric discourses in the context of triangles and quadrilaterals. A summary of ATL's changes in geometric discourse as follows:

- ATL's routines of sorting changed from grouping polygons according to their family appearances at the Pre-Interview, to classifying polygons according to their visual properties and definitions.
- ATL's identifying routines changed from self-evident visual recognition at the Pre-Interview, to identifying partial properties of the polygons (i.e. sides and angles) and recalling at the Post-Interview.

- ATL’s use of the names of parallelograms changed from describing the parallelograms as collections of unstructured quadrilaterals that share some physical appearances at the Pre-Interview, to using the names as collections of quadrilaterals that share common descriptive narratives at the Post-Interview.
- In my observations I did not find substantiation routines in the Pre-Interview or the Post-Interview. That is, ATL did not use measurement tools to prove or disprove congruent parts of the polygons at the object level; nor did she use informal or formal mathematical proofs at the abstract level.

Let me begin by introducing Task One, *Sorting Geometric Shapes.* This Task is used to analyze interviewees’ *routines of sorting*, *identifying* and *defining* polygons. This task presents eighteen polygons, including *triangles* (n=4), *quadrilaterals* (n=13) and *a hexagon*. Interviewees are asked to classify these polygons into groups, without being given measurement information. One common reaction interviewees might have is to group the polygons based on the number of their sides (e.g., 3-sided, 4 sided, etc). According to an individual interviewee’s response, I will ask her to regroup the polygons differently and to subgroup some of the large groups.

Let me begin with argument that there is a change in ATL’s routines of sorting, from using visual recognition to group quadrilaterals according to their family appearances at the pre-interview, to classifying quadrilaterals according to their common descriptive narratives (i.e., definitions and properties). I briefly describe my interviews with ATL for Task One.

At the Pre-Interview, ATL stated, “I group them solely on their amount of sides”, and sorted the polygons into three groups on her first attempt: 1) 3 sides, consisting of Fig. K, W, X, and S; 2) 4 sides, consisting of Fig. U, M, F, G, P, T, L, J, H, R and Z; and 3) Trapezoid, including Fig. V and Q. See Figure 4.2 for some examples of each group based on ATL's written response

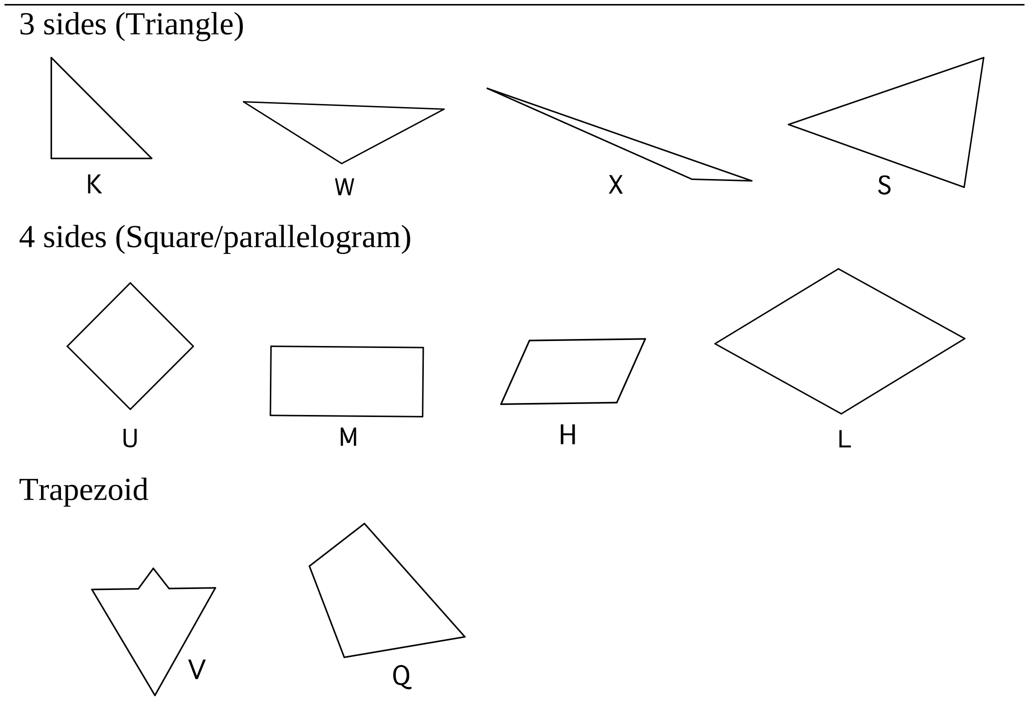


Figure 4.2 *ATL's grouping of polygons in the Pre-Interview*

ATL included all triangles in the 3-sides group and called it the *triangle group.* She included all *squares, rectangles* and *parallelograms* in the 4-sides group, and called it the *square and parallelogram group*. She grouped Fig. V (a hexagon) and Fig. Q (a quadrilateral) together as a *trapezoid* group because to her, a *trapezoid* was "a figure with five sides, varying in length", and "often make these odd shapes". Fig. N (a right trapezoid) was not included in any of these groups.

When I asked ATL to find another way to group polygons differently, she regrouped triangles "base on appearance", according to attributes of angles and sides in a triangle. ATL explained by saying, "I know this one has right angle", and "these ones demonstrate different length, they are not the same" (see Figure 4.3).

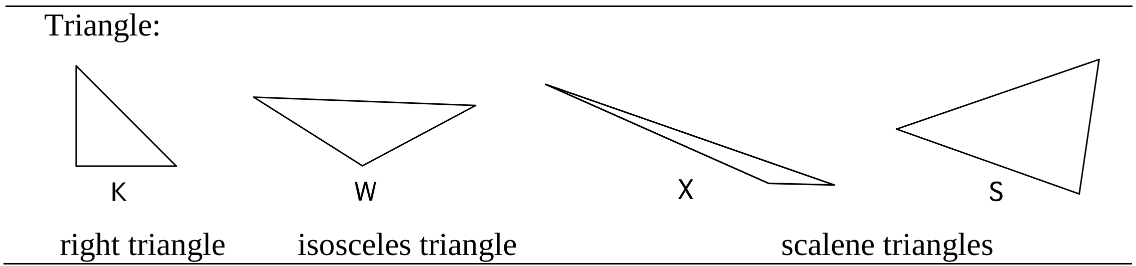


Figure 4.3 ATL's regrouping of the triangles at the Pre-Interview

ATL regrouped the triangles into *right triangles*, *isosceles triangles* and *scalene triangles* according to their visual properties. ATL then regrouped the 4-sided polygons according to their family appearances, with the names of *square, rectangle, parallelogram,* and *rhombus*. For example, Figure 4.4 shows two of the groups: the *rectangles* and the *squares.*

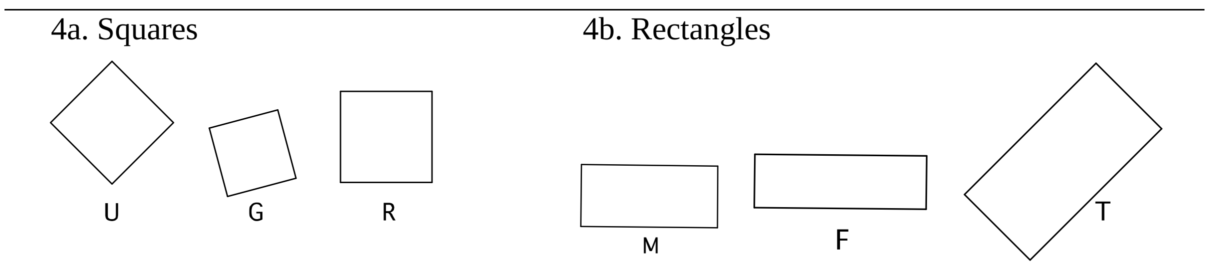
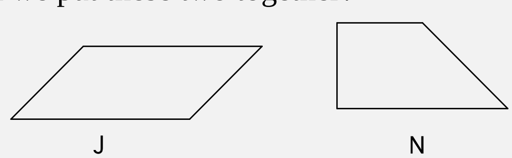


Figure 4.4 ATL's regrouping the quadrilaterals at the Pre-Interview

When I asked ATL why she regrouped the polygons in this way, she responded, "I know this figure (Fig. U, a square) and this figure (Fig. M, a rectangle) are different, but they both belong to the same quadrilateral group". Fig. N (a right trapezoid) was not included in any of these groups again at this second attempt. I asked ATL if I could put Fig. J (a parallelogram) and Fig. N together, and the following conversation took place:

Interviewer: Can we put these two together?



ATL: I wouldn't believe so… Just because this [pointing at Fig. N] shows the angle… it doesn't have the properties of a square or a rectangle, the sides…measurement… it does have four sides, but no…. congruent parts.

When ATL sorted polygons into groups, she grouped them by their visual appearances. ATL's routines of sorting polygons at the Pre-interview are illustrated in Figure 4.5.

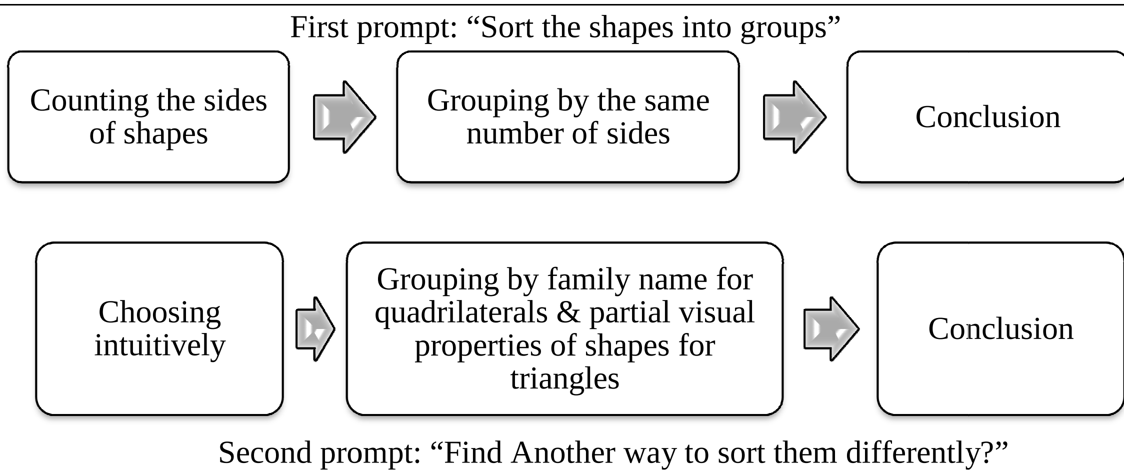


Figure 4.5 ATL's routines of sorting polygons at the Pre-Interview

ATL mentioned that all triangles and all quadrilaterals "have a broader definition of each other". When it came to classifying quadrilaterals, ATL's routine procedures focused on the appearances of the polygons and how their appearances related to their family names. It was evident that ATL identified polygons with visual recognition. However, as I explained earlier, ATL regrouped the triangles by their visual properties (e.g., angles and sides). I conclude that there was no defining routine in ATL's routines of sorting for Task One. I found identifying routines such as direct recognition and counting in ATL's routines of sorting when she sorted polygons by the numbers of their sides and grouped quadrilaterals by their visual appearances. In the case of triangles, ATL's identifying routine consisted of identifying visual properties as she regrouped triangles by the attributes of the angles and sides.

At the Post-Interview, the same task was performed. ATL grouped the polygons by "looking at the numbers of sides solely" [po1. 2], and she sorted eighteen polygons into three groups: 1) *Triangles* (n=4), including all 3-sided polygons; 2) *5-sided* (n=1), consisting of Fig. V; and 3) *Quadrilaterals* (n=13), including all 4-sided polygons in the task. Figure 4.6 compares ATL's first attempts at both interviews with some examples of each group.

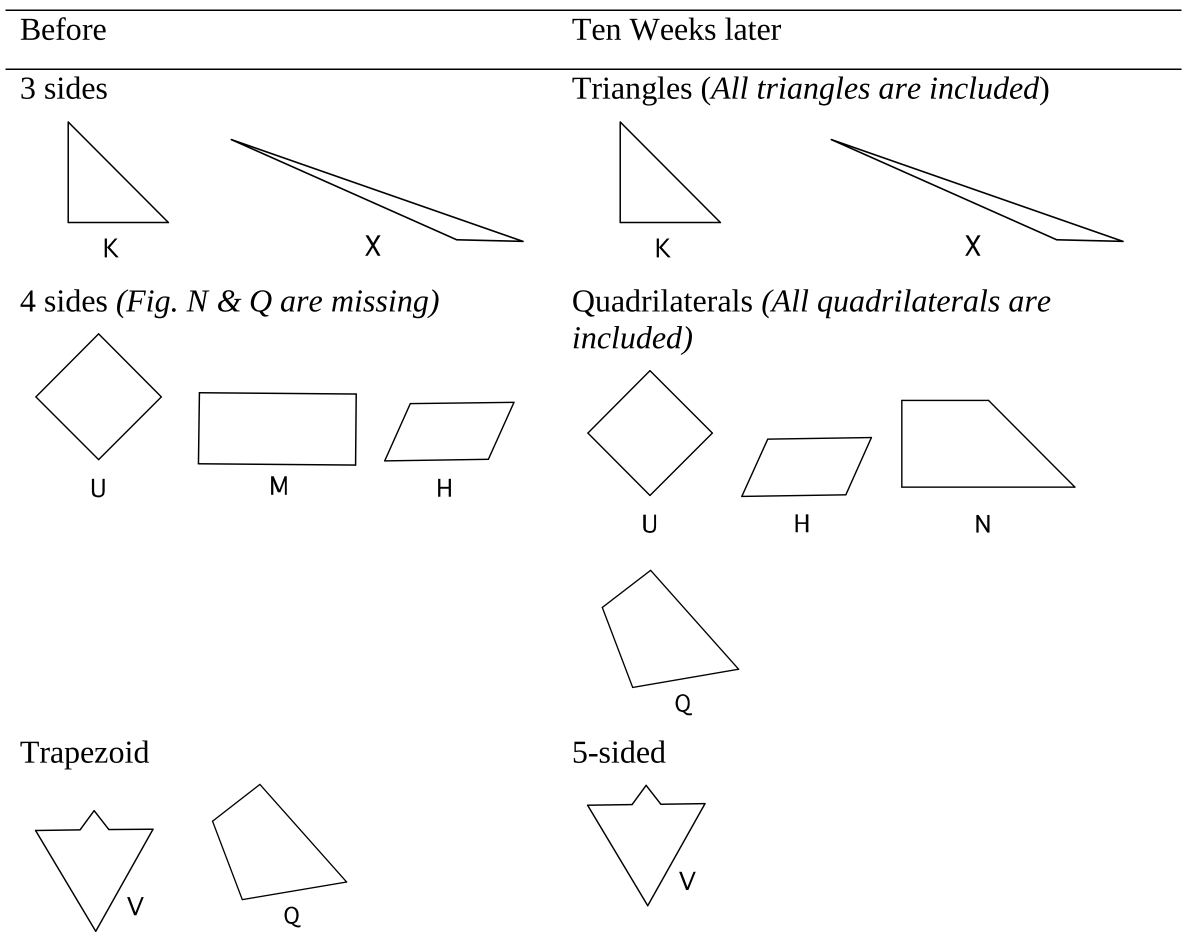


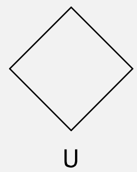
Figure 4.6 A comparison of ATL's grouping of polygons at both interviews

At the Post-Interview, ATL included both Fig. N (a right trapezoid) and Fig. Q (a 4-sided figure) into the quadrilateral group with the help of her defining routine. At the Pre-Interview

ATL did not include Fig. N in any groups because “it does have four sides, but… not congruent parts”, but at the Post-Interview she included Fig. N in the quadrilateral group because it is “a four sided figure with one distinct pair of parallel sides (pointing at Fig. N)”. I asked ATL to regroup the quadrilaterals, and her response is shown below:

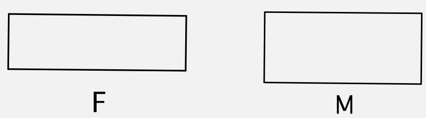
18a ATL: Quadrilaterals, you know that you have your square because … each forms 90-degree and all the side lengths are equal.

[Pointing at Fig. U]



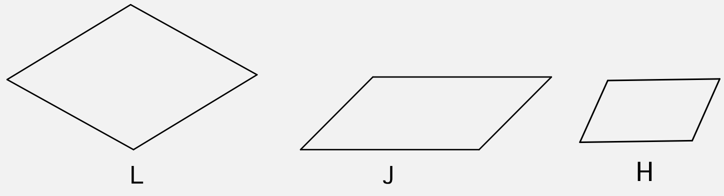
18c. ATL: these are rectangles because two sides and those two sides are the same. But again they form 90-degree angles…

[Pointing at Fig. F and Fig. M]



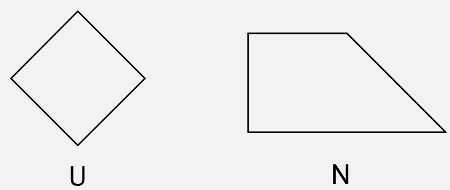
18e. ATL: opposite angles are equal and opposite sides are equal, so these three would be an example of parallelogram.

[Pointing on Fig. L, J and H]



At the Post-Interview, ATL was able to use her definitions of *square, rectangle, parallelogram* and *rhombus* to identify and to regroup the quadrilateral group. She regrouped quadrilaterals into: 1) *squares,* including Fig. U, G, and R; 2) *rectangles,* including Fig. M, F, T; 3) *rhombus*, consisting of Fig. Z; and 4) *parallelogram*, including Fig. L, J, H. When I asked ATL if I could put Fig. U (a square) and Fig. N (a right trapezoid) together, ATL responded:

Interviewer: Can Fig. U and Fig. N group together?



ATL: They can group together as both being same amount of sides… but in terms like property…no … they both have two parallel sides, but a trapezoid cannot be branched off with parallelograms into rectangles and squares….

This dialogue provides shows ATL's ability to compare Fig. U and Fig. N, not only focusing on the "same amount of sides", but also on visual properties, like "they both have parallel sides". To describe what has changed in ATL's routine of sorting polygons, Table 4.5 summarizes ATL's course of actions in response to Task One.

Note that there was no change in ATL's routines of sorting triangles. ATL's first attempt of sorting quadrilaterals at both interviews remained the same. However there was a change in routines of sorting quadrilaterals on the second attempt, when I asked her to regroup the quadrilaterals differently (see the shaded part in Table 4.5). I found defining routines when ATL sorted quadrilaterals at the Post-Interview; ATL's routine of sorting changed from only visual recognition at the Pre-Interview, to classifying polygons according to their common descriptive narratives.

Table 4.5 A comparison of ATL's routines of sorting polygons at the two interviews

Before	Ten Weeks Later
First prompt: "Sort the shapes into groups"	*First prompt: "Sort the shapes into groups"*
1. Counting the sides of shapes (*Counting*) 2. Grouping by the same number of sides 3. Conclusion	1. Counting the sides of shapes *(Counting)* 2. Grouping by the same number of sides 3. Conclusion
Second prompt: "Find another way to sort them differently"	*Second prompt: "Find another way to sort them differently"*
1. Direct recognition of possible candidates *(Visual recognition)*	1. Direct recognition of possible candidates *(Visual recognition)*
2.a Grouping by family appearance of quadrilaterals and parallelograms *(Visual recognition)*	2.a Grouping by common descriptions of quadrilaterals and parallelograms by visual properties and some mathematical definitions *(Defining routine)*
2.b Grouping by properties of angles and sides in triangles *(Defining routine)*	2.b Grouping by properties of angles and sides in triangles *(Defining routine)*
3. Conclusion	3. Conclusion

ATL's responses to the questions in Task Two also revealed changes in her geometric discourses. Task Two involves two sets of activities about parallelograms, and is designed to investigate interviewees' familiarity with the angles, sides and diagonals of a parallelogram.

The first part of Task Two asks interviewees to draw two parallelograms that are different from each other, and then to discuss the relationship between the angles, sides and diagonals of the parallelograms. In the second part of Task Two, I present pictures of parallelograms that are not included in interviewees' drawings from the first part, and then ask questions about the parallelograms. Presentation of these pictures of parallelograms is designed to elicit interviewees' thinking of "what is a parallelogram" and "what is *not* a parallelogram", and to provide a variety of parallelograms for discussions.

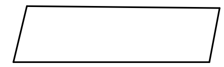
Interviewees' responses to this task vary depending on their familiarities with the properties and hierarchy of parallelograms, as well as their familiarity with the parts of parallelograms. For example, when interviewees declare a narrative such as "opposite sides are equal" regarding the sides of a parallelogram, their substantiation process can be very different. Depending on their levels of thinking, some interviewees might produce a narrative such as "because it is a parallelogram" using defining routines, whereas others might conclude, "they look like they are equal" using identifying routines.

This lead to my argument that ATL's identifying routines changed from self-evident visual recognition at the Pre-Interview, to identifying visual properties and using definitions of parallelograms to draw conclusions about the angles and the sides of parallelograms at the Post-Interview. I describe parts of my interviews with ATL for Task Two. As shown in Table 4.6, ATL drew a parallelogram and then a rectangle as a new parallelogram (see Table 4.6). ATL declared that the two parallelograms were different because "I would change the sizes of it [side]". ATL described the second drawing as, "it's a rectangle… but it's not the typical looking parallelogram". In response to the questions about the angles of the parallelograms, ATL expressed her frustrations on the angles, "I am still stuck on the question on what it means by the angles, … Usually when I'm talking about angles, we have measurements…[pausing] I feel like the angles would be the same … just based on how it looks".

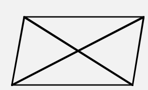
Table 4.6 summarizes ATL's declared narratives of the angles of a parallelogram and a rectangle, and her verifications of declared narratives.

Table 4.6 ATL's routines of verifying on the angles of parallelograms at the Pre-Interview

Q: "What can you say about the angles of this parallelogram?"

	 Parallelogram	 Rectangle
Conjecture (Guessing)	"I would assume that they are the same for the opposites"	"They would have to be equal…or add up to a certain amount"
	Q: "How do you know?"	
Routines	Visual recognition	Visual recognition
Declared Narrative	" No. I don't know." "just based on how it looks"	"Just looks more like a stereotypical parallelogram"

ATL made intuitive claims about the angles of a parallelogram and a rectangle using direct recognition. For example, ATL assumed that the angles were "the same for the opposites" in a parallelogram using direct recognition. In this case, the question "How do you know [they are the same]?" did not lead to any substantiations of the claim, nor lead her to endorse any narratives using mathematical definitions; instead ATL's final conclusion was reached by direct visual recognition which was self-evident. This routine pattern also appeared when ATL was discussing the diagonals of a parallelogram:

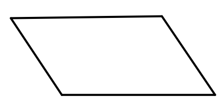
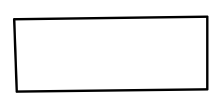
17. Interviewer	What can you say about the diagonals of this parallelogram? 
18. ATL	The diagonals would be equal…
19. Interviewer	How do you know the diagonals are equal?
20. ATL	You have to measure and make sure these were, all their sides were the same…right here [pointing on the sides], would all equal… on each side all equaling the

	same parts.	

ATL declared a narrative about the diagonals of the parallelogram stating that, “the diagonals would be equal”. The diagonals of this parallelogram are not equal, as can be detected with a ruler. However, ATL did not check because her direct recognition was intuitive and also self-evident. There was no need to substantiate the narrative, “the diagonals would be equal”, but instead ATL made her own intuitive conclusion that the “diagonals were equal” because “…all their sides were …the same”.

Ten weeks later, the same task was performed again. The change in ATL’s identifying routines was evident. Table 4.7 summarizes ATL’s course of actions in response to the question “what can you say about the angles of the parallelogram?” at the Post-Interview.

Table 4.7 ATL’s routines of verifying on the angles of parallelograms at the Post-Interview

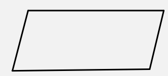
Q: “What can you say about the angles of this parallelogram?”		
	 Parallelogram	 Rectangle
Declared Narratives	“Opposite angles equal and … they don’t form 90-degreee angle”	“…you could say that the opposite angles are equal, and in this one all angles are equal”
Q: “How do you know?”		
Table 4.7 (cont’d)		

Routines	a. Visually identify partial properties of a parallelogram by checking the condition of opposite angles *(Identifying routine)* b. Describe a parallelogram with no right angles *(Defining routine-recalling)*	a. Visually identify partial properties of a rectangle by checking the condition of opposite angles *(Identifying routine)* b. Describe a rectangle with right angles *(Defining routine – recalling)*
Declared Narratives	"I would just say the property of parallelogram" [po2.118]	"It has properties of parallelogram. It's a rectangle" [po2. 4]

Recall that, at the Pre-Interview, ATL did not know how to draw a conclusion about the angles in a parallelogram without measurements. At the Post-Interview, ATL was able to discuss the angles of parallelograms using the properties of a parallelogram (*defining routine*). For example, when ATL declared the narrative "opposite angles are equal and they don't form a 90-degree angle", she identified that this 4-sided polygon was a parallelogram (*identifying routine*) and described the parallelogram, as it had no right angles using defining routines. Similarly, ATL was able to identify the differences of the angles between two parallelograms: a *parallelogram*, "opposite angles are equal and … they don't form a 90-degreee angle" and a *rectangle*, "the opposite angles are equal, and in this one [rectangle] all angles are equal".

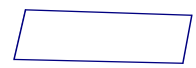
In this scenario, we begin to see the change in ATL's routines of identifying, from visual recognition, to identifying visual properties of the angles in a parallelogram. ATL's routines of defining also showed a use of definitions of parallelograms to justify her claims at the Post-Interview. However it is important to note that ATL's routine of defining was more of a *recalling*, as it appeared to be memorization of the facts.

During the Pre-interview, ATL showed more confidence in discussing the sides of the parallelogram than the angles of the parallelogram. When I asked ATL about the sides of parallelogram, the following conversation took place:

9. Interviewer	What can you say about the sides of this parallelogram? 
10. ATL	Opposite sides are equal…
11. Interviewer	How do you know they are equal?
12. ATL	Basically… on just the properties of a parallelogram. If I measure it out…if I draw it with a ruler, it would have to be the same for each side
13. Interviewer	Is there a way that you can show me that the opposite sides are equal?
14. ATL	I would draw it out with two sides having to be the same measure and these two having to be the same measure. But for one of the opposite sides, they have to be longer than others to not to make it the properties of a square.

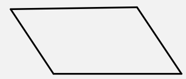
In this dialogue, ATL declared a narrative about the sides of the parallelogram, “opposite sides are equal”. When asked for substantiation, ATL justified her claim by saying “just the property of a parallelogram” [12]; and provided another explanation, “If I measure it, draw it out with a ruler...it would have to be the same” [12]. After another prompt, ATL verbally described a set of procedures to justify her claim: “draw on with a ruler”, and “draw it out with two sides having to be the same measure” [14]. ATL’s routines of verifying the sides of a parallelogram and a rectangle at the Pre-Interview are summarized in Table 4.8.

Table 4.8 ATL’s routines of verifying on the sides of parallelograms at the Pre-Interview

Q: “What can you say about the sides of this parallelogram?”		
	 Parallelogram	 Rectangle
Conjecture (Guessing)	“Opposites are equal” “Opposite sides … one longer than the other”	“Each opposite side is equal in length”
Interviewer	“How do you know?”	
Routines	a. Visual recognition b. Identify partial properties of a parallelogram *(defining routine-recalling)*	a. Visual recognition b. Identify partial properties of a parallelogram *(defining routine-recalling)*
Declared Narrative	“just on the properties of a parallelogram”	“rectangle can still have the properties of a parallelogram”

At the Pre-Interview, ATL identified the equal sides of a parallelogram and a rectangle intuitively and verified her claims using properties of parallelograms. However from ATL’s description of a parallelogram, “opposite sides one longer than the other”, I conclude that ATL was at the stage of identifying parallelograms by their visual appearances.

In contrast to her responses at the Pre-Interview, ATL declared, “Opposite sides are parallel and equal” in referring to the sides of a parallelogram at the Post-Interview. To verify her claims, ATL mentioned only “the properties of a parallelogram”.

109. Interviewer	What can you say about the sides of this parallelogram? 
110. ATL	The sides would be the same…or they should be opposite sides are parallel and they should be equal…
111. Interviewer	Why do you say ‘it should be’?
112. ATL	Because it has the properties of a parallelogram. …by looking at it, it looks as if they are, so it could be good.
113. Interviewer	Is there a way that you can show me that the opposite sides are equal and parallel?
114. ATL	Based on the properties of it [parallelogram].

ATL provided the narrative, “the [opposite] sides would be the same… or should be…parallel” [110]. After several prompts, I found that ATL’s course of actions consisted of visual recognition, “by looking at it”, and recalling using what she remembered as, “the properties of it”. Table 4.8 summarizes ATL’s routine procedures concerning the sides of a parallelogram and a rectangle at the Post-Interview.

Table 4.8 ATL’s routines of verifying on the sides of parallelograms at the Post-Interview

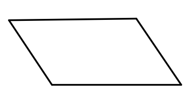
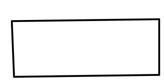
Q: “What can you say about the sides of this parallelogram?”		
	 Parallelogram	 Rectangle

Table 4.8 (cont'd)		
Declared Narratives	"opposite sides are parallel and they should be equal"	"opposite sides would be congruent" "They are parallel to one another"
	Q: "How do you know?"	
Routines	a. Visually identify partial properties of a parallelogram by checking the condition of the sides *(Identifying routine)* b. Describe a parallelogram – opposite sides are equal & parallel *(Defining routine-recalling)*	a. Visually identify partial properties of a parallelogram by checking the condition of the sides *(Identifying routine)* b. Describe a particular parallelogram-opposite sides are equal & parallel *(Defining routine-recalling)*
Declared Narratives	"It has the properties of a parallelogram"	"It's a basic property of a parallelogram" "To be a parallelogram, opposite sides have to be parallel, making them congruent"

ATL described opposite sides as parallel and equal for both parallelograms and rectangles at the Post-interview, whereas she only mentioned opposite sides as equal at the Pre-Interview. In verification, ATL's routine procedures consisted of identifying routines and recalling at the Post-Interview. ATL's changes in routines procedures were (1) from visual recognition that was self-evident, to identifying partial properties and using properties about the angles of a parallelogram, (2) from identifying partial properties (i.e., equal sides) at the Pre-Interview, to identifying more properties (i.e., equal and parallel sides) at the Post-Interview as described in Table 4.9.

Table 4.9 Summary of ATL's routines in Task Two from both Interviews

Parts of Parallelograms	Pre-Interview Routines: Identifying Routine	Pre-Interview Routines: Defining Routine	Post-Interview Routines: Identifying routine	Post-Interview Routines: Defining routine
Angles	Visual recognition Self-evident	No	Visual recognition/ Identifying partial property	Recalling
Sides	Visual recognition/ Identifying equal sides	Recalling	Visual recognition/ Identifying equal and parallel sides	Recalling
Diagonals	Visual recognition Self-evident	No	Visual recognition Self-evident	No

ATL did not think that squares and rhombi were parallelograms at the Pre-Interview; therefore my examples of ATL's identifying and defining routines for Task Two are limited to the cases of a parallelogram and a rectangle. In both interviews, I did not find a substantiation routine. ATL did not use measurement tools to verify the congruent angles and sides of parallelograms, nor did she use definitions and triangle congruency to construct newly endorsed narratives. ATL's identifying routines changed from direct recognition, to recalling and identifying partial properties of parallelograms, when discussing the angles and diagonals of parallelograms.

When searching for routine patterns, I noticed that ATL's understanding of the names of parallelograms, and the names of the parts of the parallelograms, influenced her course of action in response to the questions related to them. In the following section, I present findings on the changes in ATL's use of mathematical words.

Analyzing ATL's use of words helped me to better understand her thinking about parallelograms, and about the relations among the angles, sides and diagonals of parallelograms. In this section, I present findings on ATL's use of mathematical words related to parallelograms. Recall that, a quadrilateral is defined as "a closed shape in a plane consisting of four line segments that do not cross each other" (Beckmann, 2008). Among all quadrilaterals, five types of quadrilaterals are found predominately in school geometry textbooks: *parallelograms, trapezoids, rectangles, squares, and rhombuses* (Usiskin, 2008). Interviewees in my study are also introduced to *kite* (Parker & Baldridge, 2008) in their course work, and therefore the word search on quadrilaterals includes also *kite*s. Tables 4.10 and 4.11 summarize the frequencies of the names of quadrilaterals mentioned by ATL at the Pre-Interview and the Post-Interview:

Table 4.10 The frequencies of ATL's use of the names of quadrilaterals at the interviews

Name	Frequency					
	Pre-T1	Pos-T1	Pre-T2	Pos-T2	Pre-T3	Pos-T3
Quadrilateral	2	10	0	0	0	1
Parallelogram	13	12	17	12	6	8
Rectangle	8	6	7	7	0	1
Square	12	14	9	4	6	0
Rhombus	5	7	1	6	0	0
Trapezoid	5	7	0	0	0	0
Kite	0	1	0	5	0	0

Table 4.11 Total frequencies of ATL's use of names of quadrilaterals at the interviews

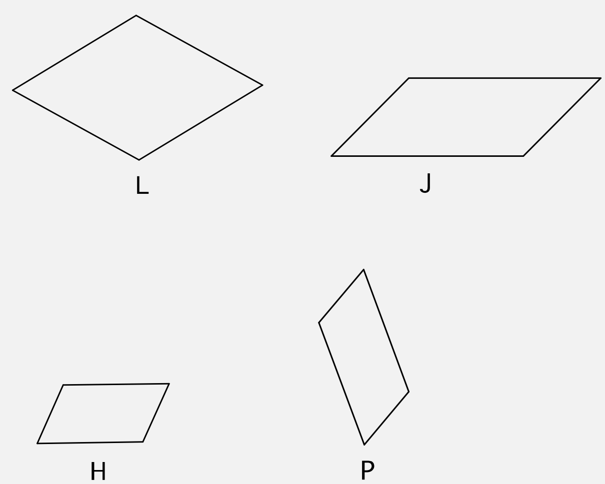
Name	Frequency	
	Pre	Post
Quadrilateral	2	11
Parallelogram	36	32
Rectangle	15	14
Square	27	18
Rhombus	6	13
Trapezoid	5	7
Kite	0	6

Table 4.11 shows that the word *parallelogram* (n=68) was the most frequently used word during the interviews and it was mentioned in all three tasks. The word *square* (n=45) was the second most frequently used, and the word *rectangle* (n=29) was third. In contrast, the word *kite* (n=6) is the least mentioned during the interviews, used only at the Post-Interview, in Task One (n=1) and Task Two (n=5). There was an increase in use of the word *quadrilateral* at the Post-Interview, and it was used mostly in Task One (n=10), and a total of eleven times in the entire interview. Also, there was an increase in use of the word *rhombus* at the Post-Interview (n=13). However, the frequencies of the words do not provide details about how and in what way those words were used. The following findings provide more information regarding ATL's word meaning in the use of *parallelogram*, *rectangle*, *square* and *rhombus*. Let me begin my analyses of ATL's word use with this conversation at the Pre-Interview:

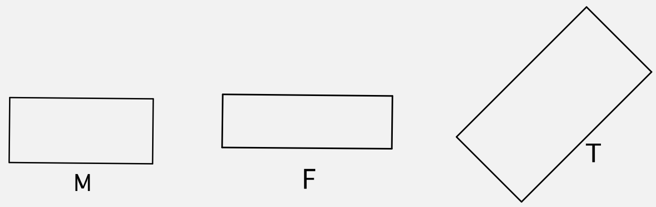
15. Interviewer	What is a parallelogram?
16. ATL	A parallelogram is when two sides of each side… all four are parallel to the opposite one…
17. Interviewer	What is a rectangle?
18. ATL	A rectangle is the two longer sides… the shorter ones … but in more technical terms, I am sure that they have congruency on both of those sides too
19. Interviewer	What is a square?
20. ATL	The square is all four of the sides are completely the same
21.Interviewer	What is a rhombus?
22. ATL	A rhombus… is a square… is just tilted [giggling]

ATL's narratives concerning *parallelogram*, *rectangle*, *square,* and *rhombus* are descriptive and visual at the Pre-Interview. ATL gave a descriptive narrative about *rectangles* based on physical appearance, "a rectangles is the two longer sides [and two] shorter ones… have congruency on both…sides". ATL made connections between *squares* and *rhombi* according to visual appearances, and declared narratives, "a rhombus is a square", because "they both have four equal sides", and "[it] is just a titled [square]". ATL's ways of defining parallelograms triggered the way she classified them. For example, when ATL was asked to identify all the parallelograms from a set of given figures, her response was as follows (see earlier analyses about the *routine of sorting*):

53. Interviewer	Can you identify all the parallelograms on this sheet? [Pointing to task One]
56a. ATL	Ok. [Marking stars on figures that are parallelograms]



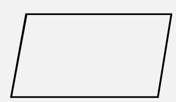
56b. ATL	…Now for these ones, these could be actually… be considered parallelograms. Based on the side measures … even though they are rectangles… they could be in the same category.



ATL’s classification was no coincidence. In ATL’s written narratives about the *rectangles*, she wrote, “rectangle is when 2 sides are differing from the other 2 sides, however, opposite sides are equal in length”, and for the *parallelograms*, she wrote, “parallelogram is when 2 parallel sides are congruent in length”. To ATL, the word *parallelogram* was a family name of figures having opposite sides that were parallel, and having two long sides and two short sides. For example, when I asked ATL to draw a parallelogram, she provided the following:

1. Interviewer Draw a parallelogram.

2. ATL [ATL Draw a figure looking like this]:



3. Interviewer How do you know this is a parallelogram?

4. ATL The opposite are equal in length… with the different sides parallel, they are the same length.

Next, I asked ATL to draw a new parallelogram different from the one she drew:

23. Interviewer Draw a new parallelogram that is different from the one you drew.

24a. ATL [ATL Drew a figure looking like this]:

24b. ATL	All I know is to change the size of it, but that's more of a rectangle…. But it's not a typical looking parallelogram…
	…….
30a. ATL	I feel it's a rectangle, but rectangles can still have the properties of a parallelogram… just a broad term for it.
33. Interviewer	Can you say a little more about why this parallelogram [rectangle] is different from this one [the parallelogram ATL drew earlier]
34. ATL	They aren't. Technically, they're probably not different, that one just looks more like a stereotypical parallelogram [Pointing on the parallelogram]. In terms of properties, there is nothing different.

ATL drew two parallelograms: "a stereotypical parallelogram" and a "not typical looking parallelogram". After ATL drew these parallelograms, I presented a picture of a square and a picture of a rhombus. ATL did not think a square and a rhombus were parallelograms because "to be a parallelogram, you have to have two long sides and two short ones, here all sides are equal and it is square". In the case of a rhombus, ATL responded, "this is similar to the square that you just showed me, … is a rhombus or just a square". From these conversations, it is evident that to ATL the word *parallelogram* signified two types of polygons, as summarized in Figure 4.7.

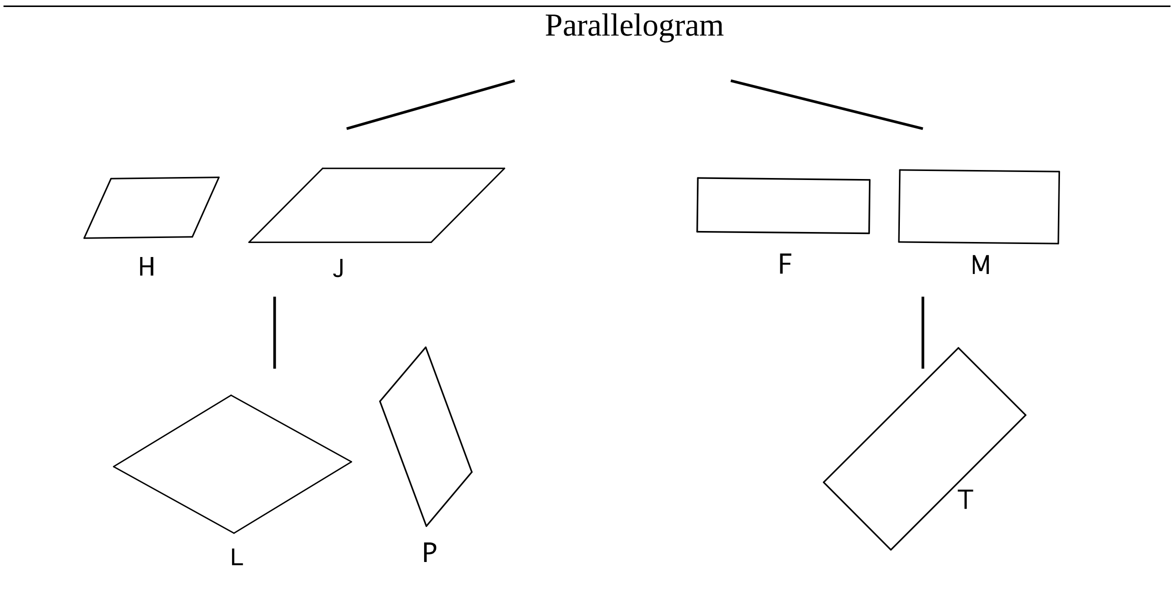
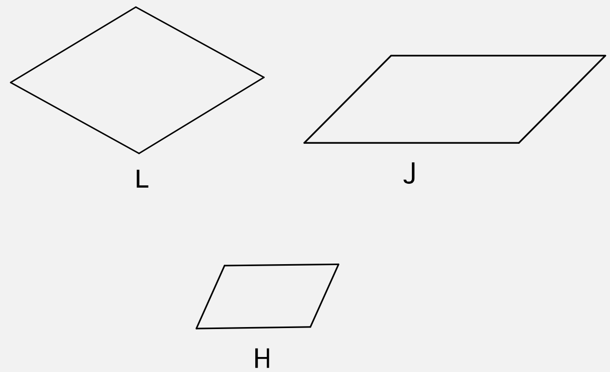


Figure 4.7 ATL's use of the word parallelogram at the Pre-Interview

ATL's use of the word *parallelogram* signified a collection of unstructured polygons by their family appearances. This family appearance included figures appearing to have opposite sides equal and parallel, and in particular, two opposite sides longer than the other two opposite sides. However there was no explicit mention of the necessary condition that these figures be 4-sided, nor of any condition on the angles in rectangles.

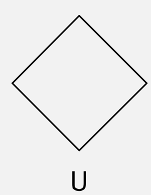
At the Post-Interview, when I asked ATL to identify all the parallelograms from eighteen polygons, her response was as follows:

19. Interviewer	What are the parallelograms here? [Pointing to Task One]
20. ATL	L and J and H will be just parallelograms, but all of these figures [pointing to figures that are squares, rectangles and rhombus] will be parallelograms, because…they all fit into the greater property of opposite angles and opposite sides to be equal.



ATL identified two groups of parallelograms: one group contained figures that were "just parallelograms", and the other group contained figures that "fit into the greater property of opposite angles and opposite sides to be equal" . As our conversation continued, ATL provided the following narratives about the parallelograms:

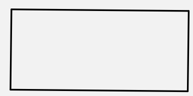
51. Interviewer	What is a square?
52. ATL	A Square is when all the angles form right angles and they are all the same they are all 90 degrees…and each side length also has to be the same. [Pointing at Fig. U]



53. Interviewer	What is a rectangle?

54. ATL — A Rectangle, each angle is 90 degrees but these sides are the same and parallel, and this one is the same and parallel, but not all 4 of them are the same, necessarily

[Pointing at Fig. M]

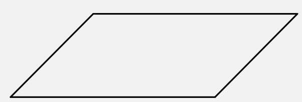


M

55. Interviewer — What is a parallelogram?

56. ATL — Um… a parallelogram is when opposite sides are equal and opposite angles are both equal…

[Pointing at Fig. J]

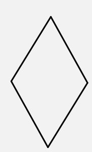


J

63. Interviewer — What is a rhombus?

64. ATL — Sides are all the same. Does not form 90-degree angle as rhombus alone.

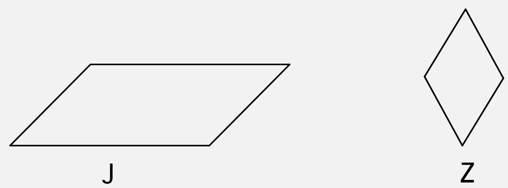
Pointing at Fig. Z:



Z

To better understand her word meaning in the context of parallelograms, I asked ATL if I could group Fig. J and Fig. Z together, and group Fig. U and Fig. M together. Her response was "yeah". The following conversation gives ATL's responses to these questions ten weeks later:

37. Interviewer: Can I group Fig. J and Fig. Z together?



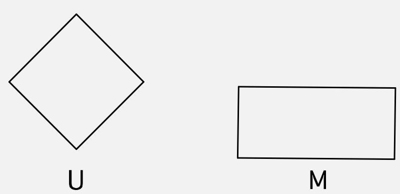
38. ATL: Mm Hmm.

39. Interviewer: Why is that?

40. ATL: Mm… because they both have opposite sides parallel and both opposite angle measures are equal.

……

45. Interviewer: Can I group Fig. U and Fig. M together?



46. ATL: Yeah, you can because U has the same property as M. The only differences is that M does not have all the same sides length, so M would not have all the properties as U, but U has all the properties of M…

In these conversations, more dimensions were added to ATL's use of the word *parallelogram*. At the Pre-Interview, the word *parallelogram* only signified polygons that fit into the physical appearances of parallelograms and rectangles; whereas at the Post-Interview, the word *parallelogram* signified a family of polygons that share common descriptive narratives.

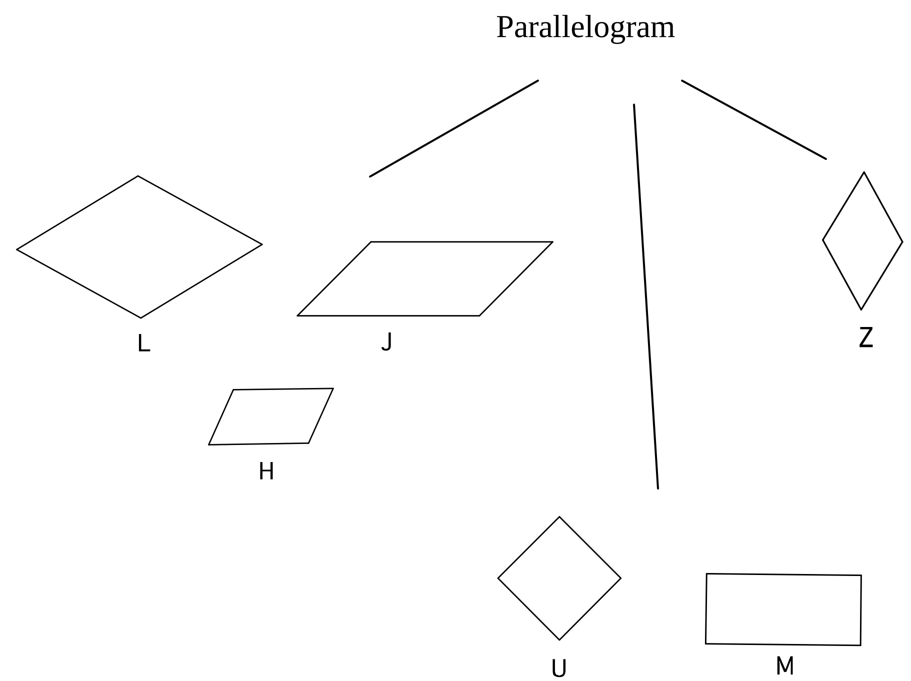


Figure 4.8 ATL's use of the word parallelogram at the Post-Interview

As shown in Figure 4.8, the word *parallelogram* signified to ATL a common family name for all figures that "have opposite sides parallel and opposite angles equal". This diagram illustrates how parallelograms were inter-connected. For example, ATL identified that "as a rhombus alone" [it] does not form a 90-degree angle, and "sides are all the same". A rhombus was different from a square with regard to the angles: "all the angles form right angles…and each side length also has to be the same". However ATL did not mention how rectangles were different from parallelograms.

To have a better understanding of ATL's familiarity of geometric terms, besides names for quadrilaterals, I searched for words describing the parts of parallelograms. Tables 4.12 and 4.13 present frequencies of the names of parts of parallelograms.

Table 4.12 The frequencies of ATL's use of the names of parts of parallelograms at the two interviews

Name	Frequency					
	Pre-T1	Pos-T1	Pre-T2	Pos-T2	Pre-T3	Pos-T3
Angle	10	17	10	75	14	39
Side	27	17	31	33	9	40
Length	2	6	3	21	11	3
Parallel side	0	3	1	1	0	0
Opposite side	1	6	9	17	3	7
Diagonal	0	1	27	20	8	10
Right angle	2	1	0	2	0	0
Opposite angle	2	5	0	13	2	0

Table 4.13 Total frequencies of ATL's use of names of parts of parallelograms at the two interviews

Name	Frequency	
	Pre	Post
Angle	34	131
Side	67	90
Length	16	30
Parallel side	1	4
Opposite side	13	30
Diagonal	35	31
Right angle	2	3
Opposite angle	4	18

Table 4.13 shows that the most frequently used words relating to the parts of parallelograms were *angle* (n=165) and *side* (n=157). There was a huge increase in use of the word *angle* at the Post-Interview (n=131) over the Pre-Interview (n=34). Besides *angle* and *side*, *diagonal* was also frequently mentioned at both interviews (n= 66). However *parallel sides* (n=5), as one of the most important characteristics of a parallelogram, was the least mentioned. There was also an increase in use of *opposite side* (n=30) and *opposite angle* (n=18) at the Post-Interview.

These changes of word use associated with parts of parallelograms were not incidental. Recall that, at the Pre-Interview, ATL showed frustration in discussing the angles without measurements, and consequently, her responses about parallelograms and parts of parallelograms focused on the *sides* (n =67) rather than *angles* (n=34). At the Post-Interview, she was more comfortable talking about angles of the different parallelograms and was able to compare the differences between angles of the parallelograms (e.g. rectangle and parallelogram, square and rhombus, etc). Moreover, ATL used a narrative, “they have opposite sides and opposite angles are equal” to classify quadrilaterals into a group of parallelograms. These changes in ATL’s *word use* and *routines* show that she gained more familiarity with the triangles and quadrilaterals, as well as with the properties of these polygons.

