THEORY OF QUASI-STATIC FLOW OF CHARGE ON SLIGHTLY CONDUCTING BODIES

Thesis for the Degree of M. S.

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Tin Oo Hlaing

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AN ABSTRACT

Submitted to the College of Science and Arts
Michigan State University of Agriculture and
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William G. Hammerle

ABSTRACT

This paper describes the motion of electric charge when placed on the surfaces of very slightly conducting bodies. It is assumed that no electric charge is initially present inside any conductor which is under consideration. For this reason it follows, from a wellknown case, that there will never be any charge observed inside the conductor at any later time. A general method is developed for finding the charge density at any time on the surface of a conducting body in terms of the initial charge density on its surface. This general method is used to find the charge density at any time for the following cases involving specific geometries and initial charge distributions:

(1) A slightly conducting cylinder with arbitrary initial two-dimensional surface charge density.

(2) Two slightly conducting coaxial cylinders with arbitrary initial two-dimensional surface charge density.

(3) A single cylinder conducting on its surface only, with arbitrary initial two-dimensional surface charge density.

(4) A similar single cylinder with a line charge on its surface initially.

(5) A slightly conducting sphere with arbitrary initial two-dimensional surface charge density.

(6) A slightly conducting sphere with a point charge on its surface initially.

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I INTRODUCTION

If a substance is rubbed against another substance, it generally becomes either positively or negatively charged. Among others, Hersh, Sharman and Montgomery[1] have measured the amount of charge produced when a filament of a given material is rubbed against another of the same or different material under controlled mechanical and ambient conditions. They used metals as well as non-metals.

Thus, electric charge may be generated on the surface of very slightly conducting non-metallic substance. Once the charge is generated, it is interesting to know how it will move from its initial position into the final equilibrium distribution which will be reached after a very long time. It is discussed in this paper how the electric charge initially placed on the surfaces of slightly conducting bodies will behave as time progresses.

In Section II, it is shown that there will never be any charge inside a conductor, if no charge is initially present inside it. Although the charge may flow through the interior of the body from one point of the surface to another, measurable amounts of charge are found only on the surface. In the same section, the general method for finding the surface charge density at any time is developed, provided that the initial surface charge density is known.

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1. Hersh, S.P., Sharman, E.P. and Montgomery, D.J., "Textile Research Journal," 24, 426 (1954)

In Section III, an arbitrary two-dimensional distribution of charge is placed on a slightly conducting cylinder. The expression for the surface charge density at any time is calculated, and is expressed in closed form.

In Section IV, an arbitrary two-dimensional distribution of charge is placed on the outer surface of two conducting coaxial cylinders. Two general expressions for the surface charge density at a later time are calculated, one for the surface of the outer cylinder, and the other for that of the inner one.

In Section V A., an arbitrary two-dimensional distribution of charge is placed on the surface of a cylinder which is conducting only within a very thin layer on the surface. This may be considered as a special case of the one discussed in Section IV. The expressions for the surface charge density at any time are calculated. In Section V B., the same cylinder is taken but the initial charge placed on the surface is a line charge. The expression for total surface charge density at any time is calculated and expressed in closed form.

In Section VI A., an arbitrary two-dimensional distribution of charge is placed on the surface of a slightly conducting sphere. An infinite series for the surface charge density is obtained. In Section VI B., the same sphere is taken but the initial charge placed on its surface is a point charge. The expression for the surface charge density at a later time is presented in a form suitable for numerical computations.

II GENERAL THEORY

Electrodynamic problems can be solved using Maxwell's equations. They are a set of differential equations which are written in the rationalized m.k.s. system of units as follows[2]:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{2.1}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \tag{2.2}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{2.3}$$

$$\vec{\nabla}\cdot\vec{D} = \rho. \tag{2.4}$$

If the medium is isotropic and homogeneous,

$$\vec{D} = K\vec{E} \tag{2.5}$$

$$\vec{B} = \mu\vec{H} \tag{2.6}$$

$$\vec{J} = \sigma\vec{E}. \tag{2.7}$$

From the conservation of charge, the relationship between $\vec{J}$ and ρ is[3]

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \tag{2.8}$$

where $\vec{E}$ = electric field intensity; $\vec{B}$ = magnetic induction; $\vec{H}$ = magnetic field intensity; $\vec{J}$ = current density; $\vec{D}$ = electric displacement; ρ volume charge density; K = dielectric constant; μ = magnetic inductive capacity; σ = electric conductivity.

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2. Pipes, Louis A.: "Applied Mathematics for Engineers and Physicists," McGraw-Hill Book Company, Inc., New York, 1946, p.364.
3. Stratton, Julius A.: "Electromagnetic Theory," McGraw-Hill Book Company, Inc., New York, 1941, p.5.

In all the cases considered in this paper, known electric charge is initially placed on various conducting bodies. If σ , the electric conductivity, is very small, the charge flows very slowly from place to place. We may, at any instant, consider that the terms involving partial derivatives with respect to time vanish in the equations, and also say that $\vec{J}$, the current density, is very small. Then, as a zeroth order approximation,

$$\vec{J} \approx 0 \tag{2.9}$$

$$\frac{\partial \rho}{\partial t} \approx 0 \tag{2.10}$$

and $\vec{D}$ and $\vec{E}$ are approximately independent of time.

By using (2.1) and considering the fact that $\vec{E}$ is independent of time, the following relations are obtained:

$$\vec{B} \approx 0 \tag{2.11}$$

$$\vec{H} \approx 0 \tag{2.12}$$

thus the magnetic field is negligible, and

$$\vec{\nabla} \times \vec{E} = 0 . \tag{2.13}$$

Thus an electric potential V exists such that

$$\vec{E} = -\vec{\nabla} V \tag{2.14}$$

where

$$\nabla^2 V = -\frac{\rho}{K} . \tag{2.15}$$

To this approximation, we have a purely electrostatic problem, that is, the electric potential and field at any instant are the same as would exist were the charge distribution at that same instant not changing with time.

As the next order of approximation, this electric potential and field may be put back into Ohm's law (2.7) to find a non-zero current density:

$$\vec{J} = \sigma\vec{E} = -\sigma\vec{\nabla}V. \qquad (2.16)$$

Then $\frac{\partial\rho}{\partial t}$, the rate of change of charge density with time, is obtained from the expression for the conservation of charge (2.8):

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J} = \nabla^2 V\sigma. \qquad (2.17)$$

But by (2.15), this gives

$$\frac{\partial\rho}{\partial t} = -\frac{\sigma\rho}{K}. \qquad (2.18)$$

The above differential equation (2.18) can be solved immediately to give

$$\underline{\rho = \rho(0)\,e^{-\frac{t}{\tau}}} \qquad (2.19)$$

where $\tau = \frac{K}{\sigma}$ and is known as the time of relaxation for the conducting material and $\rho(0)$ is the value of ρ at $t=0$, that is, it is the original charge density at the same point in space.

Two cases can be considered[4]:

(a) If $\rho(0) \neq 0$, the original charge at every point of the conducting material decays exponentially. It is clearly seen that the time of relaxation is independent of the size and the shape of the conductor. For example, the conductor may be either spherical or cylindrical in shape and the radius may be either large or small.

(b) If $\rho(0) = 0$, then $\rho = 0$, that is, if no charge is present inside a conductor initially, there will never be any charge in the interior at a later time.

4. See reference 3, p.15.

In all the problems that are considered in this paper, the original charge distribution is confined to the surfaces of the conducting objects, and thus only case (b) is applicable.

The charge density on the surface of the conducting material is defined as charge per unit area and is denoted by $\omega(t)$ at any time $t \neq 0$, and $\omega(0)$ at $t=0$. If $\omega(t)$ is known, the potential V can be obtained by solving Laplace's equation subject to the proper conditions at the boundary between the various dielectrics.

Laplace's equation is written as

$$\nabla^2 V = 0 \tag{2.20}$$

At the boundary, the boundary conditions are[5]

$$V_1 = V_2 \tag{2.21}$$

and

$$(D_n)_1 - (D_n)_2 = \omega(t) \tag{2.22}$$

where V_1 and V_2 are the potentials in the two mediums, $(D_n)_1$ and $(D_n)_2$ are the normal components of the electric displacements in the two mediums.

When V is known, $\vec{J}$ can be obtained by using (2.14) and (2.7).

By charge conservation, the relationship between $\vec{J}$ and ω is such that[6]

$$\vec{J}_1 \cdot \vec{n} - \vec{J}_2 \cdot \vec{n} = \frac{\partial \omega(t)}{\partial t} \tag{2.23}$$

where $\vec{n}$ is the normal, $\vec{J}_1$ and $\vec{J}_2$ are the current densities in the two mediums evaluated at the surface (see Fig. 1).

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5. See reference 3, p. 164.
6. See reference 3, p. 483.

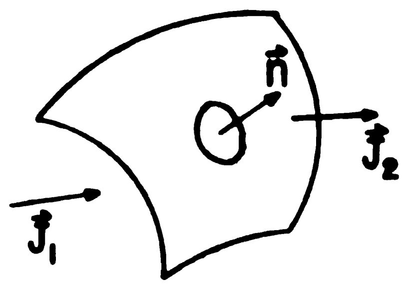

Fig. 1.

If $\omega(t)$ is known, equation (2.20) can be solved subject to the boundary conditions (2.21) and (2.22), to give V in terms of $\omega(t)$. But V in turn gives $\vec{J}$ by equations (2.14) and (2.7). Equation (2.23) then gives $\frac{\partial \omega}{\partial t}$, the time rate of change of charge density, in terms of $\vec{J}$, and thus in terms of $\omega(t)$. We may conclude, therefore, that the above set of equations is equivalent to a first order partial differential equation for the charge distribution and can be solved for $\omega(t)$ if $\omega(0)$ is known. Probably this solution for $\omega(t)$ is satisfactory if the relaxation time τ in (2.19) is much larger than the time required for light to cross the conducting objects.

It is not possible to solve these equations in general. Instead, they will be solved for $\omega(t)$ in a number of examples involving specific geometries and initial charge distributions.

III TWO-DIMENSIONAL CHARGE DISTRIBUTION ON A CONDUCTING CYLINDER

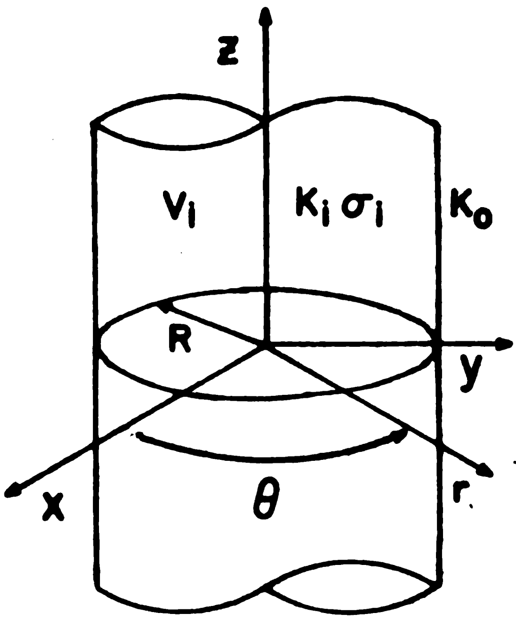

Fig.2.

