SOME PROPERTIES OF THE LAMÉ POLYNOMIAL

Thesis for the Degree of M. S.

MICHIGAN STATE UNIVERSITY

Basil C. Halkides

1961

PLACE IN RETURN BOX
to remove this checkout from your record.
TO AVOID FINES return on or before date due.

DATE DUE	DATE DUE	DATE DUE

SOME PROPERTIES

OF THE LAMÉ POLYNOMIAL

by

Basil C. Halkides

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

MASTER OF SCIENCE

Department of Physics

1961

G 19789
2/27/62

ACKNOWLEDGMENT

The author wishes to express his gratitude to Dr. A. Leitner for his considerable amount of help and guidance in this work. The author would also like to thank Dr. R. D. Spence for his suggestions.

ABSTRACT

Solution of the Schroedinger equation for a particle in a central field in sphero-conal coordinates results in Lamé functions. The properties of these functions studied in this thesis are, existence of polynomial solutions, orthogonality, relationship to spherical harmonics, normalization, and nature of the eigenvalues.

TABLE OF CONTENTS

TABLES

FIGURES

I. INTRODUCTION

This thesis may be considered a departure from a paper by Professor R. D. Spence[1], and some of the material in the first two sections following the introduction is merely a verification of results which he obtained, shown in greater detail. These results are then used in subsequent work.

Upon investigating the orthogonality property, we find the rather unexpected result that these functions are not completely orthogonal. Of special interest to us is the confluence that results when we vary certain parameters, and causes the sphero-conal coordinate system to become a spherical coordinate system. This confluence has been studied in some detail.

Some recurrence relations of Lamé polynomials have been obtained by Arscott[2], and a general introduction to Lamé functions can be found in[3]. The equivalence between the form of Lamé's equation used by Arscott and the form used in this work can be shown by the transformation $\zeta = k\,cn(z)$. Similiar work has also been done by Kraus[4].

II. THE COORDINATE SYSTEM

The sphero-conal coordinates, r , ζ , and η are defined by the following equations:

$$x = \pm \frac{r}{\beta}\left[\left(\beta^2 + \zeta^2\right)\left(\beta^2 - \eta^2\right)\right]^{1/2} \tag{1a}$$

$$y = \frac{r}{\alpha\beta}\,\zeta\,\eta \tag{1b}$$

$$z = \pm \frac{r}{\alpha}\left[\left(\alpha^2 - \zeta^2\right)\left(\alpha^2 + \eta^2\right)\right]^{1/2} \tag{1c}$$

where α and β are constant for any particular coordinate system, but always satisfy the condition $\alpha^2 + \beta^2 = 1$. The ranges of ζ and η will be discussed later.

The system can be shown to be orthogonal, with the following metric coefficients:

$$h_1 = 1 \qquad h_2 = r\left[\frac{\zeta^2 + \eta^2}{\left(\beta^2 + \zeta^2\right)\left(\alpha^2 - \zeta^2\right)}\right]^{1/2} \quad h_3 = r\left[\frac{\zeta^2 + \eta^2}{\left(\beta^2 - \eta^2\right)\left(\alpha^2 + \eta^2\right)}\right]^{1/2}$$

where h_2 and h_3 are the metrics associated with ζ and η respectively.

From equations (1a), (1b), and (1c) we can obtain, by eliminating two of the sphero-conal coordinates in each case,

$$\frac{X^2}{\beta^2+\zeta^2} + \frac{Y^2}{\zeta^2} - \frac{Z^2}{\alpha^2-\zeta^2} = 0$$

$$\frac{Z^2}{\alpha^2+\eta^2} + \frac{Y^2}{\eta} - \frac{X^2}{\beta^2-\eta^2} = 0$$

$$X^2 + Y^2 + Z^2 = r^2$$

These equations show the surfaces ζ = constant and η = constant to be elliptic cones about the Z and X axis respectively, and the surface r = constant a sphere. These are sketched in Figure 1.

Clearly setting ζ or η equal to zero puts us on the xz plane. If $\zeta = 0$, then

$$X^2(\alpha^2+\eta^2) - Z^2(\beta^2-\eta^2) = 0$$

which, for a given η is the equation of two straight lines.

$$X = \pm\left[\frac{\beta^2-\eta^2}{\alpha^2+\eta^2}\right]^{1/2} Z$$

As we vary η from 0 to β, the slope of these lines goes from $\pm\beta/\alpha$ to 0, giving us a sector of the xz plane.

If $\eta = 0$, we obtain

$$X = \pm\left[\frac{\beta^2+\zeta^2}{\alpha^2-\zeta^2}\right]^{1/2} Z$$

These two lines have slope varying from β/α at $\zeta = 0$ to ∞ as $\zeta \to \alpha$ giving the remaining sector of the xz plane. This is sketched in Figure 2.