Case 2: Changes in ANI’s Geometric Discourse

ANI was a college sophomore at the time of the interviews, having taken her last geometry class six years prior to the geometry and measurement class. The van Hiele geometry Tests showed that she was at Level 0 at the pretest, and moved to Level 3 at the posttest. I interviewed ANI after both tests, and analyzed her interview responses. ANI’s changes in geometric discourse are summarized as follows:

- ANI’s routines of sorting changed from grouping by the names of polygons, according to their family appearances and visual properties with no order, to classifying polygons according to their common descriptive narratives, and structuring quadrilaterals with a hierarchy of classification.
- ANI’s routines of substantiation changed from visual recognition and recalling at the Pre-Interview, to using endorsed narratives such as definitions and properties

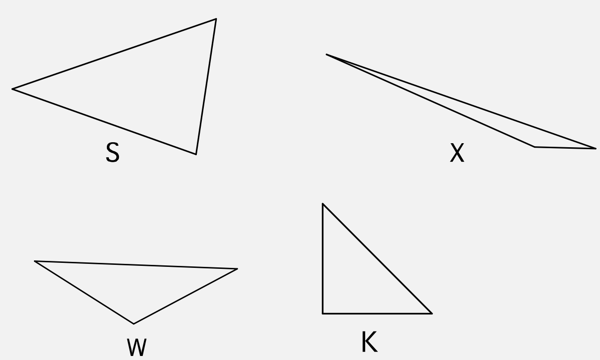
of parallelograms at the Post-Interview; and from comparing parts of parallelograms visually in the Pre-Interview, to applying various methods (e.g., Pythagorean theorem, congruence criterion, algebraic derivations) to verify claims at an object level in the Post-Interview.

- ANI's use of the names of parallelograms changed from visual recognition of their family appearances at the Pre-Interview, to using these names as collections of quadrilaterals sharing common descriptive narratives in a hierarchy of classification.

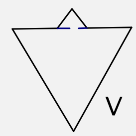
ANI's routine procedures for sorting polygons were observed and analyzed in Task One. She was asked to sort eighteen polygons into groups, consisting of triangles (n=4), quadrilaterals (n=13) and a hexagon (n=1).

During the Pre-Interview, ANI's first sorted polygons by their names based on family appearances, finding *triangles* (n=5), *squares* (n=3), *rectangles* (n=3), *parallelograms* (n=5) and *trapezoids* (n=2). She grouped Fig. V, a hexagon, with the triangles because she identified two triangles in Fig. V.

2b. ANI I made a group of triangles, which had three sides.



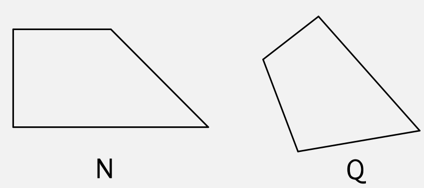
2c. ANI	I included V as two triangles, because if you draw a line here in the middle, it would make two.



drew a dashed line

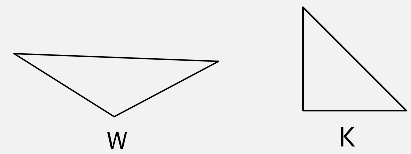
ANI put Fig. N, a right trapezoid, and Fig. Q, a quadrilateral, together as a trapezoid group because she was not sure about what to do with these two polygons. She named the group *trapezoid* by guessing.

2f. ANI	I did one that just had Q and N, which I didn't know what to classify. I guess they would be trapezoids.

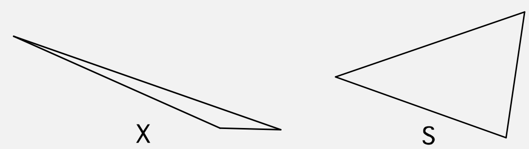


When asked for regrouping, ANI combined rectangles and squares together in one group because she thought that "every square is a rectangle". She then split the triangle group into right triangles and isosceles triangles, but did not know what to do with two other triangles.

16. ANI	That's a right triangle [pointing at Fig. K], and that's an isosceles triangle [pointing at Fig. W].

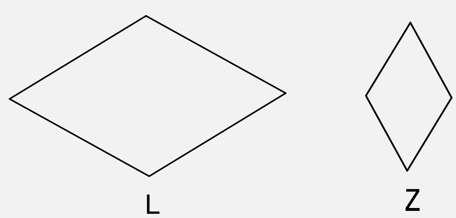


17. ANI I guess X would, I don’t know what the definition is [pointing at Fig. X], and this one [pointing at Fig. S] none of the sides on that looked even to me, so I don’t think it looked like any of the others

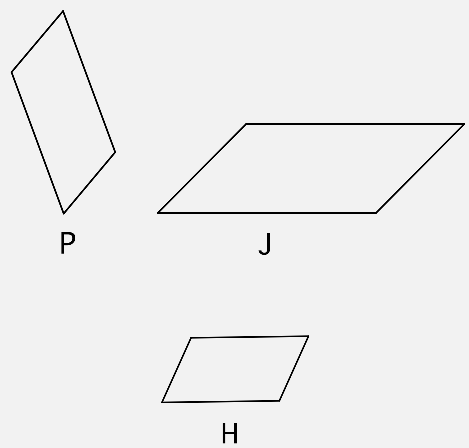


ANI grouped two triangles according to the visual properties of angles (i.e., right triangle) and sides (i.e, isosceles triangle), and left two other triangles (Fig. X and Fig. S) with no groups. When subgrouping the parallelograms (n=5), ANI explained as follows:

22a. ANI L and Z look more like square, I don't know if there are two different types of parallelograms…



22b. ANI	Where P, J, and H look more of a rectangle, like the opposite side of all of these look the same. So, I guess you could put it into two groups that way.



ANI split the parallelograms into a group of squares including Fig. L and Fig. Z, and a group of rectangles consisting of Fig. P, Fig. J and Fig. H, because they looked like squares and rectangles, respectively. When I asked for the definitions of *parallelogram*, *rectangle*, *rhombus* and *square,* ANI provided her definitions of each:

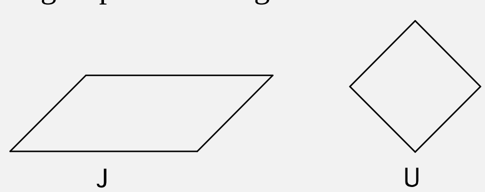
23. Interviewer	What is a parallelogram?
24. ANI	It has four sides. I think the opposite sides are the same… I guess it's kind of like a slanted rectangle.
25. Interviewer	What is a rectangle?
26. ANI	It has four sides with opposite sides being equal in length.

ANI made a connection between a parallelogram and a rectangle, as a parallelogram was a slanted rectangle, based on visual appearances. ANI next gave definitions of square and rhombus:

29. Interviewer	What is a square?
30. ANI	It has four sides of the same length.
31. Interviewer	What is a rhombus?
32. ANI	A rhombus is like a slanted square and a parallelogram is a slanted rectangle. They kind go together like that.

ANI described a square as having four sides of the same length, whereas a rhombus was a slanted square. She did not mention right angles, a defining condition of rectangles and squares among the parallelograms. To find out whether ANI considered squares as parallelograms, I continued:

51. Interviewer	Can I group J and U together?

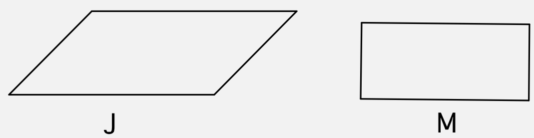


52. ANI	[Thinking] …You could if you talked about angles, I guess. You'd say this angle and this angle are equal [pointing at the opposite angles of Fig, J], and this angle and this angle are equal, where the opposite angles are

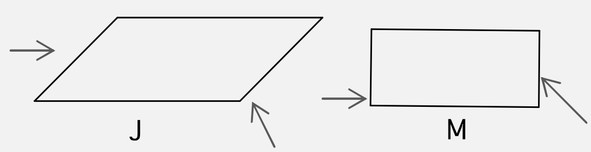
equal here [Fig. U]. So, I guess you could group it in that way.

ANI agreed that Fig. J, a parallelogram, could group with Fig. U, a square, because opposite angles were equal in both polygons [52]. Later I found that ANI did not identify a square as a parallelogram because "a square has all the same length, and a parallelogram has different sides". In this case, ANI identified polygons by visual property of their angles. However, ANI's response was different when I asked if I could group Fig. J and Fig. M, a rectangle, together:

53. Interviewer Can I group J and M together



54. ANI Yeah. Like J and U, opposite angles are equal… you also have opposite sides are parallel [pointing at the sides of Fig. J]…and this side and this side are both shorter than the other ones [pointing at the sides of Fig. M], they [Fig. J and Fig. M] have quite a bit in common.



ANI acted more positive towards the grouping of Fig. J and Fig. M, as she thought, "they have quite a bit in common" [54]. ANI explained that a rectangle was a parallelogram "because it has two opposite sides equal, a rectangle and a parallelogram go together". In this case, ANI identified polygons by visual property of their sides. ANI did not use any measurement tool to check the parts of polygons during the Pre-Interview in Task One. Her routines of regrouping polygons included direct recognition and identification of polygons by visual properties of angles and sides. ANI's routines of sorting polygons at the Pre-Interview are summarized in Figure 4.9.

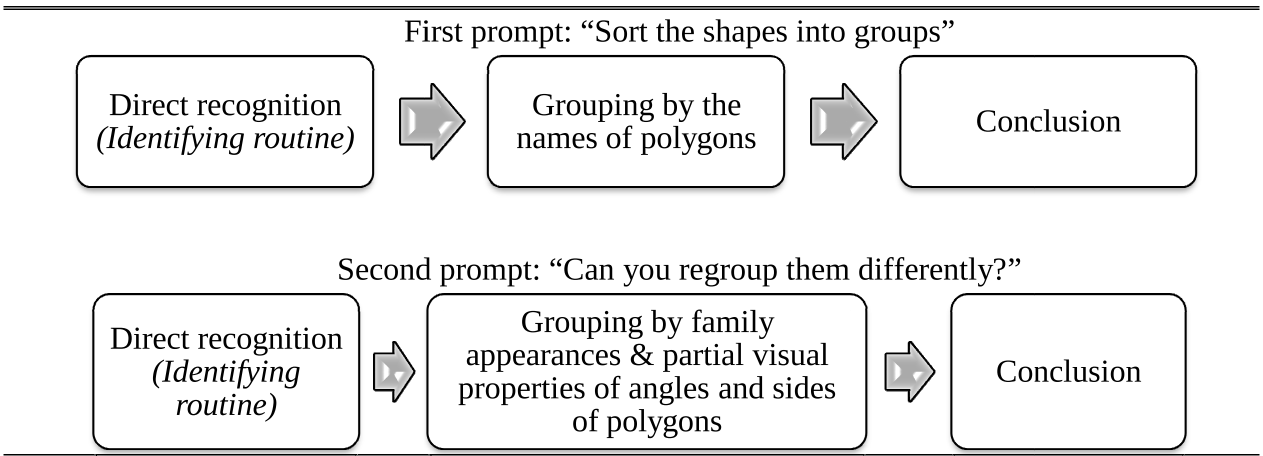


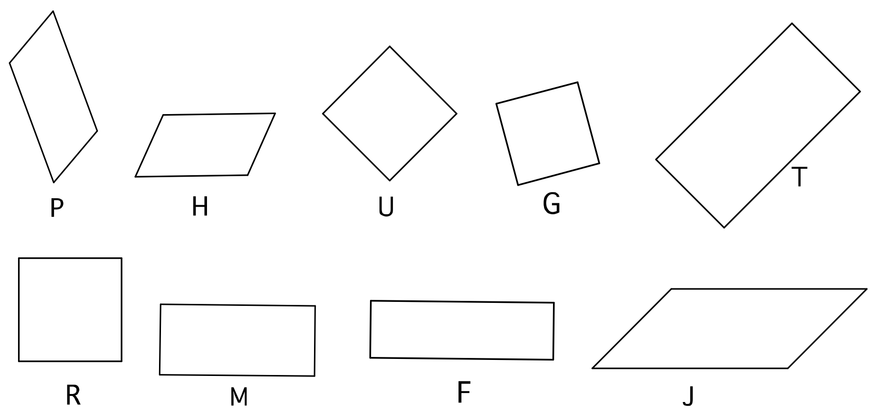
Figure 4.9 ANI's routines of sorting polygons at the Pre-Interview

In ANI's routines of sorting triangles and quadrilaterals at the Pre-Interview, I did not find a defining routine because she did not use definitions to group polygons. I found identifying routines, including direct recognition, when she grouped quadrilaterals by their visual appearances, and identifying visual properties when she identified polygons by the attributes of their angles and sides.

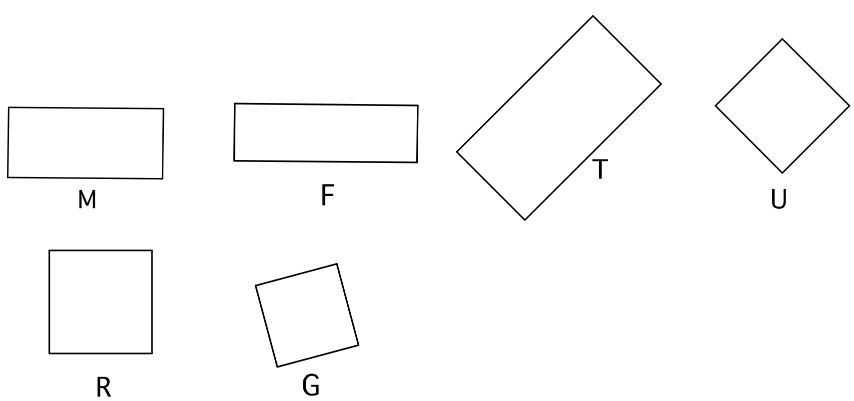
Ten weeks later when I interviewed ANI again, her routines of sorting polygons had changed from grouping polygons by their visual appearance, to classifying them by common descriptive narratives with a hierarchy of classifications.

At the Post-Interview, ANI first grouped polygons by their names, finding *quadrilaterals* (n=13), *trapezoids* (n=1), *parallelograms* (n=9), *rectangles* (n=6), *rhombi* (n=5) and *triangles* (n=4). ANI grouped all 4-sided figures into the *quadrilaterals* group, and all 3-sided figures into the *triangles* group. She included parallelograms, squares, and rectangles as *parallelograms*, but not the rhombi. The *rectangles* group consisted of rectangles and squares, and the rhombi group included squares and rhombi. Note that Fig. V, a hexagon, was not included in any of these groups. Figure 4.10 presents ANI's groups of *parallelograms, rectangles* and *rhombi.*

Parallelograms group



Rectangles group



Rhombi group

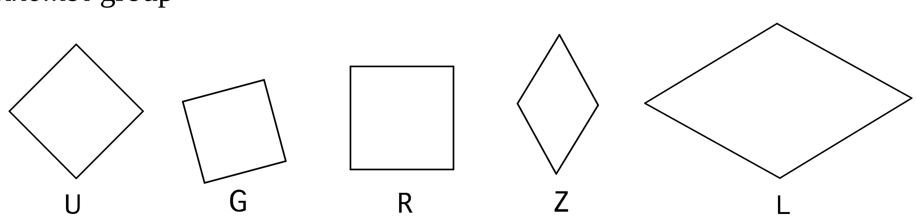


Figure 4.10 ANI's grouping of parallelograms at the Post-Interview.

As shown in Figure 4.10, ANI grouped polygons by their common descriptive narratives (i.e., definitions). For example, ANI explained that she grouped squares and rectangles together as a *rectangles* group because “all squares are rectangles, and rectangles have all 90-degree angles”. She verified that both squares and rhombi were rhombi because “that is what a rhombus is, four sides of equal length”.

There was a change in ANI’s identifying routine. Recall that at the Pre-Interview ANI grouped rhombi and squares together, because a rhombus looked like a slanted square. But at the Post-Interview, she grouped them together because they share a common narrative of having four sides of the same length. Although ANI grouped rectangles and squares together at both interviews, there was a difference as her identifying routine changed from recalling that all squares were rectangles at the Pre-Interview, to identifying common properties of rectangles and squares at the Post-Interview.

ANI’s identifying routine also changed from grouping the quadrilaterals by their names as unstructured polygons, to classifying the quadrilaterals with a hierarchy. Figure 4.11 illustrates all of ANI’s the subgroups of quadrilaterals (n=13). ANI classified the quadrilaterals beginning with the attributes of their sides. This classification included *trapezoid*, a quadrilateral with one pair of parallel sides; *parallelograms*, quadrilaterals with two sets of parallel sides; rhombus, quadrilaterals with all sides equal; and Fig. Q, a quadrilateral with no two sides equal. ANI next split parallelograms into parallelograms and rectangles, and split the rhombus group into rhombi and squares by the characteristics of right angles. ANI did not use rulers or protractors to check measurements of angles and sides at the Post-Interview, but she did explain that sides looked like they were parallel, or angles looked like they were right angles, etc. I conclude that ANI used direct recognition as an identifying routine, but it as not self-evident. ANI's routines of sorting polygons at the Post-Interview are summarized in Figure 4.12.

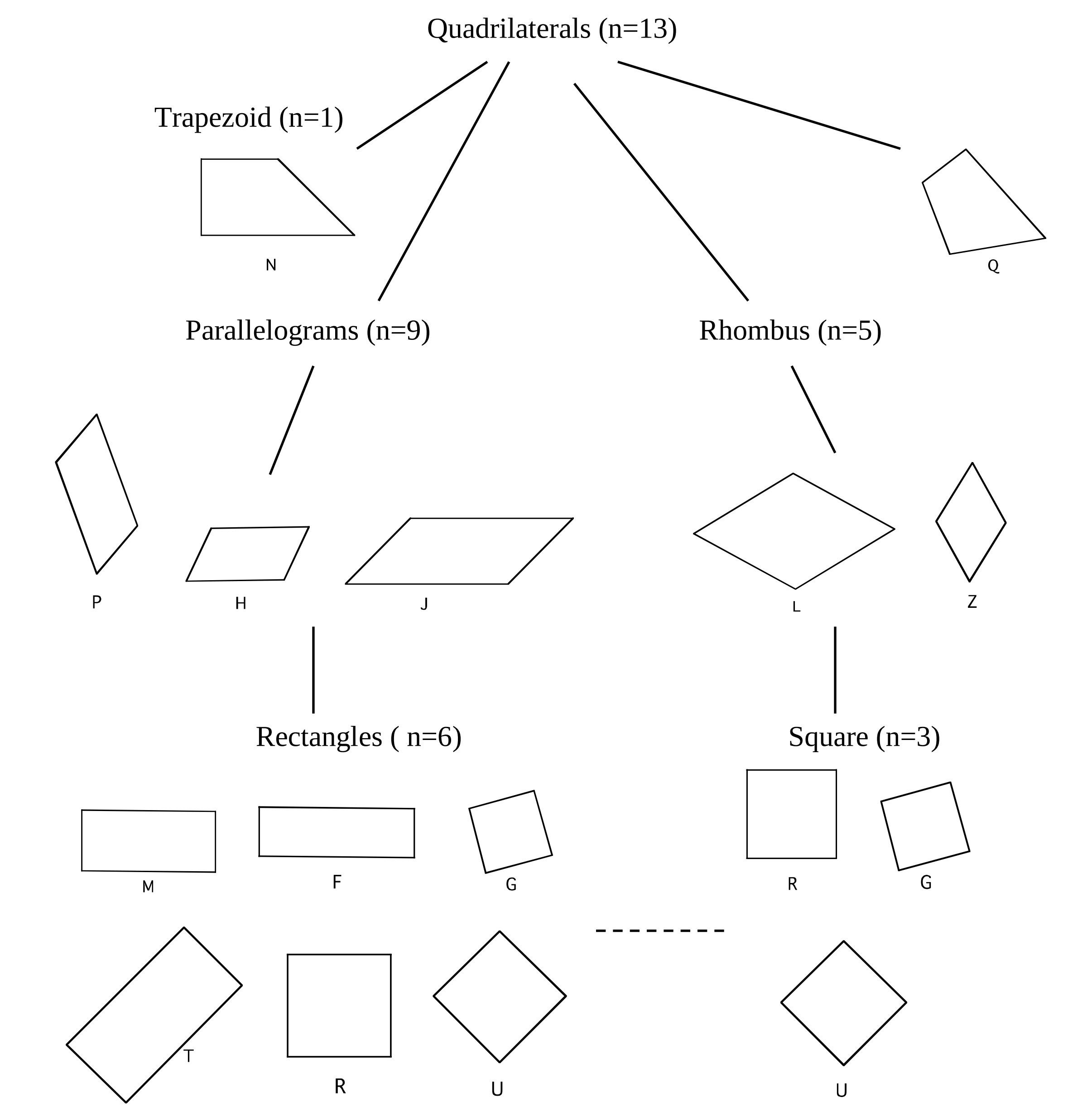


Figure 4.11 ANI's grouping of the quadrilaterals at the Post-Interview

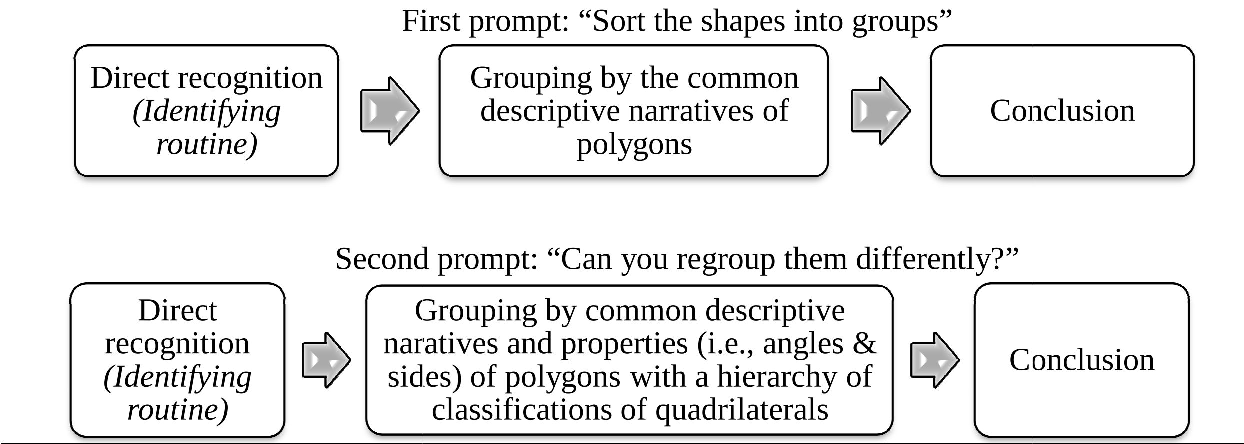
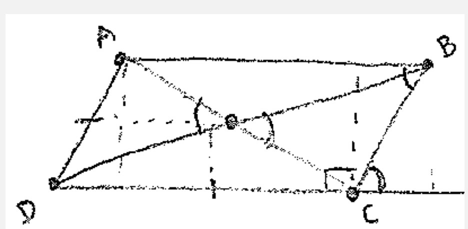


Figure 4.12 ANI's routines of sorting polygons at the Post-Interview

ANI's routines of substantiation for Task Two were analyzed and compared from both interviews. I found that ANI's routines changed from direct recognition and recalling in the Pre-Interview, to identifying properties of polygons and using triangle congruence criterion to substantiate her claims. In the Post-Interview, she also derived some statements algebraically, and used Pythagorean theorem.

In Task Two, ANI was asked to draw two different parallelograms and to discuss their properties. At the Pre-Interview she drew a parallelogram and labeled the vertices as A, B, C and D in clockwise order. She then wrote a statement regarding the angles of the parallelogram, "$\angle A = \angle C$, $\angle B = \angle D$". The following conversation took place when I asked for verification.

3. Interviewer What can you say about the angles of this parallelogram?

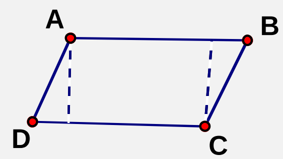


ANI's drawing of a parallelogram

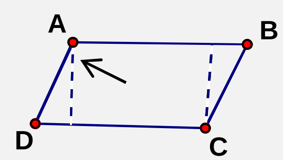
4. ANI	Angles A and C are equal, and then B and D are equal [Pointing to angles A and C, she wrote: ∠A = ∠C, ∠B = ∠D].
5. Interviewer	How do you know?
6. ANI	Mm…I just remember being taught that, I don't actually know… If you measured them, they would be equal.

ANI referred to her prior knowledge about the angles of a parallelogram to conclude that the opposite angles were equal. She was able to use mathematical symbols, such as ∠A = ∠C, to indicate the equivalence of the angles. However ANI remembered the property as a fact without knowing the explanations. After my prompt, ANI verified that ∠A = ∠C by comparing the space between the angles:

10a. ANI If you drew two lines here [adding two perpendicular lines from angles A and C]

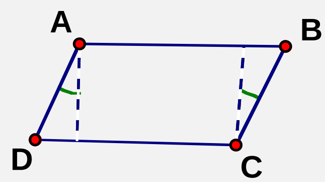


10b. ANI From this line to this line, if you know that's a 90 degree angle …[pointing at the indicated space]



10c. ANI You know what a 90-degree angle looks like, so it's easier to go off of that.

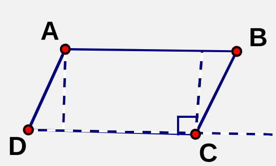
10d. ANI	…and then you look at the space here and the space here [pointing at the marked angles], and see if those are the same.

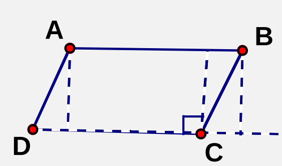


ANI drew two perpendicular lines so that both ∠A and ∠C would be the sum of a right angle and a smaller angle. [10a]. She started with comparing the right angles because they were easy to distinguish by their visual appearance [10b; 10c], and then she compared *space* between the two smaller angles to check whether they were the same. In this example, ANI's routines procedure relied on the visual appearance of the angles to verify her claims.

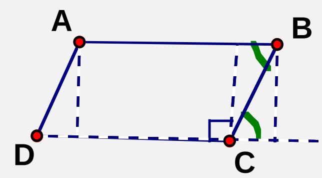
ANI made two other statements about the angles of the parallelogram, "∠A + ∠D = 180°, ∠C + ∠B = 180°". When asked for substantiation, she responded as follows:

20a. ANI	If you extend this line out, and draw another straight line there…



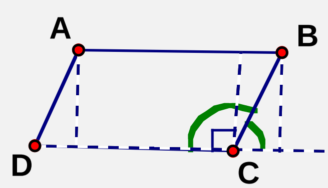


20b. ANI	You can see that this angle equals that angle.



ANI pointing at the two marked angles.

21. Interviewer	How do you know they are equal?
22a. ANI	There is a term for it. It's like a rule that I remember learning. Maybe parallel angle rule? Or, adjacent angle rule? Or something.
22b. ANI	And this is angle C [pointing at the angle BCD].
22c. ANI	So if you were to combine them, you would have a straight line, and that would make a 180-dregee angle.



ANI pointing at the two marked angles.

In this example, ANI recognized the structure of alternating interior angles formed by parallel lines and their transversals [20a]. When ANI explained, "you can *see* that this angle equals that angle" [20b], she again relied on the visual appearance of the angles. She did not know names of the angles, nor the related propositions to support her claim, but referred to a rule that she had learned. Assuming that the two marked angles were equal [20b], she verified that $\angle C + \angle B = 180°$ because they made a 180-degree angle [22c]. Using the same reasoning, ANI also stated that “∠D + ∠A = 180°”. I conclude that ANI used visual recognition to compare angles, and applied prior knowledge as a fact to verify the statements about the angles of a parallelogram.

In the Post-Interview, ANI drew a parallelogram and labeled the vertices as A, B, C, and D in a clockwise rotation, and gave the same statements about the angles of the parallelogram as in the Pre-Interview. Figure 4.13 shows some of ANI responses for Task Two in the Post-Interview.

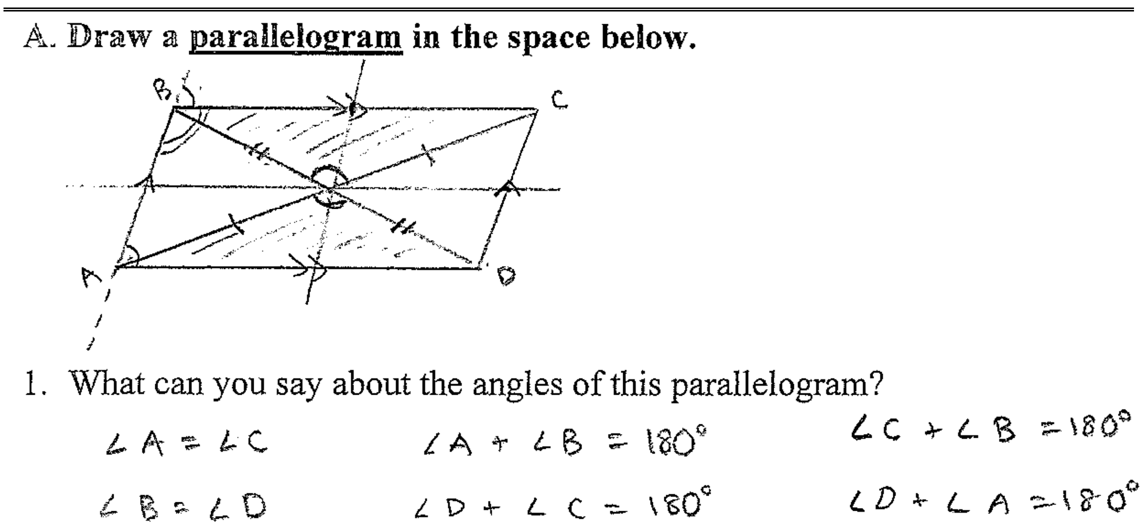
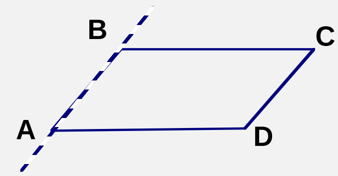


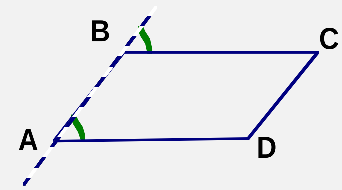
Figure 4.13 ANI’s written responses for Task Two at the Post-Interview

In contrast, ANI’s verification of her claims was different than at the Pre-Interview. When I asked for substantiation of the statement ∠A = ∠C and ∠B = ∠D, she explained that “this is a parallelogram because I drew it, so angle A is equal to angle C and angle B is equal to angle D”. ANI’s verification of ∠A + ∠B = 180° was also different than in the Pre-Interview.

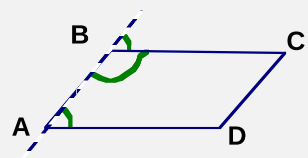
16a. ANI	If you were to extend this line…



16b. ANI You could look either way, like this angle is equal to this angle. BC and AD are parallel. They [pointing at the marked angles] are corresponding angles because they are on the parallel lines.



16c. ANI Then you could tell that if you add these two angles, it's angles on a line. So it's 180 degrees.



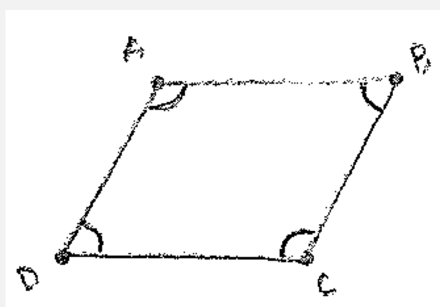
pointing at angle B and it's exterior angle.

16d. ANI So angle A and angle B add up to 180 degrees.

To verify $\angle A + \angle B = 180°$, ANI extended side AB so that the structure of the corresponding angles formed by parallel lines and their transversals was visible [16a]. She mentioned the corresponding angles were congruent *because* they were on the parallel lines [16b], and then concluded that angle A and angle B add up to 180 degrees [16c; 16d]. Although ANI verified her claim informally, it is important to see the change, as ANI justified her claim that corresponding angles were equal using an endorsed narrative that BC and AD were parallel at the Post-Interview, whereas she relied on visual appearance of the angles at the Pre-Interview.

ANI provided different narratives about the diagonals of a parallelogram at the two interviews. At the Pre-Interview, she asserted that the diagonals of a rhombus should be the same and the diagonals of a rectangle should intersect at the middle of the rectangle, whereas at the Post-Interview she stated that the diagonals of a rectangle should be equal in length, and the diagonals of a parallelogram bisect each other at the Post-Interview. Now I describe and compare the changes in ANI's routine procedures of substantiating these narratives at both interviews.

At the Pre-Interview, ANI drew a rhombus, and stated that it was a different parallelogram because all the sides were the same length. ANI later used the statement, "all the sides were the same length" to draw conclusions about the angles and diagonals of the rhombus.

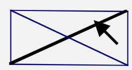
73. Interviewer	What can you say about the angles of this parallelogram
	
	ANI's drawing of a rhombus
74. ANI	I think they [angles] should all be the same… I guess if I had drawn it better, all the angles should be the same.
75. Interviewer	Why do you think they are the same?

76. ANI	Because the lengths are the same. [Wrote: $\angle A = \angle B = \angle C = \angle D$ because lengths of the sides are the same].
	…
93. Interviewer	What can you say about the diagonals of this parallelogram?
94. ANI	I think the diagonals should be equal in length.
95. Interviewer	How do you know that they should be equal in length?
96. ANI	Because the sides are all the same length and the angles are all the same. So, I think the diagonals should be the same.

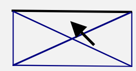
In the preceding conversation, ANI claimed that the angles in a rhombus should *all* be the same [74], as well as the diagonals [94]. When asked for verification, she explained that all the angles should be the same *because all the sides of a rhombus were the same* [76]; and diagonals should be equal in length *because all the sides and angles of a rhombus were all the same* [96]. Of course, for ANI's conclusions about the angles and diagonals of this rhombus to be correct, the rhombus had to be a square. ANI made incorrect implications from the equivalence of the sides to the equivalence of the angles, and then suggested that the diagonals must be equal because of equal sides and angles. There is no routine involved in this verification, other than making statements based on the fact that all the sides are equal in a rhombus. Obviously, ANI did not have a correct understanding of a rhombus at the Pre-Interview.

ANI stated that the diagonals of a rectangle were longer than its sides, and the intersection of the diagonals was at the middle of the rectangle.

118b. ANI The length of this [pointing at the diagonal] is longer than the length of the longest side [pointing at the longer side of the rectangle].

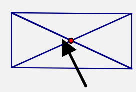


diagonal



longest side of rectangle

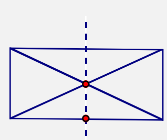
118c. ANI They [the diagonals] should intersect in the middle.



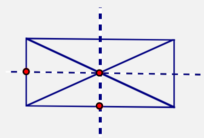
middle of the rectangle

Here, ANI's declared narratives about the diagonals of this rectangle were more like visual descriptions of what the diagonals appeared to be. She recognized the diagonal as the hypotenuse of a right triangle, and mentioned, "the Pythagorean theorem, which is how I know it" to verify the diagonals were longer than the longest side of rectangle. She argued her claim that diagonals should be at the middle of the rectangle as follows:

130a. ANI So, if you were to find the midpoint of this length… [Drew one line passing through the midpoint of the sides]



130b. ANI	…and then if you were to find the midpoint of this length, and draw a line [Drew another line passing through the midpoint of the other sides]



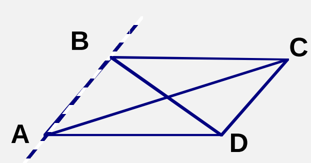
130c. ANI	… that the intersection of those two lines should be the intersection of the diagonals as well.

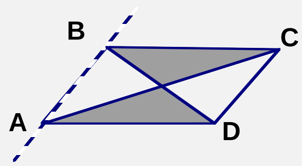
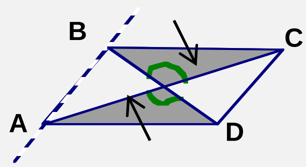
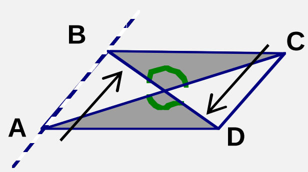
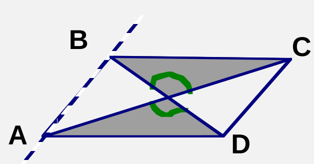
ANI verified that the diagonals intersect at the middle of the rectangle by locating the midpoints of the sides of the rectangle, and concluded that the intersection of the two medians was the same point as the intersection point of the diagonals.

In the preceding examples, ANI's understanding of the properties of parallelograms was not clearly demonstrated. To verify the claims, she mostly described what she saw about the parallelogram.

Ten weeks later I interviewed ANI again, and the same tasks were preformed. ANI first drew a parallelogram, and then stated that the diagonals of the parallelogram were not equal in length, but they cross each other at one point. She added that the diagonals create corresponding triangles.

54. ANI	They [the diagonals] cross at one point.



56. ANI	They create corresponding triangles. Well, like this triangle corresponds with this triangle [Shaded the two corresponding triangles]
	
57. Interviewer	What do you mean by “corresponding triangles”?
58a. ANI	This angle and this angle are equal, cause they're vertical angles…
	
58b. ANI	…And then, this side should equal this side…
	
58c. ANI	…and this side should equal this side. And I know they're corresponding through Side-Angle-Side. Like, that would be the rule that…
	

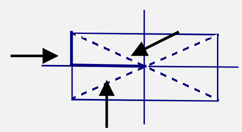
59. Interviewer	How do you know these sides are equal? [Referring to 58b & 58c]
60. ANI	Cause diagonals bisect each other.
61. Interviewer	How do you know they bisect each other?
62. ANI	I don't really know how I know… I guess it's because the sides are equal length and they're parallel, so…

In the preceding conversation ANI started with a descriptive narrative about the diagonals of a parallelogram, “they cross at one point” [54], and then she asserted that the diagonals created corresponding triangles [56]. At my request, she verified the corresponding triangles were a pair of congruent triangles with the Side-Angle-Side (SAS) criterion [58]. Here, ANI used the endorsed narrative “diagonals bisect each other” to show that the corresponding triangles were congruent [60]. However, when asked how she knew the diagonals of this parallelogram bisect each other, she responded, “I don’t really know…I guess, it’s because the sides are equal length and they’re parallel” [62].

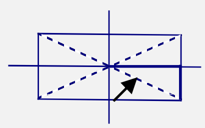
It is clear that ANI remembered how to verify congruent triangles using SAS. She identified corresponding triangles, as well as the three elements needed for verification of congruent triangles. She used that fact that “diagonals bisect each other” to justify the congruency of the sides, and used vertical angles to show the congruence of included angles. Thus, there was a change in ANI’s routine of substantiating, from no routine at the Pre-Interview, to using an endorsed narrative to identify three elements for verifying congruent triangles at the Post-Interview. In this case, ANI’s routine procedure for verifying her claim used the endorsed narrative “diagonals bisect each other”, but she did not know as an endorsed narrative, but did not know why the narrative was true. It appeared that ANI derived an informal proof that two triangles were congruent, but she did not clearly demonstrate that she knew what to substantiate and why.

When discussing the diagonals of a rectangle, ANI provided the narrative “they should be equal in length”, and tried to verify this claim using the Pythagorean theorem:

128a. ANI I know the value of this side [pointing at the shorter side], and I know the value of that side [pointing at the longer side], so I can find this side [hypotenuse] use Pythagorean theorem.

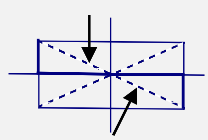


128b. ANI It would be the same over here. I know this and I know this, so I can find this.

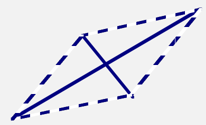


”I can find this”

128c. ANI …you can see that those two are equal [pointing at the halves of the diagonal]. And then, if I drew the other diagonal, you could do it the same way…

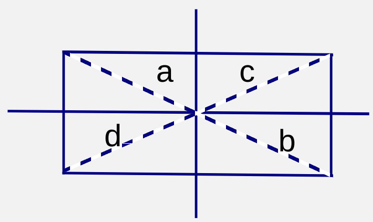


129. Interviewer	What about this case? If I add the halves of this diagonal, and add the halves of that diagonal, in my drawing here, they are not equal.

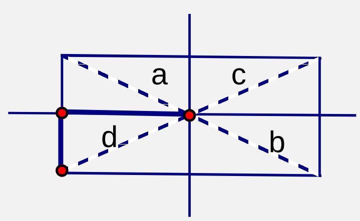


To verify the diagonals were equal, ANI attempted to show that adding the halves of a diagonal was equal to adding halves of another diagonal, and use the Pythagorean theorem. I provided a counterexample to refute her conclusion. ANI replied with another approach by first labeling the segments *a*, *b*, *c* and *d*.

134a. ANI	You could find that *a* is equal to *d*, they should be equal.

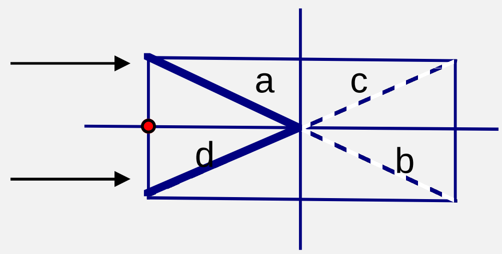


134b. ANI	I could do the Pythagorean theorem again, but with this side and this side. And then I'd find *d*.



Pointing at the two legs of the right triangle

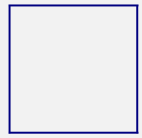
134c. ANI	*a* equal to *d* because they share this side here [pointing at the longer leg], and this point is a middle point here, so these two sides are equal [two shorter legs]. They [*a* and *d*] are equal.



ANI described informally her verification of the claim "*a* = *d*" by identifying the two triangles sharing a longer side and having equal shorter sides [134c], and then applying the Pythagorean theorem to conclude "*a* = *d*" [134a; b]. ANI did not argue that the two triangles were right triangles, an important condition of the Pythagorean theorem, nor did she give details of the algebraic derivation of "*a* = *d*". I conclude that ANI's routine procedures changed from comparing the length of diagonals with the sides visually at the Pre-Interview, to describing a process of verifying and substantiating her claims at the Post-Interview.

ANI did not think a square was a parallelogram at the Pre-interview, and there was not much to compare with what she did ten weeks later. However I do want to share ANI's routine procedure of using algebra as one way to substantiate her claims at the Post-Interview. We began with the following conversation:

161. Interviewer	What can you say about the diagonals of the square?



162. ANI	They're equal… They bisect the angles, split the angles into two-45 degree angles.
163. Interviewer	How do you know "they are equal"?
164. ANI	The same way I knew with the rectangles.
165. Interviewer	How do you know "they bisect angles"?
166. ANI	It divides the angle into two equal angles.

ANI provided two narratives about the diagonals of the square, "they're equal", and "they bisect the angles"[162]. She applied her knowledge of the diagonals in a rectangle to case of a square [164]. To verify the diagonals bisect each other, ANI explained that they divide the angle into two equal angles [166]. The following is ANI's routine procedure of verification, with corresponding transcripts.

Table 4.14 ANI's routine procedure of verification for "diagonals bisect the angles"

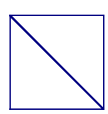
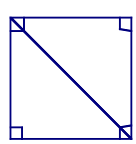
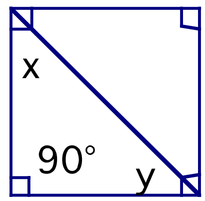
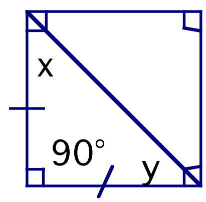
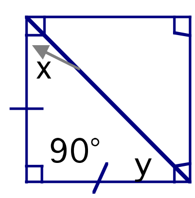
Routine Procedures	Transcripts
1. Declare narratives	
1.1Draw a diagonal	174a. I guess I'd draw a diagonal 
1.2 Identify two right triangles	174b. It splits the square into two right triangles, because all of these angles are 90-degrees.  *adding right angle sign on each angle of the square*

Table 4.14 (cont'd)

1.3 Identify the relation between the angles and sides of the right triangle.	174c. By the angle sum rule, all angles add up to 180 degrees. You already have 90 here. So, X plus Y has to equal 90. It's also an isosceles triangle.  *assigning X and Y to the two angles*
Q: How do you know it's an isosceles triangle?	
2. Verification of isosceles triangle 2.1 Identify congruent sides of the triangle	180. These two sides equal.  *Adding two marks on the sides of the triangle*
2.2 Verification of congruent angles	182a. It's an isosceles triangle. So X is equal Y.
2.3 Finding the angle measures of X and Y	182b. I know that X and Y has to equal 90 degrees. So, I know that X is 45 degrees and Y is 45 degrees.
2.4 Finding other angles measures	190. So, if you know it's 90, and Y is equal 45 degrees, and this angle is also 45 degrees. Same for X here. So diagonals splitting into two equal angles and they are 45 degrees each.  *"this angle is also 45 degrees"*
3. Conclusion	194. Yeah, diagonals bisect each other

It became clear that ANI favored algebraic reasoning in her routine procedures. She labeled the angles X and Y, and used an endorsed narrative, "all angles are 90 degrees" [174b] to justify that X and Y were the angles of a right triangle. She used another endorsed narrative, “theses two sides equal” [180] to verify that the triangle is isosceles. Finally, ANI solved X and Y algebraically, to find that they were 45 degrees each [190]. Using this newly endorsed narrative, ANI concluded that the diagonals bisect the angles [190; 194]. In this example, ANI used her knowledge in algebra to help solve a problem in geometry.

ANI’s use of the word *parallelogram* changed from describing the visual appearances of the quadrilaterals at the Pre-Interview, to using the word as a common descriptive narrative with a hierarchy of classifications at the Post-Interview. Here are the frequencies of ANI’s use of the names of quadrilaterals at two interviews:

Table 4.15 The frequencies of ANI’s use of the names of quadrilaterals at the two interviews

Name	Frequency					
	Pre-T1	Pos-T1	Pre-T2	Pos-T2	Pre-T3	Pos-T3
Quadrilateral	0	3	0	0	0	0
Parallelogram	4	3	4	0	3	1
Rectangle	12	3	2	2	3	0
Square	10	5	1	5	0	3
Rhombus	2	2	1	3	0	0
Trapezoid	2	2	0	0	0	0
Kite	0	1	0	0	0	0

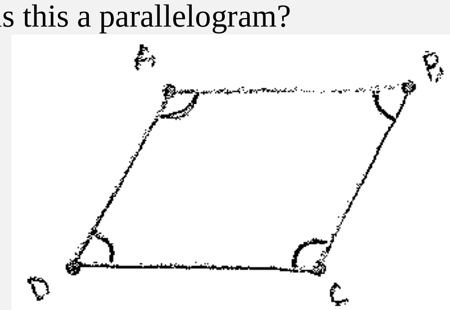
Table 4.16 Total frequencies of ANI’s use of names of quadrilaterals at the two interviews

Name	Frequency	
	Pre	Post
Quadrilateral	0	3
Parallelogram	11	4
Rectangle	17	5
Square	11	13
Rhombus	3	5
Trapezoid	2	2
Kite	0	1

Table 4.16 shows that the word *square* (n=24) was the most frequently used during the interviews. The word *rectangle* (n=22) was the second most frequently used, and *parallelogram* (n=15) was third. The names of the parallelograms were mostly mentioned in Task One, and among all the names, *rectangle* and *square* were most frequently mentioned at the Pre-Interview (see Table 4.15). There was an increase in use of the word *square* and *rhombus* in the Post-Interview. However, there was a reduction in use of the word *parallelogram* and *rectangle* in the Post-Interview. The words *kite* (n=1) and *quadrilateral* (n=3) were mentioned only at the Post-Interview, whereas the word *trapezoid* (n=4) was only mentioned in Task One. ANI's use of the names of quadrilaterals was much lower than other interviewees' use of those names.

Recall that in the Pre-Interview, while grouping in Task One, ANI referred to a rhombus as a slanted square, and a parallelogram as a slanted rectangle. Later she drew a rhombus as a parallelogram, but disagreed that a square was a parallelogram when I showed her a picture of a square in Task Two:

69. Interviewer Why is this a parallelogram?



70. ANI Because AB is parallel to DC, and AD is parallel to BC.

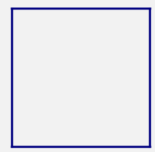
71. Interviewer Why do you think it's a different parallelogram?

72. ANI Because all the sides are the same length.

73. Interviewer What do you call this shape?

74. ANI A rhombus.

In this conversation, ANI identified a rhombus as a parallelogram because it had two pairs of parallel sides [70], and recognized that it was a different parallelogram because it had all sides of the same length. However, in the next conversation, she disqualified a square as a parallelogram.

99. Interviewer	How about this one? Is this a parallelogram? 
100. ANI	No.
101. Interviewer	Why do you think it's not a parallelogram?
102. ANI	Because all the lengths look like they are the same sides, and I think that a parallelogram has different sides.
103. Interviewer	What do you call this shape?
104. ANI	A square.

This inconsistency showed that ANI's use of the word *parallelogram* referred to visual family appearances. From ANI's grouping of parallelograms in Task One, and her ways of identifying parallelograms in Task Two, I conclude that the word *parallelogram* was used to represent *rectangles*, *parallelograms* and *rhombi*.

Figure 4.14 illustrates ANI's use of parallelograms at the pre-interview.

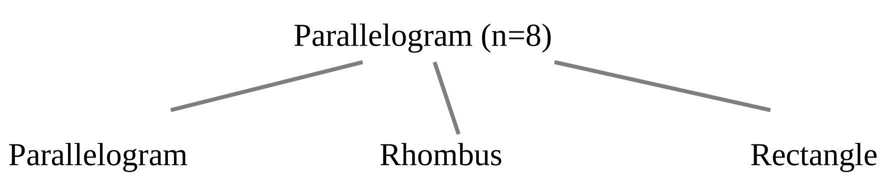


Figure 4.14 (cont'd)

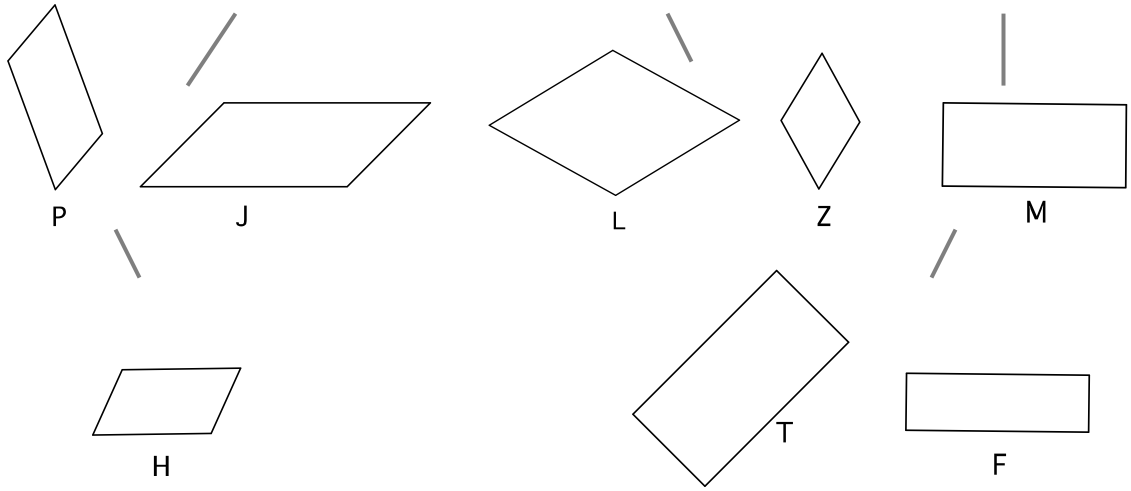


Figure 4.14. *ANI's use of the word parallelogram at the Pre-Interview*

ANI's use of the word parallelogram changed in the Post-Interview, as the word presented a class of figures sharing a common descriptive narrative, and all parallelograms presented in the task were connected in a hierarchy. In the Post-Interview, ANI used the word *quadrilateral* to extend the family of parallelograms, describing quadrilaterals with a hierarchy of classifications.

