THE ANALYSIS AND SYNTHESIS OF ROLLING CURVES

Thesis for the Degree of M. S.

MICHIGAN STATE COLLEGE

Kenneth Ward Sidwell

1954

This is to certify that the

thesis entitled

The Analysis and Synthesis of Rolling Curves

presented by

Kenneth Ward Sidwell

has been accepted towards fulfillment of the requirements for

M. S. **degree in** Mechanical Engineering

R. T. Hinkle

Major professor

Date November 24, 1954

THE ANALYSIS AND SYNTHESIS

OF ROLLING CURVES

By

Kenneth Ward Sidwell

A THESIS

Submitted to the School of Graduate Studies of Michigan
State College of Agriculture and Applied Science
in partial fulfillment of the requirements
for the degree of

MASTER OF SCIENCE

Department of Mechanical Engineering

1954

THESIS

ACKNOWLEDGMENT

The author wishes to express his sincere appreciation to Dr. Rolland T. Hinkle of the Mechanical Engineering Department for his guidance in the preparation of this thesis and for the use of his comprehensive library.

Kenneth Ward Sidwell

candidate for the degree of

Master of Science

Thesis: The Analysis and Synthesis of Rolling Curves

Outline of Studies

Major subject: Mechanical Engineering
Minor subject: Mathematics

Biographical Items

Born, March 29, 1932, Kenmore, New York

Undergraduate Studies, Michigan State College, 1949-53

Graduate Studies, Michigan State College, 1953-54

Experience: Graduate Assistant, Michigan State College, 1953-54

Membership: Tau Beta Pi
Pi Tau Sigma
Pi Mu Epsilon
Phi Kappa Phi
Michigan Society of Professional Engineers (EIT)

ABSTRACT

Rolling curves are used to produce a cyclic variation in the angular velocity of a shaft. In any application, rolling curves are used as the surfaces of non-circular cams or the pitch lines of non-circular gears.

The purpose of this thesis is:

1. To develop the conditions and the equations for rolling curves,
2. To explore some specific examples of the design and analysis of rolling curves,
3. To develop a general equation for the synthesis of rolling curves, and
4. To outline methods for manufacturing non-circular cams and gears.

The conditions and the equations for rolling curves have been derived previously. The general relationship between the angular displacements of the two rolling curves is expressed by Equ. (1).

$$\theta_f = f(\theta_d) \tag{1}$$

where θ_f = the angular displacement of the follower curve in radians (clockwise is positive)

θ_d = the angular displacement of the driver curve in radians (counterclockwise is positive)

The basic equations for rolling curves are derived from the condition that the curves must have pure rolling at the point of contact.

$$r_f = \frac{L}{f'(\theta_d)+1} \quad \text{and} \quad r_d = L - r_f \qquad \text{(2a) and (2b)}$$

$$f'(\theta_d) = \frac{d}{d\theta_d} f(\theta_d) = \frac{\omega_f}{\omega_d} \qquad (3)$$

where r_f = the radius of the follower curve

r_d = the radius of the driver curve

L = the constant distance between the axes of rotation

ω_f = the angular velocity of the follower curve

ω_d = the angular velocity of the driver curve

Several applications are discussed by showing the derivation of Equ. (1) in each case.

The most common example of rolling curves is a pair of identical ellipses. The ellipses rotate around a focus point and the distance between the axes of rotation equals the major axis. The relationships between the angular displacements and velocities are found by using the polar equation of the ellipse and Equ. (2a).

$$\omega_f = \omega_d \frac{1-e^2}{1+e^2+2e\cos\theta_d} \qquad \text{(4a)}$$

$$\theta_f = \tan^{-1}\left[\frac{(1-e^2)\sin\theta_d}{2e+(e^2+1)\cos\theta_d}\right] \qquad \text{(4b)}$$

where e = the eccentricity of the ellipses

It is proved that the equivalent linkage of rolling ellipses (a non-parallel equal crank linkage) also produces Equs. (4a) and (4b).

Using the equations for rolling curves, the problem of design reduces to the problem of finding the desired motion pattern.

The specified data can be in either of two forms. First, it could be specified in the form of Equ. (1). Here the rolling curve equations are found directly by using Equs. (2a) and (2b). Second, the motion pattern could be specified only at several points of the driver curve rotation. Here it is necessary to develop a complete motion pattern which satisfies the given data.

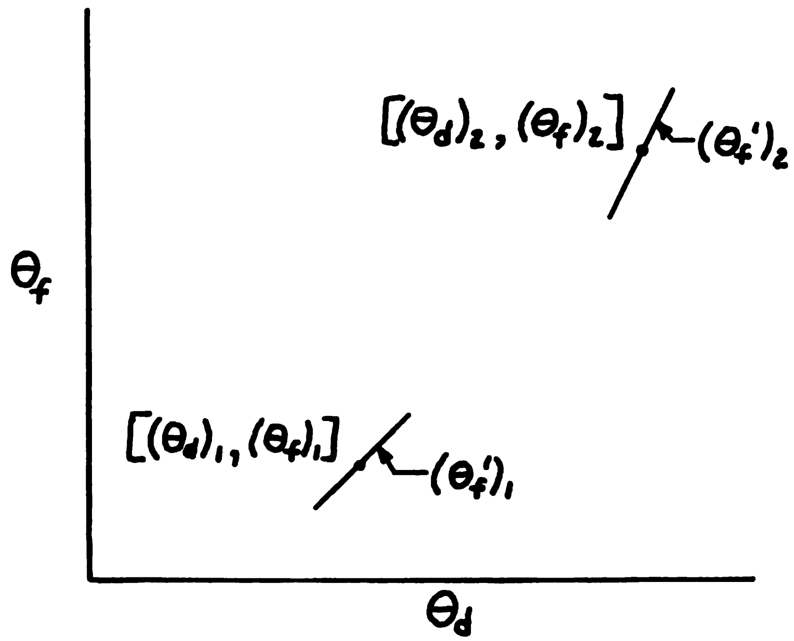


Figure 1. Displacement diagram showing the specified data

The specified data is shown in Fig. (1) in the most complete form. Only one interval of the cycle is shown as each interval is handled separately.

A general equation is developed that satisfies all the given data and the second derivative of which equals zero at the end points of the interval. With these properties the resulting composite curve is continuous and its first and second derivatives are continuous. Thus, the resulting rolling curves are continuous and have no cusps.

In some applications the values of the angular displacement of the follower curve are not specified. Further equations are developed from the general equation which enable these values to be determined such that the angular accelerations are a minimum.

The last chapter contains a brief description of a method for manufacturing non-circular cams and gear blanks. Also, a method of forming non-circular gears by using standard formed-tooth milling cutters is described.

TABLE OF CONTENTS

I.	Introduction . . . . page	1
II.	Conditions for Rolling Curves	7
III.	Representation of the Motion Pattern	13
IV.	Examples of Design	18
V.	Rolling Ellipses, an Example of Analysis	25
VI.	General Equation for Synthesis	31
VII.	Properties of the General Equation	38
VIII.	Manufacture	50
	Appendix	57

I. INTRODUCTION

One of the problems in the kinematics of machinery is producing cyclic variations in the angular velocity of a shaft. Rolling curves offer an interesting solution of this type of problem.

The use and study of rolling curves is not new. The first known design is credited to Leonardo da Vinci.[1] Some of the mathematical properties of rolling curves were stated and proved by Euler.[2,3]

However, it has been only recently that the problems of design and manufacture have been solved to an extent that enable non-circular cams and gears to be produced accurately and fairly economically.[4] This is especially true of non-circular gears.

This thesis presents a study of some of the aspects of rolling curves. More specifically, the purpose of this thesis is:

1. To develop the conditions and the equations for rolling curves,
2. To explore some specific examples of the design and analysis of rolling curves,

---

[1]Uno Olsson, Non-circular Cylindrical Gears, Acta Polytechnica, Mechanical Engineering Series, vol. 2, no. 10 (Stockholm: Esselte Aktiebolag, 1953), p. 1.

[2]Ibid., p. 7.

[3]Robert Willis, Principles of Mechanism, (2nd ed.; London: Longmans, Green, and Co., 1870), p. 62.

[4]"Non-circular" is the common spelling but "noncircular" is also used.

3. To develop a general equation for the synthesis of rolling curves, and

4. To outline methods for manufacturing non-circular cams and gears.

In any application, rolling curves are used as either non-circular gears or non-circular cams or, in some cases, a combination of non-circular gear and cam segments.

Rolling cylinders and circular gears are also rolling curves. However, it is generally more convenient to handle these separately rather than as a special case of rolling curves.

Rolling curves are used to produce a cyclic variation in the angular velocity of one of the shafts. Since they produce a variation in angular velocity, they also produce a variation in the mechanical advantage between the two shafts. This fact has been used in a number of cases, particularly in the earlier applications.

Probably the first design of rolling curves was a set of non-circular cogwheel segments which appear in the works of Leonardo da Vinci. These segments were apparently designed for use in the tensioning of crossbows, thus utilizing the variation in mechanical advantage.[5]

Another example of the use of the variation in mechanical advantage is Harfield's steering gear shown in Fig. (1). The gears are inserted between the steering wheel and the boat's rudder. As the rudder angle is increased the mechanical advantage is also increased.

[5] Olsson, op. cit., p. 1.

The change in mechanical advantage offsets the increased water forces on the rudder at the larger angles.[6]

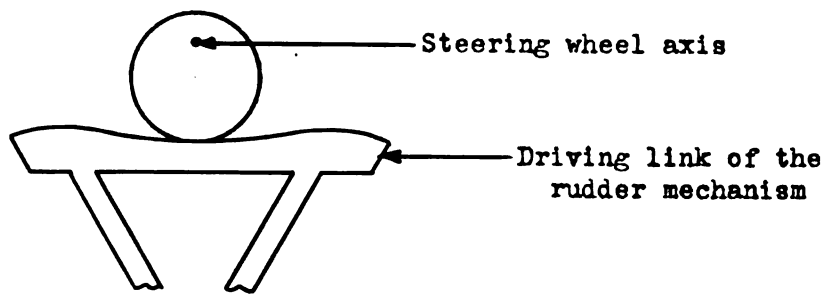


Figure 1. Harfield's steering gear

However, the main use of rolling curves is to produce a cyclic variation in the angular velocity of one shaft.

A variety of rolling curves has been developed for use in yarn and silk winding machinery. If thread is wound on a cone frustum or on any other solid of revolution with varying radius, the velocity of winding varies as the thread travels up and down the axis of revolution. Non-circular gears have been designed for controlling the angular velocity of the solid of revolution so that the thread is wound at a constant velocity.[7]

In recent years rolling curves have been widely used in multiplying mechanisms. These mechanisms are used in range finders as they are compact, quite accurate, and completely automatic.[8]

---

[6] S. Dunkerley, Mechanism, ed. by Arthur Morley (3rd ed.; London: Longmans, Green, and Co., 1912), p. 339.

[7] "Gearing for Yarn Winding Machinery," Engineering, XI (Jan. 6, 1871), p. 20.

[8] "The Works and Products of Messrs. Barr and Stroud, Limited," Engineering, CVIII (Dec. 12, 1919), pp. 778-79.

Rolling curves are used to produce a cyclic variation in angular velocity. Therefore, in the design of rolling curves there are basically two problems. First, the desired cyclic variation must be expressed mathematically. Second, the rolling curves must be designed to produce the specified variation in angular velocity.

Many authors, particularly in the field of mechanics of machinery, have preferred to design the rolling curves by graphical means. The general method is to assume one curve and to graphically plot the mating curve. This method is well adapted to drafting work but sacrifices both accuracy and the control of the cyclic variation pattern.

The mathematical conditions for rolling curves have been known for a long time. The primary condition was stated and proved by Euler.[9] Extending the mathematical conditions, it is possible to reduce the problems of rolling curve design to the problem of deriving the desired pattern of variation of the angular velocity from the physical conditions.

Once the equations for the required rolling curves have been found, the curves are used as either the surface of non-circular cams or the pitch lines of non-circular gears. The choice between the cams and the gears depends upon the particular rolling curves. If the radius of the driving curve is increasing, there is positive action and cam surfaces are used. If the radius is decreasing or constant, gear teeth must be used. In most applications it is

---

[9] Olsson, op. cit., p. 7.

possible to use a combination of gear and cam segments although usually gear teeth are used for the entire curve. Occasionally gear teeth are not used and the cams are held together by a continuous flexible connector which is wound around them.[10]

It is imperative that this discussion be limited to the most common uses of rolling curves. Therefore, the ensuing discussion of rolling curves has the following limitations:

1. The discussion is limited to plane rolling curves. These are by far the most common, but non-circular bevel gears also have been designed and manufactured.[11,12]
2. The discussion is limited to the case where one of the rolling curves rotates at a constant angular velocity. This limitation does not affect the usual application of rolling curves, but it eliminates such cases as gear trains. However, some of the discussion may easily be extended to the case where both curves have a varying angular velocity.[13]
3. The discussion is limited to the case where the ratio of the average angular velocities of a pair of mating rolling curves is unity. In other words one complete

---

[10]"Centre-turning Mobile Cranes," Engineering, 160 (Sept. 28, 1945), p. 247.

[11]Stillman W. Robinson, Principles of Mechanism, (New York: John Wiley and Sons, 1896), pp. 69-74.

[12]Olsson, op. cit., pp. 164-65.

[13]See Appendix B, pp. 63-64.

revolution of one curve produces one complete revolution of the other curve. Several rolling curve pairs have been designed where a complete revolution of one curve produces two or three revolutions of the mating curve; e. g., Harfield's steering gear. Also, non-circular rack and pinion mechanisms have been designed. However, the ratio of the average angular velocities equals unity in the usual application.

4. The discussion is limited to external rolling curves.
5. The discussion is limited to the case where the axes of the rolling curves are fixed. This limitation is generally included in the definition of rolling curves. However, the theory has been extended to non-circular planetary gears.[14]

---

[14]Olsson, op. cit., pp. 159-60.

II. CONDITIONS FOR ROLLING CURVES

The primary condition for rolling curves is that they satisfy the requirements for pure rolling. Or, in other words, they roll on each other and do not slide. Pure rolling imposes two mathematical requirements.

First, the point of contact of the rolling curves falls on the line joining the axes of rotation.[1] This requirement is expressed mathematically by considering that the point of contact is on both curves. This point is on the line joining the axes of rotation if the sum of the two radii to this point equals the fixed distance between the axes of rotation.

$$r_d + r_f = L \qquad (2.1)$$

where r_d = the radius of the driver curve

r_f = the radius of the follower curve

L = the constant center distance (distance between the axes of rotation)

Second, the rolling curves must satisfy the angular velocity ratio theorem for rolling contact. The angular velocity ratio of the driver and follower is inversely proportional to the contact radii.[2]

---

[1]Olsson, Non-circular Cylindrical Gears, p. 7.

[2]Rolland T. Hinkle, Kinematics of Mechanism, (New York: Prentice-Hall, Inc., 1953), p. 26.

This gives the second equation for rolling curves.

$$\frac{\omega_f}{\omega_d} = \frac{r_d}{r_f} \tag{2.2}$$

where ω_d = the angular velocity of the driver curve

ω_f = the angular velocity of the follower curve

The reasoning behind these two equations is based upon the velocities of the points of contact. Consider the point of contact as a pair of points, one on each curve. If the velocities of each of the two points are equal, there is no relative velocity between the two points and, therefore, no sliding between them. For the two velocities to be equal, they must be parallel (actually colinear) and numerically equal.

If the point of contact is on the line of centers, the two rolling curve radii are colinear. Since, in circular motion the velocity of a point is perpendicular to its radius, the two velocities are perpendicular to the same line and therefore parallel. The angular velocity ratio theorem assures that the velocities are numerically equal.