In this section, the investigation is done on an infinitely long dielectric cylinder (Fig.2) whose radius is R. Throughout the cylinder, the dielectric constant is K_i and the conductivity is σ_i, where σ_i is very small. Outside the cylinder, the dielectric constant is K_0 and the conductivity is zero. The radial distance from the axis of the cylinder to any point is denoted by r. θ is the angle between r and the X axis. The surface charge density ω is considered to be arbitrary at time t = 0, except that it is independent of Z and is a function of θ only. We will, furthermore, restrict ω to be symmetric about the X axis, so that

$$\omega(\theta) = \omega(-\theta). \tag{3.1}$$

To find the surface charge density $\omega(t)$ at a later time $t \neq 0$, let us expand it in the form

$$\omega(t) = \sum_{n=0}^{\infty} \beta_n(t) \operatorname{Cos} n\theta \tag{3.2}$$

where $\beta_n(t)$ is given by

$$\beta_n(t) = \begin{cases} \frac{2}{\pi}\int_0^{\pi} \omega(t) \operatorname{Cos} n\theta \, d\theta & n \geq 1 \\ \frac{1}{\pi}\int_0^{\pi} \omega(t) \, d\theta & n = 0 \end{cases} \tag{3.3}$$

The potentials about the surface - either outside or inside of the cylinder - satisfy Laplace's equation (2.20). Therefore V_i, the potential inside the cylinder, is obtained by solving the Laplace's equation, using the boundary conditions (2.21) and (2.22). It can be written as[7]

$$V_i = \sum_{n=1}^{\infty} \frac{r^n \cos n\theta \ \beta_n(t)}{n R^{n-1}(K_i + K_o)} + C_o \tag{3.4}$$

where C_o is an arbitrary constant.

At $r = R$, by using (2.23), (2.14) and (2.7) from the previous chapter, we have

$$\frac{\partial \omega(t)}{\partial t} = -\sigma_i \left(\frac{\partial V_i}{\partial r}\right)_{r=R} \tag{3.5}$$

$$= -\sigma_i \sum_{n=1}^{\infty} \frac{\beta_n(t) \cos n\theta}{K_i + K_o} \tag{3.6}$$

On differentiating both sides of (3.2) partially with respect to t, we have

$$\frac{\partial \omega(t)}{\partial t} = \sum_{n=0}^{\infty} \frac{d}{dt}\beta_n(t) \cos n\theta \tag{3.7}$$

For (3.7) to be equal to (3.6), it is necessary that

$$\frac{d\beta_n(t)}{dt} = \begin{cases} \dfrac{-\sigma_i \beta_n(t)}{K_i + K_o} & n \geq 1 \\ 0 & n = 0 \end{cases} \tag{3.8}$$

On solving (3.8) for $\beta_n(t)$, we have

$$\beta_n(t) = \begin{cases} \beta_n(0) e^{-t/\tau} & n \geq 1 \\ \beta_o(0) & n = 1 \end{cases} \tag{3.9}$$

7. See Appendix A.

where $\tau = (K_i + K_o)/\sigma_i$. (3.10)

After substituting the value of $\beta_n(t)$ from (3.9) in (3.2), the magnitude of charge per area of the surface at any time is

$$\omega(t) = \sum_{n=1}^{\infty} \beta_n(0) \cos n\theta \, e^{-\frac{t}{\tau}} + \beta_0(0) . \qquad (3.11)$$

But the sum can be evaluated in closed form because the exponential is the same for each term in the sum, and

$$\sum_{n=1}^{\infty} \beta_n(0) \cos n\theta = \omega(0) - \beta_0(0) . \qquad (3.12)$$

Thus (3.11) can now be written as

$$\underline{\omega(t) = \omega(0) e^{-t/\tau} + \beta_0(0)[1 - e^{-t/\tau}]} , \qquad (3.13)$$

where $\omega(t)$ is the surface charge density at any time t and $\omega(0)$ is the arbitrary charge density on the surface at t=0. Equation (3.13) can be interpreted in the following way: At t = 0, $\omega(t) = \omega(0)$, because $e^{-t/\tau}$ is unity. As time progresses, the initial distribution $\omega(0)$ decreases to zero exponentially, with a time constant or time of relaxation τ given by (3.10). In its place arises a charge distribution $\beta_0(0)[1 - e^{-t/\tau}]$ which is independent of θ and everywhere the same on the cylinder. After an infinite time, the charge distribution is everywhere given by $\beta_0(0)$, the average value of the charge density initially placed on the surface. From (3.3), it can be expressed as

$$\beta_0(0) = [\omega(0)]_{av} = \frac{1}{2\pi}\int_0^{2\pi} \omega(0) \, d\theta \qquad (3.14)$$

This result (3.13), incidently, is correct whether or not the symmetry condition (3.1) - viz $\omega(\theta) = \omega(-\theta)$ - is satisfied. The relaxation time τ is given by $(K_i + K_o)/\sigma_i$. It is independent of R, the radius of the cylinder, and depends only on K_i, K_o and σ_i . Thus, the relaxation time is the same for all cylinders, large or small, made from the same material.

IV TWO-DIMENSIONAL CHARGE DISTRIBUTION ON TWO CONDUCTING COAXIAL CYLINDERS

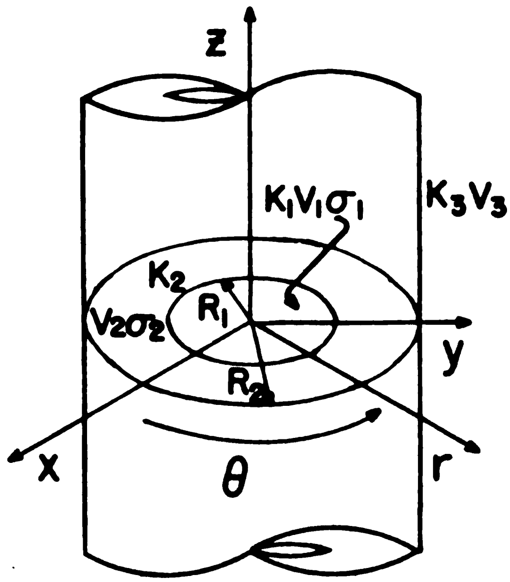

Fig.3.

In this section, the investigation is done on two infinitely long conducting coaxial cylinders (Fig. 3). The radius of the inner cylinder is R_1, and that of the outer cylinder is R_2. The conductivities are σ_1 for the inner cylinder, σ_2 for the outer cylinder, and zero outside the cylinders. The meanings of θ and r are same as in Section III.

The problem in this section is to find the surface charge density at a later time, if the charge density at t=0 is given. Initially, the charge distributions on both the surfaces are arbitrary except that they will be assumed independent of Z and a function of θ only. As before, let us expand the charge density at t=0 on the surface of the inner cylinder in the form

$$\omega'(0) = \sum_{n=0}^{\infty} \beta_n'(0) \cos n\theta \tag{4.1}$$

where $\beta_n'(0)$ is expressed as

$$\beta_n'(0) = \begin{cases} \frac{2}{\pi}\int_0^{\pi} \omega'(0) \cos n\theta \, d\theta & n \geq 1 \\ \frac{1}{\pi}\int_0^{\pi} \omega'(0) \, d\theta & n = 0 \end{cases} \tag{4.2}$$

The charge density at t=0 on the surface of the outer cylinder can be expanded in the form

$$\omega''(0) = \sum_{n=0}^{\infty} \beta_n''(0) \cos n\theta \tag{4.3}$$

where

$$\beta_n''(0) = \begin{cases} 2/\pi \int_0^{\pi} \omega''(0) \cos n\theta \, d\theta & n \geq 1 \\ 1/\pi \int_0^{\pi} \omega''(0) \, d\theta & n = 0 \end{cases} \tag{4.4}$$

At time $t \neq 0$, the charge densities can be expressed as similar series:

$$\omega'(t) = \sum_{n=0}^{\infty} \beta_n'(t) \cos n\theta \tag{4.5}$$

$$\omega''(t) = \sum_{n=0}^{\infty} \beta_n''(t) \cos n\theta \tag{4.6}$$

The only case considered here is with $\omega'(0)$ identically zero and therefore from (4.2), $\beta_n'(0) = 0$ for all n. It is impractical to put electric charge on the surface of the inner cylinder at time t=0 if the cylinders are solid.

A general solution of the Laplace's equation in cylindrical co-ordinates is[8]

$$V = a_0 \ln r + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) + \sum_{n=1}^{\infty} \frac{1}{r^n} (q_n \cos n\theta + f_n \sin n\theta) + C_0 \tag{4.7}$$

where V is the potential.

$$V = V(r, \theta) \tag{4.8}$$

because it is independent of Z.

Let us denote the potentials as V_1 for the inner cylinder, V_2 for the outer cylinder, and V_3 for the medium outside the cylinders.

8. See reference 2, p.407.

They are of the form

$$V_1 = \sum_{n=1}^{\infty} A_n r^n \cos n\theta + C_0 \tag{4.9}$$

$$V_2 = \sum_{n=1}^{\infty} B_n r^n \cos n\theta + \sum_{n=1}^{\infty} D_n r^{-n} \cos n\theta + E_0 + F_0 \ln r \tag{4.10}$$

$$V_3 = \sum_{n=1}^{\infty} G_n r^{-n} \cos n\theta + H_0 \ln r \tag{4.11}$$

where A_n, B_n, C_0, D_n, E_0, F_0, G_n and H_0 are constants whose values will be determined later. The terms containing $\sin\theta$ do not appear in any equation because the potential is symmetric about the X axis. The terms containing r^{-n} do not appear in (4.9) because the potential is finite at $r = 0$. The terms containing r^n do not appear in (4.11) because potential cannot go to infinity as $r \to \infty$ more rapidly than $\ln r$. In (4.10), both r^{-n} and r^n can appear because $r = 0$ and $r = \infty$ are excluded from the region in which V_2 is applicable.

There are two boundary conditions (2.21) and (2.22) to be satisfied at each boundary.

At $r = R_1$,

$$V_1 = V_2 \tag{4.12}$$

and

$$-K_2\left(\frac{\partial V_2}{\partial r}\right)_{r=R_1} + K_1\left(\frac{\partial V_1}{\partial r}\right)_{r=R_1} = \omega'(t) \tag{4.13}$$

Similarly at $r = R_2$,

$$V_2 = V_3 \tag{4.14}$$

and

$$-K_3\left(\frac{\partial V_3}{\partial r}\right)_{r=R_2} + K_2\left(\frac{\partial V_2}{\partial r}\right)_{r=R_2} = \omega''(t) \tag{4.15}$$

The values of V_1 and V_2 from (4.9) and (4.10) are substituted into (4.12) and (4.13). Expressions (4.12) and (4.13) are evaluated at $r = R_1$. The values of V_2 and V_3 from (4.10) and (4.11) are substituted into (4.14) and (4.15). Expressions (4.14) and (4.15) are evaluated at $r = R_2$.