As shown in Figure 4.15, this hierarchy, quadrilaterals have different names depending on attributes of their angles and sides. This hierarchy has three branches: *trapezoid*, *parallelogram* and *quadrilateral*. The word *parallelogram* includes parallelograms, rectangles, squares and rhombi by definitions. All four-sided polygons with different visual appearances some with right angles, some have all same sides, but sharing a common descriptive narrative, "opposite sides parallel and equal".

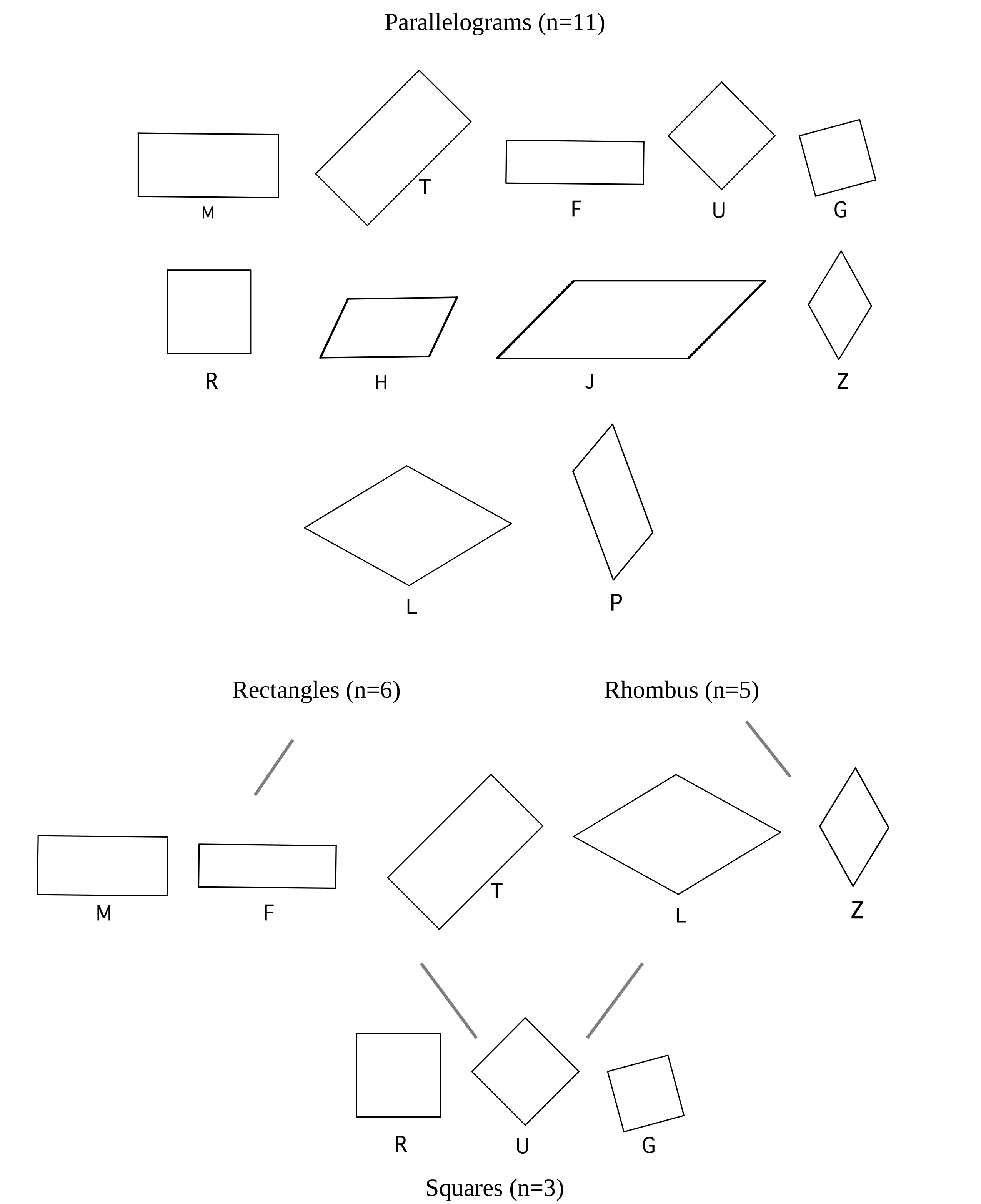


Figure 4.15 ANI's use of the word parallelogram at the Post-Interview

In the interviews, ANI mentioned more the parts of parallelogram (e.g., angles, sides, etc) than the names of parallelograms. Tables 4.17 and 4.18 give the frequencies of these words at the Pre-Interview and the Post-Interview. Table 4.18 provides total frequencies of each word at the interviews, whereas Table 4.17 presents the frequencies of each word used at each task in the interviews.

Table 4.17 The frequencies of ANI's use of names of the parts of parallelograms at the two interviews

Name	Frequency					
	Pre-T1	Pos-T1	Pre-T2	Pos-T2	Pre-T3	Pos-T3
Angle	14	7	32	37	5	3
Side	14	12	8	27	4	1
Length	3	4	13	10	8	2
Parallel side	0	3	0	3	0	0
Opposite side	2	0	1	1	2	0
Diagonal	0	0	10	12	0	0
Right angle	2	0	1	0	0	0
Opposite angle	2	0	0	0	1	1

Table 4.18 Total frequencies of ANI's use of names of the parts of parallelograms at the two interviews

Name	Frequency	
	Pre	Post
Angle	51	47
Side	26	40
Length	24	16
Parallel side	0	6
Opposite side	5	1
Diagonal	10	12
Right angle	3	0
Opposite angle	3	1

Table 4.18 shows that the most frequently used word relating to the parts of parallelograms was *angle* (n= 98), mentioned most frequently in Task Two (see Table 4). The

second most frequently used was *side* (n=66), mentioned most frequently in Task One and Task Two. The word *length* (n=40) was third, mentioned in all three tasks. The word *diagonal* (n=40) was only mentioned in Task Two. These results were expected, as Task Two asks interviewees to discuss the angles, sides and diagonals of parallelogram. *Right angle* (n=3) was least mentioned, appearing only at the Pre-Interview. There was a slight reduction in the use of the words *opposite angle* and *opposite side* in the Post-Interview. However, ANI mentioned *parallel side* (n=6) at the Post-Interview, but did not mention it at all in the Pre-Interview.

The most compelling change in ANI's word use was her use of the word *parallelogram* at the Post-Interview. She had a better understanding of parallelograms and their properties.

Case 3: Changes in ALY's Geometric Discourse

ALY was a college freshman at the time of the interviews. ALY took her last geometry class three years prior to the geometry and measurement class. The van Hiele geometry Test showed that she was at Level 3 at the pretest, and moved to Level 4 on the posttest. I interviewed ALY after both tests, and analyzed her interview responses. A summary of findings on changes in ALY's geometric discourse is as follows:

- ALY's *routines of sorting* changed from grouping polygons by the number of sides and by their names based on the attribute of their angles, at the Pre-Interview, to classifying polygons by their common descriptive narratives and arranging quadrilaterals with a hierarchy of classifications at the Post-Interview.
- ALY's *routines of substantiation* changed from verifying the congruent parts of parallelograms using *recalling, measuring and constructing routines* at the Pre-

Interview, to formulating mathematical proofs using mathematical axioms and propositions at the Post-Interview.

- ALY's word use changed from representing the word *parallelograms* as a collection of polygons sharing common descriptive narratives at the Pre-Interview, to using words with a hierarchy of classifications of parallelograms at the Post-Interview. ALY also used more mathematical terms in the Post-interview than in the Pre-Interview.

ALY's routines of sorting were analyzed in the interviews for Task One, where ALY was asked to classify eighteen polygons into groups. These polygons included triangles (n=4), quadrilaterals (n=13) and one hexagon. The following section details these interviews.

At the Pre-Interview, ALY grouped the eighteen polygons into three groups according to the number of their sides. She grouped all quadrilaterals into a group, calling it the "four-sided" group; and she grouped all three-sided polygons (n=4) together, naming the "triangles" group. ALY called Fig. V (a hexagon) "miscellaneous", and grouped it as "other".

When asked to regroup the quadrilateral group, and she regroup them by right angle versus non-right angle. After regrouping the quadrilaterals, ALY said, "I selected shapes with right angles… from looking at it. I didn't measure any of them, but I am assuming that they are right angles." ALY's regrouping of quadrilaterals is shown in Figure 4.16.

Right angle Group (n=8)

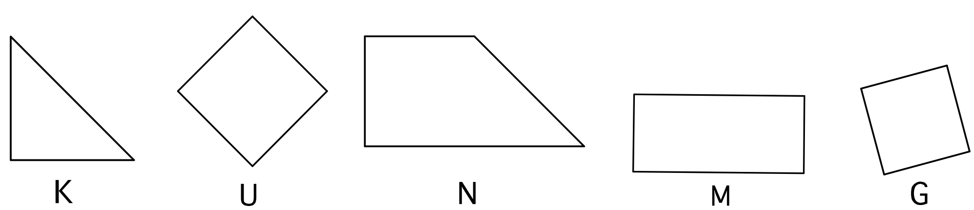
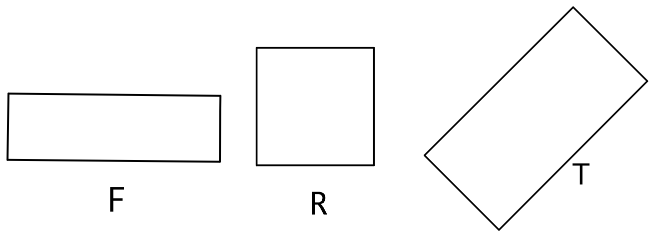


Figure 4.16 (cont'd)



Non-right angle Group (n=10)

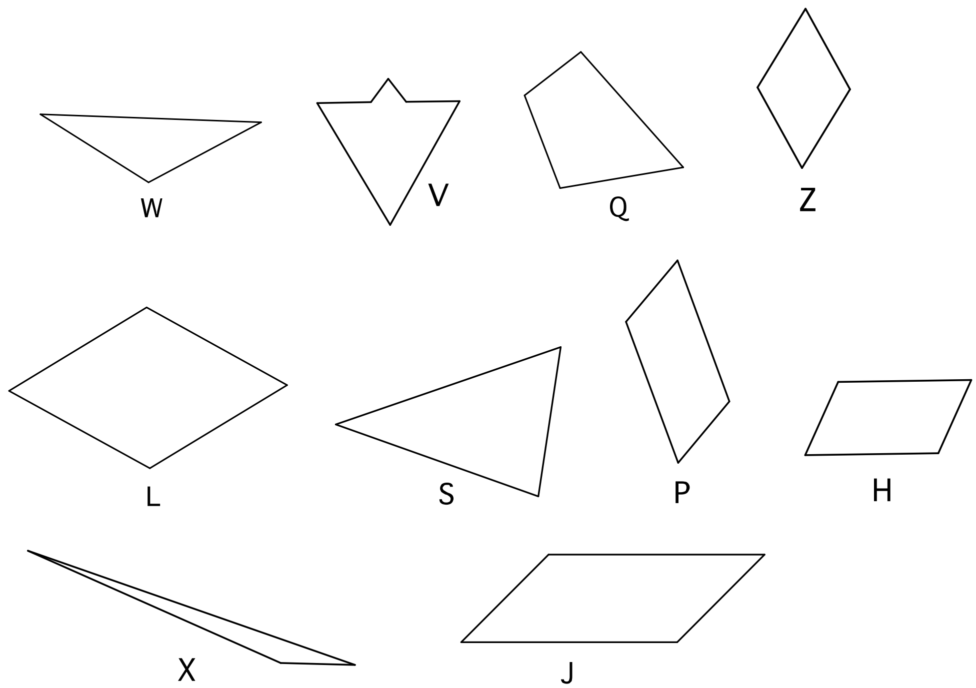


Figure 4.16 ALY's regrouping of polygons at the Pre-Interview

ALY regrouped the polygons by the attribute of having right angles or not. The *right angle* group consisted of polygons having at least one right angle, whereas the *non-right angle* group contained all other polygons. When I asked ALY to subgroup the *four-sided* group (n=13) that she initially made, she split the group into two: the *rectangle* group (n=6) and the *non-rectangle* group (n=7). Figure 4.17 illustrates ALY's subgrouping of the quadrilaterals.

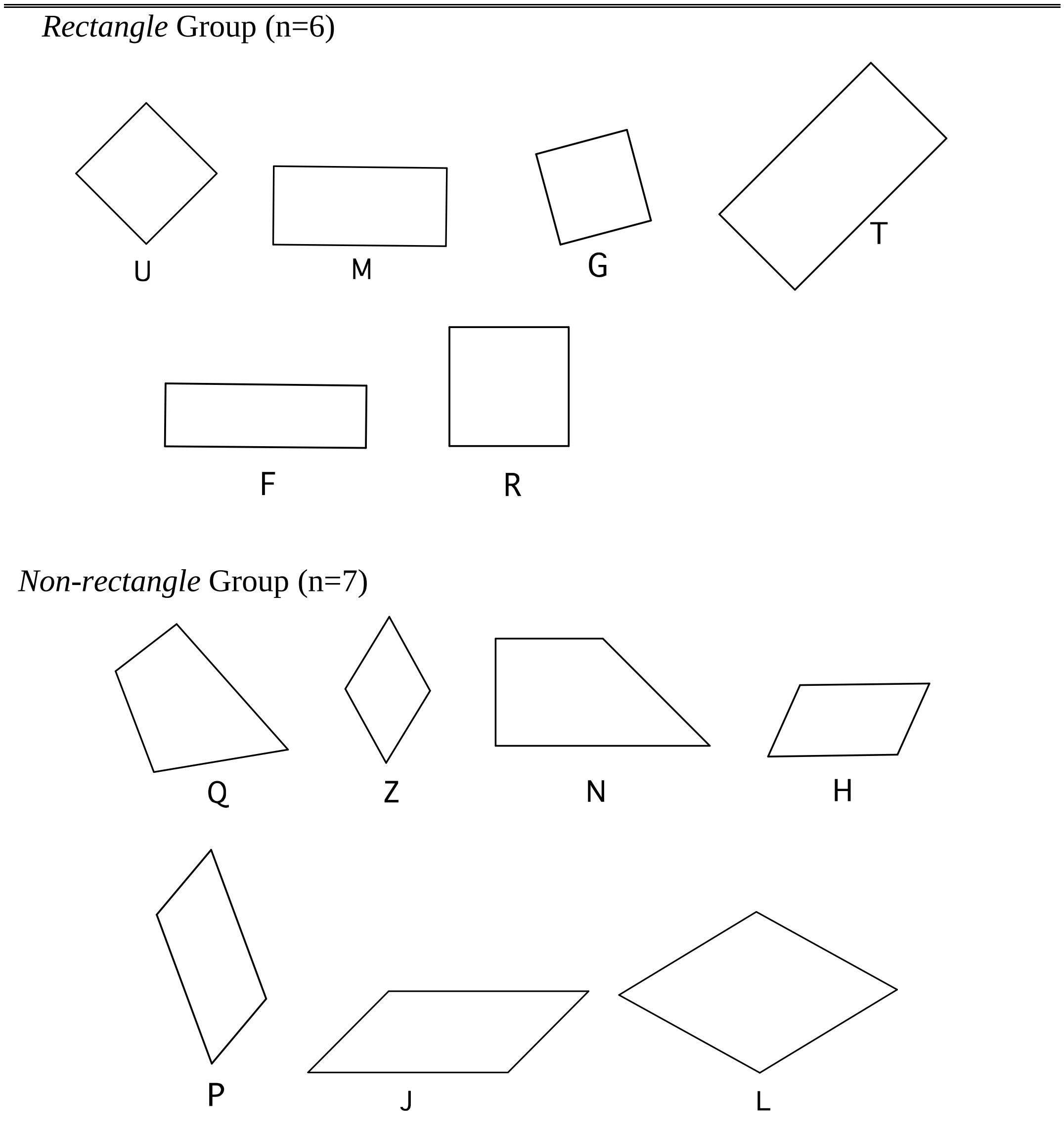
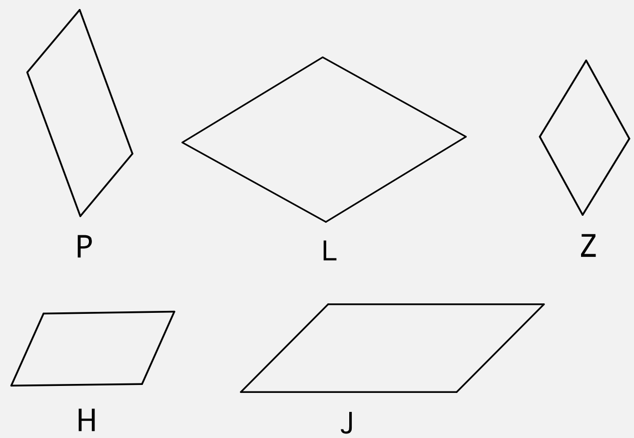


Figure 4.17 ALY's subgrouping of the quadrilaterals at the Pre-Interview

ALY used the name *rectangle* to split the quadrilaterals into two groups. In the *rectangle* group, ALY included both rectangles and squares. Consequently, the *non-rectangle* group contained all other quadrilaterals such as the parallelograms, the rhombi and a trapezoid. I also asked ALY to subgroup the *non-rectangle* group, and she responded that she could split the group into *rhombus* and *non-rhombus.*

26. ALY: I would separate them into rhombuses. P is a rhombus, L is a rhombus, J is a rhombus, Z and H are all rhombuses. [Pointing at these polygons]



In the subgrouping, ALY identified both parallelograms and rhombi as *rhombus*. To investigate further, I asked ALY to identify a parallelogram, and she pointed to Fig. H. I asked her to identify a rhombus, and she pointed to Fig. J (another parallelogram). ALY's subgrouping of the *non-rectangle* group revealed her confusion between parallelograms and rhombi.

Thus, at the Pre-Interview ALY grouped polygons by the number of sides and by the characteristics of their angles. ALY also favored dividing polygons into two groups, according to what they are and what they are not, with names of polygons such as *rectangle* and *rhombus*. ALY identified polygons by direct recognition, but it appeared to me this recognition was not self-evident. For example, ALY mentioned "just from looking at it, ... I am assuming that they are right angles" to explain her assumptions. Figure 3 summarizes ALY's *routines of sorting* at the Pre-Interview.

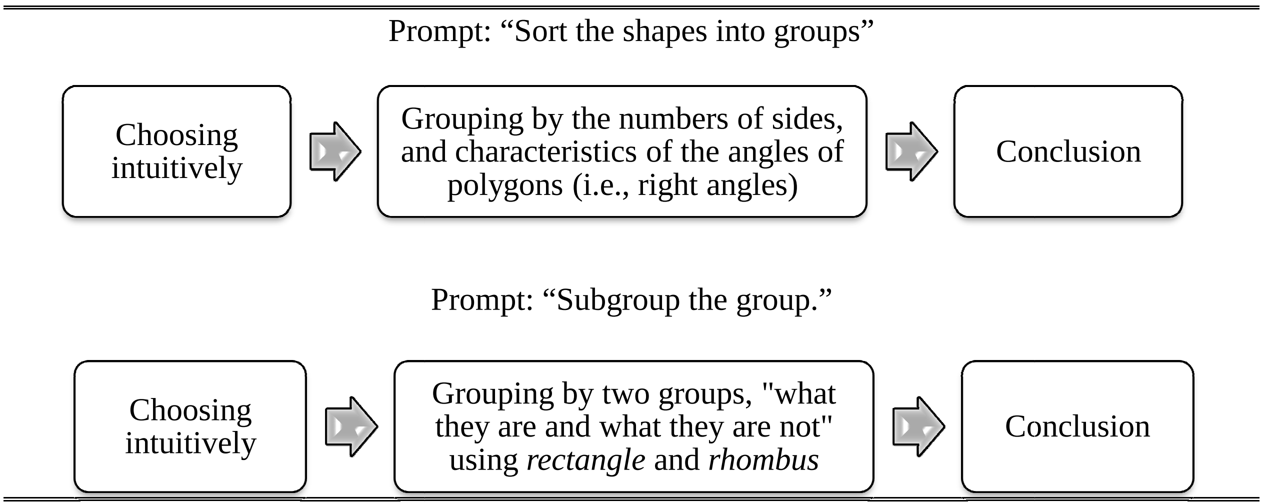


Figure 4.18 ALY’s routines of sorting at the Pre-Interview

At the Post-Interview, at my request, ALY first grouped the eighteen polygons into two groups, consisting of *triangles* (n= 4) and *non-triangles* (n=14) according to the number of their sides. When I asked ALY to regroup the eighteen polygons differently, she regrouped the polygons into two groups, including *quadrilaterals* (n=13) and *non-quadrilaterals* (n=5), again by the numbers of sides. I then asked ALY to subgroup the *quadrilaterals*, and she split the *quadrilaterals* into two groups, consisting of *parallelograms* (n=11) and *non-parallelograms* (n=2). I continued to ask ALY to subgroup the *parallelograms,* and she came up with two groups, *rectangles* (n=6) and *non-rectangles* (n=5). This pattern continued with the two subgroups consisting of *squares* (n=3) and *non-squares* (n=3) within the *rectangles* group. Figure 4.19 summarizes ALY’s subgroupings of the *quadrilaterals* at the Post-Interview. As shown in Figure 4.19, ALY’s *routines of sorting*, distinguishing two groups of polygons determined by “what it is and what it is not” proceeding through families of quadrilaterals such as *parallelograms, rectangles* and *squares.*

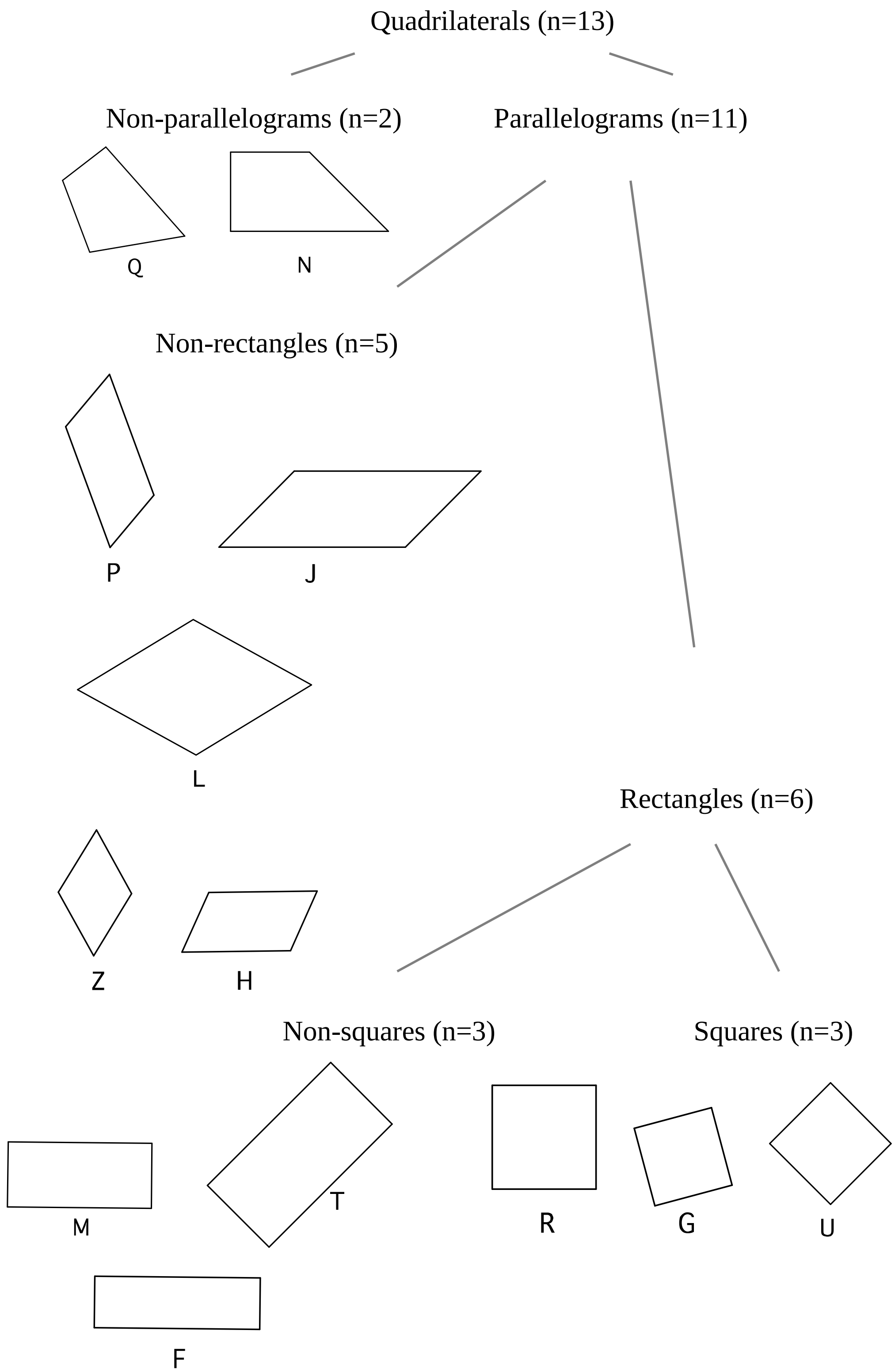
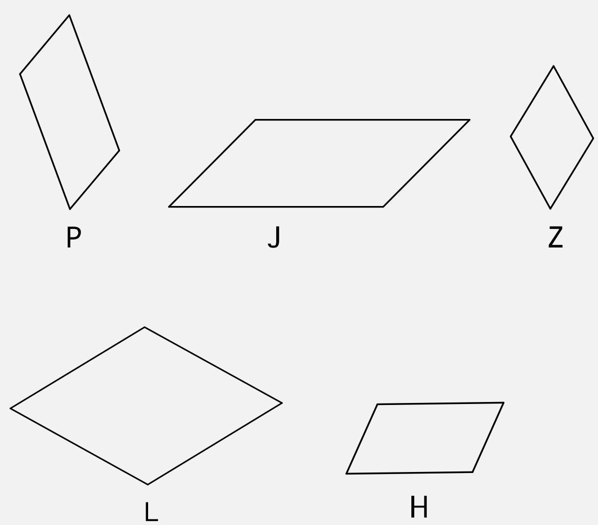


Figure 4.19 ALY's chains subgroupings of the quadrilaterals at the Post-Interview.

During our conversations in the Post-Interview, ALY demonstrated her understandings of the relations between *parallelograms*, *rhombi* and *squares* by classifying these polygons in a hierarchy. For example, ALY identified the *non-rectangles* as *parallelograms*, a group consisting of parallelograms and rhombi.

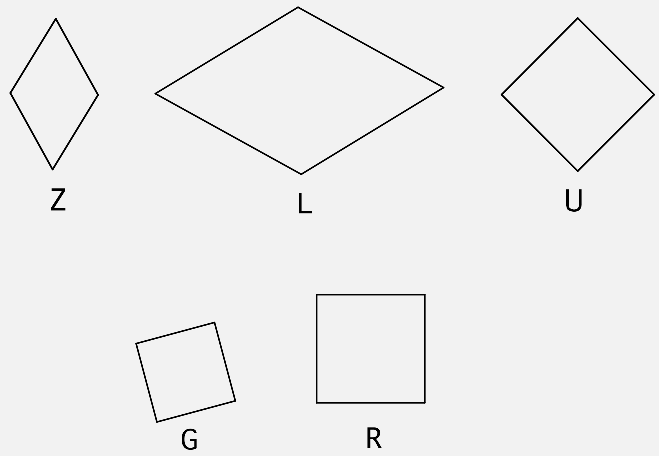
45. Interviewer	What are the non-rectangles?
46. AL	The parallelograms? P, H, Z, J and L. [Pointing at these polygons]



47. Interviewer	Why are they not the rectangles?
48. ALY	Because they don't have four right angles.

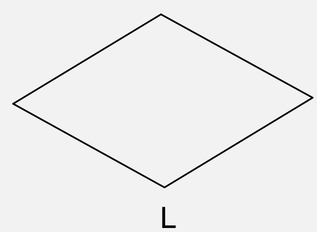
ALY also identified rhombi as a subgroup of *non-rectangles* (i.e., parallelograms), and identified a *square*, a polygon from a group of *rectangles*, as a *rhombus*.

77. Interviewer	Can you identify if there is a rhombus?
78. ALY	A rhombus? I think Z and L are rhombuses. And then U, G, and R would be rhombuses as well. [Pointing at Fig. Z and L]

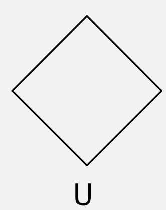


79. Interviewer How do you know that L is a rhombus?

80. ALY Because these sides look equal in length and they are also parallel to the opposite sides [pointing at the sides of Fig. L]



81. Interviewer Why is U a rhombus? [Pointing at Fig. U]



82. ALY Because all sides are equal and opposite sides parallel to each other.

At the Post-Interview, ALY first grouped the polygons by the numbers of their sides. When asked to regroup the polygons, ALY classified quadrilaterals by dividing them into two groups each time. During the interview, ALY did not use measurement tools to verify congruent angles or sides of the polygons. However, ALY did mention that she assumed the angles of the rectangles to be 90 degrees, and the sides of the parallelograms to be parallel. I have concluded that ALY identified polygons intuitively, but again it was not self-evident. ALY's *routines of sorting* at the Post-Interview are summarized in Figure 4.20.

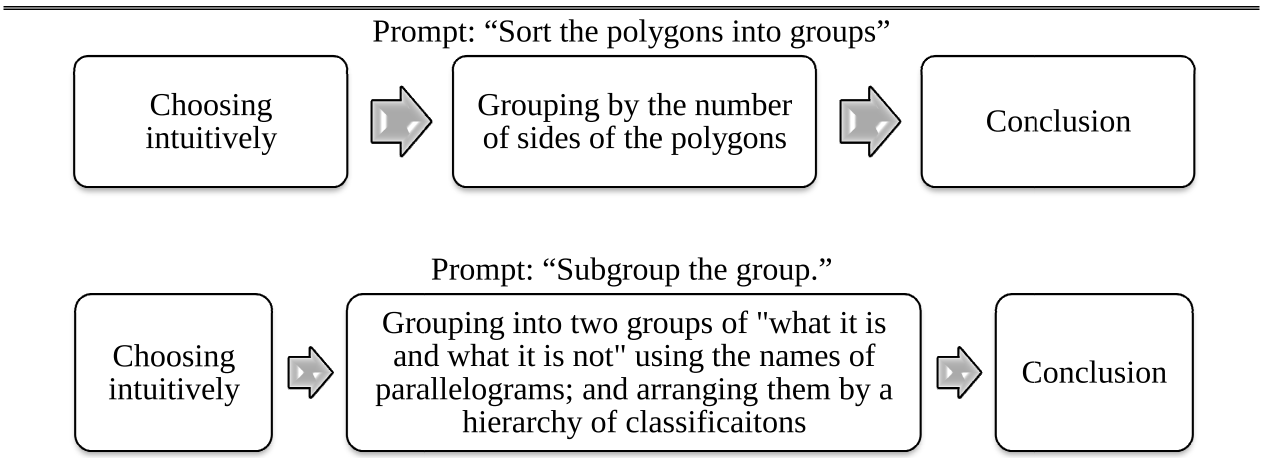


Figure 4.20 ALY's *routines of sorting* at the Post-Interview

We see a similar pattern as ALY grouped the polygons by the number of sides, and sub-grouped polygons by dividing them into two groups each time. However, there was a change in AYL's subgroupings, as her chains of subgrouping showed a hierarchy of classifications of quadrilaterals at the Post-Interview, her subgrouping was limited only to identifying the *rectangles* and the *rhombi* without a hierarchy at the Pre-Interview.

ALY's change in geometric discourse also appeared in her routines of substantiation. In the following section, I describe observations from the interviews with ALY in Task Two and Task Three.

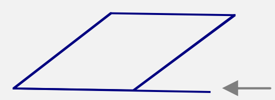
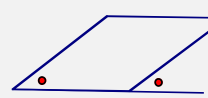
ALY explained and verified her claims about the angles and sides of parallelograms with *recalling* and *measuring* routines at the Pre-Interview; whereas she substantiated her statements using endorsed narratives at the Post-Interview. When substantiating the equivalence of two definitions, ALY constructed two polygons with angle and side measurements that fit the descriptions of the definitions at the Pre-Interview, whereas she produced a mathematical proof at the Post-Interview. To illustrate these changes, I will provide the following scenarios.

In Task Two, ALY drew a parallelogram, and stated that it was a parallelogram because the opposite sides were parallel to each other. When discussing the angles of the parallelogram, ALY responded that the opposite angles of the parallelogram were equal. When I asked why, ALY replied, “I think that is just a property of a parallelogram”. The following conversation took place after I prompted for verification:

9. Interviewer	If I ask you to convince me that the opposite angles are equal, what would you do?
10. ALY	You mean… prove it to you, that in every case it would be that way?
11. Interviewer	Yeah.
12. ALY	I could just measure the angles for you, with a protractor. I've never done a proof before, in this case. I've done lots of proofs, but not on something like that, so I don't know.

In this conversation, I noticed that writing a proof about the angles of a parallelogram was new to ALY, but she was aware of the difference between a verification of a statement “opposite angles are equal in a parallelogram” in general (i.e., a mathematical proof) and a verification of an example of a statement (i.e., check the measurements of angles), as evidenced in her asking, “prove it to you that in *every* case it would be that way?” and later proposing to measure the angles [12]. In this case, ALY explained the statement that opposite angles are equal in a parallelogram by remembering it as a property of a parallelogram.

ALY also provided another narrative about the angles of a parallelogram, namely that the adjacent angles in a parallelogram add up to 180°. When asked for substantiation, ALY replied, “I have learned it before, but I am not 100% sure…”, and produced her verification as follows:

26a. ALY	This angle looks like it would match up if I was to extend this line out like this [extending the side]
	
26b. ALY	… this angle looks equivalent to this angle. And, actually it is, because I've learned about parallel lines [pointing at the angles].
	
26c. ALY	It would, because this angle would be equivalent to this angle [pointing at the vertical angles]… and is also equal to that [pointing at alternating interior angles]
	
27. Interviewer	Why is that?
28. ALY	That's another property [giggling]... I just remember

	in high school, we learned a lot of properties about parallelograms and parallel lines, and things like that.
29. Interviewer	So… What is your conclusion?
30. ALY	That the adjacent angles add up to 180 degrees.

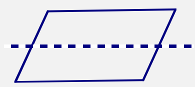
ALY reasoned her way through by drawing extended lines [26a] and by identifying parallel sides [26b] and congruent angles [26b; 26c]. ALY remembered the properties of a parallelogram and remembered the propositions of parallel lines, as well as the relation between the vertical angles [26c], and she applied them in her verification process. It is notable that ALY identified more congruent angles than she needed for the verification. More specifically, ALY needed only one of the three congruent pairs of angles, but she identified three pairs [26b; 26c]. ALY neglected to point out how the congruent angles would lead to the proof that the adjacent angles add up to 180 degrees, an argument crucial in this substantiation. In the preceding examples ALY applied her prior knowledge about the properties of parallelogram and propositions of parallel lines to identify the elements needed for verification. However, in this example, ALY verified her claim only partially and without logical order.

In contrast, at the Post-Interview ALY explained why opposite angles are equal, using the fact that the adjacent angles in a parallelogram add up to 180° as an endorsed narrative. Here is ALY's demonstration that adjacent angles in a parallelogram add up to 180 degrees.

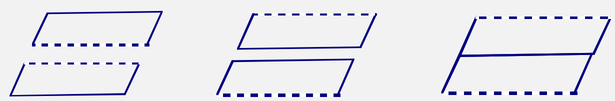
11. Interviewer	How do you know that the adjacent angles add up to 180 degrees?
12. ALY	Because that's one of the properties of a parallelogram, I can show you if you like?

13. Interviewer Go ahead.

14a. ALY If you were to rip this in half [drawing a line cut through the polygon],



14b. ALY …and then take this top half and put it down here, then it would line up, like this. Does that make sense?

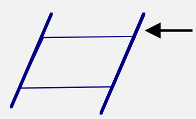


14c. ALY Right here… Let's say that this is *a* and this is *b*. If we moved this [b] down here then these would be on the same line. [Labeling the polygon with a and b representing two pieces of the polygon]

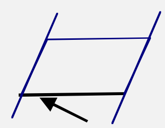
14d. ALY This angle here [pointing at the vertex angle] would be right here, and they'd be on the same line. And, angles on a line add up to 180 degrees.

To verify that adjacent angles add up to 180 degrees, ALY first referred to this statement as a property of parallelograms [12]. Then she described an activity of imagining to cut the polygon horizontally in half [14a], and moving the top half to the bottom [14b] to match the two adjacent vertex angles side by side in a line [14c; 14d]. ALY used this newly endorsed narrative that adjacent angles add to 180 degrees to justify her statement that the opposite angles in a parallelogram are equal.

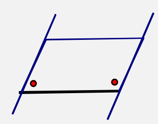
15. Interviewer	How do you know that the opposite angles are equal?
16a. ALY	Because you have two parallel lines, here, if I was to extend these lines [Extending two sides of the parallelogram].



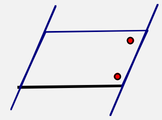
16b. ALY	Then, this would be a transversal.



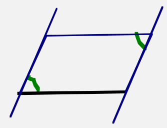
16c. ALY	So, you've got this angle here and this and this angle here [pointing at the two angles] add up to 180 degrees because they're adjacent.



16d. ALY	For the same reason, … then this angle and this angle would also add up to 180 degrees,



16e. ALY So these [pointing at the opposite angles] have to be the same.



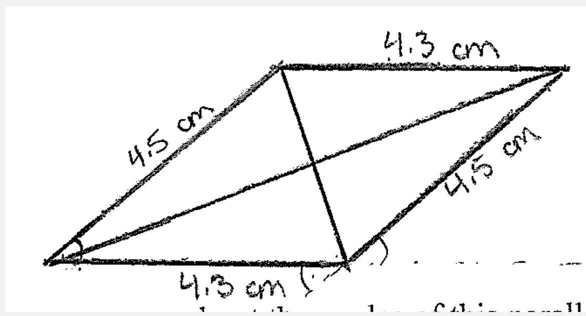
In this example, ALY first identified two parallel lines [16a] and a transversal [16b], and concluded that adjacent angles add up to 180 degrees [16c]. In order to show that opposite angles were equal, ALY identified another pair of adjacent angles, with one vertex angle included in the previous ones [16d], so that two pairs of adjacent angles shared one angle in common. Giving the same reason [16a; 16b], ALY explained that the second pair of adjacent angles also added up to 180 degrees. Finally, ALY concluded that the opposite angles had to be the same [16e].

This scenario illustrates changes in ALY's routine procedures. She progressed from referring to prior knowledge to justify her claims (i.e., *recalling routines*), to using newly endorsed narratives to substantiate her claims. ALY also moved from identifying more elements than needed for verifications, without logical order, to choosing the exact number of elements necessary to justify statements logically.

In another change, ALY went from measuring the sides of parallelograms with rulers to verify congruency at the Pre-Interview, to identifying congruent triangles by congruent criterions to verify the congruent parts of parallelograms at the Post-Interview. In Task Two, I asked ALY to draw a parallelogram and to discuss its sides. ALY drew a parallelogram and stated that it was a parallelogram because the opposite sides were parallel to each other and were equal in length.

The following conversation took place after I asked for substantiation of her claim that "the opposite sides were equal in length".

41. Interviewer	Is there a way that you can show me that they are the same length?
42. ALY	In this parallelogram? I can measure it. So this is…4.5 centimeters, this is a little less than 4.5. [Using a ruler to measure one pair of opposite sides]… Right, this looks about 4.3. Yeah, about the same. [Measuring another pair of opposite sides]



ALY's drawing

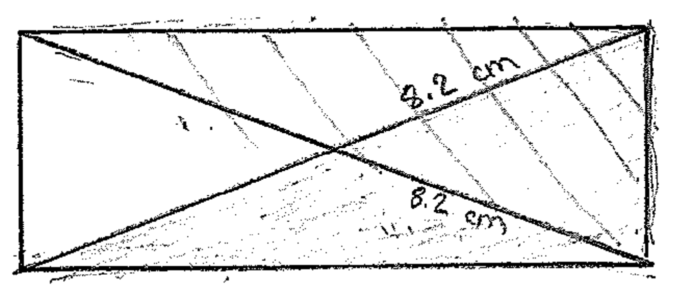
43. Interviewer	How do you know that for every parallelogram this is true?
44. ALY	You mean prove it? Well, I am not sure…but I know it's just a property of a parallelogram.

To verify the claim, ALY first referred to it as a characteristic of a parallelogram, and then used a ruler to measure the sides of the parallelogram to complete her verification. As I mentioned earlier, ALY did not have much experience in constructing a mathematical proof in this context, but my observation was that she was aware of the difference between constructing a proof at an abstract level and checking with a concrete example at an object level. For example, her awareness showed when ALY asked, "In this parallelogram? I can measure it" [42].

Consequently, ALY used a ruler to measure the opposite sides of the parallelogram to check congruency [42], and referred to this congruency as a property of a parallelogram.

This pattern of *measuring* and *checking* also appeared when ALY was verifying the equivalence of diagonals in a rectangle.

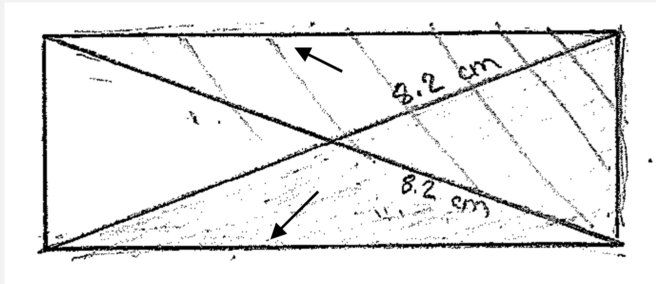
95. Interviewer	What can you say about the diagonals of this parallelogram?
96. ALY	They are of equal length.
97. Interviewer	How do you know that they are equal?
98. ALY	Because I learned it a long time ago, in a rectangle, the diagonals are the same.
99. Interviewer	Is there a way that you could convince me?
100a. ALY	I would measure them, is that O.K? [Using a ruler to measure the diagonals]
100b. ALY	Yeah, they're both 8.2 centimeters. So, the diagonals have equal length.



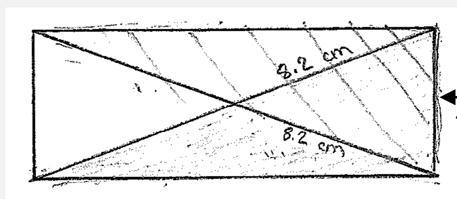
I challenged ALY by asking, "What if you don't have rulers to measure the diagonals, what would you do?" ALY replied, "If you look, the diagonals form two triangles", and identified two triangles, and explained why the two triangles were congruent.

115. ALY	[Shading the two congruent triangles]

116a. ALY This side equal to this side [pointing at the opposite sides of the paralleogram]



116b. ALY This side is obviously it's the same side.

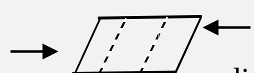


116c. ALY …Which means this side would have to be equal to this side [pointing at the diagonals]

ALY intuitively provided a general explanation about the equivalence of the diagonals using congruent triangles. ALY first identified two triangles [see shaded areas in 116a] where the diagonals were the hypotenuses of the triangles. ALY chose two congruent elements, using the opposite sides of the rectangles as one pair of corresponding sides in the triangles [116a], and noted a common side [116b]. From there, ALY concluded that the diagonals were equal based on the equivalence of the two other pairs of sides. Mathematically, to verify that two triangles are congruent we need three elements, and in this case ALY only provided two. We need information about an *included* angle of the two sides that ALY identified to complete the verification. Note that, in our earlier conversation, ALY drew this rectangle as a different parallelogram, and she knew that all angles were equal in a rectangle. However there was no mention of the equivalence of the angles, the third element needed for verification of the congruent triangles.

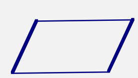
Ten weeks later I interviewed ALY again. When ALY discussed the sides of the parallelogram, she used an argument including the distance between the parallel lines to verify her claim that the opposite sides were congruent.

22. ALY	Opposite sides are congruent in a parallelogram.
23. Interviewer	How do you know that they are congruent?
24a. ALY	Because the opposite sides are parallel to each other, they have a fixed distance away from each other. So, the distance from here to here is the same as here to here, it's going to be the same all the way through.



distance

24b. ALY	So, that would mean that this would be equal to this because they never intersect [pointing at the opposite sides]



ALY's used rulers and protractors to measure and check the congruent parts of the parallelograms at the Pre-Interview. In contrast, ALY used an endorsed narrative, stating that the distance between the parallel lines were equal to demonstrate that opposite sides of the parallelogram were equal at the Post-Interview. The preceding conversation drew my attention to ALY's incorrect use of he word "distance" in the context of the distance between parallel lines; I will discuss this matter later in looking at ALY's word use.

During the Post-Interview, ALY used triangle congruent criterions to substantiate the congruent parts of parallelograms. For example, when discussing the diagonals of a parallelogram, ALY stated that the diagonals bisect each other. When asked for verification, ALY provided the following justification using the Angle-Side-Angle (ASA) triangle congruency criterion. Table 4.19 illustrates ALY's routine procedures of substantiation, with corresponding transcripts.

Table 4.19 ALY's routine procedures of substantiating that diagonals bisect each other

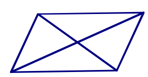
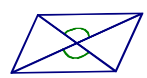
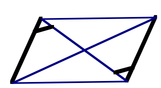
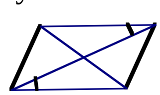
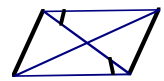
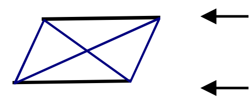
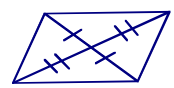
Routine Procedures	Transcripts
1. Identify triangles formed by diagonals	38a. Since I drew two diagonals, we can see that there are four triangles here. 
2. Verification of two congruent triangles 2.1 Identify first pair of corresponding angles of the triangles	38b. If you take this angle here and this angle here, they're equal to each other because they're vertical angles [pointing at the vertical angles] 
2.2 Identify the second pair of corresponding angles of the triangles	38c. Because these two lines are parallel, this angle would be equal to this angle here, because they're opposite interior angles, … 
2.3 Identify the third pair of corresponding angles of the triangles	38d. The same goes for this angle and this angle, …because of the same property. 

Table 4.19 (cont'd)

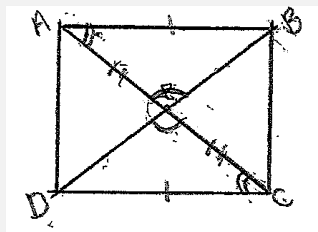
2.4 Identify the fourth pair of corresponding angles of the triangles	38e. …and then this angle and this angle. 
2.5 Identify one pair of corresponding sides of the triangles	38f. We already know that these two sides are equal because that's a property of a parallelogram [pointing at the sides of the triangles] 
4. Verify congruent triangles using A-S-A correspondence	38g. So, you have Angle-Side-Angle here. And, so, that shows that this triangle here is congruent to this triangle here 
5. Conclusion	38h. When you match up the corresponding sides, this side would be congruent to this side and then, this would be congruent to this. So, they are of equal measure, so they bisect each other. 

In this episode, ALY provided a verbal explanation of why the diagonals of this parallelogram bisect each other. More specifically, ALY used the ASA triangle criterion to verify that two triangles were congruent [38g], and concluded that the diagonals bisected each other because the parts of the diagonals were corresponding sides of two congruent triangles. Note that at each step, ALY was able to provide mathematical justifications for her conclusions. For example, ALY stated, "*because* these two lines are parallel" and "*because* they are vertical angles" as justifications to demonstrate that two pairs of angles were equal. ALY also used "a property of parallelogram" to explain the congruent sides.

During this process of verification, ALY identified four pairs of congruent angles [38b-38e] and one pair of congruent sides [38f], more elements than needed for verification. In particular, she identified one pair of vertical angles [38b] and a pair of alternating interior angles [38b] that were not needed in her final verification of two congruent triangles.

As our conversations continued, ALY drew a rectangle and stated that the diagonals also bisected each other in a rectangle. The following conversation took place when I asked for substantiation.

63. Interviewer How do you know diagonals bisect each other in this case?

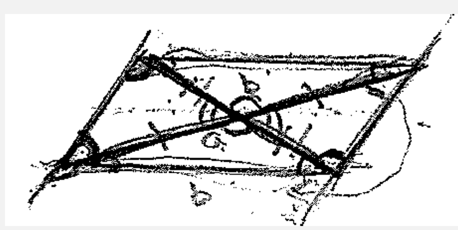


ALY's drawing

64. ALY For the same reason as last time, do you want me to explain again?

65. Interviewer When you say, "for the same reason as last time", what do you mean?

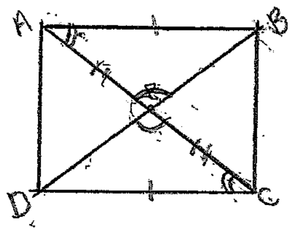
66. ALY Just, all of it, when you create these triangles and the triangles are congruent to each other…based on that property, all parallelograms have diagonals that bisect each other [pointing at the triangles in the parallelogram].



ALY's drawing

ALY referred to her newly endorsed narrative about the diagonals bisecting each other in a parallelogram [64] to justify her claim about the diagonals in a rectangle. Given this opportunity, I asked for a written proof. ALY's written proof is presented in Table 4.20.

