RADIATION OF A POINT DIPOLE LOCATED AT THE TIP OF A PROLATE SPHEROID

Thesis for the Degree of M. S.

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Eugene C. Hatcher Jr.

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Alfred Leitner
Major professor

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RADIATION OF A POINT DIPOLE LOCATED AT THE TIP OF A PROLATE SPHEROID

by

Eugene C. Hatcher Jr.

A Thesis

Submitted to the School of Graduate Studies of Michigan
State College of Agriculture and Applied Science
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The author wishes to express his sincere thanks to Dr. A. Leitner, under whose inspiration this investigation was undertaken and to whom the results are herewith dedicated.

He is also greatly indebted to Dr. R. D. Spence for the use of unpublished tables of the prolate spheroidal wave functions.

Grateful acknowledgement is also due Mrs. M. Muller for her assistance with illustrations.

TABLE OF CONTENTS

I. Introduction

The purpose of this thesis was to find the radiation pattern of an electromagnetic point dipole of unit strength located along the axis of symmetry of a perfectly conducting prolate spheroid. Maxwell's equations are solved in terms of the wave functions which have been discussed before by Stratton et.al.,[5] Page[2] and Spence.[3] Green's theorem is used to calculate the field. Our calculations were restricted to the limiting case where the dipole is imbedded in the surface of the spheroid. In that case, however, there is no radiation when the point dipole is of the magnetic type, as is easily proved.*

It is believed that these results will be of use in antenna theory and may shed some light on the mode of radiation by shells or other missiles.

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*See Appendix I.

II. Definition of Coordinates

The prolate spheroidal coordinates are defined by the transformations

$$X = a\sqrt{(\xi^2-1)(1-\eta^2)}\ \cos\varphi \qquad (1)$$

$$Y = a\sqrt{(\xi^2-1)(1-\eta^2)}\ \sin\varphi \qquad (2)$$

$$Z = a\xi\eta$$

where $\xi \geq 1, \ -1 \leq \eta \leq 1, \quad 0 \leq \varphi \leq 2\pi$ (3)

and a is the length of the semi-major axis. The metrical coefficients are defined as

$$h_\xi = a\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}} \qquad (4)$$

$$h_\eta = a\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}} \qquad (5)$$

$$h_\varphi = a\sqrt{(\xi^2-1)(1-\eta^2)} \qquad (6)$$

The practical utility of spheroidal coordinates may be surmised that as the eccentricity approaches unity the prolate spheroids become rod-shaped, which is approximately the shape of an antenna, see Fig. 1.

III. Maxwell's Equations in a Homogeneous Medium

$$\nabla \times \underline{E} + \mu \frac{\partial \underline{H}}{\partial t} = 0 \tag{7}$$

$$\nabla \times \underline{H} - \epsilon \frac{\partial \underline{E}}{\partial t} = 0 \tag{8}$$

where $\underline{E}$ and $\underline{H}$ are the electromagnetic vectors and μ, ϵ are the permeability and dielectric constants, respectively, of the medium.

We assumed a harmonic variation with time and define

$$\frac{\partial}{\partial t} \equiv -i\omega$$

Then equations (7) and (8) become

$$\nabla \times \underline{E} = i\mu\omega \underline{H} \tag{9}$$

$$\nabla \times \underline{H} = -i\epsilon\omega \underline{E} \tag{10}$$

We assumed that $\underline{E}$ and $\underline{H}$ are symmetric, i.e., they are independent of the azimuthal angle φ.

Expanded in spheroidal coordinates Maxwell's equations become

$$\frac{\partial}{\partial \xi}\sqrt{\xi^2-1}\, H_\varphi = -i\epsilon\omega a \sqrt{\xi^2-\eta^2}\, E_\eta \tag{11}$$

$$\frac{\partial}{\partial \eta}\sqrt{1-\eta^2}\, H_\varphi = i\epsilon\omega a \sqrt{\xi^2-\eta^2}\, E_\xi \tag{12}$$

$$\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{a(\xi^2-\eta^2)} \left\{ \frac{\partial}{\partial \eta}\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}\, E_\xi \right. \tag{13}$$

$$\left. - \frac{\partial}{\partial \xi}\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}\, E_\eta \right\} = i\mu\omega H_\varphi$$

$$\frac{\partial}{\partial \xi} \sqrt{\xi^2-1}\, E_\varphi = i\mu\omega a \sqrt{(\xi^2-\eta^2)}\; H_\eta \qquad (14)$$

$$\frac{\partial}{\partial \eta} \sqrt{1-\eta^2}\, E_\varphi = -i\mu\omega a \sqrt{(\xi^2-\eta^2)}\; H_\xi \qquad (15)$$

$$\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{a(\xi^2-\eta^2)} \left\{ \frac{\partial}{\partial \eta} \sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}\, H_\xi - \frac{\partial}{\partial \xi} \sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}\, H_\eta \right\} = -i\epsilon\omega E_\varphi \qquad (16)$$

These six equations have been so written as to emphasize that they separate into two independent groups which correspond to the two types of dipole excitation, electric--equations eleven through thirteen and magnetic--equations fourteen through sixteen. Upon combining we obtain the following two differential equations

$$\frac{\partial}{\partial \xi}(\xi^2-1)\frac{\partial}{\partial \xi} H_\varphi + \frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial}{\partial \eta} H_\varphi - \left[\frac{1}{1-\eta^2} + \frac{1}{\xi^2-1}\right] H_\varphi = -k^2a^2(\xi^2-\eta^2) H_\varphi \qquad (17)$$

and

$$\frac{\partial}{\partial \xi}(\xi^2-1)\frac{\partial}{\partial \xi} E_\varphi + \frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial}{\partial \eta} E_\varphi - \left[\frac{1}{1-\eta^2} + \frac{1}{\xi^2-1}\right] E_\varphi = -k^2a^2(\xi^2-\eta^2) E_\varphi \qquad (18)$$

where $\epsilon\mu = \frac{1}{c^2} = k^2/\omega^2$, $k = 2\pi/\lambda$

c being the velocity of light, λ the wavelength.

A solution of these equations is

$$H_\varphi = S_{12}(\eta)\, R_{12}(\xi) \qquad (19)$$

The functions $S_{12}(\eta)$ and $R_{12}(\xi)$ are the angular and radial prolate spheroidal wave functions of order one, respectively, and are defined by Stratton et.al.[5] We shall discuss them further in Appendix III.

IV. Solution of the Partial Differential Equation and Determination of Green's Function

We restrict ourselves to the case of an electric dipole located on the Z-axis. Let us write equation (17) as

$$L H_\varphi = 0 \tag{20}$$

where

$$L = \frac{1}{a^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi} - \left[\frac{1}{1-\eta^2} + \frac{1}{\xi^2-1}\right] + (ka)^2(\xi^2-\eta^2)\right\}$$

L is the prolate spheroidal operator leading to functions of order one. On the perfectly conducting spheroid $\xi = \xi_0$, H_φ must in addition satisfy

$$\frac{1}{h_\xi}\frac{\partial}{\partial\xi}(h_\varphi H_\varphi) = 0 \tag{21}$$

while at large distances from the origin H_φ must behave like an outgoing wave function (radiation condition).

The Green's function $G(\xi,\eta;\xi',\eta')$ satisfies

$$L G(\xi,\eta;\xi',\eta') = -\frac{\delta(\xi-\xi')\delta(\eta-\eta')\delta(\varphi-\varphi')}{h_\xi h_\eta h_\varphi}$$

and the same boundary condition as H_φ, namely

$$\frac{1}{h_\xi}\frac{\partial}{\partial \xi}(h_\varphi G) = 0, \; \xi = \xi_0 \qquad (23)$$

and the radiation condition at infinity.

Once Green's function is determined $H_\varphi(\xi,\eta)$ is found from Green's theorem

$$H_\varphi(\xi,\eta) = \int_{\sigma'} \left[H_\varphi(\xi',\eta')\frac{\partial G}{\partial n} - G\frac{\partial}{\partial n} H_\varphi(\xi',\eta') \right] d\sigma' \qquad (24)$$

in terms of the values of the integrand on the boundary. The surface $\sigma' = \sigma_1' + \sigma_2' + \sigma_3'$, where

σ_1' = boundary spheroid ξ_0

σ_2' = a small surface surrounding the point dipole

σ_3' = the great sphere, $\xi \rightarrow \infty$.

The significance of the similarity in the boundary conditions (21) and (23) is to make the integrand in (24) vanish when $\sigma' = \sigma_1'$, and $\sigma' = \sigma_2'$. Equation (24) is so written that $\partial/\partial n$ signifies normal derivatives into the field space.

The central problem of this thesis consisted of constructing this Green's function, from which it is possible to calculate the total magnetic field $\underline{H}$ at any point in space in terms of the current distribution on the antenna. The Green's function was so constructed that it satisfies the same boundary conditions on the spheroid as the electric and magnetic fields $\underline{E}$ and $\underline{H}$. Because of the symmetry of the problem, the magnetic field consists of the single azimuthal component H_φ and therefore the Green's function has a simple scalar form.