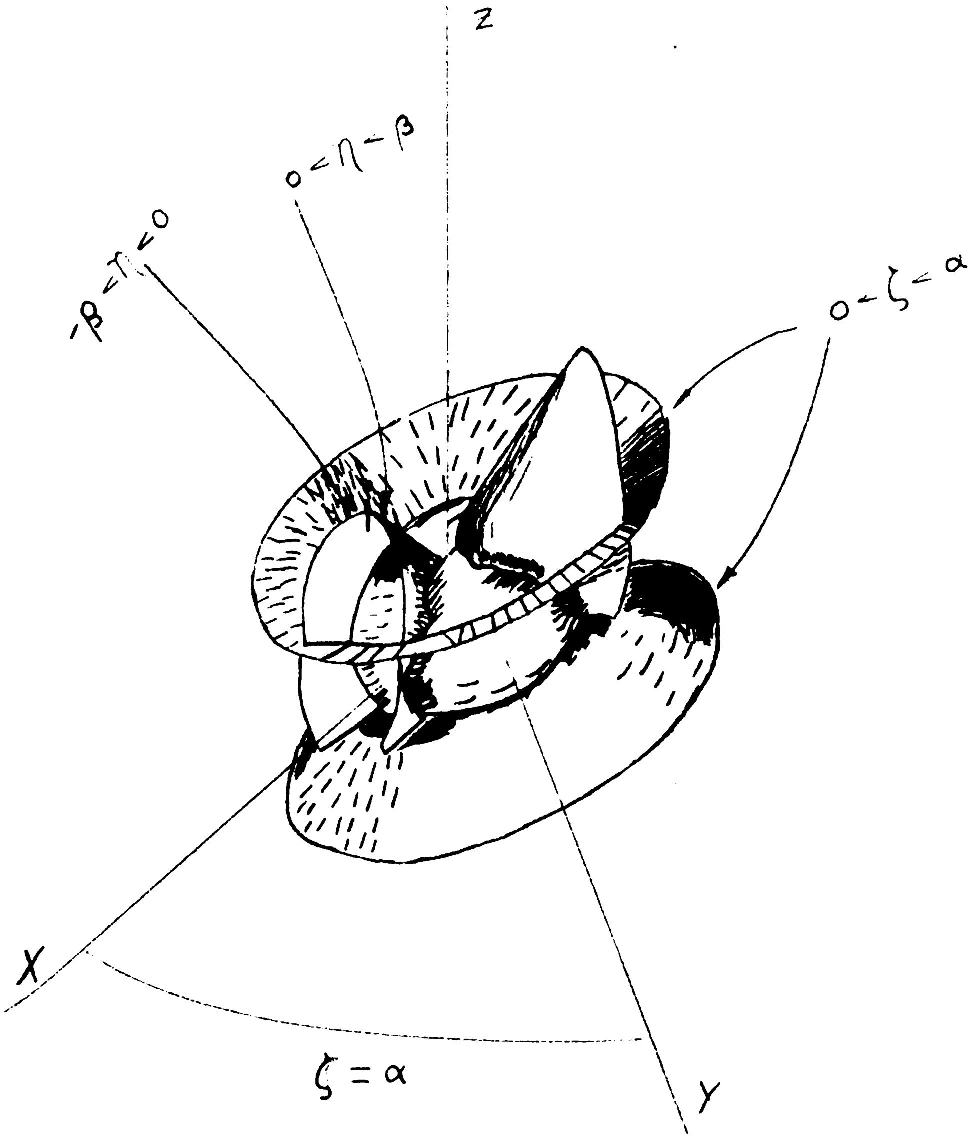

Figure 1. Coordinate Surfaces for Sphero-conal Coordinates

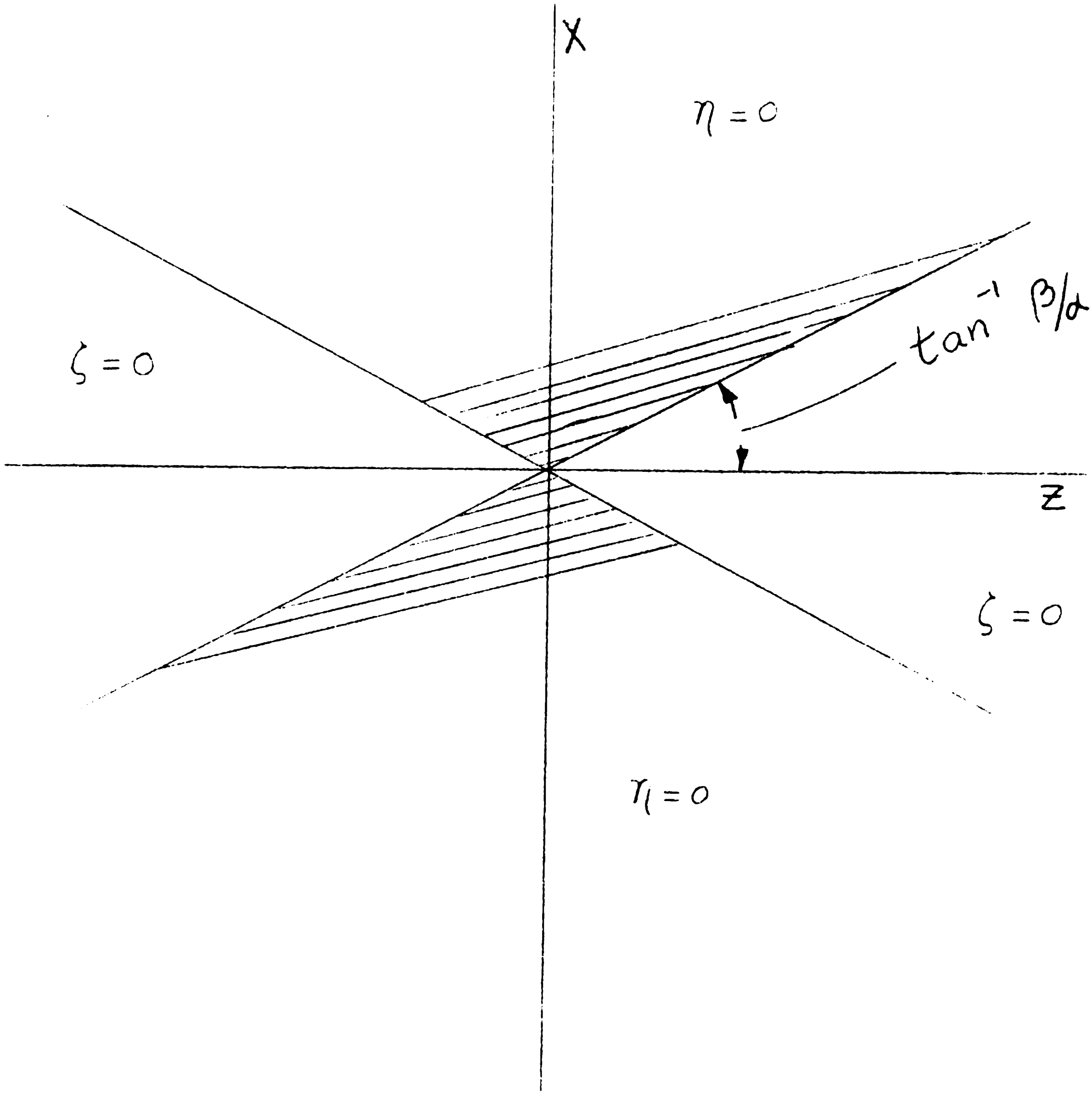

Figure 2. The sectors of the xz plane corresponding to $\zeta = 0$ and $\eta = 0$

When both ζ and η are equal to zero, we obtain the equation of the lines which border the sectors, $x = \pm \beta/\alpha\, z$.

Now at $\zeta = 0$, $x = r(\beta^2 + \zeta^2)^{1/2}$ and $z = r(\alpha^2 - \zeta^2)^{1/2}$ therefore, $x^2 + z^2 = r^2(\alpha^2 + \beta^2)$. This shows that the restriction $\alpha^2 + \beta^2 = 1$ if necessary if r is to have the meaning of "distance from the origin".

To include all points in space, x and z must be defined with a $\pm$ sign before the radical (cf. equations 1a and 1c). This creates a nonuniqueness whereby to a given r , ζ , and η there correspond four points in space. In addition to this, a given point in space may be described by two sets of sphero-conal coordinates, namely r , $-\zeta$, $-\eta$ and r , ζ , η . The former problem cannot be avoided. The latter multiplicity can be resolved as follows. We can restrict ζ to $0 \leq \zeta \leq \alpha$ and allow η somewhat greater freedom $-\beta \leq \eta \leq \beta$. Thus negative values of η will take care of points on one side of the xz plane, and positive η's the other side.

The choice of restricting ζ may seem arbitrary; however, as we shall see later, in the limit of β going to zero, ζ becomes sin θ, and η/β goes to sin ϕ. Since θ varies from 0 to π and ϕ from 0 to 2π, it seems logical to limit ζ , and not η , to nonnegative values.

If we let α (or β) go to zero, our sphero-conal coordinate system degenerates to a spherical system with the x and z axis, respectively.

being the polar axis. For example, if we let β go to zero, this forces η to go to zero also, for equation (1a) tells us that if we do not place the restriction $-\beta \leq \eta \leq \beta$, we then allow x to take on imaginary values. Equation (1c) gives us the reduction

$$\lim_{\beta \to 0} \zeta = \sin\theta \tag{2}$$

and equations (1a) and (1b) tell us that as η and β go to zero, we must have

$$\lim_{\beta \to 0} \eta/\beta = \sin\phi \tag{3}$$

This reduction brings us to the conventional spherical coordinate system.

If we let $\beta \to 1$ ($\alpha \to 0$), the reduction becomes

$$\lim_{\beta \to 1} \eta = \sin\theta' \tag{4}$$

$$\lim_{\beta \to 1} \zeta/\alpha = \sin\phi' \tag{5}$$

which leads to the spherical coordinate system with x as polar axis. The angle θ' is measured from the z axis, and ϕ' is an azimuthal angle in the yz plane.

These reductions will be of value later.

The time independent Schroedinger equation for a central field is sphero-conal coordinates is

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{(\beta^2+\zeta^2)^{1/2}(\alpha^2-\zeta^2)^{1/2}}{r^2(\zeta^2+\eta^2)}\frac{\partial}{\partial\zeta}(\beta^2+\zeta^2)^{1/2}(\alpha^2-\zeta^2)^{1/2}\frac{\partial\psi}{\partial\zeta}\right.$$

$$\left.+\frac{(\alpha^2+\eta^2)^{1/2}(\beta^2-\eta^2)^{1/2}}{r^2(\zeta^2+\eta^2)}\frac{\partial}{\partial\eta}(\alpha^2+\eta^2)^{1/2}(\beta^2-\eta^2)^{1/2}\frac{\partial\psi}{\partial\eta}\right]+V(r)\psi=E\psi \qquad (6)$$

We assume a solution of the type

$$\psi=R(r)\,Z(\zeta)\,H(\eta)$$

Separating the terms depending on ζ and η from those depending on r gives

$$\frac{\left[(\beta^2+\zeta^2)(\alpha^2-\zeta^2)\right]^{1/2}}{Z\,(\zeta^2+\eta^2)}\frac{d}{d\zeta}\left[(\beta^2+\zeta^2)(\alpha^2-\zeta^2)\right]^{1/2}\frac{dZ}{d\zeta}$$

$$+\frac{\left[(\alpha^2+\eta^2)(\beta^2-\eta^2)\right]^{1/2}}{H\,(\zeta^2+\eta^2)}\cdot\frac{d}{d\eta}\left[(\alpha^2+\eta^2)(\beta^2-\eta^2)\right]^{1/2}\frac{dH}{d\eta}=-l(l+1) \qquad (7)$$

If we now separate the terms depending on ζ from those depending on η we introduce a new separation constant, λ .

$$\frac{\left[(\beta^2+\zeta^2)(\alpha^2-\zeta^2)\right]^{1/2}}{Z}\frac{d}{d\zeta}\left[(\beta^2+\zeta^2)(\alpha^2-\zeta^2)\right]^{1/2}\frac{dZ}{d\zeta}+\ell(\ell+1)\zeta^2=\lambda \qquad (8a)$$

$$\frac{\left[(\alpha^2+\eta^2)(\beta^2-\eta)\right]^{1/2}}{H}\frac{d}{d\eta}\left[(\alpha^2+\eta^2)(\beta^2-\eta^2)\right]^{1/2}\frac{dH}{d\eta}+\ell(\ell+1)\eta^2=-\lambda \qquad (8b)$$

It is obvious from these two equations that any discussion of one also applies to the other. In fact,

$$H(\eta)=c\,Z(i\eta) \qquad (9)$$

The solutions of these two equations are called Lamé functions. We choose to discuss equation (8a), which becomes

$$Z''+\frac{\zeta(\alpha^2-\beta^2-2\zeta^2)}{(\beta^2+\zeta^2)(\alpha^2-\zeta^2)}Z'+\frac{\ell(\ell+1)\zeta^2-\lambda}{(\beta^2+\zeta^2)(\alpha^2-\zeta^2)}Z=0 \qquad (8a)$$

This equation is now in the form

$$Z''+p(\zeta)Z'+q(\zeta)Z=0$$

We see immediately that both p and q have poles of order 1 at $\zeta=\pm\alpha$ and at $\zeta=\pm i\beta$, making these points regular singular points of the differential equation. The point at infinity also turns out to be a regular singular point.

The indicial roots at the four finite singular points are 0 and 1/2 in each case. At infinity, they are $\frac{1}{2} \pm \sqrt{\frac{1}{4} + \ell(\ell+1)}$. As a check, we may note that all the indicial roots, including the two at infinity, add to a total of 3 (3, p. 203, footnote).

Of the various possible combinations of indicial roots at each singularity, we choose to eliminate some for convenience. We write the solutions to equation (8a) as follows:

$$Z(\zeta) = (\alpha^2 - \zeta^2)^{K_1/2} (\beta^2 + \zeta^2)^{K_2/2} \sum_{J=0}^{\infty} \delta_{2J+K_3} \zeta^{2J+K_3} \tag{10}$$

K_1, K_2, and K_3 can each be 1 or 0. This form eliminates, for example, solutions in which $\zeta + i\beta$ and $\zeta - i\beta$ appear explicitly in different powers.

Substitution of this solution into equation (8a) yields the recursion relation

$$c_J \delta_{2J+K_3+2} + d_J \delta_{2J+K_3} + e_J \delta_{2J+K_3-2} = 0 \tag{11}$$

$$c_J = \alpha^2 \beta^2 (2J + K_3 + 2)(2J + K_3 + 1)$$

$$d_J = (2J+K_3)\left[(2J + K_3 - 1)(\alpha - \beta^2) + (2K_2+1)\alpha^2 - (2K_1+1)\beta^2\right] + K_2\alpha^2 - K_1\beta^2 - \lambda$$

$$e_J = (2J+K_3-2)(2J + K_3 + 2K_1 + 2K_2 - 1) + 2K_1 + 2K_2 + 2K_1K_2 - \ell(\ell+1)$$

III. THE POLYNOMIAL SOLUTIONS

Referring to equation (11), we now proceed to show that, by virtue of the special form of d_J and e_J, these recursions permit certain terminated solutions. These then lead to sphero-conal functions (10) containing polynomials.

In the reduction $\beta = 0 \ (\alpha = 1)$, or $\beta = 1 \ (\alpha = 0)$ the sphero-conal coordinates lead to the spherical coordinates. It is the above referred to polynomial solutions of the sphero-conal operator which reduce to the spherical harmonics. We will call them sphero-conal harmonics.

Clearly, for the series in equation (10) to terminate, two consecutive δ's must vanish. Let us call these two δ_{2n+K_3+2} and δ_{2n+K_3+4}, thus making δ_{2n+K_3} the last non-zero coefficient.