The pair of equations can also be derived by using instant centers of velocity. If there is no relative velocity between the two curves at the point of contact, the point of contact is the instant center for the two rolling curves. Since the two curves rotate about fixed points, Kennedy's theorem dictates that the point of contact lies on the line of centers. The angular velocity theorem for instant centers gives Equ. (2.2).[3]

---

[3]Hinkle, op. cit., pp. 33, 37.

Physical considerations demand that the rolling curves must satisfy two further conditions. First, the arc lengths between any two points of contact on continuous arc segments must be equal. Second, at any point of contact the values of the angles between the radius and the tangent to the two curves must be supplements. It can be shown that both of these conditions are met if the rolling curves satisfy the conditions for pure rolling.[4]

Therefore, the only requirements for rolling curves are Equs. (2.1) and (2.2).

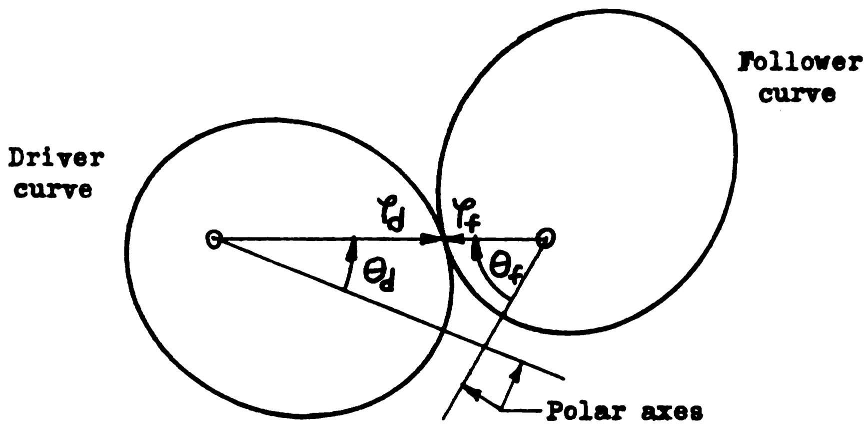


Figure 2. Sign convention for rolling curves

Using the condition that both rolling curves have external contact, a definite statement may be made about the directions of rotation. This condition dictates that the point of contact lies between the axes of rotation. When the point of contact is on the line of centers and lies between the centers, the driver and follower rotate in opposite directions.[5] This results in the sign convention

[4]See Appendix A, pp. 58–62.

[5]Hinkle, op. cit., p. 24.

shown in Fig. (2). The driver curve is assigned the usual angular sign convention used in polar coordinates; namely, a counterclockwise angular displacement is considered positive. The follower curve has the opposite sign convention in order to give positive angular values. A sign convention may be developed for angular velocities. However, it is more convenient to discuss the ratio of the angular velocities. This ratio is considered positive. Also, the condition of external contact for the rolling curves dictates that the radii values are always positive.

Using the equations for pure rolling, polar equations for both curves are derived which depend solely upon the desired motion pattern of the follower curve. The equations for the rolling curves may be expressed either as functions of time or of the angular displacement of the driver curve. The latter is the better choice for the special case of constant angular velocity of the driver curve. The main reason for this choice is that angular values are dimensionless.

Thus, the angular displacement of the follower curve is a function of the angular displacement of the driver curve.

$$\theta_f = f(\theta_d) \tag{2.3}$$

where θ_f = the angular displacement of the follower curve

θ_d = the angular displacement of the driver curve

The distinction between the driver and the follower curve depends upon Equ. (2.3). The terms "driver" and "follower" are artificial, though they do correspond to the usual application. Accordingly, in the following discussion the driver curve is the one which has a constant angular velocity.

The angular velocity of the follower curve is found by differentiating Equ. (2.3).

$$\frac{d\theta_f}{d\theta_d} = \frac{d}{d\theta_d} f(\theta_d) = f'(\theta_d) \qquad (2.4)$$

$$\frac{d\theta_f}{dt} \div \frac{d\theta_d}{dt} = f'(\theta_d)$$

$$\omega_f = f'(\theta_d) \cdot \omega_d \qquad (2.4a)$$

Equs. (2.2) and (2.4a) are combined.

$$\frac{\omega_f}{\omega_d} = \frac{r_d}{r_f} = f'(\theta_d) \qquad (2.5)$$

Equs. (2.1) and (2.5) are combined.

$$\frac{L - r_f}{r_f} = f'(\theta_d)$$

$$r_f = \frac{L}{f'(\theta_d) + 1} \qquad (2.6)$$

$$r_d = L - \frac{L}{f'(\theta_d) + 1} = \frac{L\, f'(\theta_d)}{f'(\theta_d) + 1} \qquad (2.7)$$

These equations have been derived previously in a similar manner.[6,7] The equations may also be derived by using Equ. (2.1)

---

[6] A. E. Lockenvitz, J. B. Oliphint, W. C. Wilde, and James M. Young, "Noncircular Cams and Gears," Machine Design, 24 (May, 1952), p. 142.

[7] H. E. Golber, Rollcurve Gears, Preprint of a speech presented on Dec. 6, 1938 to the Graphic Arts Section at the annual meeting of the A. S. M. E., p. 5.

and the condition that the arc lengths between any two points of contact must be equal.[8] Again, the equations may be derived by using Equ. (2.1) and the condition that the angles between the radii and the tangents to the rolling curves must be supplements.[9]

Using Equs. (2.6) and (2.7), the problem of designing rolling curves reduces to the problem of finding the desired motion pattern. The specified data of an application may be in either of two forms. First, it might be specified in the form of Equ. (2.3) or (2.4). In this case the rolling curve equations are found directly by using Equs. (2.6) and (2.7). Second, the motion pattern might only be specified at several points of the driver curve rotation. Here it is necessary to develop a satisfactory form of Equ. (2.3) or (2.4) which satisfies the prescribed data points.

---

[8]Olsson, op. cit., pp. 8-11.

[9]Julius Weisbach and Gustav Herrmann, The Mechanics of the Machinery of Transmission, Translated by J. F. Klein, Mechanics of Engineering and of Machinery, vol. III, part 1, sec. 1 (2nd ed.; New York: John Wiley and Sons, 1902), pp. 190-92.

III. REPRESENTATION OF THE MOTION PATTERN

The main problem in designing rolling curves is that of finding the motion pattern which satisfies the given application. This is especially true when the data is specified only at a few points of the driver curve rotation.

In constructing an artificial function to satisfy the given data points, it is convenient to study the motion pattern with the aid of graphs. There are two graphs which can be used. Both have the angular displacement of the driver curve as the abscissa. The two ordinates are the angular displacement of the follower curve, $f(\theta_d)$, and the ratio of the angular velocities, $\omega_f \div \omega_d$ or $f'(\theta_d)$.

In using either of these graphs, five factors must be considered from the viewpoints of how easily each may be studied on the graph and of how severely each restricts the formation of motion patterns. These five factors are:

1. The value of the angular displacement of the follower curve,
2. The value of the angular velocity of the follower curve,
3. The value of the angular acceleration of the follower curve,
4. The continuity of both curves, and
5. The smoothness of both curves or the continuity of $\frac{d\gamma}{d\theta}$ for both curves.

In discussing factors four and five it is assumed that the given data itself satisfies the continuity conditions. For discussing factor five, two equations are useful.

$$\frac{d\varphi_d}{d\theta_d} = \frac{L\,f''(\theta_d)}{[f'(\theta_d)+1]^2} \tag{3.1a}$$

$$\frac{d\varphi_f}{d\theta_f} = \frac{d\varphi_f}{d\theta_d}\cdot\frac{d\theta_d}{d\theta_f} = \frac{-L\,f''(\theta_d)}{[f'(\theta_d)+1]^2}\cdot\frac{1}{f'(\theta_d)} \tag{3.1b}$$

where $f''(\theta_d) = \dfrac{d}{d\theta_d}\,f'(\theta_d) = \dfrac{d^2}{d\theta_d^2}\,f(\theta_d)$

Fig. (3) represents a motion pattern plotted on the graph using the angular displacement of the driver curve as the abscissa and the ratio of the angular velocities of the follower and the driver curves as the ordinate. This graph is generally called a velocity diagram or a speedgraph (or "speedgraf").[1]

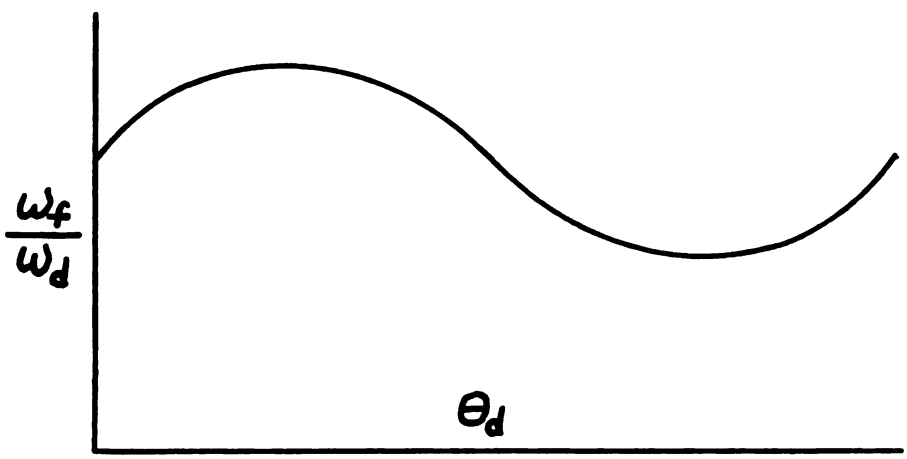


Figure 3. Velocity diagram or speedgraph

[1]Golber, Rollcurve Gears, p. 2.

Obviously, a speedgraph is very well suited for studying the value of the angular velocity of the follower curve. It is also well suited for studying the value of the angular acceleration of the follower curve, which is directly proportional to the slope of the speedgraph curve.

For the facility of studying the angular velocity and acceleration, the speedgraph sacrifices the ability to study angular displacements. The angular displacement of the follower curve is found by combining Equs. (2.3) and (2.5).

$$\theta_f = f(\theta_d)$$

$$f'(\theta_d) = \frac{\omega_f}{\omega_d}$$

$$\theta_f = \int f'(\theta_d)\, d\theta_d = \int \frac{\omega_f}{\omega_d}\, d\theta_d \qquad (3.2)$$

The right side of Equ. (3.2) is precisely the area under a speedgraph curve. Therefore, to control the values of the follower curve angular displacement it is necessary to control the area under the speedgraph curve. In practice this is fairly difficult.

It is specified that the follower curve must make exactly one revolution for each revolution of the driver curve.

$$\int_0^{2\pi} \frac{\omega_f}{\omega_d}\, d\theta_d = 2\pi \qquad (3.3)$$

This condition makes it difficult to choose a satisfactory speedgraph curve. In one application the area was computed to ten significant figures to insure accurate gears.[2]

From Equs. (2.6) and (2.7) the resulting rolling curves are continuous if the speedgraph curve is continuous. From Equs. (3.1a) and (3.1b) the rolling curves are smooth-- $d\varphi/d\theta$ is continuous--if both the speedgraph curve and its first derivative are continuous. If $d\varphi/d\theta$ is not continuous the rolling curves have sharp points or cusps. Such rolling curves have been used in practice but they are avoided if possible as the cusps cause a sudden change in the follower curve acceleration and thus prevent smooth operation.

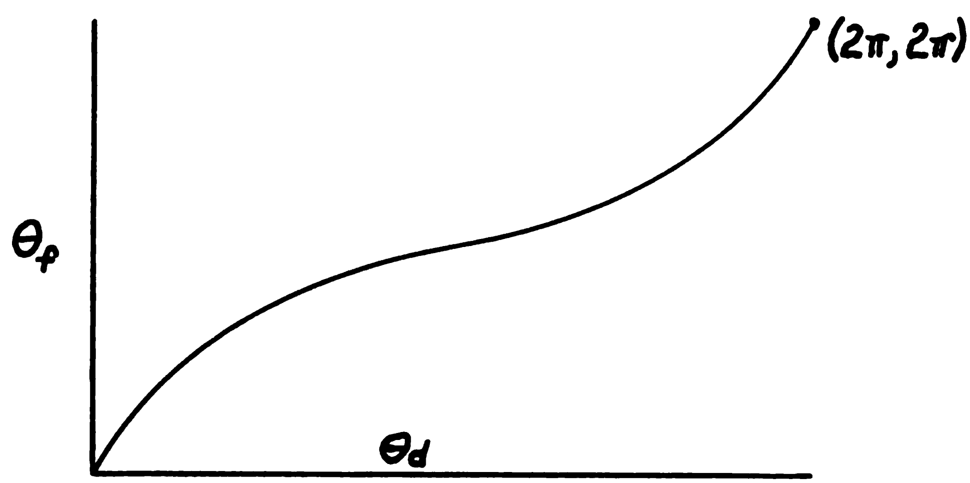


Figure 4. Displacement diagram

Fig. (4) represents a motion pattern plotted on the graph using the angular displacement of the driver curve as the abscissa and the angular displacement of the follower curve as the ordinate. This graph is generally called a displacement diagram.

---

[2]Golber, op. cit., p. 4.

The displacement diagram is very well suited for studying the value of the angular displacement of the follower curve. Also, it is well suited for studying the value of the angular velocity, which is directly proportional to the slope of the displacement diagram curve. However, the displacement diagram is not satisfactory for studying the angular acceleration of the follower curve because the acceleration is proportional to the second derivative of the displacement diagram curve.

From Equs. (2.6) and (2.7) the resulting rolling curves are continuous if the first derivative of the displacement diagram curve is continuous. Similarly, from Equs. (3.1a) and (3.1b) the rolling curves are smooth-- $d\varphi/d\theta$ is continuous--if both the first and the second derivatives of the displacement diagram curve is continuous.

IV. EXAMPLES OF DESIGN

Non-circular cams and gears have applications in various mechanical devices. This chapter develops the design of rolling curves as applied to several specific applications. No attempt is made to extensively cover any design features other than the rolling curves.

It is noted in these applications that the only problem is to express the desired motion pattern by a mathematical equation. The problem of finding the rolling curves from the motion pattern is easily met by using Equs. (2.6) and (2.7).

Sometimes Equ. (2.3) is expressed in parametric form. There is no need to eliminate the parameter as this form is satisfactory for computing purposes.

Scotch Yoke with Constant Velocity[1]

A Scotch yoke is generally used for obtaining a sinusoidal translation from a uniformly rotating shaft. It is possible to obtain a constant velocity translation by using rolling curves. In this case the Scotch yoke is driven by the follower curve.

The velocity of the Scotch yoke equals the parallel component of the velocity of a point on the follower curve.

