TOPICS IN KNOT THEORY: ON GENERALIZED CROSSING CHANGES AND THE ADDITIVITY OF THE TURAEV GENUS

By

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ABSTRACT

TOPICS IN KNOT THEORY: ON GENERALIZED CROSSING CHANGES AND THE ADDITIVITY OF THE TURAEV GENUS

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We first study cosmetic crossing changes and cosmetic generalized crossing changes in knots of genus one, satellite knots, and knots obtained via twisting operations on standardly embedded tori in the knot complement. As a result, we find obstructions to the existence of cosmetic generalized crossing changes in several large families of knots. We then study Turaev surfaces and use decomposing spheres to analyze the additivity of the Turaev genus for the summands of composite knots with Turaev genus one.

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KEY TO SYMBOLS AND ABBREVIATIONS

A'	decomposing sphere which is preferred or in standard position
B_p	the p^{th} braid group
$\mathcal{C}_{\pm},\ \mathcal{C}'_{\pm}$	$A \cap T$ or $A' \cap T$ for decomposing spheres A, A' and a Turaev surface T
$D_+(K,n)$	Whitehead double of K with n full twists and a positive clasp
$D_-(K,n)$	Whitehead double of K with n full twists and a negative clasp
$\Delta(\cdot,\cdot)$	minimal geometric instersection number
$\Delta_K(t)$	Alexander polynomial of K
Δ_p^2	central element of B_p given by one full twist
∂	boundary
$\mathbb{G}$	ribbon graph
$g(\cdot)$	genus
$g_{\text{alt}}(\cdot)$	alternating genus
$g_{\text{DA}}(\cdot)$	disk-alternating genus
$g_T(\cdot)$	Turaev genus
$\text{int}(M)$	$M - \partial M$
K	a knot
$\mathbb{K}$	class of knots which are known not to admit cosmetic generalized crossing changes
$K(q) = K_L(q)$	the knot obtained from K via an order-q generalized crossing change at the crossing circle L
$K_n = K_{n,V}$	twist knot of K

$K_{p,q}$	twist braid obtained via q full twists on p strands of the closed braid K
$\mathrm{lk}(\cdot,\cdot)$	linking number
$M(q)$	the 3-manifold obtained from S^3 by order-q Dehn surgery along a crossing circle L
$M_\pm$	handlebody bounded by $T_\pm$ lying outside of the Turaev surface T
$M_\mathcal{L}$	$\overline{S^3 - \eta(\mathcal{L})}$ for a knot or link $\mathcal{L}$
$\eta(\cdot)$	closed regular neighborhood
$\mathbb{P}$	projection of a knot corresponding to a Turaev surface
S^n	$n-$ dimensional sphere
$\mathcal{T}$	a Haken system
$T_\pm$	Turaev surface T with $T \cap \{\text{bubbles}\}$ replaced by the upper/lower hemispheres of the bubbles
$\tau(\cdot)$	Haken number
U	unknot
$V_\pm$	$\overline{S^3 - M_\pm}$
$w(\cdot,\cdot)$	winding number
$w_\pm(\cdot)$	cyclic word which records the saddles and punctures of a curve of $\mathcal{C}_\pm$
$\mathrm{wrap}(\cdot,\cdot)$	wrapping number
$\chi(\cdot)$	Euler characteristic
Y_K	double branched cover of S^3 branched over K
$\mathbb{Z}[t^{\pm 1}]$	the ring Laurent polynomials with integer coefficients
$\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$	cyclic abelian group of order m
$:=$	defined to be equal

$\cong$	isomorphic
$\doteq$	equivalent up to multiplication by a unit
$\#$	connect sum
$\sim$	$S-$equivalent matrices
$\approx$	congruent matrices
$\cdot\vert_X$	a function restricted to a domain X
$\overline{\cdot}$	closure of a manifold
$\hat{\cdot}$	closure of a braid

Chapter 1

Introduction

The primary goal of this thesis is to investigate relationships between knots and surfaces in $\mathbb{R}^3$ and use these relationships to explore properties of knots. Much of the work done here has been motivated by the nugatory crossing conjecture, Problem 1.58 on Kirby's list [1], which asks when a crossing change in a knot diagram does not change the underlying knot type. Another major goal of this work is to study the Turaev genus, a fairly new knot invariant first defined in 2008 by Dasbach, Futer, Kalfagianni, Lin and Stoltzfus [13]. We are especially interested in deciding whether the Turaev genus is additive under the operation of connect summing.

Throughout this work, we will be considering oriented tame knots and surfaces smoothly embedded in $\mathbb{R}^3 \subset S^3$, even when they are not explicitly stated as such.

1.1 Generalized crossing changes

A fundamental open question in knot theory is the question of when a crossing change on an oriented knot changes the isotopy class of the knot. Traditionally, a *crossing change* is defined in terms of a knot diagram by changing the over- and understrand of a single crossing in the diagram, as illustrated in Figure 1.1.

We would like to study the effects of crossing changes without restricting ourselves to any particular diagram of the knot in question. To this end, we define a *crossing disk* for an

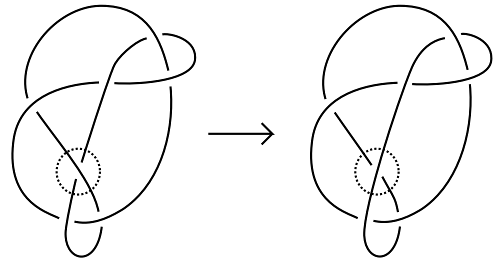


Figure 1.1: An example of a crossing change in a knot diagram.

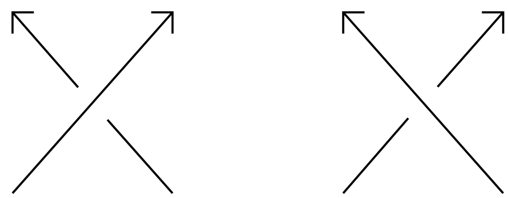


Figure 1.2: The crossing on the left has sign $+1$ and the crossing on the right has sign -1.

oriented knot $K \subset S^3$ to be an embedded disk $D \subset S^3$ such that K intersects $\text{int}(D)$ twice with zero algebraic intersection number. In other words, K meets D exactly twice and in opposite directions. A *crossing circle* is the boundary of a crossing disk.

A crossing change in K can be achieved by inserting a full twist in the appropriate direction in K at a crossing disk D. More precisely, choose an orientation on K and let C be a crossing in some diagram of K. Regardless of the choice of orientation, C is either a positive or negative crossing, as defined in Figure 1.2. Let L be a crossing circle for K which encircles the crossing C. If C has sign ε, where $\varepsilon = \pm 1$, then changing the sign of C to $-\varepsilon$ is equivalent to inserting one full twist of sign ε at L, where a twist is *positive* if it is a right-hand twist and *negative* if it is a left-hand twist. This process is illustrated in Figure 1.3.

Throughout this paper we will let $\eta(\cdot)$ denote the closed regular neighborhood of a man-

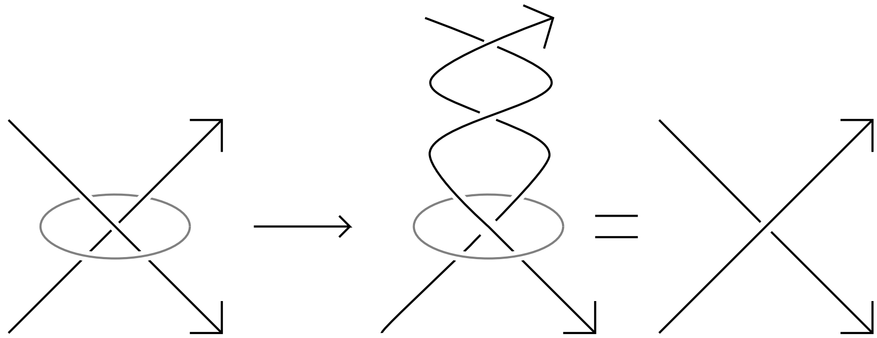


Figure 1.3: A positive crossing is changed to a negative crossing by inserting one full twist in the positive (right-hand) direction.

ifold and $M_{\mathcal{L}} := \overline{S^3 - \eta(\mathcal{L})}$ for any knot or link $\mathcal{L}$. Given a knot K, M_K is a manifold with torus boundary $\partial\eta(K)$. Given $s \in \mathbb{Q}$, a *slope-s Dehn filling* of M_K is obtained by gluing a solid torus T to ∂M_K so that the meridian of T has slope s on ∂M_K. Similarly, *s-Dehn surgery* on S^3 at K is obtained by removing a neighborhood of K from S^3 and performing a slope-s Dehn filling of M_K. Inserting one full twist of sign ε at a crossing circle L is equivalent to performing $(-\varepsilon)$-Dehn surgery on S^3 at L. (See Chapter 6 of [33] for details.)