Two conditions must be satisfied for this to happen. First, to insure $\delta_{2n+K_3+2} = 0$, we need

$$\frac{d_n}{c_n} \delta_{2n+K_3} + \frac{e_n}{c_n} \delta_{2n+K_3-2} = 0 \qquad \text{I}$$

and second, for δ_{2n+K_3+4} to vanish also, we must have

$$\frac{e_{n+1}}{c_{n+1}} = 0 \qquad . \ \text{II}$$

We can multiply out the C_n and C_{n+1}, because they are not equal to zero except when α or β is zero. Negative values of n of course, are of no significance in our Taylor series.

We examine condition II first.

$$e_{n+1} = [2(n+1)+K_3+2][2(n+1)+K_3+2K_1+2K_2-1] + 2K_1 + 2K_2 + 2K_1K_2 - \ell(\ell+1) = 0$$

This equation is quadratic in n. We keep in mind that the number n will govern the degree of the polynomial, i.e., $2n+K_3$. Since all of the K_i are one or zero, we can use the fact that $K_i^2 = K_i$. For convenience, we shall give the symbol σ to $K_1 + K_2 + K_3$.

$$4n^2 + (4\sigma+2)n + 2(K_1K_3 + K_2K_3 + K_1K_2 + \sigma) - \ell(\ell+1) = 0$$

$$n = \frac{-(4\sigma+2) \pm 2\sqrt{1+4\ell(\ell+1)}}{8}$$

One can see here why the first separation constant was called $\ell(\ell+1)$ instead of simply C. The form $\ell(\ell+1)$ is needed to form a perfect square. This gives us the following expression for n:

$$n = 1/2(\ell-\sigma) \tag{12}$$

This tells us that for the series to be a polynomial, we must have the restriction that ℓ be an integer. In fact, $\ell - \sigma$ must be an even integer.

We now examine condition I.

$$d_n \delta_{2n+K_3} + e_n \delta_{2n+K_3-2} = 0$$

From the nature of the d_n we can see that this condition places a restriction on λ. It generates eigenvalues in λ. Some examples of how this takes place follow.

If $\ell = 0$ we see that $K_1 = K_2 = K_3 = 0$ and $n = 0$. Condition I is then simply $d_0 \delta_0 = 0$, and since δ_0 is an arbitrary constant $d_0 = 0$. This gives us $\lambda = 0$.

If $\ell = 1$ again n must be zero, but now one of the K_i must be 1.

(a) If $K_3 = 1, \quad K_1 = K_2 = 0 \qquad d_0 \delta_1 = 0$

$$d_0 = \alpha^2 - \beta^2 - \lambda = 0 \qquad \lambda = \alpha^2 - \beta^2 = 1 - 2\beta^2$$

(b) If $K_1 = 1, \quad K_2 = K_3 = 0$

$$\lambda = -\beta^2$$

(c) If $K_2 = 1, K_1 = K_3 = 0$

$$\lambda = \alpha^2 = 1 - \beta^2$$

If $\ell = 2$ we can have σ either 2 or 0. If $\sigma = 0$, $n = 1$, the condition for $\delta_{2J+K_3} = 0 \quad J > n$ becomes $\dfrac{\delta_2}{\delta_0} = -\dfrac{e_1}{d_1}$.

but we know from the recursion that $c_0 \delta_2 = d_0 \delta_0$, so

$$\frac{e_1}{d_1} = -\frac{d_0}{c_0} = \frac{\lambda}{2\alpha^2\beta^2}$$

$$\frac{e_1}{d_1} = \frac{-\ell(\ell+1)}{2(2\alpha^2-2\beta^2)-\lambda} = \frac{-6}{4(\alpha^2-\beta^2)-\lambda}$$

The quadratic equation $\frac{-6}{4(\alpha^2-\beta^2)-\lambda} = \frac{\lambda}{2\alpha^2\beta^2}$ yields the two solutions

$$\lambda = 2[\alpha^2-\beta^2 \pm \sqrt{1-\alpha^2\beta^2}\,]$$

If $\sigma = 2$ we get again $n = 0$, or a one term polynomial with the condition $d_0 = 0$. This becomes

$$K_3[(2K_2+1)\alpha^2-(2K_1+1)\beta^2] + K_2\alpha^2 - K_1\beta^2 - \lambda = 0$$

There are three ways to have $\sigma = 2$

(a) $K_3 = 0 \qquad K_1 = K_2 = 1$

$$\lambda = 1 - 2\beta^2$$

(b) $K_1 = 0 \quad K_2 = K_3 = 1$

$$\lambda = 4 - 5\beta^2$$

(c) $K_2 = 0 \quad K_1 = K_3 = 1$

$$\lambda = 1 - 5\beta^2$$

We have thus obtained 5 values for λ when $\ell = 2$, 3 values for $\ell = 1$, and 1 value for $\ell = 0$.

The general condition restricting eigenvalues is obtained as follows:

Consider the following forms of the recursion.

$$\delta_{2+K_3} = -\frac{d_0}{c_0}\delta_{K_3}$$

$$\delta_{4+K_3} = -\frac{d_1}{c_1}\delta_{2+K_3} - \frac{e_1}{c_1}\delta_{K_3} = \left[-\frac{d_1}{c_1} + \frac{e_1}{c_1}\frac{c_0}{d_0}\right]\delta_{2+K_3}$$

$$\delta_{6+K_3} = -\frac{d_2}{c_2}\delta_{4+K_3} - \frac{e_2}{c_2}\delta_{2+K_3} = \left\{-\frac{d_2}{c_2} - \frac{e_2/c_2}{-\frac{d_1}{c_1} + \frac{e_1}{c_1}\frac{c_0}{d_0}}\right\}\delta_{4+K_3}$$

clearly we can always write δ_{2J+K_3} in terms of δ_{2J+K_3-2} and a finite continued fraction. The condition for termination of the series at $2n+K_3$ gives us

$$\frac{\delta_{2n+K_3}}{\delta_{2n+K_3-2}} = \frac{e_n}{d_n}$$

and application of this condition to the finite continued fraction giving δ_{2n+K_3} in terms of δ_{2n+K_3-2} results in an equation involving only the c_J, the d_J, and the e_J. This equation will also be in the form of a finite continued fraction, and it will be from this equation that we can obtain eigenvalue solutions for λ.

$$-\frac{e_n}{d_n} = -\frac{d_{n-1}}{c_{n-1}} - \cfrac{\frac{e_{n-1}}{c_{n-1}}}{-\frac{d_{n-2}}{c_{n-2}} - \cfrac{\frac{e_{n-2}}{c_{n-2}}}{-\frac{d_{n-3}}{c_{n-3}} - \ddots \cfrac{-\frac{e_2}{c_2}}{-\frac{d_1}{c_1} - \cfrac{\frac{e_1}{c_1}}{-\frac{d_0}{c_0}}}}}$$

To solve this equation for λ, we may recall that λ appears linearly in each of the d_J. If we put the right side over a common denominator, there will be a term which is a product of n of the d_J, i.e., which will contain the n th power of λ. Cross multiplication by d_n raises this to the $(n+1)$ st power, thus giving us $n+1$ values for λ.

Now let us look at what happens for fixed ℓ. For convenience, we separate the case where ℓ is odd from the case where ℓ is even.

If ℓ is odd, then σ must be odd. There are four possible combinations of the K's resulting in odd σ.

	K_1	K_2	K_3
a)	1	0	0
b)	0	1	0
c)	0	0	1
d)	1	1	1

This gives us 3 cases with $\sigma = 1$ and 1 case with $\sigma = 3$. There are $n+1$ solutions for λ for each case. Thus the total number of different eigenvalues for fixed ℓ will be, using equation (12),

$$3\left[\tfrac{1}{2}(\ell-1)+1\right] + 1\left[\tfrac{1}{2}(\ell-3)+1\right] = 2\ell+1$$

If ℓ is even, again 4 combinations of the K 's are possible, this time for even σ.

	K_1	K_2	K_3
a)	0	0	0
b)	1	1	0
c)	1	0	1
d)	0	1	1

We have 3 cases with $\sigma = 2$ and 1 case with $\sigma = 0$. The total number of eigenvalues will be

$$3\left[\tfrac{1}{2}(\ell-2)+1\right] + 1\left[\tfrac{1}{2}(\ell-0)+1\right] = 2\ell+1$$

The conclusion is that for any ℓ there will be $2\ell+1$ eigenvalues of λ.

It should be emphasized that each of these eigenvalues is a function of α or of β. Either of these parameters depends on the other through the relation $\alpha^2+\beta^2=1$ and only one of them is, therefore, significant. We will choose β as the variable.

We can now write these eigenvalues as $\lambda_\ell^\tau(\beta)$ where τ will order the $2\ell+1$ eigenvalues associated with each ℓ. We shall let τ take on integer values from $-\ell$ to ℓ.

Upon considering λ as a function of β, we find that no two λ's associated with a single ℓ cross, except at the end points, $\beta=0$ and $\beta=1$. We can thus order τ by the sizes of the eigenvalues themselves taken at any point except $\beta=0$ or $\beta=1$. This means that the largest λ_ℓ^τ will have $\tau=\ell$, the next largest $\tau=\ell-1$ etc. down to $\tau=-\ell$. Some of these eigenvalues are now listed in table 1.

To each eigenvalue λ_ℓ^τ there corresponds a Z_ℓ^τ in the form of equation (10), since a given λ_ℓ^τ has definite K's associated with it. A list of some of the eigenfunctions is given in table 2. It should be noted that until we decide upon a norm for these functions, both δ_0 and δ_1 are arbitrary constants, one corresponding to the odd and one to the even polynomials.