We found the Green's function by the use of eigenfunctions as described by Morse and Feshbach.[1] We assume $G(\xi,\eta;\xi',\eta')$ to be of the form

$$G(\xi,\eta;\xi',\eta') = \sum_{\ell} a_{1\ell}(\xi,\xi')\,S_{1\ell}(\eta)\,S_{1\ell}(\eta') \qquad (25)$$

and that it is symmetric in $(\xi,\eta;\xi',\eta')$. Substituting (25) in (22), multiplying both sides by $S_{1\ell}(\eta')$ and integrating with respect to η and φ we get

$$\frac{d}{d\xi}(\xi^2-1)\frac{da_{1\ell}}{d\xi} + \left[A_{1\ell} + (ka)^2\xi^2\right]a_{1\ell} - \frac{a_{1\ell}}{\xi^2-1} = -\frac{\delta(\xi-\xi')}{2\pi a N_{1\ell}} \qquad (26)$$

where $N_{1\ell}$ is the norm of the angular function.

By referring to equation (59) of Appendix III we see that $a_{1\ell}(\xi,\xi')$ satisfies the radial spheroidal differential equation at all points $\xi \neq \xi'$. The differential equation (26) asserts that the left hand side is infinite when $\xi = \xi'$. This means that $a_{1\ell}(\xi,\xi')$ and/or its derivatives are infinite.

If we integrate (26) with respect to ξ over any range including the point $\xi = \xi'$ we get

$$\int_{\alpha}^{\beta}\left\{\frac{d}{d\xi}(\xi^2-1)\frac{d}{d\xi} + A_{1\ell} + k^2a^2\xi^2 - \frac{1}{\xi^2-1}\right\}a_{1\ell}\,d\xi = \frac{-1}{2\pi a N_{1\ell}} \qquad (27)$$

The result is clearly independent of the range of ξ. We may therefore shrink it arbitrarily close to the point $\xi = \xi'$. In that case the derivative term in the integral outweighs all others because of the singularities discussed above. We obtain

$$\left[\frac{d}{d\xi} a_{1\ell}(\xi,\xi')\right]_{\xi'-}^{\xi'+} = \frac{-1}{2\pi a N_{1\ell}(\xi^2-1)} \tag{28}$$

where ξ'^-, ξ'^+ are two points arbitrarily close to ξ' at either side.

In addition $a_{1\ell}(\xi,\xi')$ must be symmetric in (ξ,ξ') and must satisfy Sommerfeld's radiation condition when ξ or $\xi' \to \infty$. Therefore

$$a_{1\ell}(\xi,\xi') = \begin{cases} {}^{(I)}R_{1\ell}(\xi')\,{}^{(3)}R_{1\ell}(\xi), & \xi > \xi' \\ {}^{(3)}R_{1\ell}(\xi')\,{}^{(I)}R_{1\ell}(\xi), & \xi < \xi' \end{cases} \tag{29}$$

where ${}^{(3)}R_{1\ell} = {}^{(1)}R_{1\ell} + i\,{}^{(2)}R_{1\ell}$ is the radial function with outgoing-wave behavior,* and ${}^{(I)}R_{1\ell}$ is a linear combination of the fundamental set of radial wave functions ${}^{(1)}R_{1\ell}(\xi)$, ${}^{(2)}R_{1\ell}(\xi)$, such that the condition (23) be satisfied at the boundary spheroid ξ_0. Thus we write

$${}^{(I)}R_{1\ell} \equiv d_\ell\,{}^{(1)}R_{1\ell} + g_\ell\,{}^{(2)}R_{1\ell}$$

To solve for d_ℓ, g_ℓ, two simultaneous equations are required. As the first we have the boundary condition (23), which can be written in the form

*See Appendix III.

$$ {}^{(I)}w'_{1\ell}(\xi_0) = 0 \quad \text{where} \quad {}^{(I)}w_{1\ell}(\xi) \equiv \sqrt{\xi^2-1}\; {}^{(I)}R_{1\ell}(\xi) \qquad (31) $$

As the second equation determining d_ℓ, g_ℓ we have available equation (28).* Thus one obtains the result

$$ {}^{(I)}R_{1\ell}(\xi) = -\frac{k}{2\pi N_{1\ell}\, {}^{(3)}w'_{1\ell}(\xi_0)}\, {}^{(II)}R_{1\ell}(\xi) \qquad (32) $$

where

$$ {}^{(II)}R_{1\ell}(\xi) = {}^{(2)}w'_{1\ell}(\xi_0)\, {}^{(1)}R_{1\ell}(\xi) - {}^{(1)}w'_{1\ell}(\xi_0)\, {}^{(2)}R_{1\ell}(\xi) \qquad (33) $$

The calculation of Green's function for our problem is now complete. The expansion (28) takes the final form

$$ G(\xi,\eta;\xi',\eta') = \frac{-k}{2\pi} \sum_\ell \frac{S_{1\ell}(\eta)\, S_{1\ell}(\eta')}{N_{1\ell}\, {}^{(3)}w'_{1\ell}(\xi_0)} \begin{Bmatrix} {}^{(II)}R_{1\ell}(\xi)\, {}^{(3)}R_{1\ell}(\xi'), & \xi<\xi' \\ {}^{(3)}R_{1\ell}(\xi)\, {}^{(II)}R_{1\ell}(\xi'), & \xi>\xi' \end{Bmatrix} \qquad (34) $$

*It can be easily shown from the radial differential equation (59), Appendix III that the Wronskian of its fundamental set of solutions is

$$ {}^{(1)}R_{1\ell}\, {}^{(2)}R'_{1\ell} - {}^{(2)}R_{1\ell}\, {}^{(1)}R'_{1\ell} = \frac{1}{ka(\xi^2-1)} $$

i.e., essentially the right hand side of (28)

V. Use of Green's Theorem to Calculate the Magnetic Field at any Point for an Electric Dipole on the Axis of Symmetry

In accordance with the remarks on page seven, Green's theorem reduces to

$$H_\varphi(\xi,\eta)=\int_{\sigma_2'}\left[H_\varphi(\xi',\eta')\frac{\partial G}{\partial n}-G\frac{\partial}{\partial n}H_\varphi(\xi',\eta')\right]d\sigma_2' \qquad (35)$$

where σ_2' is a small surface surrounding the dipole, and where $H_\varphi(\xi',\eta')$, the field in the immediate neighborhood of an electric dipole, is defined[4] as

$$H_\varphi(\xi',\eta')=-\frac{i\omega}{4\pi}\frac{e^{ikR}}{R^2}\sin\theta \qquad (36)$$

where R is the distance from the dipole to the field point.

The surface σ_2' over which we are integrating is infinitesimally small and approximates a circular cylinder, see Fig. Two. The dipole is located at D. The side of the cylinder σ_2' surrounding D is constructed from the coordinate hyperboloid η equals constant and nearly unity. The bases of σ_2' are cut out by this hyperboloid from two coordinate spheroids ξ_1^+ and ξ_1^-, symmetrically above and below the point D. Substituting (34) and (36) in (35) and integrating we find that the singularities