Table 4.20 ALY's written proof that diagonals bisect each other at the Post-Interview



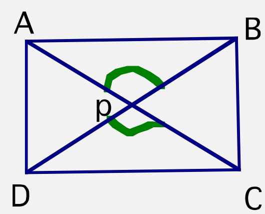
p is the intersection of the diagonals

∠APB ≅ ∠DPC (vertical angles)
∠APD ≅ ∠BPC (vert. ∠s)
∠BAC ≅ ∠DCP (alt. int ∠s, $\overline{AB} /\!/ \overline{DC}$)
AB = DC (prop. of ll-ogram)
ΔABP ≅ ΔCDP (ASA)
AP = PC (corr. sides in ≅ Δs) DP = BP (corr. sides in ≅ Δs)

Before ALY started to write the proof, she labeled the vertices of the rectangle with A, B, C, and D in a clockwise order. ALY used mathematical symbols like ∠, ≅, Δ, ⫽ to replace the words *angle*, *congruent*, *triangle*, and *parallel*, respectively, in her written proof. For example, ALY wrote "∠APB ≅ ∠DPC", "AB = DC" and "ΔABP ≅ ΔCDP" to indicate two congruent angles, sides and triangles accordingly. I asked ALY for further clarification.

77. Interviewer	Can you explain to me what you wrote?
78a. ALY	I have angle APB, so this angle right here, is equal to DPC,

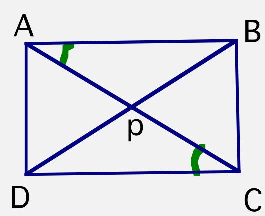
because they are vertical angles, And then also angle APD, so this here, well I guess that's not really important



∠APB ≅ ∠DPC (vertical angles)

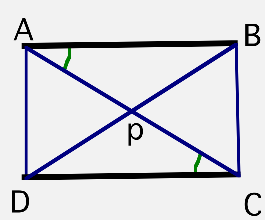
∠APD ≅ ∠BPC (vert. ∠s)

78b. ALY Angle BAC is congruent to DCP because AB and DC are parallel to each other, so these two angles are alternate interior angles, and they're always congruent.



∠BAC ≅ ∠DCP (alt. int ∠s, AB//DC)

78c. ALY AB is equal to DC because that is a property of a parallelogram. So, we have two angles on this side, and that is enough information to conclude that triangle ABP is congruent to triangle CDP.

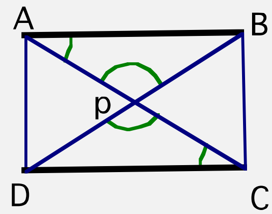


AB = DC (prop. of //ogram)

ΔABP ≅ ΔCDP (ASA)

79. Interviewer When you say two angles and a side, what do you mean?

80. ALY	This angle and this angle. And then we have a side.



$\angle APB \cong \angle DPC$

(vertical angles) –*angle*

$\angle BAC \cong \angle DCP$

(alt. int $\angle$s, AB//DC) –*angle*

AB = DC (prop. of //ogram) –

side

It is evident that there was a change in ALY’s routines of verifying. She previously measured and compared the sides of the parallelogram to verify her claims at an object level in the Pre-Interview, whereas, at the Post-Interview, she constructed a mathematical proof using mathematical symbols and justifications at an abstract level. However, ALY incorrectly used Angle-Angle-Side (AAS) to verify congruent triangles. Recall that, in an earlier substantiation, ALY mentioned “two angles and a side” as ASA to verify the congruent triangles correctly. However, in this example the same phrase, “two angles and a side” appeared again but with an indication of AAS. Thus ALY displayed ambiguity in using congruent criterions.

When proving the equivalence of the two definitions, ALY’s routine procedures also changed. At the Pre-Interview, ALY verified the equivalence of the definitions by *constructing* the parts of the parallelograms with measurements that fit the descriptions of the definitions, whereas she constructed a mathematical proof of the equivalence of the definitions at the Post-Interview. The following illustrates ALY’s routine procedure that I observed in Task Three.

In Task Three, interviewees were asked to construct mathematical proofs from given definitions. In particular, these interviewees were given two definitions of a parallelogram and were asked to substantiate that these definitions were equivalent.

- Definition #1: A quadrilateral is a parallelogram if and only if both pairs of opposite sides have the same length.
- Definition #2: A quadrilateral is a parallelogram if and only if both pairs of opposite angles have the same measure.

In order to verify the equivalence of the definitions mathematically, interviewees needed to use *deduction* to prove the following implications:

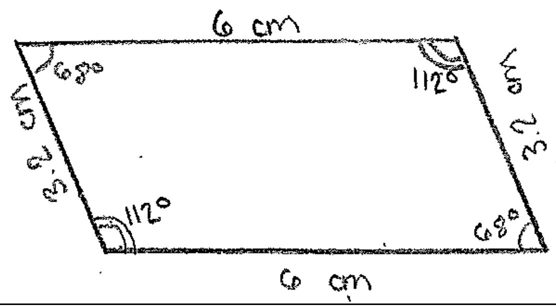
- If a quadrilateral has both pairs of opposite sides of the same length, then both pairs of opposite angles have the same measure; and
- If a quadrilateral has both pairs of opposite angles of the same measure, then both pairs of opposite sides have the same length.

Deduction takes place when a newly endorsed narrative is obtained from previously endorsed narratives with the help of well-defined inferring operations.

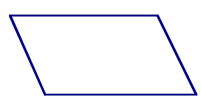
At the Pre-Interview, I asked ALY what she would do to prove the two statements were equivalent. She replied, “I would demonstrate what each definition is saying, then show how it results in the same thing”. In trying to prove the implication “If a quadrilateral has both pairs of opposite sides of the same length, then both pairs of opposite angles have the same measure”, ALY explained, “I would draw a parallelogram where the opposite sides have the same length. And, because they have the same length, they're going to have the same angles”. ALY used a ruler to construct a polygon with opposite sides parallel and equal, so that the parallelogram fit the description in Definition #1. ALY next used a protractor to check all the angles of the parallelogram, and found that the opposite angles had the same measures, matching the descriptions in Definition #2. ALY finished this part of the substantiation with the conclusion "it's going to be that case every time, where the angles will be equal every time." ALY's routine procedure of constructing a parallelogram with both pairs of opposite sides of the same length in order to prove the implication, is shown in Table 4.21.

Table 4.21 ALY's proof by constructing a parallelogram with opposite sides equal

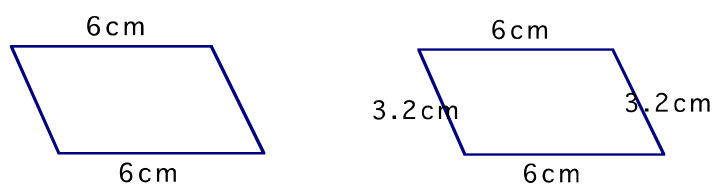
ALY's drawing of the parallelogram



1. *Constructing a parallelogram with opposite sides equal*



2. *Verifying the given conditions (Definition #1):* Using a ruler to measure the opposite sides of the parallelogram



3. *Verifying the results (Definition #2)*: Using a protractor to measure the opposite angles of the parallelogram

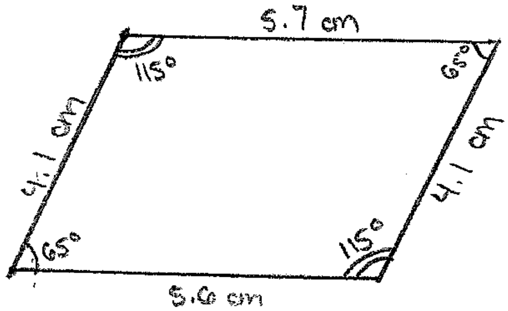
4. *Conclusion*

ALY: So, it's going to be that case every time, where the angles will be equal every time. I wasn't measuring the angles when I drew it. I was just focusing on the lengths, but it just came out that they were the same. And, it will come out that way every time.

To prove the other implication, “If a quadrilateral has both pairs of opposite angles of the same measure, then both pairs of opposite sides have the same length.” ALY began by drawing a quadrilateral with opposite angles equal then she measured the opposite sides. She used a ruler to check all the sides of the parallelogram, and found that opposite sides had the same lengths matching the descriptions in Definition #1. ALY completed her verification with the conclusion “based on opposite angles being the same. I ended up with sides that were almost identical in length”. ALY’s routine procedure of constructing a quadrilateral with both pairs of opposite angles of the same measure to prove the implication is shown in Table 4.22.

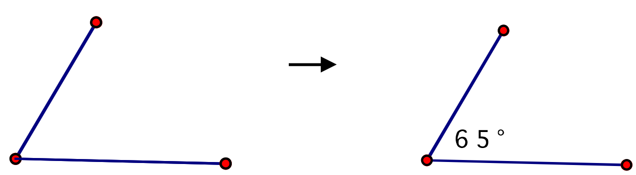
Table 4.22 ALY’s proof by constructing a parallelogram with opposite angles equal

ALY’s drawing of the parallelogram



1. *Constructing a parallelogram with opposite angles equal*

1.1 Constructing an angle with 65°



1.2 Constructing a parallel line to the side and measuring the angle

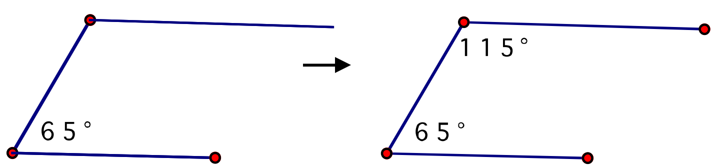
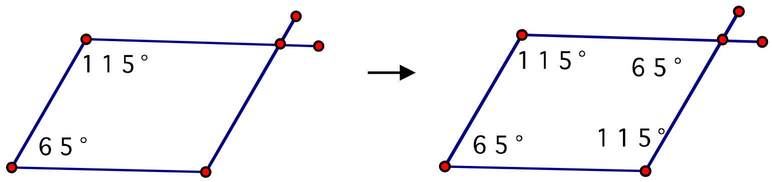
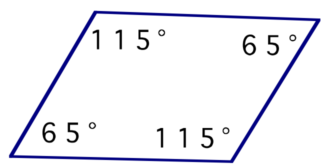


Table 4.22 (cont'd)

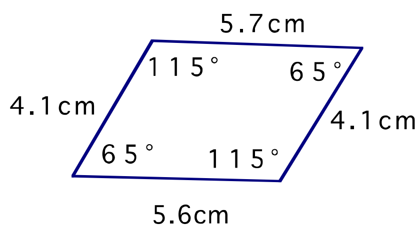
1.3 Constructing the fourth side that is parallel to its opposite side and measuring the rest of the angles



2. *Verifying the given conditions (Definition #2):* Checking the angle measure of the opposite angles



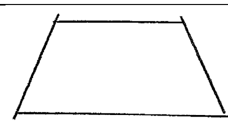
3. *Verifying the results (Definition #1)*: Using a ruler to measure the opposite sides of the parallelogram



4. *Conclusion*
ALY: This time, when I drew it, I focused on the angles. I drew the lines, then I created the shape, based on opposite angles being the same. And, I ended up with sides that were almost identical in length.

At the Pre-Interview, ALY's routine procedures of proving the equivalence of the two definitions involved the constructing a parallelogram, measuring the angles or the sides, and comparing the measurements, all mathematical activities operating at an object level. It is important to note that ALY did not know how to construct a mathematical proof and her argument was to generalize the result from a particular example, assuming that "it will came out that way every time."

Ten weeks later, ALY's routine procedures for substantiating the equivalence of the two definitions changed. First of all, ALY proved the implication "If a quadrilateral has both pairs of opposite sides of the same length, then both pairs of opposite angles have the same measure" by providing an example of the *inverse* of the implication. That is, ALY drew a polygon that fit the implication "if a quadrilateral does not have both pairs of opposite sides of the same length, then both pairs of the opposite angles don't have the same measure." ALY's drawing and her explanation are shown in Figure 4.21.



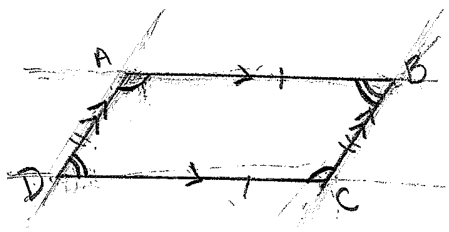
ALY's drawing

ALY: …if they [pointing at the parallel sides] didn't have the same length, if one of the sides had a different length than the other, ……then they [pointing at the legs of the trapezoid] would not be a fixed distance away from each other, they could not be parallel to each other, eventually, they would intersect.

ALY: So, opposite angles are not equal.

Figure 4.21 ALY's verification of the first implication at the Post-Interview

To my request, "Can you prove that opposite angles are equal, knowing that two opposite sides have the same measure?" ALY drew a parallelogram and constructed a mathematical proof by assuming the parallel sides of the polygon as given, and applied the proposition that when parallel lines are cut by a transversal, the adjacent interior angles add up to 180 degrees. She then solved the equations algebraically to justify that opposite angles have the same measure. ALY's drawing of a parallelogram and her construction of the proof is shown in Figure 4.22.



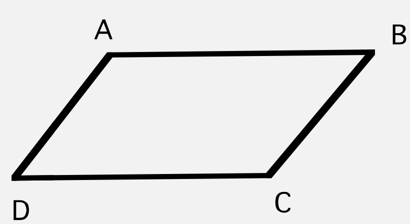
ALY's drawing

$\overline{AB} \parallel \overline{DC}$ (given)
$\overline{AD} \parallel \overline{BC}$
$\angle A + \angle B = 180^\circ$ (prop. of //-ogram)
$\angle A + \angle D = 180^\circ$ (prop of //-ogram)
$\angle B \cong \angle D$
$\angle C + \angle B = 180^\circ$ (prop of //-ogram)
$\angle C \cong \angle A$

Figure 4.22 ALY's proof that opposite angles have the same measure

To gain a better understanding of ALY's thinking, I asked for clarification, and her response to that as follows:

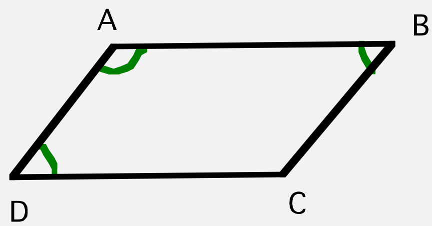
13. Interviewer	Can you explain to me what you wrote?
14. ALY	We're given that these opposite sides are parallel to each other. So it's a parallelogram. We know that adjacent angles add up to 180 degrees.



AB // DC
AD // BC (given)

15. Interviewer	How do you know this is a parallelogram?
16a. ALY	Because it's a four-sided figure and opposite sides are parallel to each other and equal to each other, in length.

16b. ALY	A and D add up to 180 degrees, and also A and B add up to 180 degrees. So, that must mean that D and B are equal to each other, because they are both supplementary to A. So, the same is true for angles C and A, because B and C add up to 180 degrees and B and A add up to 180 degrees. So, C and A must be equal too.



$\angle A + \angle B = 180°$ (prop. of //.ogram)

$\angle A + \angle D = 180°$ (prop. of //.ogram)

$\angle B \cong \angle D$

$\angle C + \angle B = 180°$ (prop. of //.orgam)

$\angle C \cong \angle A$

ALY's routines of substantiation changed from constructing parallelograms that match the descriptions in the definitions at the Pre-Interview, to formulating new narratives "angle B equal to angle D, angle A equal to angle C" using the endorsed narratives "adjacent angles add up to 180" and "two angles supplement to the same angle are the same" at the Post-Interview. Although ALY was able to use mathematical symbols in her proofs and she developed some skills of proving in geometry, she still could not make a clear distinction between what was given and what was to be proved in the Post-Interview. For example, ALY was to assume a quadrilateral with opposite sides of the same length as the given, but instead she incorrectly assumed the polygon was a parallelogram and used that to begin her substantiation. It appeared that ALY was at the beginning stage of constructing mathematical proofs. More evidence of the change in ALY's geometric discourse was in her word use. In the following section, I describe the change in ALY's use of mathematical terms during the two interviews.

ALY's use of the word *parallelogram* changed from indicating both *rhombi* and *parallelograms* in the Pre-Interview, to representing a hierarchy of classifications of parallelograms in the Post-Interview. ALY also used more mathematical terms between the Post-Interview than during the Pre-Interview. Let us look at ALY's use of the general words *quadrilateral*, *parallelogram*, *rectangle*, *square*, *rhombus*, *trapezoid* and *kite*. The total frequencies of these categories of quadrilaterals at the Pre-interview and the Post-Interview are listed in Table 4.23 and Table 4.24.

Table 4.23 The frequencies of ALY's use of the names of quadrilaterals at the two interviews

Name	Frequency					
	Pre-T1	Pos-T1	Pre-T2	Pos-T2	Pre-T3	Pos-T3
Quadrilateral	0	1	0	4	0	1
Parallelogram	3	4	6	14	3	4
Rectangle	9	3	4	4	0	1
Square	1	3	3	8	0	0
Rhombus	11	3	0	3	0	0
Trapezoid	2	2	0	0	0	0
Kite	0	0	0	0	0	0

Table 4.24 shows the word *parallelogram* (n=34) was the most frequently used during the interviews, being mentioned in all three tasks (see Table 4.24). The word *rectangle* (n=21) was the second most frequently used, and *rhombus* (n=17) the third. The word *kite* (n=0) was not mentioned at all in both interviews, and *trapezoid* (n=4) was the second least mentioned. Table 5

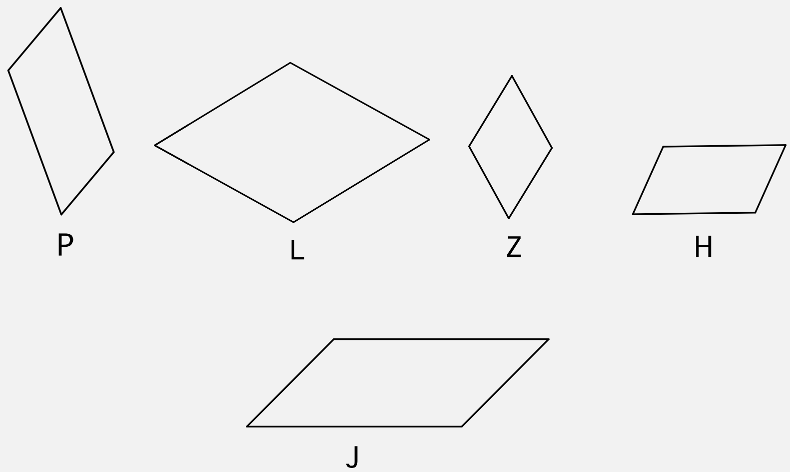
shows that the words *rhombus* and *trapezoid* were mostly mentioned in Task One, where interviewees were asked to group the polygons. There was an increase in use of the words *quadrilateral*, *parallelogram* and *square* from the Pre-Interview to the Post-Interview. In particular, the word *quadrilateral* was used only during the Post-Interview. The word *parallelogram* almost doubled in the Post-Interview, while the word *square* almost tripled. There was a reduction in the use of the words *rectangle* and *rhombus* at the Post-Interview. The following section will discuss findings of ALY's use of the word *parallelogram* at the interviews.

Table 4.24 Total frequencies of ALY's use of names of quadrilaterals at the two interviews

Name	Frequency	
	Pre	Post
Quadrilateral	0	5
Parallelogram	12	22
Rectangle	13	8
Square	4	11
Rhombus	11	6
Trapezoid	2	2
Kite	0	0

In an earlier section, I described ALY's routine procedures for sorting quadrilaterals in Task One. In the Pre-Interview, ALY identified all 4-sided polygons with opposite sides parallel and without right angles, as *rhombi*. My conversations with ALY revealed that she did not have clear concepts of a rhombus and a parallelogram.

30. ALY: P is a rhombus, L is a rhombus, J is a rhombus, Z and H are all rhombuses. [Pointing at these polygons]



31. Interviewer: What is a rhombus?

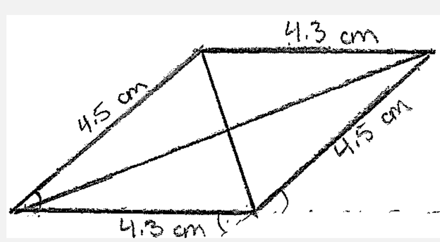
32. ALY: a rhombus would be the opposite sides and the opposite angles are equal and it's a four-sided figure.

33. Interviewer: What is a parallelogram?

34. ALY: A parallelogram with opposite sides and angles are equal. … This is a parallelogram [pointing at Fig. J].

In this conversation ALY made no distinction between a *rhombus* and a *parallelogram*, as she described both as a four-sided figure with opposite sides and angles equal. In Task Two, ALY drew a picture of a parallelogram with the properties of a rhombus.

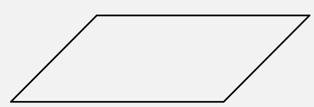
3. Interviewer Why do you think this is a parallelogram?



4. ALY	Because opposite sides are equal and parallel.
	… …
49. Interviewer	What can you say about the diagonals of this parallelogram?
50. ALY	They are perpendicular.

ALY's drawing of a parallelogram looked like a rhombus. Our conversations led ALY to investigate the sides and the diagonals of the parallelogram. She confirmed that the figure was a parallelogram with opposite sides equal, and the diagonals of the parallelogram were perpendicular. Of course, a parallelogram with diagonals perpendicular is a rhombus. These observations suggest that ALY was not aware of the differences between a *parallelogram* and a *rhombus*, and treated them as the same entity. To ALY at the time of the Pre-Interview, a parallelogram *was* a rhombus.

57. Interviewer   Can you identify a rhombus here?



J

58. ALY   J. [Pointing at Fig. J]

My interview with ALY showed that her use of the word *parallelogram* applied of *rhombi* and *rectangles*, a family of a four-sided figures that have opposite sides equal and parallel. Figure 4.23 illustrates ALY's use of the word *parallelogram* at the Pre-Interview

Parallelogram

Rhombus/Parallelogram   Rectangle

Figure 4.23 (cont'd)

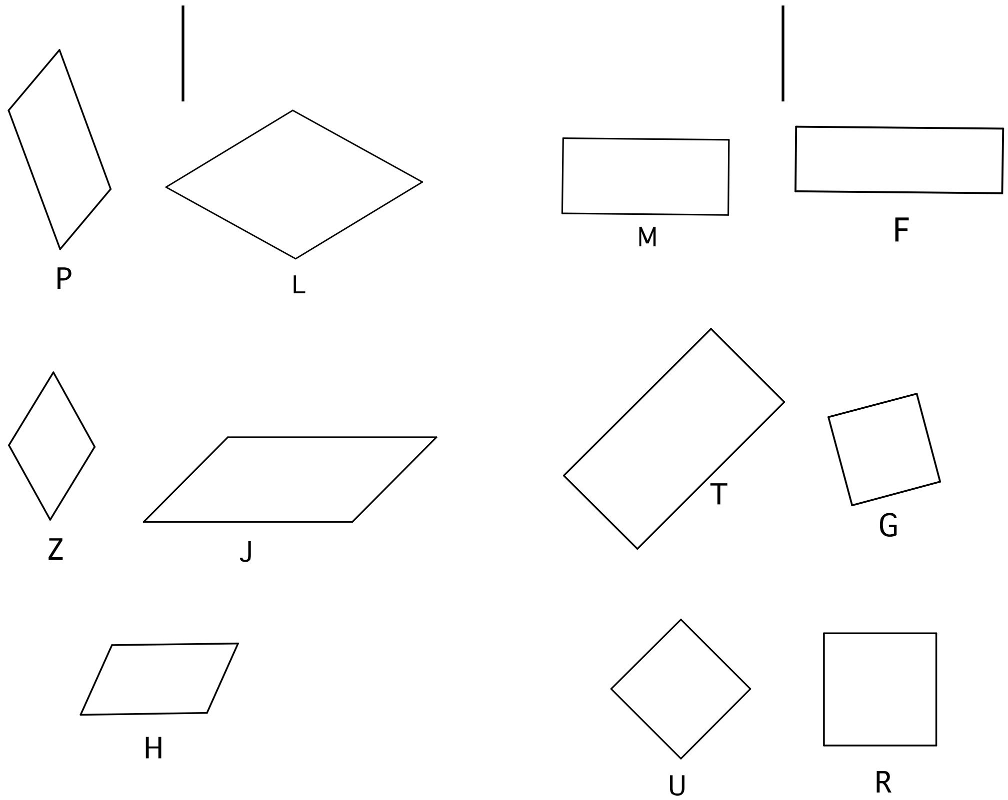


Figure 4.23 (Previous Page) ALY's use of the word parallelogram at the Pre-Interview

ALY's use of the words *parallelogram* and *rhombus* refers to a family of four-sided polygons that share a common descriptive narrative: opposite sides are equal and parallel. During the Pre-Interview, ALY did consider squares as rectangles, but she made no connections between a rhombus and a square. Her grouping and identification of quadrilaterals suggests that she had no clear understanding of a hierarchy of classifications of quadrilaterals.

At the Post-Interview, the change in ALY's use of the names of quadrilaterals showed an understanding of the word *parallelogram*, as revealed in her hierarchy of the classifications of parallelograms. In this hierarchy, the word *parallelogram* describes a collection of quadrilaterals with different appearances and names, and arranged by the characteristics of their angles (i.e., right angle versus non-right angle) and sides (i.e., all sides equal versus opposite sides equal). Although given different *names* such as *rectangles*, *parallelograms*, *rhombi* and *squares*, they are all called *parallelograms* because they fit the description of opposite sides being equal and parallel. This hierarchy of classification is analyzed in Figure 4.24.

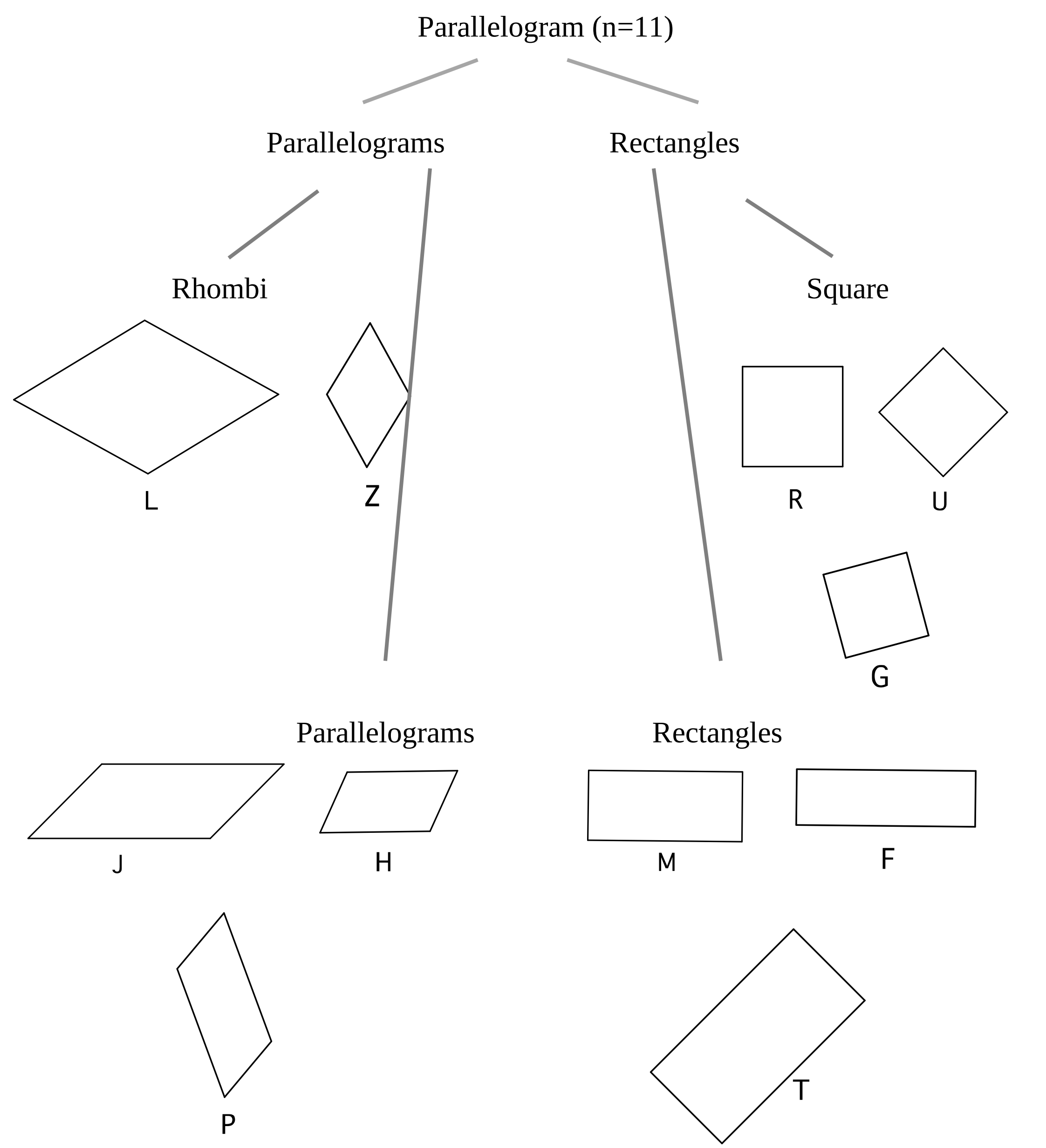


Figure 4.24 ALY's use of the word parallelogram at the Post-Interview

There was also a change in ALY's use of the names of the parts of parallelograms. She used more mathematical terms describing the relations between angles and sides in the Post-Interview than in the Pre-Interview. A word search of the names of the parts of parallelograms included *angle*, *sides, length*, *parallel side*, *opposite side*, *opposite angle*, *right angle* and *diagonal*. Findings show that ALY used more words describing the parts of parallelograms than the names of parallelograms. Tables 4.25 and 4.26 provide word usage frequency of word usage at the two interviews.

Table 4.25 The frequencies of ALY's use of the names of the parts of parallelograms at the two interviews.

Name	Frequency					
	Pre-T1	Pos-T1	Pre-T2	Pos-T2	Pre-T3	Pos-T3
Angle	6	9	23	52	5	16
Side	13	10	15	27	4	16
Length	0	3	3	12	9	7
Parallel side	1	2	2	1	1	1
Opposite side	3	3	7	9	3	5
Diagonal	0	0	5	13	1	0
Right angle	3	1	0	1	0	0
Opposite angle	1	0	5	1	3	4

Table 4.26 Total frequencies of ALY's use of names of the parts of parallelograms at the two interviews

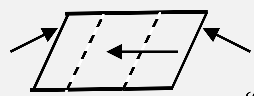
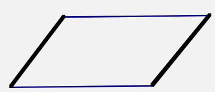
Name	Frequency	
	Pre	Post
Angle	34	77
Side	32	53
Length	12	22
Parallel side	4	4
Opposite side	13	17
Diagonal	6	13
Right angle	3	2
Opposite angle	9	5

Table 4.26 shows that the most frequently used word relating to the parts of parallelograms was *angle* (n= 111), with its usage of the word doubling in the Post-Interview over the Pre-interview. The word *angle* (n=75) was mentioned mostly in Task Two (see Table 4.25). The word *side* (n=85) was the second most frequently mentioned. The words *angle* and *side* were mentioned in all tasks. After *angle* and *side*, the words *length* (n=34) and *opposite side* (n=30) were the next most frequently mentioned at both interviews. The word *length* was mostly used in Task Two and Task Three, whereas the term *opposite side* was mentioned in all three tasks. Likewise the word *diagonal* (n=19) was mostly used in Task Two. These results were expected, as Task Two asks interviewees about the relations of the angles, sides and diagonals of a parallelogram. The term *right angle* (n=5) was the least mentioned, and it was used mostly in Task One. The term *parallel side* (n=8) was the second least mentioned, and it was mentioned only one or two times during each task. The term *opposite angle* was mentioned fourteen times and it was used mostly in Task Two and Task Three. The words *parallel side* and *opposite angle* were not frequently mentioned during the interviews.

During the Post-Interview, ALY used more mathematical terms to describe the relations between the angles of the parallelograms than at the Pre-Interview. More specifically, she used the terms *alternating interior angle* (n=6) and *adjacent angles* (n=3) in the Post-Interview, whereas she used “this angle” and “that angle” to refer to such angles in the Pre-Interview. ALY also used the word *quadrilateral* (n=5) in the Post-Interview, whereas she used the term “four-sided figures” to describe such polygons in the Pre-Interview.

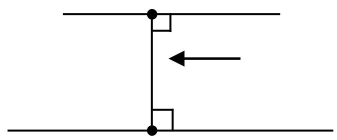
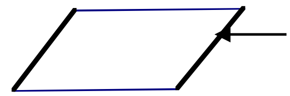
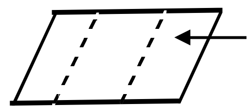
Lastly, I want to draw attention to ALY’s use of the word *distance*. The word *distance* was mentioned eight times during the Post-Interview, two times for Task Two and six times for

Task Three. Let us look at one case of ALY's use of the word *distance* during the Post-Interview.

24a. ALY	Because the opposite sides are parallel to each other, they have a fixed distance away from each other. So, the distance from here to here is the same as here to here, it's going to be the same all the way through.
	 "fixed distance"
	… …
6c. ALY	Because they have a fixed distance away from each other and they always have a fixed distance away from each other [pointing at the opposite sides of the parallelograms]
	 "fixed distance"

ALY's use of the word *distance* was ambiguous. She indicated that the distance between two parallel lines was the length of segments parallel to the other pair of parallel sides of the parallelogram [24a; 6c]. Mathematically, we define *distance* differently in this context. Table 4.27 illustrates the mathematical definition of *distance* and ALY's use of the word *distance*.

Table 4.27 The word distance in geometry and used by ALY

Distance between two parallel lines: 	 ALY’s use of “fixed distance” 

In this section I have shown evidence of changes in ALY’s geometric discourse. ALY changed from relying on the measurements of angles and sides to verify their congruency, to using axioms and propositions to substantiate claims about the congruent parts of the parallelograms. During the Post-Interview, although ALY’s mathematical proofs were not all correct, she demonstrated an ability to construct mathematical proofs using symbols and justifications. ALY’s word use regarding names of parallelograms also changed as we see a structured hierarchy of classifications of parallelograms at the Post-Interview.

Among the twenty interviewees, five of them showed no change in their van Hiele levels in the van Hiele Geometry Test conducted at the pretest and posttest. These five interviewees were AYA (2-2), ARI (3-3), AJA (3-3), ALI (3-3) and ARA (3-3). Therefore, in this section I describe AYA and ARI’s geometric discourses, and I point out the differences and changes in their geometric discourses in the context of quadrilaterals and triangles.

Case 4: Changes in AYA’s Geometric Discourse

AYA was a college sophomore at the time of the interviews. AYA took her last geometry class seven years prior to the geometry and measurement class. The van Hiele Geometry Test showed that she was at Level 2 at the pretest, and stayed at Level 2 according to the posttest ten weeks later. I interviewed AYA after both tests, and analyzed both her interview responses. A summary of findings about AYA's geometric discourse follows:

- My analyses from the Pre-Interview and the Post-Interview show that AYA's routines of sorting polygons remained the same.
- AYA's routine of substantiation changed from colloquial mathematical discourse, where her substantiation was a set process of an activity at object-level, towards a mathematical discourse using previously endorsed narratives about mathematical objects at an abstract-level. AYA's routine procedures were descriptions about the processes of activities using transformations such as reflection, translation, and rotation at the Pre-Interview, but were constructions of newly endorsed narratives using propositions and definitions at the Post-Interview.
- When verifying congruent figures, AYA chose Side-Side-Angle (SSA) as the conditions of verification at the Pre-Interview, which was incorrect, whereas at the Post-Interview AYA chose angle-side-angle and side-angle-side, valid congruent criterions for verification of congruent triangles.
- When substantiating the equivalence of two definitions, AYA did not know how to construct newly endorsed narratives from given definitions at the Pre-Interview, whereas at the Post-Interview AYA constructed a newly endorsed narrative using deduction.
- There were changes in AYA's use of mathematical terminology such as the names of polygons and their parts.

AYA's routine procedures for sorting polygons were observed and analyzed in Task One. During the Pre-Interview, when AYA was asked to sort polygons into groups, her first question was, “Am I doing it on the assumption that those are right angles [pointing at the angles of a square], and by itself can I assume anyway?” AYA’s first attempt at sorting geometric shapes “in terms of the numbers of sides they had” resulted in the following: 1) 3-sided figures (n=4) consisting of all triangles; 2) 4-sided figures (n=13) consisting of all quadrilaterals; and 3) 6-sided (n=1) figures, which is Fig. V (a hexagon). When I asked AYA to subgroup the 4-sided group, her first reaction was, “If I can assume that the sides appear to be parallel to each other”, while pointing to the opposite sides of a parallelogram. AYA then rearranged this 4-sided group into three subgroups, and subsequently, she rearranged the 3-sided group into three subgroups as well. See Figure 4.25 for details of AYA’s subgrouping of the quadrilaterals and the triangles on the first attempt.

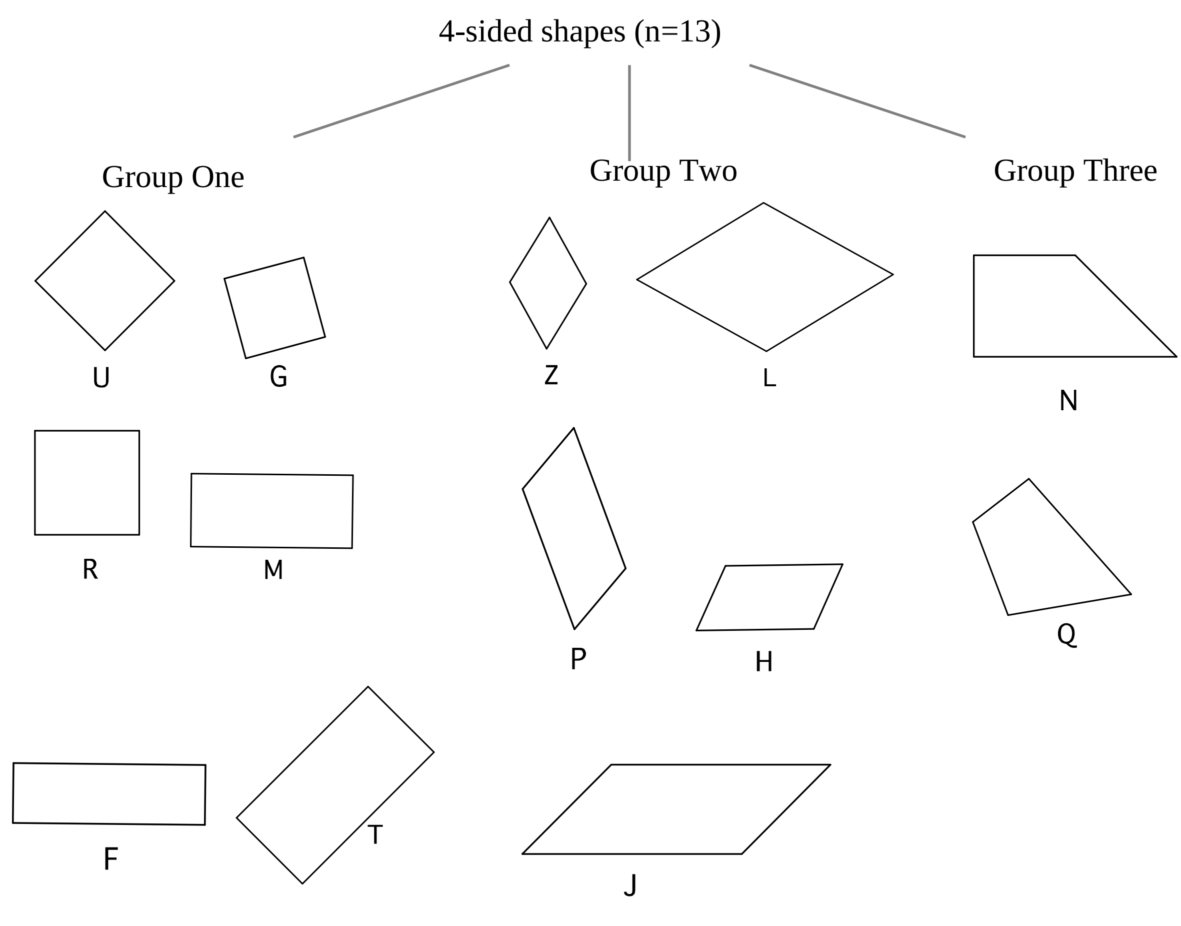


Figure 4.25 (cont'd)

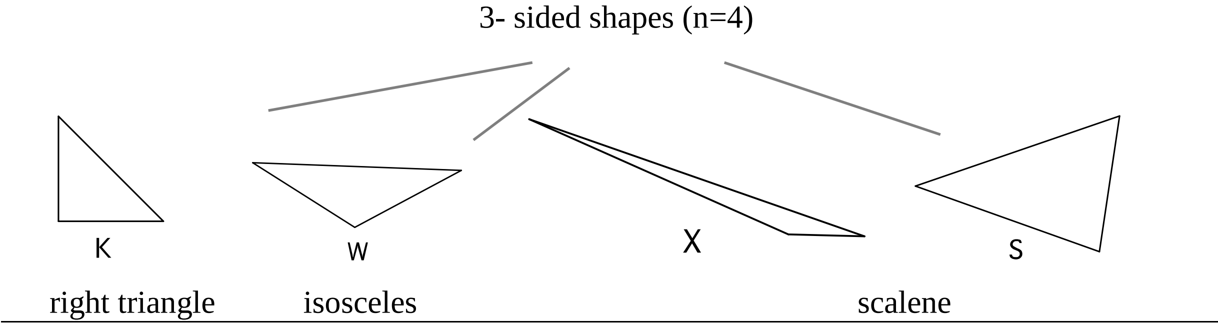
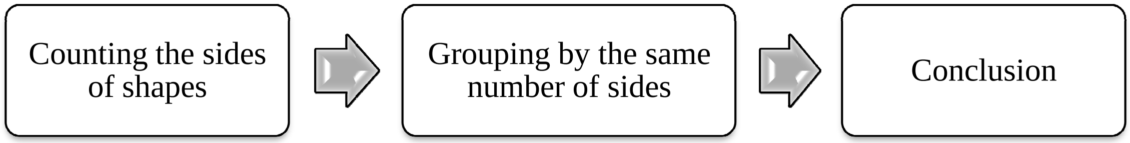


Figure 4.25 AYA's first attempt at sorting polygons at the Pre-Interview

Figure 4.25 presents three subgroups for quadrilaterals: squares/rectangles, parallelograms and a group of 4-sided figures that do not fit into the descriptions of the two previous groups. AYA made it very clear about the characteristics of each group. For example, AYA talked about the *parallelograms group* consisting only of the parallelograms that "don't have right angles", and the *squares/rectangles group* consisting of figures that "have four sides, all right angles, pairs of sides are parallel and have the same length". Similarly, AYA sorted triangles into three groups by the characteristics of their angles or sides. On the first attempt, AYA's courses of actions in response to the questions about sorting geometric figures focused on characteristics of angles (e.g., right angles) and sides (e.g., parallel sides or equal sides). During the interview, AYA did not use measurement tools such as rulers or protractors to check the angles and sides of the figures, but instead she chose geometric figures under the assumptions that "the sides appeared parallel" and "angles are right angles". Figure 4.26 illustrates AYA's routine procedures of sorting geometric shapes into different groups.

First prompt: “Sort the shapes into groups”



First prompt continued: “Can you subgroup 3-sided and 4-sided group?”

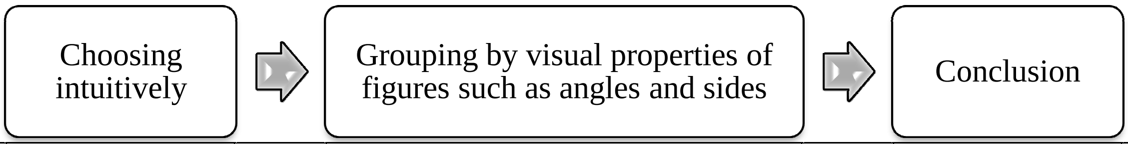
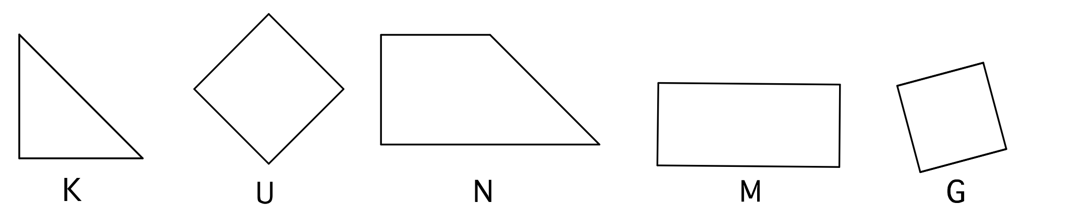


Figure 4.26 AYA’s first attempt of the *routine of sorting* at the Pre-Interview

When I asked AYA to regroup the figures differently, her first response was, “I want to separate them into shapes containing right angles and shapes that do not contain right angles”. Among the eighteen geometric figures in Task One, AYA included figures (n=8) with at least one right angle in Group One, and included the figures (n=10) with no right angles in Group Two. See Figure 4.27 for some examples.

Group One: shapes contain right angles (n=8)



Group Two: shapes do not contain right angles (n=10)

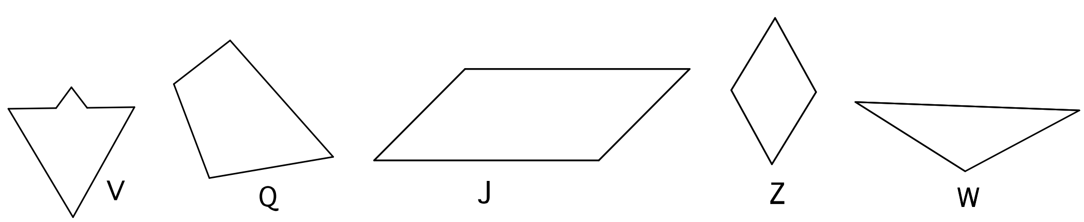
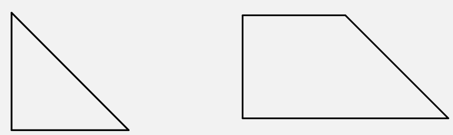


Figure 4.27 Examples of AYA’s regrouping at the Pre-Interview

In Figure 4.27 we see a variety of figures in each group, where AYA simply divided figures that have right angles from those figures that do not have right angles. I then asked AYA to subgroup Group One, and she provided the following response:

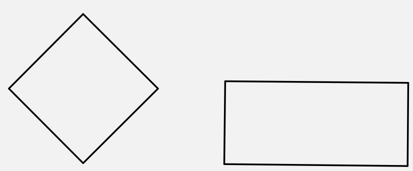
Interviewer: Can you subgroup Group One?

AYA: I guess for Group One [subgroup 1]…I could take squares and non-squares …I could take shapes that have acute angles…like triangle…like K and N [pointing at Fig. K and N]



K N

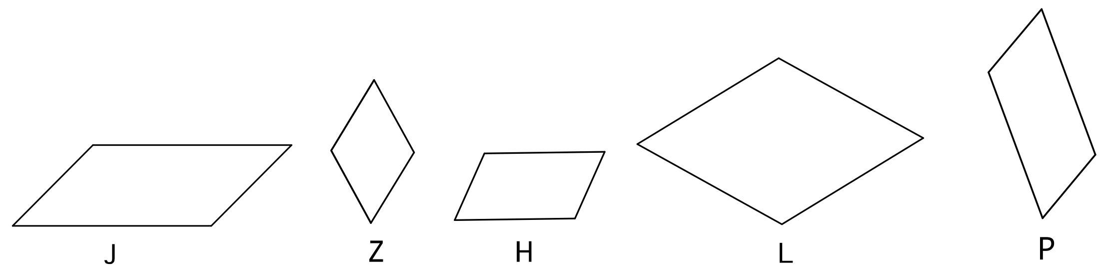
AYA: I am sure that by defining the second group [subgroup 2] …I don't know they kind seem exclusive.... will follow figures that don't contain acute angles [pointing at Fig. U and M]



U M

AYA continued to talk about her strategies of subgrouping Group Two. She divided Group Two into two subgroups that do not contain right angles: one with figures that have at least one set of parallel sides, and the other with figures that have no parallel sides. Figure 4.28 illustrates the two subgroups in Group Two.

Subgroup 1: with at least one set of parallel sides



Subgroup 2: with no parallel sides

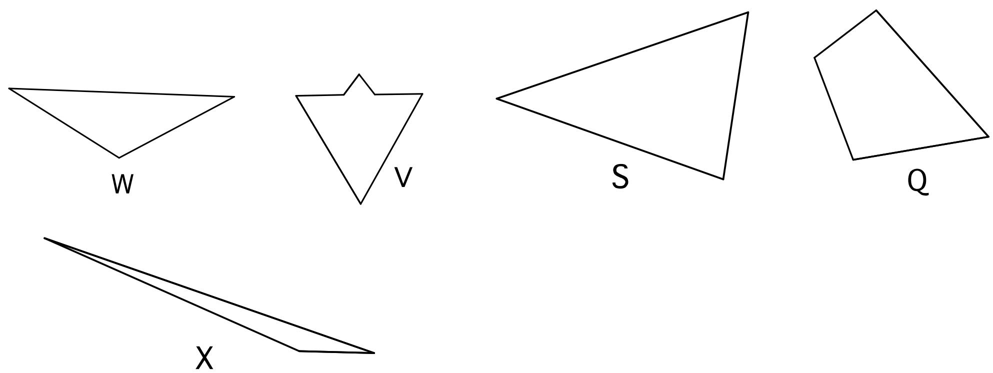


Figure 4.28 *The two subgroups of Group Two at the Pre-Interview*

During the regrouping, AYA's courses of actions for sorting geometric shapes focused mostly on the characteristics of the angles of figures, *not* the sides of figures. That is, AYA first divided the entire group of figures (n=18) into two groups, depending on whether the figures had a right angle or not; and then divided Group One into two subgroups based on whether the figures had an acute angle or not. AYA divided Group Two according to whether the figures had parallel sides or not; half of the figures in Group Two are parallelograms. AYA's routine procedures in sorting geometric figures at the second attempt are summarized in Figure 4.29.

Second prompt: "Find another way to sort them differently?"

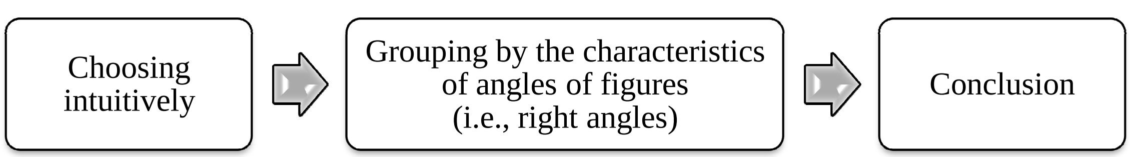


Figure 4.29 (cont'd)

Second prompt: "Can you subgroup the two groups?"