---

[1]Willis, Principles of Mechanism, pp. 231-33. These rolling curves were originally designed to be used with a slider crank mechanism to give a constant velocity to the slider. Based upon Equ. (4.1) the application to the slider crank mechanism is theoretically incorrect although it is a good approximation for a high connecting rod to crank ratio (about 8.5:1 in this text). However, if the example is applied to a Scotch yoke, Equ. (4.1) is exact.

$$V = r\,\omega_f \sin\theta_f \tag{4.1}$$

where V = the constant velocity of the Scotch yoke

r = the length of the driving crank on the follower curve

$$f'(\theta_d) = \frac{\omega_f}{\omega_d} = \frac{d\theta_f}{d\theta_d} = \frac{V}{r\sin\theta_f}\cdot\frac{1}{\omega_d} = \frac{K}{\sin\theta_f}$$

where $K = \dfrac{V}{r\cdot\omega_d}$

$$\theta_d = \int \frac{1}{K}\sin\theta_f\cdot d\theta_f = \frac{-1}{K}\cos\theta_f + C$$

Boundary conditions: $\theta_d = 0$ when $\theta_f = 0$

$\theta_d = \pi$ when $\theta_f = \pi$

Therefore, $C = \dfrac{\pi}{2}$ and $K = \dfrac{2}{\pi}$

$$\theta_d = \frac{\pi}{2} - \frac{\pi}{2}\cos\theta_f = \frac{\pi}{2}\operatorname{versin}\theta_f$$

$$r_f = \frac{L}{f'(\theta_d)+1} = \frac{L\sin\theta_f}{2/\pi + \sin\theta_f}$$

$$r_d = L - r_f = \frac{2L}{2+\pi\sin\theta_f}$$

In one complete revolution of the follower curve the Scotch yoke travels in two directions. For the velocity to be linear for both directions, the yoke must have infinite accelerations at the dead center positions. In any application this is impossible. Therefore, the actual rolling curves only approximate the theoretical curves near the dead center positions.

The rolling curves must be used as non-circular gears. In this application the gears are designed as a pin and cog arrangement.

The rolling curve equations are based upon only one direction of yoke travel. Therefore, they are valid for only one half of a revolution. The second half of the rolling curves are easily obtained because in this case the rolling curves are symmetrical to the polar axis.

Non-circular Cams for Obtaining Linear Measurements[2]

In any continuous measurement it is advantageous to use an instrument which gives a linear record of the desired variable. Sometimes this is not possible. Either a linear instrument can not be used or it is easier to use a non-linear instrument. A couple of possibilities are:

1. Measuring flow by means of the differential pressure, and
2. Measuring temperature by means of the saturated vapor pressure by a specific liquid.

The non-linear variation of the instrument is changed to a linear variation by using rolling curves. This is actually a reverse application. Rolling curves are generally used to obtain a varying motion pattern from a linear rotation.

---

[2] F. W. Hannula, "Designing Noncircular Surfaces for Pure Rolling Contact," Machine Design, 23 (July, 1952), pp. 111-14.

In using the equations developed in Chapter II, it makes no difference which curve drives and which curve follows. The terms "driver" curve and "follower" curve are purely artificial. Since the driver curve generally rotates at a constant angular velocity, this distinction is used here.

For a specific example, consider a case where it is necessary to measure the temperature of a fluid by indirect means. One method of obtaining a linear measurement of temperature is:

1. To measure the saturated vapor pressure exerted by another fluid in a closed system subject to the fluid in question, and
2. To convert the pressure variation to a linear temperature variation.

The variation in pressure is converted to rotation of the follower curve (actually the driving shaft in this case) by means of a Bourdon tube or a similar mechanism. The shaft rotation must be linear with respect to the pressure.

$$\theta_f = KP$$

where P = the pressure in psi.

K = a constant

It is necessary that the relationship between the pressure and temperature in step one must be known. In this example

$$P = l e^{\frac{mt}{t+n}}$$

where t = the temperature in degrees Fahrenheit

Let $\ell = 10$, $m = 6$, and $n = 470$.

The driver shaft is to rotate linearly with respect to temperature.

$$\theta_d = K_1 t$$

$$\theta_f = K_2 P = K_2 \cdot 10 \cdot e^{\frac{6t}{t+470}}$$

Let $K_1 = 0.15$ and $K_2 = 0.91$.

The constants K_1 and K_2 are used to control the maximum variation in the angular rotations.

$$f'(\theta_d) = \frac{d\theta_f}{dt} \div \frac{d\theta_d}{dt} = \frac{171{,}174\, e^{\frac{6t}{t+470}}}{(t+470)^2}$$

The rolling curves are determined by using Equs. (2.6) and (2.7).

In this case they are used as non-circular cams. The cams are held together by gravity. The constants limit the maximum possible angular variation to forty-five and thirty degrees for the driver and follower respectively. Thus, it is not necessary to use gear teeth to keep the cams in contact during rotation in either direction.

Spiral Gears in a Multiplying Instrument[3]

Spiral gears are used in a multiplying instrument which has wide application in range finders. The instrument was used as a part of British naval range finders in World War I.[4] A similar

[3]The non-circular spiral gears should not be confused with spiral bevel gears.

[4]"The Works and Products of Messrs. Barr and Stroud, Limited," Engineering, pp. 778-79.

instrument is used in the United States Army T41 range finder.[5]

The motion pattern is a logarithmic function.

$$\theta_f = f(\theta_d) = a \log \theta_d + b \tag{4.2}$$

$$f'(\theta_d) = \frac{a}{\theta_d}$$

$$r_d = \frac{La}{\theta_d + a} \qquad r_f = \frac{L\theta_d}{\theta_d + a}$$

where a and b = constants

Three pairs of these rolling curves are used in the multiplying mechanism shown in Fig. (5). The values of the variables x and y linearly control the driver curve rotation. The two sets of spiral gears convert the linear rotation into a rotation which is a logarithmic function of the variable by Equ. (4.2). These logarithmic functions are added by a differential mechanism like rolling cones or bevel gears. This produces a shaft rotation which is a logarithmic function of the product of x and y. The logarithmic function is eliminated by using a reversed set of spiral gears; i. e., the "follower" gear drives the "driver." This produces a rotation which is the product of x and y.[6]

As an example of the use of spiral gears in range finders, consider the problem of finding the altitude of an approaching airplane.

---

[5]"Non-circular Gears Generated Automatically," Iron Age, 171 (Feb. 19, 1953), p. 123.

[6]Francis J. Murray, The Theory of Mathematical Machines, (rev. ed.; New York: King's Crown Press, 1948), Part II, p. 18.

The range finder gives the distance and the angle of inclination of the airplane. From trigonometry

$$h = r \sin \alpha \quad \text{AND}$$

$$\log h = \log r + \log \sin \alpha,$$

where h = the unknown altitude

r = the distance between the range finder and the airplane

α = the angle between r and the horizontal

The range finder produces two shaft rotations, one proportional to r and one proportional to the sine of α. It is generally necessary to use gear reductions to eliminate proportionality constants.[7]

This type of multiplying mechanism has several advantages. It is small. It is completely automatic. By using spiral gears, it has sufficient accuracy for fire control purposes.

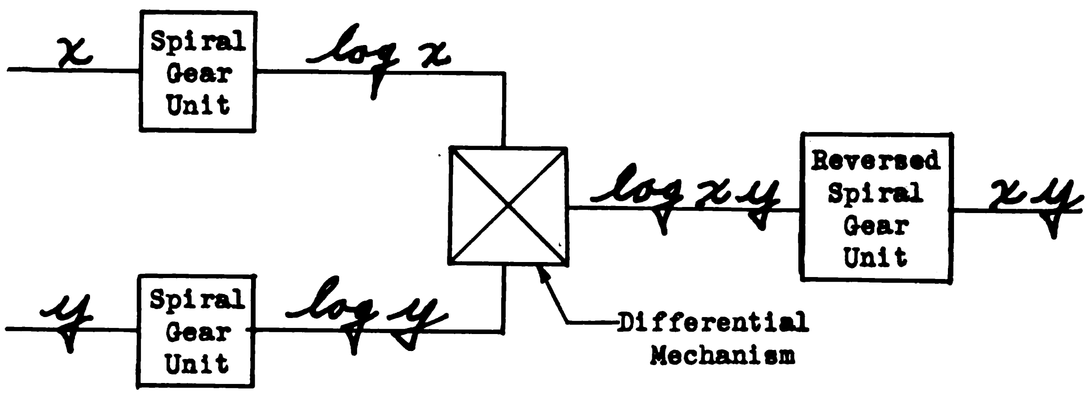


Figure 5. Spiral gears in a multiplying mechanism

[7] "The Works and Products of Messrs. Barr and Stroud, Limited," op. cit., pp. 778-79.

V. ROLLING ELLIPSES, AN EXAMPLE OF ANALYSIS

The majority of the literature on rolling curves and non-circular gears concerns elliptical gears. Elliptical gears have wide application as quick-return mechanisms for shapers, planers, and machine tools. Elliptical gears have the advantage of being efficient, positive, and comparatively cheap.[1]

A pair of identical ellipses which rotate about one of the foci satisfy the conditions for rolling curves if the center distance is equal to the major axis. Several other combinations of simple geometric curves satisfy the conditions for rolling curves. However, a pair of identical ellipses is the only combination where both curves are continuous and closed.

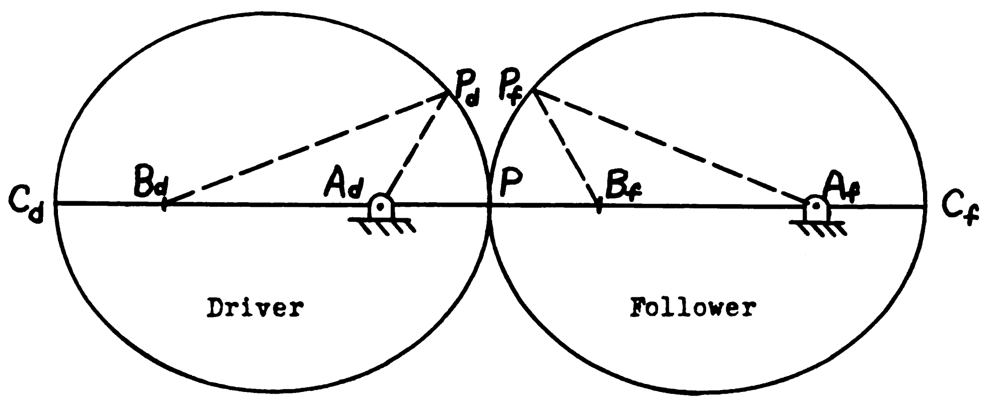


Figure 6. Rolling ellipses

[1]Reginald Trautschold, "Machine-cut Elliptical Gears. Laying Out and Machining Elliptic and Oval Gears," Machinery (New York), XXIII (Aug., 1917), p. 1049.

A pair of identical ellipses is shown in Fig. (6). The distance between the axes of rotation is A_dA_f. This distance is equal to the major axis of each ellipse because A_dP equals A_fC_f for identical ellipses. From the properties of an ellipse

$$A_d P_d + P_d B_d = C_d P \text{ AND}$$

$$A_f P_f + P_f B_f = C_f P.$$

Therefore,

$$A_d P_d + P_d B_d = A_f P_f + P_f B_f = A_d A_f. \tag{5.1}$$

The two points P_d and P_f are chosen such that

$$\text{ARC } P P_d = \text{ARC } P P_f .$$

Therefore, from the symmetry of the ellipses shown in Fig. (6)

$$P_d B_d = A_f P_f \text{ AND} \tag{5.2a}$$

$$A_d P_d = B_f P_f . \tag{5.2b}$$

Combine Equs. (5.1) and either (5.2a) or (5.2b).

$$A_d P_d + P_f A_f = A_d A_f \tag{5.3}$$

Equ. (5.3) is a special case of Equ. (2.1). Thus, the point of contact is on the line of centers and the curves satisfy the conditions of pure rolling.[2]

The question naturally arises, what motion pattern does a pair of rolling ellipses produce?

---

[2]Hinkle, Kinematics of Mechanism, p. 129-30.

The equation of an ellipse in polar coordinates with the origin at a focus point is used.[3]

For the driver ellipse

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{5.4}$$

where a = half of the major axis

e = the eccentricity

This equation corresponds to the angular values shown in Fig. (7).

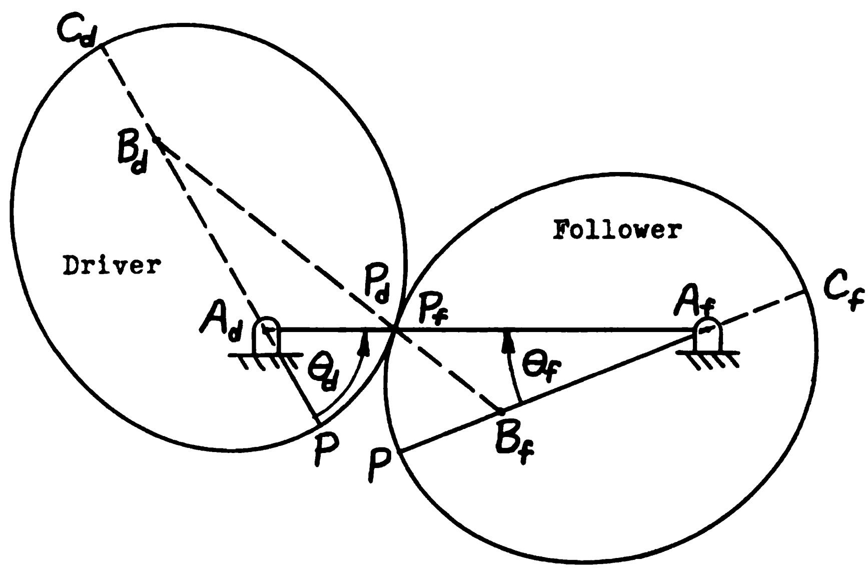


Figure 7. Sign convention for rolling ellipses

Combine Equs. (5.4) and (2.7) and solve for the ratio of angular velocities.

$$r_d = \frac{a(1-e^2)}{1+e\cos\theta_d}$$

[3]Olsson, Non-circular Cylindrical Gears, p. 27.

$$r_d = 2a - \frac{2a}{f'(\theta_d)+1}$$

where $2a = L$ = the major axis (the distance between the axes of rotation)

$$\frac{a(1-e^2)}{1+e\cos\theta_d} = 2a - \frac{2a}{f'(\theta_d)+1}$$

$$f'(\theta_d) = \frac{\omega_f}{\omega_d} = \frac{1-e^2}{1+e^2+2e\cos\theta_d} \qquad (5.5)$$

One of the main considerations in an application of elliptical gears is the maximum and minimum values of the ratio of angular velocities. From geometrical considerations the maximum and minimum values are reciprocals. These values are found from Equ. (5.5).

$$\frac{\omega_f}{\omega_d}\begin{matrix}\text{MAX.}\\\text{MIN.}\end{matrix} = \frac{1-e^2}{1+e^2\pm 2e} = \frac{(1+e)(1-e)}{(1\pm e)^2}$$

$$n = \frac{1+e}{1-e} \qquad \frac{1}{n} = \frac{1-e}{1+e}$$

$$e = \frac{n-1}{n+1} \qquad (5.6)$$

where n = the maximum value of $\frac{\omega_f}{\omega_d}$

$\frac{1}{n}$ = the minimum value of $\frac{\omega_f}{\omega_d}$

Solve for the angular displacement of the follower ellipse by integration.[4]

$$\theta_f = f(\theta_d) = \tan^{-1}\left[\frac{(1-e^2)\sin\theta_d}{2e+(e^2+1)\cos\theta_d}\right] \quad (5.7)$$

Again consider the symmetry of the ellipses in Fig. (6). Using this symmetry

$$\angle A_d P_d B_d = \angle A_f P_f B_f. \quad (5.8)$$

From Equ. (5.8) the line $B_d B_f$ in Fig. (7) is a straight line containing the point of contact. Extend the discussion used to prove that the point of contact lies on the line of centers for rolling ellipses. Combine Equs. (5.1) and (5.2b).

$$B_f P_f + P_d B_d = A_d A_f \quad (5.9)$$

Therefore, it is possible to connect the moving foci with a rigid link without changing the motion pattern. Further, it is possible to replace the rolling ellipses with the linkage shown in Fig. (8). The lengths of the links correspond to the distances between the foci of the rolling ellipses.

$$\text{LINK } A_d B_d = \text{LINK } A_f B_f$$

$$\text{LINK } A_d A_f = \text{LINK } B_f B_d$$

The linkage is called the equivalent linkage for rolling ellipses.[5]

---

[4]Richard S. Burington, comp., Handbook of Mathematical Tables and Formulas, (2nd ed.; Sandusky, Ohio: Handbook Publishers, Inc., 1940), p. 74.