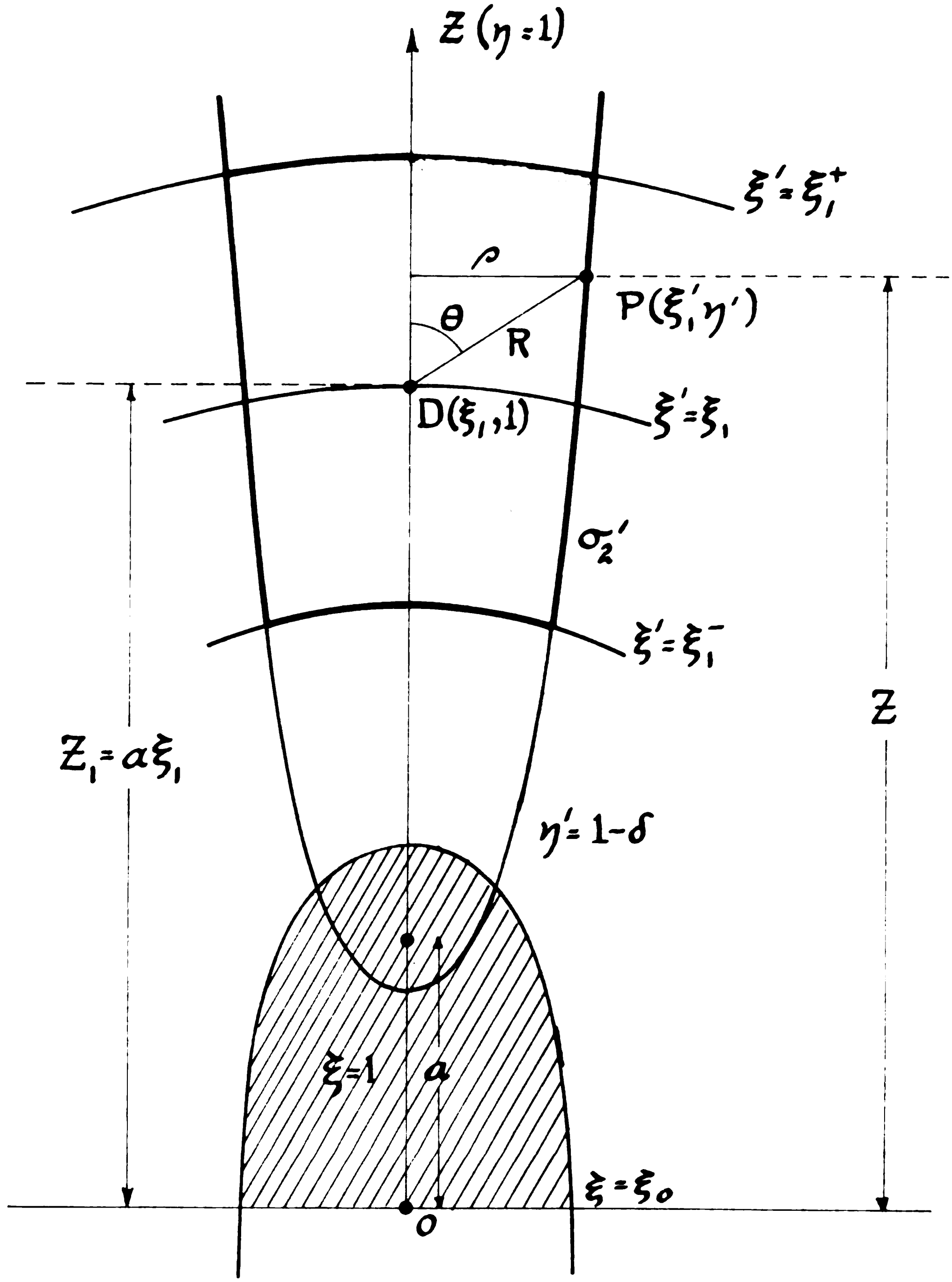

FIG. 2. $P(\xi', \eta')$ is the variable point of integration; δ is an infinitesimally small positive number.

are integrated out, with the final result*

$$H_\varphi(\xi,\eta)=\frac{ik}{a^2\pi\sqrt{\epsilon\mu}}\sum_\ell B_\ell S_{1\ell}(\eta)\,{}^{(3)}R_{1\ell}(\xi) \tag{37}$$

where

$$B_\ell = \frac{ka\, C_0^{1\ell}\,{}^{(\mathrm{II})}R_{1\ell}(\xi_1)/\sqrt{\xi_1^2-1}}{N_{1\ell}\,{}^{(3)}w_{1\ell}'(\xi_0)} \tag{38}$$

As $\xi_1 \to \xi_0$ this coefficient simplifies to

$$B_\ell = \frac{C_0^{1\ell}}{N_{1\ell}(\xi_0^2-1)\,{}^{(3)}w_{1\ell}'(\xi_0)} \tag{39}$$

which applies when our dipole is imbedded in the spheroid. This coefficient is singular when $\xi_0 \to 1$. Since the spheroid in the limit $\xi_0 = 1$ becomes a rod of zero thickness and length $2a$, this result means that the system would radiate an infinite amount in this limiting case. At large distance $\xi \to \infty$, ${}^{(3)}R_{1\ell}(\xi)$ becomes $(-i)^{\ell+1} e^{ika\xi}/ka\xi$ and $ka\xi$ becomes kr. Thus, for the radiation field we find

$$H_\varphi(\xi,\eta)=\frac{1}{a^2\pi\sqrt{\epsilon\mu}}\,\frac{e^{ikr}}{r}\sum_\ell (-i)^\ell B_\ell S_{1\ell}(\eta) \tag{40}$$

*Details of this calculation are described in Appendix II

VI. Energy Radiated

The radiation intensity is defined as the real part of the complex Poynting vector

$$S^* = \frac{1}{2}\, \underline{E} \times \underline{\hat{H}} \tag{41}$$

where $\underline{\hat{H}}$, is the complex conjugate of $\underline{H}$. When we substitute (40) in (10) we find that

$$\left\{ \begin{aligned} E_\eta &= \sqrt{\mu/\epsilon}\; H_\varphi \\ E_\xi &= 0 \end{aligned} \right\} \quad \text{when } \xi \text{ is large} \tag{42}$$

Thus we obtain the Poynting flux

$$R_e(S^*) = \frac{(\epsilon\mu)^{-1/2}}{2\pi^2 \epsilon a^4}\, \frac{1}{r^2}\, f(\theta, ka, \xi_0) \tag{43a}$$

where

$$f(\theta, ka, \xi_0) = \left| \sum_{l=0}^{\infty} (-i)^l B_l S_{1l}(\eta) \right|^2 \tag{43}$$

Here θ is the polar angle measured from the positive z-axis. The coordinate η becomes $\cos\theta$ at large distance as the hyperboloids approach the cones of constant θ.

The function f represents the variation with angle of the radiated energy, and depends in addition on the parameters $ka = 2\pi a/\lambda$ and ξ_0.

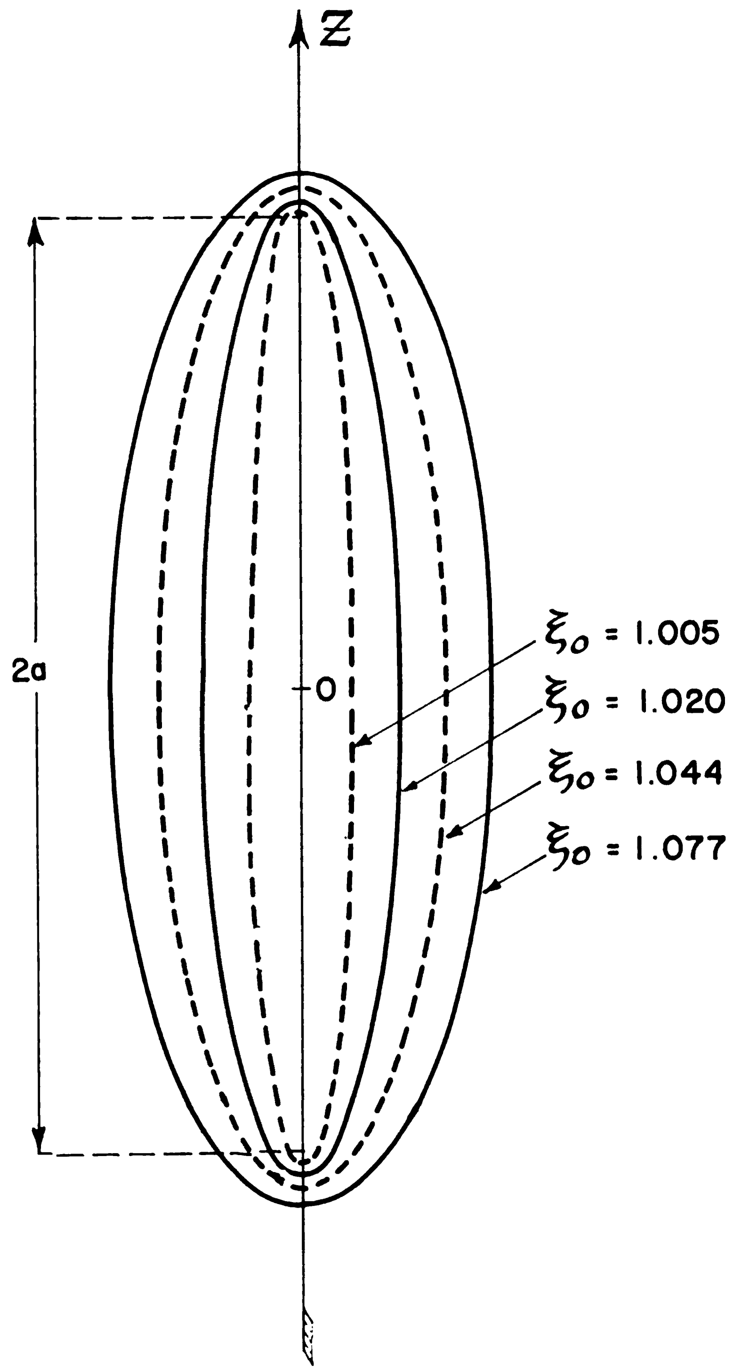

FIG. 3. Proportions of spheroids

VII. Results

The radiation pattern for a number of values of ξ_0 and ka are plotted against angle in figures four to fifteen.

The values chosen for ξ_0 were 1.005, 1.020, 1.044 and 1.077. These represent prolate spheroidal radiators whose eccentricity decreases in the order given, and their proportions are illustrated by figure three.