Choosing intuitively → Grouping by the characteristics of angles and sides of figures (i.e., acute angles and parallel sides) → Conclusion

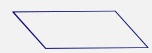
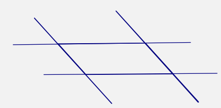
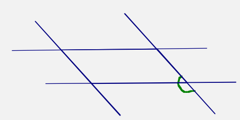
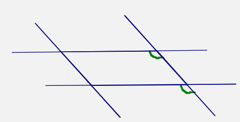
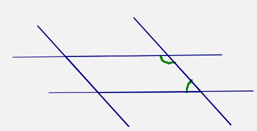
Figure 4.29 AYA's second prompt of the routine of sorting at the Pre-Interview

Ten weeks later I interviewed AYA again, and found *no change* in her routine procedures for sorting geometric shapes when compared to those of the Pre-interview. For example, at the Post-Interview, when I asked AYA to group the figures she said, "the first thing I want to do is separate them by numbers of sides, like I did last time [at the Pre-Interview]". When I asked AYA to regroup the figures, she replied, "This [group] just assumes that all figures appeared to have right angles … Group Two could just be all the figures that don't contain right angles". Therefore AYA's routine procedures for grouping polygons at the Post-Interview were similar to what she did at the Pre-Interview.

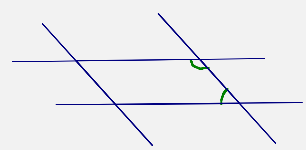
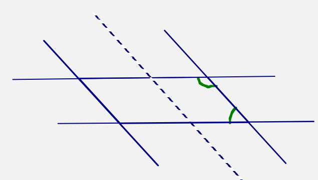
Although I did not find any changes in AYA's routine procedures in classifying geometric figures between the time of the Pre-Interview and the Post-Interview, I noted changes in her routine procedures of substantiation of narratives. In the following, I describe changes in AYA's substantiation routines in Task Two and Task Three between the Pre-interview and the

Recall that a *routine of substantiation* is a set of patterns describing a process of using endorsed narratives to produce new narratives that are true. For instance, in the context of this study, a *routine of substantiation* describes what an interviewee did, step-by-step, to substantiate her/his declared statements that opposite sides are equal in a parallelogram. One important finding in AYA's geometric discourse was the changes in her *routines of substantiation* as observed and analyzed in Task Two and Task Three.

Recall that Task Two asks interviewees to draw two parallelograms that are different from each other, and then to discuss the angles, sides and diagonals of these two parallelograms. At my request, AYA drew a parallelogram and declared, “in this parallelogram all angles should add up equal to 360°”. After my prompt for substantiation, “how do you know that all angles add up to 360°?” AYA produced the following:

14a. AYA	Well, when you have parallel sides, you can extend all the sides…	AYA’s drawing:  AYA extended the sides of parallelogram: 
14b. AYA	…it's 180 degrees and they’re complementary angles …	Pointing at the two angles that form straight angles: 
14c. AYA	…but you can see that this angle really just match this angle	Pointing at the two angles: 
14d. AYA	…so you know that these two angles together are gonna equal 180 degrees	Pointing at the two angles: 

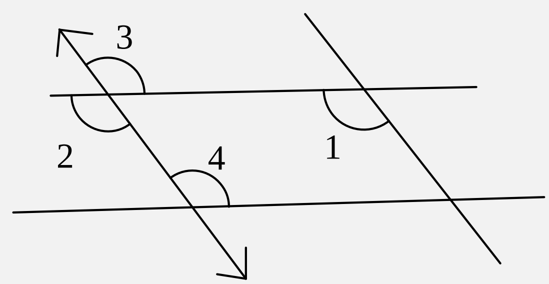
In the preceding substantiation, AYA first drew extended lines on the sides of the parallelogram, in saying “you have parallel sides… you can extend…”[14a], and identified a vertex angle and its corresponding exterior angle forming a “complementary angle”[14b]; and she then identified an adjacent vertex angle transversal to the same exterior angle and made an intuitive claim about the two angles, “you …see this angle…matches this angle” [14c]. AYA concluded that the two adjacent vertices of a parallelogram added up to 180 degrees [14d]. Using this endorsed narrative, “two angles equal 180 degrees”, AYA continued her substantiation to the final step:

24a. AYA	From this diagram and the parallel sides, these two angles add up to 180 degrees…	Pointing at the two angles: 
24b. AYA	… the fact that it’s just like a mirror image, the two 180 sets of angles are just gonna add up to 360 degrees.	Making an invisible line: 

AYA used her previously endorsed narrative, “two angles add up to 180”, and then endorsed a new narrative, “two sets of 180 degrees angles add up to 360 degrees” because these are a “mirror image” (i.e., a reflection) of each other, and drew a reflection line (i.e., the dashed line in 24b). AYA’s substantiation of the narrative, “all angles add up to 360 degrees” was intuitive and self-evident because the reflection line that AYA drew was not a line of reflection of the parallelogram. Mathematically, this parallelogram only has point symmetry, symmetry with respect to the center of the parallelogram (i.e., where the diagonals intersect), and not line symmetry. In this example, AYA used a “mirror image”(i.e., reflection) to draw a conclusion that all the angles add up to 360 degrees.

During the Pre-Interview, AYA frequently used reflections, rotations and translations in her substantiations of narratives. For example, when I asked AYA to verify her claim that “two opposite angles (i.e., ∠ 1 and ∠ 4) are equal”, she provided the following response:

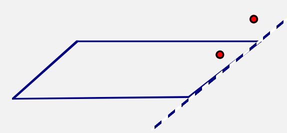
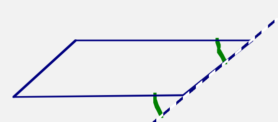
37.Interviewer:	How do you know this angle is equal to this? [Pointing at ∠1 and ∠4]

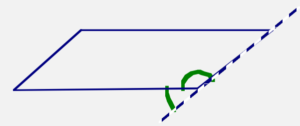
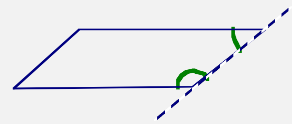
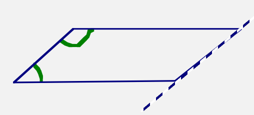


38a. AYA:	…this angle [pointing at ∠1] can just be slid over to this position and create this angle [pointing at ∠2] …
38b. AYA:	…this line [drawing arrowhead on the line] can be rotated so that this angle [pointing at ∠2] now becomes this angle [pointing at ∠3].
38c. AYA:	…this angle [pointing at ∠3] at this intersection, can just be slid down and then be in this angle’s position [pointing at ∠4].
38d. AYA	So these two angles are equal [Pointing at ∠1 and ∠4]

In this case, AYA used words such as “slid over”, “rotated” and “slid down” to indicate a sequence of movements preformed to substantiate the claim that “two opposite angles are equivalent”. Lines and angles are static mathematical objects, but AYA used these sequences of imaginary movements to complete her substantiation; and through AYA’s description, these imaginary movements became visible to me. AYA’s substantiation was intuitive and visual. AYA’s substantiation focused on the processing of the activities of mathematical objects, rather than on discussions about these mathematical objects. I conclude that AYA’s routine procedures operated at the object level at the time of the Pre-Interview.

Ten weeks later when asked for substantiation, AYA used mathematical axioms and propositions to verify her claims. The following brief substantiation was typical at the Post-Interview:

16a. AYA	… angles on a straight line add up to 180 degrees…	Extending one side of the parallelogram with a dashed line, and pointing at the two angles: 
16b. AYA	…this angle here is the same as this angle… Because parallel lines meet a third line at the same angle.	Pointing at the angles: 
16c. AYA	By the same reason [referring to	Pointing at the two angles:

	16a], this angle added to this angle equals 180 degrees…	
16d. AYA	…these two also add up to 180 degrees	Pointing at the two angles: 
16e. AYA	…for a similar reason, these two angles add up to 180 degrees...	Pointing at the two angles: 
16f. AYA	Together they equal 360 degree…	

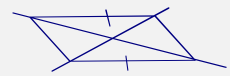
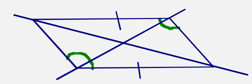
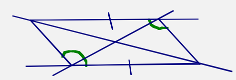
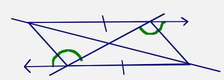
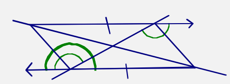
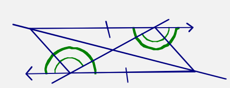
AYA made the same statement as previously about the angles of a parallelogram, “all added together they equal 360°”. In contrast to AYA’s routines of substantiation at the Pre-Interview, this example shows two changes that are evident: The first is that in each step of substantiation, AYA provided endorsed narratives (e.g., mathematical axioms and propositions, etc) as evidence instead of reasoning intuitively. For example, AYA explained how two transversal angles are equivalent, not because you “can see it” as in the Pre-Interview, but as a result of “two parallel lines meet a third line at the same angle” in the Post-Interview. The second change occurs in AYA’s conclusion that “all angles add up to 360 degree”. At the Pre-Interview, she argued on the assumption of this “mirror image”, whereas at the Post-Interview AYA reached her conclusion in a repeat of a similar proof that “two angles add up to 180 degrees” for two adjacent angles in a parallelogram [16e-f]. Thus, one change in AYA’s *routine of substantiation* was the shift from descriptions about the processes of activities at the object level towards the abstract level. I will suggest that the maturity of this abstract level of substantiation is revealed in AYA's substantiation of congruent triangles.

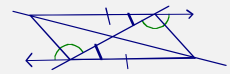
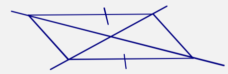
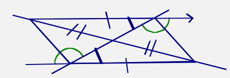
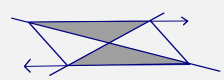
In the following example, I will describe the changes in AYA's *routines of substantiation* of two congruent triangles that I observed between the Pre-Interview and the Post-Interview. To describe these changes I looked at two aspects: 1) change from the use of transformations in the process of substantiation at the object level, to the use of mathematical axioms at the abstract level; and 2) the change in the choices of elements needed for verification of congruent triangles.

During the interviews, participants were asked to substantiate their declared narratives about the angles, sides and diagonals of a parallelogram. For example, when asked for substantiation of the narratives, "opposite sides are equal", "opposite angles are equal" and /or "diagonals bisect each other", some interviewees would support their narratives by using rulers and protractors to measure the corresponding sides and angles, whereas other interviewees would try to use mathematical proofs to verify their statements. Using triangle congruency to substantiate the corresponding sides and angles are congruent in a parallelogram is a common method students utilize.

During my interviews with AYA, when asked for substantiation of declared narratives about the sides and angles of a parallelogram, AYA's first response was, "other than just measuring them?" AYA expected to substantiate her declared narratives without using the measurement tools at both the Pre-Interview and the Post-Interview. As an example, the following are AYA's routine procedures for the narrative, "diagonals bisect each other in a parallelogram", using the triangle congruency method at the Pre-Interview.

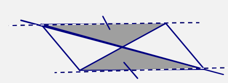
When AYA discussed the diagonals of the parallelogram, she talked about diagonals creating two pairs of congruent triangles. After my prompt for substantiation, AYA identified one pair of such congruent triangles, and then identified two corresponding sides and two corresponding angles from the two triangles to verify their congruency:

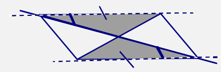
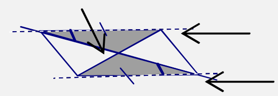
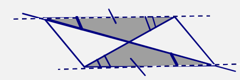
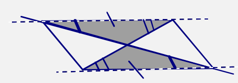
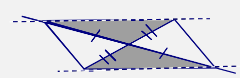
64a. AYA	Because I previously established that, it is given that these are parallel sides…	AYA added two marks: 
64b. AYA	And, these angles are equal and when lines intersect…	AYA added two angle signs: 
64c. AYA	… it's essentially the same intersection, translated to a new position…	AYA drew extended lines: 
64d. AYA	… I was suggesting that this angle is the same as this angle here.	AYA added an arrowhead on the two extendded sides, and two angle signs: 
64e. AYA	… And that likewise, the complementary angles, the smaller angle that makes it add up to 180 degrees…	AYA identified two angles that form a striaght angle: 
64f. AYA	… is the same over here…	AYA identified another two angles that form a straight angle: 

64g. AYA	… So, now I know that the angle here of this triangle is equivalent to the angle here of this triangle…	Pointing at the alternating interior angles: 
64h. AYA	… and this side length is, the same of this side length… So, I've already shown how a side length and an angle match of each…	Referring to the two sides: 
64i. AYA	… And then diagonals bisect themselves equally. I can't really prove that, but I'm suggesting that this side length is the same as this side length…	AYA added two marks on the diagonal: 
64j. AYA	…this triangle is equivalent to this triangle here.	The shaded area indicates two congruent triangles: 

AYA's substantiation included two parts: the first was the substantiation of the equivalence of alternating interior angles [64b-64g], and the second was the verification of congruent triangles. The first part of substantiation, "this angle is equivalent to this angle" (i.e., alternating interior angles), was intuitive and self-evident. To show that opposite angles are equivalent in a parallelogram [64b], AYA used an instinctive process of translating the intersection to a new position [64c], and then "suggested" that the corresponding alternating exterior angles were equivalent [64d], another intuitive act. The second part of substantiation involved the verification of the two congruent triangles that she identified. During the process of verification, AYA did not use measurement tools to measure the angles and sides (an object level of verification) to check equivalence, but instead chose three elements of the triangles to verify congruent triangles abstractly. However it is important to note that AYA's choice of these three elements (angle, side, side) for verification of congruent triangles was incorrect, because this criterion does not guarantee congruent triangles. I conclude that AYA's substantiation was a combination of an objective level of substantiation (e.g., these angles are equal), and an abstract level of verification (e.g., two triangles are congruent), even though her choice of the elements for verification was not all correct.

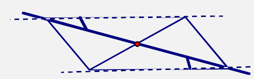
Ten weeks later, I interviewed AYA again, and the same tasks were performed. At the Post-Interview AYA was able to use triangle congruency to substantiate most of her declarations stating that "opposite angles are equivalent", "opposite sides are equivalent" and "diagonals bisect each other" in a parallelogram. AYA was able to choose three exact elements of the six, such as Side-Angle-Side and Angle-Side-Angle, to verify congruent triangles, and she was comfortable using the triangle congruency method. The following response illustrates AYA's substantiation that "diagonals bisect each other":

54a. AYA	… I'm looking at this triangle as compared to this one here…	Pointing at the shaded area: 
54b. AYA	And I know that these two angles are congruent…	AYA marked the angle signs on the two angles in the shaded triangles:

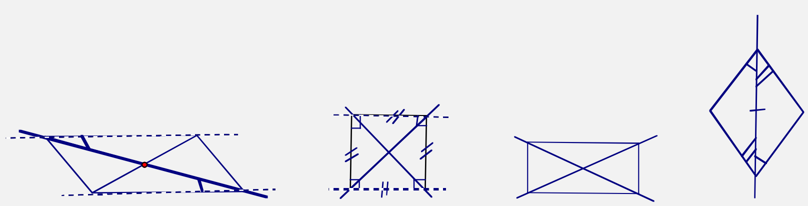
		
54c. AYA	…And between these parallel lines, and now this diagonal,	Pointing at the two parallel lines and one diagonal: 
54d. AYA	… these angles are also congruent.	AYA marked the angle signs on the two angles of shaded triangles: 
54e. AYA	… So, by the triangle test, angle, side, angle, these two triangles are congruent.	Pointing at the corresponding angles, sides and angles of shaded triangles: 
54f. AYA	…which means that this side corresponds with this side and that this side corresponds with that side. That's probably the most roundabout way to find that answer.	Pointing at each half of the diagonals: 

In the preceding substantiation AYA first verified that the “two triangles are congruent” [54e] using the angle-side-angle criterion. She identified the exact three elements (i.e., two angles and their included sides) needed for verification; and used the endorsed narrative "the two triangles are congruent" to construct a new narrative that "diagonals bisect each other", by saying "this side corresponds with this side…."[54f] as a result of congruent triangles. AYA also made no intuitive claim about the equivalence of alternating interior angles at the Post-Interview, as she clearly explained:

46d. AYA …And, we know that between parallel lines, if you take a third line and cross both lines, then it will have angles that are congruent. In this case, this angle and this angle.
[AYA extended the two parallel lines, and marked angle signs on the two alternating interior angles]



During the Post-Interview, AYA applied the same substantiation to other similar situations. For instance, when I asked AYA why diagonals bisect each other in a rectangle, she responded, "the same as what I did in parallelogram, I already established that." When I asked AYA at the end of Task Two, "is it true that in all parallelograms diagonals bisect each other?" AYA responded, "Yes, that's true" and then shared her thinking about this conclusion:



148. AYA	…because when you draw the diagonals in a figure, there is an intersection point and it divides the figure into four triangles. And, regardless of the figure, if it's a parallelogram, these two triangles will be congruent and these two triangles will be congruent [pointing at the two pairs of congruent triangles in the rectangle] So, it can be found that in congruent triangles, corresponding sides will be equal [therefore diagonals bisect each other in all these cases].

In summary I conclude that there was a change in AYA’s routine procedures, from using translation intuitively in the process of substantiation about the equivalence of the angles at the object level, to using mathematical axioms to substantiate the same claim at the abstract level. I am convinced that AYA was more rigorous at the Post-Interview, when she made choices of three elements needed for verification of congruent triangles, than at the Pre-Interview.

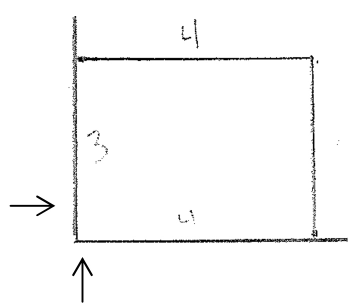
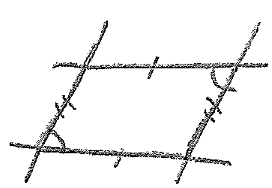
To illustrate AYA’s routines of substantiation, and how that substantiation helped to produce newly endorsed narratives, the following scenario points out a further change in AYA’s geometric discourse in the context of her routine procedures of constructing new narratives from previously endorsed narratives.

Recall that Task Three involves a mathematical proof, in discursive terms, where interviewees were asked to construct new narratives from endorsed narratives (i.e., definitions of a parallelogram). In particular, interviewees were given two definitions of a parallelogram and were asked to substantiate that these definitions are equivalent.

- Definition #1: A quadrilateral is a parallelogram if and only if both pairs of opposite sides have the same length.
- Definition #2: A quadrilateral is a parallelogram if and only if both pairs of opposite angles have the same measure.

My conversations with AYA at both the Pre-Interview and the Post-Interview show that she did not know the mathematical meaning of "two definitions are equivalent"; she thought the statement meant, "they're both saying the same thing" about the parallelograms. At the Pre-Interview AYA did not know how to substantiate the equivalence of two definitions; however she did attempt an analysis by drawing two quadrilaterals that fit the description in definition #1. AYA's drawings from the Pre-Interview are shown in Table 4.27.

Table 4.27 AYA's attempts at proving the equivalence of two definitions

a. AYA' first attempt	
AYA's drawing	Transcripts
	AYA: I wanted to draw a figure… making opposite sides the same length. I thought I was gonna measure the angles, except that I cheated and used a right angle. So, that's not a very good example.
b. AYA's second attempt	
	AYA: Well, this equal this side length, and that equal this side length … and pairs of the angles of the same measure. I think the definitions are equivalent, but how do I prove it without any numbers?

AYA first drew a figure as a concrete example of what is described in Definition #1: a rectangle has opposite sides equal, and in particular the rectangle that AYA drew has opposite sides with measurements of 4 and 3 respectively (Table 4.27a). AYA suggested that “it was not a good example” because the conclusion of “opposite angles are equal” is obvious given that it is a rectangle. As a second attempt, AYA drew an arbitrary quadrilateral with no specific length measurements and she assumed that it had opposite sides equal as described in Definition #1 (Table 4.27b); but she could not continue the proof because she did not know how to prove the angles were equivalent without measurements, as she wondered “what is proof of this?”

In contrast, at the Post-Interview AYA was able to complete the substantiation of “if a quadrilateral has both pairs of opposite sides of the same length, then both pairs of opposite angles have the same measure”, using the Side-Side-Side triangle congruence criterion to verify two congruent triangles. Using this endorsed narrative, AYA then identified the corresponding angles in the two congruent triangles to construct the newly endorsed narrative that “both pairs of opposite angles have the same measure”. AYA’s routine procedures are analyzed and the corresponding transcripts from the Post-Interview are provided in Table 4.28.

Table 4.28 AYA’s substantiation of two congruent triangles at the Post-Interview

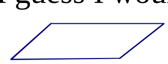
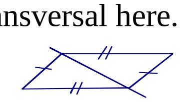
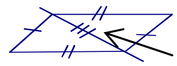
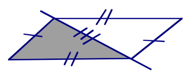
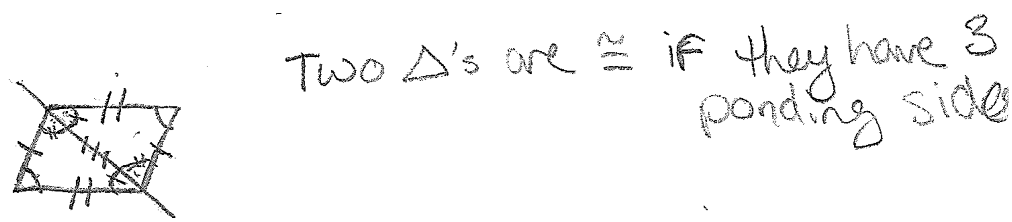
Routine Procedures	Transcripts
1. Draw a figure	4a. I guess I would start with a figure… 
1.1. Identify the given	4c. …just assume that this is a figure that this side length is equal to this side length and this side length is equal to this side length, and that's all we know.
1.2 Draw a transversal	12a. I first would just draw a line from these angles, a transversal here. 
2. Verification of two congruent triangles	

Table 4.28 (cont'd)

2.1 Identify corresponding sides of the triangles	12b. … this is a common side in both of these triangles  common side
2.2 Identifying three elements needed for verification of congruent triangles	12b. It's enough to say that two triangles with three sides congruent to one another, to a corresponding side in another figure, are congruent figures.
3. Conclusion congruent triangles using S-S-S correspondence	 two congruent triangles: shaded vs. non-shaded

AYA's drawing and writing



AYA wrote: "Two Δ's are ≅ if they 3 corresponding sides"

AYA's construction of a new endorsed narrative continued as she was trying to show that "opposite angles in this quadrilateral of the same measure":

Table 4.29 AYA's newly constructed endorsed narrative at the Post-Interview

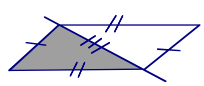
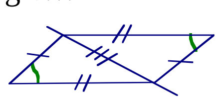
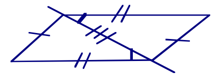
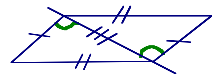
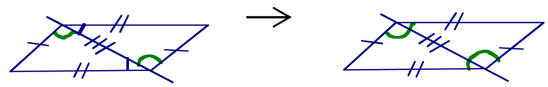
Routine procedure	Transcripts
1. Use previously endorsed narrative	18a. using the fact that these two triangles are congruent 
2. Identify corresponding angles of the congruent triangles	
2.1 Identifying one pair of corresponding angles (i.e., the first pair of opposite angles)	18a. I know that this angle is equal in measure to this angle… 

Table 4.29 (cont'd)

2.2 Identify second pair of corresponding angles	18b. … and this angle is equal to measure to this angle… 
2.3 Identify the third pair of corresponding angles	18c. …And that, this one is equal to this one… 
2.4 If all the parts of two angles are equal then two angles are equal	18d. …therefore, these two added together is gonna equal these two added together, which would make them still the same angle, corresponding… 
3. Conclusion	18d. … a four-sided figure with opposite angles equal…

It is notable that there was a change in AYA's routine procedures of constructing new narratives from previous endorsed narratives. At the Pre-Interview, AYA was not able to finish the proof because she was unsure how to prove the equivalence of the angles without checking their measurements. However at the Post-interview, AYA was able to verify congruent triangles by choosing three elements of six elements needed for verification, and using deduction to construct a new narrative. Thus, AYA's geometric discourse made a transition from a geometric discourse with only a partial properties check at an object level at the Pre-Interview, towards a geometric discourse with routine procedures capable of constructing new endorsed narratives using mathematical axioms and definitions.

I have described AYA's change in routine procedures of substantiation and the changes in her routine procedures of constructing new narratives; now let me describe the changes in AYA's use of mathematical terminologies.

I begin with my general analyses of words that AYA used in naming geometric figures in various conversations; in particular, the words *quadrilateral*, *parallelogram*, *rectangle*, *square*, *rhombus*, *trapezoid* and *kite*. The total frequencies of the names of these geometric figures from the Pre-Interview and the Post-Interview are shown in Table 4.30.

Table 4.30 The frequencies of AYA's use of the names of quadrilaterals at the two interviews

Name	Frequency					
	Pre-T1	Pos-T1	Pre-T2	Pos-T2	Pre-T3	Pos-T3
Quadrilateral	3	1	0	0	2	1
Parallelogram	7	5	33	19	4	6
Rectangle	3	4	0	1	0	0
Square	5	4	6	7	0	0
Rhombus	6	0	0	0	0	0
Trapezoid	3	0	0	0	1	0
Kite	0	0	0	0	0	0

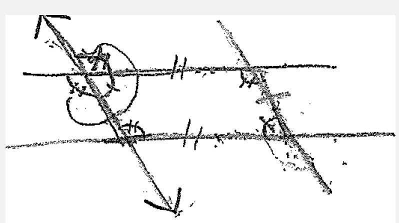
Table 4.31 Total frequencies of AYA's use of names of quadrilaterals at the two interviews

Name	Frequency	
	Pre	Post
Quadrilateral	5	2
Parallelogram	44	30
Rectangle	3	5
Square	11	11
Rhombus	6	0
Trapezoid	4	0
Kite	0	0

Table 4.31 shows that the word *parallelogram* (n=74) is the most frequently used during both interviews, being mentioned in all three tasks. The word *square* (n=22) is the second most frequently used, and the word *rectangle* (n=8) is third. Note that the large difference in the frequency of the words *parallelogram* and *rectangle* (n=66), and between the words

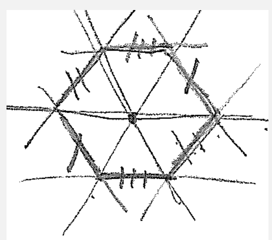
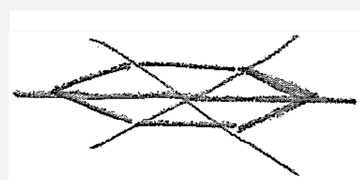
parallelogram and *square* (n=52). The word *kite* (n=0) was not mentioned at all in both interviews. Table 4.30 shows that the word *rhombus* (n=6) and *trapezoid* were only mentioned at the Pre-Interview, and mostly were mentioned at Task One. There was a reduction in use of the words *quadrilateral, parallelogram, rhombus* and *trapezoid* at the Post-Interview, and there was a slight increase in use of the word *rectangle* (n=2) at the Post-Interview. However, the frequencies of the word counts do not provide details about how and in what way these words were used. The following analyses looks at AYA's word meaning in the use of *parallelogram*, *rectangle*, *square*, *trapezoid* and *rhombus*.

In an earlier section, I described my observations of AYA's routine of sorting for Task One. It is important to note that the natures of the tasks designed for the interviews were limited and pre-constructed. For example, when AYA identified geometric figures among given figures in Task One, the pool of choices was limited to eighteen figures and those figures were pre-drawn. Consequently, AYA's misunderstandings about some of the geometric figures were not detected in Task One. It is in Task Two, when I asked AYA to draw two different parallelograms at the Pre-Interview, that I began to understand AYA's misconstrued definition of *parallelogram*, and it was quite different from what I expected:

2. AYA	*AYA's drawing*:  *Note: AYA drew a parallelogram first, and extended sides of the parallelogram later*

3. Interviewer	Why is this a parallelogram?
4a. AYA	I believe that this is a parallelogram because I drew it so that this side… would be parallel to this side [pointing at the two longer sides of the parallelogram]…
4b. AYA	… and this side would be parallel with this side [pointing at the two shorter sides of the parallelogram]

Later I asked AYA to draw a new parallelogram different from the one she drew, and she provided the following responses:

86. AYA	AYA's drawing of new parallelograms:   *Note: AYA drew a hexagon first, and she extended sides of the hexagon later*
87. Interviewer	Why is this a parallelogram?
88. AYA	I think it's a parallelogram… because all the sides are parallel to another side.
89. Interviewer	Why is it a different parallelogram?
90. AYA	It's different…because there are more sides and because the angles are different.

The preceding conversations present an interpretive description of *parallelogram* when AYA used that word at the Pre-Interview. During our earlier conversation, I asked AYA "what is a parallelogram", and she responded, "it [parallelogram] is any figure that has at least one pair of parallel sides. I think trapezoid [pointing at Fig. N, a right trapezoid] is considered a parallelogram". When I asked AYA to write down the definition, she wrote, "A parallelogram is a figure with all sides being pairs of parallel line segments", and that was inconsistent with her verbal statement. Neither AYA's written narrative nor her verbal narrative about parallelograms mentioned the necessary condition of a parallelogram being a quadrilateral. Because of this missing condition, AYA chose a hexagon as an example of a different parallelogram. When identifying and defining parallelograms, AYA focused on the necessary condition of parallel sides. At the Pre-Interview, AYA's concept of a parallelogram was unclear, as she expressed, "I actually don't know if parallelograms are strictly four-sided figures… or many shapes should be parallelograms".

AYA's use of the word *parallelogram* (see Figure 4.30) signifies a collection of figures that share this visual property of parallel sides. Based on AYA's definition, this collection of figures could include figures that have one pair of parallel sides such as *trapezoids*, two pairs of parallel sides such as *parallelograms,* or figures that have more than two pairs of parallel sides such as *hexagons*. We notice that rectangles and squares are not included in the family tree of parallelograms. According to what I observed during the Pre-Interview, AYA did not include rectangles and squares as parallelograms, but rather considered them as a separate group of figures that have right angles.

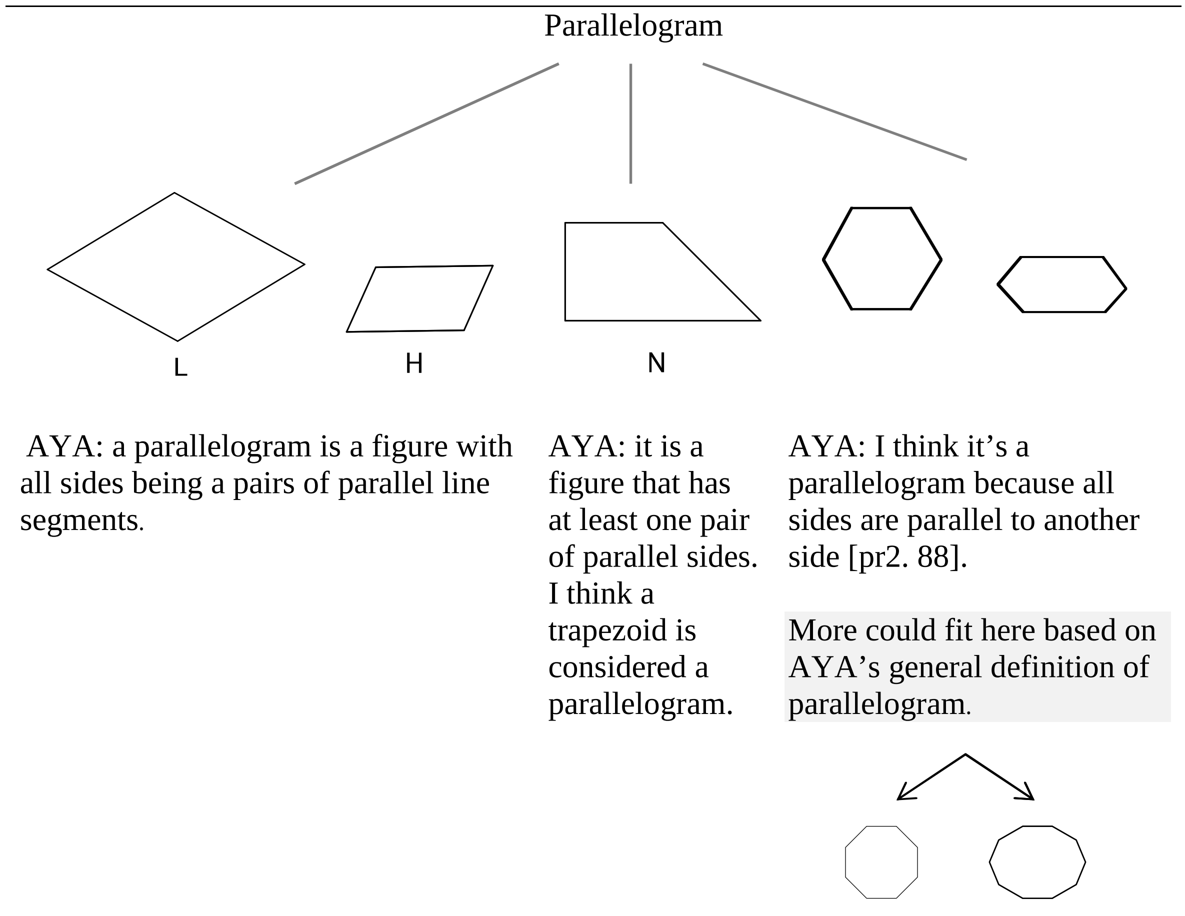
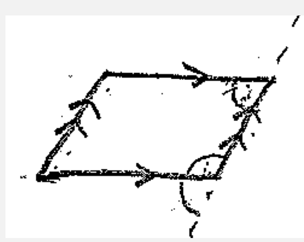
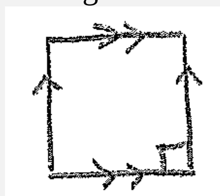


Figure 4.30 AYA's use of the word parallelogram at the Pre-Interview

At the Post-Interview the most important change in AYA's word use is her use of the word *parallelogram*. Although AYA showed very similar routine procedures when identifying geometric figures in both interviews, her concept of a *parallelogram* was different from that of the Pre-interview. For example, when I asked AYA to draw two different parallelograms in Task Two, she drew a parallelogram and a square:

2. AYA: AYA's drawing of a parallelogram



3. Interviewer:	Why is this a parallelogram?
4. AYA:	Because it has four sides and each opposing side is parallel to one another.
	… …
60. AYA:	AYA's drawing of a different parallelogram 
61. Interviewer:	Why is this a parallelogram?
62. AYA:	It's a square… it has four sides of equal measure and all angles are 90 degrees.
63. Interviewer:	Why is this different from the one you drew?
64. AYA	This one is different because all the angles in this figure are equal.

AYA's use of the word *parallelogram* changed with regard to this added necessary condition of "four-sided" figure, and the necessary condition of "parallel sides". AYA considered rectangles and squares as figures with 90-degree angles and as parallelograms. AYA's use of the word *parallelogram* now signified a collection of figures sharing this common descriptive narrative, "a four-sided figure with two sets of parallel sides".

It is notable that AYA included all quadrilaterals with two sets of parallel sides in this family tree of parallelograms, relating these quadrilaterals because "they have two sets of parallel sides". AYA did not provide any explicit information about how these figures were related other than being parallelograms. For example, AYA grouped rhombi together with parallelograms because all rhombi have two sets of parallel sides; however there were no connections made between rhombi and squares, although AYA defined a rhombus as a “four-sided figure with all side length equal in measure”. Moreover, AYA did not mention any relations between squares and rectangles other than that they have four right angles. Therefore, I argue that at the Post-Interview AYA had a good grasp of the concept of parallelograms in general, but her understanding of the hierarchy of parallelograms was missing, or not clearly demonstrated in the interviews. Figure 4.31 illustrates AYA’s understanding of definition of a *parallelogram* at the Post-Interview.

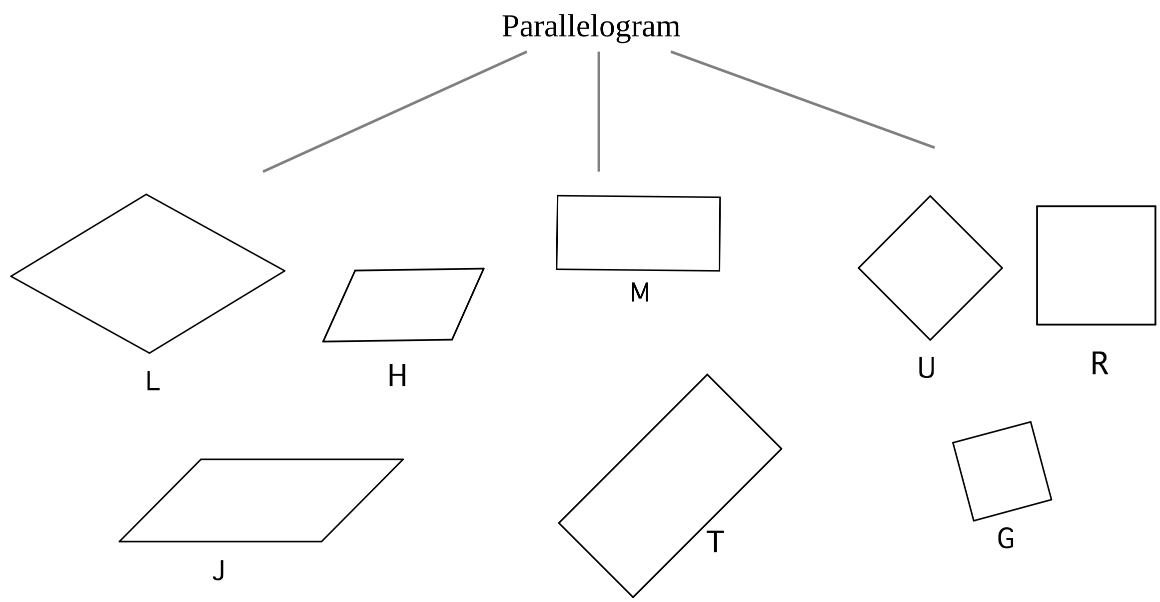


Figure 4.31 AYA’s use of the word parallelogram at the Post-Interview

According to my observations, AYA focused on the angles and sides of geometric figures when identifying and defining geometric figures. When asked to group figures AYA’s first reaction was to group them by the numbers of sides. When asked for subgrouping, AYA looked at differentiating the figures by their angles, such as by right angles versus acute angles. Table 4.32 provides the frequencies of the names of the parts of parallelograms that AYA mentioned in each task during the two interviews, and Table 4.33 provides the total frequencies of the names of the parts of parallelograms at the Pre-Interview and the Post-Interview:

Table 4.32 The frequencies of AYA's use of the names of the parts of parallelograms at the two interviews

Name	Frequency					
	Pre-T1	Pos-T1	Pre-T2	Pos-T2	Pre-T3	Pos-T3
Angle	1	4	60	69	4	24
Side	21	18	41	53	7	13
Length	4	4	18	34	7	6
Parallel side	9	6	4	3	1	1
Opposite side	0	0	1	3	3	0
Diagonal	0	0	27	13	4	0
Right angle	9	10	2	4	2	0
Opposite angle	0	0	1	3	0	3

Table 4.33 Total frequencies of AYA's use of names of the parts of parallelograms at the two interviews

Name	Frequency	
	Pre	Post
Angle	65	97
Side	69	84
Length	29	44
Parallel side	14	10
Opposite side	4	3
Diagonal	31	13
Right angle	13	14
Opposite angle	1	6

Table 4.33 shows that the most frequently used words relating to the parts of parallelograms were *angle* (n= 162) and *side* (n=153). The words *angle* (n=129) and *side* (n=94) were mentioned mostly in Task Two during both interviews. After *angle* and *side*, the word *length* is the next most frequently mentioned at both interviews (n= 73), and it was mostly mentioned during Task Two (n= 52). Likewise the word diagonal (n=44) was mostly used in task

Two. These results are expected, as Task Two asks interviewees about the relations of the angles, sides and diagonals in a parallelogram. I want to draw a little attention here to words such as *parallel side* (n=24), *opposite side* (n=7) and *opposite angle* (n=7); these words describe important characteristics of a parallelogram, but were least mentioned at both interviews.

To support my claims of AYA's change in geometric discourse, I described her word use and routine procedures during the interviews, and analyzed the changes in AYA's *routines of substantiation, routines of constructing new narratives* and her use of the word *parallelogram*. We see a dynamic change in her geometric discourse, from a colloquial mathematical discourse towards a mathematical one.

<u>Case 5: Changes in ARI's Geometric Discourse</u>

ARI was a college sophomore at the time of the interviews. ARI took her last geometry class in 9th grade, about five years prior to the geometry and measurement class. The van Hiele Geometry Test showed that ARI was at Level 3 at the pretest, and stayed at Level 3 according to the posttest ten weeks later. I interviewed ARI after both tests, and analyzed her interview responses. Based on my observations, findings about ARI's geometric discourses from the Pre-Interview and the Post-Interview are as follows:

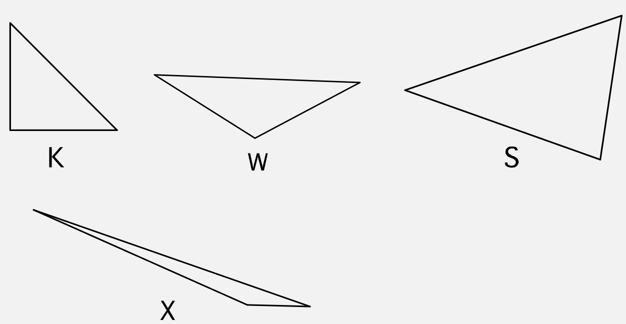
- ARI changed her routines of sorting polygons, from focusing on the names and attributes of the quadrilaterals at the Pre-Interview, to focusing on the hierarchy of the classifications of the quadrilaterals at the Post-Interview.
- ARI's routine of substantiation changed from a combination of recalling and measuring, a routine procedure using measurement tools to check the results at an object-level at the Pre-Interview, to routine procedures using previously endorsed narratives to construct new narratives at an abstract-level at the Post-Interview.

- ARI applied congruence criterions such as angle-side-angle, side-side-side for verification of congruent triangles; and used the dissection method to verify the sum of the interior angles of parallelograms.
- ARI used new mathematical terminology to describe and to justify the congruent parts of triangles and parallelograms at the Post-Interview, whereas she explained her claims informally at the Pre-Interview.

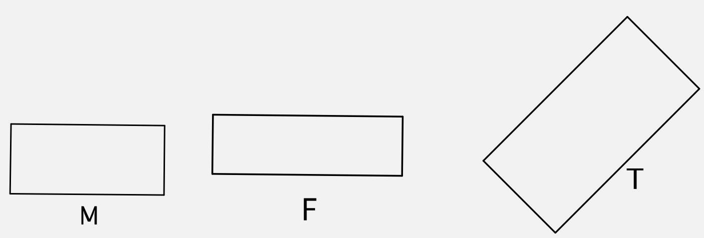
In the following sections let me begin with ARI's *routine of sorting*. The findings of ARI's routine procedures of sorting polygons are mostly observed in Task One. Task One has eighteen polygons, consisting of triangles, quadrilaterals and a hexagon. During the interviews, ARI was asked to place these figures into groups, and then to regroup them differently.

At the beginning of the Pre-Interview, when ARI was asked to sort the polygons into groups, she sorted them based on their names. She grouped eighteen polygons into groups of *triangles*, *rectangles*, *squares*, *parallelograms*/*rhombi*, *quadrilaterals*, and *other*. ARI's method of grouping was as follows:

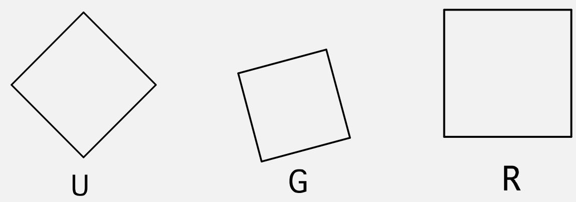
2a. ARI This group is triangles:



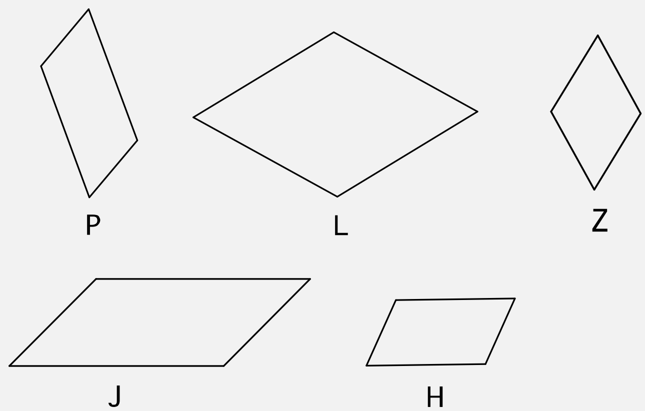
2b. ARI … And then these ones are rectangles [pointing at the Fig. M, F, G]:



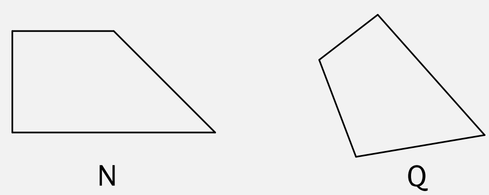
2c. ARI U, G, R are squares [pointing at Fig. U, G, R]:



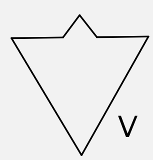
2d. ARI … and these P, L, J, Z and H are parallelograms… rhombuses…



2e. ARI And then N and Q just are quadrilaterals…



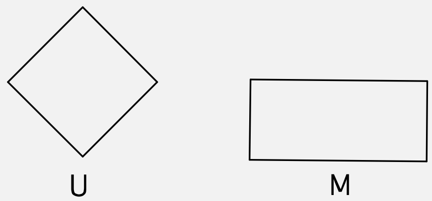
2f. ARI And then V is … just a weird shape.



ARI grouped all 3-sided polygons together and called them *triangles*; she also sorted all *rectangles* together, as well as *squares*. She grouped *parallelograms* and *rhombi* together, the only group with two names. ARI put Fig. N (a right trapezoid) and Fig. Q (a quadrilateral with no parallel or equal sides) together because both have just four sides. Fig. V, a hexagon, was grouped by itself. ARI called it “other” because “it is a just weird shape”.

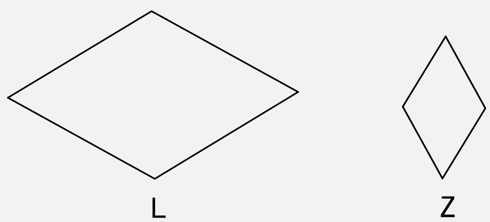
I asked ARI to regroup the polygons differently, and she regrouped them by combining rectangles and squares together. ARI called that group *rectangles*:

22. ARI	These are just rectangles because squares can be also rectangles…yeah [pointing at Fig. U and M]

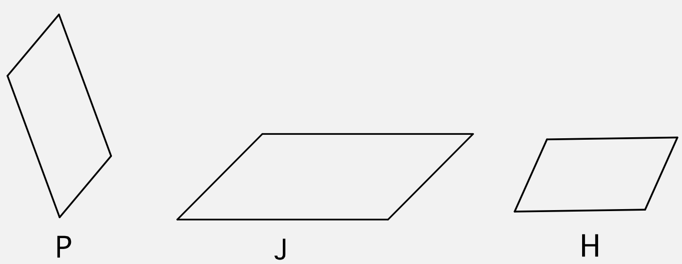


ARI continued her regrouping, and she split the *parallelograms/rhombuses* group into a group of *parallelograms* and a group of *rhombi*

24a. ARI	And then L, Z those are rhombuses



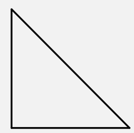
24b. ARI	And then, those are parallelograms



ARI sorted the polygons based on the characteristics of their angles. She grouped the polygons into a *right-angled shape* group (n= 8), consisting of polygons with at least one right angle, and an *obtuse triangle* (n=2) group containing triangles with an obtuse angle.

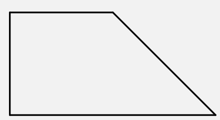
28. ARI And these are right-angled shapes.

30a. ARI ..the right angle triangle [pointing at Fig. K]



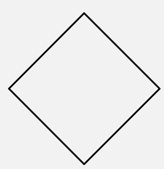
K

30b. ARI …and this has a right angle here [pointing at a right angle in Fig. N]

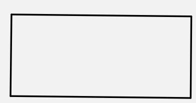


N

30c. ARI All the squares and rectangles [pointing at Fig. U and Fig. M again]

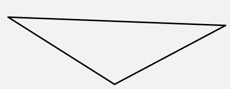


U

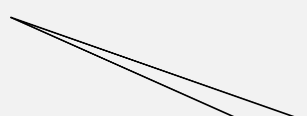


M

30d. ARI These ones I put as obtuse triangles

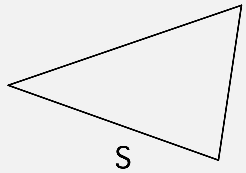
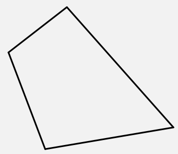
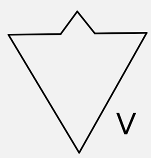


W



X

Three polygons were left: Fig. S, Fig. Q and Fig. V. Then the following conversation took place:

33. Interviewer	Why doesn’t Fig. S go with any other groups? [Pointing at Fig. S]
	 S
34. ARI	It doesn’t have a right angle, it’s not four-sided…so it’s not a square, a parallelogram, and then …it’s not obtuse either.
35. Interviewer	How about Fig. Q? Why is Q left out? [Pointing at Fig. Q]
	 Q
36. ARI	It doesn’t have parallel sides and it doesn’t have a right angle.
37. Interviewer	How about Fig. V? [Pointing at Fig. V]
38. ARI	V? …that’s just a weird shape…
	 V

ARI grouped figures according to their common descriptive narratives (i.e., definitions) by direct recognition. For example, when I asked ARI if I could put Fig. U (a square) and Fig. L (a rhombus) together, she responded, “yes, because they both can be seen as rhombuses”. When I followed by asking ARI why she thought the two polygons were rhombuses, she said, “because all their sides are equal lengths”. ARI did not use any measurement tools to check the angles or the sides of any polygon for verification while working on Task One. Therefore, ARI’s judgments about the attributes of the angles and sides were direct recognition and intuition. Figure 4.32 summarizes ARI’s routine procedures for sorting polygons at the Pre-Interview.