Table 1. List of Eigenvalues

$$\lambda_0^0 = 0$$

$$\lambda_1^{-1} = -\beta^2$$

$$\lambda_1^0 = 1 - 2\beta^2$$

$$\lambda_1^1 = 1 - \beta^2$$

$$\lambda_2^{-2} = 2(1 - 2\beta^2 - \sqrt{1 - \beta^2 + \beta^4})$$

$$\lambda_2^{-1} = 1 - 5\beta^2$$

$$\lambda_2^0 = 1 - 2\beta^2$$

$$\lambda_2^1 = 4 - 5\beta^2$$

$$\lambda_2^2 = 2(1 - 2\beta^2 + \sqrt{1 - \beta^2 + \beta^4})$$

$$\lambda_3^{-3} = 2 - 7\beta^2 - 2\sqrt{1 - \beta^2 + \beta^4}$$

$$\lambda_3^{-2} = 5 - 10\beta^2 - 2\sqrt{4 - 7\beta^2 + 4\beta^4}$$

$$\lambda_3^0 = 4 - 8\beta^2$$

$$\lambda_3^1 = 2 - 7\beta^2 + 2\sqrt{1 - \beta^2 + \beta^4}$$

$$\lambda_3^2 = 5 - 10\beta^2 + 2\sqrt{4 - \beta^2 + \beta^4}$$

$$\lambda_3^3 = 5 - 7\beta^2 + 2\sqrt{4 - 7\beta^2 + 4\beta^4}$$

Table 2. List of Eigenfunctions

$$Z_0^0 = \delta_0$$

$$Z_1^{-1} = (\alpha^2 - \zeta^2)^{1/2} \delta_0$$

$$Z_1^0 = \zeta \delta_1$$

$$Z_1^1 = (\beta^2 + \zeta^2)^{1/2} \delta_0$$

$$Z_2^{-2} = \delta_0 \left[1 + \frac{\lambda_2^{-2} \zeta^2}{\beta^2(1-\beta^2)} \right]$$

$$Z_2^{-1} = (\alpha^2 - \zeta^2)^{\frac{1}{2}} \zeta \delta_1$$

$$Z_2^0 = \left[(\alpha^2 - \zeta^2)(\beta^2 + \zeta^2) \right]^{1/2} \delta_0$$

$$Z_2^1 = (\beta^2 + \zeta^2)^{1/2} \zeta \delta_1$$

$$Z_2^2 = \delta_0 \left[\frac{1 + \lambda_2^2 \zeta^2}{\beta^2(1-\beta^2)} \right]$$

$$Z_3^{-3} = (\alpha^2 - \zeta^2)^{1/2} \delta_0 \left[1 + \frac{\beta^2 + \lambda_3^{-3}}{2\beta^2(1-\beta^2)} \zeta^2 \right]$$

$$Z_3^{-2} = \zeta \delta_1 \left[1 + \frac{2\beta^2 - 1 + \lambda_3^{-2}}{6\beta^2(1-\beta^2)} \zeta^2 \right]$$

Table 2. (continued)

$$Z_3^{-1} = (\beta^2+\zeta^2)^{1/2}\,\delta_0\left[1+\frac{\beta^2-1+\lambda_3^{-1}}{2\beta^2(1-\beta^2)}\zeta^2\right]$$

$$Z_3^{0} = \left[(\beta^2+\zeta^2)(\alpha^2-\zeta^2)\right]^{1/2}\zeta\,\delta_1$$

$$Z_3^{1} = (\alpha^2-\zeta^2)^{1/2}\,\delta_0\left[1+\frac{\beta^2+\lambda_3^{1}}{2\beta^2(1-\beta^2)}\right]$$

$$Z_3^{2} = \zeta\,\delta_1\left[1+\frac{2\beta^2-1+\lambda_3^{2}}{6\beta^2(1-\beta^2)}\right]$$

$$Z_3^{3} = (\beta^2+\zeta^2)^{1/2}\,\delta_0\left[1+\frac{\beta^2-1+\lambda_3^{3}}{2\beta^2(1-\beta^2)}\right]$$

IV. ORTHOGONALITY

The Lamé functions do not form a completely orthogonal set of functions, as we shall now show.

In the first part of this section we study the conditions for the mutual orthogonality of a set of functions $Z_\ell^\tau(\zeta)$ with ℓ fixed, $\tau = -\ell, \ldots \ell$ over the ranges $-\alpha \le \zeta \le \alpha$ and $0 \le \zeta \le \alpha$.

The second part of the section will consider the orthogonality over the surface of a sphere of the spheroconal harmonics $Z_\ell^\tau(\zeta) H_\ell^\tau(\eta)$ when the ℓ's differ.

Let us write equation (8a) for two functions, Z_ℓ^τ and $Z_\ell^{\bar\tau}$.

$$f_1 = \left[(\alpha^2-\zeta^2)(\beta^2+\zeta^2)\right]^{1/2}$$

$$f_1 \frac{d}{d\zeta} f_1 \frac{d Z_\ell^\tau}{d\zeta} + \left[\ell(\ell+1)\zeta^2 - \lambda_\ell^\tau\right] Z_\ell^\tau = 0 \tag{8a}$$

$$f_1 \frac{d}{d\zeta} f_1 \frac{d Z_\ell^{\bar\tau}}{d\zeta} + \left[\ell(\ell+1)\zeta^2 - \lambda_\ell^{\bar\tau}\right] Z_\ell^{\bar\tau} = 0 \tag{8a}$$

We multiply the first equation by $Z_\ell^{\bar\tau}$, the second by Z_ℓ^τ , and subtract.

$$f_1 \frac{d}{d\zeta} f_1 \left(Z_\ell^{\bar\tau} Z_\ell^{\tau\prime} - Z_\ell^\tau Z_\ell^{\bar\tau\prime}\right) + \left(\lambda_\ell^{\bar\tau} - \lambda_\ell^\tau\right) Z_\ell^\tau Z_\ell^{\bar\tau} = 0$$

If we now integrate from $-\alpha$ to α

$$\left(\lambda_\ell^\tau - \lambda_\ell^{\bar\tau}\right) \int_{-\alpha}^{\alpha} \frac{Z_\ell^\tau Z_\ell^{\bar\tau}}{f_1}\, d\zeta = \left[f_1\left(Z_\ell^{\bar\tau} Z_\ell^{\tau\prime} - Z_\ell^\tau Z_\ell^{\bar\tau\prime}\right)\right]_{-\alpha}^{\alpha} \tag{13}$$

Consider the form of the Z_ℓ^τ given by equation (10). Let us say that Z_ℓ^τ has $K_1=1$. Then $Z_\ell^{\tau'}$ will have terms containing $(\alpha^2-\zeta^2)^{1/2}$ and $(\alpha^2-\zeta^2)^{-1/2}$. If $Z_\ell^{\bar\tau}$ also has $K_1=1$, the + 1/2 power of $\alpha^2-\zeta^2$ from the function will cancel the -1/2 power from the derivative. The +1/2 power in f_1 will thus send the bracket to zero at both limits.

If both Z_ℓ^τ and $Z_\ell^{\bar\tau}$ have $K_1=0$, again f_1 will force the bracket to vanish at both limits.

If, however, one of the functions, say Z_ℓ^τ, has $K_1=1$ and the other one has $K_1=0$, the term $Z_\ell^\tau Z_\ell^{\bar\tau'}$ vanishes at the limits when it is multiplied by f_1, but the term $Z_\ell^{\bar\tau} Z_\ell^{\tau'}$ has one term containing $(\alpha^2-\zeta^2)^{-1/2}$. When multiplied by the $(\alpha^2-\zeta^2)^{1/2}$ in f_1, this term will not, in general, be zero at either limit.

We may still have orthogonality if the bracket in equation (13) is even in ζ, so that it will vanish inside symmetric limits. It will be noted that the oddness or evenness of the Z's is controlled entirely by K_3. The condition for the bracket to be even is that one of the functions Z_ℓ^τ be odd and the other even, i.e. that their K_3's differ.

To summarize, two Z_ℓ^τ's are orthogonal when they have their K_1's alike, and also when their K_1's differ provided, then, that their K_3's also differ. This leaves as non-orthogonal only those functions which differ in K_1, but have alike K_3's.

It should be quite clear that the same can be said of the $H_\ell^\tau(\eta)$, except that K_2 will take on the role of K_1 and β the role of α .

Some examples follow which confirm these conclusions.

a) Both K's and K_3's unlike

$$\int_{-\alpha}^{\alpha} \frac{Z_1^{-1} Z_1^{0}}{f_1} d\zeta = \delta_0 \delta_1 \int_{-\alpha}^{\alpha} \frac{\zeta \; d\zeta}{(\beta^2+\zeta^2)^{1/2}} = 0$$

b) K_1's alike, K_3's unlike

$$\int_{-\alpha}^{\alpha} \frac{Z_1^{0} Z_1^{1}}{f_1} d\zeta = \delta_0 \delta_1 \int_{-\alpha}^{\alpha} \frac{\zeta \; d\zeta}{(\alpha^2-\zeta^2)^{1/2}} = 0$$

c) K_1's unlike, K_3's alike

$$\int_{-\alpha}^{\alpha} \frac{Z_1^{1} Z_1^{-1}}{f_1} d\zeta = \delta_0^2 \int_{-\alpha}^{\alpha} d\zeta = \delta_0^2 \, 2\alpha \neq 0$$

d) Both K_1's and K_3's unlike

$$\int_{-\alpha}^{\alpha} \frac{Z_2^{0} Z_2^{1}}{f_1} d\zeta = \delta_0 \delta_1 \int_{-\alpha}^{\alpha} (\beta^2+\zeta^2)^{1/2} \zeta \, d\zeta = 0$$

e) K_1's unlike, K_3's alike

$$\int_{-\alpha}^{\alpha} \frac{Z_2^{1} Z_2^{-1}}{f_1} d\zeta = \delta_1^2 \int_{-\alpha}^{\alpha} \zeta^2 d\zeta = \delta_1^2 \, \tfrac{2}{3}\alpha^3 \neq 0$$

If we restrict the coordinate ζ to vary from 0 to α, we obtain different results as to the orthogonality of the Z_ℓ^τ's. The bracket in equation (13) still vanishes at the upper limit for like K_1's. However the function in the bracket must be odd in ζ to ensure its being zero at the origin. This requires that the K_3's be alike. We thus have orthogonality reduced to only Z_ℓ^τ with both K_1's and K_3's alike.