[5]Hinkle, op. cit., p. 130.

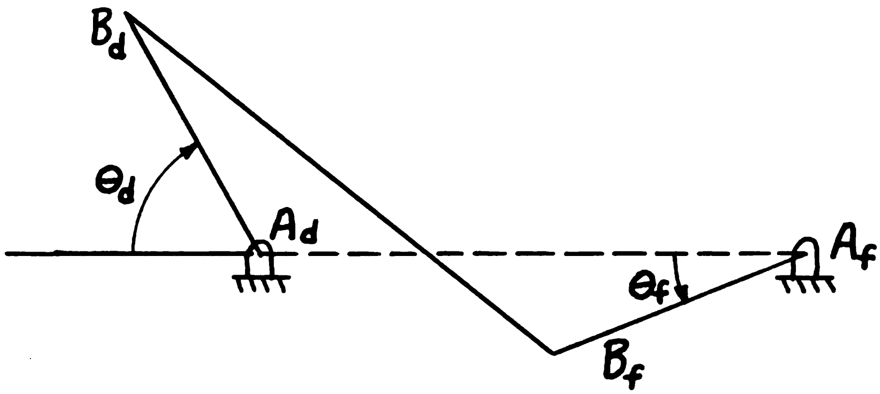


Figure 8. Equivalent linkage for rolling ellipses

A relationship between the angular displacements of the driver and follower ellipses is obtained from the equivalent linkage. The derivation is based entirely upon trigonometric considerations.[6]

$$e\,(1+\sin\theta_d\cdot\sin\theta_f-\cos\theta_d\cdot\cos\theta_f) \\ -\cos\theta_f+\cos\theta_d=0 \tag{5.10}$$

where $e = A_d B_d \div A_d A_f$

The definition of eccentricity corresponds to that for the ellipse.

It can be proved that Equ. (5.10) is identical to Equ. (5.7).[7]

Continuing the analysis of the equivalent linkage, a relationship is obtained for the ratio of angular velocities. This derivation uses Equ. (2.2) (the angular velocity ratio theorem) and trigonometric considerations. The resulting expression is identical to Equ. (5.5).[8]

---

[6]See Appendix C, pp. 65-67.

[7]See Appendix C, pp. 67-70.

[8]See Appendix C, pp. 70-72.

VI. GENERAL EQUATION FOR SYNTHESIS

In an application where data is given for several positions of the rolling curves, it is necessary to connect these positions with a proper curve which can be used to design the complete rolling curves. This has been done by plotting the data upon a speedgraph and connecting these points by a series of algebraic curve segments.[1] Another possibility is to derive a general analytical expression for connecting given data points. The interval between each pair of points can be handled by a separate expression which, except for its end points, is independent of the expressions for all the other intervals.

For developing an analytical expression, it is easiest to work with the displacement diagram. The main reason for this choice is that it eliminates any conditions for the integral of the general equation. Also, it is possible to directly control the angular displacement of the follower curve.

The given data consists of three values at each data point: the driver curve displacement, the follower curve displacement, and the ratio of the angular velocities of the follower and the driver curves (which equals the slope of the displacement diagram curve). Referring to the requirements for curves on the displacement diagram, the general equation for an interval must meet the following conditions:

---

[1]Golber, Rollcurve Gears, p. 4.

1. It must connect two given end points,
2. It must have specific slopes at each of the end points,
3. It must have the same second derivatives as the adjacent curves at its end points, and
4. The follower curve accelerations should be as small as possible.

The fourth condition is general and is the main criterion in choosing between possibilities which meet the other three conditions.

It is very desirable to have the curve for each interval completely independent of the curves for adjacent intervals, except for the given data at the end points. This can be done by replacing the third condition by specifying a particular value for the second derivative at the end points. The best choice is to have the second derivative equal zero at the end points. Any other choice would eliminate the special case where two points might be connected by a straight line, the case which gives circular gear segments.

Thus, the general formula must meet the following conditions:

1. It must connect two given points,
2. It must have specific slopes at each of the end points,
3. Its second derivative must equal zero at the end points, and
4. The follower curve accelerations should be as small as possible.

The given data for the general case is shown in Fig. (9).

Condition one is easily met by a simple linear expression. Also, the second derivative of a linear expression is zero.

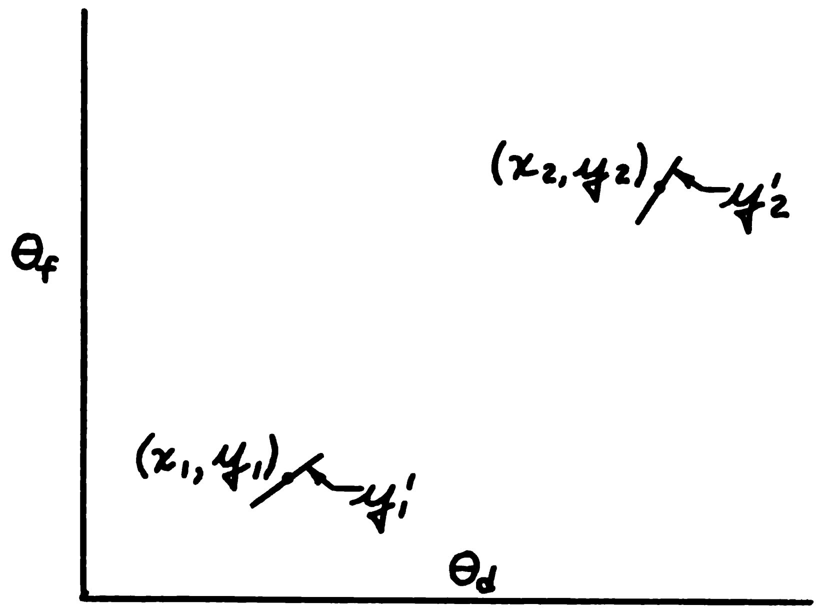


Figure 9. Displacement diagram with the given data

The general formula must be the sum of several expressions. Thus, the remaining expressions have the following properties at both end points: the value and the second derivative both equal zero and the first derivative does not equal zero. These conditions are met by any sine curve whose period or the integer multiple of that period is equal to twice the given interval. Since there are two slope constants that the curve must satisfy, the general expression contains two sine terms.

$$\theta_f = K_1 \cdot \sin m\pi \frac{\theta_d - x_1}{x_2 - x_1} + K_2 \cdot \sin n\pi \frac{\theta_d - x_1}{x_2 - x_1} + \frac{y_2 - y_1}{x_2 - x_1}(\theta_d - x_1) + y_1 \tag{6.1}$$

where K_1 and K_2 = constants which depend upon the given data

m and n = unequal integers

x_1 and x_2 = the value of the driver curve displacement at points one and two respectively

y_1 and y_2 = the value of the follower curve displacement at points one and two respectively

It is convenient to use a different set of coordinate axes for each interval as the curve for each interval is independent of all other intervals. By using a set whose axes are parallel to the original set but whose origin is at the given point one, Equ. (6.1) is simplified.

$$\theta_f = K_1 \sin\frac{m\pi\theta_d}{x} + K_2 \sin\frac{n\pi\theta_d}{x} + \frac{y}{x}\theta_d \qquad (6.1a)$$

where $x = x_2 - x_1$

$y = y_2 - y_1$

The first derivative of Equ. (6.1a) is used to determine the data constants K_1 and K_2.

$$\theta_f' = \frac{K_1 m\pi}{x} \cos\frac{m\pi\theta_d}{x} + \frac{K_2 n\pi}{x} \cos\frac{n\pi\theta_d}{x} + \frac{y}{x} \qquad (6.2)$$

It is apparent that m and n cannot be either both even or both odd. Therefore, let m be odd and n be even. The values of the data constants are found by substituting the given data at the end points into Equ. (6.2).

$$y_1' = \frac{K_1 m\pi}{x} + \frac{K_2 n\pi}{x} + \frac{y}{x} \qquad (6.2a)$$

$$y_2' = \frac{-K_1 m\pi}{x} + \frac{K_2 n\pi}{x} + \frac{y}{x} \tag{6.2b}$$

where y_1' and y_2' = the value of the ratio of the angular velocities of the follower and driver curves (also the slope on the displacement diagram) at points one and two

K_1 and K_2 are found by solving Equs. (6.2a) and (6.2b) simultaneously.

$$K_1 = \frac{x}{2m\pi}(y_1' - y_2') \tag{6.3a}$$

$$K_2 = \frac{x}{2n\pi}\left(y_1' + y_2' - \frac{2y}{x}\right) \tag{6.3b}$$

The angular acceleration of the follower curve should be as small as possible. Condition four is used to find the best values for the integers m and n. The angular acceleration of the follower curve is the second derivative of Equ. (6.1a) with respect to time.

$$\alpha_f = -\left(\frac{\pi \omega_d}{x}\right)^2\left(K_1 \cdot m^2 \sin\frac{m\pi}{x}\theta_d + K_2 \cdot n^2 \sin\frac{n\pi}{x}\theta_d\right) \tag{6.4}$$

$$\alpha_f = \frac{-\pi\omega_d^2}{2x}\left[m(y_1' - y_2')\sin\frac{m\pi}{x}\theta_d \right. \tag{6.4a}$$

$$\left. + \left(y_1' + y_2' - \frac{2y}{x}\right) n \sin\frac{n\pi}{x}\theta_d\right]$$

The integers appear both within the sine terms and as linear factors in Equ. (6.4a). Thus, for minimum follower curve accelerations the values of m and n should be as small as possible. Let m equal one and n equal two. The general equation can now be written in definite form as the terms depend only upon the given data.

$$\theta_f = K_1 \sin\frac{\pi}{\chi}\theta_d + K_2 \sin\frac{2\pi}{\chi}\theta_d + \frac{\psi}{\chi}\theta_d \tag{6.5}$$

$$K_1 = \frac{\chi}{2\pi}\left(\psi_1' - \psi_2'\right) \tag{6.5a}$$

$$K_2 = \frac{\chi}{4\pi}\left(\psi_1' + \psi_2' - \frac{2\psi}{\chi}\right) \tag{6.5b}$$

The general equations for the angular velocity and angular acceleration of the follower curve are written in their final form.

$$\frac{\omega_f}{\omega_d} = K_1\frac{\pi}{\chi}\cos\frac{\pi}{\chi}\theta_d + 2K_2\frac{\pi}{\chi}\cos\frac{2\pi}{\chi}\theta_d + \frac{\psi}{\chi} \tag{6.6}$$

$$\alpha_f = -\left(\frac{\pi}{\chi}\omega_d\right)^2\left(K_1 \sin\frac{\pi}{\chi}\theta_d + 4K_2 \sin\frac{2\pi}{\chi}\theta_d\right) \tag{6.7}$$

Referring to Equs. (2.6) and (2.7), the general equations for the rolling curves are written.

$$r_f = \frac{L}{\frac{\pi}{\chi}\left(K_1\cos\frac{\pi}{\chi}\theta_d + 2K_2\cos\frac{2\pi}{\chi}\theta_d\right) + \frac{\psi}{\chi} + 1} \tag{6.8a}$$

$$\varphi_d = L - \frac{L}{\frac{\pi}{\chi}\left(K_1 \cos\frac{\pi}{\chi}\theta_d + 2K_2 \cos\frac{2\pi}{\chi}\theta_d\right) + \frac{\mu}{\chi} + 1} \qquad (6.8b)$$

It should be remembered that all of the angular values used in this chapter are in radian units. In applications it is generally convenient to convert to degree values.

VII. PROPERTIES OF THE GENERAL EQUATION

Special Cases of the General Equation

The conditions for either of the constants to equal zero are found from Equs. (6.5a) and (6.5b).

$$K_1 = 0 \quad \text{IF} \quad y_1' = y_2'$$

$$K_2 = 0 \quad \text{IF} \quad y_1' + y_2' = 2\frac{y}{x}$$

If both of these conditions are met, the displacement curve is linear. This case gives circular rolling curve segments.

Equations of the General Equation and Its Derivatives

In the design of rolling curves it is important to know the arc length, the angle between the radius and the tangent, and several other properties. These properties may be found for the general equation by substitution into the calculus formulas. However, for the purposes of computation the resulting equations are burdensome. It is generally easier to compute the value of the general equation and its derivatives as intermediate values. These values are then substituted into the necessary equations. This method is especially advantageous because the intermediate values appear in several equations.

$$f(\theta_d) = K_1 \sin\frac{\pi}{x}\theta_d + K_2 \sin\frac{2\pi}{x}\theta_d + \frac{y}{x}\theta_d \tag{7.1a}$$

$$f'(\theta_d) = \frac{\pi}{\chi} K_1 \cos\frac{\pi}{\chi}\theta_d + \frac{2\pi}{\chi} K_2 \cos\frac{2\pi}{\chi}\theta_d + \frac{\psi}{\chi} \qquad (7.1b)$$

$$f''(\theta_d) = -\left(\frac{\pi}{\chi}\right)^2 \left(K_1 \sin\frac{\pi}{\chi}\theta_d + 4K_2 \sin\frac{2\pi}{\chi}\theta_d\right) \qquad (7.1c)$$

$$f'''(\theta_d) = -\left(\frac{\pi}{\chi}\right)^3 \left(K_1 \cos\frac{\pi}{\chi}\theta_d + 8K_2 \cos\frac{2\pi}{\chi}\theta_d\right) \qquad (7.1d)$$

It is noted that the above equations have mainly sine and cosine terms. With a proper choice of values of the driver curve displacement the computing work is cut in half.

Arc Length of the General Equations for Rolling Curves

In the design of non-circular cams and gears it is desirable to know the arc length of the rolling curves. This is especially true in the design and manufacture of non-circular gears.

Since rolling curves satisfy the conditions for pure rolling, the length of arc segments is the same for both curves. The equation for the arc length is obtained by using either Equ. (2.6) or (2.7) and the expression for the arc length of a curve in polar coordinates.[1]

$$S = L\int_{\alpha}^{\beta}\left\{\frac{[f'(\theta_d)]^2[f'(\theta_d)+1]^2 + [f''(\theta_d)]^2}{[f'(\theta_d)+1]^4}\right\}^{1/2} d\theta_d \qquad (7.2)$$

where S = the arc length of the curve between α and β

Substitute into Equ. (7.2) to find the arc length of the rolling curves of the general equation.

---

[1]See Appendix A, pp. 58–60.

$$S = L\int_0^{\chi}\left[\frac{\left(K_1\frac{\pi}{\chi}\cos\frac{\pi}{\chi}\theta_d + 2K_2\frac{\pi}{\chi}\cos\frac{2\pi}{\chi}\theta_d + \frac{\psi}{\chi}\right)^2}{\left(K_1\frac{\pi}{\chi}\cos\frac{\pi}{\chi}\theta_d + 2K_2\frac{\pi}{\chi}\cos\frac{2\pi}{\chi}\theta_d + \frac{\psi}{\chi} + 1\right)^2} + \frac{\left(K_1\frac{\pi^2}{\chi^2}\sin\frac{\pi}{\chi}\theta_d + 4K_2\frac{\pi^2}{\chi^2}\sin\frac{2\pi}{\chi}\theta_d\right)^2}{\left(K_1\frac{\pi}{\chi}\cos\frac{\pi}{\chi}\theta_d + 2K_2\frac{\pi}{\chi}\cos\frac{2\pi}{\chi}\theta_d + \frac{\psi}{\chi} + 1\right)^4}\right]^{1/2} d\theta_d \tag{7.3}$$

Since the rolling curves of the general equation correspond to particular intervals only, the arc length of each interval must be computed separately.