The parameter ka was given the values one, two and three. Since $ka = 2\pi a/\lambda_y$ and $2a$ is roughly the length of the radiator, these values of ka correspond to wavelengths equal to π, $\pi/2$ and $\pi/3$. In other words, in no case considered is the wavelength shorter than the spheroid, being roughly equal to it in the shortest case, $ka = 3$.

We realize that it would be of interest to compute radiation patterns for wavelengths shorter than the spheroids. However, the results are given as infinite series representations in terms of spheroidal wave functions, cf. equation (43), and the convergence deteriorates as ka increases. We limited ourselves to cases where the computation was accurate while using no more than four terms in the series (43), i.e., $l = 0, 1, 2, 3$. As it was, this required an extension of the tables of spheroidal wave functions over the existing ones of Stratton et.al.,[5] and of Spence.[3] Some of these results are tabulated in Appendix III.

As regards to the radiation patterns of Figs. 4-15, the spheroids have the effect of radiating into directions where $\theta > 90°$. Whereas for the dipole alone, the radiation is symmetric about the z-axis. This type of radiation pattern becomes more pronounced as ka increases.

A greater amount of energy is radiated as the spheroids approach the limiting case, where $\xi_0 = 1$, as was predicted by equation (39).

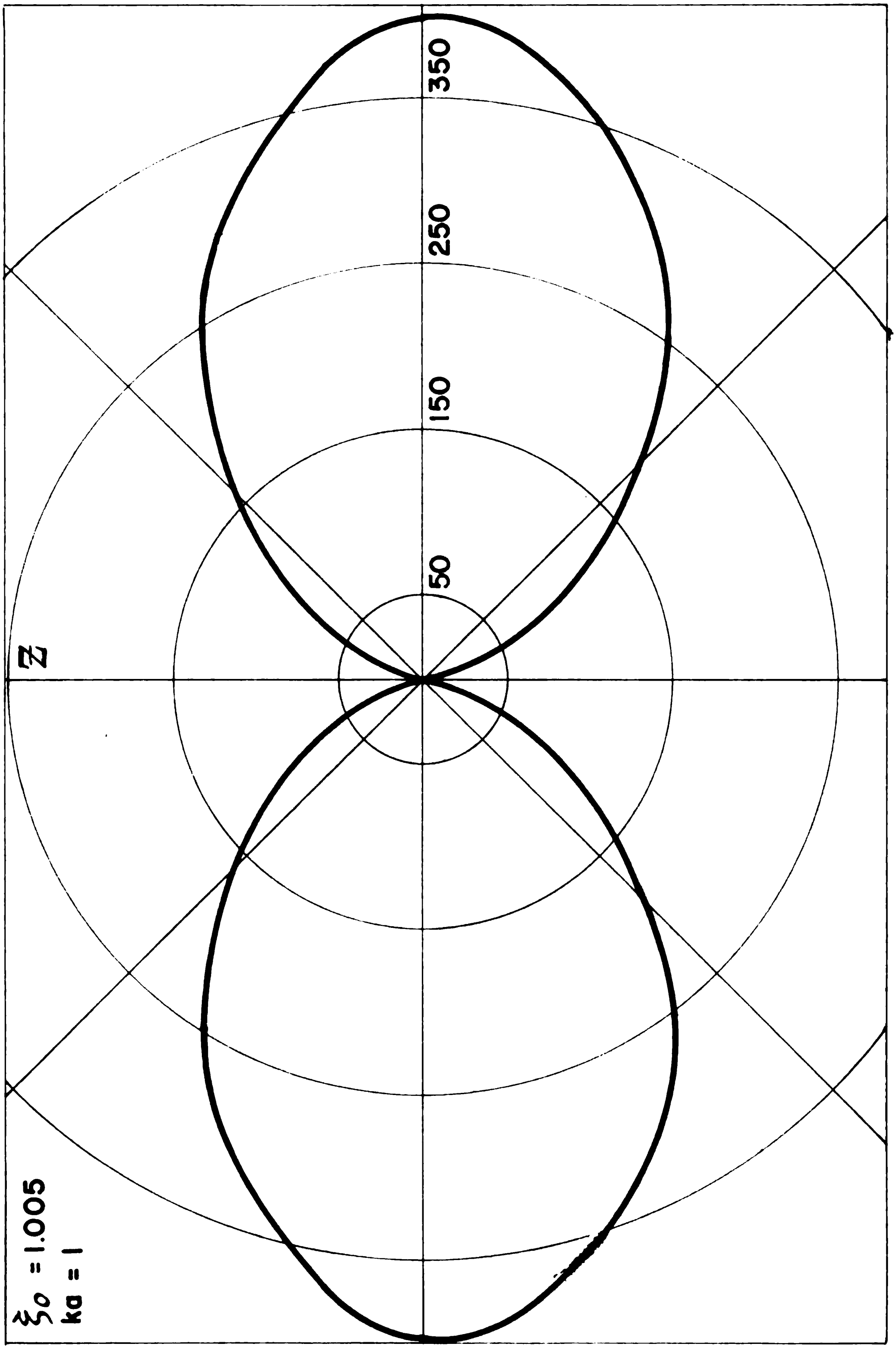

FIG. 4

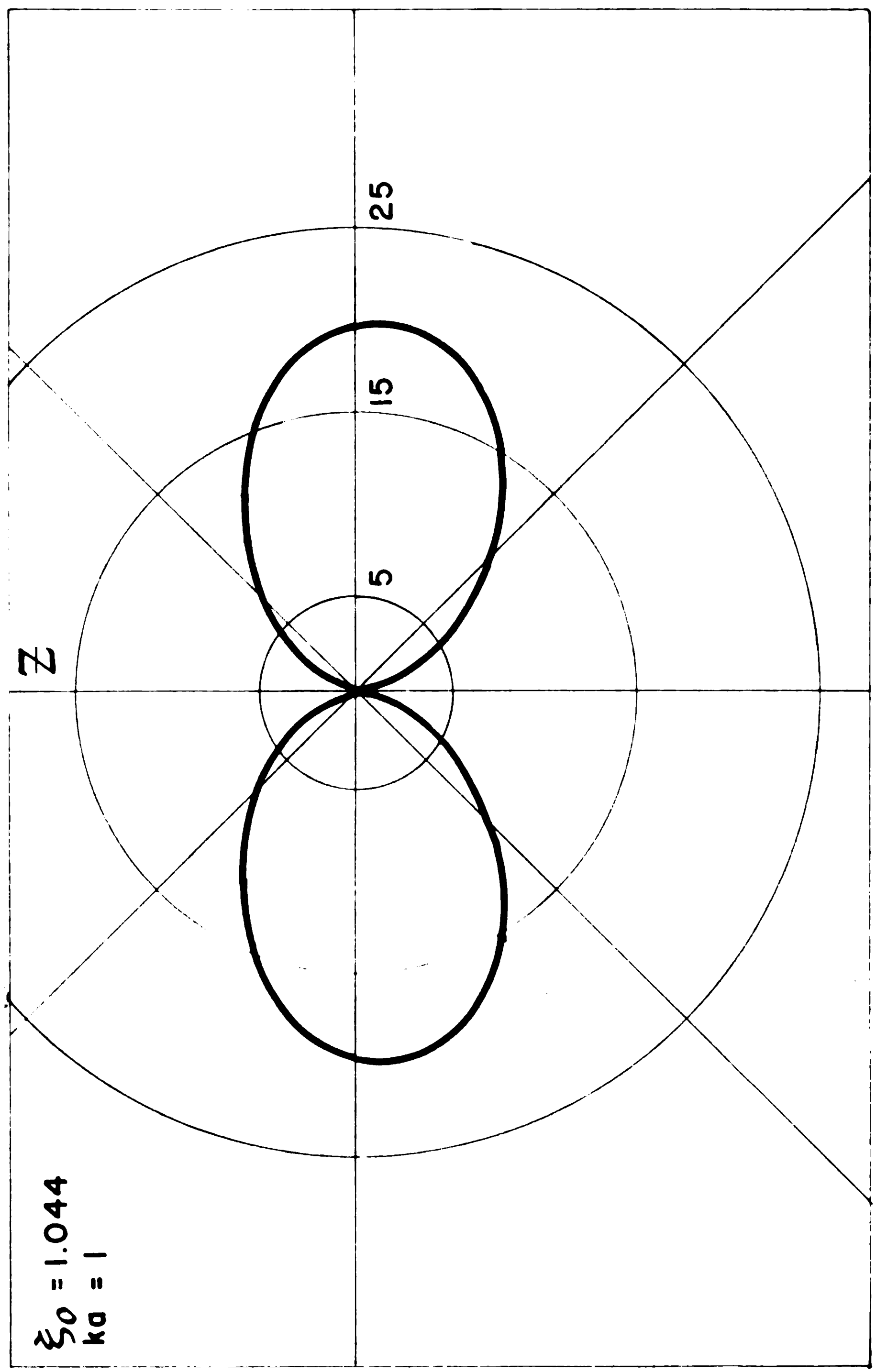

FIG. 6

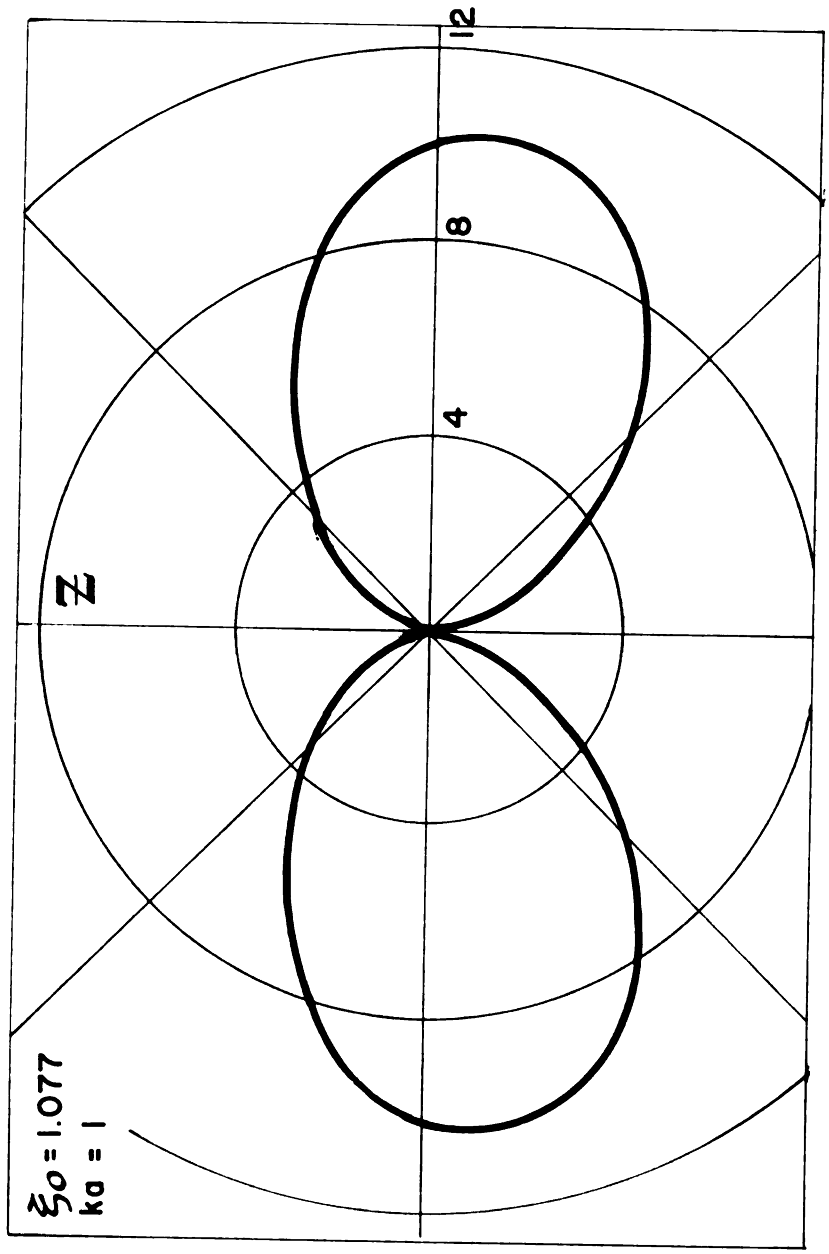

FIG. 7

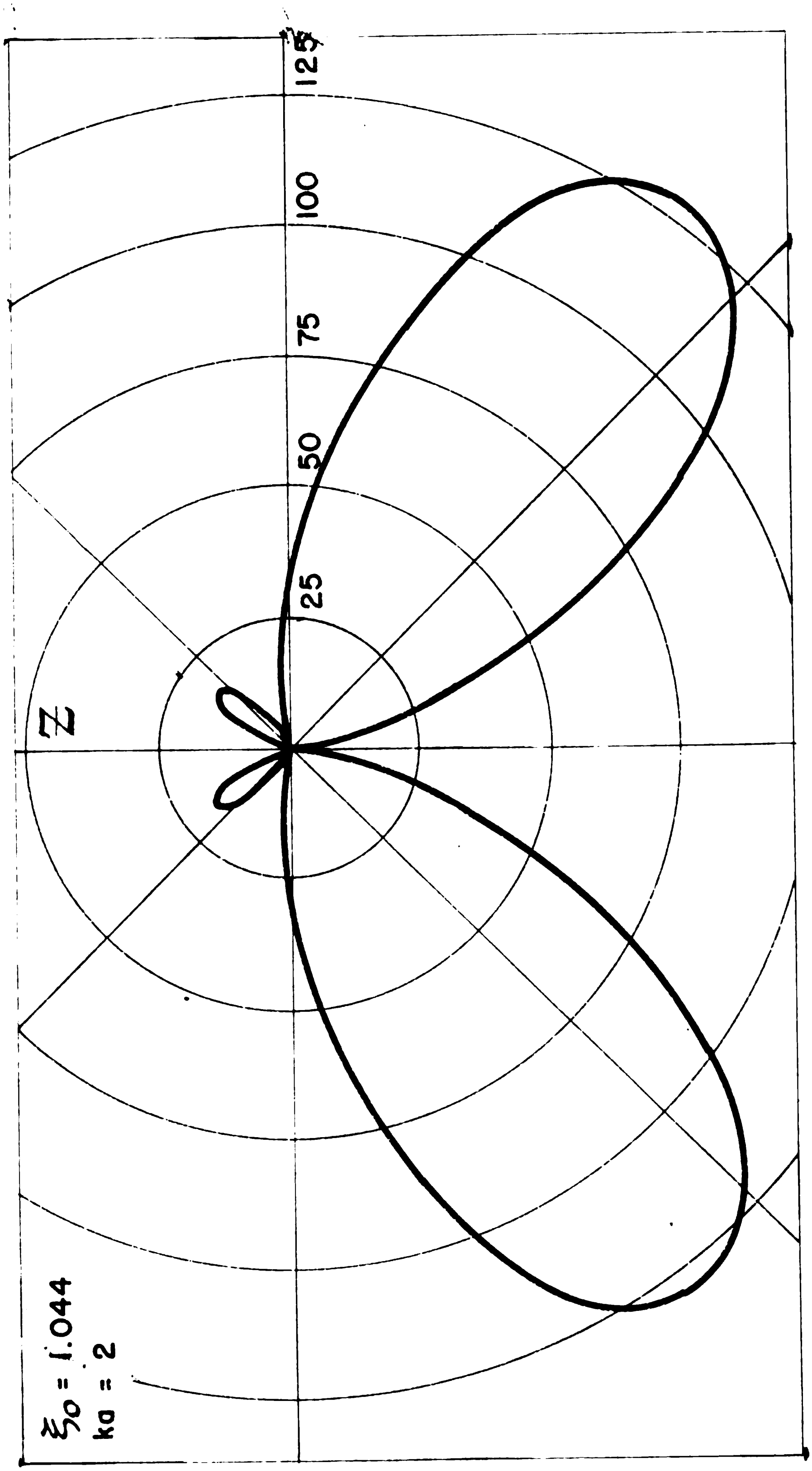

FIG. 10

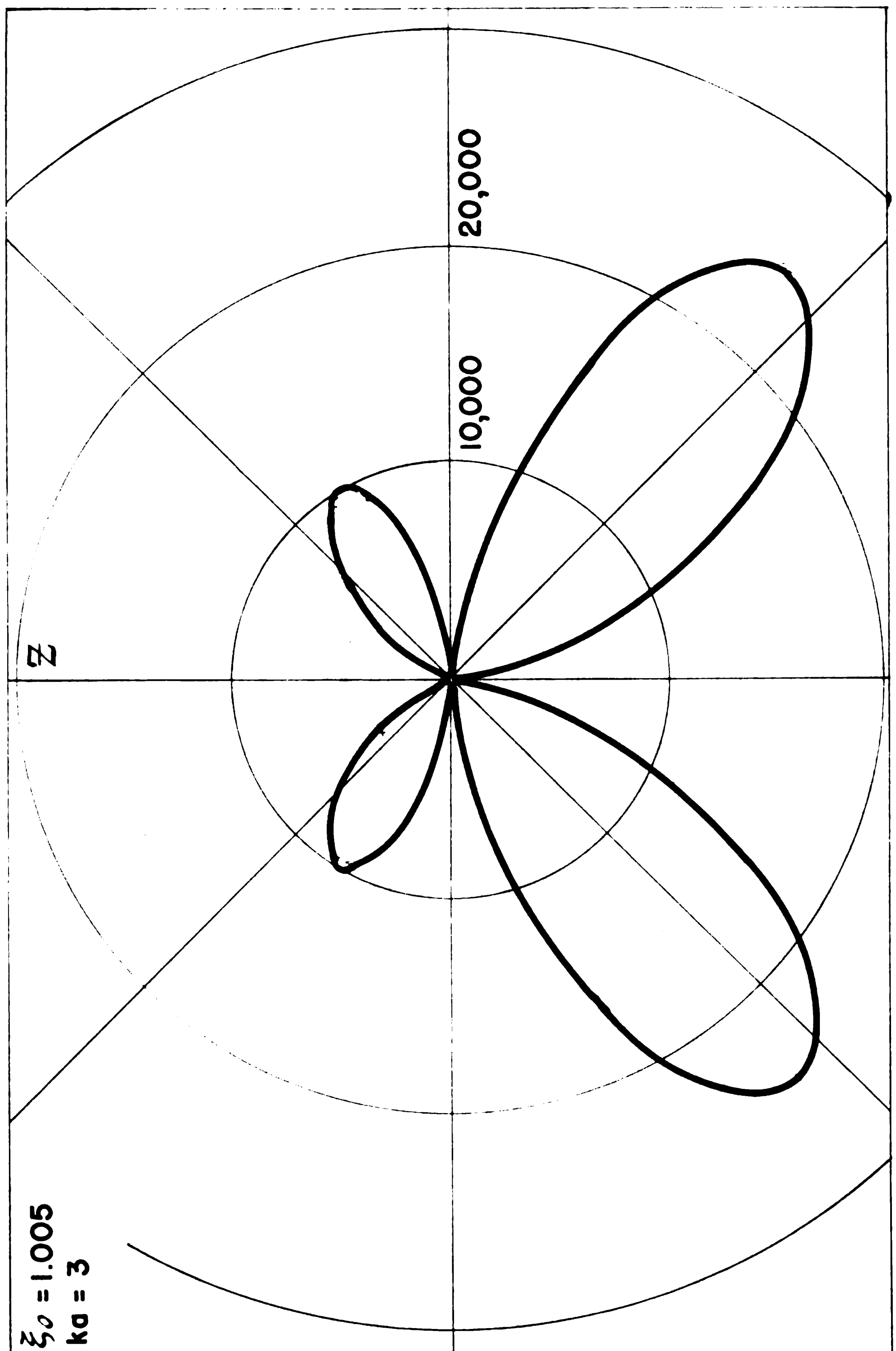

FIG. 12

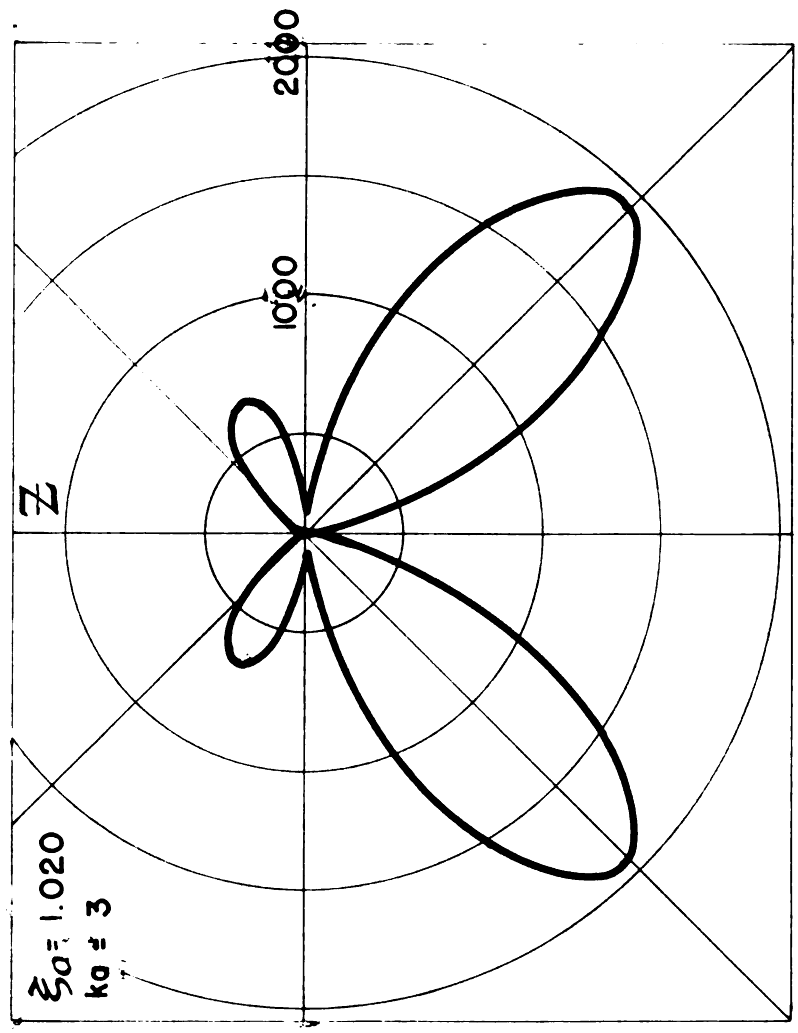

FIG. 13

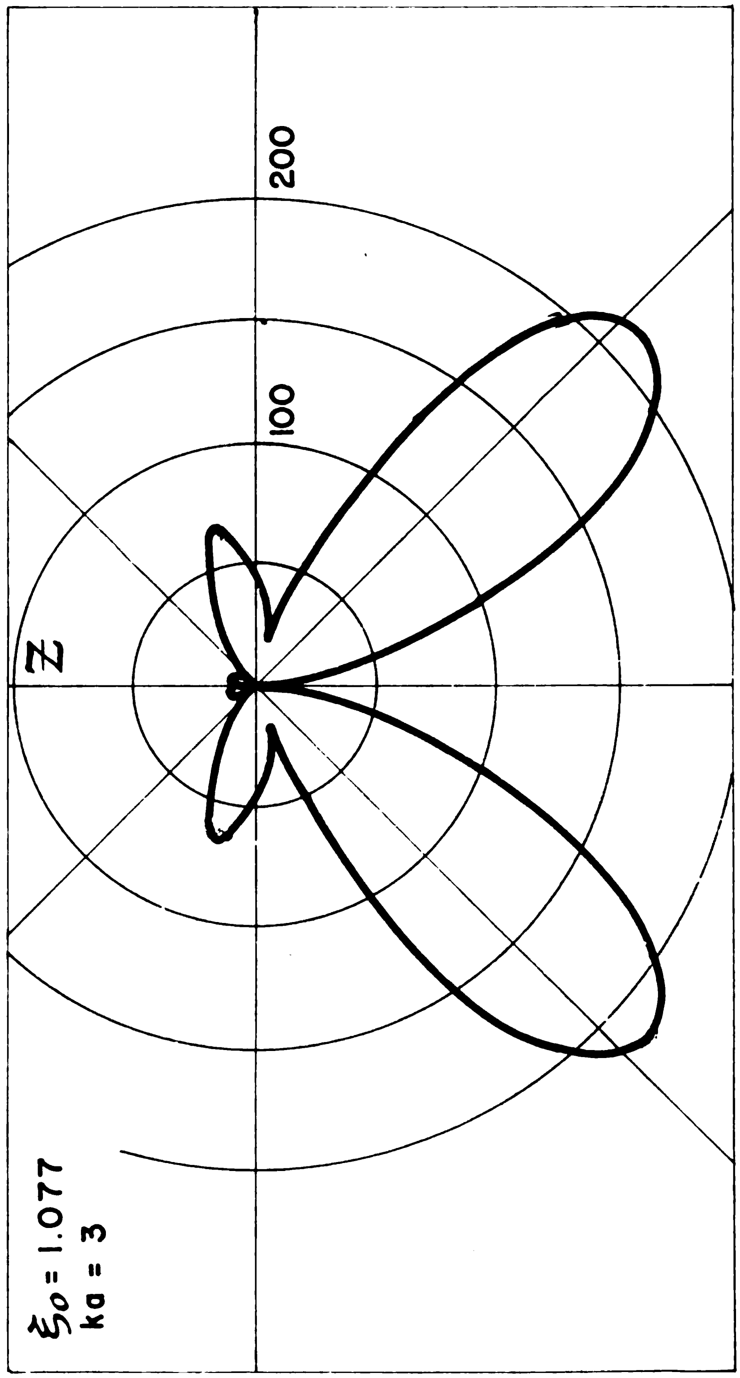

FIG. 15