More generally, if we perform $(-1/q)$-Dehn surgery along the crossing circle L for some $q \in \mathbb{Z} - \{0\}$, we twist K q times at the crossing circle in question. We will call this an *order-q generalized crossing change* denoted by $K_L(q)$, or simply $K(q)$ if there is no chance of confusion regarding the crossing circle L. Note that if q is positive, then we give K q right-hand twists when we perform $(-1/q)$-surgery, and if q is negative, we give K q left-hand twists.

Definition 1.1.1. We will use the term *crossing change* to refer to a crossing change in the diagrammatic sense, which is equivalent to a generalized crossing change of order ± 1, and the term *generalized crossing change* to refer to such changes of any order $q \neq 0$.

We are interested in studying generalized crossing changes which do not change the

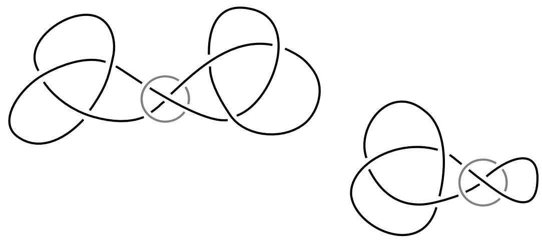


Figure 1.4: Examples of nugatory crossing circles.

underlying knot. To this end, we have the following definition.

Definition 1.1.2. A crossing of K and its corresponding crossing circle L are called *nugatory* if L bounds an embedded disk in $S^3 - \eta(K)$. Examples of nugatory crossing circles are shown in Figure 1.4. Clearly a generalized crossing change of any order at a nugatory crossing of K yields a knot isotopic to K. A generalized crossing change on K is called *cosmetic* if the crossing change yields a knot isotopic to K and is performed at a crossing of K which is *not* nugatory. We will also use the term cosmetic to refer to the corresponding crossing circle and the crossing itself.

The following question, often referred to as the *nugatory crossing conjecture*, is Problem 1.58 on Kirby's list [1].

Problem 1.1.3. Does there exist a knot K which admits a cosmetic crossing change? Conversely, if a crossing change on a knot K yields a knot isotopic to K, must the crossing be nugatory?

Since a crossing change is the same as an order-(± 1) generalized crossing change, one can ask the following stronger question concerning cosmetic generalized crossing changes.

Problem 1.1.4. Does there exist a knot K which admits a cosmetic generalized crossing change of any order?

Problem 1.1.3 was answered for the unknot by Scharlemann and Thompson when they showed that the unknot admits no cosmetic crossing changes in [37] using work of Gabai [14]. The proof in [37] can be easily generalized to show that the unknot admits no cosmetic generalized crossing changes of any order. It has been shown by Kalfagianni that the answer to Problem 1.1.4 is no for fibered knots [18] and by Torisu that the answer is no for 2-bridge knots [43]. Torisu also reduces Problem 1.1.4 to the case where K is a prime knot in [43]. More precisely, suppose that the knot K is composite and admits a cosmetic generalized crossing change. Then the corresponding crossing disk meets exactly one summand of K.

In Chapters 3 and 4 we will use crossing circles to find obstructions to cosmetic generalized crossing changes in several families of knots. Chapter 3 is based on results obtained jointly by the author with Friedl, Kalfagianni and Powell in [4], with the main difference being that many of the results in [4] have been generalized here to include cosmetic generalized crossing changes and not just crossing changes of order-(± 1). The results of Chapter 4 first appeared in work by the author in [3] and joint work with Kalfagianni in [5].

In Chapter 3 we focus on genus-one knots. Two of the main results of this chapter are the following.

Theorem 3.2.1. *Let K be a genus-one knot. If K admits a cosmetic generalized crossing change, then K is algebraically slice. In particular, there is a linear polynomial $f(t) \in \mathbb{Z}[t]$ such that the Alexander polynomial of K is of the form $\Delta_K(t) \doteq f(t)f(t^{-1})$.*

Theorem 3.3.1. *Let K be a genus-one knot and let Y_K denote the double branched cover of S^3 branched over K. If K admits a cosmetic crossing change, then the homology group $H_1(Y_K;\mathbb{Z})$ is a finite cyclic group.*

These results allow us to find obstructions to cosmetic generalized crossing changes in certain pretzel knots and knots with low crossing number, including those in the following theorem and corollaries.

Theorem 3.6.2. *Let K be a genus-one knot that has a diagram with at most 12 crossings. Then K admits no cosmetic crossing changes.*

Corollary 3.6.3. *Let K be a genus-one knot that has a diagram with at most 12 crossings. If K admits an order-q cosmetic generalized crossing change, then one of the following must be true.*

1. *$K = 9_{46}$ and $q = 3n$ for some $n \in \mathbb{Z}$*
2. *$K = 11n_{139}$*

Corollary 3.5.1. *The pretzel knot $P(p, q, r)$ with p, q and r odd does not admit a cosmetic generalized crossing change of any order if $pq + qr + pr \neq -m^2$, for every $m \in \mathbb{Z}$.*

In Chapter 4 we turn our attention to potential cosmetic generalized crossing changes in satellite knots and other knots embedded in solid tori. Our first main result of this chapter is the following.

Theorem 4.3.1. *Suppose K is a satellite knot which admits a cosmetic generalized crossing change of order q with $|q| \geq 6$. Then K admits a pattern knot K' which also has an order-q cosmetic generalized crossing change.*

This leads us to a very nice corollary.

Corollary 4.3.7. *If there exists a knot admitting a cosmetic generalized crossing change of order q with $|q| \geq 6$, then there must be such a knot which is hyperbolic.*

We then go on to study knots contained in standardly embedded solid tori, including twist knots and fibered m-braids. This leads to the following two results, in which $\mathbb{K}$ denotes the class of knots which are known not to admit cosmetic generalized crossing changes of any order, and $K_{n,V}$ and $K_{p,q}$ are defined in Definitions 2.5.1 and 2.5.2, respectively.

Theorem 4.4.4. *Let $K' \in \mathbb{K}$ be contained in a standardly embedded solid torus V' with $w(K', V') = \mathrm{wrap}(K', V') \geq 3$. Then, for every $n \in \mathbb{Z}$, the twist knot $K'_{n,V'}$ does not admit a cosmetic generalized crossing change of any order.*

Corollary 4.4.5. *Let K be a fibered m-braid with $m \geq 3$. Then for every $3 \leq p \leq m$ and $q \in \mathbb{Z}$, there is no twisted fibered braid $K_{p,q}$ which admits a cosmetic generalized crossing change of any order.*

We close Chapter 4 with several results that obstruct cosmetic generalized crossing changes in certain classes of Whitehead doubles, including the following.

Corollary 4.6.3. *Let K be a prime knot that is not a cable knot. Then no Whitehead double of K admits a cosmetic generalized crossing change of any order.*

1.2 Additivity of the Turaev genus

Another question of great interest in knot theory is whether the Turaev genus of a knot is additive under the operation of connect sum. The Turaev genus is a generalization of the concept of alternating knots. In [13], the construction of a Turaev surface for a knot $K \subset S^3$ is given as follows.