Equ. (7.3) must be computed by using the trapezoidal rule or Simpson's rule.

Angles Between the Radius and Both the Tangent and Normal to the Rolling Curves of the General Equation

The expressions for the angles between the radius and both the tangent and normal to the rolling curves are used in the manufacture of both non-circular cams and gears. The angle between the radius and the normal is especially important in producing non-circular gears. This angle is used in adding the addendum to the rolling curve to find the size of the gear blank. Also, the center lines of the gear teeth are normals to the pitch line.

These angles are found by using the appropriate calculus formulas in polar coordinates.

The angle between the radius and the tangent may be computed for either the driver or the follower rolling curve. Since these values are supplements, it is only necessary to compute one of them.

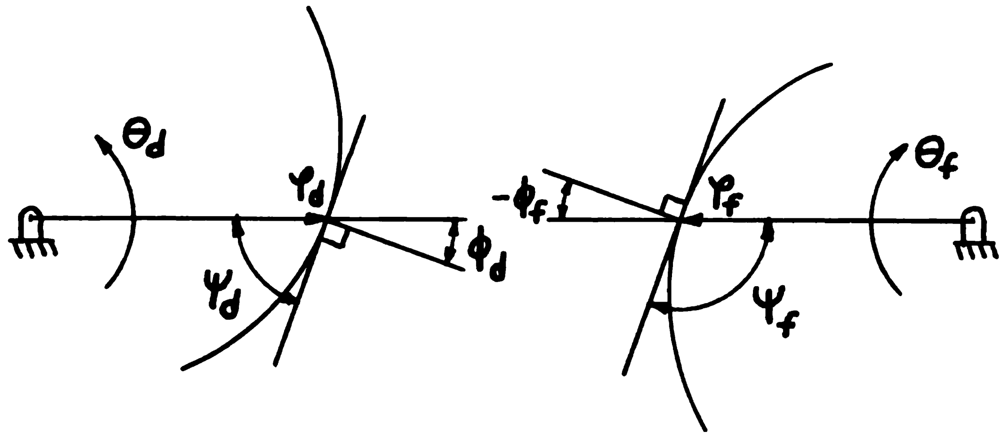


Figure 10. Angles between the radius and both the tangent and normal

$$\tan \psi_d = \frac{f'(\theta_d)}{f''(\theta_d)}\left[f'(\theta_d)+1\right] \qquad (7.4)^{2}$$

where ψ_d = the angle between the radius and the tangent to the driver curve as shown in Fig. (10)

$$\tan \psi_d = \frac{-\left(K_1\frac{\pi}{\chi}\cos\frac{\pi}{\chi}\theta_d + 2K_2\frac{\pi}{\chi}\cos\frac{2\pi}{\chi}\theta_d + \frac{\psi}{\chi}\right) \times \left(K_1\frac{\pi}{\chi}\cos\frac{\pi}{\chi}\theta_d + 2K_2\frac{\pi}{\chi}\cos\frac{2\pi}{\chi}\theta_d + \frac{\psi}{\chi} + 1\right)}{\frac{\pi}{\chi}\left(K_1\frac{\pi}{\chi}\sin\frac{\pi}{\chi}\theta_d + 4K_2\frac{\pi}{\chi}\sin\frac{2\pi}{\chi}\theta_d\right)} \qquad (7.5)$$

$$\psi_f = 180^\circ - \psi_d^\circ \qquad (7.6)$$

where ψ_f = the angle between the radius and the tangent to the follower curve as shown in Fig. (10) in degree units

The value of the angle between the radius and the normal to the rolling curve is found from Fig. (10).

---

[2]See Appendix A, pp. 61-62.

$$\phi_d = 90^\circ - \psi_d^\circ \tag{7.7a}$$

$$\phi_f = 90^\circ - \psi_f^\circ = \psi_d^\circ - 90^\circ \tag{7.7b}$$

where ϕ_d = the angle between the radius and the normal to the driver curve as shown in Fig. (10) in degree units

ϕ_f = the angle between the radius and the normal to the follower curve as shown in Fig. (10) in degree units

Angular Acceleration of the Follower Curve

A previous method for obtaining rolling curves from given data consists of plotting the given data upon the speedgraph and connecting the given points by a combination of simple algebraic curves.[3] A main advantage of this method is that it provides a method for controlling the angular accelerations of the follower curve.

In working with a general analytical equation, the control of angular accelerations is lost. In the case where the complete data is specified for all the given points, the value of the acceleration follows directly from the general equation and there is no possibility of changing it. However, in the case where the specified data is incomplete, the value of the angular acceleration of the follower curve is undetermined. Thus, the maximum angular acceleration of the follower shaft furnishes a criterion for determining the incomplete values.

---

[3]Golber, Rollcurve Gears, p. 2.

For example, consider the case where the given data is complete except for the follower curve position at both end points of an interval. In order to find the general equations for the interval, it is first necessary to specify the follower curve displacement increment. The best choices are the positions which give the smallest possible maximum angular acceleration of the follower curve. Since the general equation has two sinusoidal terms, this condition occurs when the maximum angular accelerations for all intervals are numerically equal.

In any problem there are four values which determine the general equation for an interval. These are: the slope constants (ratios of angular velocities), ψ_1' and ψ_2'; the change in driver curve displacement, χ ; and the change in follower curve displacement, ψ. Thus, four types of problems are possible depending upon which factor is not specified. Generally, the follower curve increment, ψ, is the adjustable factor. This type of problem corresponds to the speedgraph method for synthesis. The acceleration expression is developed for this type of problem, but the equations may be used in solving the other types of problems.

The angular acceleration of the follower curve is given by Equ. (6.7).

$$\alpha_f = -\left(\frac{\omega_d \cdot \pi}{\chi}\right)^2 \left(K_1 \sin\frac{\pi}{\chi}\theta_d + 4K_2 \sin\frac{2\pi}{\chi}\theta_d\right) \qquad (7.8)$$

$$\alpha_f = \omega_d^2 \frac{\pi}{2\chi}\left[-(\psi_1' - \psi_2')\sin\frac{\pi}{\chi}\theta_d - 4\left(\frac{\psi_1'+\psi_2'}{2} - m\right)\sin\frac{2\pi}{\chi}\theta_d\right] \qquad (7.8a)$$

where $m = \frac{y}{x}$

Assuming the type of problem where the follower increment is not specified, x is given. Therefore, Equ. (7.8a) may be written

$$\alpha_f = \frac{\pi}{2x} \omega_d^2 \cdot F. \qquad (7.9)$$

$$F = \frac{2\pi}{x}\left[-K_1 \sin\frac{\pi}{x}\theta_d - 4K_2 \sin\frac{2\pi}{x}\theta_d\right] \qquad (7.9a)$$

$$F = -(y_1' - y_2')\sin\frac{\pi}{x}\theta_d - 4\left(\frac{y_1' + y_2'}{2} - m\right)\sin\frac{2\pi}{x}\theta_d \qquad (7.9b)$$

The maximum angular acceleration of the follower curve occurs where the first derivative of the factor F equals zero.

$$\frac{dF}{d\theta_d} = 0 = \frac{-2\pi^2}{x^2}\left[K_1 \cos\frac{\pi}{x}\theta_d + 8K_2\left(2\cos^2\frac{\pi}{x}\theta_d - 1\right)\right]$$

$$16K_2 \cos^2\frac{\pi}{x}\theta_d + K_1 \cos\frac{\pi}{x}\theta_d - 8K_2 = 0 \qquad (7.10)$$

Solve for the location of the maximum follower curve acceleration by using the formula for the roots of a quadratic equation.

$$\cos\beta = \frac{-K_1}{32K_2} \pm \frac{\sqrt{K_1^2 + 512K_2^2}}{32K_2} \qquad (7.11a)$$

$$\cos\beta = \frac{-(y_1' - y_2') \pm \sqrt{(y_1' - y_2')^2 + 512\left[\frac{1}{2}(y_1' + y_2') - m\right]^2}}{32\left[\frac{1}{2}(y_1' + y_2') - m\right]} \qquad (7.11b)$$

where β = the value of $\frac{\pi}{\chi}\theta_d$ which gives the maximum value of the follower curve acceleration

Equ. (7.11b) gives a pair of values for the location of the maximum follower curve acceleration. The value which gives the largest numerical value of the factor F is used. Table I shows the location of these values and the sign of the maximum value of the factor F for various values of the data constants.

TABLE I. LOCATION AND SIGN OF THE MAXIMUM ACCELERATION

K_1 AND $\psi_1' - \psi_2'$	K_2 AND $\frac{1}{2}(\psi_1' + \psi_2') - m$	β	F AND α_f
+	+	45°-90°	−
+	−	90°-135°	−
−	+	90°-135°	+
−	−	45°-90°	+

Figs. (11) and (12) show the location of the maximum follower curve acceleration and the maximum value of the factor F for particular values of ψ_1' and ψ_2' and for various values of the ratio m. The location curve shows the predominance of the accelerations of the double angle term over those of the single angle term. This is also apparent in Equ. (7.11b).

The process of computing the maximum value of the factor F for a satisfactory range of data constants becomes very involved. It is therefore necessary to use an approximation. Several are possible.

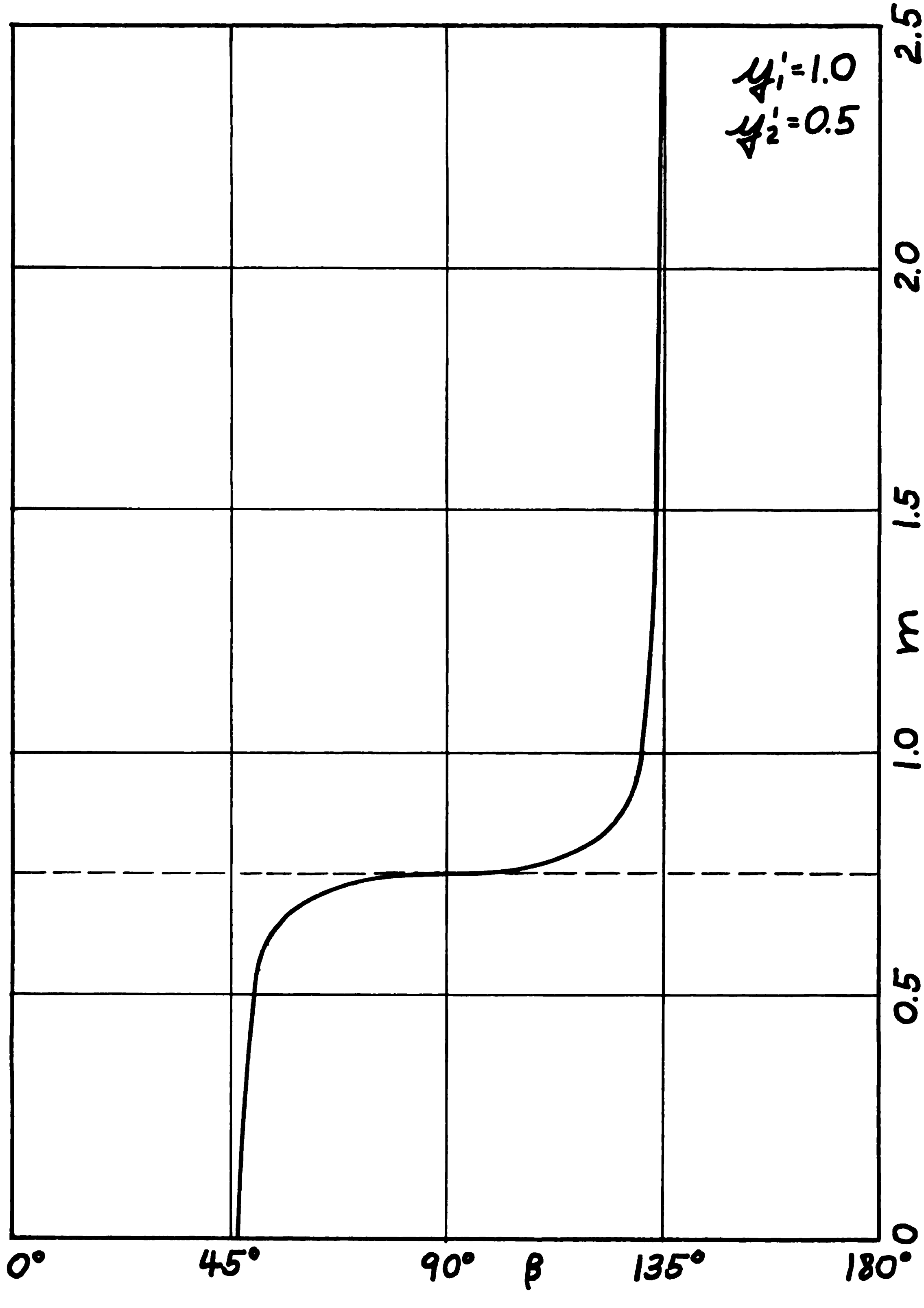


Figure 11. Location of the maximum follower curve acceleration versus the ratio m for particular values of y_1' and y_2'

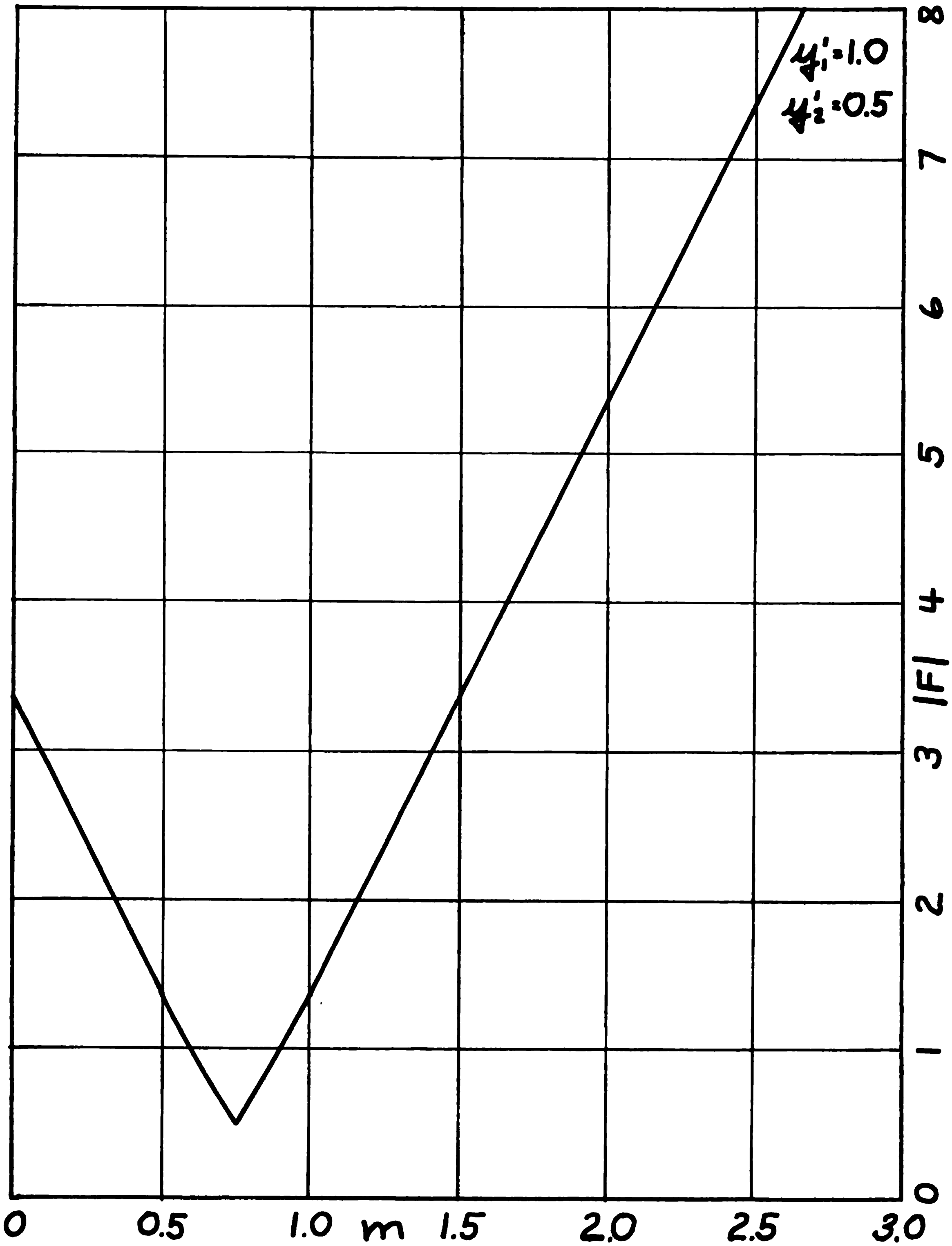


Figure 12. Maximum value of the factor F versus the ratio m for particular values of ψ_1' and ψ_2'