APPENDIX

TABLE OF CONTENTS

I. A Magnetic Dipole Near the Tip of the Spheroid

The electromagnetic field can in this case be deduced from the component E_φ alone. It must satisfy the boundary condition

$$E_\varphi = 0 \qquad \text{when } \xi = \xi_0 \tag{45}$$

The calculation for Green's function yields

$$G(\xi,\eta;\xi',\eta') = \frac{-k}{2\pi} \sum_{\ell} \frac{S_{1\ell}(\eta)\, S_{1\ell}(\eta')}{N_{1\ell}\, {}^{(3)}R_{1\ell}(\xi_0)} \begin{Bmatrix} {}^{(\mathrm{II})}R_{1\ell}(\xi)\, {}^{(3)}R_{1\ell}(\xi'), & \xi < \xi' \\ {}^{(3)}R_{1\ell}(\xi)\, {}^{(\mathrm{II})}R_{1\ell}(\xi'), & \xi > \xi' \end{Bmatrix} \tag{46}$$

where now

$${}^{(\mathrm{II})}R_{1\ell}(\xi) = {}^{(2)}R_{1\ell}(\xi_0)\, {}^{(1)}R_{1\ell}(\xi) - {}^{(1)}R_{1\ell}(\xi_0)\, {}^{(2)}R_{1\ell}(\xi) \tag{47}$$

in order that $G(\xi,\eta;\xi',\eta')$ satisfy the same boundary condition as E_φ.