Thus the following four equations are obtained from (4.12),(4.13),(4.14) and (4.15) together with (4.5) and (4.6)

$$\sum_{n=1}^{\infty} A_n R_1^n \cos n\theta + C_0 = \sum_{n=1}^{\infty} B_n R_1^n \cos n\theta + \sum_{n=1}^{\infty} D_n R_1^{-n} \cos n\theta + E_0 + F_0 \ln R_1 \tag{4.16}$$

$$-K_2\left(\sum_{n=1}^{\infty} B_n n R_1^{n-1} \cos n\theta + \sum_{n=1}^{\infty} D_n (-n) R_1^{-n-1} \cos n\theta + \frac{F_0}{R_1}\right) + K_1 \sum_{n=1}^{\infty} A_n n R_1^{n-1} \cos n\theta = \sum_{n=0}^{\infty} \beta_n'(t) \cos n\theta \tag{4.17}$$

$$\sum_{n=1}^{\infty} B_n R_2^n \cos n\theta + \sum_{n=1}^{\infty} D_n R_2^{-n} \cos n\theta + E_0 + F_0 \ln R_2 = \sum_{n=1}^{\infty} G_n R_2^{-n} \cos n\theta + H_0 \ln R_2 \tag{4.18}$$

$$-K_3\left(\sum_{n=1}^{\infty} G_n (-n) R_2^{-n-1} \cos n\theta + \frac{H_0}{R_2}\right) + K_2\left(\sum_{n=1}^{\infty} B_n n R_2^{n-1} \cos n\theta + \sum_{n=1}^{\infty} D_n (-n) R_2^{-n-1} \cos n\theta + \frac{F_0}{R_2}\right) = \sum_{n=0}^{\infty} \beta_n''(t) \cos n\theta \tag{4.19}$$

On equating the coefficients of $\cos n\theta$ in the above four equations, we have

$$A_n R_1^n - B_n R_1^n - D_n R_1^{-n} = 0 \tag{4.20}$$

$$K_1 A_n R_1^{n-1} + K_2 D_n R_1^{-n-1} - K_2 B_n R_1^{n-1} = \beta_n'(t)/n \tag{4.21}$$

$$B_n R_2^n + D_n R_2^{-n} - G_n R_2^{-n} = 0 \tag{4.22}$$

$$K_2 B_n R_2^{n-1} - K_2 D_n R_2^{-n-1} + K_3 G_n R_2^{-n-1} = \beta_n''(t)/n \tag{4.23}$$

Also,

$$F_0 = -\frac{R_1}{K_2}\beta_0'(t) \tag{4.24}$$

$$\dot{H}_0 = -\left(\frac{\beta_0'(t)R_2 + \beta_0'(t)R_1}{K_3}\right) \tag{4.25}$$

$$E_0 = \left[-\beta_0''(t)\frac{R_2}{K_3} - \beta_0'(t)R_1\frac{(K_2-K_3)}{K_2K_3}\right]\ln R_2 \tag{4.26}$$

$$C_0 = E_0 + F_0 \ln R_1 \tag{4.27}$$

The values of A_n, B_n, D_n and G_n can be obtained by solving (4.20), (4.21), (4.22) and (4.23) using determinants:

$$A_n = \frac{\begin{vmatrix} 0 & -R_1^{n} & -R_1^{-n} & 0 \\ \beta_n'(t)/n & -K_2R_1^{n-1} & K_2R_1^{-n-1} & 0 \\ 0 & R_2^{n} & R_2^{-n} & -R_2^{-n} \\ \beta_n''(t)/n & K_2R_2^{n-1} & -K_2R_2^{-n-1} & K_3R_2^{-n-1} \end{vmatrix}}{\begin{vmatrix} R_1^{n} & -R_1^{n} & -R_1^{-n} & 0 \\ K_1R_1^{n-1} & -K_2R_1^{n-1} & K_2R_1^{-n-1} & 0 \\ 0 & R_2^{n} & R_2^{-n} & -R_2^{-n} \\ 0 & K_2R_2^{n-1} & -K_2R_2^{-n-1} & K_3R_2^{-n-1} \end{vmatrix}}$$

or

$$A_n = \left\{\frac{\beta_n'(t)}{n}\left[R_1^{n}R_2^{-2n-1}(K_3-K_2) - R_1^{-n}R_2^{-1}(K_3+K_2)\right] - 2\frac{\beta_n''(t)}{n}R_1^{-1}R_2^{-n}K_2\right\}\Big/|\text{deno}| \tag{4.28}$$

where $|\text{deno}|$ means the value of the denominator:

$$|\text{deno}| = R_1^{2n-1}R_2^{-2n-1}(K_1K_3 - K_1K_2 - K_2K_3 + K_2^2) - R_1^{-1}R_2^{-1}(K_2K_3 + K_2^2 + K_1K_3 + K_1K_2) \tag{4.29}$$

Similarly the values of B_n, D_n and G_n are obtained as follows:

$$B_n = \left\{\frac{\beta_n'(t)}{n}R_1^{n}R_2^{-2n-1}(K_3-K_2) - \frac{\beta_n''(t)}{n}R_1^{-1}R_2^{-n}(K_1+K_2)\right\}\Big/|\text{deno}| \tag{4.30}$$

$$D_n = \left\{-\frac{\beta_n'(t)}{n}R_1^{n}R_2^{-1}(K_3+K_2) + \frac{\beta_n''(t)}{n}R_1^{2n-1}R_2^{-n}(K_1-K_2)\right\}\Big/|\text{deno}| \tag{4.31}$$

$$G_n = \left\{ -\frac{2\beta_n'(t)}{n} R_1^{n} R_2^{-n} K_2 + \frac{\beta_n''(t)}{n}\left[R_1^{2n-1} R_2^{-n}(K_1 - K_2) - R_1^{-1} R_2^{n}(K_1 + K_2)\right]\right\} \Big/ |\text{deno}| \quad (4.32)$$

The next step is to find the time rate of change of charge density by using (2.23).

Thus

$$\frac{\partial \omega'(t)}{\partial t} = -\sigma_1 \left(\frac{\partial V_1}{\partial r}\right)_{r=R_1} + \sigma_2 \left(\frac{\partial V_2}{\partial r}\right)_{r=R_1} \quad (4.33)$$

$$\frac{\partial \omega''(t)}{\partial t} = -\sigma_2 \left(\frac{\partial V_2}{\partial r}\right)_{r=R_2} \quad (4.34)$$

The values of V_1 and V_2 from (4.9) and (4.10) are substituted into (4.33) and evaluated at $r = R_1$. Similarly the value of V_2 from (4.10) is substituted into (4.34) and evaluated at $r = R_2$. After substitution, we have

$$\frac{\partial \omega'(t)}{\partial t} = -\sigma_1 \left(\sum_{n=1}^{\infty} A_n n R_1^{n-1} \cos n\theta\right) + \sigma_2 \left(\sum_{n=1}^{\infty} B_n n R_1^{n-1} \cos n\theta - \sum_{n=1}^{\infty} D_n n R_1^{-n-1} \cos n\theta + F_0/R_1\right) \quad (4.35)$$

and

$$\frac{\partial \omega''(t)}{\partial t} = -\sigma_2 \left(\sum_{n=1}^{\infty} B_n n R_2^{n-1} \cos n\theta - \sum_{n=1}^{\infty} D_n n R_2^{-n-1} \cos n\theta + \frac{F_0}{R_2}\right) \quad (4.36)$$

But from (4.5) and (4.6)

$$\frac{\partial \omega'(t)}{\partial t} = \sum_{n=0}^{\infty} \frac{d}{dt} \beta_n'(t) \cos n\theta \quad (4.37)$$

$$\frac{\partial \omega''(t)}{\partial t} = \sum_{n=0}^{\infty} \frac{d}{dt} \beta_n''(t) \cos n\theta \quad (4.38)$$

After substituting A_n, B_n, D_n and F_0 into (4.35) and (4.36), the coefficient of $\cos n\theta$ in (4.35) is equated to that in (4.37). Similarly the coefficient of $\cos n\theta$ in (4.36) is equated to that in (4.38). By this procedure, we obtain

$$\frac{d\beta_n'(t)}{dt} = \alpha_{11} \beta_n' + \alpha_{12} \beta_n'' \qquad (n \geq 1) \quad (4.39)$$

$$\frac{d\beta_n''(t)}{dt} = \alpha_{21}\beta_n' + \alpha_{22}\beta_n'' \qquad (n \geq 1) \tag{4.40}$$

$$\frac{d\beta_0'(t)}{dt} = -\frac{\sigma_2}{K_2}\beta_0'(t) \tag{4.41}$$

$$\frac{d\beta_0''(t)}{dt} = \frac{\sigma_2}{K_2}\frac{R_1}{R_2}\beta_0'(t) \tag{4.42}$$

where

$$\alpha_{11} = \left[R_1^{2n-1}R_2^{-2n-1}(K_3-K_2)(\sigma_2-\sigma_1) + R_1^{-1}R_2^{-1}(K_3+K_2)(\sigma_2+\sigma_1)\right] / |\mathrm{deno}| \tag{4.43}$$

$$\alpha_{12} = \left[R_1^{n-2}R_2^{-n}(2K_2\sigma_1 - 2K_1\sigma_2)\right] / |\mathrm{deno}| \tag{4.44}$$

$$\alpha_{21} = \left[R_1^{n}R_2^{-n-2}(K_3-K_2)(-\sigma_2) - R_1^{n}R_2^{-n-2}(K_3+K_2)(\sigma_2)\right] / |\mathrm{deno}| \tag{4.45}$$

$$\alpha_{22} = \left[R_1^{-1}R_2^{-1}(K_1+K_2)(\sigma_2) + R_1^{2n-1}R_2^{-2n-1}(K_1-K_2)(\sigma_2)\right] / |\mathrm{deno}| \tag{4.46}$$

The solutions of the two differential equations (4.39) and (4.40) are assumed to be of the form

$$\beta_n' = J_n' e^{-\lambda t} \tag{4.47}$$

and

$$\beta_n'' = J_n'' e^{-\lambda t} \tag{4.48}$$

where the J_n' and J_n'' are independent of time. Whenthe solutions (4.47) and (4.48) are substituted into (4.39) and (4.40), the following equations are obtained:

$$(\alpha_{11}+\lambda)J_n' + \alpha_{12}J_n'' = 0 \tag{4.49}$$

$$\alpha_{21}J_n' + (\alpha_{22}+\lambda)J_n'' = 0 \tag{4.50}$$

In order to be consistent, the coefficients of J_n' and J_n''

have to satisfy the determinant given below:

$$\begin{vmatrix} (\alpha_{11} + \lambda) & \alpha_{12} \\ \alpha_{21} & (\alpha_{22} + \lambda) \end{vmatrix} = 0 \qquad (4.51)$$

Expanding the above determinant gives two values of λ, which are

$$\lambda_1 = \frac{-(\alpha_{11} + \alpha_{22}) + \sqrt{(\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}\alpha_{21}}}{2} \qquad (4.52)$$

$$\lambda_2 = \frac{-(\alpha_{11} + \alpha_{22}) - \sqrt{(\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}\alpha_{21}}}{2} \qquad (4.53)$$

From equation (4.49), the value of J_n'' is

$$J_n'' = -\left(\frac{\alpha_{11} + \lambda}{\alpha_{12}}\right) J_n'. \qquad (4.54)$$

Because there are two values for λ (4.52) and (4.53), the general solutions (of the differential equations) are as follows:

$$\beta_n' = C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t} \qquad (4.55)$$

$$\beta_n'' = -\left(\frac{\alpha_{11} + \lambda_1}{\alpha_{12}}\right) C_1 e^{-\lambda_1 t} - \left(\frac{\alpha_{11} + \lambda_2}{\alpha_{12}}\right) C_2 e^{-\lambda_2 t} \qquad (4.56)$$

where C_1 and C_2 are constants whose values are obtained from the requirement that, at t=0, $\beta_n' = 0$ and $\beta_n'' = \beta_n''(0)$. After evaluating C_1 and C_2, (4.55) and (4.56) become

$$\beta_n'(t) = \beta_n''(0)\left(\frac{\alpha_{12}}{\lambda_2 - \lambda_1}\right)\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \qquad (4.57)$$

$$\beta_n''(t) = \beta_n''(0)\left[-\left(\frac{\alpha_{11} + \lambda_1}{\lambda_2 - \lambda_1}\right) e^{-\lambda_1 t} + \left(\frac{\alpha_{11} + \lambda_2}{\lambda_2 - \lambda_1}\right) e^{-\lambda_2 t}\right] \qquad (4.58)$$

On solving (4.41) for $\beta_0'(t)$, it is found that

$$\beta_0'(t) = \beta_0'(0)\, e^{-\sigma_2 t / K_2} = 0 \qquad (4.59)$$

because $\quad \beta_0'(0) = 0$

Also, on solving (4.42) for $\beta_0''(t)$, it is found that

$$\beta_0''(t) = \beta_0''(0) \tag{4.60}$$

On substituting (4.57) and (4.59) into (4.5), and (4.58) and (4.60) into (4.6), we have

$$\omega'(t) = \sum_{n=1}^{\infty} \beta_n''(0) \left(\frac{\alpha_{12}}{\lambda_2 - \lambda_1} \right) \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) \cos n\theta \tag{4.61}$$

$$\omega''(t) = \beta_0''(0) + \sum_{n=1}^{\infty} \beta_n''(0) \left[-\frac{(\alpha_{11} + \lambda_1)}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{(\alpha_{11} + \lambda_2)}{\lambda_2 - \lambda_1} e^{-\lambda_2 t} \right] \cos n\theta \tag{4.62}$$

These equations (4.61) and (4.62) complete the general solution for $\omega'(t)$ and $\omega''(t)$, the charge densities at a later time on the inner and outer surfaces respectively. For a particular known $\omega''(0)$, the initial charge distribution on the surface, $\beta_n''(0)$ can be found from (4.4). Then α_{11}, α_{12}, α_{21} and α_{22} can be found from (4.43), (4.44), (4.45) and (4.46) respectively. λ_1 and λ_2 are obtained from (4.52) and (4.53). These results are then put into (4.61) and (4.62) to give the charge distribution at any later time t.