Let P be a projection of K onto $\mathbb{R}^2$ such that P has no nugatory crossings. At each crossing of P we can consider the A and B smoothings of the crossing, as defined in Figure

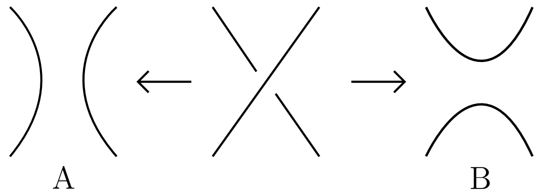


Figure 1.5: The A and B smoothings of a crossing.

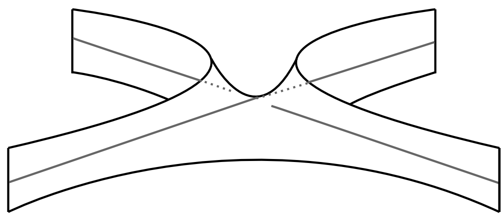


Figure 1.6: A saddle at a crossing of P in the construction of the Turaev surface T_P, where the projection $\mathbb{P}$ is shown in gray.

1.5. Thicken the projection plane to $\mathbb{R}^2 \times [-1, 1]$ so that P lies in $\mathbb{R}^2 \times \{0\}$. Away from any crossings, thicken P to $P \times [-1, 1]$. In a neighborhood of each crossing, insert a saddle so that the boundary circles in $\mathbb{R}^2 \times \{1\}$ correspond to the all-A smoothing of P and the boundary circles in $\mathbb{R}^2 \times \{-1\}$ correspond to the all-B smoothing. (See Figure 1.6.) Finally, cap off each boundary component with a disk to obtain a *Turaev surface* T_P for K. Lemma 4.1 of [13] shows that for all P, T_P is unknotted in S^3, and Lemma 4.5 of [13] shows that K has an alternating projection $\mathbb{P}$ on T_P coming directly from P and the construction of T_P. Further, $\mathbb{P}$ has no nugatory crossings on T_P in the sense that there is no simple closed curve on T_P which meets $\mathbb{P}$ exactly once at a crossing of $\mathbb{P}$ and bounds a disk on T_P.

Definition 1.2.1. The *Turaev genus* of a knot K is given by

$$g_T(K) := \min\{g(T_P) \mid T_P \text{ is a Turaev surface coming form a projection } P \text{ of } K\}$$

where $g(\cdot)$ denotes the genus of a surface.

If $g_T(K) = 0$, then K has an alternating projection on the sphere S^2, which clearly can happen if and only if K is alternating. Note that although a knot K has an alternating projection on any Turaev surface for K, the Turaev genus of K is not the same as the minimal genus Heegard splitting of S^3 on which K has an alternating projection, which is called the alternating genus of K, since any Turaev surface for K must be obtained through the procedure described above and hence satisfies properties besides simply admitting an alternating projection. In particular, $g_T(K)$ is bounded below by the alternating genus of K.

Although it is very natural to ask whether the Turaev genus is additive in the sense that $g_T(K_1 \# K_2) = g_T(K_1) + g_T(K_2)$, very little is known toward answering this question. Abe has shown in [2] that $g_T(K_1 \# K_2) = g_T(K_1) + g_T(K_2)$ if K_1 and K_2 are both in a class of knots called adequate knots by using the heavy machinery of Khovanov homology.

As a first step towards answering the question of the additivity of the Turaev genus for all knots, we will study composite knots of Turaev genus one. Suppose K is such a knot and T is a torus which is a Turaev surface for K. If $K = K_1 \# K_2$, then there is a 2-sphere A which meets K exactly twice and decomposes K into K_1 and K_2. Using arguments similar to those used by Menasco in [29], we can add a bubble to T at each crossing of P, as shown in Figure 5.2. W can then construct T_+ and T_- from T by replacing each disk of T inside a bubble by the upper and lower hemisphere of the bubble, respectively. By analyzing the intersections of $A \cap T_\pm$, we can study the Turaev genera of the summands K_1 and K_2. Section 5.5 is devoted to such analysis when A and $A \cap T_\pm$ are especially nice, as in the following corollary, where $\mathcal{C}'_- := A' \cap T_-$ and g_{alt} and g_{DA} are defined in Definition 5.5.3.

Corollary 5.5.8. *Let K be a composite knot with $g_T(K) = 1$ and let A' be a preferred decomposing sphere for K with respect to the genus-one Turaev surface T such that A' decomposes K into $K_a \# K_b$. If no bubble of T meets A' more than once, then either $g_T(K_a) + g_T(K_b) = 1$ or $|\mathcal{C}'_-| = 1$ and, up to relabelling, K_a is alternating and $g_{alt}(K_b) = g_{DA}(K) = 1$.*

The question of the additivity of the Turaev genus is especially interesting because of the role played by the Turaev genus in studying knot homologies. In 2000, Khovanov introduced an important new knot invariant refining the Jones polynomial, which is now referred to as *Khovanov homology* [20]. Then, in 2003, Ozsváth, Szabó and Rasmussen defined *knot Floer homology*, a knot invariant which categorifies the Alexander polynomial [31, 35]. For a knot K, both the width of the knot Floer homology and the width of the reduced Khovanov homology of K, denoted $w_{HF}(K)$ and $w_{Kh}(K)$, respectively, are bounded above by the inequality $g_T(K) \geq w_*(K) - 1$. (See [24, 12, 27] for details.) We will say a knot is *thin* if its width in *both* of these homologies is one.

The class of quasi-alternating knots was defined in [32], and all knots in this class are known to be thin [26]. There is an infinite family of quasi-alternating knots with Turaev genus one [11]. For these knots, $g_T(K) - w_*(K) = 0$ and the above inequality is not sharp. This leads one to ask whether $g_T(K) - w_*(K)$ can be arbitrarily large for one or both of these homology theories. In particular, are there quasi-alternating knots with arbitrarily high Turaev genus? Since the connect sum of quasi-alternating knots is quasi-alternating, if the Turaev genus is shown to be additive, this would give a family of knots $\{K_n\}$ such that $g_T(K_n) - w_*(K_n) = n - 1$ for each positive integer n.

Chapter 2

Preliminaries

2.1 Surfaces and 3-manifolds

Throughout this work we will be using orientable surfaces embedded in 3-manifolds to study properties of knots. Hence all knots, links, surfaces and 3-manifolds will be assumed to be orientable, even if they are not specifically stated as such. A 3-manifold M is called *irreducible* if any 2-sphere $S^2 \subset M$ bounds a 3-ball on at least one side. Otherwise, we say M is *reducible*. A surface $\Sigma \subset M$ is *compressible* if there is a disk $D \subset M$ smoothly embedded such that $D \cap \Sigma = \partial D$ (where ∂ denotes the boundary of a manifold) and ∂D does not bound a disk in Σ. If no such *compressing disk* exists, then Σ is *incompressible*.

Let C be a simple closed curve on some surface Σ and let $A \subset \Sigma$ be an open annular neighborhood of C. A *Dehn twist* of Σ at C is a diffeomorphism $\phi : \Sigma \to \Sigma$ such that ϕ is the identity map on $\Sigma - A$ and $\phi|_A$ is given by a full twist around C. More specifically, if $A \cong S^1 \times (0, 1)$ is given the coordinates $(e^{i\theta}, x)$, then $\phi|_A : (e^{i\theta}, x) \to (e^{i(\theta+2\pi x)}, x)$.

Given a closed surface $T \subset M$, by M *cut along* T we mean the 3-manifold $\overline{M - \eta(T)}$. Similarly, given a surface Σ and another surface $F \subset M$ with $F \cap \Sigma = \partial F$, we can *surger* Σ *at* F by removing an open neighborhood of ∂F from Σ and gluing 2 copies of F to Σ along the newly created boundary components of $\overline{\Sigma - \eta(\partial F)}$.

Much of the work in Chapter 4 utilizes tori in knot complements in S^3. A torus T in a 3-manifold M is called *essential* if it is incompressible and is not boundary parallel, or *peripheral*, in M. The manifold M is *atoroidal* if it does not contain any embedded essential tori.