Now we consider two sphero-conal harmonics of unequal ℓ and T .

$$\left[f_1\frac{\partial}{\partial\zeta}f_1\frac{\partial}{\partial\zeta}+f_2\frac{\partial}{\partial\eta}f_2\frac{\partial}{\partial\eta}+\ell(\ell+1)(\zeta^2+\eta^2)\right]Z_\ell^T H_\ell^T=0 \quad (7)$$

$$\left[f_1\frac{\partial}{\partial\zeta}f_1\frac{\partial}{\partial\zeta}+f_2\frac{\partial}{\partial\eta}f_2+\bar\ell(\bar\ell+1)(\zeta^2+\eta^2)\right]Z_{\bar\ell}^{\bar T} H_{\bar\ell}^{\bar T}=0 \quad (7)$$

$$f_2=\left[(\alpha^2+\eta^2)(\beta^2-\eta^2)\right]^{1/2}$$

Again, we multiply the first equation by $Z_{\bar\ell}^{\bar T}\,H_{\bar\ell}^{\bar T}$, the second by $Z_\ell^T\,H_\ell^T$ and subtract.

$$\frac{f_1}{\zeta^2+\eta^2}\frac{\partial}{\partial\zeta}\,f_1\left[Z_{\bar\ell}^{\bar T}H_{\bar\ell}^{\bar T}Z_\ell^{T'}H_\ell^{T}-Z_\ell^{T}H_\ell^{T}Z_{\bar\ell}^{\bar T'}H_{\bar\ell}^{\bar T}\right]$$

$$+\frac{f_2}{\zeta^2+\eta^2}\frac{\partial}{\partial\eta}\,f_2\left[Z_{\bar\ell}^{\bar T}H_{\bar\ell}^{\bar T}Z_\ell^{T}H_\ell^{T'}-Z_\ell^{T}H_\ell^{T}Z_{\bar\ell}^{\bar T}H_{\bar\ell}^{\bar T'}\right]$$

$$-\left[\bar\ell(\bar\ell+1)-\ell(\ell+1)\right]Z_\ell^T\,H_\ell^T Z_{\bar\ell}^{\bar T}\,H_{\bar\ell}^{\bar T}=0$$

$$\left[\frac{\bar\ell(\bar\ell+1)-\ell(\ell+1)}{f_1\,f_2}\right](\zeta^2+\eta^2)\,Z_\ell^T H_\ell^T Z_{\bar\ell}^{\bar T} H_{\bar\ell}^{\bar T}=$$

$$\frac{H_\ell^T H_{\bar\ell}^{\bar T}}{f_2}\,\frac{d}{d\zeta}\,f_1 W(Z_\ell^T,\,Z_{\bar\ell}^{\bar T})+\frac{Z_\ell Z_{\bar\ell}^{\bar T}}{f_1}\,\frac{d}{d\eta}\,f_2 W(H_\ell^T,H_{\bar\ell}^{\bar T})$$

W here signifies the Wronskian determinant.

Integrating over the sphere, we obtain

$$\left[\bar{\ell}(\bar{\ell}+1)-\ell(\ell+1)\right]\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha}\frac{Z_\ell^{\tau} H_\ell^{\tau} Z_{\bar{\ell}}^{\bar{\tau}} H_{\bar{\ell}}^{\bar{\tau}}}{f_1 f_2}(\zeta^2+\eta^2)\,d\zeta\,d\eta$$

$$=\left[f_1 W(Z_\ell^{\tau}\, Z_{\bar{\ell}}^{\bar{\tau}})\right]_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\frac{H_\ell^{\tau} H_{\bar{\ell}}^{\bar{\tau}}}{f_2}\,d\eta+\left[f_2 W(H_\ell^{\tau}\, H_{\bar{\ell}}^{\bar{\tau}})\right]_{-\beta}^{\beta}\int_{-\alpha}^{\alpha}\frac{Z_\ell^{\tau} Z_{\bar{\ell}}^{\bar{\tau}}}{f_1}\,d\zeta$$

$\dfrac{(\zeta+\eta^2)}{f_1 f_2}$ may be identified as $\dfrac{h_2 h_3}{r^2}$

We discussed the conditions necessary for a bracket like the one appearing above to vanish at the limits $-\alpha$ and α in the earlier part of this section. Here we need to have the two brackets vanish simultaneously.

Therefore, orthogonality over the sphere exists between spheroconal harmonics of different ℓ whenever the associated K_3 are unlike, or, if they are alike, whenever both pairs of K_1 and K_2 are also alike.

V. AN ADDITION THEOREM

Now since the sphero-conal harmonics $Z_\ell^\tau H_\ell^\tau$ and the spherical harmonics $P_\ell^s e^{is\phi}$ are solutions to the same partial differential equation in different coordinate systems, we expect that $Z_\ell^\tau H_\ell^\tau$ may be expressible as a linear combination of spherical harmonics with the same ℓ. We assume

$$Z_\ell(\zeta) H_\ell^\tau(\eta) = \sum_{s=-\ell}^{\ell} a_s P_\ell^s e^{is\phi} \qquad \tau = -\ell \ldots \ell-1\, \ell \tag{14}$$

Let us take the right side of equation (14) and substitute it for Z in equation (8a). It should satisfy the equation, since Z is a solution, and H is a constant with respect to the $\frac{\partial}{\partial\zeta}$ operator.

$$f_1 \frac{\partial}{\partial\zeta} f_1 \frac{\partial}{\partial\zeta} \sum_{s=-\ell}^{\ell} a_s P_\ell^s e^{is\phi} + [\ell(\ell+1)\zeta^2 - \lambda] \sum_{s=-\ell}^{\ell} a_s P_\ell^s e^{is\phi} = 0$$

$$\sum_{s=-\ell}^{\ell} a_s \left[f_1 \frac{\partial}{\partial\zeta} f_1 \frac{\partial}{\partial\zeta} + \ell(\ell+1)\zeta^2 - \lambda\right] P_\ell^s e^{is\phi} = 0$$

To evaluate $\frac{\partial P_\ell^s}{\partial\zeta}$ we set it equal to $\frac{d P_\ell^s}{d\cos\theta} \frac{\partial \cos\theta}{\partial\zeta}$

We also use $\frac{\partial e^{is\phi}}{\partial\zeta} = \frac{d}{d\phi} e^{is\phi} \frac{\partial\phi}{\partial\zeta}$

To evaluate terms like $\frac{\partial\cos\theta}{\partial\zeta}$ and $\frac{\partial\phi}{\partial\zeta}$ we turn to equations (1a), (1b), and (1c), and write the cartesian coordinates in terms of spherical coordinates. We then have

$$\sin\theta\cos\phi = \pm\frac{1}{\beta}\left[(\beta^2+\zeta^2)(\beta^2-\eta^2)\right]^{\frac{1}{2}} \tag{15a}$$

$$\sin\theta\cos\Phi = \frac{\zeta\eta}{\alpha\beta} \tag{15b}$$

$$\cos\theta = \pm\frac{1}{\alpha}\left[(\alpha^2-\zeta^2)(\alpha^2+\eta^2)\right]^{\frac{1}{2}} \tag{15c}$$

From equation (15c) we can write

$$\frac{\partial\cos\theta}{\partial\zeta} = -\left[\frac{\alpha^2+h^2}{\alpha^2-\zeta^2}\right]^{1/2}\frac{\zeta}{\alpha} \tag{16}$$

And dividing equation (15b) by equation (15a) gives us

$$\tan\Phi = \frac{\zeta\eta}{\alpha}\left[(\beta^2+\zeta^2)(\beta^2-\eta^2)\right]^{-\frac{1}{2}}$$

Therefore

$$\frac{\partial\Phi}{\partial\zeta} = \frac{\alpha\eta(\beta^2-\eta^2)^{1/2}}{(\beta^2+\zeta^2)^{1/2}\left[\zeta^2\eta^2+\alpha^2(\beta^2+\zeta^2-\eta^2)\right]} \tag{17}$$

Thus we have introduced terms into equation (8a) involving P_ℓ^s, $P_\ell^{s'}$, $e^{is\varphi}$, $ise^{is\varphi}$, and, after the second $\frac{\partial}{\partial\zeta}$ operation, also $-s^2e^{is\varphi}$ and $P_\ell^{s''}$. The details will not be shown here; however we will write one of the transitional forms.

$$\sum_s a_s e^{is\phi}\Big\{ P_\ell^{s''}(\alpha^2+\eta^2)(\beta^2+\zeta^2)\frac{\zeta^2}{\alpha^2} + P_\ell^{s'}\Big[-\cos\theta(2\zeta^2+\beta^2) - \frac{2is\zeta^2\eta^2\cos\theta\cot\phi}{\alpha^2\sin^2\theta}\Big] + P_\ell^{s}\Big[-\frac{is\zeta^2\eta^2\cot\phi}{\alpha^2\sin^2\theta} - \frac{2is\zeta^2\eta^2\cot\phi\cos^2\theta + s^2\eta^2(\beta^2-\eta^2)(\alpha^2-\zeta^2)}{\alpha^2\sin^4\theta} + \ell(\ell+1)\zeta^2+\lambda\Big]\Big\}=0$$

We make use of well known recursion formulae for Legendre functions* to eliminate $P_\ell^{s'}$ and $P_\ell^{s''}$, introducing instead P_ℓ^{s+2} and P_ℓ^{s-2}. Also, since the equation must hold for all values of θ and ϕ, we set, for convenience, $\phi = 0$.

$$\sum_s a_s\Big\{\frac{\beta^2}{4}P_\ell^{s+2} - \Big[s^2(1-\beta^2/2) - \beta^2\frac{\ell(\ell+1)}{2} - \lambda\Big]P_\ell^s + \beta^2/4\,(\ell-s+1)(\ell-s+2)(\ell+s-1)(\ell+s)P_\ell^{s-2}\Big\}=0$$

* $$(1-x^2)P_\ell^{s''} - 2xP_\ell^{s'} + \Big[\ell(\ell+1) - \frac{s^2}{1-x^2}\Big]P_\ell^s = 0$$

$$P_\ell^{s'} = \frac{-x}{1-x^2}P_\ell^s - \frac{P_\ell^{s+1}}{(1-x^2)^{\frac{1}{2}}} \qquad x=\cos\theta$$

$$P_\ell^{s+2}\ 2(s+1)\frac{x}{(1-x^2)^{\frac{1}{2}}}P_\ell^{s+1} + (\ell-s)(\ell+s+1)P_\ell^s = 0$$

This may be rewritten as

$$\sum_s \rho_\ell^s \left\{ \beta^2 a_{s-2} - \left[4(s^2-\lambda) - 2\beta^2 \langle s^2 + \ell(\ell+1) \rangle \right] a_s \right.$$

$$\left. + \beta^2 (\ell-s-1)(\ell-s)(\ell+s+1)(\ell+s+2) a_{s+2} \right\} = 0$$

and finally

$$\beta^2 a_{s-2} - \left\{ 4(s^2-\lambda) - 2\beta^2 \left[s^2 + \ell(\ell+1) \right] \right\} a_s$$

$$+ (\ell-s-1)(\ell-s)(\ell+s+1)(\ell+s+2) a_{s+2} = 0 \qquad (18)$$

giving us the desired relation between coefficients.