The series involved in (4.61) and (4.62) cannot be summed because the relaxation times of the terms in the series are not all the same as they depend on n. No general conclusions are drawn from these expressions in this section. In Section V, however, we will solve a more specific problem for the surface charge density using the expressions obtained in this section.

V TWO-DIMENSIONAL CHARGE DISTRIBUTION ON A SINGLE CYLINDER WITH SURFACE CONDUCTIVITY

A. Arbitrary Charge Distribution on a Single Cylinder

The case considered in this section is that of an infinitely long cylinder which is not conducting in its interior, but only conducting slightly in a very thin layer on its surface. The expressions obtained in the previous section can be used to solve this problem directly. The only changes necessary are the following:

(a) It is assumed that the outer cylinder of the two conducting coaxial cylinders, discussed in the previous section, is very thin, such that

$$R_1 = R_2(1-\delta) \tag{5.1}$$

where R_1 is the radius of the inner cylinder, R_2 is the radius of the outer cylinder, and δ is very small $(\delta \ll 1)$. Then all the expressions involving δ can be expanded in powers of δ, and the δ^2 and higher terms can be neglected. For example

$$\frac{1}{1-\delta} = 1+\delta+\delta^2+\cdots \approx 1+\delta \tag{5.2}$$

Also,

$$\sqrt{a+b\delta} = \sqrt{a}\left[\sqrt{1+\frac{b}{a}\delta}\right] \approx \sqrt{a}\left[1+\frac{b}{2a}\delta\right] \tag{5.3}$$

$$\frac{a+b\delta}{c+d\delta} = \frac{(a+b\delta)(c-d\delta)}{c^2-\delta^2 d^2}$$

$$\approx a/c + \left(b/c - \frac{ad}{c^2}\right)\delta \tag{5.4}$$

(b) It is assumed that the dielectric constants of the two cylinders have the same value and the inner cylinder is not conducting.

Thus

$$K_1 = K_2 \tag{5.5}$$

and $$\sigma_1 = 0 \tag{5.5a}$$

when the equations of the previous section are rewritten after substituting (5.1),(5.5) and (5.5a) in appropriate places, we have:

From (4.43)

$$\alpha_{11} = \left\{R_2^{-2}[1-(2n-1)\delta](K_3-K_1)\sigma_2 + R_2^{-2}(1+\delta)(K_3+K_1)\sigma_2\right\}\Big/|deno| \tag{5.6}$$

From (4.44)

$$\alpha_{12} = \left\{R_2^{-2}[1-(n-2)\delta](-2K_1\sigma_2)\right\}\Big/|deno| \tag{5.7}$$

From (4.45)

$$\alpha_{21} = \left\{R_2^{-2}(1-n\delta)(-2K_3\sigma_2)\right\}\Big/|deno| \tag{5.8}$$

From (4.46)

$$\alpha_{22} = \left\{R_2^{-2}(1+\delta)(2K_1\sigma_2)\right\}\Big/|deno| \tag{5.9}$$

Here

$$|deno| = -R_2^{-2}(1+\delta)(2K_1K_3 + 2K_1^2) \tag{5.10}$$

By using (4.43),(4.44),(4.45),(4.46) and putting $K_1 = K_2$ and $\sigma_1 = 0$, the values of λ's in (4.52) and (4.53) become

$$\lambda_1 = \frac{n\sigma_2\delta}{(K_1+K_3)} \tag{5.11}$$

$$\lambda_2 = \frac{\sigma_2}{K_1} + \delta\left(\frac{-n\sigma_2K_3}{K_1(K_1+K_3)}\right) \tag{5.12}$$

Also

$$\frac{\alpha_{11}+\lambda_1}{\lambda_2-\lambda_1} = \frac{-K_3}{K_1+K_3} \tag{5.13}$$

$$\frac{\alpha_{11}+\lambda_2}{\lambda_2-\lambda_1} = \frac{K_1}{K_1+K_3} \tag{5.14}$$

$$\frac{\alpha_{12}}{\lambda_2-\lambda_1} = \left(\frac{K_1}{K_1+K_3}\right)(1+\delta) \qquad (5.15)$$

On substituting (5.15),(5.12),(5.11) into (4.57) and (5.14),(5.13),(5.12),(5.11) into (4.58), we have

$$\beta_n'(t) = \beta_n''(0)\left[\left(\frac{K_1}{K_1+K_3}\right)(1+\delta)e^{-\frac{n\sigma_2\delta t}{K_1+K_3}} - \left(\frac{K_1}{K_1+K_3}\right)(1+\delta)e^{-\frac{\sigma_2 t}{K_1}+\frac{n\sigma_2 K_3\delta t}{K_1(K_1+K_3)}}\right] \qquad (5.16)$$

$$\beta_n''(t) = \beta_n''(0)\left[\frac{K_3}{K_1+K_3}e^{-\frac{n\sigma_2\delta t}{K_1+K_3}} + \frac{K_1}{K_1+K_3}e^{-\frac{\sigma_2 t}{K_1}+\frac{n\sigma_2 K_3\delta t}{K_1(K_1+K_3)}}\right] \qquad (5.17)$$

Then substituting (5.16) and (5.17) into (4.5) and (4.6) respectively, we have

$$\omega'(t) = \sum_{n=1}^{\infty}\beta_n''(0)\left[\left(\frac{K_1}{K_1+K_3}\right)(1+\delta)e^{-\frac{n\sigma_2\delta t}{K_1+K_3}} - \left(\frac{K_1}{K_1+K_3}\right)(1+\delta)e^{-\frac{\sigma_2 t}{K_1}+\frac{n\sigma_2 K_3\delta t}{K_1(K_1+K_3)}}\right]\cos n\theta \qquad (5.18)$$

$$\omega''(t) = \beta_0''(0) + \sum_{n=1}^{\infty}\beta_n''(0)\left[\frac{K_3}{K_1+K_3}e^{-\frac{n\sigma_2\delta t}{K_1+K_3}} + \frac{K_1}{K_1+K_3}e^{-\frac{\sigma_2 t}{K_1}+\frac{n\sigma_2 K_3\delta t}{K_1(K_1+K_3)}}\right]\cos n\theta \qquad (5.19)$$

Expressions (5.18) and (5.19) can be rewritten as

$$\omega'(t) = \frac{K_1}{K_1+K_3}(1+\delta)\sum_{n=1}^{\infty}\beta_n'(0)\,a_1^n \cos n\theta - \frac{K_1}{K_1+K_3}(1+\delta)\,e^{-\sigma_2 t/K_1}\sum_{n=1}^{\infty}\beta_n'(0)\,a_2^n \cos n\theta \quad (5.20)$$

$$\omega''(t) = \beta_0''(0) + \frac{K_3}{K_1+K_3}\sum_{n=1}^{\infty}\beta_n''(0)\,a_1^n \cos n\theta + \frac{K_1}{K_1+K_3}\,e^{-\sigma_2 t/K_1}\sum_{n=1}^{\infty}\beta_n''(0)\,a_2^n \cos n\theta \quad (5.21)$$

where

$$a_1 = e^{-\sigma_2 \delta t/(K_1+K_3)} \qquad a_2 = e^{\sigma_2 \delta t K_3/K_1(K_1+K_3)} \quad (5.22)$$

The expressions (5.20) and (5.21) are the general results for any two-dimensional charge distribution. $\omega'(t)$ is the charge density on the surface of the inner cylinder at any time and $\omega''(t)$ is that on the outer cylinder.

B. Line Charge on a Single Cylinder

The case considered in Section V A is a general case, where the original charge placed on the cylinder may be any arbitrary two-dimensional distribution. A special case is considered in this section, where the original charge placed on the cylinder is a line charge. In this case, $\beta_n''(0)$ in (5.20) and (5.21) becomes[9]

$$\beta_n''(0) = \begin{cases} \rho/\pi R_2 & n \geq 1 \\ \rho/2\pi R_2 & n = 0 \end{cases} \quad (5.23)$$

where ρ is the charge per unit length of the original line charge. The value of $\beta_0''(0)$ in (5.21) can be obtained from (5.23). On substituting the value of $\beta_n''(0)$ from (5.23) into (5.20) and (5.21), we have

9. See Appendix B.

$$\omega'(t) = \frac{K_1}{K_1+K_3}(1+\delta)\frac{p}{\pi R_2}\sum_{n=1}^{\infty} a_1^n \cos n\theta - \frac{K_1}{K_1+K_3}(1+\delta)e^{-\frac{\sigma_2 t}{K_1}}\frac{p}{\pi R^2}\sum_{n=1}^{\infty} a_2^n \cos n\theta \quad (5.24)$$

$$\omega''(t) = \frac{p}{2\pi R_2} + \frac{K_3}{K_1+K_3}\frac{p}{\pi R_2}\sum_{n=1}^{\infty} a_1^n \cos n\theta + \frac{K_1}{K_1+K_3}e^{-\frac{\sigma_2 t}{K_1}}\frac{p}{\pi R_2}\sum_{n=1}^{\infty} a_2^n \cos n\theta \quad (5.25)$$

The series $\sum_{n=1}^{\infty} a^n \cos n\theta$ can be summed in a closed form to give[10]

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \left[\frac{1-a^2}{2(1-2a\cos\theta+a^2)} - \frac{1}{2}\right]$$

We obtain two orders of magnitude for relaxation times in (5.24) and (5.25): The ones appearing in a_1 and a_2 are of the order of some dielectric constant K divided by $\delta\sigma_2$, while the last two terms are multiplied by an exponential which has a relaxation time of K_1/σ_2. If $\delta \ll 1$ and the surface layer is very thin, the relaxation associated with a τ of the order of K_1/σ_2 will occur much more rapidly than the others. Experimentally, one can probably observe only the slower relaxations associated with a_1 and a_2, because the other relaxation will have already taken place. Also, if δ is small, it is not possible to distinguish experimentally between charges placed on the surfaces of the inner and outer cylinders. Mathematically we may therefore restrict our solution to determining $\omega(t)$, the sum of $\omega'(t)$ and $\omega''(t)$ for the case when σ_2 is very large, δ is very small, while $\sigma_2\delta$ has some finite non-zero value. The factor $\sigma_2\delta$ will be designated by σ', which may be called the surface conductivity. From the definition of

10. See Appendix C.

δ , it may be seen that $\sigma' = \frac{\sigma T}{R}$, where σ is the conductivity of the surface layer of thickness T , while R is the radius of the cylinder.