Definition 2.1.1. Given a compact, irreducible, orientable 3-manifold M, let $\mathcal{T}$ be a collection of disjointly embedded, pairwise non-parallel, essential tori in M, which we will call an *essential torus collection* for M. By Haken's Finiteness Theorem (Lemma 13.2 of [17]) the number

$$\tau(M) := \max\{|\mathcal{T}| \mid \mathcal{T} \text{ is an essential torus collection for } M\}$$

is well-defined and finite, where $|\mathcal{T}|$ denotes the number of tori in $\mathcal{T}$. We will call such a collection $\mathcal{T}$ with $|\mathcal{T}| = \tau(M)$ a *Haken system* for M. Note that any essential torus $T \subset M$ is part of some Haken system $\mathcal{T}$.

A torus $T \subset S^3$ always bounds a solid torus on at least one side. If T cuts S^3 into two solid tori, then we say T is *standardly embedded* or *unknotted* in S^3. Let V be a solid torus bounded by T. A *meridian* of T is a simple closed curve $\mu \subset T$ such that μ bounds a disk in V called a *meridional disk*. A *longitude* of T is a simple closed curve $\lambda \subset T$ such that λ generates $H_1(V) \cong \mathbb{Z}$. Let C be the core of the solid torus V. Then a longitude λ of T is said to be *canonical* or *standard* if $\operatorname{lk}(\lambda, C) = 0$. The torus T is unknotted if and only if a standard longitude on T bounds a disk in the complement of T.

Given a connected surface Σ, define $\chi_-(\Sigma) := \max(0, -\chi(\Sigma))$, where $\chi(\cdot)$ denotes the Euler characteristic. If $\Sigma := \bigsqcup_{i=1}^n \Sigma_i$ is disconnected, then $\chi_-(\Sigma) := \sum_{i=1}^n \chi_-(\Sigma_i)$. Given a 3-manifold with boundary M and an element $a \in H_2(M, \partial M)$, we define the *Thurston norm* $x : H_2(M, \partial M) \to \mathbb{Z}$ by $x(a) := \min\{\chi_-(\Sigma) \mid a = [\Sigma] \in H_2(M, \partial M)\}$. We say a surface $(\Sigma, \partial\Sigma) \subset (M, \partial M)$ is *Thurston norm-minimizing* if $\chi_-(\Sigma) = x([\Sigma])$.

An orientable manifold is called a *Haken manifold* if it is a compact, irreducible 3-manifold that contains an orientable, incompressible surface. The following well-known result of Gabai will be used in Chapters 3 and 4.

Theorem 2.1.2 (Gabai, Corollary 2.4 of [14])**.** *Let M be a Haken manifold whose boundary is a nonempty union of tori. Let Σ be a Thurston norm-minimizing surface representing an element of $H_2(M, \partial M)$ and let P be a component of ∂M such that $P \cap \Sigma = \emptyset$. Then with at most one exception (up to isotopy) Σ remains Thurston norm-minimizing in each manifold $M(s)$ obtained by a slope-s Dehn filling of M along P. In particular Σ remains incompressible in all but at most one manifold obtained by a Dehn filling of P.*

2.2 Seifert surfaces and the Alexander polynomial

Given a knot $K \subset S^3$, and *Seifert surface* for K is an orientable surface $F \subset S^3$ such that $\partial F = K$. The genus of K, denoted $g(K)$ is then the minimal genus of a Seifert surface for K. Given a Seifert surface F, we can choose a basis $\{a_i\}$ for the first homology group $H_1(F) := H_1(F; \mathbb{Z})$ where each generator a_i is represented by an oriented simple closed curve $\bar{a}_i$ on F. By choosing an orientation on F, we can define $\bar{a}_i{}^+$ to be the curve $\bar{a}_i$ pushed slightly off F in the positive normal direction. The *Seifert matrix* V associated to F and $\{a_i\}$ is then given by $V_{i,j} := \mathrm{lk}(\bar{a}_i, \bar{a}_j{}^+)$.

Definition 2.2.1. The *Alexander polynomial* of a knot K is a polynomial in $\mathbb{Z}[t^{\pm 1}]$ given by $\Delta_K(t) \doteq \det(tV - V^T)$ where V is a Seifert matrix for K and $\doteq$ denotes equality up to multiplication by a unit in $\mathbb{Z}[t^{\pm 1}]$. The *knot determinant* $\det(K) := |\Delta_K(-1)|$.

The Alexander polynomial $\Delta_K(t)$ is a well-defined invariant of K, but, in general, a knot K does not have a unique Seifert matrix.

Definition 2.2.2. Two integral square matrices V and V' are S-*equivalent*, denoted $V \sim V'$, if V' is obtained from V by a finite sequence of the following moves or their inverses.

1. Replacing V by PVP^T for some integral unimodular matrix P.

2. A *column expansion*, where we replace the $n \times n$ matrix V with an $(n+2) \times (n+2)$ matrix of the form

$$\left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & V & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline u_1 & \cdots & u_n & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{array}\right)$$

   where $u_1, \ldots, u_n \in \mathbb{Z}$.

3. A *row expansion*, which is defined analogously to the column expansion, with the roles of rows and columns reversed.

A Seifert matrix for a knot K is not unique since it depends on the choices of the Seifert surface F, the basis $\{a_i\}$ and the orientations. However, any two Seifert matrices for K are always S-equivalent. If K has a unique minimal-genus Seifert surface F, up to isotopy, and $g(F) = g$, then a $(2g \times 2g)$ Seifert matrix for K depends only on the choice of basis for $H_1(F)$ and orientations. In this situation, any two $(2g \times 2g)$ Seifert matrices V and V' for K are *congruent*, so $V' = PVP^T$ for some integral unimodular matrix P, and we write $V \approx V'$.

Given a Seifert surface F for a knot K, the *Seifert form* associated to F is the map $\theta : H_1(F) \times H_1(F) \to \mathbb{Z}$ is given by $\theta(\alpha, \beta) := \mathrm{lk}(\bar{\alpha}, \bar{\beta}^+)$, where $\bar{\alpha}$ and $\bar{\beta}$ are simple closed curves on F representing α and β, respectively. (See [23] for details.)

Definition 2.2.3. A knot K is called *algebraically slice* if it admits a Seifert surface F such that the Seifert form $\theta : H_1(F) \times H_1(F) \to \mathbb{Z}$ vanishes on a half-dimensional summand of $H_1(F)$. This half-dimensional summand is called a *metabolizer* of $H_1(F)$.

If F has genus one, then the existence of a metabolizer for $H_1(F)$ is equivalent to the existence of an essential oriented simple closed curve on F that has zero self-linking number. If K is algebraically slice, then $\Delta_K(t)$ is of the form $\Delta_K(t) \doteq f(t)f(t^{-1})$, where $f(t) \in \mathbb{Z}[t]$ is a linear polynomial.

2.3 Alternating knots

An alternating knot diagram for a knot K is a projection P such that, as we traverse K, each time we come to a crossing of P, we alternate between meeting an overstrand and an understrand. A knot is *alternating* if it has an alternating diagram. Since the Turaev genus is a generalization of this property, the following proposition will be helpful in Chapter 5.

Proposition 2.3.1. *Given knots K_1 and K_2, $K_1 \# K_2$ is alternating if and only if both K_1 and K_2 are.*

Proof. One direction holds obviously since the connect sum of alternating diagrams is an alternating diagram. For the other direction, let $K := K_1 \# K_2$ and assume K is alternating. Let P be a projection of K on $\mathbb{R}^2$. We will call P a *composite diagram* if there is a simple closed curve $S^1 \subset \mathbb{R}^2$ meeting P transversely exactly twice and not at crossings such that each of the components of $\mathbb{R}^2 - \eta(S^1)$ contains a nontrivial summand of K. If P is not composite, then P is called a *prime diagram.*

In [29], Menasco shows that for any alternating knot K with an alternating projection P such that P has no nugatory crossings, K is composite if and only if P is a composite

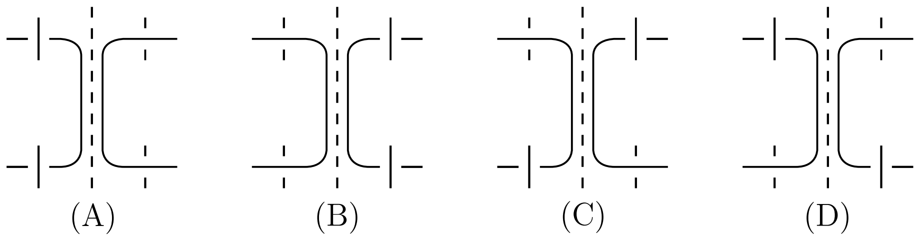


Figure 2.1: Possible decomposition sites for a composite alternating link diagram.