The calculation of E_φ yields an expression analogous to equation (37), but now the coefficients of this representation have the form

$$B_\ell = \frac{ka\, C_0^{1\ell}\, {}^{(\mathrm{II})}R_{1\ell}(\xi_1)/\sqrt{\xi_1^2 - 1}}{N_{1\ell}\, {}^{(3)}R_{1\ell}(\xi_0)} \tag{48}$$

where ξ_1, is the coordinate of the dipole at any point of the positive z-axis such that $\xi_1 \geq \xi_0$. Now as $\xi_1 \rightarrow \xi_0$, ${}^{(III)}R_{1\ell}$, and therefore B_ℓ as well as the entire field must vanish.

A simple physical argument can be advanced to account for this fact. When a magnetic dipole is approached to a perfectly conducting surface, with its magnetic moment perpendicular to it, there will arise surface currents producing an opposing magnetic dipole field vanish, as required by general theory. In the limit of zero distance the effect is complete cancellation.

II. The Integration in Equation (35)

We substitute from equations (34) and (36) into the integral (35). The integration variables are the primed coordinates ξ', η', and all quantities are represented in terms of these. Thus one finds, cf. Fig. 3 and equations (1), (2) and (3), that

$$\sin\theta = \rho/R \tag{49}$$

where

$$R = a\sqrt{(\xi'^2-1)(1-\eta'^2)+(\xi'-\xi_1)^2} \tag{50}$$

and

$$\rho = a\sqrt{(\xi'^2-1)(1-\eta'^2)} \tag{51}$$

also $z_1 = a\xi_1$, and $z = a\xi'$ since $\eta = 1-\delta$ (52)

where δ is an arbitrarily small positive number. In substituting from (34) it will be convenient to write down here just a typical term of the infinite series, and furthermore only that part of the term which varies during the integration, i.e.,

$$S_{1\ell}(\eta')\,{}^{(\mathrm{III})}R_{1\ell}(\xi') \tag{53}$$

This clearly is the function to be used in finding the field at points (ξ,η) more distant from the origin than the dipole D itself $(\xi > \xi')$. (If the near field, such as the current on the spheroid, were desired, ${}^{(\mathrm{III})}R_{1\ell}$ would be replaced by ${}^{(3)}R_{1\ell}$).

The integral breaks up into three parts, the top, bottom and sleeve of the cylinder. Since the cylinder is infinitesimal in size, those terms with the highest negative power in R predominate, all others can be neglected. Thus one obtains

$$-\frac{3i\omega}{2}\Big\{ {}^{(\mathrm{II})}R_{1\ell}(\xi^+)\int_1^{1-\delta}\frac{z-z_1}{R^5}\rho e^{ikR}S_{1\ell}(\eta')h_{\eta'}h_{\varphi'}d\eta' - {}^{(\mathrm{II})}R_{1\ell}(\xi^-)\int_1^{1-\delta}\frac{z-z_1}{R^5}\rho e^{ikR}S_{1\ell}(\eta')h_{\eta'}h_{\varphi'}d\eta' + 2S_{1\ell}(1-\delta)\int_{\xi_1^-}^{\xi^+}\rho^2/R^5\, e^{ikR}\,{}^{(\mathrm{II})}R_{1\ell}(\xi')h_{\xi'}h_{\varphi'}d\xi'\Big\} \qquad (54)$$

The choice of proportions of the cylinder is arbitrary, and if we assume it to be long and slender such that

$$\rho \ll a(\xi^+ - \xi_1)$$

then it is easily shown that the first two integrals cancel each other, while in the third

$$\frac{1}{R^5} = \frac{1}{a^5\left[(1-\eta'^2)(\xi'^2-1)+(\xi^+-\xi_1)^2\right]^{5/2}}$$

can be expanded by the binominal theorem.

Since the value of the integral over σ_2' must be independent of its size so long as it enclosed the dipole, we may now shrink it arbitrarily close to the point D and obtain the result that only the first term of the binomial expansion will contribute an amount which is not arbitrarily small. The calculation yields the result

$$-\frac{2i\omega}{a}\left[\frac{S_{1\ell}(1-\delta)}{\sqrt{1-(1-\delta)^2}}\right]_{\delta\to 0}\frac{{}^{(\mathrm{II})}R_{1\ell}(\xi_1)}{\sqrt{\xi_1^2-1}} \qquad (57)$$

One finds that the bracketed expression becomes the coefficient $C_0^{1\ell}$, defined in (67).

Remembering that the actual term of the expansion (34) is not (53) , but $S_{1\ell}(\eta)\,{}^{(3)}R_{1\ell}(\xi)/N_{1\ell}\,{}^{(3)}w_{1\ell}'(\xi_0)$ times (53) , and imposing the summation appearing in (34), one obtains the final result embodied by equations (37) and (38) .

III. Formulas Useful in Computation with Prolate Spheroidal Wave Functions

The partial differential equations (17) and (18) upon separation of the variables yield two ordinary differential equations

$$\frac{d}{d\eta}(\eta^2-1)\frac{dS_{1\ell}}{d\eta}+\left[A_{1\ell}+k^2a^2\eta^2-\frac{1}{1-\eta^2}\right]S_{1\ell}=0 \qquad (58)$$

$$\frac{d}{d\xi}(\xi^2-1)\frac{dR_{1\ell}}{d\xi}+\left[A_{1\ell}+k^2a^2\xi^2-\frac{1}{\xi^2-1}\right]R_{1\ell}=0 \qquad (59)$$

The solution of (58) is given by Stratton et.al.[5] as

$$^{(1)}S_{1\ell}(ka,\eta)=\sum_{n=0,1}^{\infty}{}' d_n^{\ell}\, P_{1+n}^{1}(\eta) \qquad (60)$$

The prime on this summation and those which follow indicate that the sum contains only terms even in n if ℓ is even and odd in n if ℓ is odd.

The norm of the angular functions is given by

$$N_{1\ell}\equiv\int_{-1}^{1}\left[{}^{(1)}S_{1\ell}(\eta)\right]^2 d\eta = 2\sum_{n=0,1}^{\infty}{}'(d_n^{\ell})^2\frac{(n+2)!}{n!\,(2n+3)} \qquad (61)$$

The functions ${}^{(2)}S_{1\ell}(\eta)$ possess singularities at $\eta = \pm 1$ and therefore must be eliminated as a possible solution in problems where the z— axis lies in free space.

The radial equation (59) has two fundamental solutions, ${}^{(1)}R_{1\ell}(\xi)$ and ${}^{(2)}R_{1\ell}(\xi)$. The radial function of the first kind is given by

$$ {}^{(1)}R_{1\ell}(\xi) = \frac{1}{\lambda_{1\ell}} \left(\frac{1}{\xi} \right) \sum_{K=0}^{\infty} C_K^{1\ell} \, (\xi^2 - 1)^{K+1/2} \tag{62} $$

where the factor 1 or ξ are used when ℓ is even or odd, respectively and where

$$ \lambda_{1\ell} = \frac{6\, C_0^{1\ell}}{\sqrt{\pi}\, d_0^{\ell}\, ka} \; \frac{\left(\frac{\ell-1}{2}\right)!}{\left(\frac{\ell}{2}\right)!} \qquad \ell \text{ even} \tag{63} $$

$$ \lambda_{1\ell} = \frac{20\, C_0^{1\ell}}{\sqrt{\pi}\, d_1^{\ell}\, k^2 a^2} \; \frac{\left(\frac{\ell+2}{2}\right)!}{\left(\frac{\ell-1}{2}\right)!} \qquad \ell \text{ odd} \tag{64} $$

The coefficients $C_K^{1\ell}$ are found from the recursion formulas

$$ 2K(2K+2)C_K^{1\ell} + \left\{ (2K+1)(2K-2) + A_{1\ell} + 2 + k^2a^2 \right\} C_{K-1}^{1\ell} + k^2a^2 C_{K-2}^{1\ell} = 0 \qquad \ell \text{ even} \tag{65} $$

$$2K(2K+2)C_K^{1\ell} + \{(2K+1)2K + A_{1\ell} + k^2a^2\} C_{K-1}^{1\ell} + k^2a^2 C_{K-2}^{1\ell} \qquad \ell \text{ odd} \tag{66}$$

and
$$C_0^{1\ell} = \sum_{n=0,1}^{\infty}{}' d_n^{\ell} \frac{(n+2)!}{2\,n!} \tag{67}$$

We did not use the expansion of the radial function of the second kind as defined by Spence[3] for two significant reasons, namely

(i) his expansion is a power series in the variable $\xi^2-1=(\xi+1)(\xi-1)$. If one wishes to compute the value of this series in the range ξ just beyond $+1$ the use of the variable $\xi-1$ alone is clearly of advantage;

(ii) the coefficients in the expansion given by Spence are determined as infinite series, which do not always converge rapidly, whereas the coefficients of an expansion in the powers of $\xi-1$ are determined in closed form from the set of $C_K^{1\ell}$ previously defined.