Thus after substituting 0 for δ ; ∞ for σ_2 ; σ' for $\delta\sigma_2$; and $\left[\frac{1-a^2}{2(1-2a\cos\theta+a^2)}-\frac{1}{2}\right]$ for $\sum_{n=1}^{\infty} a^n \cos n\theta$ into (5.24) and (5.25), we have

$$\omega(t) = \omega'(t) + \omega''(t) = \left[\frac{1-a_1^2}{2(1-2a_1\cos\theta+a_1^2)}\right]\frac{p}{\pi R_2} \tag{5.26}$$

After substituting the value of a_1 in (5.26), it can be rewritten as

$$\omega(t) = \left(\frac{1 - e^{-\frac{2t}{\tau}}}{\left(1 - 2e^{-t/\tau}\cos\theta + e^{-\frac{2t}{\tau}}\right)}\right)\frac{p}{2\pi R_2} \tag{5.27}$$

where

$$\tau = \frac{K_1 + K_3}{\sigma'} \tag{5.28}$$

When both the numerator and the denominator of the right hand side of expression (5.27) are multiplied by $e^{t/\tau}$, we have

$$\omega(t) = \left(\frac{e^{t/\tau} - e^{-t/\tau}}{e^{t/\tau} - 2\cos\theta + e^{-t/\tau}}\right)\frac{p}{2\pi R_2} \tag{5.29}$$

or

$$\omega(t) = \left(\frac{\sinh t/\tau}{\cosh t/\tau - \cos\theta}\right)\frac{p}{2\pi R_2} \tag{5.30}$$

where $\omega(t)$ is the total charge density or the total charge per unit area on the surface at any time.

In Fig.4, a set of curves is plotted with $\omega(t)$ as ordinate and θ , the angular position, as abscissa for various values of t/τ . The graph is normalized (by setting $\rho = 2\pi R_2$) so that the charge distribution at infinite time is unity everywhere on the cylinder. The charge density at any time $\omega(t)$ for any ρ can be obtained by multiplying the graph by $\rho/2\pi R_2$. The values of t/τ used are 0.2, 0.5, 1, 2 and 5. From the curves the following conclusions may be drawn: For the intermediate times between $t = 0$ and $t = \infty$, it is found that, at $\theta = 0$ (the original position of the line charge), the charge decays as time progresses. At places where θ is small [$\theta < 90°$], the charge density increases at first and falls off later. At places where θ is large [$\theta > 90°$], the charge density increases monotonely with time. The flow of charge ceases when the surface charge density is the same everywhere. Thus the line charge placed on the cylinder at t=0 spreads out until the charge density is everywhere the same at $t = \infty$.

TABLE I. NUMERICAL VALUES FOR FIGURE 4

θ in degrees	$\omega(t)$ at $t/\tau = 0.2$	$\omega(t)$ at $t/\tau = 0.5$	$\omega(t)$ at $t/\tau = 1$	$\omega(t)$ at $t/\tau = 2$	$\omega(t)$ at $t/\tau = 5$
0	10.0149	4.0839	2.1639	1.3130	1.0136
6	7.8633	-	-	-	-
10	5.7025	-	-	-	-
15	3.7140	3.2226	-	-	-
30	1.3063	1.9920	1.7356	1.2523	1.0117
60	0.3870	0.8303	1.1266	1.1118	1.0067
90	0.1973	0.4621	0.7616	0.9640	0.9999
120	0.1324	0.3202	0.5752	0.8509	0.9932
150	0.1067	0.2614	0.4878	0.7837	0.9884
180	0.0996	0.2449	0.4621	0.7616	0.9866

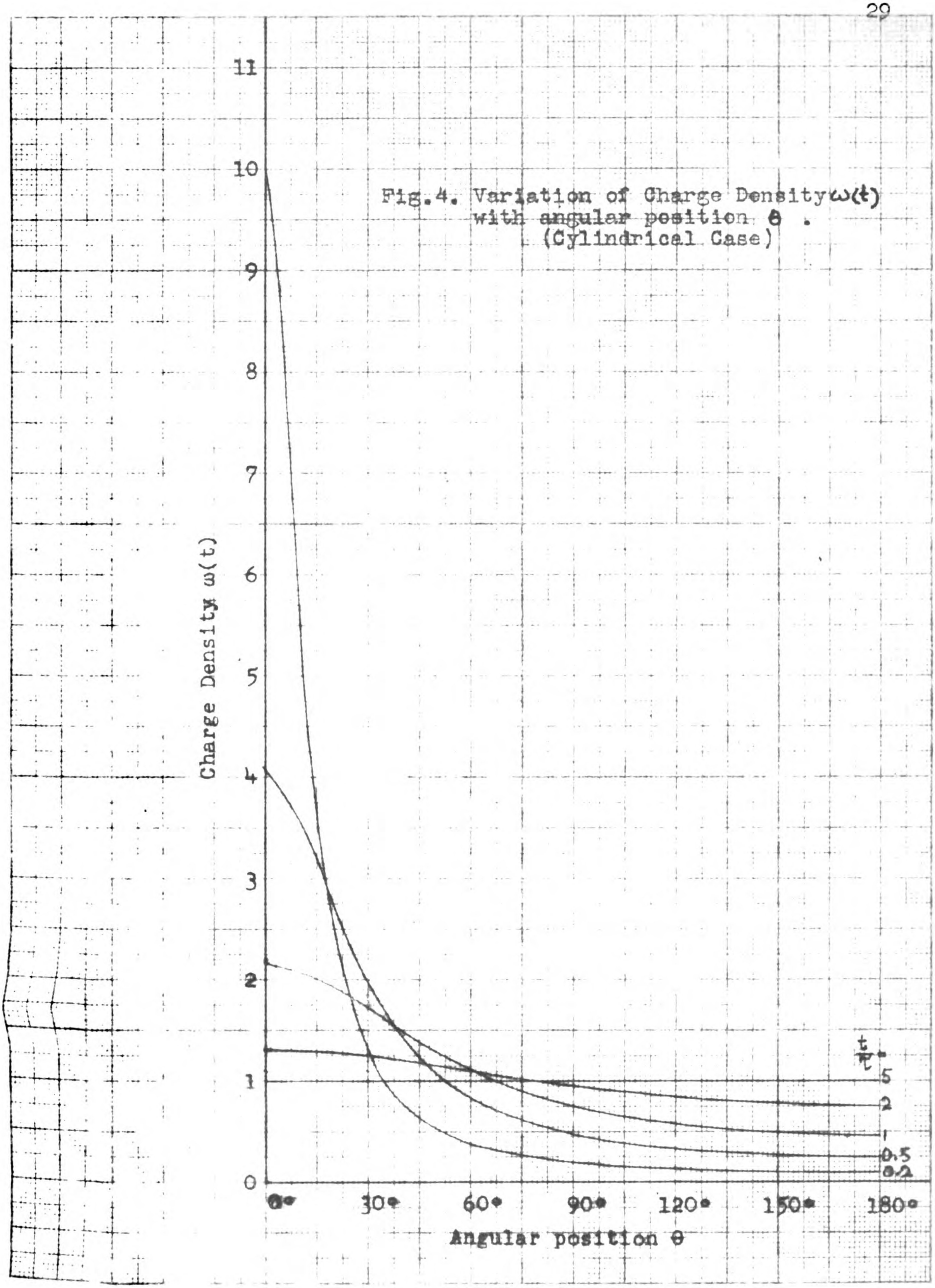

Fig.4. Variation of Charge Density ω(t) with angular position θ . (Cylindrical Case)
Charge Density ω(t)
0
1
2
3
4
5
6
7
8
9
10
11
0°
30°
60°
90°
120°
150°
180°
Angular position θ
t/τ =
5
2
1
0.5
0.2

VI TWO-DIMENSIONAL CHARGE DISTRIBUTION ON A CONDUCTING SPHERE

A. Arbitrary Charge Distribution on a Sphere

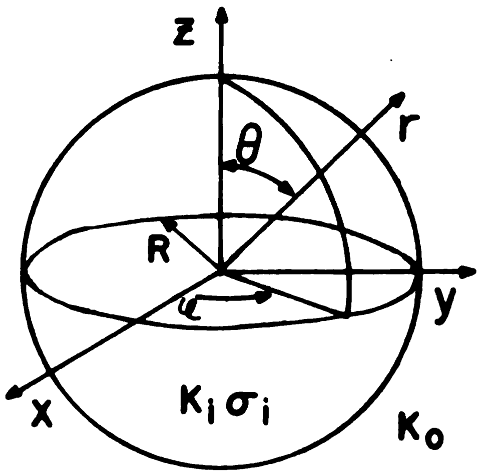

Fig.5.

In this section, the investigation is done on a conducting sphere. The radius of the sphere is R. Inside the sphere, the dielectric constant is K_i and the conductivity is σ_i. Outside the sphere the dielctric constant is K_0 and the conductivity is zero. Radial distance from the center of the sphere to any point is denoted by r and θ is the angle between r and the Z axis (See Fig.5). θ is sometimes known as the co-latitude angle.

The charge density at time t=0 is a two-dimensional distribution and arbitrary except for the fact that it is cylindrically symmetric. We will denote it as $\omega(0)$.

It can be expanded as

$$\omega(0) = \sum_{n=0}^{\infty} \beta_n(0) P_n(\cos\theta) \tag{6.1}$$

where $P_n(\cos\theta)$ are Legendre Polynomials and $\beta_n(0)$ is expressed as[11]

$$\beta_n(0) = \frac{2n+1}{2} \int_0^{\pi} \omega(0) P_n(\cos\theta) \sin\theta \, d\theta \tag{6.1a}$$

11.See reference2,p.418 and also Appendix E.

$\omega(0)$ is considered to be cylindrically symmetric and independent of φ .

It has already being proved that if there is no charge in the interior of a conductor at t=0, there will never be any chage in the interior at a later time. Although charge flows in the interior, it appears only on the surface.

The problem is to study the way the charge is distributed on the surface of the sphere at a later time if the original charge on it is of the form given by (6.1). The charge density $\omega(t)$ at any time t, can be expanded as

$$\omega(t) = \sum_{n=0}^{\infty} \beta_n(t)\, P_n(\cos\theta) \tag{6.2}$$

then the potential inside the sphere is[12]

$$V_i = \sum_{n=0}^{\infty} \frac{\beta_n(t)\, r^n P_n(\cos\theta)}{R^{n-1}\{n(K_0+K_i)+K_0\}} \tag{6.3}$$

The above expression (6.3) is the particular solution of Laplace's equation in spherical polar coordinates when the boundary conditions are satisfied. Using (2.23), the time rate of change of charge density is

$$\frac{\partial\omega(t)}{\partial t} = -\sigma_i\left(\frac{\partial V_i}{\partial r}\right)_{r=R}$$

$$= -\sigma_i \sum_{n=0}^{\infty} \frac{\beta_n(t)\, n\, P_n(\cos\theta)}{n(K_0+K_i)+K_0} \tag{6.4}$$

But using (6.2), the time rate of change of charge density is

$$\frac{\partial\omega(t)}{\partial t} = \sum_{n=0}^{\infty} \frac{d}{dt}\beta_n(t)\, P_n(\cos\theta) \tag{6.5}$$

12. See Appendix D.

Since (6.4) equals (6.5), the coefficients of $P_n(\cos\theta)$ can be equated. Thus

$$\frac{d\beta_n(t)}{dt} = -\frac{\sigma_i n \beta_n(t)}{n(k_0+k_i)+k_0} . \qquad (6.6)$$

The expression (6.6) can be solved for $\beta_n(t)$ to give

$$\beta_n(t) = \beta_n(0)\, e^{-\frac{\sigma_i n t}{n(k_0+k_i)+k_0}} \qquad (6.7)$$

where $\beta_n(0)$ is the value of $\beta_n(t)$ at t=0. When (6.7) is substituted into (6.2), we have,

$$\omega(t) = \sum_{n=0}^{\infty} \beta_n(0)\, e^{-t/\tau_n} P_n(\cos\theta) \qquad (6.8)$$

where $\omega(t)$ is the charge per unit area at any time and

$$\tau_n = \frac{n(k_0+k_i)+k_0}{\sigma_i n} \qquad (6.9)$$

The expression (6.8) is the general result for any two-dimensional charge distribution initially placed on the sphere.