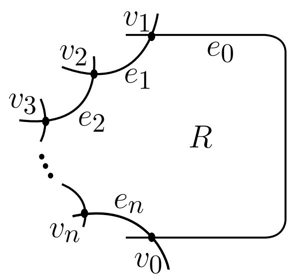


Figure 2.2: The four-valent graph P'.

diagram. Let P be such a composite alternating diagram for K, and let K' and K'' be two summands of K which are visible in P. At the site where we decompose K into K' and K'', the projection must look locally like one of the possibilities shown in Figure 2.1.

In each of the Figures 2.1(C) and 2.1(D), K' and K'' are alternating. Consider Figure 2.1(A) and let K' be the summand on the left. We will denote this projection of K' by P'. Thinking of P' as a four-valent graph with over/under information at each of the vertices, let e_0 be the edge of P' which was created in decomposing K, and let R be the bounded region of $\mathbb{R}^2 - \eta(P')$ which is adjacent to e_0. (See Figure 2.2.) Label the other edges of P' adjacent to R as $e_1, ..., e_n$, in order, and the vertices of P' such that each e_i is adjacent to v_i and $v_{i+1 \pmod n}$.

Since each of the e_i for $1 \leq i \leq n$ is an edge of P, it must be alternating. This causes a contradiction, since e_0 contributes an undercrossing at both v_0 and v_n. Hence Figure 2.1(A) cannot occur. An identical argument applies to Figure 2.1(B), so K' and K'' must

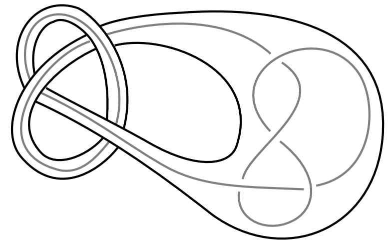


Figure 2.3: A follow-swallow companion torus for a composite knot.

be alternating. We can iterate this argument to show that each prime summand of K is alternating and hence so are K_1 and K_2. $\square$

2.4 Classification of knots

In [42], Thurston classified all knots in S^3 as being either hyperbolic knots, torus knots, or satellite knots. *Hyperbolic knots* are those which admit a complete hyperbolic metric on their complements in S^3. The *torus knot* $T_{p,q}$ is the knot embedded on an unknotted torus with slope p/q with $\gcd(p, q) = 1$.

Definition 2.4.1. A knot K is a *satellite knot* if $M_K := \overline{S^3 - \eta(K)}$ contains a knotted torus T which is essential in M_K and K is contained in the solid torus V bounded by T in S^3. Such a torus T is called a *companion torus* for K. Further, there exists a homeomorphism $f : (V', K') \to (V, K)$, called the *satellite map*, where V' is an unknotted solid torus in S^3 and K' is contained in $\mathrm{int}(\mathrm{V}')$. The knot K' is called a *pattern knot* for the satellite knot K.

Any composite knot $K = K_1 \# K_2$ is a satellite knot. This can be seen by choosing K_1 as the pattern knot and K_2 as the core of the companion torus, as illustrated in Figure 2.3. Such a companion torus for a composite knot is called a *follow-swallow* torus.

Another well-known class of satellite knots are the *Whitehead doubles.* The n-twisted,

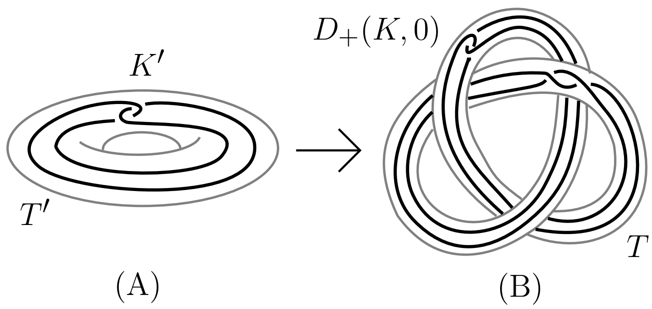


Figure 2.4: (A) The pattern knot for any positive Whitehead double $D_+(K,n)$. (B) The knot $D_+(K,-2)$ for the left-handed trefoil K.

positively-clasped Whitehead double of K, denoted $D_+(K,n)$ is obtained using the pattern knot K' in Figure 2.4(A), where the core of the companion torus for $D_+(K,n)$ is K and the canonical longitude of the unknotted torus T' is sent to the longitude γ_n of T, where $\text{lk}(K,\gamma_n) = n$. The n-twisted, negatively-clasped Whitehead double of K, written $D_-(K,n)$ is defined similarly, except both of the crossings in the pattern knot K' are changed from positive to negative. This class of knots is the focus of Section 4.6.

Another family of knots discussed in this work is that of *cable knots*. This class of knots contains all torus knots as well as satellite knots obtained from torus knots in the following way. Let $K' := T_{p,q}$ be a torus knot sitting on the unknotted torus T'. By pushing K slightly into the solid torus bounded by T', K' can be thought of as a pattern knot and the resulting satellite knot is a cable knot.

2.5 Twisted knots and braids

Let K be a knot in a solid torus V. The *winding number*, $w(K,V)$, is the minimal algebraic intersection number of K with a meridional disk of V. Similarly, the *wrapping number*, $\text{wrap}(K,V)$, is the minimal geometric intersection of K with a meridional disk. If V is

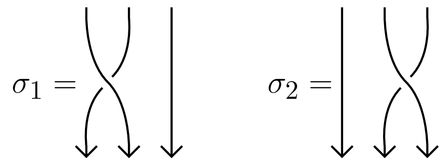


Figure 2.5: The generators σ_1 and σ_2 of B_3.

knotted, then K is a satellite knot. If V is unknotted, we have the following definition.

Definition 2.5.1. Let K be a knot that is *geometrically essential* in a solid torus V, so every meridional disk of ∂V meets K at least once. Given $n \in \mathbb{Z}$, let $K_{n,V}$ be the knot obtained from K via n full twists along a meridional disk $D \subset V$ such that the geometric intersection of D with K is at least two and and cannot be reduced by isotopy. We call such a disk a *twisting disk*, and $K_{n,V}$ is called a *twist knot of* K. When there is no danger of confusion, we will simply write K_n instead of $K_{n,V}$.

For $p > 0$, let B_p denote the p-string braid group. Then B_p is generated by $\{\sigma_i\}_{i=1}^{p-1}$ where σ_i is given by a single crossing between the i^{th} and $(i+1)^{\text{th}}$ strand of a p-braid, and we will let Δ_p^2 denote the full twist braid, which is the central element in B_p (See [6] for details.) For example, the generators of B_3 are shown in Figure 2.5, and $\Delta_3^2 = (\sigma_1\sigma_2)^3$. Given a braid β, we will let $\hat{\beta}$ denote the knot or link obtained as the standard closure of β. Note that $\hat{\beta}$ depends only on the conjugacy class of β in B_p.

Definition 2.5.2. A *fibered m-braid* is a closed braid on m strands which is also a fibered knot in the sense that the complement of the knot in S^3 fibers over S^1. Given a fibered m-braid K with $2 \le p \le m$ and $q \in \mathbb{Z}$, a new knot can be obtained by inserting a copy of Δ_p^{2q} (q full twists) into p strings of K. We will call such a knot a *twisted fibered braid* and denote it by $K_{p,q}$. Note that the knot $K_{p,q}$ is not unique and depends on the choice of the p strings which are twisted.