It is observed that the curve of Fig. (12) has a pair of asymptotes.

$$\left[|F| - \frac{\sqrt{2}}{2}(\psi_1' - \psi_2')\right] = \pm 4\left[m - \frac{1}{2}(\psi_1' + \psi_2')\right] \quad (7.12)$$

One approximation is an hyperbola whose axis is parallel to the F-axis, whose asymptotes are Equ. (7.12), and which passes through the minimum point.

$$\frac{\left[|F| - \frac{\sqrt{2}}{2}(\psi_1' - \psi_2')\right]^2}{\left(1 - \frac{\sqrt{2}}{2}\right)^2(\psi_1' - \psi_2')^2} - \frac{\left[m - \frac{1}{2}(\psi_1' + \psi_2')\right]^2}{\frac{1}{16}\left(1 - \frac{\sqrt{2}}{2}\right)^2(\psi_1' - \psi_2')^2} = 1 \quad (7.13)$$

The analytical expression describes a pair of hyperbolas. One of these is extraneous. Solving for the factor F, the particular hyperbola used in the approximation is found.

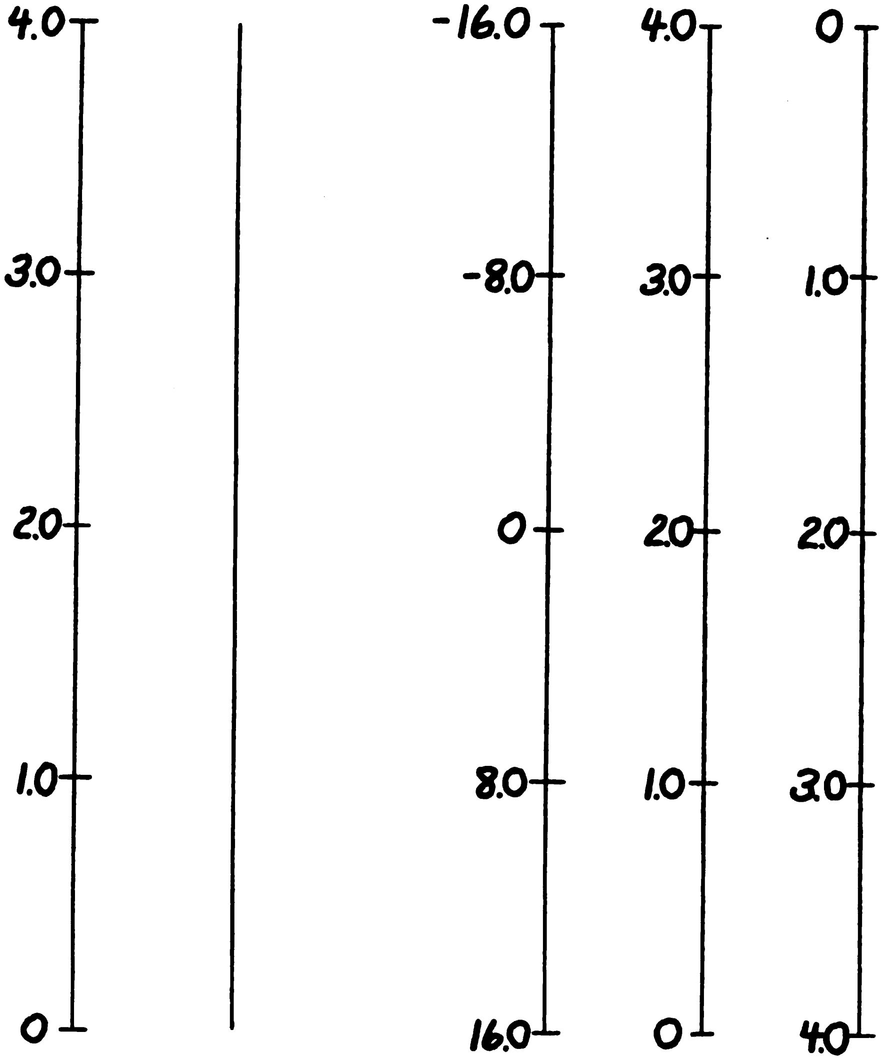
$$|F| = \left|\frac{\sqrt{2}}{2}(\psi_1' - \psi_2')\right| + \sqrt{\left(1 - \frac{\sqrt{2}}{2}\right)^2(\psi_1' - \psi_2')^2 + 16\left[m - \frac{1}{2}(\psi_1' + \psi_2')\right]^2}$$

Another expression is possible which enables a graphical determination of the factor F. As an approximation, consider the maximum value of F to be equal to the sum of the maximum values of both of the terms. The resulting vee shaped curve is easily represented by an alignment chart. The resulting maximum error is the difference between the approximation and the asymptotes.

$$E = \left|\left(1 - \frac{\sqrt{2}}{2}\right)\left[\frac{1}{2}(\psi_1' + \psi_2') - m\right]\right| \quad (7.14)$$

where E = the maximum possible error in the alignment chart shown in Fig. (13)

β	A	C		E	B	D
$<90^\circ$	y_1'			$+F$	y_2'	m
$>90^\circ$	y_2'			$-F$	y_1'	m



1. Use Table I to find the location of β.
2. Plot data on lines A, B, and D.
3. Use lines A and B to find the point on line C.
4. Use lines C and D to find the point on line E.

Figure 13. Alignment chart for an approximation of the angular acceleration of the follower curve

VIII. MANUFACTURE

Using the conditions for pure rolling, rolling curves are designed with theoretically perfect accuracy. Therefore, the accuracy of non-circular cams or gears depends entirely upon their manufacture from the specified equations.

In general there are two methods of manufacture, namely continuous cutting and increment cutting.[1] Increment cutting is best adapted for either small quantity production or production of masters for large quantity production. Only increment cutting is discussed here.

It should be remembered that the follower curve equations are not based upon the usual sign convention for polar coordinates. Any confusion on this point may be easily avoided by manufacturing the follower curve in the same manner as the driver curve and simply inverting the follower curve before use.

Manufacture of Non-circular Cams

The manufacture of non-circular cams is developed for increment cutting with a milling cutter. The theory may easily be adapted to other cutting tools.

Fig. (14) shows a method of milling a non-circular cam. The values used in the theory apply to either the driver or the follower cam and, therefore, the subscripts have been dropped.

---

[1]Lockenvitz, Oliphint, Wilde, and Young, "Noncircular Cams and Gears," p. 143.

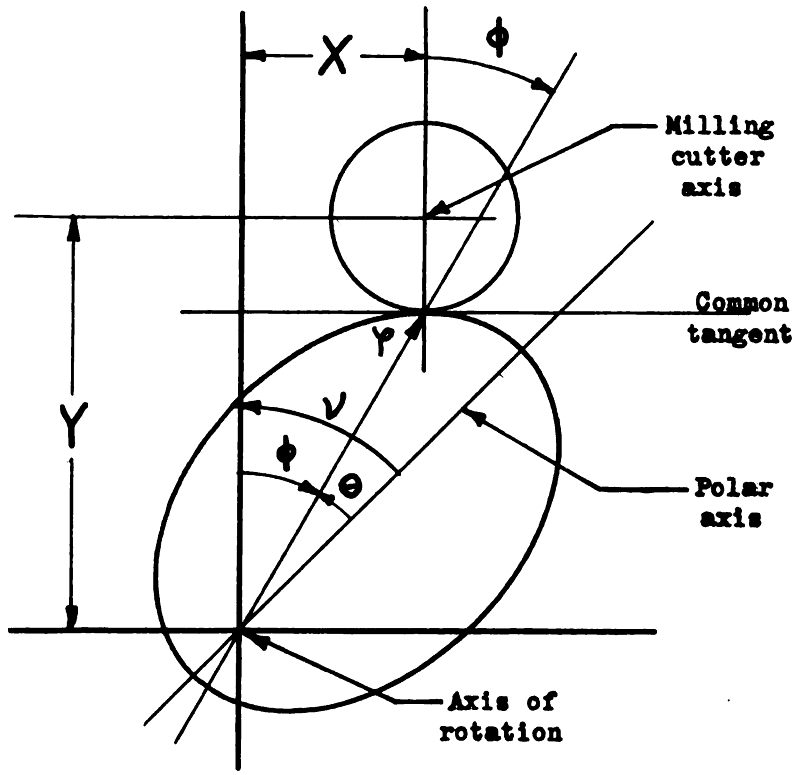


Figure 14. Machining non-circular cams with a milling cutter

The cams are easily cut if the location of the center line of the milling cutter is known as a function of the angular displacement of the driver curve. It is easiest to specify the milling cutter position by using Cartesian coordinates.

$$\tan \phi = \frac{d\rho}{d\theta} \div \rho \tag{8.1}$$

where ϕ = the angle between the normal and the radius as shown in Fig. (14)

$$\nu = \theta - \phi \tag{8.2}$$

where ν = the angle between the vertical and the polar axis as shown in Fig. (14)

$$X = -r \sin \phi \tag{8.3a}$$ [2,3]

$$Y = r \cos \phi + C \tag{8.3b}$$

where X and Y = the Cartesian coordinates of the center line of the milling cutter

C = the radius of the milling cutter

Manufacture of Blanks for Non-circular Gears

Blanks for non-circular gears are produced by the same method used for non-circular cams.

The specified rolling curve forms the pitch line of the non-circular gear. To find the gear blank it is necessary to add the addendum to the rolling curve. Since the center lines of the gear teeth are the normal lines, the addendum is added normal to the pitch line. The equations for the position of the milling cutter are similar to those for non-circular cams since the cutter radius is also measured along the normals.[4]

$$X = -r \sin \phi \tag{8.4a}$$

$$Y = r \cos \phi + C + A \tag{8.4b}$$

where A = the addendum

---

[2]Lockenvitz, Oliphint, Wilde, and Young, op. cit., p. 143.

[3]Olsson, Non-circular Cylindrical Gears, p. 132.

[4]Ibid., p. 128.

As with ordinary gears the addendum depends upon the diametral pitch and the tooth form.

Manufacture of Non-circular Gears

There are several methods of forming teeth in the non-circular gear blanks. The method described here uses formed-tooth milling cutters and is well adapted to small quantity production.

Involute teeth are used on non-circular gears. Theoretically the involutes are drawn from a non-circular base curve. However, the true involutes closely approximate circular involutes and ordinary formed-tooth milling cutters are used.

Since the radius of curvature is not constant, it is generally impossible to use the same formed cutter for the entire gear. This requires computing the radius of curvature at the center line of each tooth space and then specifying the proper formed cutter.

In Fig. (15) the formed cutter center lines are located by the same method used for non-circular cams. The settings of the blank must be found so that the teeth are evenly spaced. This requires that the distance between every tooth space center line be equal to the circular pitch.

$$P = \frac{S}{n} \qquad (8.5)$$

where P = the circular pitch

S = the total circumference of the pitch curve

n = the number of teeth

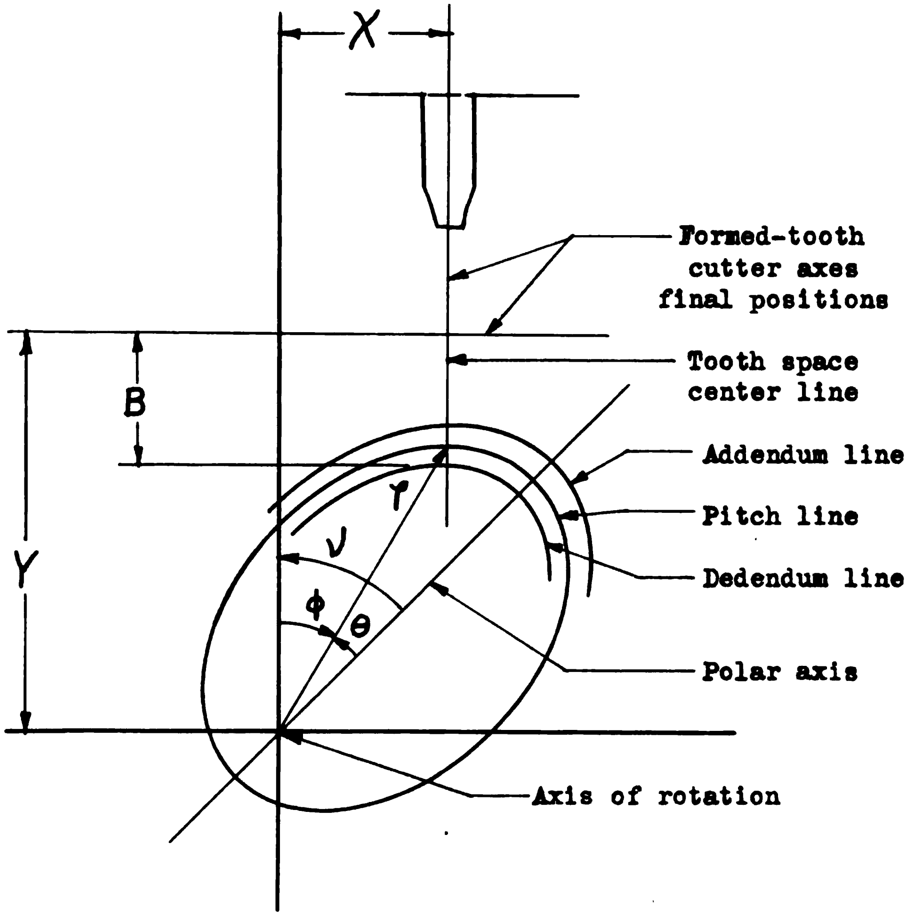


Figure 15. Cutting non-circular gears with a formed-tooth cutter

TABLE II. SELECTION OF STANDARD FORMED-TOOTH MILLING CUTTERS[5]

Value of Z	12	13	14	15	17	19	21	23
Cutter No.	8	7½	7	6½	6	5½	5	4½
Value of Z	26	30	35	42	55	80	135	
Cutter No.	4	3½	3	2½	2	1½	1	

[5]Olsson, op. cit., p. 134.

The first tooth space center line is located arbitrarily. This also locates a gear tooth and consequently all of the tooth space center lines on the mating gear.

Using Equ. (8.5), it is possible to find all the proper values of the angular displacement and accordingly the settings for the gear blanks.