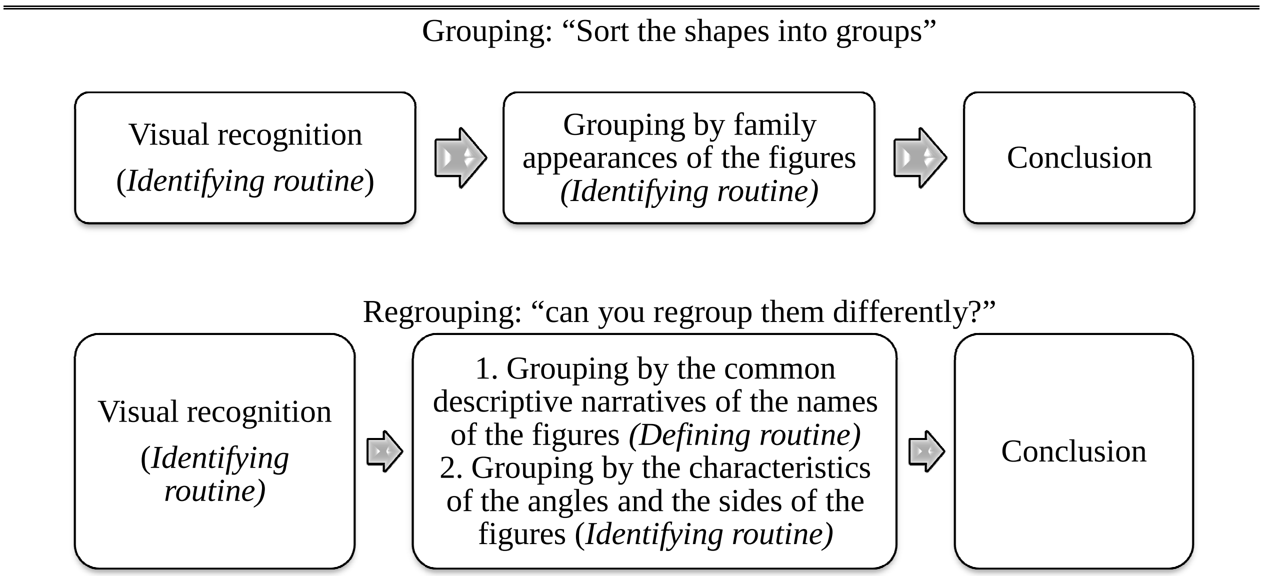


Figure 4.32 ARI's routine procedures for sorting polygons at the Pre-Interview

Ten weeks later I interviewed ARI again. In her routine procedures of sorting quadrilaterals she arranged them with a hierarchy of classifications. Let me begin with ARI's responses in grouping the eighteen polygons. ARI first grouped polygons (n=18) by the numbers of their sides: *triangles* (n=4*)*, *quadrilaterals* (n=13) and a *six-sided figure* (n=1). ARI then divided the *quadrilateral* group into subgroups consisting of *squares*, *rectangles*, *parallelograms*, *trapezoids* and *quadrilaterals.* Figure 4.33 illustrates all the subgroups of the *quadrilateral* group.

Figure 4.33 ARI's grouping of the subgroups of the quadrilaterals at the Post-Interview.

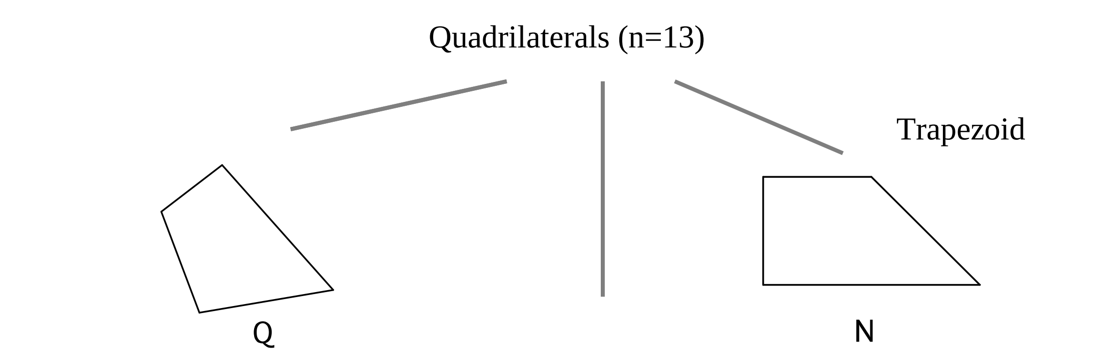
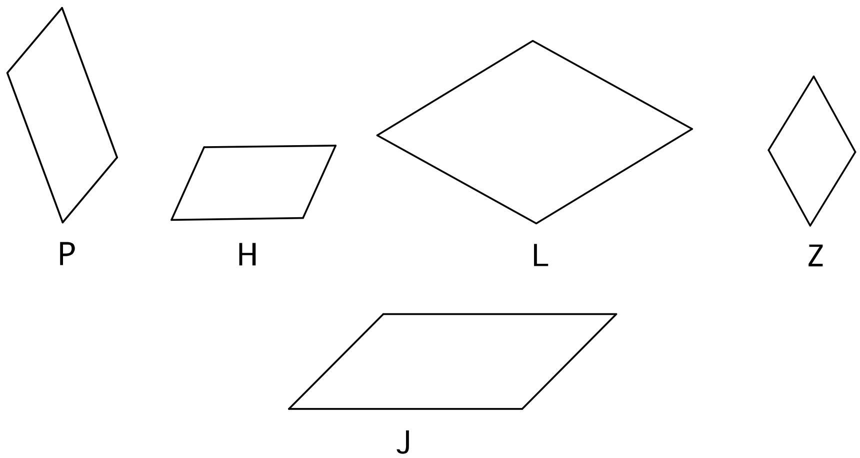
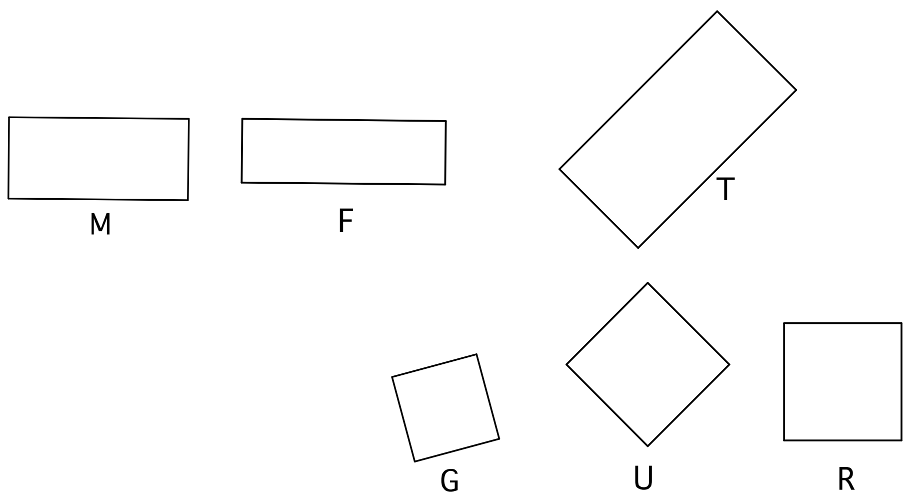


Figure 4.33 (cont'd)

Parallelograms (n=11)



ARI: "…then P, L, J, Z, H, R, T G, F, M and U are all parallelograms"



ARI: " M, F T, U, G, R are rectangles"

" U, G, R are squares"

Figure 4.33 ARI's grouping of the subgroups of the quadrilaterals at the Post-Interview.

At the time of the Post-Interview, ARI was able to group the polygons using a hierarchy of classifications of parallelograms. In her grouping, a polygon could be placed multiple times because it was identified in several subgroups by different common descriptive narratives of the polygons. For example, in the family of *parallelograms*, Fig. R, a *square,* was not only classified as a *parallelogram* but it was also identified as a *rectangle*.

When I asked for regrouping, ARI regrouped triangles according to the characteristics of their angles, splitting the *triangle* group into three subgroups consisting of *obtuse triangles*, a *right triangle* and an *acute triangle*. For the *parallelograms*, the only change ARI made from her previous grouping was to put the rhombuses and the squares into the same group, which she called the *rhombuses* group. When I asked ARI why she made this change, she responded, “they are all rhombuses because they have equal sides”. In this regrouping process, a *square* was also grouped as a *rhombus* by this common descriptive narrative, “a rhombus is a four sided figure with all equal sides”. During the Post-Interview, ARI again did not use measurement tools. ARI’s *routines of sorting* polygons at the Post-Interview are summarized in Figure 4.34.

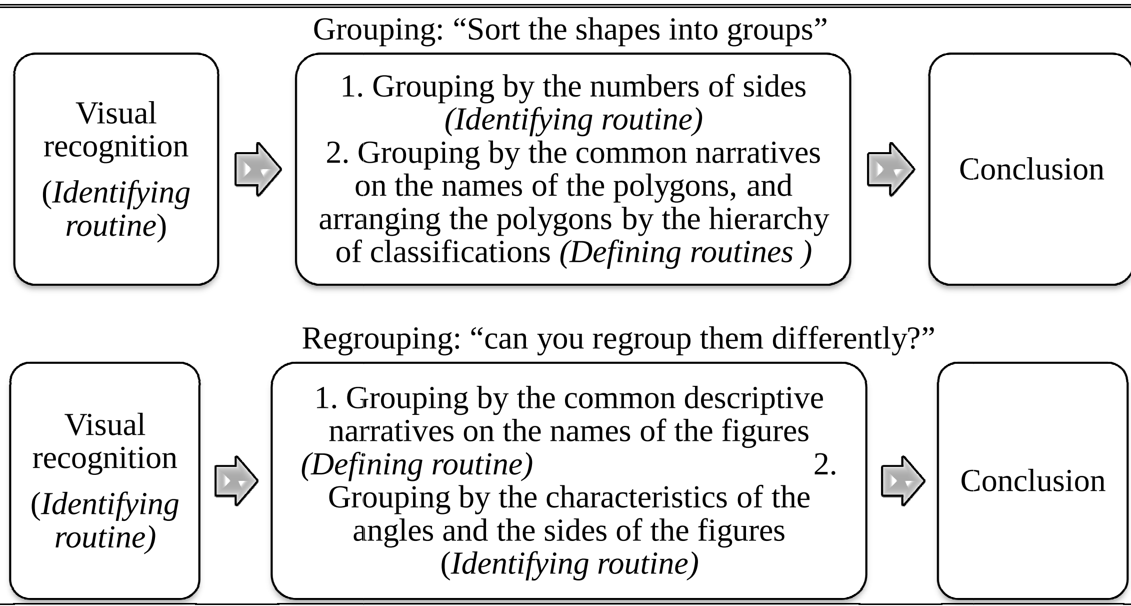
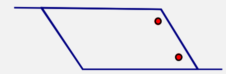
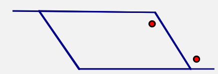
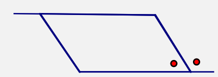
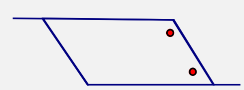


Figure 4.34 ARI’s routines of sorting polygons at the Post-Interview

In ARI's responses for Task One in the Pre-Interview and the Post-Interview, she was able to group geometric shapes by their common descriptive narratives. Also, she was able to classify quadrilaterals into a hierarchy with the help of her defining routines at the Post-Interview. At the Pre-Interview, ARI's defining routines focused on the necessary conditions of the definitions. Thus figures were categorized by their common names. In contrast, at the Post-Interview, ARI's defining routines focused on both necessary and sufficient conditions of the definitions, and thus, the quadrilaterals were grouped with a hierarchy of classifications. Change in ARI's geometric discourse also occurred in her routines of substantiation. In the following section I describe ARI's routine procedures of substantiating her claims.

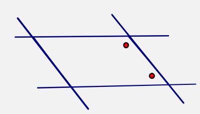
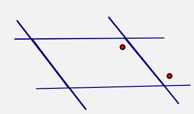
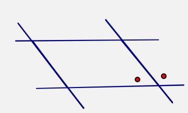
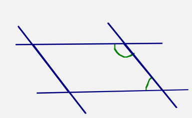
I observed ARI's routine patterns of verifying and justifying declared narratives while working on Task Two. ARI was able to use endorsed narratives to construct new narratives at the Post-Interview, whereas she depended on recalling and measuring routines at the Pre-Interview. A recalling routine is a course of action using previously endorsed narratives, and is more about remembering what one learned previously. In this study, A measurement routine is a set of repetitive actions where participants measure the parts of polygons, and use these measurements in their identifying, verifying and substantiating processes. In the next example, ARI tried to verify that all angles add up to 360 degrees in a parallelogram.

In Task Two all interviewees were asked to draw two parallelograms different from each other, and to discuss the angles of the parallelograms. ARI drew a parallelogram and wrote that all of the angles added together equal 360°. When asked for substantiation, ARI first showed that two adjacent angles added together equal 180°, and next used that result to justify her claim that "all of the angles added together equal 360°":

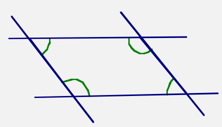
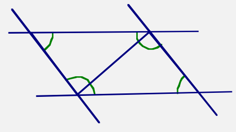
26. ARI	If you add this angle and this angle together, it’s equal 180…	Pointing at the angle: 
27. Interviewer	Why is that?	
28. ARI	There is a name for it. I forget the term…	
29. ARI	I think this angle is suppose to equal this outside angle here…	Pointing at the two angles: 
30. ARI	…and this line would equal 180	Pointing at the two angles: 
31. Interviewer	How do you know these angles are equal?	Pointing at the two angles in [29]
32. ARI	…because I learned it in school? I don’t know how to explain it.	
33a. ARI	If that equals 180 added together,	Pointing at the two angles: 
33b. ARI	…then this would equal 180 added together. So all the angles all together would equal 360.	Pointing at the other two angles

In this episode, ARI’s routine procedure is a recalling routine, where she remembered related mathematical rules, without knowing their mathematical terms or why these rules work in a given situation. For example, in looking at the two adjacent vertex angles in a parallelogram, ARI concluded that they added up to 180 degrees [26]. This conclusion is correct because we know by definition that a parallelogram has opposite sides parallel, and we can use propositions about parallel lines and their transversals to conclude that the two angles add up to 180 degrees. Similarly, ARI was able to recognize that the alternating interior angles were congruent [29], but she did not know why these two angles were congruent, a consequence of two parallel lines cut by a transversal. Note that ARI utilized logical thinking in the statement, "*if* that equals 180…*then* all the angles…" [33a-b].

In contrast to ARI's response at the Pre-Interview to the declared narrative that the angles add up to 180 degrees, her substantiation of this narrative was different at the Post-Interview.

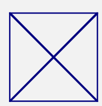
33a. ARI	These are adjacent angles…	Pointing at the angles: 
33b. ARI	… and these are alternate interior angles and they are equal, because… by the parallel lines	Pointing at the angles: 
33c. ARI	… then this plus this angle is 180 because angles on a line.	Pointing on the angles: 
33d. ARI	… And so this and this, the adjacent angles equal 180.	Referring to the adjacent angles: 

In this argument ARI began to use mathematical terms to express her ideas. ARI used the words “adjacent angles” and “alternating interior angles” to replace her informal use of “this angle” and “that angle”, and “angle inside” and “angle outside” at the Pre-Interview. More importantly, ARI substantiated her claims using endorsed narratives. For example, when ARI produced the narrative that alternate interior angles were equal, she used a phrase *by … the parallel lines* [33b]; and when ARI declared another narrative that two angles add up to 180 degrees, she used the phrase *because angles on a line* [33c]. When I asked ARI why she thought all angles in a parallelogram add up to 360°, the following conversation took place:

34. Interviewer	Why do all the angles add up to 360? What are the angles that add together to equal 360?
35. ARI	All the interior angles, this plus this plus this plus this equal 360. [Pointing at these angles]: 
36. Interviewer	Why is that?
37. ARI	Because if you draw a line, the diagonal, there are two triangles, and the interior angles of a triangle equal 180. So, two triangles would equal 180 plus 180. That equals 360. [Draw a diagonal on the figure] 

I expected ARI to use a newly endorsed narrative, such as two adjacent angles add up to 180° [33d], to make the substantiation. She surprised me with her dissection method, drawing a diagonal to cut the parallelogram into two triangles. ARI then used the endorsed narrative that the sum of the interior angles of a triangle equals 180° [37] to complete her proof of the claim that all interior angles in a parallelogram add up to 360°. ARI's routine procedures also changed in verifying the congruent parts of a parallelogram.

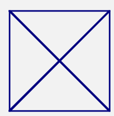
ARI was aware of the abstraction of congruent parts of the parallelograms at the Post-Interview, whereas she only used *measurement routines* to check the congruent parts at the Pre-Interview. That is, when recalling routines did not seem to help, ARI used measurements to check her claims about the sides and angles of a parallelogram. In the next conversation, I asked ARI to substantiate her declared narrative about the diagonals of a square:

182. Interviewer	What can you say about the diagonals of this parallelogram? [Pointing at the square]
	
183. ARI	They're gonna be equal lengths
184. Interviewer	How do you know?
185a ARI	Because they are all equal sides and I am pretty sure they would be all equal diagonals…[Pointing at the sides of the square]
185b. ARI	I can check…
185c. ARI	Let's do it in inches…this one is 2.7… that's also 2.7 [Using a ruler to measure the length of the diagonals of the

	square].
185d. ARI	Yeah, diagonals are equal lengths.

When asked for substantiation, ARI first made an intuitive claim about why the diagonals would be equal [185a], and then used a ruler to measure the length of the diagonals, getting measurements for each diagonal of 2.7 inches [185c]. With this confirmation, ARI concluded that the diagonals were of equal length [185d], completing substantiation.

Our conversation went on, and ARI declared another narrative about the diagonals of a square being perpendicular to each other.

189. Interviewer	What can you say about the diagonals of this parallelogram? [Pointing at the square]
	
190. ARI	I think they're perpendicular to each other.
191. Interviewer	What do you mean when you say perpendicular?
192. ARI	At the intersection, they create a 90-degree angle [Pointing at the intersection of the diagonals]
193. Interviewer	How do you know they are 90-degree angles?
194. ARI	I can measure it… Yeah… it's 90-degree. [Using a protractor to measure one of the angles at the intersection]
195. Interviewer	How do you know they are all 90-degrees? [Pointing at the other angles at the intersection]

196. ARI	I am pretty sure that they are all 90-degrees. Yeah.. this is 90… that’s 90… they are all equal. [Giggling, and use a protractor to measure two other angles at the intersection]

When I asked, “how do you know that they are all 90-degrees?”, I was looking for a routine procedure operating at an abstract level. However, in ARI’s response to this question, she focused on the concreteness of the congruent angles, in using a protractor to check the angles one by one [194; 196], thereby using measurement routines to verify her claims at an object level.

As in her responses at the Pre-Interview, ten weeks later ARI declared the same narrative about the diagonals of a square, stating that the diagonals were equal length. When asked for substantiation, ARI this time did not measure the length of the diagonals but responded with “I can prove again that the triangles are congruent”. ARI had just substantiated that the diagonals in a rectangle were equal, so she applied that argument. Table 4.34 summarizes ARI’s routine procedures of this verification with corresponding transcripts:

Table 4.34 ARI’s routine procedures of substantiation for “two diagonals are equal”

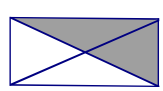
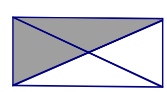
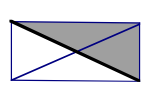
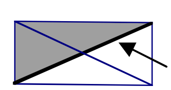
Routine Procedures	Transcripts
1. Identify two congruent triangles	111a. ARI: this triangle and this triangle are congruent [pointing at the shaded triangles]:  
2. Declared narrative	111b. ARI: and then that side equals this side [pointing at the diagonals]:  

Table 4.34 (cont'd)

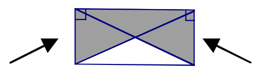
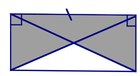
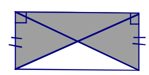
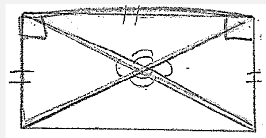
Prompt for verification 3. Verification of two congruent triangles	*112. Interviewer: How do you know these two triangles are congruent? [Referring to 111a]*
3.1 Identify corresponding angles of the triangles	113a. ARI: well, these are 90 degrees [pointing at the two right angles] 
3.2 Identify corresponding sides of the triangles	113b. ARI: they have a common side so that would be the same for both triangles… [Marked a tally on the common side] 
3.3 Identify another corresponding sides of the triangles	113c. ARI: opposite sides that are parallel…are equal [Marked two tallies on the opposite sides of the rectangle] 
4. Verify congruent triangles using S-S-S correspondence	113d. ARI: that gives you side, angle, side … and makes these two triangles congruent.
5. Conclusion	113e. ARI: by that, you can conclude that this side and this side are equal [pointing at the diagonals of the rectagnle].

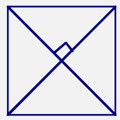
Table 4.34 shows that ARI first identified a pair of congruent triangles with diagonals as one set of corresponding sides of the triangles, and drew a conclusion about the equivalence of the diagonals [111a-b]. After my prompt for substantiation, ARI provided a sequence of steps of selecting three elements needed for the verification of congruent triangles [113a-c]. During this selection, ARI did not use a ruler or protractor to check the measurements of the sides and angles, but instead used identifying routines and defining routines. For example, ARI used the definition of a rectangle to identify two corresponding angles that "are 90 degrees" [113a], and two corresponding sides that "are equal" [113c] in the triangles.

86. Interviewer	Why is this a parallelogram? [Pointing at ARI's drawing]
	
87. ARI	Because opposite sides are parallel and the opposite sides are equal.
88. Interviewer	Why is this a different parallelogram from the one you drew?
89a. ARI	Because they all form 90 degree angles, all the angles are equal, not just the opposite angles.
89b. ARI	This is a rectangle.

ARI used an identifying routine to identify the right angles and opposite sides as parts of the rectangle; and then used a *defining routine* to confirm her choice of the elements of the congruent triangles needed for verification. Because the polygon was a rectangle, all angles were equal and opposite sides were equal. It is legitimate for ARI to apply this proof in the case of a square as in the next example, because she considered a square as a rectangle (see my earlier analyses for routines of sorting).

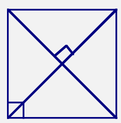
We continued to discuss the diagonals of a square. ARI produced further narratives such as "diagonals are perpendicular to each other", "diagonals bisect each other" and "diagonals bisect the angles". As noted earlier, ARI made the connection between the diagonals of a square and the diagonals of a rectangle, as in both cases their "diagonals were equal". Later ARI made another connection between the diagonals of a square and the diagonals of a parallelogram, as in both cases their "diagonals bisect each other". ARI used the Angle-Side-Angle criterion to substantiate two congruent triangles in a parallelogram, and applied that result to draw the same conclusion in the case of a square. To avoid redundancy, ARI's substantiation of "the diagonals bisect each other" is not presented here. Instead I share ARI's substantiation of "diagonals are perpendicular to each other":

133. ARI They are perpendicular to each other. [Pointing at the intersection of the diagonals]:

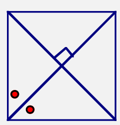


134. Interviewer How do you know?

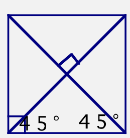
135a. ARI [Thinking]… because the angles of the square are 90 degrees [Pointing at the angle]



135b. ARI And the diagonal cut this angle in equal halves. [Pointing at the two angles]



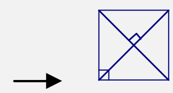
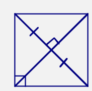
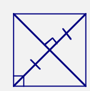
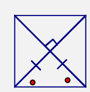
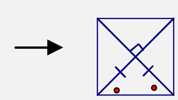
135c. ARI So this would be 45 degrees and this would be 45 degrees. [Writing 45-degree at each angles]

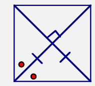
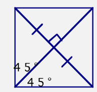


135d. ARI So that would equal to 90 degrees … [Adding the two 45-degree angles]

135e. ARI For angles a triangle it would be 180 total, so it would have to be 90 degrees. [Pointing at the intersection of the diagonals]

To verify that the diagonals are perpendicular to each other, ARI tried to show that the diagonals form 90-degree angles [135a]. ARI did not use a protractor to measure all the angles at the intersection, but used endorsed narratives to verify her claims [135e]. In the process of this verification, ARI made one assumption, that “the diagonal cuts this angle in half” [135b], that was not mentioned before. Therefore, I asked for substantiation:

142. Interviewer	How do you know that the diagonal cuts the angle in two halves?	Pointing at the angle 
143a. ARI	Because, if the diagonals are bisecting each other, …. Then the halves are all equal lengths too … Because the diagonals are equal lengths.	Pointing at the halves of one diagonal:  
143b. ARI	… So from there, if these sides are equal, and it would be an isosceles triangle…by the definition of an isosceles triangle, then these angles would have to be equal.	Pointing at the base angles of the triangle: 
143c. ARI	And then the same with this triangle, it would also be an isosceles.	Pointing at the adjacent triangle: 
143d. ARI	So, these are the same isosceles triangles. So, the triangles are all	Pointing at the angles:

	congruent, and the angles would all have to be congruent.	
143e. ARI	But this is 90 degrees. So, if these angles are equal… then they are all 45 degrees.	Pointing at the angle of the square: 
143f. ARI	So the diagonals cut the angles in half.	

ARI applied several endorsed narratives to substantiate her declared narrative “the diagonals cut the angles in half”. ARI first used “diagonals bisect each other” and “diagonals are equal”, both newly endorsed, to conclude that “the halves are all equal” [143a], and used this newly endorsed narrative to identify congruent isosceles triangles [143b-d]. By using the properties of isosceles triangles, ARI showed that all corresponding [base] angles were equal [143b]. Knowing that the figure was a square, with 90-degree angles, ARI concluded that the diagonals cut the angles in halves and they were all 45 degrees [143e-f].

In summary, I conclude that ARI changed her routine of substantiation, when she moved away from an object level of measuring and checking the congruent parts of the parallelogram at the Pre-Interview, towards an abstract level of substantiation, using endorsed narratives to verify the congruent parts at the Post-Interviews.

The use of words and language are important when we study ones mathematical discourse. Different interviewees show different developments in their use of mathematical word use. In ARI’s case, she was able to use more mathematical terms in the Post-Interview than in the Pre-Interview; she also developed conceptual understandings of quadrilaterals and parallelograms after the Pre-Interview. Let me begin with some general findings of words that ARI used in the names of the polygons. More specifically, these words were *quadrilateral*, *parallelogram*, *rectangle*, *square*, *rhombus*, *trapezoid* and *kite*. The total frequencies of the names of quadrilaterals at the two interviews are shown in Table 4.35 and Table 4.36.

Table 4.35 The frequencies of ARI's use of the names of quadrilaterals at the two interviews

Name	Frequency					
	Pre-T1	Pos-T1	Pre-T2	Pos-T2	Pre-T3	Pos-T3
Quadrilateral	3	4	0	0	1	0
Parallelogram	5	3	1	4	3	4
Rectangle	5	2	0	1	0	7
Square	6	1	1	3	0	7
Rhombus	5	3	0	1	0	2
Trapezoid	3	2	1	0	0	0
Kite	0	0	0	0	0	0

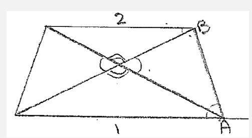
Table 4.36 Total frequencies of ARI's use of names of quadrilaterals at the two interviews

Name	Frequency	
	Pre	Post
Quadrilateral	4	4
Parallelogram	9	11
Rectangle	5	10
Square	7	11
Rhombus	5	6
Trapezoid	4	2
Kite	0	0

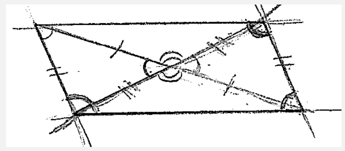
Table 4.36 shows that the word *parallelogram* (n=20) was the most frequently used during the interviews, and being mentioned in all three tasks (see Table 4.35). The word *square* (n=18) was the second most frequently used, and *rectangle* (n=15) was third. The word *kite* (n=0) was not mentioned at all in both interviews, and *trapezoid* (n=6) was the second least

mentioned. Table 4.35 shows that the words *quadrilateral*, *rhombus* and *trapezoid* were mostly mentioned in Task One, where interviewees were asked to group the polygons. There was an increase in use of the words *parallelogram*, *square* and *rhombus* in the Post-Interview, over the Pre-Interview, and use of the word *rectangle* doubled in the Post-Interview. ARI's use of the names of quadrilaterals was much lower than other interviewees' use of those names.

In an earlier section, I described ARI's routine procedures for sorting quadrilaterals for Task One. At the Pre-Interview, ARI first grouped the figures by their names and then by the characteristics of their angles. However ARI was confused about how a *trapezoid* and a *parallelogram* were related. ARI's confusion was not detected in Task One, but rather her choice in drawing a different parallelogram at Task Two shed light on her understanding of the words *trapezoid* and *parallelogram*.

1a. ARI	I don't know if this is right, but I am going to draw it. [ARI drew a figure]
	
1b. ARI	I think this is wrong.
3. Interviewer	Why? I just want to know what bothers you.
4. ARI	I know these are parallel sides [pointing at the two parallel sides], but I don't know if a trapezoid is also a parallelogram. I am not sure.
5. Interviewer	What is a trapezoid?
6. ARI	I am not sure, cause I think trapezoids need two parallel sides…

7. Interviewer	What is a parallelogram?
8. ARI	[Thinking]…A four-sided shape with like this [Pointing at her first drawing], opposite sides that are parallel?



The preceding conversation shows that ARI's concept of *trapezoid* was unclear, as she explained, "I don't know if a trapezoid is also a parallelogram" [4]. ARI produced a verbal narrative to my request "what is a trapezoid", stating that a trapezoid needs two parallel sides [6]. We see that ARI did know something about trapezoids, four-sided polygons with two parallel sides, but had no clear understanding that a pair of parallel sides is a necessary condition in the definition of trapezoid, confusing *trapezoid* with *parallelogram*, a polygon requiring two pairs of parallel sides.

In Task One, ARI created the *parallelogram* group consisting only of *parallelograms* and *rhombi*. The *parallelogram* group was later split into parallelograms and rhombi as two subgroups. ARI created the *rectangle* group consisting of rectangles and squares then divided them into a *rectangle* group and a *square* group. At that time ARI did not group the *rectangles* with the *parallelograms,* and did not provide information about how *rectangles* and *parallelograms* were related. In Task Two, ARI identified a rectangle and a square as parallelograms, and explained, "they are parallelograms, because they have both opposite sides parallel". ARI's use of the word *parallelogram* described a collection of figures having parallel sides. As noted, I included ARI's drawing of a trapezoid in this diagram because she did consider at one point that a trapezoid could be a parallelogram, displaying her confusion about what a trapezoid was, and how it related to a parallelogram. Figure 4.35 illustrates ARI's understanding that all squares are rectangles, as well as her understanding of the relation between a rhombus and a parallelogram. However, ARI did not clearly demonstrate the connections between a rhombus and a square, but she knew that both were a "four-sided figure with all sides equal". I conclude that ARI's uses of the names of parallelograms were limited to their common descriptive narratives at the local level, while her understanding of the hierarchy of parallelograms at the global level was missing in the Pre-Interview.

To me, ARI's use of the word *parallelogram* referred to a family of parallelograms with two branches: *parallelograms* and *rectangles*, illustrated as follows:

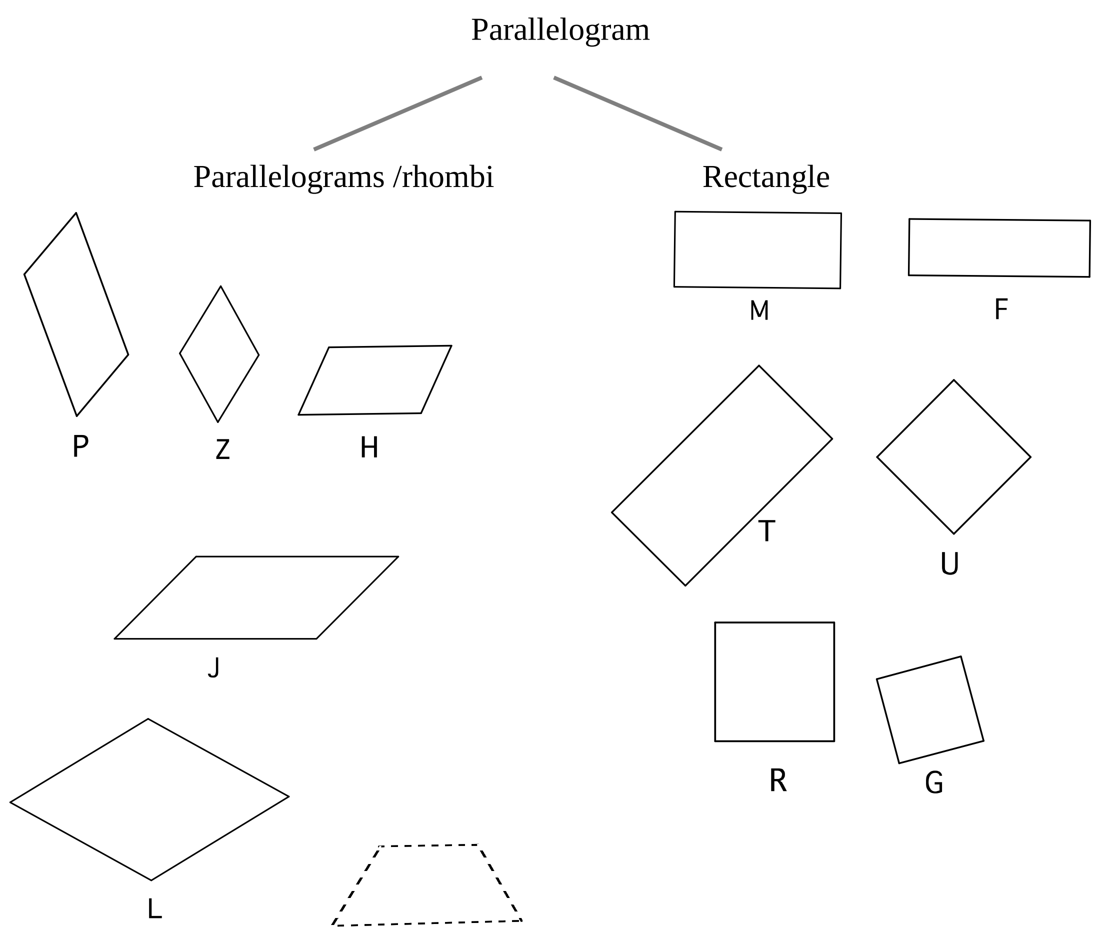


Figure 4.35 ARI's use of the word parallelogram at the Pre-Interview

At the Post-Interview the change in ARI’s word use regarding the names of quadrilaterals was revealed in the hierarchy of classifications of parallelograms. In ARI’s grouping and regrouping for Task One in the Post-Interview, the word *parallelogram* described a collection of figures that were 4-sided with two pairs of parallel sides. In the family of parallelograms, ARI also recognized two subgroups: a *rectangle* group consisting of parallelograms with right angles, and a *rhombus* group consisting of parallelograms with all equal sides. Based on the characteristics of the sides, ARI also split the *rectangle* group into *squares*, a group of rectangles with all equal sides, and *rectangles*. Lastly, ARI grouped *rhombi* and *squares* together because all their sides were equal.

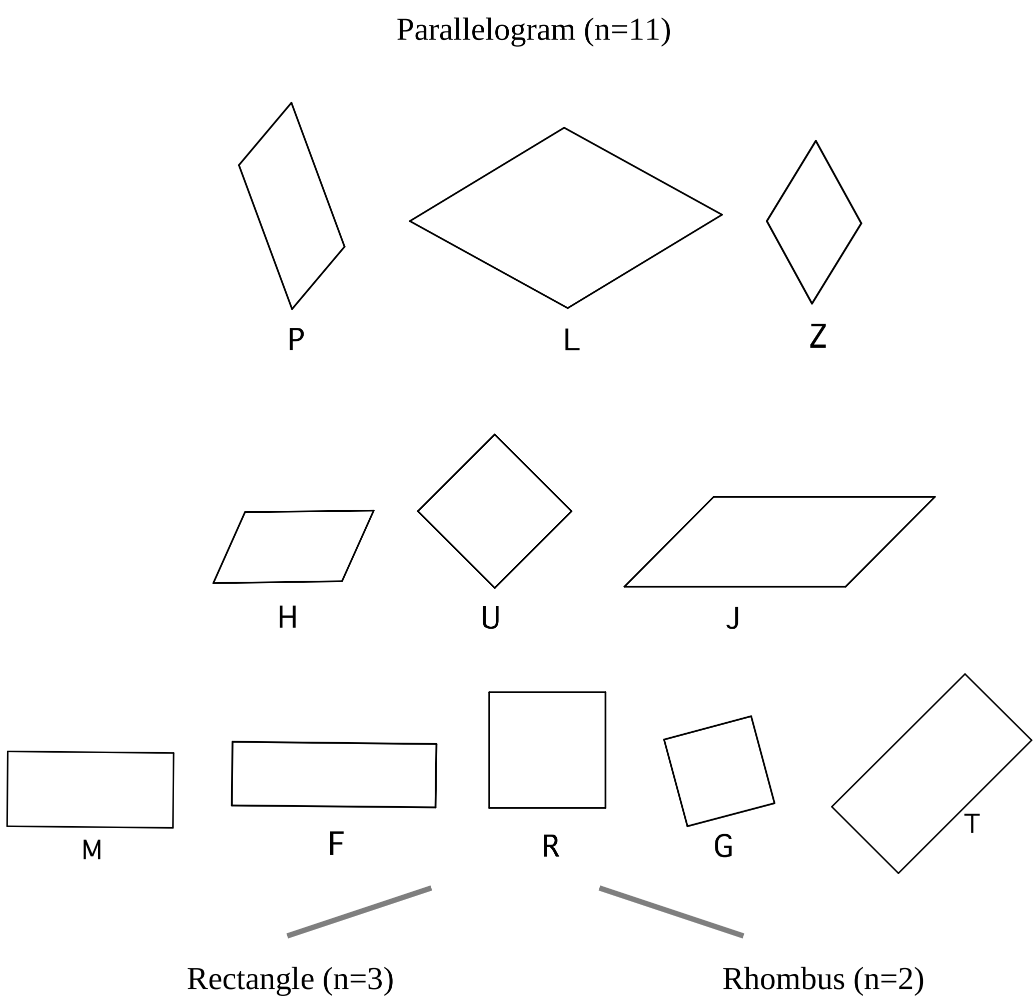


Figure 4.36 (cont'd)

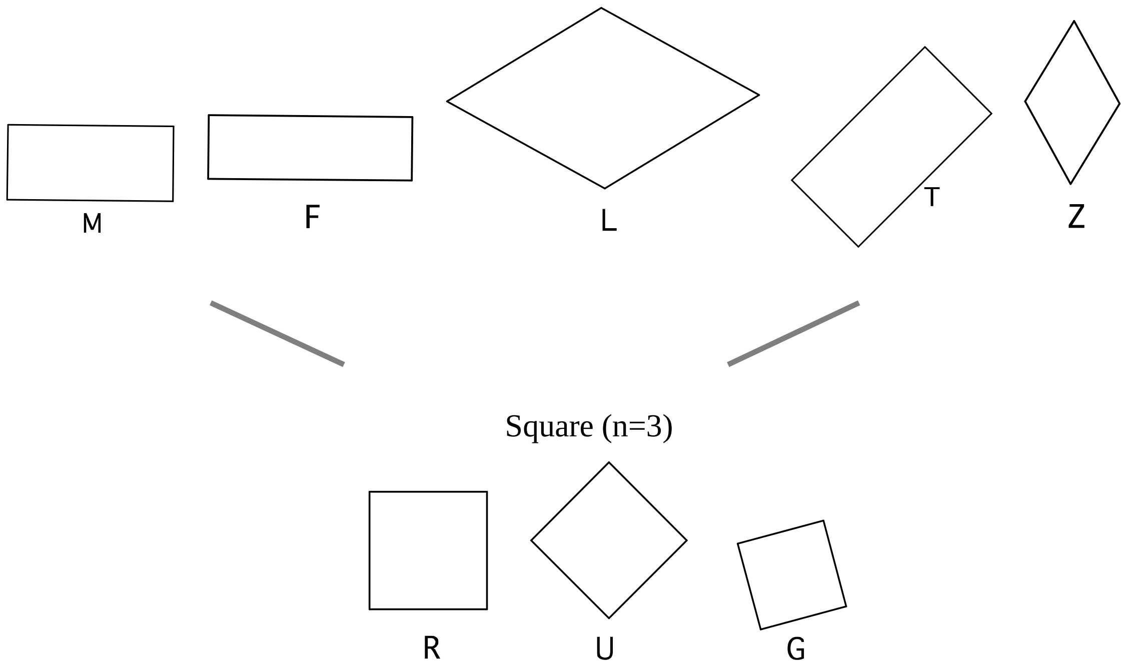


Figure 4.36 ARI's classifications of parallelograms at the Post-Interview

Figure 4.36 shows a hierarchy of classifications of the parallelograms in ARI's use of the word *parallelogram*. In this hierarchy, the word *parallelograms* denoted a collection of quadrilaterals with different *names,* and these names were *parallelograms, rectangles, squares* and *rhombi*. Although these *names* identify polygons with different physical appearances and attributes of their angles and sides, they are all *parallelograms*.

There was also a change in ARI's use of the names of the parts of parallelograms. In this study, the names of the parts of parallelograms considered were *angle, sides, length, parallel side*, *opposite side, opposite angle, right angle* and *diagonal*. Findings show that ARI used more words describing the parts of the parallelograms than the names of the parallelograms. Tables 4.37 and 4.38 provide information on the frequencies of these words' usage at the Pre-Interview and the Post-Interview.

Table 4.37 The frequencies of ARI's use of the names of the parts of parallelograms at the two interviews

Name	Frequency					
	Pre-T1	Pos-T1	Pre-T2	Pos-T2	Pre-T3	Pos-T3
Angle	1	2	22	26	4	3
Side	10	10	11	25	0	7
Length	1	0	18	14	5	5
Parallel side	2	1	2	1	0	0
Opposite side	2	3	6	11	3	7
Diagonal	2	0	4	12	0	0
Right angle	4	1	0	1	0	0
Opposite angle	0	0	4	4	2	4

Table 4.38 Total frequencies of ARI's use of names of the parts of parallelograms at the two interviews

Name	Frequency	
	Pre	Post
Angle	27	31
Side	22	42
Length	24	19
Parallel side	4	2
Opposite side	11	21
Diagonal	6	12
Right angle	4	2
Opposite angle	6	8

Table 4.38 shows that the most frequently used words relating to the parts of parallelograms were *side* (n= 64) and *angle* (n= 58). The words *angle* (n= 48) and *side* (n=36) were mentioned mostly in Task Two during both interviews (see Table 7). After *angle* and *side*, the word *length* was the next most frequently mentioned at both interviews (n= 43), and it was mostly mentioned at Task Two (n= 32). Likewise the word *diagonal* (n=16) was mostly used in task Two. These results were expected, as Task Two asks interviewees about the relations of the angles, sides and diagonals of a parallelogram. Among the terms *opposite sides*, *parallel sides*

and *opposite angles* (n=14) that describe important characteristics of parallelograms, *opposite sides* (n=33) was mostly mentioned, and *parallel sides* (n=6) the least mentioned.

During the interviews, I noticed that ARI used the words *sides* and *lengths* interchangeably. For example, ARI identified Fig. Z as a rhombus because “it has all equal *lengths*”[pr1. 50], and later she stated a rhombus “has equal *sides*”. Also, ARI used the phrases “same *length*” and “same *side*” frequently during the interviews when referring to figures with the same length measures.

Another important change in ARI’s use of words was in her use of formal mathematical terminology at the Post-Interview. These formal mathematical terms describe attributes of lines and angles, as well as the relations between them. In particular, she used the words *adjacent angle*, *alternating interior angles*, *complementary*, *supplementary*, *transversal* and *congruent*. In an earlier section, I briefly mentioned that ARI used the terms “adjacent angles” and “alternating interior angles” to replace her informal use of “this angle” and “that angle”, and “angle inside” and “angle outside” in her substantiation that angles add up to 360°.

Tables 4.39 and 4.40 provide information on the frequencies of these new mathematical words used at the interviews.

Table 4.39 The frequencies of ARI’s use of formal mathematical words at the two interviews

Name	Frequency					
	Pre-T1	Pos-T1	Pre-T2	Pos-T2	Pre-T3	Pos-T3
adjacent angle	0	0	0	4	0	1
alt. Interior	0	0	0	8	0	0
complementary	0	0	0	2	0	1
supplementary	0	0	0	1	0	0
vertical angle	0	0	0	2	0	0
transversal	0	0	0	1	0	0
congruent	0	0	0	12	0	0

Table 4.40 Total frequencies of ARI's use of formal mathematical words at the two interviews

Name	Frequency	
	Pre	Post
adjacent angle	0	5
alt. Interior	0	8
complementary	0	3
supplementary	0	1
vertical angle	0	2
transversal	0	1
congruent	0	12

Table 4.40 shows that all these mathematical words were used only at the Post-Interview, and most of them were mentioned in Task Two, where interviewees discussed the angles, sides and diagonals of a parallelograms. The word *congruent* (n=12) was the most frequently mentioned, as ARI used congruent triangles to substantiate the congruent parts of the parallelograms in the Post-Interview. During the process of verification of congruent triangles, ARI often identified *alternating interior angles* (n=8) as one of the elements for verification, and consequently this term was the second most mentioned (see Example Two).

ARI's transition from colloquial mathematics discourse towards mathematical discourse also appeared in her use of logical justification. For example, at the Post-Interview, ARI produced narratives with justifications such as "those sides are equal *by corresponding parts in congruent triangles*", "angles are equal *because vertical angles are equal*" and "these sides are equal *by the definition of a parallelogram*". To me, ARI's use of such mathematical language was a step closer to the language of proof.

The van Hiele Levels as Geometric Discourse

In this section I introduce the model, *the Development of Geometric discourse* (See Appendix F) that I developed for this study. In order to have a theoretical basis of the model, I translated each van Hiele level into discursive terms using four mathematical features described in the commognitive framework. These translations are presented in Chapter 2.

The model consists of four components: (1) *Geometric Objects* are utilized in the participants' use of mathematical words, saming criterions, realizations and systems of objects, (2) *Routines* are used in participants' courses of actions in response to mathematical tasks, (3) *Endorsed Narratives* are collected from participants' written or verbal narratives in this study, (4) *Visual Mediators* are collections of symbolic artifacts, geometric figures and their parts, all involved in the study.

Let me describe *Geometric Objects* and *Routines* in a bit more detail. In a mathematical discourse, a mathematical object constitutes "this thing" that we discuss. In this study, "this thing" very often is a triangle or a quadrilateral. Perhaps "this thing" also could be parts of a triangle or a quadrilateral (e.g., sides, angles, etc). It is important to pay attention to the mathematical objects involved in a given discourse.

In this study, at different van Hiele levels the same geometric figure discussed may not be the same the *geometric object* in the corresponding geometric discourse. For example, at Level 1 the word *square* is used as a label to a picture of a square, just a matching of a word with a shape (signifier). All squares can be grouped together because they all fit this family appearance of four sides equal; however, at this level students will not group a rhombus and a square together because they do not have the same family appearance even though they both have four sides equal (saming criterion). All figures are grouped by their names only, because each name represents a family appearance (realization), and there is no hierarchy connecting geometric figures at Level 1 (system of objects). At Level 3, the word *square* can also represent a parallelogram, a rectangle or a rhombus because a square fits the common descriptive narratives "opposite sides parallel", "a parallelogram with four right angles", and "parallelogram with four sides equal" (saming criterion). Thus, a square can be grouped with parallelograms, rectangles and rhombi (realization).

If a student moves from Level 1 to Level 3, the polygon called "square" plays different roles as a geometric object in the two different geometric discourses. A square in Level 1 is a picture of "a thing" called a square, whereas a square in Level 3 is an abstract object with required properties that can have different names. It is important in this study to compare geometric objects at each van Hiele level.

Routines are discursive patterns that repeat themselves in similar situations. In this study, routines consist of identifying routines and defining routines, helping me pay attention to the role of definitions played in van Hiele levels 1 to 3. The model helps me to be more explicit about how students identify a polygon, and with what evidence, as a repetitive pattern. For example, at Level 1 a student identified a square because it looked like one (visual recognition), whereas at Level 2, this student identified a square because it has four equal sides. But the same could also be rhombus because right angles were not mentioned (identifying partial properties). The defining routines provide clues to the students' use of definitions.

This model helped to identify participants' geometric discourses with regard to their van Hiele levels. The model was revised during the process of analyzing participants' geometric discourses.

CHAPTER FIVE: DISCUSSION

Efforts in the mathematics education research community toward an understanding of students' learning have defined mathematical learning as actively building new knowledge from experience and prior knowledge (NCTM, 2000), moving to a higher level of thinking (van Hiele, 1959), or as changes in discourse (Sfard, 2008). Other researchers have developed methods to measure learning quantitatively (Floden, 2002). In this chapter I return to the questions that motivated this study and guided my analyses of learning. The question that served as the impetus for the study was: "What do prospective teachers learn in geometry from their preparation for the work of teaching geometry?" It can be argued that this study does little to answer the question because of the complexity of participants' learning, and of the context in which these students were observed. However, my effort is to conceptualize these participants' mathematical thinking through their mathematical discourses as an evidence of their learning, thereby adding some information to the views of learning as moving to a higher level of thinking, and as changes in discourse. I will begin with a brief summary of the participants' changes in levels of thinking as well as changes in their geometric discourses.

Summary of the Results

To investigate changes in students' mathematical learning, my study focused on comparisons of students' competencies in the topics of triangles and quadrilaterals at the beginning of a semester and at its end. I conducted the analysis using van Hiele levels (1959) to investigate changes in students' geometric thinking, and also used Sfard's (2008) discursive framework in which mathematical features of discourse (i.e., word use, visual mediators, narratives and routines) are analyzed.

Changes in van Hiele Levels

Changes in participants' (n=63) competencies were revealed in their overall performances on the van Hiele geometry pretest and posttest. There were improvements in answering questions related to van Hiele Levels 1 to 3 at the posttest. In particular, most participants did better in the following:

- More than ninety-five percent of the participants correctly named triangles, squares, rectangles, and parallelograms at the posttest.
- More than ninety-five percent of the participants at the posttest correctly identified the properties of isosceles triangles, squares, rectangles, and rhombi related to their sides, angles and diagonals.
- About ninety percent of the posttest participants correctly used logical statements regarding triangles, squares, rectangles, and parallelograms.

These changes show that participants gained familiarity with figures like triangles, squares, rectangles, rhombi and parallelograms, and with their properties. Participants mentioned more about the properties of angles and sides in a parallelogram, but less on the properties of diagonals. The comparisons of van Hiele pretest and posttest levels revealed students' weaknesses in using deductive reasoning to construct proofs (Level 4) and abstract thinking (Level 5).

Given these test results, one conclusion is that the geometry course helped students to move from a lower van Hiele level to Level 3. However, the van Hiele test also showed that a student entering the class at Level 3 likely would stay at Level 3. But that was expected, as the course was designed for future elementary and middle school teachers, and the course materials emphasized activities mostly at Levels 1 to Level 3 of geometric thinking, and included only a brief introduction to constructing proofs. The study did not look at how teaching or the use of the textbook affected these students' learning, but certainly the textbook and course instructions contributed in some degree to these prospective teachers' learning about geometric figures and their properties.