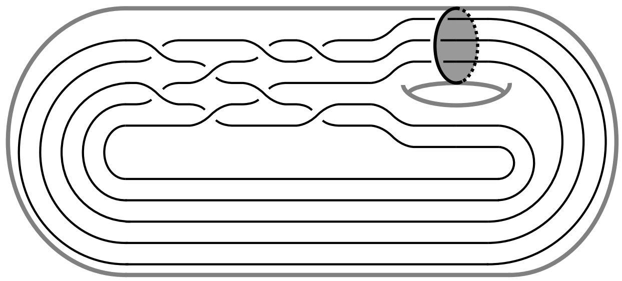


Figure 2.6: A fibered 5-braid K. By inserting q full twists at the shaded meridional disk, one obtains a twisted fibered knot $K_{3,q}$.

Twisted fibered braids are a family of twist knots, as illustrated in Figure 2.6. A well-studied class of twisted fibered braids is produced from torus knots by inserting full twists along a number of strands. We call such knots *twisted torus knots* (see [8, 10]).

Definition 2.5.3. A braid $\beta = \sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\cdots\sigma_{i_n}^{\epsilon_n} \subset B_p$, where each $\epsilon_i \in \mathbb{Z}$, is called *homogeneous* if both of the following hold.

1. Every σ_i occurs at least once for each $1 \leq i \leq p-1$.

2. The exponent of σ_i is of the same sign for each occurance of σ_i in β.

In particular, a braid β is *positive* if β is homogeneous and all exponents are positive.

The closure of a homogeneous braid is known to be fibered by [41], so by adding full twists to closed homogenous braids we may obtain broad classes of twisted fibered braids.

Chapter 3

Crossing changes and genus-one knots

Recall that throughout this paper, $M_{\mathcal{L}} := S^3 - \eta(\mathcal{L})$ for any knot or link $\mathcal{L}$. We first prove the following lemma, which will be used several times in this chapter and the next.

Lemma 3.0.4. *Let K be a knot, and let L a crossing circle for K. Suppose that $M_{K\cup L}$ is reducible. Then L is nugatory.*

Proof. An essential 2-sphere in $M_{K\cup L}$ must separate $\eta(K)$ and $\eta(L)$. Thus if $M_{K\cup L}$ is reducible, then L lies in a 3-ball in S^3 disjoint from K. Since L is unknotted, L bounds a disc in the complement of K. □

3.1 Crossing disks and Seifert surface

Let K be an oriented knot and $L = \partial D$ be a crossing circle for K. Since the linking number of L and K is zero, K bounds a Seifert surface in the complement of L. Let F be a Seifert surface that is of minimal genus among all such Seifert surfaces in the complement of L. Since F is incompressible, after an isotopy we may assume that the closed components of $F \cap D$ are homotopically essential in $D - \eta(K)$. In particular, suppose there is a component of $F \cap D$ which bounds a disk in $D - \eta(K)$ and let C be an innermost such component, so C bounds a disk $\Delta_1 \subset D - (\eta(K) \cup F)$. Since F is incompressible, C also bounds a disk $\Delta_2 \subset F$ and $\Delta_1 \cup \Delta_2 \cong S^2$ bounds a 3-ball $B \subset S^3 - F$. By isotoping Δ_2 through B we can remove C from $F \cap D$. Hence we may assume each simple closed curve of $F \cap D$ is

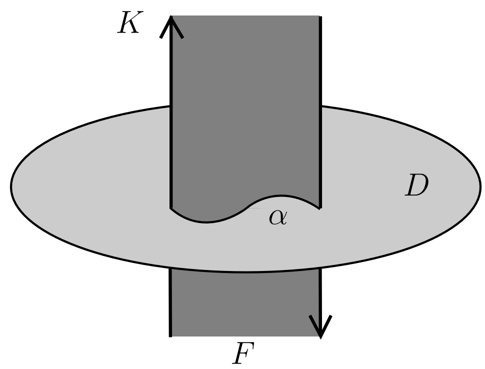


Figure 3.1: The crossing arc $\alpha = F \cap D$.

parallel in $D - \eta(K)$ to ∂D and, by further isotopy, we can arrange so that $F \cap D$ contains no closed components. Any component of $F \cap D$ which has boundary must have its endpoints at $K \cap D$, so $F \cap D$ must now consist of a single arc α that is properly embedded in F, as illustrated in Figure 3.1. The surface F gives rise to a Seifert surface $F(q)$ for $K_L(q)$ by twisting F q times at α.

Proposition 3.1.1. *Suppose that L is an order-q cosmetic crossing circle for a knot K. Let F be a minimal-genus Seifert surface for K in the complement of L. Then F and $F(q)$ are Seifert surfaces of minimal genus for K and $K(q)$, respectively, in S^3.*

Proof. If L is nugatory, then L bounds a disc in the complement of F and the conclusion is clear.

Suppose L is cosmetic. By Lemma 3.0.4, $M_{K \cup L}$ is irreducible. We can consider the surface F properly embedded in $M_{K \cup L}$ so that F is disjoint from $\partial\eta(L) \subset \partial M$. The assumptions on irreducibility of $M_{K \cup L}$ and on the genus of F imply that the foliation machinery of Gabai [14] applies. In particular, F is Thurston norm minimizing in $M_{K \cup L}$. The manifolds M_K and $M_{K(q)}$ are obtained by Dehn fillings of $M_{K \cup L}$ along $\partial\eta(L)$. By Theorem 2.1.2, F can fail to remain Thurston norm minimizing (i.e. genus minimizing) in at most one of M_K and $M_{K(q)}$. Since we have assumed that L is cosmetic, M_K and $M_{K(q)}$ are homeomorphic (by an orientation-preserving homeomorphism). Thus F remains taut in both of M_K and $M_{K(q)}$. This implies that F and $F(q)$ are Seifert surfaces of minimal genus for K and $K(q)$, respectively. □

By Proposition 3.1.1, a generalized crossing change in a knot K that produces an isotopic knot corresponds to a properly embedded arc α on a minimal-genus Seifert surface F of K. This leads to the following.

Lemma 3.1.2. *Let F be a minimal-genus Seifert surface for a knot K, and let $\alpha \subset F$ be an embedded arc corresponding to a crossing circle L of K. If α is inessential on F, then L is nugatory.*

Proof. Recall that α is the intersection of a crossing disc D with F, where $\partial D = L$. Since α is inessential, it separates F into two pieces, one of which is a disc E. Consider D as properly embedded in a regular neighborhood $\eta(F)$ of the surface F. The boundary of a regular neighborhood of E in $\eta(F)$ is a 2-sphere S that contains the crossing disc D. Then $S - \text{int}(D)$ is a disk bounded by the crossing circle L with its interior disjoint from the knot $K = \partial F$. □

3.2 Obstructing cosmetic crossings in genus-one knots

The following results use Seifert matrices and the Alexander polynomial to find obstructions to genus-one knots admitting a cosmetic generalized crossing changes.

Theorem 3.2.1. *Let K be a genus-one knot. If K admits a cosmetic generalized crossing change, then K is algebraically slice. In particular, there is a linear polynomial $f(t) \in \mathbb{Z}[t]$*

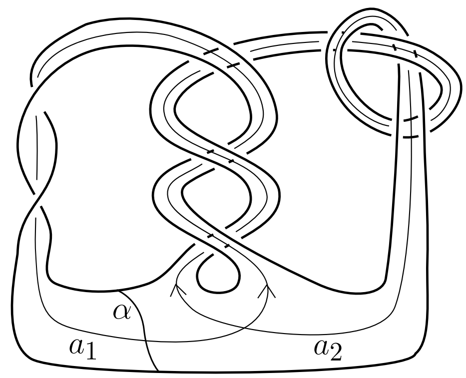


Figure 3.2: A genus-one surface F with generators a_1 and a_2 of $H_1(F)$ and a non-separating arc α.

such that the Alexander polynomial of K is of the form $\Delta_K(t) \doteq f(t)f(t^{-1})$.

Proof. Let $K(q)$ be a knot that is obtained from K by a cosmetic generalized crossing change at a crossing disk D. By Proposition 3.1.1, there is a genus-one Seifert surface F such that D intersects F in a properly embedded arc $\alpha \subset F$. Let $F(q)$ denote the result of F after the order-q generalized crossing change at α.