Again, the cutter positions is located by Cartesian coordinates.

$$X = -r \cdot \sin\phi \qquad (8.6a)$$

$$Y = r\cos\phi + B - D \qquad (8.6b)$$

$$V = \theta - \phi \qquad (8.6c)$$

where B = the outside radius of the formed cutter

D = the dedendum

The specific formed cutter depends upon the radius of curvature and the diametral pitch.

$$Z = 2P_d \cdot R \qquad (8.7)$$

where Z = "The number of teeth for which the cutter is designed when milling cylindrical gears."[6]

P_d = the diametral pitch

R = the radius of curvature

To use Table II, round down the algebraic value of Z. The value of Z is negative for internal gears.

The radius of curvature is computed directly from the calculus.[7]

---

[6]Olsson, op. cit., p. 128.

[7]See Appendix D, pp. 73-75.

$$R_d = \frac{L}{[f'(\theta_d)+1]^3} \cdot \frac{\left\{[f'(\theta_d)]^2[f'(\theta_d)+1]^2 + [f'''(\theta_d)]^2\right\}^{3/2}}{[f'(\theta_d)]^2[f'(\theta_d)+1] + 2[f''(\theta_d)]^2 - f'(\theta_d)f'''(\theta_d)} \quad (8.8a)$$

$$R_f = \frac{L}{[f'(\theta_d)+1]^3} \cdot \frac{\left\{[f'(\theta_d)]^2[f'(\theta_d)+1]^2 + [f'''(\theta_d)]^2\right\}^{3/2}}{[f'(\theta_d)]^4 + [f'(\theta_d)]^3 - [f''(\theta_d)]^2 + f'(\theta_d)f'''(\theta_d)} \quad (8.8b)$$

where R_d = the radius of curvature of the driver curve

R_f = the radius of curvature of the follower curve

For the purposes of selecting a formed cutter, compute the radius of curvature at the midpoint of the tooth space.

Generally it is far simpler to use an approximation for computing the radius of curvature. The portion of the pitch circle between two tooth spaces is replaced by a circle with radius equal to the radius of curvature.[8]

$$P \approx \Delta\nu \cdot R$$

$$\frac{P}{\Delta\nu} = \frac{\pi}{P_d \cdot \Delta\nu} \approx R = \frac{Z}{2P_d}$$

$$Z \approx \frac{2\pi}{\Delta\nu} \quad (8.9)$$

where $\Delta\nu$ = the angular difference between two adjacent blank positions for cutting tooth spaces

---

[8]Olsson, op. cit., p. 134.

APPENDIX

A. Proof That the Arc Lengths Are Equal and the Angles Between the Radius and the Tangent Are Supplements for Any Pair of Rolling Curves . . . . page 58

B. Extension of the Rolling Curve Equations to the Case of a Driver Curve with a Varying Angular Velocity 63

C. Derivation of the Equations of Motion for Rolling Ellipses by the Use of the Equivalent Linkage and Proof that These Equations Are Identical to the Equations Obtained from the Theory of Rolling Curves 65

D. Derivation of the Equations for the Radius of Curvature of Rolling Curves 73

A. PROOF THAT THE ARC LENGTHS ARE EQUAL AND THE ANGLES BETWEEN THE RADIUS AND THE TANGENT ARE SUPPLEMENTS FOR ANY PAIR OF ROLLING CURVES

Physical considerations demand that a pair of rolling curves satisfy two conditions. First, the arc lengths of both curves between any two particular points of contact on continuous arc segments must be equal. Second, at any point of contact the values of the angle between the radius and the tangent for both curves must be supplements. It was stated in Chapter II that both of these conditions are met if the rolling curves satisfy the conditions for pure rolling.

Using only the conditions for pure rolling, the equations for the rolling curves are found.

$$r_f = \frac{L}{f'(\theta_d)+1} \tag{2.6}$$

$$r_d = \frac{L \cdot f'(\theta_d)}{f'(\theta_d)+1} \tag{2.7}$$

The arc length expression for a curve in polar coordinates is found by calculus.[1]

$$S = \int_{\alpha}^{\beta} \left[r^2 + \left(\frac{dr}{d\theta} \right)^2 \right]^{\frac{1}{2}} d\theta \tag{A.1}$$

where S = the arc length of the curve between α and β

[1]William A. Granville, Percey F. Smith, and William R. Longley, Elements of Calculus, (Boston: Ginn and Co., 1941), p. 292.

The proof that the arc lengths of a pair of rolling curves are equal consists of separately evaluating Equ. (A.1) for both Equs. (2.6) and (2.7) and then showing that the expressions for the two arc lengths are identical.

Let two points of contact on continuous arcs of a pair of rolling curves be:

Point 1: $\theta_d = \alpha$ AND

Point 2: $\theta_d = \beta$.

$$\frac{d\gamma_d}{d\theta_d} = \frac{L f''(\theta_d)}{[f'(\theta_d)+1]^2} \tag{2.7a}$$

$$S_d = \int_\alpha^\beta \left\{ \left[\frac{L f'(\theta_d)}{f'(\theta_d)+1}\right]^2 + \left[\frac{L f''(\theta_d)}{[f'(\theta_d)+1]^2}\right]^2 \right\}^{\frac{1}{2}} d\theta_d$$

$$S_d = L \int_\alpha^\beta \left\{ \frac{[f'(\theta_d)]^2}{[f'(\theta_d)+1]^2} + \frac{[f''(\theta_d)]^2}{[f'(\theta_d)+1]^4} \right\}^{\frac{1}{2}} d\theta_d \tag{A.2a}$$

where S_d = the arc length of the driver curve between α and β

$$S_f = \int_\alpha^\beta \left[\gamma_f^2 + \left(\frac{d\gamma_f}{d\theta_f}\right)^2\right]^{\frac{1}{2}} d\theta_f$$

where S_f = the arc length of the follower curve between α and β

$$\frac{d\gamma_f}{d\theta_f} = \frac{-L f''(\theta_d)}{[f'(\theta_d)+1]^2} \cdot \frac{1}{f'(\theta_d)} \tag{2.6a}$$

$$d\theta_f = f'(\theta_d) \cdot d\theta_d \tag{2.4}$$

$$S_f = \int_\alpha^\beta \left\{ \left[\frac{L}{f'(\theta_d)+1} \right]^2 + \left[\frac{L\, f''(\theta_d) \cdot 1/f'(\theta_d)}{[f'(\theta_d)+1]^2} \right]^2 \right\}^{1/2} f'(\theta_d)\, d\theta_d$$

$$S_f = L \int_\alpha^\beta \left\{ \frac{1}{[f'(\theta_d)+1]^2} + \frac{[f''(\theta_d)]^2 \left[1/f'(\theta_d) \right]^2}{[f'(\theta_d)+1]^4} \right\}^{1/2} f'(\theta_d)\, d\theta_d$$

$$S_f = L \int_\alpha^\beta \left\{ \frac{[f'(\theta_d)]^2}{[f'(\theta_d)+1]^2} + \frac{[f''(\theta_d)]^2}{[f'(\theta_d)+1]^4} \right\}^{1/2} d\theta_d \tag{A.2b}$$

Therefore, $S_d = S_f$.

It is noted that the arc length is directly proportional to the distance between the axes of rotation. This fact is used in applications of non-circular gears. The gears are designed for an arbitrary distance between the axes of revolution. This value is then adjusted to give a convenient value of the arc length, the circular pitch and, accordingly, the diametral pitch.

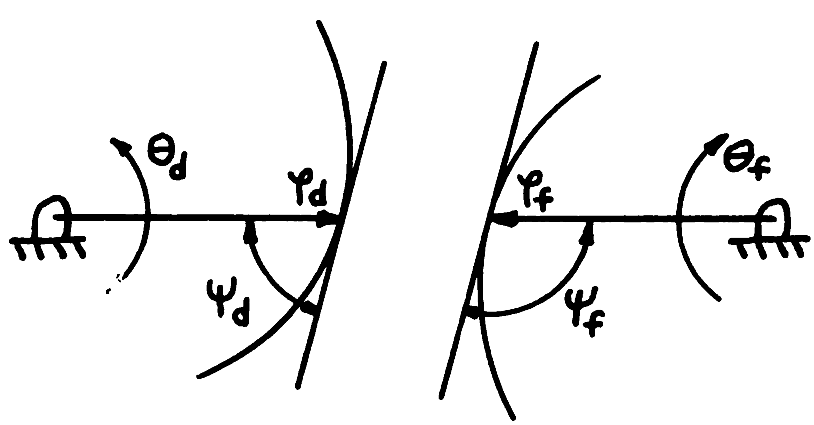


Figure 16. Sign convention for the angle between the radius and the tangent

The proof that the values of the angles between the radius and the tangent to the curve are supplements follows in a similar manner.

$$\tan \psi = \Upsilon \div \frac{d\Upsilon}{d\theta} \qquad \text{(A.3)}^2$$

where ψ = the angle between the radius and the tangent to the curve as shown in Fig. (16)

The two angles must be supplements of each other. In mathematical terms

$$\tan \psi_f + \tan \psi_d = 0. \qquad \text{(A.4)}$$

where ψ_f = the angle between the radius and the tangent of the follower curve

ψ_d = the angle between the radius and the tangent of the driver curve

From here the proof consists of using Equ. (A.3) for both curves and showing that the resulting equations satisfy Equ. (A.4).

$$\tan \psi_f = \Upsilon_f \div \frac{d\Upsilon_f}{d\theta_f}$$

$$\tan \psi_f = \frac{L}{f'(\theta_d)+1} \cdot \frac{[f'(\theta_d)+1]^2}{-L\, f''(\theta_d)} \cdot f'(\theta_d)$$

$$\tan \psi_f = \frac{-f'(\theta_d)}{f''(\theta_d)} \cdot [f'(\theta_d)+1] \qquad \text{(A.5a)}$$

---

[2]Granville, Smith, and Longley, op. cit., p. 207.

$$\tan \Psi_d = \gamma_d \div \frac{d\gamma_d}{d\theta_d}$$

$$\tan \Psi_d = \frac{L \cdot f'(\theta_d)}{f'(\theta_d)+1} \cdot \frac{[f'(\theta_d)+1]^2}{L \cdot f''(\theta_d)}$$

$$\tan \Psi_d = \frac{f'(\theta_d)}{f''(\theta_d)} \cdot [f'(\theta_d)+1] \qquad \text{(A.5b)}$$

Therefore, $\tan \Psi_f + \tan \Psi_d = 0.$

B. EXTENSION OF THE ROLLING CURVE EQUATIONS TO THE CASE OF A DRIVER CURVE WITH A VARYING ANGULAR VELOCITY

In the usual application the driver curve rotates at a constant angular velocity. However, this is not a necessary condition. For example when a very high maximum value of the ratio of angular velocities is required, non-circular gear trains are used.

The theory is easily extended to cover the case where both curves rotate with a varying angular velocity.

Consider the case where the complete motion pattern of both curves is known.

$$\theta_d = g(t) \tag{B.1a}$$

$$\theta_f = f(t) \tag{B.1b}$$

where t = time

It is possible to use the equations developed in Chapter II if the variable t can be eliminated so that θ_f is expressed as a function of θ_d. A different set of equations must be used if the variable t can not be eliminated.

$$f'(\theta_d) = \frac{\omega_f}{\omega_d} = \frac{f'(t)}{g'(t)} \tag{B.2}$$

$$r_f = \frac{L\, g'(t)}{g'(t) + f'(t)} \tag{B.3a}$$

$$\gamma_d = \frac{L\, f'(t)}{g'(t) + f'(t)} \tag{B.3b}$$

The general equation can be used for a synthesis problem. The only requirement is that the required data can be specified on a displacement diagram.

All of the properties of the general equation remain the same except the acceleration expression.

$$\begin{aligned} \alpha_f = &-\left(\frac{\pi}{\chi}\omega_d\right)^2\left(K_1 \sin \frac{\pi}{\chi}\theta_d + 4K_2 \sin \frac{2\pi}{\chi}\theta_d\right) \\ &+ \alpha_d\left(K_1 \frac{\pi}{\chi}\cos \frac{\pi}{\chi}\theta_d + 2K_2 \frac{\pi}{\chi}\cos \frac{2\pi}{\chi}\theta_d + \frac{\psi}{\chi}\right) \end{aligned} \tag{B.4}$$

The discussion of angular accelerations developed in Chapter VII is invalid for the case of varying angular velocity of the driver curve.

C. DERIVATION OF THE EQUATIONS OF MOTION FOR ROLLING ELLIPSES BY THE USE OF THE EQUIVALENT LINKAGE AND PROOF THAT THESE EQUATIONS ARE IDENTICAL TO THE EQUATIONS OBTAINED FROM THE THEORY OF ROLLING CURVES

In Chapter V it was proved that rolling ellipses may be replaced by an equivalent linkage. This linkage is used to derive a relationship between the angular displacements of the driver and follower curves. The derivation depends only upon trigonometry.

Fig. (17) shows the equivalent linkage.

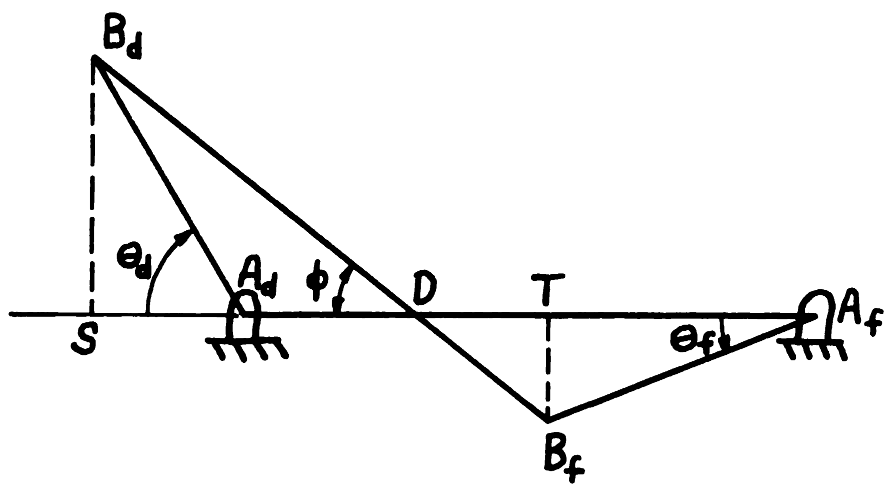


Figure 17. Equivalent linkage for rolling ellipses

Let $\phi = \angle B_d D A_d = \angle B_f D A_f$

$$A_d B_d = A_f B_f = R$$

$B_d D = x$ and $B_f D = 2a - x$

It is convenient to use the eccentricity from the rolling ellipses.

$$e = \frac{R}{2a}$$

The following relationships are obtained from trigonometric considerations.

$$B_d S = R \sin\theta_d$$

$$B_f T = R \sin\theta_f$$

$$\sin\phi = \frac{R \sin\theta_d}{x}$$

$$\sin\phi = \frac{R \sin\theta_f}{2a - x}$$

$$\frac{R \sin\theta_d}{x} = \frac{R \sin\theta_f}{2a - x}$$

$$x = \frac{2a \sin\theta_d}{\sin\theta_d + \sin\theta_f}$$

$$\sin\phi = \frac{R \sin\theta_d}{x} = e(\sin\theta_d + \sin\theta_f) \qquad \text{(C.1)}$$

$$ST = 2a - R\cos\theta_f + R\cos\theta_d$$

$$\cos\phi = \frac{ST}{B_d B_f}$$

$$\cos\phi = \frac{2a - R\cos\theta_f + R\cos\theta_d}{2a}$$

$$\cos\phi = 1 + e(\cos\theta_d - \cos\theta_f) \qquad (C.2)$$

Equs. (C.1) and (C.2) are combined by using an elementary trigonometric identity.

$$\sin^2\phi + \cos^2\phi = 1$$

$$e^2\sin^2\theta_d + 2e^2\sin\theta_f\sin\theta_d + e^2\sin^2\theta_f$$
$$+1 + 2e\cos\theta_d - 2e\cos\theta_f + e^2\cos^2\theta_f$$
$$-2e^2\cos\theta_f\cos\theta_d + e^2\cos^2\theta_d = 1$$

$$e^2(\sin^2\theta_d + \cos^2\theta_d + \sin^2\theta_f + \cos^2\theta_f)$$
$$+2e(\cos\theta_d - \cos\theta_f)$$
$$+2e^2(\sin\theta_f\sin\theta_d - \cos\theta_f\cos\theta_d) = 0$$

$$e(1 + \sin\theta_f\sin\theta_d - \cos\theta_f\cos\theta_d)$$
$$+\cos\theta_d - \cos\theta_f = 0 \qquad (C.3)$$

Equ. (5.7) gives the relationship between the angular displacements for two rolling ellipses when considered as rolling curves.

$$\tan\theta_f = \frac{(1-e^2)\sin\theta_d}{2e + (e^2+1)\cos\theta_d} \qquad (C.4)$$

Equ. (C.3) gives the relationship between the angular displacements for two rolling ellipses as derived from their equivalent linkage.