The van Hiele Geometry pretest and posttest served as a frame to gather general information about students' competencies and their thinking as a whole, but it did not provide details on changes in students' thinking at an individual level. For this purpose, I also analyzed changes in participants' geometric discourses with in-depth interviews.

<u>Changes in Geometric Discourses</u>

I begin with brief remarks on the discursive framework, in order to set up my later comments. Recall that the discursive framework conceptualizes mathematics as a discourse, and defines "learning mathematics" as changing of participation in mathematical discourse. The four key mathematical features highlighted in the discursive framework (along with their definitions) are:

- Word use: Mathematical words that signify mathematical objects or process
- Visual mediators: Symbolic artifacts, created specially for the sake of this particular communication
- Narratives: Any text, spoken or written, which is framed as a description of objects, of relations between processes with or by objects, and which is subject to endorsement or rejection; that is, to being labeled as true or false.
- Routines: Repetitive patterns characteristic of the given mathematical discourse (Sfard, 2008)

These features interact with one other in a variety of ways. For example, endorsed narratives contain mathematical words, and provide the context in which mathematical words are used; mathematical routines are apparent in the use of visual mediators and produce narratives; visual mediators are used in the construction of endorsed narratives, etc. The most interesting part of this investigation was the way in which the examination of these mathematical features contributed to understanding the richness and detail of the participants' geometric discourse and their thinking.

In an earlier chapter, I described in detail the uniqueness of each participant's geometric discourse at the beginning of the semester, and again at the end of the semester. The main changes in participants' geometric discourses occurred in the following two features of mathematics discourse: *word use* and *routines*.

Word use. This study focused on participants' use of the names of quadrilaterals, and of the parts of quadrilaterals. Participants' use of mathematical terminology changed from describing parallelograms as collections of unstructured quadrilaterals based on family appearances, to using the names as collections of quadrilaterals sharing common descriptive narratives. Some participants used of names of quadrilaterals with a hierarchy of classifications in the Post-Interview.

Word use is a key element in identifying objectification in the discursive framework. Sfard (2008) defines objectification as "a process in which a noun begins to be used as if it signifies an extradiscursive, self-sustained entity (object), independent of human agency" (p.412).

The use of the word *parallelogram* illustrates the importance of objectification. When a student states, "this is a *parallelogram*" based on its family appearance, she uses the word *parallelogram* as a label to match polygons fitting this visual description, rather than a definition. It is perhaps in course work, through the process of identifying, defining and generalizing properties of quadrilaterals, that this student has developed a concept of *parallelogram,* so that the word is used as a collection of quadrilaterals sharing a common descriptive narrative (i.e., definition), such as "a *parallelogram* is a quadrilateral with opposite sides of the same measure" That said, the student uses the word *parallelogram* to include 4-sided polygons such as squares, rectangles, and rhombi. The change in discourse counting as learning is a transition from non-objectified speaking to objectified speaking.

Objectification is not straight-forward to detect; however there are clues in the ways students speak that provide hints about how they are thinking. In the example in the previous paragraph, "a *parallelogram* is a quadrilateral with opposite sides of the same measure", the word "parallelogram" is used with "is", "quadrilateral" and "opposite sides of the same measure". One clue that "parallelogram" has been objectified is that "is" is used with the object. That is, "parallelogram" is a noun. Also, in this discourse "quadrilateral" and "opposite sides of the same measure" are used exclusively with geometric shapes. Thus, "a parallelogram is a quadrilateral with sides of the same measure" is stated as a mathematical fact in geometry. At the non-objectified stage of the use of "parallelogram", we would be more likely to hear something like "this parallelogram has two long sides equal and two short sides equal". The use of "has", "long sides", "short sides" and "equal" describes what a student *sees* about a parallelogram, but does not have to describe a mathematical fact.

The objectification of *parallelogram* is perhaps even more complex when we consider the hierarchy of classifications. I suggest that the objectification of *parallelogram* is complete when students can map out a hierarchy of classifications. Only then is the word *parallelogram* recognized in all its connections and relations, and its diversity realized.

Routines. I analyzed changes in participants' routine procedures, including routines of classifying, identifying, defining, verifying and substantiating. Briefly stated, participants' routine procedures changed from identifying polygons using visual recognition, to identifying them using endorsed narratives. In verifying claims, participants' routine procedures changed from recalling, measuring and/or constructing routines, to formulating proofs using mathematical propositions and axioms. Some participants' routine procedures also included algebraic reasoning to verify claims in geometry.

Some participants' routines of verifying were descriptions of processes of mathematical activity. For example, one participant verified that diagonals in a rectangle have equal measure by explaining, "They are the same because I measured it". The term "I measured it" reveals that the participant's routine of verifying relies on comparing and checking measurements, and is a description of what *she did*. Another participant verified that two angles were congruent by asserting, "the angle can slide over to this position and create this angle, and the line can be rotated so that this angle now becomes this angle". The use of "angle" with "slide over" and "create", and the use of "line" with "be rotated" and "becomes", indicate that this participant's routine of verifying was a description of what *the lines and angles did* in an imaginary way, but was not rigorously based on mathematical facts. Changes in discourse that count as learning are transitions from an object level way of speaking to an abstract level ways of speaking.

In this study, the participants' routine procedures were less polished than those of professional mathematicians. However the analyses of their routine procedures shed light on participants' reasoning and problem solving skills, and on their abilities in constructions of mathematical proofs.

What Can We Generalize and Why

Although, much support exists for van Hiele levels, researchers have questioned and modified certain aspects of these levels. To answer the question "What additional information does the discursive framework provide with regard to the levels of geometric thinking?", I need to discuss how and in what way my study contributes to mathematics education based on what we know *so far* about van Hiele theory. In this section I begin with a brief review of what we know about so far the van Hiele theory, and follow with descriptions of geometric discourses at each van Hiele level, and then discuss the usefulness of revisiting the van Hiele levels using a discursive lens.

To elaborate more on "what do we know *so far* about the van Hiele theory?", let me refer to the following three questions raised by Clements (1992):

1. Are the levels discrete?
2. Do students reason at the same van Hiele levels across topics?
3. Should other characteristics of the levels be considered?

First, according to the van Hieles (1959), the levels are discrete in the sense that they are qualitatively different from one another, and the "discontinuities are … jumps in the learning curve, and these jumps reveal the presence of the levels" (p. 76). Research confirmed that the five van Hiele levels are distinct qualitatively from each other. However, many studies have questioned whether the van Hiele levels are discrete, because some students' levels of geometry thinking are in transition between two levels (e.g., Fuys et al., 1988; Burger & Shuaghnessy, 1986). For example, when assigning students to van Hiele levels in Burger and Shaughnessy's project, reviewers could not resolve disagreements on whether students should be assigned to van Hiele Level 0 or Level 1 (Burger & Shaughnessy, 1986, p.18). These results encouraged researchers to provide evidence for a more dynamic and continuous model (e.g., Fuys et al., 1988; Gutierrez et al., 1991; Usiskin, 1982).

Going in that direction, with the assumption that van Hiele levels are not discrete, Gutierrez et al. (1991) developed a model, "degree of acquisition of a van Hiele level" (p. 238). This model assigned a numerical value to indicate one's acquisition of a van Hiele level. For instance, when students have no trace of the thinking methods specific to a new level, they have *no acquisition* to this level of reasoning. Once the students begin to be aware of the methods of thinking at a given level, with some attempts to work on this level, they have a *low degree* of acquisition of the level. Improvement goes on until students attain complete acquisition of the new level, when they have complete mastery of this way of thinking and are able to use it without difficulties. Figure 5.1 shows both the quantitative and qualitative interpretations of the model (p.238).

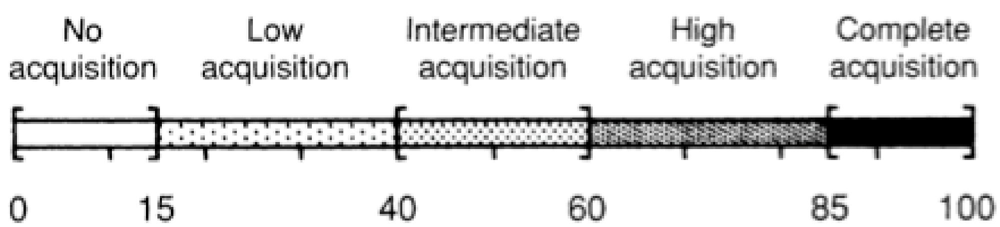


Figure 5.1 Degree of acquisition of a van Hiele level

Using this model, Gutierrez et al. found that the possibility that a student can develop two consecutive levels of reasoning at the same time, and the acquisition of the lower level is more complete than the acquisition of the upper level. Their study inferred possible continuity within a van Hiele level.

More recently, when describing the levels of reasoning, Battista (2009) illustrated students' reasoning at each level with different phases. For example, he suggests that "Students at the beginning of Level 1 [Visual-Holistic reasoning] might identify figures…as squares" (p.93); "Students at the end of Level 1 might reject… as a square"(p.93); and finally, "Before reaching the last phase of Level 2 [Descriptive-Analytic reasoning], most students would identify … as a square"(p.93). Battista's descriptions of students' levels of reasoning showed the continuity of the development within levels. More researchers are convinced intuitively that the levels are "dynamic rather then static" (Burger & Shaughnessy, 1986, p.45) and refer to "continuity rather than jumps" (Clements, 1992, p.429) in the process of learning. However, very little study has been done in this area to verify these claims.

Several studies using van Hiele levels to categorize students' levels of geometry thinking across different topics indicate that students may not be working at the same level on all concepts (e.g., Burger & Shaughnessy, 1986; Mayberry, 1983). For example, Mayberry (1983) assessed nineteen undergraduate pre-service teachers' levels of thinking using seven geometry concepts: squares, right triangles, isosceles triangles, circles, parallel lines, similarity and congruence. The study found that "the determination of the success criterion for a given topic and level was rather subjective" (p.68). This conclusion can be understood to mean that pre-service teachers were at different levels for different concepts. However the study did not provide information concerning in what way they are different. For instance, one might suspect that a more difficult concept such as similarity would require a higher van Hiele level of thinking than the classification of a quadrilateral.

In Burger and Shaughnessy's project (1986), interview tasks consisted of drawing, sorting, identifying, and defining geometric shapes such as triangles and quadrilaterals. With regard to different tasks, some students operated at different levels. For example, one student was reasoning at Level 3 (Abstraction) on the sorting task, but was assigned to Level 4 (Deduction) on the identifying and defining tasks because he was able to conjecture and attempt to verify his conjecture by means of formal proof (p.42).

The ways of identifying students' levels of geometric thinking suggest that we should adapt van Hiele levels to the complexity of the human reasoning process because students do not behave in a simple, linear manner.

The van Hieles argued that a learning process is also a process of learning a new language, because "each level has its own linguistic symbols" (van Hiele, 1959/1985, p.4). The van Hiele levels reveal the importance of language use, and language is a critical factor in the movement through the levels. Van Hiele (1986) provides an explanation of the language use at each level. For example, at Basic level there is a *language*, but the use of this *language* is limited to the indication of configurations that have been made clear based on observation. At Level 1, students need to develop the *language* that belongs to the descriptive level. At Level 2, the *language* has a much more abstract character then the descriptive level, and reasoning about logical relations between theorems begins at this level. At Level 3, students use the *language* of proof (p.43-53). However, the word " language" is not clearly defined in the broad use of it. Some researchers would consider "language" in the comparisons of informal language versus formal language, whereas others would refer to it as the different use of mathematical vocabulary at different van Hiele levels.

In discussing the assessment of students' reasoning in van Hiele levels, Battista (2007) argues about the validity of the reasoning, which involves the accuracy and precision of students' identifications, descriptions, conceptions, explanations and justifications. Researchers must determine which, if any, validity or mere uses of proprieties are critical characteristics of van Hiele levels (p.855). The analysis of van Hiele levels depends largely on illustrating students' verbal expressions. Many researchers found that activities such as sorting shapes and drawing pictures of polygons can also provide evidences of students' levels of thinking (e.g., Burger & Shaughnessy; 1986; Mayberry, 1983). These activities work well especially with younger children (Clements et al., 1999), while their language skills are still under development. Therefore we need to take a greater consideration of what students "say" and "do" when working on a geometry task in order to better understand their geometric thinking.

My study took a different direction, in examining students' geometric thinking through their geometric discourse. The results of these examinations revealed small fractions of the richness of human thinking, while helping to add a little more data on what we know *so far* about the van Hiele Theory.

The van Hiele Levels: Discrete or Continuous?

This ongoing discussion about the continuity and discreteness of van Hiele theory motivates the study to revisit the van Hiele levels with a different lens. I will use two of the most revealing characters of geometric discourse, *geometric object* and *routine of substantiation,* at each van Hiele level to argue that my study confirms the discrete nature of the levels, and adds as well more information on the continuity of the levels. I use the term "geometric object(s)" to refer to all the mathematical objects involved in a particular geometric discourse, whereas the term "geometric figure(s)" refers to all the 3 or 4-sided polygons. My discussions about students' substantiation at each van Hiele level focus on two types of substantiations: the object level and abstract level substantiations.

The object level substantiation emphasizes students' routines of substantiation, looking at descriptions of how geometric figures are being investigated. Describing static lines, angles and polygons as movable entities under transformations (i.e., rotation, translation and reflection), as a way of substantiation, is an example of the object level of substantiation. With regard to definitions of different quadrilaterals, however, routines of substantiation depending on measurement routines to check the sides and angles of quadrilaterals, without thinking about how quadrilaterals are connected, are other examples of the object level of substantiation. Object level substantiation is a routine of substantiation, where students focus on the concreteness of geometric figures.

Abstract level substantiation emphasizes students' routines of substantiation using endorsed narratives to endorse new narratives. That is, students use mathematical definitions and axioms to construct mathematical proofs. During my interviews with students, I noticed that students with an abstract level of substantiations also used object level substantiations to modify their justifications. For example, a student used the Angle-Side-Angle congruence criterion to construct a proof at an abstract level that opposite angles of a parallelogram are congruent, and could also justify why this congruence criterion works using rotations at an object level.

<u>Geometric Discourse at Level 1</u>

At Level 1, students name geometric figures based on their appearance. In geometric discourse at this level, the word use is passive. That is, the process of naming a polygon is an act of matching a picture of a polygon with its given name. When a student is asked for verification of why such polygons are called "rectangles", or why "opposite sides and angles of a parallelogram are equal", the course of actions include direct recognition that are self-evident. Some students use their prior experiences to draw conclusions, but the course of actions are known as rote memorizations, such as “I learned it in school”, or “I know it is a square”. At this level, grouping quadrilaterals into different groups (i.e., rectangle, rhombus, parallelogram, square, etc.) is about putting them together by their names. The geometric objects at this level of discourse are collections of concrete, unstructured, discursive objects, and there is no routine of substantiation.

<u>Geometric Discourse at Level 2</u>

At Level 2, students are able to identify some properties of geometric figures, but properties are not yet ordered. In geometric discourse at this level, word use is routine driven, which means that naming a polygon involves not just matching a polygon with a name, but referring to it with a common descriptive narrative according to some visual properties. When a student is asked for an explanation of why a polygon is called a “rectangle”, or why “opposite sides and angles of a parallelogram are equal”, the courses of actions include direct recognition, as well as counting the number of sides, or measuring the sides and angles. The student might respond with, “It looks like it has four right angles”, or “I measured and all the angles are 90 degrees”. At this level, grouping quadrilaterals into different groups involves organizing them by their names and by some of their visual properties. The geometric objects at this level of discourse are collections of concrete, unstructured discursive objects which might be placed into disjoint categories (i.e., they all have right angles, or parallel sides, etc.). The routines of substantiations in this geometric discourse focus on checking and verifying partial visual properties of geometric figures.

<u>Geometric Discourse at Level 3</u>

Level 3 is a level where the properties of geometric figures are ordered, and they are deduced one from another. A student thinking at this level does not understand deduction, but the definitions of figures come into play. In geometric discourse at this level, word use is still object driven, as the naming of a polygon depends on its visual properties, and a common descriptive narrative accompanying the name of the figure (i.e., a definition of a quadrilateral). When a student is asked why a polygon is called a “rectangle”, the course of action is to check the defining conditions of the polygon by counting the number of sides, and measuring and comparing the sides or angles. A possible response is, “It is a rectangle because it is a parallelogram, and it has four right angles”. At this level, when the student groups quadrilaterals, a polygon could belong to multiple groups at the same time by definition. For example, a student could identify a square as a rectangle, a parallelogram, and a rhombus because it fits the descriptions of these polygons. Geometric objects at Level 3 are collections of concrete discursive objects and they begin to connect with joint categories. In the case of quadrilaterals, all 4-sided polygons begin to fall into a hierarchy of classification.

It is important to note that objectification can be found in geometric discourse at Level 3. That is, a concrete discursive object, such as a 4-sided polygon labeled as a “square” at a lower geometric discourse (i.e., a lower van Hiele level), becomes an abstract discursive object at this geometric discourse, as the word “square” presents this multi-dimensional thing with definitions and relations to other quadrilaterals. Geometric discourse at Level 3 also reveals the details of substantiation as a beginning stage of deductive reasoning.

Geometric Discourse at Level 4

At Level 4, students are able to reason deductively. In geometric discourse at this level, word use is object driven. That is, the naming of a polygon or using a mathematical term (e.g., angle bisector, supplement angle, etc.) is guided by common descriptive narratives (i.e., definitions). Grouping quadrilaterals into different groups means arranging them by definitions with a hierarchy of classification. Routines of substantiations lead to the constructions of new endorsed narratives. At this level, students are more fluent in using definitions in their substantiation, and in making connections among endorsed narratives (axioms, propositions, etc.) to construct new endorsed narratives. Geometric objects at Level 4 are collections of abstract discursive objects, and the main activity of substantiation is to produce newly endorsed narratives, or commonly, to construct mathematical proofs.

After the description of geometric discourse at each van Hiele level, the discreteness, in terms of qualitatively different geometric thinking at each level, is evident. For years, researchers have examined the possibility that levels are continuous without jumps. Let me move on to my earlier claim that my study provides evidence of the continuity of the levels; in particular, the continuity of levels made visible through the variability of participants' changes in their geometric discourse at the same van Hiele level, as well as at two consecutive levels.

Continuity becomes more evident once we realize that changes in students' geometric discourses are forms of change in thinking and communication, and that thinking is developed continuously towards, rather than in jumps, to a higher van Hiele level. I wish to discuss in more detail the development of geometric discourse, highlighting the development of word use and the changes in routines of substantiation.

<u>Continuity Within a van Hiele Level</u>

AYA and ARI showed no change in van Hiele levels in their responses in the pretest and posttest, but I found changes in their geometric discourses.

AYA's van Hiele pretest and posttest responses suggested that her thinking operated at Level 2 (descriptive). I analyzed AYA's word use, and found that her use of the word "parallelogram" changed from the pre-interview to the post-interview. When she spoke the word “parallelogram” at the beginning of the semester, she meant *any* polygon having pairs of parallel sides, in using a definition of parallelogram with only a necessary condition. Later in the semester, AYA developed more understanding of parallelograms in the geometry class, and she was able to use definitions of parallelograms with both necessary and sufficient conditions. Her thinking at the time fit more towards the descriptions of geometric discourse at Level 3. Figures 5.2 and 5.3 illustrate the characteristics of AYA’s geometric discourse at the pre-interview and at the post-interview, respectively.

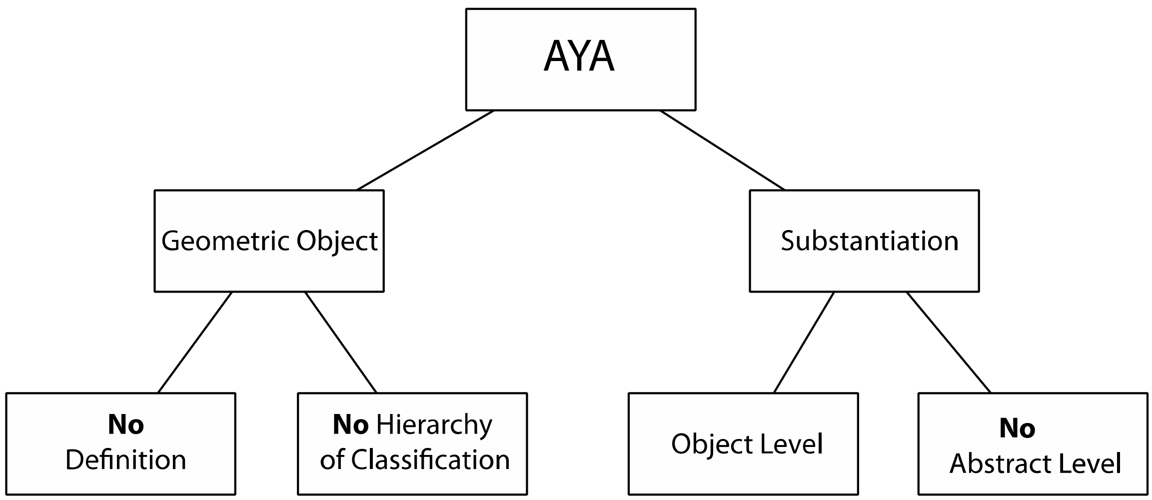


Figure 5.2 Characteristics of AYA’s geometrics discourses at Level 2 at the Pre-interview.

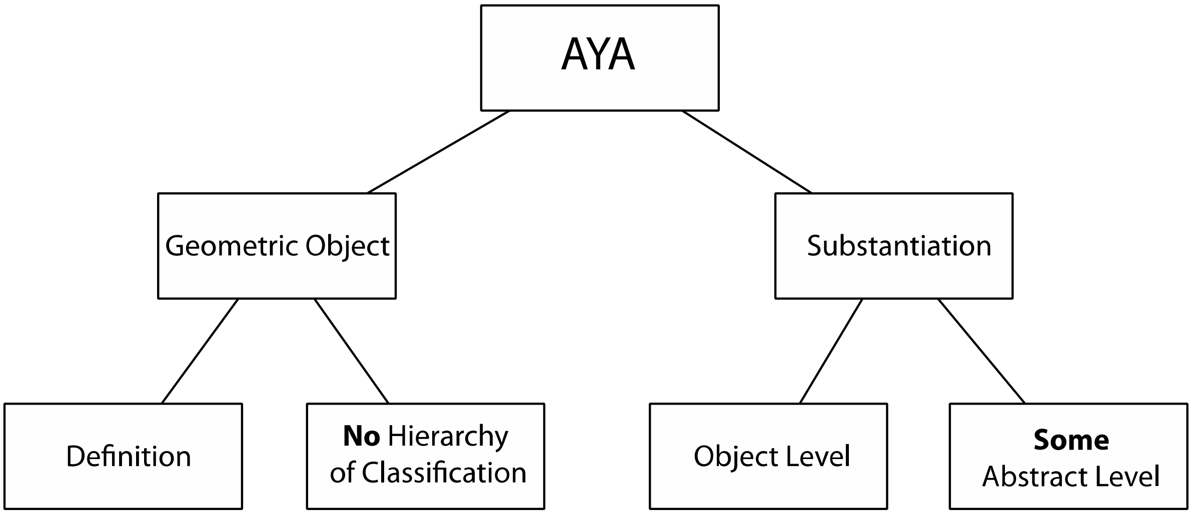


Figure 5.3 *Characteristics of AYA's geometrics discourses at Level 2 at the Post-interview.*

Figures 5.2 and 5.3 show two main changes in AYA's geometric discourse, a change in word use and a change in reasoning. AYA had developed competence in using definitions to identify and group polygons with no hierarchy of classification, and had developed some informal deductive reasoning as her geometric thinking moved towards Level 3. Here, I am not trying to contradict the findings from AYA's paper-pencil pretest and posttest with her interview results, but rather to compile the results and to treat her thinking more dynamically. Her progress illustrates a student's geometric thinking developing continuously within Level 2 and in transition between Level 2 and Level 3, as she was more competent in using definitions to name polygons, and her routines of substantiation began to operate at an abstract level in using definitions and axioms to construct mathematical proofs.

ARI's geometric discourse presents another example of such continuity, but within a different van Hiele level. Her van Hiele pretest and posttest responses suggested that her thinking operated at Level 3. ARI came in with the ability to identify and group polygons using definitions at the beginning of the semester. By the end of the semester, she began to reason more abstractly by constructing mathematical proofs using definitions and axioms. Figures 5.4 and 5.5 illustrate characteristics of ARI's geometric discourse at Level 3, in the pre-interview and post-interview, respectively.

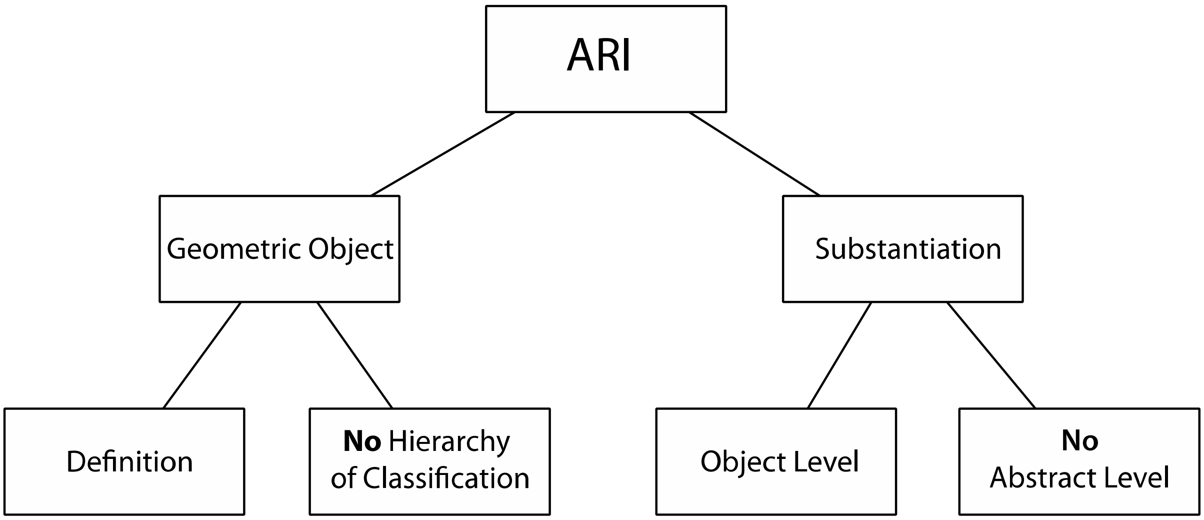


Figure 5.4 Characteristics of ARI's geometric discourse at Level 3 at the Pre-interviews.

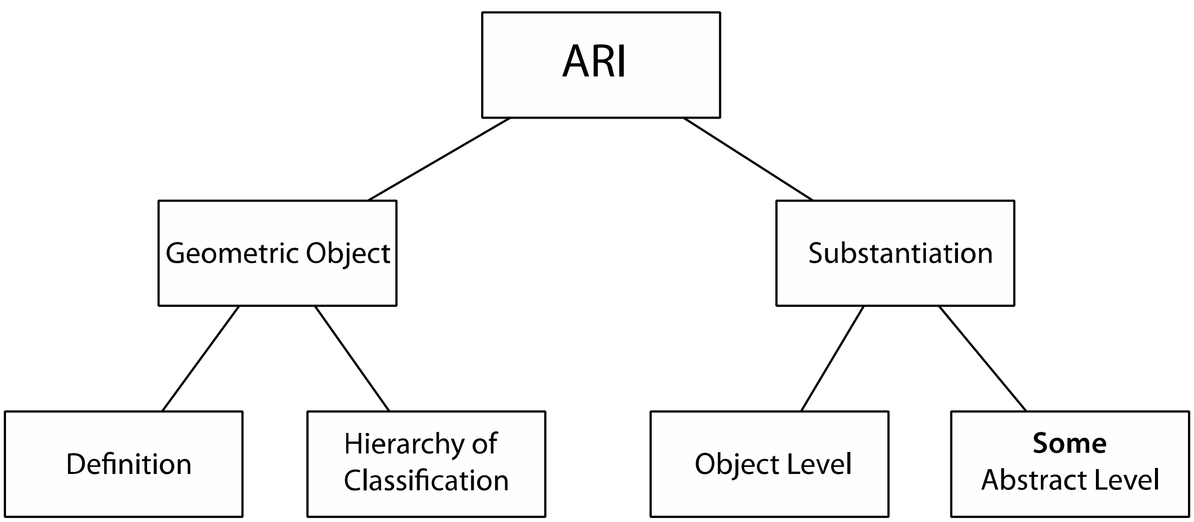


Figure 5.5 Characteristics of ARI's geometric discourse at Level 3 at the Pre-interviews.

Clearly, there are changes in ARI's geometric discourse within Level 3. Building on her familiarity with using definitions of quadrilaterals, ARI later showed understanding of how quadrilaterals are connected. After ARI was more fluent in using definitions, she developed routines of substantiations using informal deductive reasoning to substantiate her claims.

In comparing ARI's geometric discourse at Level 3 with AYA's at Level 2, we find similarities between ARI's geometric discourse at the pre-interview and AYA's at the post-interview. Both shared familiarity with using definitions in identifying and grouping quadrilaterals, and were able to reason at an object level. Perhaps this observation indicates the continuity of learning in transitioning between two consecutive levels, Level 2 to Level 3. I will next look at ALY's geometric discourse to make another case that the learning process is continuous.

Continuity Within Two Consecutive Levels

ALY was one of two students of the study who reached Level 4 at the end of the semester. At Level 3, ALY demonstrated a typical behavior at this level, as illustrated in Figure 5.6.

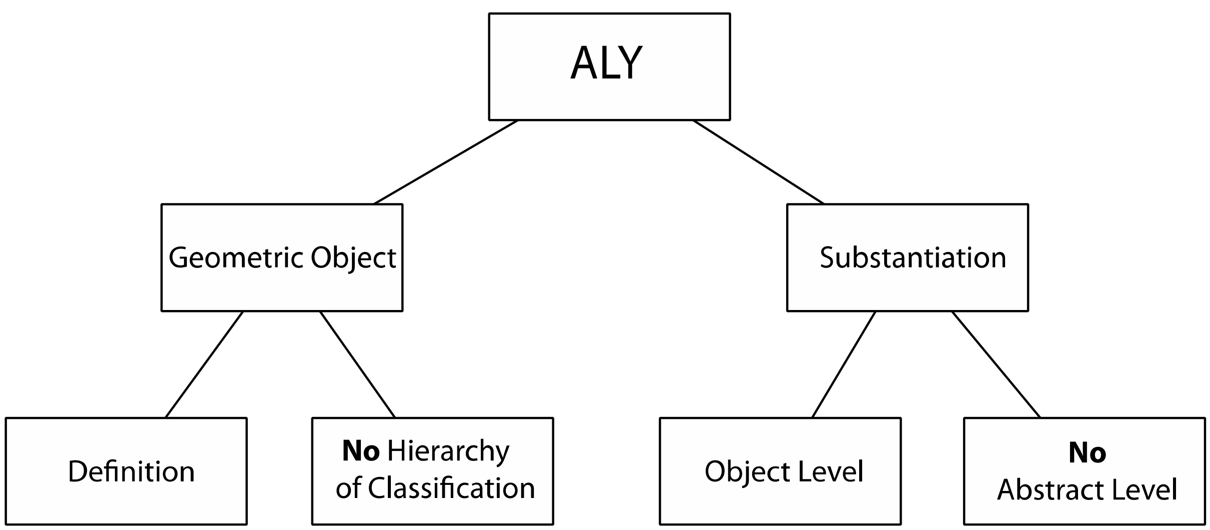


Figure 5.6 Characteristics of ALY's geometric discourse at Level 3 at the Pre-Interview.

Similar to other students at Level 3, ALY came in with the ability to use definitions to identify and group polygons, but she did not show how quadrilaterals were connected, and performed object level substantiation. At the end of the semester, she was able to draw connections among quadrilaterals and to use propositions and axioms to construct mathematical proofs, as illustrated in Figure 5.7.

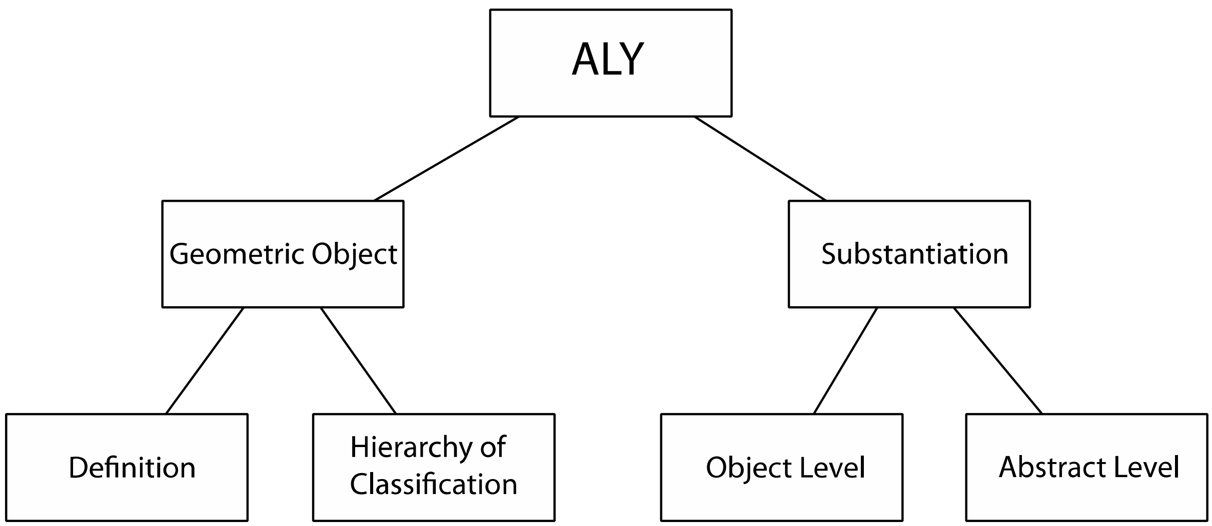


Figure 5.7 Characteristics of ALY's geometric discourse at Level 3 at the Post-interview.

Figure 5.7 presents a main characteristic of a Level 4 discourse that is absent in Level 3: abstract level of substantiations. At this level, ALY showed familiarities with using definitions and axioms to construct proofs and with using algebraic symbols to write a formal mathematical proof. However, at Level 4 we also expect to see behavior where students are able to apply inductive reasoning in an unfamiliar situation, and to connect the knowledge they learned. In ALY's case, she was able to apply her knowledge of quadrilaterals to construct mathematical proofs in a familiar situation (e.g., to prove opposite angles or sides are congruent using congruent criterions), having carried out similar proofs in her geometry class. When ALY was asked to prove two definitions were equivalent, she did not finish the proof because the task was new to her, and she did not know how to use the same axioms in a new situation. I argue that ALY was at the beginning stage of Level 4 thinking, starting to gain the skills and languages needed for mathematical proofs, but needing more practice to move forward to an advanced abstract level.

AYA, ARI and ALY each made a case that the development of geometric discourse within a level is continuous, and the development of geometric discourse from one van Hiele level towards a geometric discourse at the next van Hiele level is also continuous. Using a discursive lens in this study allowed me to unpack students' thinking, and to better understand what students said about geometric figures and what they did when they asked for justifications. As a result, my study also contributes to answering another question raised by Clements, on whether there are *other* characteristics that *should be* considered at each van Hiele level.

The van Hieles wished to note language differences and different linguistic symbols at each level, in the study of language in geometric thinking, but were never explicit about it. The language of mathematics I wish to discuss here does not refer to a list of vocabulary words or grammar rules, but rather to the communicative competence necessary and sufficient for competent participation in mathematical discourse.

The van Hiele descriptions of the levels focus largely on how a student reasons about geometric figures in a language, for instance, in response to what is a rectangle versus what is not a rectangle, in applying a definition. What is missed or not clearly emphasized is the meaning of a mathematical term when used by a student. When I consider each van Hiele level as its own geometric discourse with characteristics of word use, narratives, routines and visual mediators, I regard word use as all-important, revealing facts concerning how a concept is formed. In this study, students' word use provided significant information about how a concept of a geometric figure is formed at different van Hiele levels among different students. Moreover, a careful analysis of students' mathematical word use in geometric discourse also shed light on how words are used and whether the words are used correctly for the sake of communications. If any other characteristics should be considered at van Hiele levels, I recommend adding *word use* to the list.

Discursive routines do not determine students' actions, but only constrain what they can reasonably say or do in a given situation, as negotiated conventions. However, discursive routines offer valuable information about what students do and say as a course of action to make conjectures and justifications in a pattern at a geometric discourse. I find it very useful to see the details of students' routines of identifying, defining and justifying when working on a task about geometric figures and their properties, where the roles of definitions are demonstrated at the first three van Hiele levels. Discursive routines are associated with students' creativity when they apply routines in non-routine ways; that is, in applying familiar routines in an unfamiliar discursive context. In my study, some participants used algebraic reasoning (familiar routines) to construct geometric proofs (unfamiliar discursive context), without using geometric axioms. Therefore, if any other characteristics should be considered in van Hiele levels, I also recommend adding *routine*, a repetitive discursive action to the list.

I have tried in several ways to explain what additional information this discursive framework provides with regard to the levels of geometric thinking, as well as how this additional information adds to what we know about van Hiele levels. There is no closure to what we know about the van Hiele levels. Looking at what we do know, we are led to ask what future studies are needed in this important area of research.

What Can Be Asked and Why

Have you ever had an experience of sitting alone at your desk thinking, lost in thoughts as if engaged in a conversation with someone? We must concede that thinking is an individualized form of interpersonal communication, and whatever is created is a product of collective doing. As a teacher, most of the time, I wish to know what is in my students' minds and their thoughts in mathematics. This empirical study in some way gives us an opportunity to analyze students' thinking at an individual level. One question natural to ask is, "If anything, what do we learn from students' thinking?"

What Do We Learn From Students' Thinking?

First, I want to discuss the existence of Level 0. Some researchers suspected the existence of a level prior to the Base Level (Level 1). My empirical study shows that students can reason at a higher van Hiele level, but their lack of knowledge in geometry, or simply forgetting what they learned in geometry, has kept them from giving correct answers. In ANI's case, we learned that the geometric pretest placed her at Level 0. However, my interview with ANI after the pretest revealed that she was able to group parallelograms by their names, but did not know the differences between a rhombus and a square, as well as the differences between a parallelogram and a rectangle; this geometric discourse fit more to the descriptions at Level 1 than Level 0.

It was quite common during the interviews that a student could not identify a "trapezoid", or a "rhombus", because they did not learn these names, or forgot the names. So, if we consider the existence of Level 0 (a pre-level to Level 1), then it is likely that we include the possibility of the kind of reasoning students perform in a domain of knowledge that they have not yet explored. This observation led us to consider: “What does van Hiele theory serve to assess?” and “What do we wish to find out using van Hiele theory?”

Next I want to think about this scenario, when students were prompted to show that two opposite angles were congruent in a parallelogram. First, visually they were convinced that the two angles were the same, but further verification was required. One student responded that the two angles were congruent because she used a protractor to measure the angles and they had the same measurement. This course of action is typical in geometric discourse at Level 2, where a student’s reasoning depends on checking and verifying the conditions for being congruent. From this response we learn that this student has mastered knowing what are “opposite angles” in a parallelogram, but needs to explore what we call “congruent”, a property of opposite angles, in a concrete way. For the same task, another student described a sequence of transformations where a rotation was followed by a translation, to show that the two angles were the same. She stated that she could rotate one angle, and moved the angle to match the other one, and was sure that the two angles would match exactly. This course of action is typical in geometric discourse at Level 3, where a student is familiar with the term “congruent” and tries to explore whether opposite angles are congruent concretely. From this response, we might infer that students need to explore the properties of parallelograms through hands-on activities before they reach the conclusion that “*all* opposite angles are congruent in *any* parallelograms”; then inductive reasoning starts to make sense.

Recall that van Hiele levels are sequential, in that students pass through the levels in the same order, although varying at different rates, and it is not possible to skip levels. In my study, we noticed a sequence where a student needs to understand “opposite angles” in the case of a parallelogram and the meaning of “congruent” first; and then move on to explore the properties and relations regarding opposite angles in a parallelogram; and then, perhaps, based on these concrete experiences, she begins to develop some abstract thinking such as inductive reasoning. Thus, the object level of substantiations clouds important points van Hiele made that students must explore domains before describing them, that elaborate descriptions of concrete properties and relations must be made before abstract relations are explored.

One challenge our students face is the development of abstract relations, because the abstract relations in geometry may never be fully understood by some students. It takes time for students to get used to new mathematical terms, as well as to digest the hands-on activities relating to a particular property, before they can generalize it. When students are introduced to more advanced thinking in deductive reasoning, some mimic the proofs without fully understanding them. When we rush to the stage of constructing proofs that a student is not ready for, it creates obstacles. It is important to give students enough opportunities to explore a sequence of activities at a level built on other activities at a previous level before abstract relations are explored.

Next, let us discuss the breath of Level 3. Recall that among the sixty-three students who took the van Hiele posttest, thirty of them were considered at van Hiele Level 3; and among the twenty students who participated in the interviews, ten of them were being placed at Level 3 in the van Hiele posttest, and the interview analyses confirmed that these students were competent to use definitions to justify their conclusions. It was quite surprising to have more students at Level 3 than students at the lower levels. However, we also learned that students' geometric discourses seemed to develop at different rates, and geometric discourse at Level 3 varied from person to person and varied in the same person at different times of the semester. Two main variations of the discourse at Level 3 are 1) how profound a student uses definitions (geometric object) and 2) the way she reasons about the geometric figures (substantiation). Figure 5.8 illustrate these two variations in Level 3.

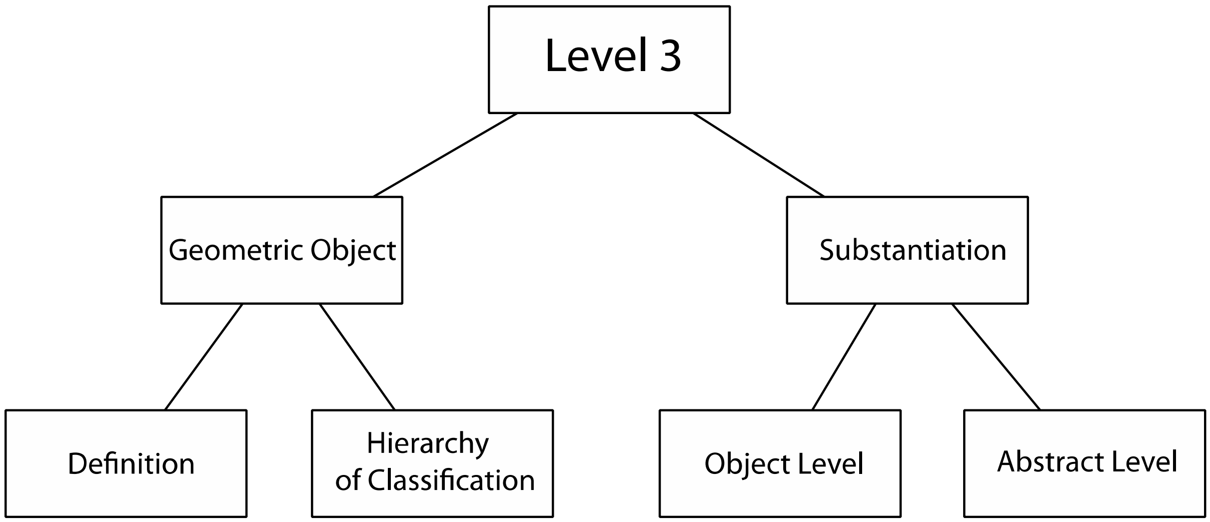


Figure 5.8 Characteristics of Level 3 geometric discourse.

Figure 5.8 highlights possible variations of geometric discourse at Level 3. Having a geometric discourse at Level 3 indicates that a student has developed competence in applying definitions in their identifying and justifying routines. In this study, such a student may or may not make connections among the quadrilaterals, where a hierarchy of classification is presented and depending on how profoundly the student understands the definitions and uses them adequately.

ATL's geometric discourse shows that a student at Level 3 could have competence in using definitions to identify quadrilaterals, but still needs to develop other stills needed at this level. In contrast to ATL's geometric discourse at the same level, AYA and ALY represented a group of students who could use definitions fluently, as well as reason at the object level. ANI and ARI represented a group of students who were more advanced at Level 3, when they used definitions to show a hierarchy of classification among quadrilaterals; and based on their experiences of reasoning at object level, they also tried to substantiate their conclusions using some abstract relations.

In our analysis of the variety of geometric discourses at Level 3, we have learned that Level 3 thinking is more complicated than previously thought. At this level, students need to be familiar with and feel comfortable using the definitions fluently; and at the same time, they also are developing informal reasoning by describing what they observe from exploration of concrete properties of geometric figures, so that they can see or feel the particularity of figures before abstract relations take place. All these mathematical activities become students' prior experiences in the development of abstract thinking.

There is a challenge for teachers when teaching a group of students whose geometric discourses vary at different places. What kinds of activities will be suitable to all students so that they can move toward a higher level of thinking? It is also a challenge for researchers to identify kinds of activities that will help students in their development of geometric discourse at the same level and at different levels.

Practical Information About Teaching and Assessment

The empirical data of the study offers a dialogue between a learner and a researcher in a designed environment. The researcher carefully chooses the tasks, and prompts students' thinking with well-designed protocols. However, this procedure is not too far from what a teacher might do in preparing instructional materials. So one question we ask is, "What practical information does this study offer about teaching and assessment?"

We need more instructional interventions in our classrooms. Students need to explore "unfamiliar situations" using their existing knowledge. In an instructional sequence, a student first is introduced to a new concept or new way of presenting her idea in mathematics, such as in constructing a proof by imitating what others say and do, and then she is asked to solve similar problems or construct similar proofs in a similar context. It is a practice of repeating the same process, the process of exploring the domain of polished tasks and well-designed mathematics activities, and it is an important part of learning. However, we do want our students to move beyond this stage and to be more creative. There are opportunities in the classrooms where students can be creative. Their discursive routines show that sometimes they prefer to use algebraic reasoning to derive a geometric proof. We need to encourage such thinking and create more activities to help students make connections between geometry and other domains of mathematics without losing the goals of introducing definitions and axioms in geometry.

Our concerns about “communication,” “language,” and “discourse” in the mathematics classroom are not new. Fifty years ago, the same concerns motivated the van Hieles to develop their theory. Surprisingly, we still don’t have much to recommend what needs to be done in a geometry classroom for the sake of communication. My study can add some information about the need to clarify the mathematical terms we use in the classroom, and to be specific about the context in which they are used.

When a student mentions the word “parallelogram”, she says nothing unless she makes explicit what she means. My empirical data shows that the word “parallelogram” could mean quite different things to our prospective teachers. Some thought parallelograms are four sided figures having two pairs of parallel sides with two sides longer; whereas some thought parallelograms are tilted rectangles and squares. A few students thought that a parallelogram is a polygon with pairs of parallel sides, and then of course, hexagons and octagons are parallelograms. I also found that the use of the word “bisector” was confusing, because of the several contexts where it was used. Some referred to it as “angle bisector”, whereas others thought it “a mid-point of a segment”. A few thought “all bisectors cross each other in the middle of a parallelogram”. All these are correct ideas of “bisector” in a particular context. Just imagine a group of students working together, using the same word in their own way, and definitely creating some miscommunications. Teachers need to be very cautious about the way students use mathematical terms, in order to make the classroom discussions more understandable. During the interviews, I found it helpful to get to the bottom of what students meant with a mathematical term by asking questions such as, “Can you say more about what you mean by…?”, “Will you give me an example of what you just said about…?” and “Can you show me in this picture where is (are)…?”.

Every classroom is different. I only can offer what worked in my study with prospective teachers. But the principle is that classroom teachers need to ask questions and to ask different questions, and to listen carefully to students’ responses, and give rapid feedback to ensure that mathematics subjects are communicated well.

Limitation of the Study

This study contributes to the field of mathematics education in several ways. It pilots an analytic method for investigating students’ geometric thinking using a discursive framework looking in particular at changes in prospective teachers’ learning about triangles and quadrilaterals in Euclidean geometry. The discursive framework presents another way to view van Hiele levels as qualitatively different geometric discourses.

The study illustrates the usefulness of the discursive framework for highlighting the opportunities for rich description at each van Hiele level through a discursive lens. A detailed analysis can discover differences in participants’ geometric discourse at each van Hiele level, which may impact the ways in which students do mathematics, speak about mathematics, and therefore learn mathematics. This study also provides information for thought and discussion to teacher educators interested in geometry more specifically. For example, should the development of a concept in geometry begin with a word? Should the introduction of a quadrilateral begin with the necessary condition of a polygon with four sides? Finally, this study acknowledges students' participation by analyzing their thinking processes as the union of the words and actions observed during the interviews.

The primary limitations of this study have been stated throughout this dissertation, but will be reiterated here briefly. First, analyzing students' thinking processes (i.e., in video recordings and interview transcripts) is challenging. In particular, when participants' thinking was not yet consistent and logically ordered, analyzing their geometric discourse was harder. Hopefully, my descriptions and interpretations are clear. Secondly, the analytic decisions regarding what to present and to compare for each participant were made to illustrate similarities and differences. Thus, the primary perspective represented here is mine. I acknowledge that another individual using the same data (i.e., reading the transcriptions) may see things quite differently. It is possible that some important responses from participants during the interviews may have been missed. Finally and most importantly, I interviewed twenty students, and presented my analyses for five of them. Thus many claims are based on the cases of five students, or of twenty students. In addition, this study focused only on the "two end points" of students' thinking during the time of their course work. I did not consider the individual aptitudes of students' learning, and the study may miss some advanced thinking gained after the course is over. Therefore, there is no closure to my study, but it opens the door for me to pursue further investigations.

The study leaves me a long list of looming questions and ideas for future research. First, there is a group of students whose van Hiele levels could not be determined by their test results, but who participated in the interviews. I decided not to include their interview responses in my analyses in order to narrow the scope of the study. To continue the study, one could analyze these data to find out how the model of development of geometric discourse would help to identify participants' levels of thinking. And how would these data help to refine the model of development of geometric discourse?

This study focused on students' geometric discourse, and how this discourse helps us to learn more about their thinking. However, one open question asks what this mathematics discourse looks like when a student works on different mathematical tasks that include different content domains of mathematics, and how the subsets of mathematics discourse interact with each other. Perhaps it will help to gain more information regarding one of Clement's questions, "do students reason at the same van Hiele levels across topics?"

As mentioned previously, for those interested in geometry or teaching geometry, an investigation using a discursive lens into students' use of mathematical terminology in geometry would be a next step. This analysis could also be extended to other mathematical topics. More discussions regarding classroom interactions are needed. What can we do to help students use mathematical terminology more precisely for the sake of communication and development of a mathematical concept?