This relation has been tested and verified for all cases for which $Z_\ell^\tau H_\ell^\tau$ was worked out independently, and expressed as a series of $\rho_\ell^s e^{i\omega t}$. Some of these are listed in table 3.

It will be noted that certain of the ρ_ℓ^s are absent in these series. In particular, for a given ℓ, all the ρ_ℓ^s with s even, or all the ρ_ℓ^s with s odd are absent. For example, $Z_2^2 H_2^2$ contains ρ_2^0, ρ_2^2, and ρ_2^{-2}, but not ρ_2^1 or ρ_2^{-1}.

If we attempt to use equation (18) to relate these missing coefficients, e.g. if we let $s=-1$ in the above mentioned case, and relate a_{s+2} and a_s, and then let $s=1$ and relate a_s and a_{s-2}, we obtain a set of contradictory conditions, resolved only by setting a_1 and a_{-1} equal to zero.

Table 3. Examples of the Addition Theorem

$$Z_0^0 H_0^0 = \delta_0^2 P_0^0$$

$$Z_1^{-1} H_1^{-1} = \delta_0^2 \alpha P_1^0$$

$$Z_1^0 H_1^0 = \delta_1^2 \alpha\beta\left(-P_1^{-1} e^{-i\varphi} - \tfrac{1}{2} P_1^1 e^{i\varphi}\right)$$

$$Z_1^1 H_1^1 = \delta_0^2 \beta\left(P_1^{-1} e^{-i\varphi} - \tfrac{1}{2} P_1^1 e^{i\varphi}\right)$$

$$Z_2^{-2} H_2^{-2} = \frac{\delta_0^2}{\alpha^2\beta^2}\left\{\left[\alpha^4 - 2\alpha^2 + 1 + \beta^2\sqrt{1-\beta^2+\beta^4}\,\right] P_2^0 + \tfrac{1}{4}\left[1 - 2\beta^2 - \sqrt{1-\beta^2+\beta^4}\right]\left[\alpha^2 - \sqrt{1-\beta^2+\beta^4}\right]\left[3P_2^{-2} e^{-2i\varphi} + \tfrac{1}{3} P_2^2 e^{2i\varphi}\right]\right\}$$

$$Z_2^{-1} H_2^{-1} = \delta_1^2 \alpha^2\beta\left(-P_2^{-1} e^{-i\varphi} - \tfrac{1}{6} P_2^1 e^{i\varphi}\right)$$

$$Z_2^0 H_2^0 = \delta_0^2 \alpha\beta\left(P_2^{-1} e^{-i\varphi} - \tfrac{1}{6} P_2^1 e^{i\varphi}\right)$$

$$Z_2^1 H_2^1 = \delta_1^2 \beta^2\alpha\left(-2 P_2^{-2} e^{-2i\varphi} + \tfrac{1}{12} P_2^2 e^{2i\varphi}\right)$$

$$Z_2^2 H_2^2 = \frac{\delta_0^2}{\alpha^2\beta^2}\left\{\left[\alpha^4 - 2\alpha^2 + 1 - \beta^2\sqrt{1-\beta^2+\beta^4}\,\right] P_2^0 + \tfrac{1}{4}\left[1 - 2\beta^2 + \sqrt{1-\beta^2+\beta^4}\right]\left[\alpha^2 + \sqrt{1-\beta^2+\beta^4}\quad 3P_2^{-2} e^{-2i\varphi} + \tfrac{1}{3} P_2^2 e^{2i\varphi}\right]\right\}$$

One can, without reference to the recursion, predict which P_ℓ^s will be absent. It is a matter of symmetry. Consider the form of the Legendre polynomials.

$$P_\ell^s(\cos\theta) = (\sin\theta)^s \quad \text{(polynomial of degree } \ell - s \text{ in } \cos\theta)$$

Now $\sin\theta$ is even about the equator, $\theta = \pi/2$, whereas $\cos\theta$ is odd. Thus P_ℓ^s is odd or even about the equator according to whether $\ell - s$ is odd or even.

The sphero-conal harmonics possess like symmetry. The functions $I_\ell^\tau H_\ell^\tau$ are odd or even about the equator according to whether K_1 is 1 or 0. This can be seen from the following arguments: the xy plane corresponds to $\zeta = \alpha$ in the sphero-conal system, cf. equations (1c) and (15c), due to the factor $(\alpha^2 - \zeta^2)^{\frac{1}{2}}$. This factor is present in the sphero-conal harmonic whenever $K_1 = 1$, and absent when $K_1 = 0$, cf. equation (10). A $\pm$ sign must be attached before this radical, respectively, according to whether $\theta \lessgtr \pi/2$.

Therefore, depending on the choice of ℓ, and depending on the value of K_1, even s only, or odd s only will be present in the addition theorem.

VI. REDUCTION OF SPHERO-CONAL INTO SPHERICAL HARMONICS AS $\beta \to 0$

In the limit $\beta \to 0$ and in the limit $\beta \to 1$ the sphero-conal coordinates reduce to spherical coordinates, as has been mentioned in the first section of this thesis. We may expect in these limits that each of the sphero-conal harmonics $Z_\ell^\tau(\zeta) H_\ell^\tau(\eta)$ and the associated eigenvalues $\lambda_\ell^\tau(\beta)$ reduce to a single spherical harmonic and eigenvalue, respectively. This is indeed the case, as we will show in the present section.

Moreover, we shall choose - arbitrarily - to normalize the sphero-conal harmonics in such a manner that they reduce exactly to their associated limiting spherical harmonic as $\beta \to 0$.

Recalling equations (2), (3), (4), and (5), the confluence of our coordinates into spherical coordinates in the limits $\beta \to 0, 1$ respectively takes the form

$$\lim_{\beta \to 0} \zeta = \sin \theta \tag{2}$$

$$\lim_{\beta \to 0} \eta/\beta = \sin \phi \tag{3}$$

$$\lim_{\beta \to 1} \eta = \sin \theta' \tag{4}$$

$$\lim_{\beta \to 1} \zeta/\alpha = \sin \phi' \tag{5}$$

The first system, (r, θ, ϕ), corresponds to polar axis Z , the second, (r, θ', ϕ'), to polar axis X.

When we let $\beta \to 0$, equation (18) becomes

$$4(s^2-\lambda_\ell^\tau)a_s=0 \tag{19}$$

Indicating that all the a_s are equal to zero except the two for which $\lambda_\ell^\tau = s^2$. This value of s we will call m.

The two surviving terms involve $\rho_\ell^m e^{im\phi}$ and $\rho_\ell^{-m} e^{-im\phi}$. ρ_ℓ^{-m} is linearly dependent upon ρ_ℓ^m. The two surviving terms combine into one, which can be expressed as $\rho_\ell^m \sin m\phi$ or $\rho_\ell^m \cos m\phi$. This can easily be verified.

In the case $\beta \to 1$, $Z_\ell^\tau H_\ell^\tau$ reduce to a single $\rho_\ell^s \binom{\sin}{\cos} m\phi$, but not the same s as the reduction when $\beta \to 0$. This value of s we shall refer to as m'.

We choose a normalization for the Z's (and H's), such that in the limit $\beta \to 0$, $Z_\ell^\tau H_\ell^\tau$ will go to $\rho_\ell^m \sin m\phi$ or $\rho_\ell^m \cos m\phi$ with coefficient 1.

We can, at the same time, include the proper power of α, so that $Z_\ell^\tau H_\ell^\tau$ will remain finite in the $\alpha \to 0$ reduction as well, even though, as we find upon detailed calculation, we cannot simultaneously make the coefficient 1 for both reductions. We will choose the $\beta \to 0$ reduction, simply because this takes us to a more familiar coordinate system. A list of some of the sphero-conal harmonics is given in table 4 along with the K_i associated with each, and the particular $\rho_\ell^m \cos m\phi$ or $\rho_\ell^m \sin m\phi$ to which they reduce in the limit $\beta \to 0$.

Table 4. Reduction of Sphero-conal Harmonics

Sphero-conal Harmonic	K_1	K_2	K_3	Reduction as $\beta \rightarrow 0$
$Z_1^{-1} H_1^{-1}$	1	0	0	P_1^0
$Z_1^0 H_1^0$	0	0	1	$P_1^1 \sin \phi$
$Z_1^1 H_1^1$	0	1	0	$P_1^1 \cos \phi$
$Z_2^{-2} H_2^{-2}$	0	0	0	P_2^0
$Z_2^{-1} H_2^{-1}$	1	0	1	$P_2^1 \sin \phi$
$Z_2^0 H_2^0$	1	1	0	$P_2^1 \cos \phi$
$Z_2^1 H_2^1$	0	1	1	$P_2^2 \sin 2\phi$
$Z_2^2 H_2^2$	0	0	0	$P_2^2 \cos 2\phi$

Two things may be noted in table 4. One is the relation K_1 (being 0 or 1) to $l-m$ in the P_l^m being even or odd, as was predicted earlier. The other is the fact that the $\cos m\phi$ reduction seems to be related to $K_3 = 0$, and the $\sin m\phi$ reduction to $K_3 = 1$. This latter fact will be discussed further in the next section.

VII. NORMALIZATION

As indicated in the foregoing section, we choose, for convenience, a normalization factor for any given sphero-conal harmonic $Z_\ell^\tau(\zeta)H_\ell^\tau(\eta)$ as follows: In the limit $\beta \to 0$, the sphero-conal coordinates flow into the spherical coordinates whose polar axis is z. We now cause $Z_\ell^\tau H_\ell^\tau$ in the limit $\beta \to 0$ to reduce exactly to its associated spherical harmonic.