For any given $\omega(0)$, $\beta_n(0)$ can be found by using (6.1a). Then if $\beta_n(0)$ is substituted into (6.8), the surface charge density at a later time can be found. The value of τ_n can be obtained by using (6.9). It is interesting to note that the values of τ_n are independent of the radius R although they depend on **n**.

Because the relaxation times of the terms of the series are not all the same, it is not possible to sum the series in expression (6.8). Therefore no conclusion will be drawn from this expression at present. However, (6.8) will be used in Section VI B, where we will solve for the surface charge density in a specific example.

B. Point Charge on a Sphere

As a specific example of the results of VI A, the case of a conducting sphere with a point charge on it is considered in this section.

Let a point charge of total charge q be placed on the sphere at $\theta = 0$ instead of the arbitrary charge distribution. The coefficients of $P_n(\cos\theta)$ in (6.1) is then[13]

$$\beta_n(0) = \frac{(2n+1)}{4\pi R^2} q$$

The above value of $\beta_n(0)$ is substituted into (6.8) and the general expression for $\omega(t)$ is now

$$\omega(t) = \frac{q}{4\pi R^2} \sum_{n=0}^{\infty} (2n+1)\, e^{-t/\tau_n} P_n(\cos\theta) \qquad (6.10)$$

From the above equation it is seen that at t=0,

$$\omega(0) = \frac{q}{4\pi R^2} \sum_{n=0}^{\infty} (2n+1)\, P_n(\cos\theta) \qquad (6.11)$$

and at $t=\infty$,

$$\omega(\infty) = \frac{q}{4\pi R^2} \qquad (6.12)$$

The original charge is a point charge whose magnitude is q and the area of the sphere is $4\pi R^2$. It shows that the total charge present at the begining is present at the end. It also shows that the point charge placed on the sphere at the begining is uniformly distributed throughout the surface of the sphere after a very long time.

13. See Appendix E.

To determine how the charge is distributed at times $0 < t < \infty$, (6.10) can be rewritten as

$$\omega(t) = \frac{q}{4\pi R^2} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta)\, e^{-t/\tau_\infty}$$
$$+ \frac{q}{4\pi R^2} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta) \left[e^{-t/\tau_n} - e^{-t/\tau_\infty} \right]$$
$$= \frac{q}{4\pi R^2} \delta(0)\, e^{-t/\tau_\infty} + \frac{q}{4\pi R^2} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta) \left[e^{-t/\tau_n} - e^{-t/\tau_\infty} \right] \quad (6.13)$$

where
$$\tau_\infty = (K_0 + K_i)/\sigma_i \quad (6.14)$$

and $\delta(0)$, the Dirac delta function, from (6.11), is equal to $\sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta)$ in this case. The first term of the right hand side of equation (6.13) means that the point charge decreases exponentially with relaxation time τ_∞.

Also we can write

$$\sum_{n=0}^{\infty} (2n+1) \left[e^{-t/\tau_n} - e^{-t/\tau_\infty} \right] P_n(\cos\theta) = \sum_{n=1}^{\infty} 2\alpha \left(\frac{t}{\tau_\infty} \right) e^{-t/\tau_\infty} P_n(\cos\theta)$$
$$+ \sum_{n=1}^{\infty} \left[2\alpha \left(\frac{1}{2} - \alpha \right) \left(\frac{t}{\tau_\infty} \right) + \alpha^2 \left(\frac{t}{\tau_\infty} \right)^2 \right] \frac{e^{-t/\tau_\infty}}{n} P_n(\cos\theta)$$
$$+ (1 - e^{-t/\tau_\infty}) + \sum_{n=1}^{\infty} R_n P_n(\cos\theta) \quad (6.15)$$

where $\alpha = K_0/(K_0 + K_i)$;

$$R_n = (2n+1) \left[e^{-t/\tau_n} - e^{-t/\tau_\infty} \right] - 2\alpha \left(\frac{t}{\tau_\infty} \right) e^{-t/\tau_\infty}$$
$$- \left[2\alpha \left(\frac{1}{2} - \alpha \right) \left(\frac{t}{\tau_\infty} \right) + \alpha^2 \left(\frac{t}{\tau_\infty} \right)^2 \right] \frac{e^{-t/\tau_\infty}}{n} \quad (6.16)$$

The expression (6.15) can be rewritten as

$$\sum_{n=0}^{\infty} (2n+1) \left[e^{-t/\tau_n} - e^{-t/\tau_\infty} \right] P_n(\cos\theta) = (1 - e^{-t/\tau_\infty}) - 2\alpha \left(\frac{t}{\tau_\infty} \right) e^{-t/\tau_\infty} \left(1 - \frac{1}{2\sin\frac{\theta}{2}} \right)$$
$$- \left[2\alpha \left(\frac{1}{2} - \alpha \right) \left(\frac{t}{\tau_\infty} \right) + \alpha^2 \left(\frac{t}{\tau_\infty} \right)^2 \right] e^{-t/\tau_\infty} \ln_e \left[\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2} \right] + \sum_{n=1}^{\infty} R_n P_n(\cos\theta) \quad (6.17)$$

because[14]

$$\sum_{n=0}^{\infty} P_n(\cos\theta) = \frac{1}{2\sin\theta/2} \qquad \text{(F.2)}$$

and

$$\sum_{n=1}^{\infty} \frac{P_n(\cos\theta)}{n} = -\ln_e\left[\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right] \qquad \text{(F.3)}$$

The particular division of (6.17) into several terms plus $\sum_{n=1}^{\infty} R_n P_n(\cos\theta)$ is necessary because $\sum_{n=0}^{\infty} P_n(\cos\theta)$, $\sum_{n=1}^{\infty} \frac{P_n(\cos\theta)}{n}$, and the original series (6.10) do not converge. The series $\sum_{n=1}^{\infty} R_n P_n(\cos\theta)$ does converge. Thus $\sum_{n=0}^{\infty}(2n+1)\left[e^{-\frac{t}{\tau_n}} - e^{-\frac{t}{\tau_\infty}}\right]P_n(\cos\theta)$ may be evaluated numerically by eliminating the non-convergent series and expressing them in closed forms.

The expression (6.13) can now be rewritten, after (6.17) is substituted into it, as

$$\omega(t) = \frac{q}{4\pi R^2}\,\delta(0)\,e^{-t/\tau_\infty} + \frac{q}{4\pi R^2}\Big\{(1-e^{-t/\tau_\infty}) - 2\alpha\left(\frac{t}{\tau_\infty}\right)e^{-\frac{t}{\tau_\infty}}\left(1-\frac{1}{2\sin\frac{\theta}{2}}\right)$$
$$-\left[2\alpha\left(\frac{1}{2}-\alpha\right)\left(\frac{t}{\tau_\infty}\right) + \alpha^2\left(\frac{t}{\tau_\infty}\right)^2\right]e^{-t/\tau_\infty}\ln_e\left[\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right]$$
$$+\sum_{n=1}^{\infty} R_n P_n(\cos\theta)\Big\} \qquad (6.18)$$

where $\omega(t)$ is the surface charge density at any time. The series $\sum_{n=1}^{\infty} R_n P_n(\cos\theta)$ is a relatively small fraction of the entire expression for $\omega(t)$. For t/τ_∞ of the order of unity, it is about 1% of $\omega(t)$. Also, the series converges rapidly, with the first ten terms constituting about 99% of the sum. Thus, in numerical computation, limiting the series to ten terms gives $\omega(t)$ to an accuracy of about 0.1%.

For sample calculation, the value of α is chosen to be $\frac{1}{4}$, that is, $K_i/K_0 = 3$. In Fig.6, a set of curves is plotted with $\omega(t)$ as ordinate and θ as abscissa for various values

14. See Appendix F.

of t/τ_∞. The values of t/τ_∞ used are t/τ_∞ = 1,2,4 and 10. In Table II, the values of $\sum_{n=0}^{\infty}(2n+1)[e^{-t/\tau_n} - e^{-t/\tau_\infty}]P_n(\cos\theta)$ are tabulated against various θ's for different values of t/τ_∞. In Table III, $\sum_{n=1}^{10} R_n P_n(\cos\theta)$ is tabulated against various θ's for different values of t/τ_∞. The value of $\sum_{n=1}^{10} R_n P_n(\cos\theta)$ for $\theta = 15^\circ$ is not computed because it is very small and can be ignored. The following conclusions can be drawn from the curves (Fig. 6):

As time increases, the magnitude of the point charge, placed at $\theta = 0$ initially, decreases exponentially and the charge density increases at all other points. There is, however, one exception. At points close to $\theta = 0$, charge density increases at first, reaches a maximum, and then decreases afterwards. The flow of charge ceases when the charge density is everywhere the same. In general, the point charge placed on the surface of the sphere at the begining spreads out in all directions as time progresses until the charge density is the same everywhere.

TABLE II. NUMERICAL VALUES FOR FIGURE 6.