We need to develop frameworks for analyzing activities from both textbooks and classrooms, and to identify mathematical activities that help students move to higher van Hiele levels. The rise and popularity of computer software created a new learning environment for students, and presented an important instructional and learning tool in school curriculum. Many researchers and curriculum developers want to design pre-constructed activities using software, and they hope that these activities will serve as mediators to help students learn geometry. In response, there is a need to develop instruments to examine these activities, with the goal of helping students develop more advanced levels of thinking.

Finally, we need to revisit van Hiele levels with multiple lenses, in order to have a better picture of human thinking, and to improve communication through classroom interaction. How can teachers better facilitate classroom discussions at various levels and in various contexts?

In summary, this study provides opportunities for conversations among mathematics education researchers, curriculum developers, and teacher educators and teachers, on the learning and teaching of geometry. Such conversations would address students' levels of thinking in Euclidean geometry through the lens of a discursive framework, in hopes of improving research and teaching, and therefore better serving mathematics learners.

APPENDICES

APPENDIX A

RESEARCH STUDY CONSENT FORM

Appendix A: Research Study Consent Form

Dear Participant:
You are invited to participate in a research study (Research title: The van Hiele Theory Through the Discursive Lens: Prospective Teachers’ Geometric Discourses). Researchers are required to provide a consent form to inform you about the study, to convey that participation is voluntary, to explain risks and benefits of participation, and to empower you to make an informed decision. You should feel free to ask the researcher any questions you may have.

From this study, the researcher hopes to learn about prospective teachers’ knowledge of basic geometric shapes; their abilities to make conjectures and their abilities to construct mathematical proofs. Participating in the study will involve discussing your understanding about geometric shapes and their properties. You will also be asked to explain your reasoning for statements you make about geometric shapes. The entire study will take you about four hours: the first two-hour will take place at the beginning of the semester. This part includes a pretest (35 minutes) and an interview (85 minutes); the second 2-hour will take place at the end of the semester, and it includes a posttest (35 minutes) and an interview (85 minutes). You will receive10 extra points for the Math 202 class you are taking after you complete both pretest and posttest. If you are selected for an interview, you will be asked to work on three geometric tasks, and will discuss your results with the researcher or other students during the interview. You will receive $20 as compensation for your time from the researcher one week after you complete the 85-minutes interview.

In the study, your written work (e.g., pretest, posttest, exercise worksheet, interview tasks etc.) will be collected, and your activities and conversations with the researcher and with other students will be audio-recorded and/or video-recorded. The researcher may take notes during the exercise and interview. The potential benefit for you to participate in this study is that you will receive the opportunity to learn properties of basic geometric shapes and to learn to formulate conjectures and geometric proofs, which is a part of the course content of Math 202 you are taking. Additionally, research indicates that self-reflection on one's thinking aids in increasing the level of sophistication of that thinking. Therefore, the researcher on this study will encourage such self-reflection.

The results of this research study might be published in professional publications for teachers and researchers. Only the research team will use the audio, videotapes or transcripts for analysis. Your confidentiality will be protected to the maximum extent allowable by law. The data collected will be coded such that your name and personal information will not show up in or be linked with any reports and presentations of research project. The data will be stored and locked in a steel cabin in the office of the researcher in Wells Hall at the Michigan State University. Only the research term of the project or the Institutional Review Board (IRB) of the Michigan State University have access to the research records about you and the data collected from you. All research data will be retained for a minimum of 3 years following closure of project.

Participation in this research study is entirely voluntary. You may choose not to participate at all. You may also refuse to participate in certain procedures or answer certain questions. Furthermore, you may decide to discontinue your participation at any time without penalty.

There are some minimal risks associated with your participation in this research study in that you may feel tired and embarrassed when you do not know how to justify your thinking. Thinking out loud may be an unfamiliar process to you and thus you might feel uncomfortable at first.

Please sign your initials if you consent to either of the following:

- I agree to participate in the pretest and posttest but not the interviews.
  ☐ Yes ☐ No Initials____________
- I agree to participate in the pretest and posttest with the possibility of being chosen for the interviews.
  ☐ Yes ☐ No Initials__________

Please sign your initials if you consent to participate in the videotaping:

- I agree to allow my image on the videotapes to be included in presentations.
  ☐ Yes ☐ No Initials____________
- I agree to allow audio recording/video recording of the interview.
  ☐ Yes ☐ No Initials____________

If you have concerns or questions about this study, such as scientific issues, how to do any part of it, or to report an injury, please contact Dr. Glenda Lappan at (517) 432-3635, or e-mail glappan@math.msu.edu or regular mail at A718 Wells Hall, MSU, East Lansing, MI, 48824.

If you have any questions or concerns about your role and rights as a research participant, would like to obtain information or offer input, or would like to register a complaint about this study, you may contact, anonymously if you wish, the Michigan State University Human Research Protection Program at 517-355-2180, Fax 517-432-4503, or e-mail irb@msu.edu or regular mail at 207 Olds Hall, MSU, East Lansing, MI 48824.

Participant's Printed Name ____

Participant's Signature__________________ Date __________________

Sincerely,

Glenda Lappan, Ph.D.
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APPENDIX B

BEHAVIORS AT EACH VAN HIELE LEVEL

Appendix B: Behaviors at Each van Hiele Level

<u>Level 1 (van Hieles' Basic level)</u>
(P.M., 1958-59)

1. " Figures are judged according to their appearance."

2. "A child recognizes a rectangle by its form, shape.

3. … and the rectangle seems different to him from a square."

4. "When one has shown to a child of six, a six year old child, what a rhombus is, what a rectangle is , what a square is, what a parallelogram is , he is able to produce those figures without error on a geoboard of Gattegno, even in difficult situations."

5. "a child does not recognize a parallelogram in a rhombus."

6. "the rhombus is not a parallelogram. The rhombus appears … as something quite different."

(P.M., 1968)

7. "when one says that one calls a quadrilateral whose four sides are equal a rhombus, this statement will not be enough to convince the beginning student [from which I deduce that this is his level 0] that the parallelograms which he calls squares are part of the set of rhombuses."

(P.M., 1979)

8. (on a question involving recognition of a titled square as a square)
   "basic level, because you can see it."

<u>Level 2 (van Hieles' first level)</u>
(P.M., 1957)

1. "He is able to associate the name 'isosceles triangle' with s specific triangle, knowing that two of its sides are equal, and draw the subsequent that the two corresponding angles are equal."

(P.M., 1957; P.M. and Dina, 1958)

2. " … a pupil who knows the properties of the rhombus and can name them, will also have a basic understanding of the isosceles triangle = semirhombus."

3. "The figures are the supports (lit. 'supports' in French) of their properties."

4. "That a figure is a rectangle signifies that it has four right angles, it is a rectangle, even if the figure is not traced very carefully."

5. "The figures are identified by their properties. (E.g.) If one is told that the figure traced on the blackboard possesses four right angles, it is a rectangle, even if the figure is not traced very carefully."

6. "The properties are not yet organized in such a way that a square is identified as being a rectangle."

(P.M., 1959)

7. "The child learns to see the rhombus s an equilateral quadrangle with identical opposed angles and inter-perpendicular diagonals that bisect both each other and the angles."

8. (a middle ground between this and the next level) "once the child gets to the stage where it knows the rhombus and recognizes the isosceles triangle for a semi-rhombus, it will also be ale to determine of hand a certain number of properties of the equilateral triangle."

9. "Once it has been decided that a structure is an 'isosceles triangle' the child will also know that a certain number of governing properties must be present, without having to memorize them in this special case."

(P.M., 1976)

10. "The inverse of a function still belongs to the first thought level."

11. "Resemblance, rules of probability, powers, equations, functions, revelations, sets – with these you can go from zero to the first thought level."

<u>Level 3 (van Hieles' second level)</u>

(Dina, 1957)

1. "Pupils … can understand what is meant by 'proof' in geometry. They have arrived at the second level of thinking."

(P.M., 1957)

2. "He can manipulate the interrelatedness of the characteristics of geometric patterns."

3. "e.g., if on the strength of general congruence theorem, he is able to deduce the equality of angles or linear segments of specific figures."

(P.M., 1958-59)

4. "The properties are ordered [lit. 'ordonnent']. They are deduced from each other: one property precedes or follows another property."

5. "The intrinsic significance of deduction is not understood by the student."

6. "The square is recognized as being a rectangle because at this level definitions of figures come into play."

(P.M., 1959)

7. “the child… [will] recognize the rhombus by means of certain of its properties,… because , e.g., it is a quadrangle whose diagonals bisect each other perpendicularly.

8. “It [the child] is not capable of studying geometry in the strictest sense of the word.”

9. “The child knows how to reason in accordance with a deductive logical system… this is not however, identical with reasoning <u>on the strength of</u> formal logical.”

(P.M., 1976)

10. “the question about whether the inverse of a function is a function belongs to the second thought level.”

11. “The understanding of implication, equivalence, negation of implication belongs to the second thought level.”

(P.M., 1978)

12. “they are able to understand more advanced thought structure, such as: ‘the parallelism of the lines implies (according to their signal character) the presence of a saw, and therefore (according to their symbolic character) equality of the alternate-interior angles’.”

13. “ I [the student] can learn a definition by heart. No level. I can understand that definitions may be necessary: second level.”

14. “… you know that is meant by it [the use of ‘some’ and ‘all’] second level.

<u>Level 4 (van Hieles’ third level)</u>

(P.M., 1957)

1. “He will reach the third level of thinking when he starts manipulating the intrinsic characteristics of relations. For example: if he can distinguish between a proposition and its reverse” [sic. Meaning our converse]

(Dina, 1957)

2. We can start studying a deductive system of propositions, i.e., the way in which the interdependency of relations is affected. Definitions and propositions now come within the pupil’s intellectual horizon.”

3. “Parallelism of the lines implies equality of the corresponding angles and vice versa.”

(P.M. and Dina, 1958)

4. “The pupil will be able, e.g., to distinguish between a proposition and its converse.”

5. “it (is) … possible to develop an axiomatic system of geometry.”

(P.M., 1958-59)

6. “The mind is occupied with the significance of deduction, of the converse of a theorem, of an axiom, of the conditions necessary and sufficient.”

(P.M., 1968)

7. “ … one could tell him (the student) that in proof it is really a question of knowing whether these theses are true or not, or rather of the relationship between the truth of these theses and of some others. Without their understanding such relationships we cannot explain to the student that one has to have recourse to axioms.” [I include the level from the first part of the statement; he never identified the level.]

<u>Level 5 (van Hieles’ fourth level)</u>

(Dina, 1957)

1. “A comparative study of the various deductive syatems within the field of geometric relations is … reserved for those, who have reached the fourth level…”

(P.M. and Dina, 1958)

2. “finally at the fourth level (hardly attainable in secondary teaching) logical thinking itself can become a subject matter.”
3. “the axiomatic themselves belong to the fourth level.”

(P.M., 1958-59)

4. “one doesn’t ask such question as: what are the points, lines, surfaces, etc.?... figures are defined only by symbols connected by relationships. To find the specific meaning of the symbols, one must turn to lower levels where the specific meaning of these symbols can be seen.”

APPENDIX C

VAN HIELE GEOMETRY TEST

Appendix C: van Hiele Geometry Test

Please print

Name ______________________________Section _____________ .

Directions

Do not open this test booklet until you are told to do so.

This test contains 25 questions. It is not expected that you know everything on this test.

When you are told to begin:

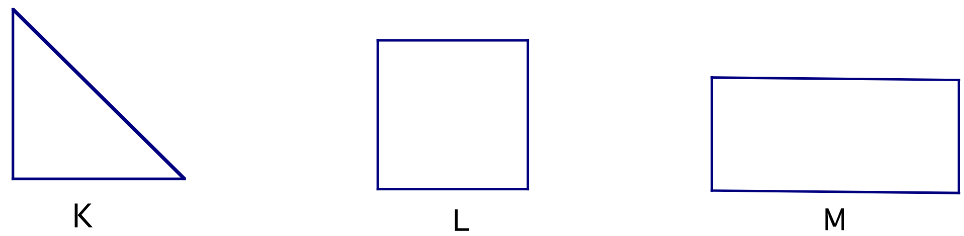
1. Read each question carefully.
2. Decide upon the answer you think is correct. There is only one correct answer to each question. Check the letter corresponding to your answer on the answer sheet.
3. Use the space provided on the answer sheet for figuring or drawing. Do not mark on this test booklet.
4. If you want to change an answer, completely erase the first answer.
5. If you need a pencil and an eraser, raise your hand.
6. You will have 35 minutes for this test.

Wait until the instructor says that you may begin.

This test is based on the work of P.M. van Hiele.

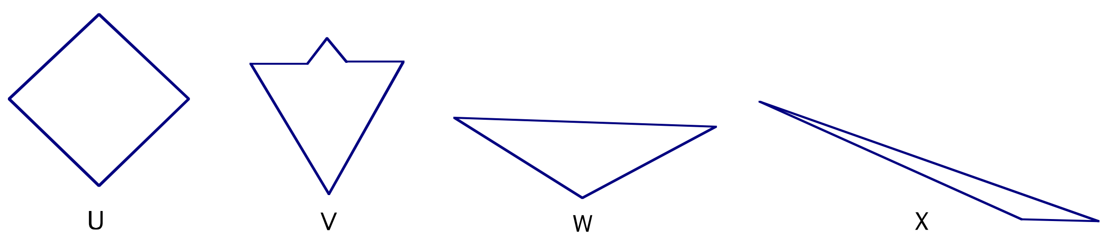
VAN HIELE GEOMETRY TEST

1. Which of these are squares?



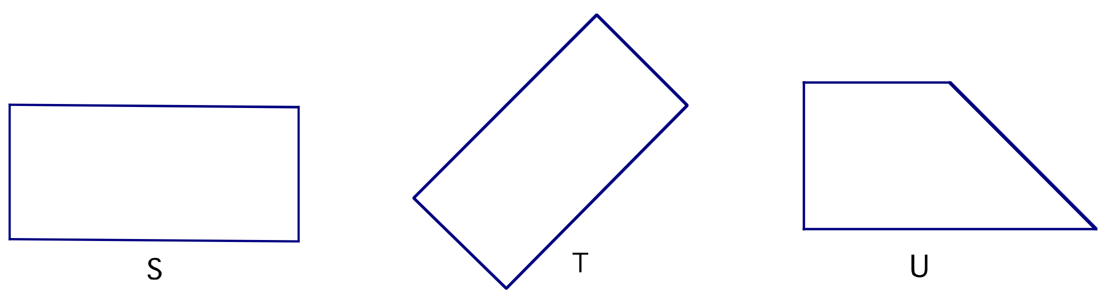
(F) K only
(G) L only
(H) M only
(I) L and M only
(J) All are squares

2. Which of these are triangles?



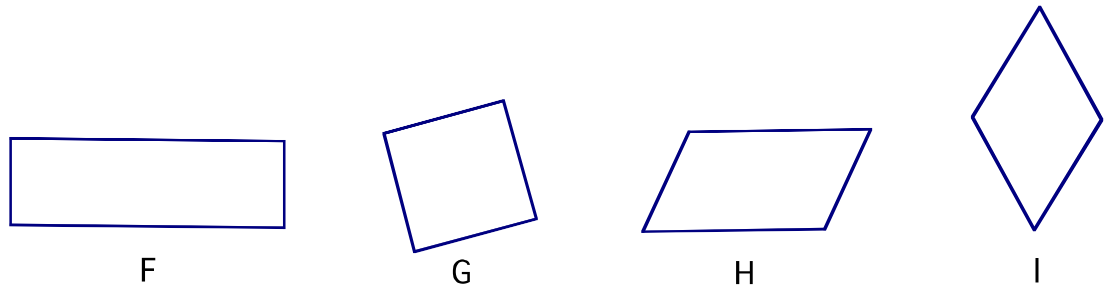
(A) None of these are triangles.
(B) V only
(C) W only
(D) W and X only
(E) V and W only

3. Which of these are rectangles?



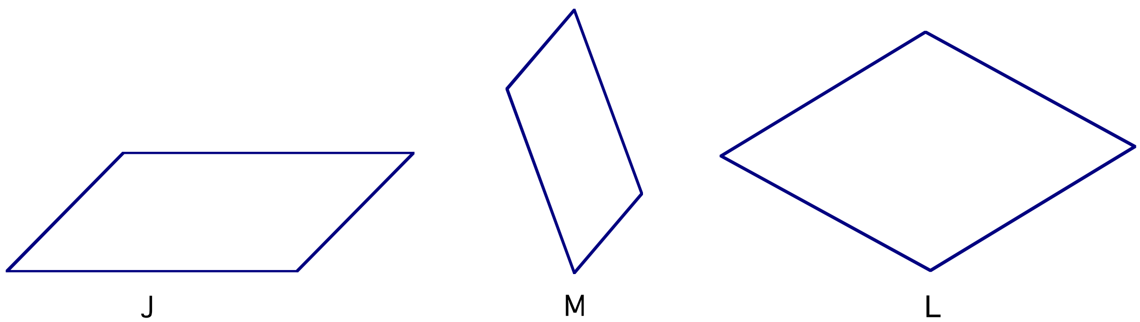
(A) S only
(B) T only
(C) S and T only
(D) S and U only
(E) All are rectangles

4. Which of these are squares?



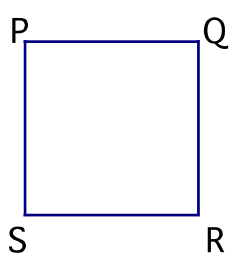
(A) None of these are squares.
(B) G only
(C) F and G only
(D) G and I only
(E) All are squares.

5. Which of these are parallelograms?



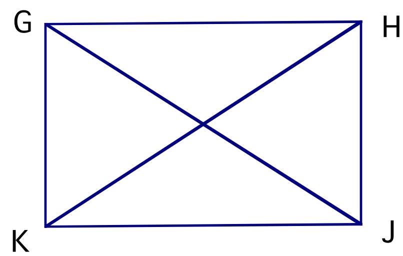
(A) J only
(B) L only
(C) J and M only
(D) None of these are parallelograms.
(E) All are parallelograms.

6. PQRS is a square.
Which relationship is true in all squares?



(A) $\overline{PR}$ and $\overline{RS}$ have the same length.
(B) $\overline{QS}$ and $\overline{PR}$ are perpendicular.
(C) $\overline{PS}$ and $\overline{QR}$ are perpendicular.
(D) $\overline{PS}$ and $\overline{QS}$ have the same length.
(E) Angle Q is larger than angle R.

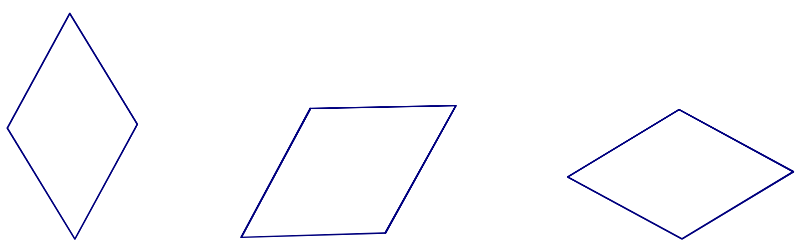
7. In a rectangle, GHJK, $\overline{GJ}$ and $\overline{HK}$ are the <u>diagonals.</u>



Which of (A) – (D) is <u>not</u> true in <u>every</u> rectangle?

(A) There are four right angles.
(B) There are four sides.
(C) The diagonals have the same length.
(D) The opposite sides have the same length.
(E) All of (A) – (D) are true in every rectangle.

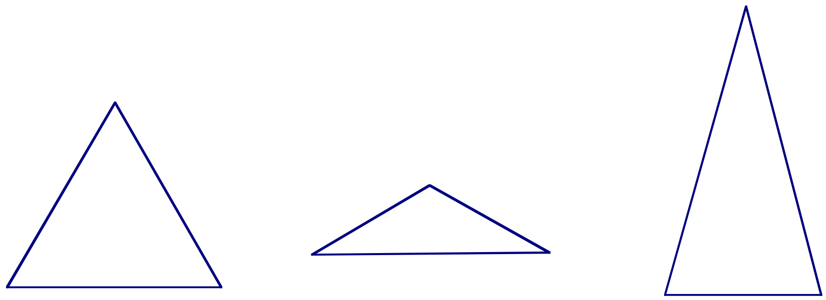
8. A <u>rhombus</u> is a 4-sided figure with all sides of the same length. Here are three examples.



Which of (A) – (D) is <u>not</u> true in every rhombus?

(A) The two diagonals have the same length.
(B) Each diagonal bisects two angles of the rhombus.
(C) The two diagonals are perpendicular.
(D) The opposite angles have the same measure.
(E) All of (A) – (D) are true in every rhombus.

9. An isosceles triangle is a triangle with two sides of equal length.
Here are three examples.



Which of (A) – (D) is true in every isosceles triangle?

(A) The three sides must have the same length.
(B) One side must have twice the length of another side.
(C) There must be at least two angles with the same measure.
(D) The three angles must have the same measure.
(E) None of (A) – (D) is true in every isosceles triangle.

10. Two circles with centers P and Q intersect at R and S to form a 4-sided figure PRQS.
Here are two examples.

Which of (A) – (D) is not always true?

(A) PRQS will have two pairs of sides of equal length.
(B) PRQS will have at least two angles of equal measure.
(C) The lines $\overline{PQ}$ and $\overline{RS}$ will be perpendicular.
(D) Angles P and Q will have the same measure.
(E) All of (A) – (D) are true.

11. Here are two statements.

Statement 1: Figure F is a rectangle.
Statement 2: Figure F is a triangle.

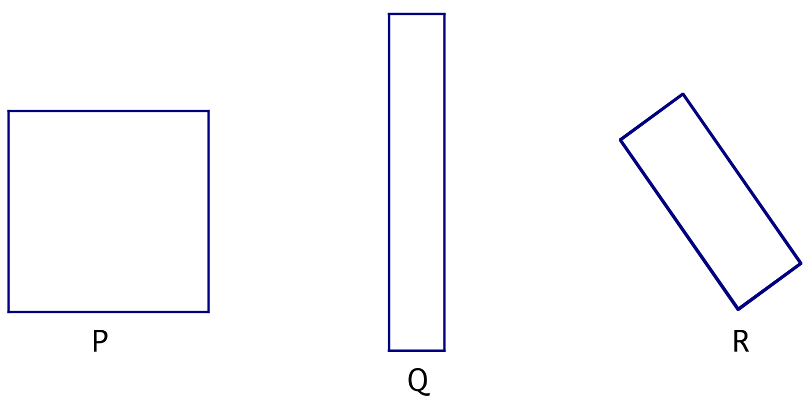
(A) If 1 is true, then 2 is true.
(B) If 1 is false, then 2 is true.
(C) 1 and 2 cannot both be true.
(D) 1 and 2 cannot both be false.
(E) None of (A) – (D) is correct.

12. Here are two statements.

Statement S: ΔABC has three sides of the same length.
Statement T: In ΔABC, $\angle B$ and $\angle C$ have the same measure.

Which is correct?
(A) Statements S and T cannot both be true.
(B) If S is true, then T is true.
(C) If T is true, then S is true.
(D) If S is false, then T is false.
(E) None of (A) – (D) is correct.

13. Which of these can be called rectangles?



(A) All can.
(B) Q only
(C) R only
(D) P and Q only
(E) Q and R only

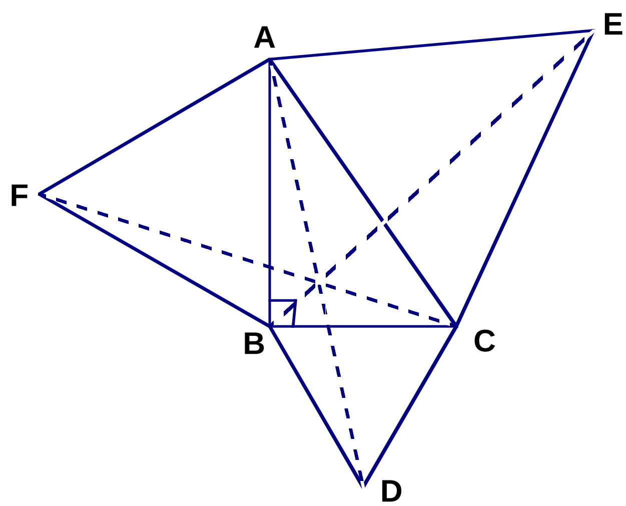
14. Which is true?

(A) All properties of rectangles are properties of all squares.
(B) All properties of squares are properties of all rectangles.
(C) All properties of rectangles are properties of all parallelograms.
(D) All properties of squares are properties of all parallelograms.
(E) None of (A) – (D) is true.

15. What do all rectangles have that some parallelograms do not have?

(A) Opposite sides equal
(B) Diagonals equal
(C) Opposite sides parallel
(D) Opposite angles equal
(E) None of (A) – (D)

16. Here is a right triangle ABC. Equilateral triangles ACE, ABF, and BCD have been constructed on the sides of ABC.



From this information, one can prove that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common. What would this proof tell you?

(A) Only this triangle drawn can we be sure that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common.
(B) In some but not all right triangles, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common.
(C) In any right triangle, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common.
(D) In any triangle, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common.
(E) In any equilateral triangle, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ have a point in common.

17. Here are three properties of a figure.

Property D: It has diagonals of equal length.
Property S: It is a square.
Property R: It is a rectangle.

Which is true?
(A) D implies S which implies R.
(B) D implies R which implies S.
(C) S implies R which implies D.
(D) R implies D which implies S.
(E) R implies S which implies D.

18. Here are two statements.

I: If a figure is a rectangle, then its diagonals bisect each other.
II: If the diagonals of a figure bisect each other, the figure is a rectangle.

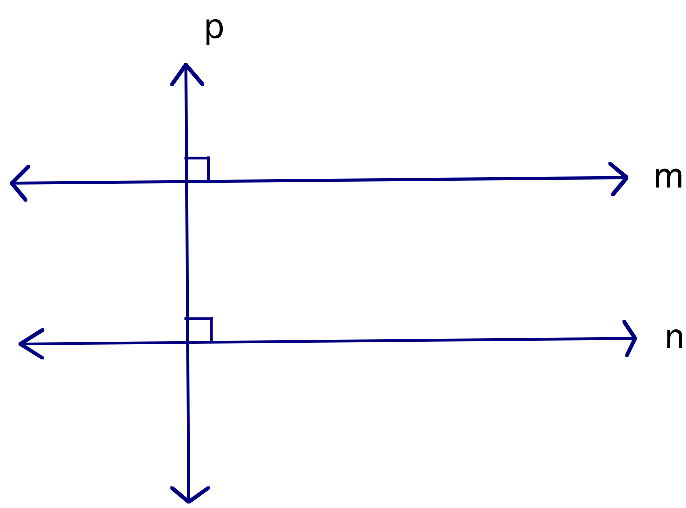
Which is correct?
(A) To prove I is true, it is enough to prove that II is true.
(B) To prove II is true, it is enough to prove that I is true.
(C) To prove II is true, it is enough to find one rectangle whose diagonal bisect each other.
(D) To prove II is false, it is enough to find one non-rectangle whose diagonals bisect each other.
(E) None of (A) – (D) is correct.

19. In geometry:

(A) Every term can be defined and every true statement can be proved true.
(B) Every term can be defined but it is necessary to assume that certain statements are true.
(C) Some terms must be left undefined but every true statement can be proved true.
(D) Some terms must be left undefined and it is necessary to have some statements, which are assumed true.
(E) None of (A) – (D) is correct.

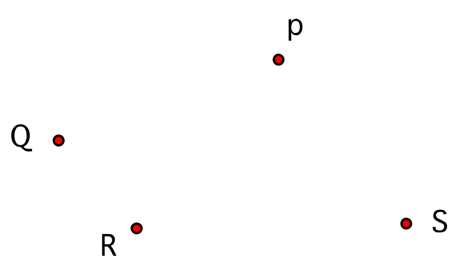
20. Examine these three sentences.
(1) Two lines perpendicular to the same line are parallel.
(2) A line that is perpendicular to one of two parallel lines is perpendicular to the other.
(3) If two lines are equidistant, then they are parallel.

In the figure below, it is given that lines m and p are perpendicular and lines n and p are perpendicular. Which of the above sentences could be the reason that line m and is parallel to line n?



(A) (1) only
(B) (2) only
(C) (3) only
(D) Either (1) or (2)
(E) Either (2) or (3)

21. In F-geometry, one that is different from the one you are used to, there are exactly four points and six lines. Every line contains exactly two points. If the points are P, Q, R and S, the lines are {P, Q}, {P, R}, {P, S}, {Q, R}, {Q, S}, and {R, S}



Here are how the words "intersect" and "parallel" are used in F-geometry.
The lines {P, Q} and {P, R} intersect at P because {P, Q} and {P, R} have P in common.
The lines {P, Q} and {R, S} are parallel because they have no points in common.

From this information, which is correct?
(A) {P, R} and {Q, S} intersect.
(B) {P, R} and {Q, S} are parallel.
(C) {Q, R} and {R, S} are parallel.
(D) {P, S} and {Q, R} intersect.
(E) None of (A) – (D) is correct.

22. To <u>trisect</u> an angle means to divide it into three parts of equal measure. In 1847, P.L. Wantzal proved that, in general, it is impossible to trisect angles using only a compass and an unmarked ruler. From his proof, what can you conclude?

    (A) In general, it is impossible to <u>bisect</u> angles using only a compass and unmarked ruler.
    (B) In general, it is impossible to trisect angles using only a compass and <u>marked</u> ruler.
    (C) In general, it is impossible to trisect angles using any drawing instruments.
    (D) It is still possible that in the future someone may find a general way to trisect angles using only a compass and an unmarked ruler.
    (E) No one will ever be able to find a general method for trisecting angles using only a compass and an unmarked ruler.

23. There is a geometry invented by a mathematician J in which the following is true:

    The sum of the measures of the angles of a triangle is less than 180º.

    Which is correct?

    (A) J made a mistake in measuring the angles of the triangle.
    (B) J made a mistake in logical reasoning.
    (C) J has a wrong idea of what is meant by "true."
    (D) J started with different assumptions than those in the usual geometry.
    (E) None of (A) – (D) is correct.

24. Two geometry books define the word rectangle in different ways.
    Which is true?

    (A) One of the books has an error.
    (B) One of the definitions wrong. There cannot be two different definitions for rectangle.
    (C) The rectangles in one of the books must have different properties from those in the other book.
    (D) The rectangles in one of the books must have the same properties at those in the other book.
    (E) The properties of rectangles in the two books might be different.

25. Suppose you have proved statements I and II.

I. If p, then q.
II. If s, then not q.

Which statement follows from statements I and II?

(A) If p, then s.
(B) If not p, then not q.
(C) If p or q, then s.
(D) If s, then not p.
(E) If not s, then p.

APPENDIX D

INTERVIEW TASKS

Appendix D: Interview Tasks

Task One

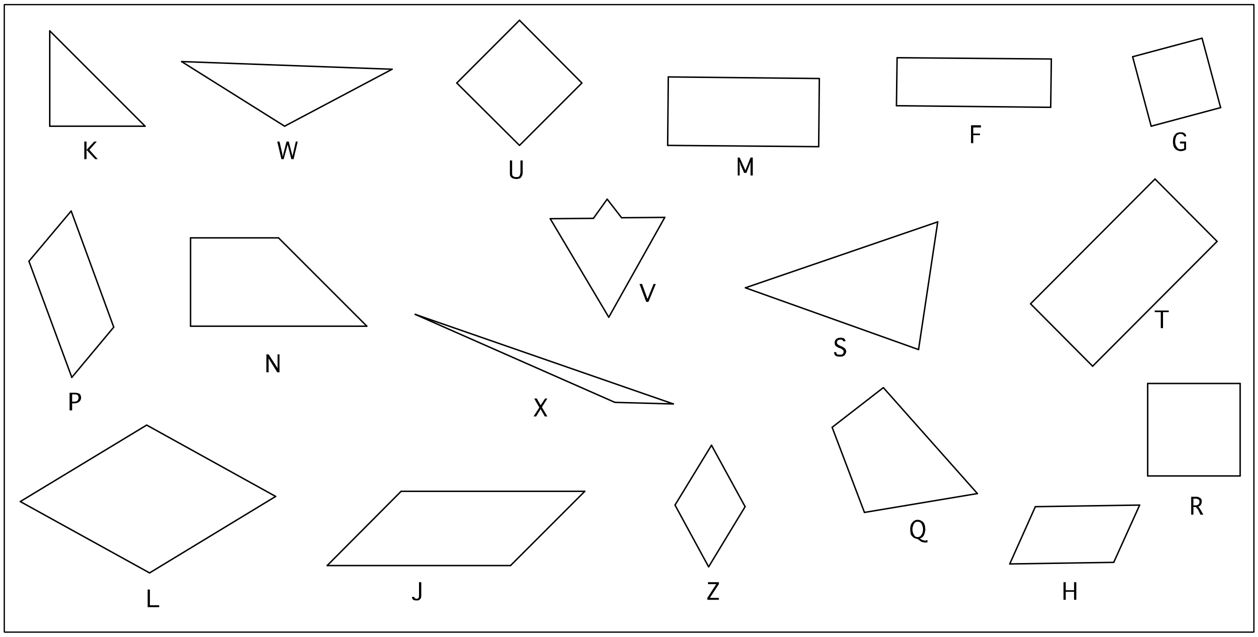


Figure Appendix D. 1. Task One: Sorting Geometric Figures

Task Two

A. Draw a parallelogram in the space below.

1. What can you say about the angles of this parallelogram?

2. What can you say about the sides of this parallelogram?

3. What can you say about the diagonals of this parallelogram?

B. In the space below, draw a new parallelogram that is different from the one you drew previously.

1. What can you say about the angles of this parallelogram?

2. What can you say about the sides of this parallelogram?

3. What can you say about the diagonals of this parallelogram?

Task Three

Two geometry books define the word *parallelogram* in different ways.

1: A quadrilateral is a *parallelogram* if and only if two pairs of opposite sides of the same length.

2: A quadrilateral is a *parallelogram* if and only if two pairs of opposite angles of the same measure.

Show me that these two definitions are equivalent. To verify that two definitions are equivalent, you must show that each set of defining conditions implies the other.

APPENDIX E

INTERVIEW PROTOCOLS

Appendix E: Interview Protocols

Before beginning the interview, provide the student with the following materials:

Pencils, ruler, protractor, blank sheets of paper

Turn on both video cameras.

Task One

<u>Present Task One and turn the page to face the student.</u>

1. Say: These are geometric shapes. Sort these shapes into groups. You can sort them any way you want. Write down your answers at the bottom of the task, and make notes about why you group them in such a way. Let me know when you are finished.

While the student is working on the task, check the positions of the cameras and see if they are recording appropriately. Monitor the student while she/he is working on the task, and make notes to prepare possible questions.

After the student has finished the task, **turn on the audiotape.**

2. Ask: Can you describe each group to me?

After the student has finished describing her/his results, ask one of the following:

If the student sorts the shapes as all rectangles together, all triangles together, all squares together, etc, then

- Ask: Can you find another way to sort these shapes into groups? Try it.
- Ask: Why?

If the student sorts the shapes as all triangles together, all quadrilaterals together, etc., then

- Ask: Can you sort these shapes into subgroups? Try it.
- Ask: Why?

If the student says that he/she doesn't know any other way to sort the shapes, then

- Ask: Can "this" (e.g., a rectangle, or a parallelogram) and "this" (e.g, a rhombus, or a trapezoid) go together?
- Ask: Why, or why not?

3. Ask: What is a parallelogram?

After the student has answered the questions verbally, then give the student a piece of blank paper, and Say: write it down. Do the same for the following questions.

4. Ask: What is a rectangle?
5. Ask: What is a square?
6. Ask: What is a rhombus?
7. Ask: What is a trapezoid?
8. Ask: What is an isosceles triangle?

Turn off the cameras and audio recorder. Remind the student to write the date and his/her name on all the worksheets.

Say: I will collect all your worksheets.

Put all Task One materials away, give the student three minutes break and get ready for Task Two.

Task Two

Turn on both video cameras and audio recorder.

<u>Present Task Two – "A. Draw a parallelogram …" and turn the page to face the student</u>

Say: Draw a parallelogram in this empty space here.

Once the student has finished drawing, then

1. Ask: What can say about the angles of this parallelogram?

- If the student says, "the opposite angles are equal", or "all the vertex angles add up to 360°, or "the adjacent angles add up to 180°", then
    - Say: Write down your answer(s), and convince me.

After the student has finished explaining his/her conclusion, then

Ask: Is there any other relationship among the angles of this parallelogram?

- If the student says, "all the vertex angles add up to 360°", then
    - Say: Write down your answer(s), and convince me.
- If the student says, "no, that's all", then

2. Ask: What can you say about the sides of this parallelogram?

- If the student says, " Opposite sides are equal", or "opposite sides are parallel", then
    - Say: Write down your answer(s) and convince me.

After the student has finished explaining his/her conclusion, then

Ask: Is there any other relationship involving the sides of this parallelogram?

<u>Present Task Two – "B. Draw a new parallelogram …" and turn the page face to the student</u>

Say: In the empty space here, draw a new parallelogram that is different from the one you drew previously.

Once the student finished drawing, then

1. Ask: Why is this a different parallelogram from the first one you drew?
2. Ask: What can you say about the angles of this parallelogram?

- If the student draws another parallelogram, then his/her answer to this question might be identical to Task Two A. No need to repeat the process as in Task Two A.
- If the student draws a rectangle, or a square, or a rhombus, and provides the same answer as he/she did in Task Two A., then
    - Say: Convince me.

3. Ask: What can you say about the sides of this parallelogram?

- If the student draws another parallelogram, then his/her answer to this question might be identical to Task Two A. If so, then ask question 4, "what can you say about the diagonals of this parallelogram?"
- If the student draws a rectangle, or a square, or a rhombus, and provides the same answer as he/she did in Task Two A., then

  Say: Convince me.

4. What can you say about the diagonals of this parallelogram?

- If the student draws a parallelogram, after she/he has finished describing the diagonals of the parallelogram,
  - Ask: Why?

  (Present a drawing of a rectangle), and then
  - Ask: What can you say about the diagonals of this one?
  - Ask: Why?

  (Present a drawing of a square), and then
  - Ask: What can you say about the diagonals of this one?
  - Ask: Why?

  (Present a drawing of a rhombus), and then
  - Ask: What can you say about the diagonals of this one?
  - Ask: Why?
- If the student draws a rectangle as a new parallelogram, after she/he has finished describing the diagonals of the rectangle,
  - Ask: Why?

  (Present a drawing of a square), and then
  - Ask: What can you say about the diagonals of this one?
  - Ask: Why?

  (Present a drawing of a rhombus), and then
  - Ask: What can you say about the diagonals of this one?
  - Ask: Why?
- If the student draws a square as a new parallelogram, after he/she has finished describing the diagonals of the square,
  - Ask: Why?

  (Present a drawing of a rectangle), and then

- Ask: What can you say about the diagonals of this one?
- Ask: Why?

(Present a drawing of a rhombus), and then

- Ask: What can you say about the diagonals of this one?
- Ask: Why?

- If the student draws a rhombus as a new parallelogram, after he/she has finished describing the diagonals of the rhombus,
  - Ask: Why?

  (Present a drawing of a square), and then

  - Ask: What can you say about the diagonals of this one?
  - Ask: Why?

  (Present a drawing of a rectangle), and then

  - Ask: What can you say about the diagonals of this one?
  - Ask: Why?

5. Is it true that in every parallelogram the diagonals have the same midpoint (bisect each other)?

- Ask: Why? Or Why not?

After the student has finished describing his/her conclusion, then

- Say: write it down

Turn off the cameras and audio recorder. Remind the pair to write the date and their names on all the worksheets.

Say: I will collect all your worksheets.

Put all Task Two materials away, give the pairs three minutes break and get ready for Task Three.

Task Three

Turn on both video cameras and audio recorder.

Present Task Three and turn the page to face the student

Say: Read the task carefully, and show your work. Let me know if you have any questions.

If the student shows difficulty understanding the task, and doesn't know what to do, then

Say: To show that the two definitions are equivalent, you need to show:

1. If in a quadrilateral, two pairs of opposite sides of the same length, then two pairs of opposite angles of the same measure.

And,

2. If in a quadrilateral, two pairs of opposite angles of the same measure, then two pairs of opposite sides of the same length.

When the interview is finished, turn off both cameras and audio recorder.

Say: "Thank you" to the student, and let him/her know that you will share the results with them if he/she is interested.

APPENDIX F

THE DEVELOPMENT OF GEOMETRIC DISCOURSES AT EACH VAN HIELE LEVEL

Appendix F: Development of Geometric Discourse at Each van Hiele Level

Table 4.41 Development of Geometric Discourse at Each van Hiele Level

Level of dis-course	Geometric Objects (figures)				Routines				
	Signifier - Word Use	Saming criterion	Realiz-ations	System of objects (figures)	Identifying Routines: Attributed	Identifying Routines: Examples of Declared narratives ("How do you know it's X?")	Examples of Justifying Identification ("Why is this X?")	Defining Routines: How ("What is X?")	Defining Routines: When
1	proper name (passive use)	according to family appearance s	primary d-objects,	unstructured collection of concrete d-objects	visual recognition, self-evident	"It looks like…"	"Because it is"	--	--
2	common name (routine driven and/or phrase driven)	according to visual properties (with no order)	primary d-objects,	unstructured collection of disjoint categories of concrete d-objects	Step 1. visual recognition Step 2. *Substantiation*. Identifying by partial properties check (e.g. counting, measuring, comparing, etc);	"It looks like they are *parallel*…" "I measured (sides & angles)…"	"Because I can see it" "Because I measured... [some visual properties, no superordinate]	describe *the* figure by visual properties, or by recalling (no superor-dinate);	serves as necessary condition for the use of word

Table 4.41 (Cont'd)									
3	common name (object driven)	1. common descriptive narrative on the *name* of the figures 2*. common descriptive narrative on the *properties* of figures (e.g., equal, bisector, etc)	concreted-object (Objectification occurred) *two levels of realization tree: definitions and properties of geometric figures, and the relations about how one implies the other.	May or may not have hierarchy of classifications	Step 1. visual recognition Step 2. *Substantiation* Identifying by definitions (check of defining conditions by counting, measuring, comparing, etc.) Step 3. *Construction* of new narratives (informal proving equality, congruency, etc.)	"All squares are rectangles" "All these figures (e.g., squares, rectangle, rhombus) are parallelograms. [critical conditions are fulfilled]	"If it is a square, then it has to be a rectangle, it fulfills the definition." "Because they all have two pairs of parallel sides."	describe *a* figure or a mathematical term by definition	serves as a necessary & sufficient condition for the use of word

Table 4.41 (Cont'd)

4	1. common name (object driven)	Common descriptive narrative on the *name* of figures	abstract d-object	hierarchy of classifications	*Construction* of new narratives (formal proving using definitions, axioms, theorems, etc)	"they are alternating interior angles, and they are equal."	"If two parallel lines are cut by a transversal, their alternating interior angles are equal."	describe a figure or a mathematical term by definition.	serves as a necessary & sufficient condition for the use of word
	2. common relations among definitions, axioms, theorems, etc.	Common descriptive narrative on the *properties* of figures (e.g., equal, bisector, etc)	* two-level realization tree: relations among definitions, properties, axioms, theorems about geometric figures.						serves as a necessity for the use of available axioms, theorem, etc.

BIBLIOGRAPHY

BIBLIOGRAPHY

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal, 93*(4), 373-397.

Battista, M.T. (2007). The development of geometric and spatial thinking. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 2, pp. 483-908). Charlotte, NC: Information Age Publishing.

Battista, M.T. (2009). Highlights of research on learning school geometry. In T.V. Craine & R. Rubenstein (Eds.), *Understanding geometry for a changing world* (pp.91-108). Reston, VA: National Council of Teachers of Mathematics.

Battista, M.T., & Clements, D. H. (1995). Geometry and proof. *The Mathematics Teacher*, 88(1), 48-54.

Burger, W. F., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education, 17*(1), 31-48.

Clements, D.H. (1991). Elaborations on the levels of geometric thinking. Paper presented at the III International Symposium for Research in Mathematics Education, Valencia, Spain.

Clements, D. H. (2003). Teaching and learning geometry. In J. Kilpatrick (Ed.), *A research companion to principles and standards for school mathematics* (pp. 151-178). Reston, VA: National Council of Teachers of Mathematics.

Clements, D. H., & Battista, M. T. Geometry and spatial reasoning. In A.G. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 420-464). Reston, VA: National Council of Teachers of Mathematics.

Clements, D. H., Swaminathan, S., Hannibal, M. A. Z., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education, 30*, 192-212.

Crowley, M. L. (1987). The van Hiele model of the development of geometric thought. In M. Lindquist (Ed.), *Learning and teaching geometry, K-12* (pp. 1-16). Reston, VA: National Council of Teachers of Mathematics.

Crowley, M. L. (1990). Criterion referenced reliability indices associated with the van Hiele geometry test. *Journal for Research in Mathematics Education, 21*, 238-241.

Darken, B. (2007, May/June). Educating future elementary school and middle school teachers. *Focus*, 20-21.

De Villires, M. (1999). The Future of Secondary School Geoemtry. http://www.lettredlapreuve.it/Resume/deVilliers/deVilliers98/deVilliers982.html

Floden, R. E. (2002). The measurement of opportunity to learn. In Board on International Comparative Studies in Education, A. C. Porter & A. Gamoran (Eds.), *Methodological advances in cross-national surveys of educational achievement*. Washington, DC: National Academy Press.

Fuys, D., Geddes, D., & Tischler, R (1988). *The van Hiele model of thinking in geometry among adolescents.* (Journal for Research in Mathematics Education Monograph No. 3). Reston, VA: National Council of Teachers of Mathematics.

Gutierrez, A., Jaime, A., & Fortuny, J. M. (1991). An alternative paradigm to evaluate the acquisition of the van Hiele levels. *Journal for Research in Mathematics Education, 22*, 237-251.

Hoffer, A. (1981). Geometry is more than proof. *Mathematics Teacher,74,* 11-18.

Hoffer, A. (1983). Van Hiele—based research. In R.Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp.205-227). New York, NY: Academic Press.

Kepner, H. S. (2009, March). Don’t forget geometry. *NCTM New Bulletin.*

Kerslake, D. (1991). The language of fractions. In K.Durkin & B. Shire (Ed.). Language in mathemtical education: Research and practice. Chapter 8. Open UNivesity Press: Bristol, PA. 85-94.

Lampert, M. (1990). When the problem is not the question and the answer is not the solution: Mathematical knowing and teaching. *American Educational Research Journal* 27 (1), 29-63.

Lampert, M. (1998) Introduction. In *Talking mathematics in school*. M. Lampert & M.L. Blunk (Ed.). Cambridge University Press.

Mayberry, J. (1983). The van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education, 14*(1), 58-69

McClintock, E., Jiang, Z. & July, R.(2002). *Students’ development of Three-dimensional Visualization in the Geometer’s Sketchpad Environment.* PME geometry.

Morgan, C. (1996). ‘The language of mathematics’: Towards a critical analysis of mathemtics texts. *For the learning of Mathematics, 16 (3),* 2-10.

Morgan, C. (1998). Writing Mathematically: The Discourse of Investigation. London: Falmer Press.

Morgan, C. (2005). Words, definitions and concepts in discourses of mathematics, teaching and learning. *Language and Education. 19,103-117.*

Moschkovich, J. (2002). An intrioduction to examining everyday and academic mathematical practices. In M. Brenner & J. Moschkovvich (Eds), *Everyday and academic mathematics in the classroom*. JRME Monograph Number 11. (pp.1-11). Reston, VA: NCTM

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (2003). *A research companion to principles and standards for school mathematics.* Reston, VA: National Council of teachers of Mathematics.

Newton, J. (in press). K-8 geometry state standards: A look through the lens of van Hiele's levels of geometric thinking. In J. P. Smith & J. E. Tarr (Eds.), The intended mathematics curriculum as represented in state-level curriculum standards: Consensus or confusion? (Vol. 2). Charlotte, NC: Information Age Publishing.

Pimm, D. (1987). Speaking mathematically: Communication in mathematics classrooms. London: Routledge & Kegan Paul.

Rowland, T. (1995). Between the lines: The languages of mathematics. In J. Anghileri (Ed.), Children's Mathematical Thinking in the Primary Years (pp. 54-73). London: Cassell.

Schleppegrell, M.J. (2007). Language in mathematics teaching and learning: A research review. *Prepared for the Spencer Foundation*.

Senk, S. L. (1983). Proof-writing achievement and van Hiele levels among secondary school geometry students (Doctoral dissertation, The University of Chicago). *Dissertation Abstracts International, 44*, 417A.

Senk, S. L. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education, 20*(3), 309-321.

Sfard, A. (2000). On reform movement and the limits of mathematical discourse, *Mathematical Thinking and Learning, 2*(3). 157-189.

Sfard, A. (2002). Learning mathematics as developing a discourse. In R. Speiser, C. Maher, &C. Walter (Eds.), *Proceedings of 21st Conference of PME-NA*. Columbus, Ohio: Clearing House for Science, Mathematics and Environmental Education. 23-44.

Sfard, A. (2005). Why cannot children see as the same what grwn-ups cannot see as different/ Early numerical thinking revisited. *Cognition and Instruction, 23*(2), 237-309.

Sfard, A. (2007). When rules of discourse change, but nobody tells you: making sense of mathematics learning from a commogntive standpoint. *The Journal of the Learning Science*. 16(40) 567-615.

Sfard, A. (2008). *Thinking as Communicating: human development, the growth of discourses, and mathematizing*: Cambridge.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry* (Final report of the Cognitive Development and Achievement in Secondary School Geometry Project: ERIC Document Reproduction Service No. ED 220 288). Chicago: University of Chicago.

Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Orlando, FL: Academic Press.

Van Hiele, P. M. (1959/1985). The child's thought and geometry. In D. Fuys, D. Geddes & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. Van Hiele* (pp. 243-252). Brooklyn, NY: Brooklyn College, School of Education.

Van Hiele, P. M. (1999). Developing geometric thinking through activities that begin with play. Teaching Children Mathematics, 5(6), 310-315.

Wilson, M. (1990). Measuring a van Hiele geometry sequence: A reanalysis. *Journal for Research in Mathematics Education, 21,* 230-237.

Wilson, S. M., Floden, R. F., & Furrini-Mundy, J. (2001, February). *Teacher preparation research: Current knowledge, gaps, and recommendations.* Center for the Study of Teaching Policy, University of Washington, Seattle, WA.

Wirszup, I. (1976). Breakthroughs in the psychology of learning and teaching geometry. In J. L. Martin (Ed.), *Space and geometry: Papers from a research workshop* (pp. 75-97). Columbus, OH: ERIC/SMEAC.