To achieve the normalization, we study the form of the sphero-conal harmonics as obtained from equations (9) and (10). We keep in mind that we are considering the polynomial solutions, i.e. the summation goes to n, not to infinity. **We also set C of equation (9) equal to 1 for reasons of symmetry.**

$$Z_\ell^\tau H_\ell^\tau = \left[(\alpha^2-\zeta^2)(\alpha^2+\eta^2)\right]^{K_1/2}\left[(\beta^2+\zeta^2)(\beta-\eta^2)\right]^{K_2/2}(\zeta i\eta)^{K_3}\left(\sum_{J=0}^{n}\delta_{2J+K_3}\zeta^{2J}\right)\left(\sum\delta_{2J+K_3}(-1)^J\eta^{2J}\right) \quad (20)$$

$$\lim_{\beta\to 0} Z_\ell^\tau H_\ell^\tau = \beta^{K_2+K_3}(\cos\theta)^{K_1}(\sin\theta)^{K_2+K_3}(\cos\phi)^{K_2}(i\sin\phi)^{K_3}\left(\sum(\sin\theta)^{2J}\lim_{\beta\to 0}\delta_{2J+K_3}\right)\left(\sum(-1)^J(\beta\sin\phi)^{2J}\lim_{\beta\to 0}\delta_{2J+K_3}\right) \quad (21)$$

We know that this expression in equation (21) must be $P_\ell^m \cos m\phi$ or $P_\ell^m \sin m\phi$. Thus we can match up the θ and ϕ dependence. We start with the ϕ dependence.

$\cos m\phi$ with even m is an even polynomial of degree m in $\cos\phi$
or an even polynomial of degree m in $\sin\phi$

$\cos m\phi$ with odd m is an odd polynomial of degree m in $\cos\phi$
or is $\cos\phi$ (even polynomial of degree $m-1$ in $\sin\phi$)

$\sin m\phi$ with even $m = \sin\phi\cos\phi$ (even polynomial of degree $m-2$ in $\cos\phi$)
$= \sin\phi\cos\phi$ (even polynomial of degree $m-2$ in $\sin\phi$)

$\sin m\phi$ with odd $m = \sin\phi$ (even polynomial of degree $m-1$ in $\cos\phi$)
= odd polynomial of degree m in $\sin\phi$

Looking at the form of the $\sum_{\ell}^{\tau} H_{\ell}^{\tau}$, it seems better to choose the polynomials in $\sin\phi$ for our representation.

In equation (21), the summation can only contribute an even polynomial in $\sin\phi$. This leaves $(\cos\phi)^{K_2}(\sin\phi)^{K_3}$, and results in the following four possibilities:

a) If both K_2 and K_3 are zero, we have an even polynomial in $\sin\phi$, and the reduction must be to $\cos m\phi$ with m even.

b) If $K_2=0$ and $K_3=1$ we get an odd polynomial in $\sin\phi$, or reduction to $\sin m\phi$, with m odd.

c) If $K_2=1$ and $K_3=0$ we have $\cos\phi$ (even polynomial in $\sin\phi$) meaning a reduction to $\cos m\phi$ with m odd.

d) Finally, if both K_2 and K_3 are equal to 1 we get
$\sin\phi\cos\phi$ (even polynomial in $\sin\phi$), i.e. reduction to $\sin m\phi$, m even.

Summarizing, if $K_3 = 0$, the reduction is always to $\cos m\phi$. If $K_2 = 0$, the summation in equation (21) is of degree m in $\sin\phi$, and if $K_2 = 1$, it is of degree $m-1$.

If $K_3 = 1$, the reduction is always to $\sin m\phi$. If $K_2 = 0$, the above mentioned summation is of degree $m-1$ in $\sin\phi$, and if $K_2 = 1$, it is of degree $m-2$.

In general, the summation in equation (21) is of degree $m-K_2-K_3$, and thus $m-K_2-K_3$ must always be an even integer. Clearly, for this to happen, all the terms higher than $m-k_2-k_3$ must contain a greater power of β, and go to zero in the reduction. Let us examine the form of δ_{2J+k_3} to verify this. The following is the recursion relation satisfied by these coefficients, and was obtained earlier:

$$\delta_{2J+k_3} = -\frac{d_{J-1}}{c_{J-1}}\,\delta_{2J+k_3-2} - \frac{e_{J-1}}{c_{J-1}}\,\delta_{2J+k_3-4} \qquad (11)$$

Repeated application of this relation will give us δ_{2J+k_3} in terms of δ_{k_3}, which is an arbitrary constant, and the c's, d's, and e's. Now all the c's are of order β^2, while the e's do not contain β. The important term, i.e. the term containing the highest negative power of β, in the relation just mentioned is

$$(-1)^J \frac{d_{J-1}\, d_{J-2} \cdots d_1\, d_0}{c_{J-1}\, c_{J-2} \cdots c_1\, c_0}$$

Let us now look at the nature of the d's.

$$d_J = (2J+K_2+K_3)^2 - \lambda_\ell^\tau(0) + O(\beta^2) \tag{22}$$

We know, of course, that $\lambda_\ell^\tau(0) = m^2$. For any given Z_ℓ^τ (and thus a particular λ_ℓ^τ) there is, according to equation (22) precisely one d_J which is of order β^2 - the one whose J satisfies the relation $2J+K_2+K_3=m$. The term

$$\frac{d_{J-1} \cdots d_1 d_0}{c_{J-1} \cdots c_1 c_0}$$

and thus δ_{2J+K_3}, is then clearly either of order β^{-2J} or β^{-2J+2}, depending on whether the d of order β^2 is contained in the above term. $\delta_{2J+K_3}(\beta \sin\phi)^{2J}$ is then either of order β^0 or β^2. In particular,

$$\sum_{J=0}^{n} \delta_{2J+K_3} (\beta \sin\phi)^{2J}$$

in the limit $\beta \to 0$ will break off at the point where the d of order β^2 appears in the δ_{2J+K_3}. The last non-zero coefficient will then be $\delta_{\frac{m-K_2-K_3}{2}}$, corresponding to $2J = m-K_2-K_3$. This will contain d_{J-1}, but not d_J, which is of order β^2. The highest power of $\sin\phi$ will be $m-K_2-K_3$, in agreement with the conclusion reached on the previous page. We also find that multiplication by a power of β is not needed for the η series.

The series in ζ have coefficients δ_{2J+K_3} which behave like the ones in the η series; however ζ reduces to $\sin\phi$ (whereas η reduced to $\beta\sin\phi$) and thus multiplication by some power of β will be necessary for finite, non-zero reduction.

Let us consider, once again, the θ dependence of the P_ℓ^m .

$$P_\ell^m = \sin^m\theta \text{ (a polynomial of degree } \ell-m \text{ in } \cos\theta)$$

This must be equal to the θ dependence of the right side of equation (21) i.e.

$$(\cos\theta)^{k_1}(\sin\theta)^{k_2+k_3}\left(\sum_{J=0}^{n}(\sin\theta)^{2J}\lim_{\beta\to 0}\delta_{2J+k_3}\right)$$

The lowest surviving power of $\sin\theta$ must be m , and so the lowest surviving power in the summation must be $m-k_2-k_3$. The highest surviving power must be $\ell-m-k_1$ (always an even number) higher than that. This means that the highest surviving power must be of degree $\ell-k_1-k_2-k_3$ in $\sin\theta$. By equation (12), this is equal to $2n$, the original highest power of J in the summation.

The power of β necessary to send the powers of $\sin\theta$ lower than $m-k_2-k_3$ in the summation to zero is then, clearly, the negative of the power of β in δ_{2J+k_3} with $2J=m-k_2-k_3$.

$$\delta_{m-k_2} \approx (-1)^{\frac{m-k_2-k_3}{2}}\frac{d_{\frac{m-k_2-k_3}{2}-1}\cdots d_1 d_0}{c_{\frac{m-k_2-k_3}{2}-1}\cdots c_1 c_0} \tag{23}$$

The d of order β^2 is not contained here. Thus δ_{m-k_2} possesses as its highest negative power of β , $\beta^{-(m-k_2-k_3)}$, and the series has to be multiplied by $\beta^{m-k_2-k_3}$.

In addition, there appears in equation (21) a factor $\beta^{\kappa_2+\kappa_3}$. This requires multiplication by $\beta^{-\kappa_2-\kappa_3}$, making the full power of β necessary for normalization $\beta^{m-2\kappa_2-2\kappa_3}$. We can achieve this by multiplying each of the Z_ℓ^τ and H_ℓ^τ by $\beta^{m/2-\kappa_2-\kappa_3}$.

A completely analogous investigation gives the not too surprising result that a factor of $\alpha^{m'/2-\kappa_1-\kappa_3}$ is required for finite, non-zero reduction of the sphero-conal harmonics in the limit $\alpha \to 0$. m' is defined by

$$\lim_{\beta \to 1} \lambda_\ell^\tau(\beta) = -m'^2$$

This leaves us only to consider the factor necessary to make the coefficient of $P_\ell^m \cos\left(\frac{\pi}{2}\kappa_3 - m\varphi\right)$ equal to 1 in the $\beta \to 0$ reduction. From equation (21) and subsequent work, we see that

$$\lim_{\beta \to 0} Z_\ell^\tau H_\ell^\tau = i^{\kappa_3} \sin^m\theta \cos^{\kappa_1}\theta \cos^{\kappa_2}\phi \sin^{\kappa_3}\phi \left[\delta_{\kappa_3} + O(\sin^2\phi)\right] \left[\beta^{m-\kappa_2-\kappa_3}\delta_{m-\kappa_2} + O(\sin^2\theta)\right]$$

which for small values of θ and ϕ becomes

$$i^{\kappa_3} \sin^m\theta \sin^{\kappa_3}\phi \, \delta_{\kappa_3} \beta^{m-\kappa_2-\kappa_3} \delta_{m-\kappa_2}$$

whereas the spherical harmonics for small θ and ϕ take the following forms:

$$P_\ell^m \cos m\varphi \approx \frac{(-1)^m (\ell+m)!}{2^m m! (\ell-m)!} \sin^m \vartheta$$

$$P_\ell^m \sin m\varphi \approx \frac{(-1)^m (\ell+m)! \, m\varphi}{2^m m! (\ell-m)!} \sin^m \theta$$

Clearly the factor by which $Z_\ell^\tau H_\ell^\tau$ needs to be multiplied (let us call it $(A_\ell^\tau)^2$) is

$$(A_\ell^\tau)^2 = \frac{(-1)^m (\ell+m)! \, (m\varphi)^{\kappa_3}}{2^m m! (\ell-m)! \sin^{\kappa_3}\varphi \, i^{\kappa_3} \delta_{\kappa_3} \beta^{m-\kappa_2-\kappa_3} \delta_{m-\kappa_2}}$$

$$= \frac{(-1)^m (\ell+m)!}{2^m (m-\kappa_3)! (\ell-m)! \, i^{\kappa_3} \delta_{\kappa_3} \beta^{m-\kappa_2-\kappa_3} \delta_{m-\kappa_2}}$$