θ in degrees	$\omega(t)$ at $t/\tau_\infty=1$	$\omega(t)$ at $t/\tau_\infty=2$	$\omega(t)$ at $t/\tau_\infty=4$	$\omega(t)$ at $t/\tau_\infty=10$
15	1.28495	1.51	1.13802	1.
30	0.87202	1.06329	1.04998	1.00085
60	0.64848	0.88423	0.99088	1.00017
90	0.56623	0.81263	0.96541	0.99979
120	0.52555	0.77514	0.95139	0.99953
150	0.50546	0.75845	0.94403	0.99939
180	0.49929	0.75284	0.94172	0.99936

$\omega(t)$ in this table is $\sum_{n=0}^{\infty}(2n+1)\left[e^{-t/\tau_n} - e^{-t/\tau_\infty}\right]P_n(\cos\theta)$

TABLE III. TABLE FOR $\sum_{n=1}^{\infty} R_n P_n(\cos\theta) = Z_n$

θ in degrees	Z_n at $t/\tau_\infty=1$	Z_n at $t/\tau_\infty=2$	Z_n at $t/\tau_\infty=4$	Z_n at $t/\tau_\infty=10$
30	-0.00885	-0.00337	0.00336	0.00031
60	-0.00348	0.00010	0.00130	0.00012
90	0.00097	0.00035	-0.00037	-0.00003
120	0.00428	0.00016	-0.00162	-0.00016
150	0.00631	0.00247	-0.00236	-0.00023
180	0.00695	0.00275	-0.00260	-0.00024

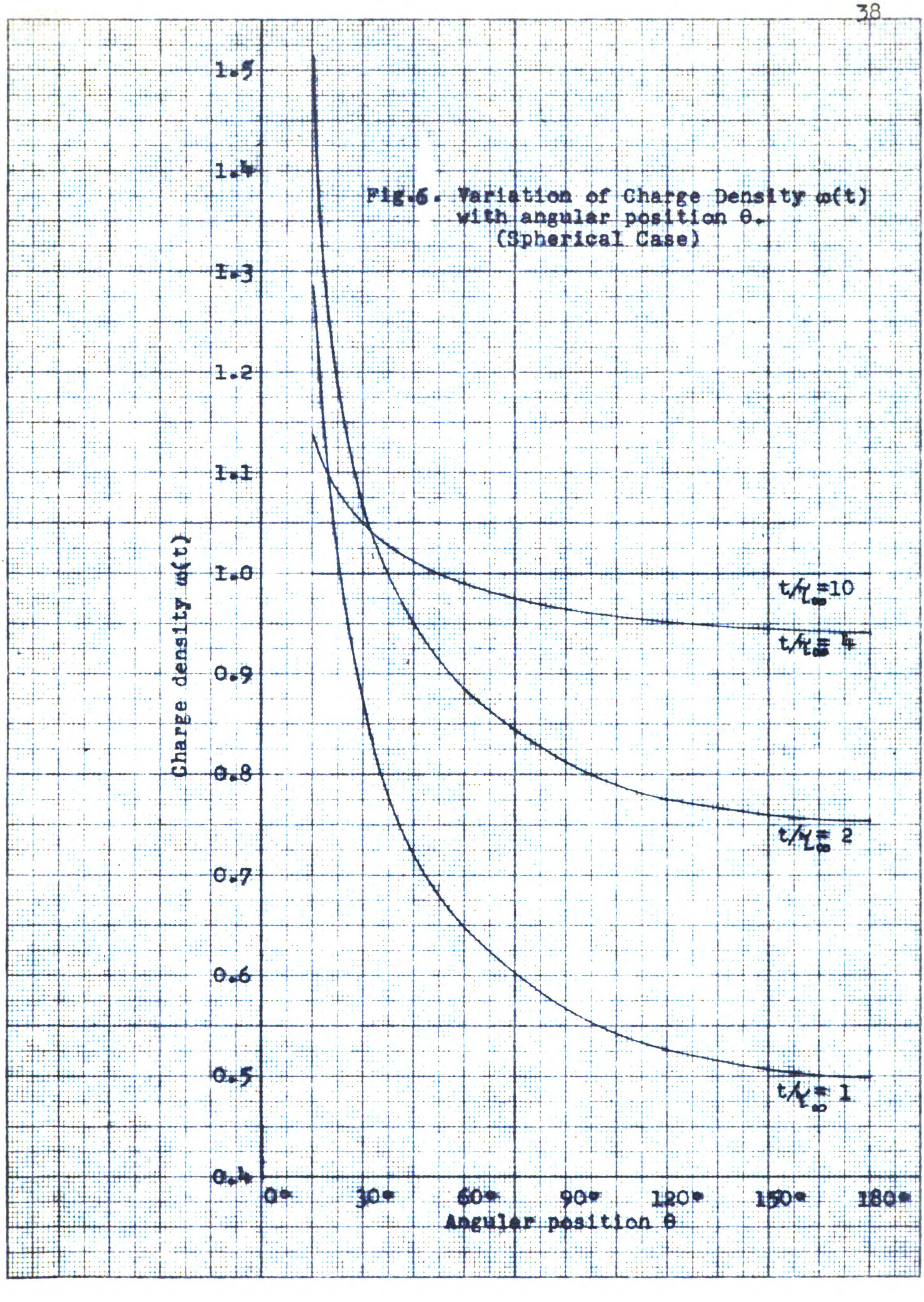

Fig. 6. Variation of Charge Density ω(t) with angular position θ. (Spherical Case)
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
Charge density ω(t)
t/τ∞ =10
t/τ∞ = 4
t/τ∞ = 2
t/τ∞ = 1
0°
30°
60°
90°
120°
150°
180°
Angular position θ

VII SUMMARY

(1) A general approximate method for solving Maxwell's equations is given for the case when the conductivities of all materials present are very small. This method of solution permits the charge distribution at any time to be calculated if the charge distribution at any preceeding time is known.

(2) To the above approximation, it is proved that if no charge is present initially inside a conductor, there will never be any charge inside it at a later time. If charge is originally present inside the conductor, the charge at every point will decay exponentially with a relaxation time equals to $\frac{K}{\sigma}$, where K is the dielectric constant and σ is the conductivity. This relaxation time is independent of the size and the shape of the conductor.

(3) It is found that if any two-dimensional distribution of charge $\omega(0)$ is placed on the surface of a slightly conducting cylinder, the surface charge density at any later time is given by $\omega(t) = \omega(0)e^{-t/\tau} + \beta_0(0)[1 - e^{-t/\tau}]$ (3.13), where $\omega(0)$ is the arbitrary two-dimensional charge density at t=0, $\beta_0(0)$ is the average value of the original distribution of charge $\omega(0)$, and τ, the time of relaxation, is given by $\frac{K_i + K_0}{\sigma_i}$, where K_i is the dielectric constant inside the cylinder, K_0 is the dielectric constant outside the cylinder and σ_i is the conductivity of the cylinder. The relaxation time is independent ofthe size of the cylinder.

(4) For two conducting coaxial cylinders with an arbitrary two-dimensional distribution of charge initially on the surface of the outer cylinder, the general formula for $\omega(t)$, the charge density at any time, is given in equations (4.61) and (4.62) as infinite Fourier series.

(5) For a single cylinder conducting only within a very thin layer on its surface and with a line charge on it initially, it is found that the surface charge density at any time t can be expressed as

$$\omega(t) = \frac{p}{2\pi R_2}\left(\frac{\sinh t/\tau}{\cosh t/\tau - \cos\theta}\right) \qquad (5.30)$$

where $\omega(t)$ is the total charge density residing on the surfaces of the conducting layer at any time t, p is the line charge per unit length initially placed on the cylinder, R_2 is the radius of the cylinder, and $\tau = \frac{K_1 + K_3}{\sigma'}$, where K_1 is the dielectric constant of the cylinder, K_3 is the dielectric constant of the outside medium. σ' is the product of the bulk conductivity of the conducting layer and its thickness divided by the radius of the cylinder. A set of curves are plotted in Fig. 4 for the variation of charge density $\omega(t)$ with time t and θ, angular position measured from the line charge.

(6) For a conducting sphere, with a two-dimensional charge distribution on its surface, $\omega(t)$, the charge density at any time, is given by

$$\omega(t) = \sum_{n=0}^{\infty} \beta_n(0)\, e^{-t/\tau_n} P_n(\cos\theta) \qquad (6.8)$$

where $\beta_n(0)$ are the expansion coefficients of $\omega(0)$ in Legendre Polynomials $P_n(\cos\theta)$. $\tau_n = \frac{n(K_0 + K_i) + K_0}{\sigma_i n}$, where K_0 and K_i are the dielectric constants of the mediums outside and inside the sphere, and σ_i is the conductivity of the sphere.

(7) For a conducting sphere with a point charge on its surface initially, the surface charge density at any later time is given by equation (6.18). A set of curves are plotted in Fig.6 for the variation of charge density $\omega(t)$ with time t and θ , angular position measured from the original position of the point charge. It is found that the rate of relaxation of the charge density is independent of the size of the sphere.

APPENDIX A

POTENTIAL AROUND A CYLINDER

The problem is to find the potentials inside and outside a cylinder with a two-dimensional distribution of charge on its surface. The charge density on the surface will be denoted by $\omega(\theta)$. It is an arbitrary function of θ only, and can be expanded as

$$\omega(\theta) = \sum_{n=0}^{\infty} B_n \cos n\theta \tag{A.1}$$

The radius of the cylinder is R and the dielectric constant of the cylinder is K_i. The dielectric constant of the medium outside the cylinder is K_o and r is the radial distance from the axis of the cylinder to any point.

The potentials V_i inside the cylinder and V_o outside the cylinder are particular solutions of Laplace's equation in cylindrical coordinates. They can be written in the form[15]

$$V_i = \sum_{n=1}^{\infty} B_n r^n \cos n\theta + C_o \tag{A.2}$$

$$V_o = \sum_{n=1}^{\infty} \frac{A_n \cos n\theta}{r^n} + A_o \ln r \tag{A.3}$$

A_n, B_n, A_o and C_o are constants whose values are to be determined so that the boundary conditions are satisfied.

The first boundary condition states that $V_i = V_o$ at

15. Smythe, William R.: "Static and Dynamic Electricity," McGraw-Hill Book Company, Inc., New York, 1939, p. 67.

$r = R$, therefore,

$$\sum_{n=1}^{\infty} B_n R^n \cos n\theta + C_0 = A_0 \ln R + \sum_{n=1}^{\infty} \frac{A_n \cos n\theta}{R^n} \qquad (A.4)$$

or

$$B_n R^n = A_n / R^n \qquad (A.5)$$

Also,

$$A_0 \ln R = C_0 \qquad (A.6)$$

The second boundary condition states that at $r = R$,

$$K_i \left(\frac{\partial V_i}{\partial r}\right)_{r=R} - K_0 \left(\frac{\partial V_0}{\partial r}\right)_{r=R} = \omega$$

therefore,

$$K_i \left(\sum_{n=1}^{\infty} B_n n R^{n-1} \cos n\theta \right) - K_0 \left(\sum_{n=1}^{\infty} \frac{A_n \cos n\theta \, (-n)}{R^{n+1}} + \frac{A_0}{R} \right)$$

$$= \sum_{n=0}^{\infty} \beta_n \cos n\theta \qquad (A.7)$$

or

$$K_i B_n n R^{n-1} + \frac{K_0 A_n n}{R^{n+1}} = \beta_n \qquad (A.8)$$

Also,

$$A_0 = -\frac{\beta_0 R}{K_0} \qquad (A.9)$$

and

$$C_0 = -\frac{\beta_0 R}{K_0} \ln R \qquad (A.10)$$

Using (A.5) and (A.8), the values of A_n and B_n can be computed

$$A_n = (\beta_n R^{n+1}) / n(K_i + K_0) \qquad (A.11)$$

$$B_n = \beta_n / n R^{n-1} (K_i + K_0) \qquad (A.12)$$

On substituting the values of the arbitrary constants into (A.2) and (A.3), we have

$$V_i = \sum_{n=1}^{\infty}\left(\frac{\beta_n}{n R^{n-1}(K_i+K_o)}\right) r^n \cos n\theta - \frac{\beta_o R}{K_o} \ln R \qquad (A.13)$$

$$V_o = \sum_{n=1}^{\infty}\left(\frac{\beta_n R^{n+1}}{n(K_i+K_o)}\right) \frac{\cos n\theta}{r^n} - \frac{\beta_o R}{K_o} \ln R \qquad (A.14)$$

APPENDIX B

EXPANSION OF A LINE CHARGE IN CYLINDRICAL HARMONICS

Let us again expand the charge density $\omega(\theta)$ as in (A.1):

$$\omega(\theta) = \sum_{n=0}^{\infty} \beta_n \cos n\theta \qquad \text{(A.1)}$$

To find β_n, both sides of (A.1) are multiplied by $\cos p\theta \, d\theta$ and integrated from $\theta = 0$ to $\theta = \pi$.