Since $L = \partial D$ is cosmetic, by Lemma 3.1.2, α is essential. Further, since the genus of F is one, α is non-separating. We can find a simple closed curve a_1 on F that intersects α exactly once. Let a_2 be another simple closed curve so that a_1 and a_2 intersect exactly once, $a_2 \cap \alpha = \emptyset$, and the homology classes of a_1 and a_2 form a basis for $H_1(F) \cong \mathbb{Z} \oplus \mathbb{Z}$. Note that $\{a_1, a_2\}$ forms a corresponding basis of $H_1(F(q))$. (See Figure 3.2.)

The Seifert matrices of F and $F(q)$ with respect to these bases are

$$V := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathrm{V_q} := \begin{pmatrix} a+q & b \\ c & d \end{pmatrix},$$

respectively, where $a, b, c, d \in \mathbb{Z}$. The Alexander polynomials of K and $K(q)$ are given by

$$\Delta_K(t) \doteq ad(1-t)^2 - (b-ct)(c-tb) \tag{3.1}$$

and

$$\Delta_{K(q)}(t) \doteq (a+q)d(1-t)^2 - (b-ct)(c-tb). \tag{3.2}$$

Since $K \cong K(q)$, we must have $\Delta_K(t) \doteq \Delta_{K(q)}(t)$, which easily implies that $d = \mathrm{lk}(a_2,\ a_2^+) = 0$. Hence K is algebraically slice and

$$\begin{aligned} \Delta_K(t) &\doteq (b-ct)(c-tb) = (-t)(b-ct)(b-ct^{-1}) \\ &\doteq (b-ct)(b-ct^{-1}). \end{aligned} \tag{3.3}$$

Setting $f(t) := b - ct$ we obtain $\Delta_K(t) \doteq f(t)f(t^{-1})$, as desired. Note that since $|b-c|$ is the intersection number between a_1 and a_2, by suitable orientation choices, we may assume that $c = b+1$. □

As a corollary of Theorem 3.2.1, we have the following.

Corollary 3.2.2. *Let K be a genus-one knot. If* $\det(K)$ *is not the square of an integer, then K admits no cosmetic generalized crossing change of any order.*

Proof. Suppose that K admits a cosmetic generalized crossing change. By Theorem 3.2.1, $\Delta_K(t) \doteq f(t)f(t^{-1})$ where $f(t) \in \mathbb{Z}[t]$ is a linear polynomial. Thus, if K admits a cosmetic generalized crossing change, we have $\det(K) = |\Delta_K(-1)| = [f(-1)]^2$. □

Recall that any two Seifert matrices for a knot K are S-equivalent as defined in Definition 2.2.2. Hence the following corollary is an immediate consequence of the proof of Theorem

3.2.1.

Corollary 3.2.3. *Let K be a genus-one knot. If K admits an order-q cosmetic generalized crossing change, then K has a Seifert matrix of the form* $\begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix}$ *which is S-equivalent to* $\begin{pmatrix} a+q & b \\ b+1 & 0 \end{pmatrix}$.

3.3 Crossing changes and double branched covers

In this section we primarily restrict our attention to crossing changes, i.e. generalized crossing changes of order ± 1. We derive obstructions to cosmetic crossing changes in terms of the homology of the double branched cover of S^3 branched over the knot K. More specifically, we will prove the following.

Theorem 3.3.1. *Let K be a genus-one knot and let Y_K denote the double branched cover of S^3 branched over K. If K admits a cosmetic crossing change, then the homology group $H_1(Y_K;\mathbb{Z})$ is a finite cyclic group.*

To prove Theorem 3.3.1 we need the following lemma. Here, given $m \in \mathbb{Z}$, we denote by $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ the cyclic abelian group of order $|m|$.

Lemma 3.3.2. *If H denotes the abelian group given by the presentation*

$$H \cong \left\langle \, c_1, c_2 \, \middle| \, \begin{array}{l} 2xc_1 + (2y+1)c_2 = 0 \\ (2y+1)c_1 = 0 \end{array} \right\rangle,$$

then we have the following.

1. $H \cong 0$, *if* $y = 0$ *or* $y = -1$.

2. $H \cong \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{(2y+1)^2}{d}}$, *if* $y \neq 0,\ -1$ *and* $d := gcd(2x,\ 2y+1)$ *with* $1 \leq d \leq |2y+1|$.

Proof. If $y = 0$ or $y = -1$, clearly we have $H \cong \{0\}$. Suppose that $y \neq 0,\ -1$ and set $d := \gcd(2x,\ 2y+1)$ with $1 \leq d \leq |2y+1|$. Then there are integers A and B such that $2x = dA,\ 2y+1 = dB$, and $\gcd(A, B) = 1$. Let α and β be such that $\alpha A + \beta B = 1$. Since $\begin{pmatrix} dA & dB \\ dB & 0 \end{pmatrix}$ is a presentation matrix of H and $\begin{pmatrix} \alpha & \beta \\ -B & A \end{pmatrix}$ is invertible over $\mathbb{Z}$, we get that $\begin{pmatrix} \alpha & \beta \\ -B & A \end{pmatrix}\begin{pmatrix} dA & dB \\ dB & 0 \end{pmatrix} = \begin{pmatrix} d & d\alpha B \\ 0 & -dB^2 \end{pmatrix}$ is also a presentation matrix for H. So

$$H \cong \left\langle \begin{array}{c|l} c_1, c_2 & dc_1 + d\alpha B c_2 = 0 \\ & dB^2 c_2 = 0 \end{array} \right\rangle \tag{3.4}$$

Now letting $c_3 = c_1 + \alpha B c_2$, we have

$$H \cong \left\langle \begin{array}{c|l} c_2, c_3 & dc_3 = 0 \\ & dB^2 c_2 = 0 \end{array} \right\rangle \tag{3.5}$$

Hence $H \cong \mathbb{Z}_d \oplus \mathbb{Z}_{dB^2} = \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{(2y+1)^2}{d}}$. □

With this lemma, we are now ready to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Suppose that a genus-one knot K admits a cosmetic crossing change yielding an isotopic knot K'. The proof of Theorem 3.2.1 shows that K and K' admit Seifert matrices of the form

$$V := \begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix} \text{ and } V' := \begin{pmatrix} a+\varepsilon & b \\ b+1 & 0 \end{pmatrix} \tag{3.6}$$

respectively, where $a, b \in \mathbb{Z}$ and $\varepsilon = 1$ or -1 according to whether K' is obtained via (-1)- or $(+1)$-Dehn surgery, respectively. In particular we have

$$\Delta_K(t) \doteq \Delta_{K'}(t) \doteq b(b+1)(t^2+1) - (b^2 + (b+1)^2 t). \tag{3.7}$$

Presentation matrices for $H_1(Y_K)$ and $H_1(Y_{K'})$ are given by

$$V + V^T = \begin{pmatrix} 2a & 2b+1 \\ 2b+1 & 0 \end{pmatrix} \text{ and } V' + (V')^T = \begin{pmatrix} 2a+2\epsilon & 2b+1 \\ 2b+1 & 0 \end{pmatrix}, \tag{3.8}$$

respectively, as shown in [36]. It follows that Lemma 3.3.2 applies to both $H_1(Y_K)$ and $H_1(Y_{K'})$. By that lemma, $H_1(Y_K)$ is either cyclic or $H_1(Y_K) \cong \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{(2b+1)^2}{d}}$, with $b \neq 0,\ -1$ and $\gcd(2a,\ 2b+1) = d$ where $1 < d \leq |2b+1|$. Similarly, $H_1(Y_{K'})$ is either cyclic or $H_1(Y_K') \cong \mathbb{Z}_{d'} \oplus \mathbb{Z}_{\frac{(2b+1)^2}{d'}}$, with $\gcd(2a+2\varepsilon,\ 2b+1) = d'$ where $1 < d' \leq |2b+1|$. Since K and K' are isotopic, we have $H_1(Y_K) \cong H_1(Y_{K'})$. One can easily verify this can only happen in the case that $\gcd(2a,\ 2b+1) = \gcd(2a+2\varepsilon,\ 2b+1) = 1$ in which case $H_1(Y_K)$ is cyclic. □

It is known that for an algebraically slice knot of genus one every minimal-genus Seifert surface F contains a metabolizer. After completing the metabolizer to a basis of $H_1(F)$ we have a Seifert matrix V as in equation (3.6) above.