The only surviving term in $\delta_{m-\kappa_2}$ will be the one containing $\beta^{\kappa_2+\kappa_3-m}$ cf. equation (23). Let us examine this term.

$$\lim_{\beta\to 0} \beta^{m-\kappa_2-\kappa_3} \delta_{m-\kappa_2} = (-1)^{\frac{m-\kappa_2-\kappa_3}{2}} \frac{d_{\frac{m-\kappa_2-\kappa_3}{2}-1} \cdots d_1 d_0}{c_{\frac{m-\kappa_2-\kappa_3}{2}-1} \cdots c_1 c_0}$$

From equation (22) we can see that the product of the d's can be written as

$$[(m-2)^2 - m^2][(m-4)^2 - m^2] \quad \cdots \quad [(\kappa_2+\kappa_3)^2 - m^2]$$

From equation (11) we can write the C 's as $(m-\kappa_2)(m-\kappa_2-1)\ldots(\kappa_3+2)(\kappa_3+1)$

so that

$$\lim_{\beta\to 0} \beta^{m-\kappa_2-\kappa_3} C_{m-\kappa_2} = (-1)^{\frac{m-\kappa_2-\kappa_3}{2}} \frac{(2m-2)(-2)(2m-4)(-4)\ldots(\kappa_2+\kappa_3+m)(\kappa_2+\kappa_3-m)}{(m-\kappa_2)(m-\kappa_2-1)\quad(\kappa_3+2)(\kappa_3+1)}$$

$$= (-1)^{m-\kappa_2-\kappa_3} \frac{(2m-2)(2m-4)\ldots\ldots(m+\kappa_2+\kappa_3)}{(1)(3)(5)\ldots(m-\kappa_2-1+\kappa_3)}$$

It will be recalled that $m-\kappa_2-\kappa_3$ is always even.

$$\left|A_\ell^\tau\right|^2 = \frac{(-1)^m(\ell+m)!\,(1)(3)(5)\ldots(m-\kappa_2-1+\kappa_3)}{2^m(m-\kappa_3)!\,(\ell-m)!\,i^{\kappa_3}\delta_{\kappa_3}(m+\kappa_2+\kappa_3)(m+\kappa_3+\kappa_2+2)\ldots(2m-2)} \qquad (24)$$

Now if we define N_ℓ^τ by

$$\lim_{\beta\to 0} \left(N_\ell^\tau\right)^2 \vec{Z}_\ell^\tau H_\ell^\tau = P_\ell^m \cos\left(\frac{\pi}{2}\kappa_3 - m\varphi\right)$$

$$N_\ell^\tau = A_\ell^\tau \beta^{\frac{m}{2}-\kappa_2-\kappa_3} \alpha^{\frac{m}{2}-\kappa_1-\kappa_3} \qquad (25)$$

VIII. EIGENVALUES

We discussed briefly how the eigenvalues $\lambda_\ell^\tau(\beta)$ were obtained in section III, and a partial list was given in table 1. In figure 3, the 7 eigenvalues associated with $\ell = 3$ have been sketched. It may be noted from figure 3 that the eigenvalues come together in pairs at both ends of the range of β (0 and 1), where they take on values of squares of integers. Only the value 0 is taken on by a single eigenvalue. These values can be easily seen from equation (19) to be m^2 at $\beta = 0$ (analogously $(m')^2$ at $\beta = 1$).

These degeneracies correspond to the degeneracy of the eigenvalues of the pair of spherical harmonics $P_\ell^m(\cos\Theta)\sin m\varphi$ and $P_\ell^m(\cos\Theta)\cos m\varphi$ when $\beta \to 0$, and of the pair $P_\ell^{m'}(\cos\Theta')\ \sin m'\varphi'$ and $P_\ell^{m'}(\cos\Theta')\cos m'\varphi'$ when $\beta \to 1$, as was already shown on pp. 33 and 34.

We find, empirically, that the λ_ℓ^τ with the larger τ will correspond to $P_\ell^m \cos m\varphi$ (and $\kappa_3 = 0$), whereas the smaller τ will correspond to the $P_\ell^m \sin m\varphi$ reduction (and $\kappa_3 = 1$). This enables us to write an empirical formula for τ. It will be recalled that the τ's are ordered by the size of the corresponding λ_ℓ^τ, and that τ takes on integer values from $-\ell$ to ℓ. We can order the τ's, then, in terms of the values of the λ_ℓ^τ when $\beta = 0$ (i.e. by ordering the m's), and resolve the duplicity resulting from two λ_ℓ^τ having the same value, m^2, by using the empirical result just mentioned, and distinguish the two by their κ_3's. The expression we want then is: $\tau = 2m - \ell - \kappa_3$

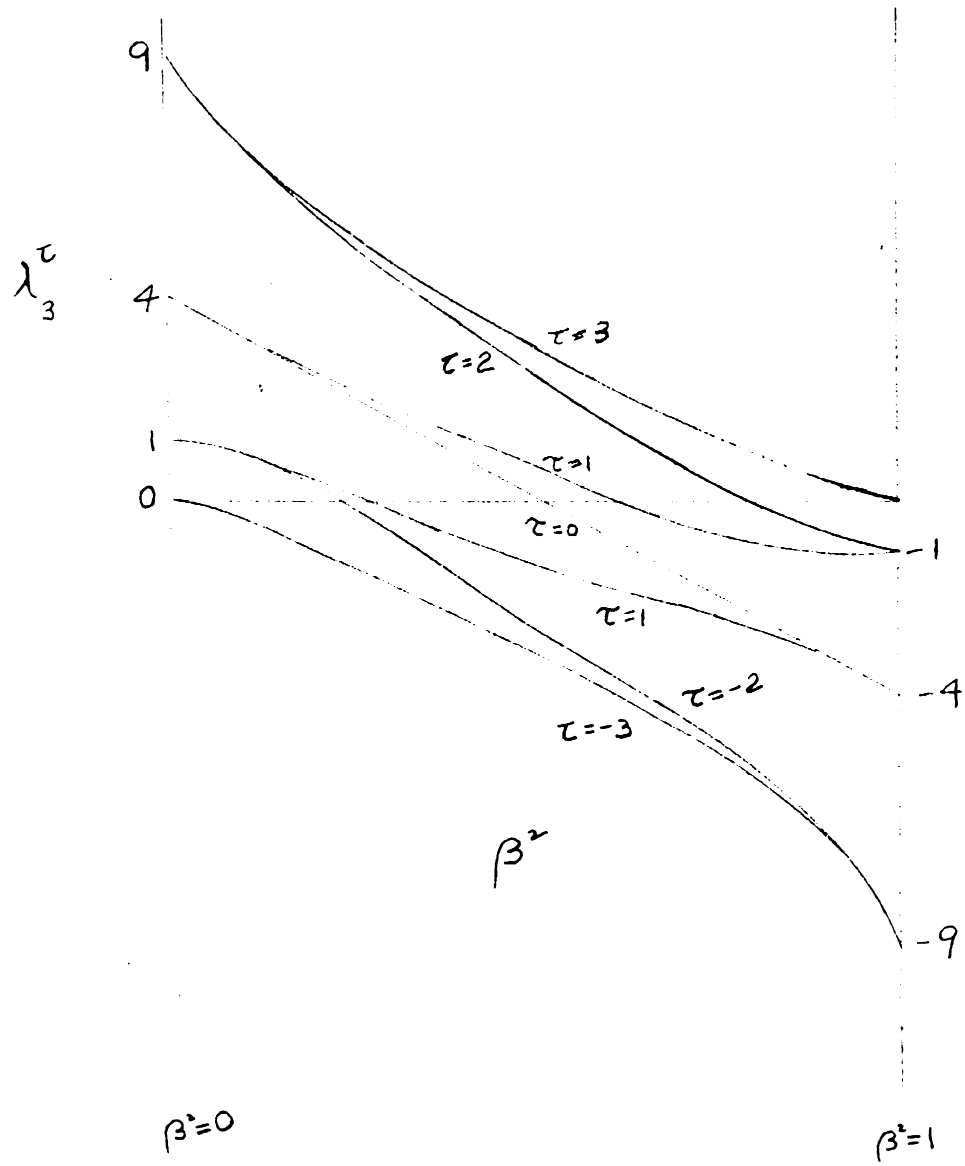

Figure 3. $\lambda_\ell^\tau(\beta^2)$, $\ell=3$.

It would be desirable to prove the empirical fact that, of the two $\lambda_\ell^\tau(\beta)$ which reduce to the same m^2, the one with $K_3 = 0$ is the larger.

In order to show this, we embarked on a calculation of a series expansion for $\lambda_\ell^\tau(\beta^2)$ in the powers of β^2. It is permissible to assume that

$$\lambda_\ell^\tau(\beta^2) = m^2 + \Lambda_1 \beta^2 + \Lambda_2 \beta^4 + \ldots \tag{26}$$

where $0 \leq m \leq \ell$, and where to each value of m are associated two eigenvalues of different τ.

The values of Λ_j can be found by successive approximations of the finite continued fraction determining the eigenvalues, as given on p. 16.

It is interesting to note that for the pairs of eigenvalues reducing to the same m^2 as $\beta \to 0$, Λ_1 and Λ_2 come out to be the same. For example

$$\Lambda_1 = -\tfrac{1}{2}\left[m^2 - \ell(\ell+1)\right] \qquad n \neq 0$$

$$= K_3 - (2K_3 + 1)\ell \qquad n = 0$$

Λ_2 also does not differ for the pair of eigenvalues, except for similar special cases. We suspect strongly that a difference will appear in Λ_3, but the calculations for higher coefficients becomes extremely laborious, and no results have yet been obtained for this case.

BIBLIOGRAPHY

1. R. D. Spence, Am. J. Phys., Vol. 27, No. 5, 329-335, (1959).

2. F. M. Arscott, J. London Math. Soc. 31, 360364, (1956).

3. Whittaker and Watson, A Course in Modern Analysis, 4th edition, Cambridge University Press (1958).

4. O. Kraus, Thesis M.S., Michigan State University, (1952).