Equs. (C.3) and (C.4) must be proved to be identical. The method is to find the expressions for the sine and cosine of the angular displacement of the follower curve from both Equ. (C.3) and (C.4).

$$\tan x = \frac{\sin x}{\cos x}$$

$$\sin \theta_f = \frac{(1-e^2)\sin\theta_d}{H} \tag{C.5}$$

$$\cos \theta_f = \frac{2e + (e^2+1)\cos\theta_d}{H} \tag{C.6}$$

The factor H is found by two methods. First, it is found by the use of a trigonometric identity. Second, it is found by substituting Equs. (C.6) and (C.5) into Equ. (C.3). The proof consists of showing that these two methods give the same value for H and consequently the same values for the sine and cosine of θ_f.

The first method uses Equs. (C.5) and (C.6) and a trigonometric identity.

$$\sin^2\phi + \cos^2\phi = 1$$

$$\left[(1-e^2)\sin\theta_d\right]^2 + \left[2e + (e^2+1)\cos\theta_d\right]^2 = H^2$$

$$\begin{aligned} H^2 = e^4 + 2e^2 + 1 + 4e^2\cos^2\theta_d \\ + 4e^3\cos\theta_d + 4e\cos\theta_d \end{aligned} \tag{C.7}$$

The second method uses Equs. (C.5) and (C.6) which are substituted into Equ. (C.3).

$$e\left\{1+\frac{\sin\theta_d\,(1-e^2)\sin\theta_d}{H}-\frac{\cos\theta_d\left[2e+(e^2+1)\cos\theta_d\right]}{H}\right\}$$

$$-\frac{2e+(e^2+1)\cos\theta_d}{H}+\cos\theta_d=0$$

$$(e+\cos\theta_d)\,H=e\sin^2\theta_d\,(e^2-1)+2e^2\cos\theta_d+2e$$
$$+e(e^2+1)\cos^2\theta_d+(e^2+1)\cos\theta_d$$
$$=e(e^2-1)(1-\cos^2\theta_d)+2e^2\cos\theta_d+2e$$
$$+e(e^2+1)\cos^2\theta_d+(e^2+1)\cos\theta_d$$
$$=e^3-e-e^3\cos^2\theta_d+e\cos^2\theta_d+2e^2\cos\theta_d$$
$$+e^3\cos^2\theta_d+e\cos^2\theta_d+2e+e^2\cos\theta_d+\cos\theta_d$$
$$=(e^3+e)+\cos\theta_d\,(3e^2+1)+2e\cos^2\theta_d$$
$$=2e\cos^2\theta_d+2e^2\cos\theta_d+\cos\theta_d\,(e^2+1)$$
$$+e^3+e$$
$$=(\cos\theta_d+e)(2e\cos\theta_d+e^2+1)$$

$$H=2e\cos\theta_d+e^2+1$$

$$H^2 = 4e^2\cos^2\theta_d + e^4 + 1 + 4e^3\cos\theta_d + 4e\cos\theta_d + 2e^2 \qquad (C.8)$$

Therefore, the value of H^2 in Equ. (C.7) equals the value in Equ. (C.8). This proves only that the two methods give the same relation between the numerical values of the angular displacements of the two ellipses.

The proof is incomplete because two values of the factor H may be obtained from Equ. (C.7). These two values are numerically equal and differ only in sign. One of these values is eliminated by physical considerations. For a small rotation of the driver ellipse in the positive direction from the polar axis both θ_d and θ_f assume first quadrant values. This also applies to the equivalent linkage for a small positive rotation of the driver link. From this consideration one of the values of the factor H obtained from Equ. (C.7) is eliminated.

Therefore, the two methods give the identical relationship between the angular displacements of rolling ellipses.

Also, Equs. (C.5) and (C.6) may now be completed.

$$\sin\theta_f = \frac{(1-e^2)\sin\theta_d}{2e\cos\theta_d + e^2 + 1} \qquad (C.5a)$$

$$\cos\theta_f = \frac{2e + (e^2+1)\cos\theta_d}{2e\cos\theta_d + e^2 + 1} \qquad (C.6a)$$

The equivalent linkage is used to obtain a relationship between the angular velocities of the driver and follower ellipses. The derivation uses the angular velocity ratio theorem and trigonometric considerations. The angular velocity ratio theorem states that the angular velocities of the driver and follower vary inversely as the segments on the line of centers cut by the line of transmission.[1] For the equivalent linkage the line of transmission is link $B_d B_f$ in Fig. (17).

$$\frac{\omega_f}{\omega_d} = \frac{r_d}{r_f} = \frac{A_d D}{A_f D} \tag{C.9}$$

The following relationships are obtained from trigonometric considerations.

$$SD = x \cos\phi = \frac{2a \sin\theta_d}{\sin\theta_d + \sin\theta_f} \cdot \cos\phi$$

$$TD = (2a - x)\cos\phi = \frac{2a \sin\theta_f}{\sin\theta_d + \sin\theta_f} \cdot \cos\phi$$

$$A_d D = SD - SA_d$$

$$A_d D = \frac{2a \sin\theta_d}{\sin\theta_d + \sin\theta_f} \cdot \cos\phi - R\cos\theta_d$$

$$A_d D = \frac{2a \sin\theta_d \left[1 + e(\cos\theta_d - \cos\theta_f)\right] - R\cos\theta_d(\sin\theta_d + \sin\theta_f)}{\sin\theta_d + \sin\theta_f}$$

$$A_d D = \frac{2a \sin\theta_d - R\cos\theta_f \sin\theta_d - R\cos\theta_d \sin\theta_f}{\sin\theta_d + \sin\theta_f}$$

[1]Hinkle, Kinematics of Mechanism, p. 22.

$$A_f D = A_f T + TD$$

$$A_f D = R\cos\theta_f + \frac{2a\sin\theta_f}{\sin\theta_d + \sin\theta_f}\cdot\cos\phi$$

$$A_f D = \frac{R\cos\theta_f(\sin\theta_d + \sin\theta_f) + 2a\sin\theta_f\left[1 + e(\cos\theta_d - \cos\theta_f)\right]}{\sin\theta_d + \sin\theta_f}$$

$$A_f D = \frac{2a\sin\theta_f + R\cos\theta_f\sin\theta_d + R\sin\theta_f\cos\theta_d}{\sin\theta_d + \sin\theta_f}$$

$$\frac{A_d D}{A_f D} = \frac{2a(\sin\theta_d - e\cos\theta_f\sin\theta_d - e\cos\theta_d\sin\theta_f)}{2a(\sin\theta_f + e\cos\theta_f\sin\theta_d + e\cos\theta_d\sin\theta_f)} \quad \text{(C.10)}$$

Substitute Equs. (C.5a) and (C.6a) into Equ. (C.10) and simplify.

$$\frac{A_d D}{A_f D} = \frac{\sin\theta_d(2e\cos\theta_d + e^2 + 1) - e\left\{\sin\theta_d\left[2e + (e^2+1)\cos\theta_d\right] + \cos\theta_d(1-e^2)\sin\theta_d\right\}}{\sin\theta_d(1-e^2) + e\left\{\sin\theta_d\left[2e + (e^2+1)\cos\theta_d\right] + \cos\theta_d(1-e^2)\sin\theta_d\right\}}$$

$$\frac{A_d D}{A_f D} = \frac{(2e\cos\theta_d + e^2 + 1 - 2e^2 - 2e\cos\theta_d)\sin\theta_d}{\left[1 - e^2 + 2e^2 + \cos\theta_d(e^3 + e + e - e^3)\right]\sin\theta_d}$$

Equ. (C.11) is identical to Equ. (5.5).

$$\frac{\omega_f}{\omega_d} = \frac{A_d D}{A_f D} = \frac{1-e^2}{1+e^2+2e\cos\theta_d} \quad \text{(C.11)}$$

D. DERIVATION OF THE EQUATIONS FOR THE RADIUS OF CURVATURE OF ROLLING CURVES

The equations for the radius of curvature of rolling curves are derived by using the general equation for the radius of curvature in polar coordinates.

$$R = \frac{\left[\rho^2 + (\rho')^2\right]^{3/2}}{\rho^2 + 2(\rho')^2 - \rho \cdot \rho''} \qquad (D.1)[1]$$

where $\rho' = \dfrac{d\rho}{d\theta}$

$\rho'' = \dfrac{d^2\rho}{d\theta^2}$

The driver curve equation is used.

$$\rho_d = \frac{L f'}{f'+1} \qquad (D.2a)$$

$$\rho_d' = \frac{L f''}{(f'+1)^2} \qquad (D.2b)$$

$$\rho_d'' = L \cdot \frac{f'''(f'+1) - 2(f'')^2}{(f'+1)^3} \qquad (D.2c)$$

---

[1]Granville, Smith, and Longley, Elements of Calculus, p. 222.

where $$f' = \frac{d\theta_f}{d\theta_d} = \frac{d}{d\theta_d} f(\theta_d)$$

$$f'' = \frac{d^2\theta_f}{d\theta_d^2} = \frac{d^2}{d\theta_d^2} f(\theta_d)$$

$$f''' = \frac{d^3\theta_f}{d\theta_d^3} = \frac{d^3}{d\theta_d^3} f(\theta_d)$$

Equs. (D.1) and (D.2a-c) are combined.

$$R_d = \frac{L}{(f'+1)^3} \frac{\left[(f')^2(f'+1)^2 + (f'')^2\right]^{3/2}}{\left[(f')^3 + (f')^2 + 2(f'')^2 - f''' \cdot f'\right]} \qquad \text{(D.3)}$$

where R_d = the radius of curvature of the driver curve

The follower curve equation is used.

$$r_f = \frac{L}{f'+1} \qquad \text{(D.4a)}$$

$$r_f' = \frac{L(-f'')}{(f'+1)^2 \cdot f'} \qquad \text{(D.4b)}$$

$$r_f'' = \frac{d}{d\theta_d}\left(\frac{dr_f}{d\theta_d} \cdot \frac{d\theta_d}{d\theta_f}\right)\frac{d\theta_d}{d\theta_f} = \frac{d^2 r_f}{d\theta_d^2}\left(\frac{d\theta_d}{d\theta_f}\right)^2 + r_f'\left(\frac{d}{d\theta_d} \cdot \frac{d\theta_d}{d\theta_f}\right) \qquad \text{(D.5)}^2$$

$$r_f'' = \frac{L}{(f'+1)^3 (f')^3}\left[3(f'')^2 f' + (f'')^2 - (f')^2 f''' - f' \cdot f'''\right] \qquad \text{(D.4c)}$$

---

[2] Ivan S. Sokolnikoff, Advanced Calculus, (New York and London: McGraw-Hill Book Company, Inc., 1939), p. 48.

Equs. (D.1) and (D.4a-c) are combined.

$$R_f = \frac{L}{(f'+1)^3} \cdot \frac{\left[(f'+1)^2(f')^2 + (f'')^2\right]^{3/2}}{\left[(f')^4 + (f')^3 - (f'')^2 + f''' \cdot f'\right]} \qquad \text{(D.6)}$$

where R_f = the radius of curvature of the follower curve

BIBLIOGRAPHY

Burington, Richard S., comp. Handbook of Mathematical Tables and Formulas. 2nd ed. Sandusky, Ohio: Handbook Publishers, Inc., 1940. 275 pp.

"Centre-turning Mobile Cranes," Engineering, 160 (Sept. 28, 1945), 246-48.

Dunkerley, S. Mechanism, 3rd ed. Edited by Arthur Morley. London: Longmans, Green, and Co., 1912. 448 pp.

"Gearing for Yarn Winding Machinery," Engineering, XI (Jan. 6, 1871), 20.

Golber, H. E. Rollcurve Gears, Preprint of a speech presented on Dec. 6, 1938 to the Graphic Arts Section at the annual meeting of the A. S. M. E. 7 pp. This work was also published. Golber, H. E. "Rollcurve Gears," Transactions of the A. S. M. E., 61 (April, 1939), 223-31.

Grandville, William A., Percey F. Smith, and William R. Longley. Elements of Calculus. Boston: Ginn and Co., 1941. 549 pp.

Hannula, F. W. "Designing Noncircular Surfaces for Pure Rolling Contact," Machine Design, 23 (July, 1952), 111-14, 190-92.

Hinkle, Rolland T. Kinematics of Machines. New York: Prentice-Hall, Inc., 1953. 231 pp.

Lockenvitz, A. E., J. B. Oliphint, W. C. Wilde, and James M. Young. "Noncircular Cams and Gears," Machine Design, 24 (May, 1952), 141-45.

Murray, Francis J. The Theory of Mathematical Machines. rev. ed. New York: King's Crown Press, 1948. 139 pp.

"Non-circular Gears Generated Automatically," Iron Age, 171 (Feb. 19, 1953), 122-23.

Olsson, Uno. Non-circular Cylindrical Gears. Acta Polytechnica, Mechanical Engineering Series, Vol. 2, no. 10. Stockholm: Esselte Aktiebolag, 1953. 216 pp.

Peyrebrune, H. E. "Application and Design of Noncircular Gears," Machine Design, 25 (Dec., 1953), 185-93.

Robinson, Stillman W. Principles of Mechanism. New York: John Wiley and Sons, 1896. 309 pp.

Sokolnikoff, Ivan S. Advanced Calculus. New York and London: McGraw-Hill Book Company, Inc., 1939. 446 pp.

Temperley, Clive. "Intermeshing Non-circular Algebraic Gears," Engineering, 165 (Jan. 16, 1948), 49-52.

Trautschold, Reginald. "Machine-cut Elliptical Gears," Machinery (New York), XXIII (Aug., 1917), 1049-55.

Weisbach, Julius, and Gustav Herrmann. The Mechanics of the Machinery of Transmission. Translated by J. F. Klein. Mechanics of Engineering and of Machinery, Vol. III, part 1, sec. 1. 2nd ed. New York: John Wiley and Sons, 1902. 544 pp.

Willis, Robert. Principles of Mechanism. 2nd ed. London: Longmans, Green, and Co., 1870. 463 pp.

"The Works and Products of Messrs. Barr and Stroud, Limited," Engineering, CVIII (Aug. 1, Aug. 15, Aug. 29, Sept. 19, Oct. 24, Nov. 21, Dec. 12, and Dec. 26, 1919), 133-34, 198-200, 263-66, 363-65, 536-40, 669-72, 776-79, 846-49.