Thus,

$$\int_0^{\pi} \cos p\theta \, \omega(\theta) \, d\theta = \sum_{n=0}^{\infty} \beta_n \int_0^{\pi} \cos p\theta \cos n\theta \, d\theta$$

$$= \begin{cases} \beta_n \frac{\pi}{2} & n \geq 1 \\ \beta_n \pi & n = 0 \end{cases} \qquad \text{(B.1)}$$

and

$$\beta_n = \begin{cases} \frac{2}{\pi} \int_0^{\pi} \omega(\theta) \cos n\theta \, d\theta & n \geq 1 \\ \frac{1}{\pi} \int_0^{\pi} \omega(\theta) \, d\theta & n = 0 \end{cases} \qquad \text{(B.2)}$$

If the initial charge placed on the surface of the cylinder (at $\theta = 0$) is a line charge, then we have

$$\omega(\theta) = \begin{cases} \text{LARGE} & \theta = 0 \\ 0 & \theta \neq 0 \end{cases}$$

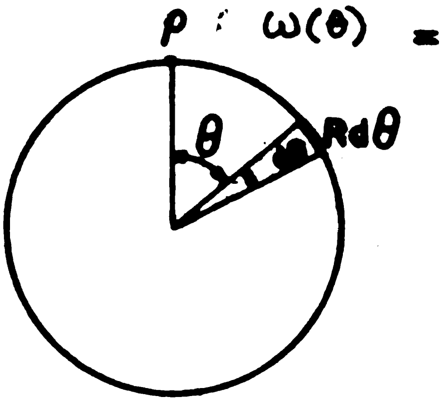

Fig.7.

From Fig.7, we have

$$\int_0^{\text{SMALL ANGLE}} \omega(\theta)(R \, d\theta) = \frac{1}{2} \rho \qquad \text{(B.3)}$$

where ρ is the charge per unit length.

In this case (B.2) can be written as

$$\beta_n = \begin{cases} \frac{2}{\pi} \int_0^{\text{SMALL ANGLE}} \omega(\theta) \cos n\theta \, d\theta = \frac{2}{\pi} \int_0^{\text{SMALL ANGLE}} \omega(\theta) d\theta = \frac{\rho}{\pi R_2} & n \geq 1 \\ \frac{1}{\pi} \int_0^{\text{SMALL ANGLE}} \omega(\theta) \cos n\theta \, d\theta = \frac{1}{\pi} \int_0^{\text{SMALL ANGLE}} \omega(\theta) d\theta = \frac{\rho}{2\pi R_2} & n = 0 \end{cases}$$

because when the angle is small, $\cos n\theta = 1$.

APPENDIX C

THE VALUE OF $\sum_{n=1}^{\infty} a^n \cos n\theta$ IN CLOSED FORM

Let $\sigma(\theta)$ be an arbitrary function of θ that can be expanded as

$$\sigma(\theta) = \sum_{n=0}^{\infty} \beta_n \cos n\theta \qquad (C.1)$$

Then,

$$\int_0^{\pi} \sigma(\theta) \cos m\theta \, d\theta = \sum_{n=0}^{\infty} \beta_n \int_0^{\pi} \cos m\theta \cos n\theta \, d\theta$$

$$= \beta_m \int_0^{\pi} \cos^2 m\theta \, d\theta$$

$$\beta_m = \begin{cases} \frac{2}{\pi} \int_0^{\pi} \sigma(\theta) \cos m\theta \, d\theta & n \geq 1 \\ \frac{1}{\pi} \int_0^{\pi} \sigma(\theta) \, d\theta & n = 0 \end{cases} \qquad (C.2)$$

Now we note that[16]

$$\int_0^{\pi} \frac{1}{(1 - 2a\cos\theta + a^2)} \cos m\theta \, d\theta = \frac{\pi a^m}{1 - a^2}$$

$$\frac{1 - a^2}{2} \int_0^{\pi} \frac{1}{(1 - 2a\cos\theta + a^2)} \cos m\theta \, d\theta = \frac{\pi}{2} a^m \qquad (C.3)$$

Thus if

$$\sigma(\theta) = \frac{1 - a^2}{2(1 - 2a\cos\theta + a^2)}$$

it follows that $\beta_m = a^m$ provided that $m \neq 0$.

When $m = 0$, by using (C.2) we have,

$$\frac{1 - a^2}{2} \int_0^{\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} = \beta_0 \pi \qquad (C.4)$$

But from (C.3)

$$\frac{1 - a^2}{2} \int_0^{\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} = \pi/2 \qquad (C.5)$$

therefore

$$\beta_0 = 1/2$$

or from (C.1)

$$\frac{1 - a^2}{2(1 - 2a\cos\theta + a^2)} = \sum_{n=1}^{\infty} a^n \cos n\theta + \frac{1}{2} \qquad (C.6)$$

Consequently

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{1 - a^2}{2(1 - 2a\cos\theta + a^2)} - \frac{1}{2} \qquad (C.7)$$

16. Dwight.: "Tables of Integrals and other Mathematical Data," #860.1.

APPENDIX D

POTENTIALS AROUND A SPHERE

The problem is to find the potentials inside and outside of a sphere with a two-dimensional distribution of charge on its surface at the begining..

The charge on the surface will be denoted by $\omega(\theta)$. It is arbitrary except that it is a function of only. It can be expanded as

$$\omega(\theta) = \sum_{n=0}^{\infty} \beta_n P_n(\cos\theta) \tag{D.1}$$

The radius of the sphere is R and the dielectric constant of the sphere is K_i . The dielectric constant of the medium outside of the sphere is K_0 and r is the radial distance from the centre of the sphere to any point. The potential, either inside or outside, is the particular solution of the Laplace's equation in spherical coordinates. It can be written in the form[17]

$$V_i = \sum_{n=0}^{\infty} B_n r^n P_n(\cos\theta) \tag{D.2}$$

$$V_0 = \sum_{n=0}^{\infty} A_n \frac{P_n(\cos\theta)}{r^{n+1}} \tag{D.3}$$

where V_i is the potential inside the sphere and V_0 is the potential outside the sphere. A_n and B_n are the constants whose values are to be determined so that the boundary conditions are satisfied. The first boundary condition states that $V_i = V_0$ at $r = R$. Therefore

$$\sum_{n=0}^{\infty} B_n R^n P_n(\cos\theta) = \sum_{n=0}^{\infty} A_n \frac{P_n(\cos\theta)}{R^{n+1}} \tag{D.4}$$

17. See reference 2, p. 417-418.

or

$$B_n R^n = \frac{A_n}{R^{n+1}} \qquad (D.5)$$

The second boundary condition states that

$$K_i\left(\frac{\partial V_i}{\partial r}\right)_{r=R} - K_o\left(\frac{\partial V_o}{\partial r}\right)_{r=R} = \omega$$

at $r = R$. Therefore,

$$K_i\left(\sum_{n=0}^{\infty} B_n n R^{n-1} P_n(\cos\theta)\right) - K_o\left(\sum_{n=0}^{\infty} \frac{A_n(-n-1)P_n(\cos\theta)}{R^{n+2}}\right)$$

$$= \sum_{n=0}^{\infty} \beta_n P_n(\cos\theta) \qquad (D.6)$$

or

$$\left[K_i B_n n R^{n-1} + K_o A_n(n+1)/R^{n+2}\right] = \beta_n \qquad (D.7)$$

Using (D.5) and (D.7), the values of A_n and B_n can be obtained:

$$A_n = \frac{\beta_n R^{n+2}}{n(K_i + K_o) + K_o} \qquad (D.8)$$

$$B_n = \frac{\beta_n}{[n(K_i + K_o) + K_o]R^{n-1}} \qquad (D.9)$$

On substituting the values of the constants into (D.2) and (D.3) we have

$$V_i = \sum_{n=0}^{\infty}\left(\frac{\beta_n}{[n(K_i + K_o) + K_o]R^{n-1}}\right) r^n P_n(\cos\theta) \qquad (D.10)$$

$$V_o = \sum_{n=0}^{\infty}\left(\frac{\beta_n R^{n+2}}{n(K_i + K_o) + K_o}\right)\frac{P_n(\cos\theta)}{r^{n+1}} \qquad (D.11)$$

APPENDIX E

EXPANSION OF A POINT CHARGE IN SPHERICAL HARMONICS

Let us expand the charge density $\omega(\theta)$ as in (D.1).

Thus
$$\omega(\theta) = \sum_{n=0}^{\infty} \beta_n P_n(\cos\theta) \qquad \text{(D.1)}$$
[18]

where β_n is given by

$$\beta_n = \frac{2n+1}{2}\int_0^{\pi} \omega(\theta)\, P_n(\cos\theta)\sin\theta\, d\theta \qquad \text{(E.1)}$$

If the charge placed on the surface of the sphere (at $\theta = 0$) is a point charge,

$$\beta_n = \frac{2n+1}{2}\int_0^{\text{SMALL ANGLE}} \omega(\theta)\sin\theta\, d\theta \qquad \text{(E.2)}$$

because for small angles $P_n(\cos\theta) = 1$.

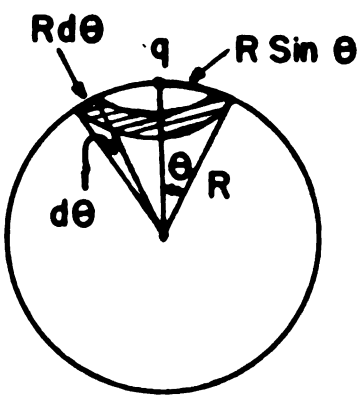

Fig.8.

From Fig.8, it is clearly seen that area dA is given by

$$dA = 2\pi R^2 \sin\theta\, d\theta \qquad \text{(E.3)}$$

But
$$\int_0^{\text{SMALL ANGLE}} \omega(\theta)\, dA = q \qquad \text{(E.4)}$$
where q is the magnitude of the point charge.

Therefore
$$\beta_n = \frac{2n+1}{2}\cdot\frac{1}{2\pi R^2}\int_0^{\text{SMALL ANGLE}} \omega(\theta)\sin\theta\, d\theta\, 2\pi R^2 \qquad \text{(E.5)}$$
or
$$\beta_n = \frac{2n+1}{4\pi R^2}\, q \qquad \text{(E.6)}$$

18. See reference 2, p.418.

APPENDIX F

THE VALUES OF $\sum_{n=0}^{\infty} P_n(\cos\theta)$ AND $\sum_{n=1}^{\infty} \frac{P_n(\cos\theta)}{n}$ IN CLOSED FORM

In terms of the generating function, the Legendre polynomials may be defined by the following equation:[19]

$$\sum_{n=0}^{\infty} t^n P_n(\cos\theta) = \frac{1}{(1-2t\cos\theta+t^2)^{\frac{1}{2}}} \tag{F.1}$$

When $t = 1$,

$$\sum_{n=0}^{\infty} P_n(\cos\theta) = \frac{1}{(2-2\cos\theta)^{\frac{1}{2}}} = \frac{1}{2\sin\theta/2} \tag{F.2}$$

From (F.1), we also have

$$\sum_{n=1}^{\infty} t^n P_n(\cos\theta) = \frac{1}{(1-2t\cos\theta+t^2)^{\frac{1}{2}}} - 1$$

Then

$$\int_0^1 \sum_{n=1}^{\infty} t^{n-1} P_n(\cos\theta)\, dt = \int_0^1 \frac{1}{t}\left[\frac{1}{(1-2t\cos\theta+t^2)^{\frac{1}{2}}} - 1\right] dt$$

Performing the integrations[20] gives

$$\sum_{n=1}^{\infty} \frac{P_n(\cos\theta)}{n} = -\ln_e\left[\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right] \tag{F.3}$$

19. See reference 2, p.327.
20. Dwight.: "Tables of Integrals and other Mathematical Data," #380.111