Consequently, we found the following expansion for ${}^{(2)}R_{1\ell}$ useful:

$${}^{(2)}R_{1\ell}(\xi) = \frac{\lambda_{1\ell}^2}{ka}\,{}^{(1)}R_{1\ell}(\xi)\left\{-\frac{G_0^{\ell}}{\tau} + G_1^{\ell}\ln\tau + K_{\ell} + \sum_j \frac{G_{j+2}}{j+1}\tau^{j+1}\right\} \tag{68}$$

where

$$\tau = \xi - 1$$

$$G_0^{\ell} = 1/e_0^2$$

$$G_1^{\ell} = -2e_1/e_0^3$$

$$G_2^{\ell} = (3e_1^2 - 2e_0e_2)/e_0^4$$

$$G_3^{\ell} = -(4e_1^3 - 6e_0e_1e_2 + 2e_0^2e_3)/e_0^5$$

. . . .

and where

$$e_0 = 2C_0'^{\ell}$$

$$e_1 = C_0'^{\ell} + 4C_1'^{\ell} \quad \text{or} \quad 3C_0'^{\ell} + 4C_1'^{\ell}$$

$$e_2 = 4C_1'^{\ell} + 8C_2'^{\ell} \quad \text{or} \quad C_0'^{\ell} + 8C_1'^{\ell} + 8C_2'^{\ell}$$

$$e_3 = C_1'^{\ell} + 12C_2'^{\ell} + 16C_3'^{\ell} \quad \text{or} \quad 5C_1'^{\ell} + 20C_2'^{\ell} + 18C_3'^{\ell}$$

. . . .

where the first alternative corresponds to even ℓ, the second to odd ℓ. Finally, the constant K_{ℓ} is defined in terms of infinite series over the coefficients d_{η}^{ℓ} which, however, rapidly converge in the range of our calculation, viz.

$$K_l = \frac{C_1^{1l}}{2(C_0^{1l})^3} + \frac{k^2a^2 d_0^l}{6(C_0^{1l})^3 d_{-2}^l}\left\{\frac{\ln 2+1}{2}C_0^{1l} - \frac{1}{2}\sum_{n=2}^{\infty}{}'\left[d_n^l(n+1)(n+2)\sum_{\lambda=1}^{n}\frac{1}{\lambda}\right] - \frac{1}{4}\sum_{n=0}^{\infty}{}'(2n+3)d_n^l - \frac{d_{-2}^l}{4} + \frac{1}{2}\sum_{\mu=4}^{\infty}{}'\left(\frac{d_{-\mu}}{\rho}\right)_{\rho=0}(\mu-1)(\mu-2)\right\} \quad l \text{ even} \tag{69}$$

$$K_l = \frac{1}{4(C_0^{1l})^2} + \frac{C_1^{1l}}{2(C_0^{1l})^3} + \frac{k^2a^2 d_1^l}{10(C_0^{1l})^3 d_{-1}^l}\left\{\frac{\ln 2+1}{2}C_0^{1l} - \frac{1}{2}\sum_{n=1}^{\infty}{}'\left[d_n^l(n+1)(n+2)\sum_{\lambda=1}^{n}\frac{1}{\lambda}\right] - \frac{1}{4}\sum_{n=1}^{\infty}{}' d_n^l(2n+3) + \frac{d_{-1}^l}{4} + \frac{1}{2}\sum_{\mu=3}^{\infty}{}'\left(\frac{d_{-\mu}}{\rho}\right)_{\rho=0}(\mu-1)(\mu-2)\right\} \quad l \text{ odd} \tag{70}$$

Other radial functions arise in the solution of boundary value problems involving prolate spheroids. They are linear combinations of ${}^{(1)}R_{1l}$ and ${}^{(2)}R_{1l}$ given in (62) and (68). Among these is ${}^{(3)}R_{1l}(\xi)$

$${}^{(3)}R_{1l}(\xi) = {}^{(1)}R_{1l}(\xi) + i\,{}^{(2)}R_{1l}(\xi) \tag{71}$$

which for large ξ takes the form

$${}^{(3)}R_{1l}(\xi) = (-i)^{l+2}\frac{e^{ika\xi}}{ka\xi} \tag{72}$$

In conjunction with the assumed time dependence this function satisfies the Sommerfeld radiation condition at infinity.

The extensions of the tables of prolate spheroidal constants and functions by Stratton et.al.[5] and by Spence[3] are given in the following tables.

	A_{13}	A_{14}
$ka =$ 2	-21.94014372	-31.96273891
3	-24.408312175	-34.45404438
4	-27.91179	

d_n^3	ka = 2	ka = 3
n = -9		-0.00003 ρ
-7	0.000004 ρ	0.0004587 ρ
-5	-0.000053 ρ	-0.0026301 ρ
-3	-0.0008113 ρ	-0.019033 ρ
-1	0.0038639	0.042681
1	0.036649	0.18268
3	0.97072	0.93036
5	-0.035648	-0.07719
7	0.0005902	0.00288
9	-0.000002	-0.00002

d_n^4		
n = -8		0.000010 ρ
-6	-0.0000011 ρ	-0.0000664 ρ
-4	-0.000016829 ρ	-0.00045367 ρ
-2	0.00030497	0.0036564
0	0.0031979	0.015612
2	0.068002	0.15410
4	1.0138	1.0246
6	-0.032729	-0.074684
8	0.0004799	0.00246
10	-0.000052	-0.00006

$^{(1)}S_{l\ell}(\cos\Theta)$	$\ell = 3$, ka = 2	$\ell = 3$, ka = 3	$\ell = 4$, ka = 3
$\Theta = 0^\circ$	0.00000000	0.000000000	0.000000000
5°	0.793116	0.716983	1.22076
10°	1.508531	1.37358	2.27047
15°	2.07587	1.913126	2.99461
20°	2.43897	2.28670	3.30957
25°	2.561776	2.46921	3.15275
30°	2.432795	2.40483	2.55889
35°	2.0673668	2.12939	1.62687
40°	1.507390	1.65492	0.514635
45°	0.8179867	1.02993	-0.58363
50°	0.080981	0.32501	-1.47123
55°	-0.6142452	-0.37405	-1.98655
60°	-1.18183	-0.990093	-2.03822
65°	-1.55095	-1.39807	-1.62865
70°	-1.67634	-1.58209	-0.85874
75°	-1.545167	-1.49921	0.9015
80°	-1.1803748	-1.16531	0.99475
85°	-0.638058	-0.63606	1.64097
90°	0.0000000	0.0000000	1.87500

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$C_0^{1\ell}$	$\ell = 3$	$\ell = 4$
ka = 2	9.2397	14.720
3	8.3329	14.324

$\lambda_{1\ell}$		
ka = 2	999.69	12,946
3	190.06	1731.5

$N_{1\ell}$		
ka = 2	4.2142	5.6301
3	3.9656	

$^{(1)}R_{1\ell}$	$\ell=3$, ka=2	$\ell=3$, ka=3	$\ell=4$, ka=2	$\ell=4$, ka=3
ξ = 1.005	.00094393	.004461	.00011757	.00085277
1.020	.0020082	.0094068		.0018632
1.044	.0032753	.015113		.0031594
1.077	.004801,4	.022153		.0049581

$^{(1)}R'_{1\ell}$				
ξ = 1.005	.098346	.46222	.012538	.090413
1.020	.058350	.26741		.057322
1.044	.049859	.22060		.052962
1.077	.049563	.21013		.056820

${}^{(2)}R_{1\ell}$	$\ell=3$, ka=2	$\ell=3$, ka=3	$\ell=4$, ka=2	$\ell=4$, ka=3
ξ=1.005	-262.46	-31.977	-1792.1	-166.62
1.020	-152.07	-10.39		-60.65
1.044	-146.3	-2.26		-28.
1.077	-2×10^{2}	2.8		

${}^{(2)}R'_{1\ell}$				
ξ=1.005	52838.	4.1×10^{3}	2.3309×10^{5}	21325.
1.020	1743.2	581.		2560.
1.044	-536.5	2.1×10^{2}		7×10^{2}
1.077	-1×10^{3}	1.2×10^{2}		

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