Corollary 3.3.3. *Let K be an algebraically slice knot of genus one. Suppose that a genus-one Seifert surface of K contains a metabolizer leading to a Seifert matrix* $V = \begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix}$ *so that $b \neq 0, -1$ and $\gcd(2a,\ 2b+1) \neq 1$. Then K cannot admit a cosmetic crossing change.*

Proof. Let $d = \gcd(2a,\ 2b+1)$. As in the proof of Theorem 3.3.1, we use Lemma 3.3.2

to conclude that $H_1(Y_K) \cong \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{(2b+1)^2}{d}}$ and hence is not cyclic unless $d = 1$. Now the conclusion follows by Theorem 3.3.1. □

We also have the following corollary with regard to cosmetic generalized crossing changes.

Corollary 3.3.4. *Let K be an algebraically slice knot of genus one. Suppose that a genus-one Seifert surface of K contains a metabolizer leading to a Seifert matrix $V = \begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix}$ so that $b \neq 0, -1$. Let $d := \gcd(2a,\ 2b+1)$. Then K cannot admit an order-q cosmetic generalized crossing change if d does not divide q.*

Proof. First suppose V is the matrix coming from the basis defined in the proof of Theorem 3.2.1. Again we use Lemma 3.3.2 to see that $H_1(Y_K) \cong \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{(2b+1)^2}{d}}$ and $H_1(Y_{K(q)}) \cong \mathbb{Z}_{d'} \oplus \mathbb{Z}_{\frac{(2b+1)^2}{d'}}$ where $d' := \gcd(2a+2q,\ 2b+1)$. Hence $H_1(Y_K) \cong H_1(Y_{K(q)})$ if and only if $d = d'$. Since $2b+1$ is odd, d divides a and can only be a divisor of $2(a+q)$ if it also divides q. Hence for any Seifert matrix V for K of the form $V = \begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix}$, if K admits an order-q cosmetic generalized crossing change, then $d := \gcd(2a,\ 2b+1)$ divides q. □

3.4 S-equivalence of Seifert matrices

In light of the last section, it is natural to ask the following question.

Problem 3.4.1. *Given $a, b, q \in \mathbb{Z}$, when is* $\begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix} \sim \begin{pmatrix} a+q & b \\ b+1 & 0 \end{pmatrix}$*?*

A first observation is that if $b = 0$ or -1, then the two given matrices given in Problem 3.4.1 are congruent and, in particular, S-equivalent. We therefore restrict ourselves to matrices with non-zero determinant. In terms of Seifert matrices, this is equivalent to considering knots of genus one whose Alexander polynomial $\Delta_K(t) \doteq \det(V - tV^T)$ is non-trivial.

Proposition 3.4.2 is an auxiliary algebraic result about congruences of Seifert matrices. As an application of of this proposition, we prove cosmetic crossing changes do not exist in genus-one knots with non-trivial Alexander polynomial and with a minimal-genus Seifert surface which, up to isotopy, is unique.

Proposition 3.4.2. *Suppose that the matrices* $\begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix} \approx \begin{pmatrix} c & b \\ b+1 & 0 \end{pmatrix}$ *are congruent over* $\mathbb{Z}$*, where* $a, b, c \in \mathbb{Z}$*. Then there is an integer* n *such that* $a + n(2b+1) = c$*.*

Proof. To begin, we suppose that an integral congruence exists as hypothesized. That is, suppose that there exist integers x, y, z, t such that

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix} \begin{pmatrix} x & z \\ y & t \end{pmatrix} = \begin{pmatrix} c & b \\ b+1 & 0 \end{pmatrix}. \tag{3.9}$$

The left hand side multiplies out to give

$$\begin{pmatrix} x^2a + xy(2b+1) & xza + yz(b+1) + xtb \\ xza + xt(b+1) + zyb & z^2a + zt(2b+1) \end{pmatrix} = \begin{pmatrix} c & b \\ b+1 & 0 \end{pmatrix}. \tag{3.10}$$

By the bottom right entry of equation (3.10), we must consider each of the following three cases.

- $z = 0$
- $z \neq 0$ and $a = 0$
- $z \neq 0$ and $a \neq 0$

First, if $z = 0$, then equation (3.10) becomes

$$\begin{pmatrix} x^2a + (2b+1)xy & xtb \\ xt(b+1) & 0 \end{pmatrix} = \begin{pmatrix} c & b \\ b+1 & 0 \end{pmatrix}. \tag{3.11}$$

We need $x = t = 1$ or $x = t = -1$ for the top right and bottom left entries to be correct. Then setting $n = xy$ proves the proposition in this case.

Now, suppose $z \neq 0$ and $a = 0$. Then by the bottom right entry of equation (3.10), $zt(2b+1) = 0$. Hence equation (3.10) becomes

$$\begin{pmatrix} (2b+1)xy & yz(b+1) \\ zyb & 0 \end{pmatrix} = \begin{pmatrix} c & b \\ b+1 & 0 \end{pmatrix}. \tag{3.12}$$

The equations $zyb = b+1$ and $zy(b+1) = b$ imply that $b^2 = (b+1)^2$, which has no integral solutions.

Finally, suppose $z, a \neq 0$. Then $z = -t(2b+1)/a$, which we substitute into equation (3.10), to yield

$$\begin{pmatrix} xk & -t(b+1)k/a \\ -tbk/a & 0 \end{pmatrix} = \begin{pmatrix} c & b \\ b+1 & 0 \end{pmatrix} \tag{3.13}$$

where $k := ax + y(2b+1)$. The equations $(-tk/a)(b+1) = b$ and $(-tk/a)b = b+1$ imply again that $(b+1)^2 = b^2$ since neither t nor k can be 0. Since this does not have integral solutions, we also rule out this case. Hence the only congruences possible are those in the statement of the proposition, which occur when $z = 0$ and $x = t = \pm 1$. □

If K is a knot with, up to isotopy, a unique minimal-genus Seifert surface, then the Seifert matrix corresponding to that surface only depends on the choice of basis for the first homology. Put differently, the integral congruence class of the Seifert matrix corresponding to the unique minimal-genus Seifert surface is an invariant of the knot K. As a consequence of Proposition 3.4.2, we have the following theorem.

Theorem 3.4.3. *Let K be a genus-one knot with a unique minimal-genus Seifert surface, and suppose K admits a cosmetic crossing change. Then $\Delta_K(t) \doteq 1$.*

Proof. Let K be a genus-one knot with a unique minimal-genus Seifert surface, which admits a cosmetic crossing change (i.e. a cosmetic generalized crossing change of order-(± 1)). It follows from Corollary 3.2.3 and from the paragraph preceding the statement of this theorem that K admits a Seifert matrix $\begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix}$ which is congruent to $\begin{pmatrix} a+\varepsilon & b \\ b+1 & 0 \end{pmatrix}$ for some $\varepsilon \in \{-1, 1\}$. For $b \neq 0, -1$, Proposition 3.4.2 precludes such congruences from being possible. If $b = 0$ or -1, then the Alexander polynomial is 1. □

We also have the following corollary regarding cosmetic generalized crossing changes and unique genus-one Siefert surfaces.

Corollary 3.4.4. *Let K be a genus-one knot with a unique minimal-genus Seifert surface and suppose K admits an order-q cosmetic generalized crossing change. Then K has a Seifert matrix $V := \begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix}$ such that $(2b+1)$ divides q.*

Proof. Given such a K, it again follows from Corollary 3.2.3 that K admits a Seifert matrix $V := \begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix}$ which is S-equivalent to $\begin{pmatrix} a+q & b \\ b+1 & 0 \end{pmatrix}$. If $b = 0$ or -1, $|2b+1| = 1$ and the result follows immediately. For $b \neq 0, -1$, Proposition 3.4.2 implies that there exists some $n \in \mathbb{Z}$ such that $a + n(2b+1) = a + q$ and hence $q = n(2b+1)$. □

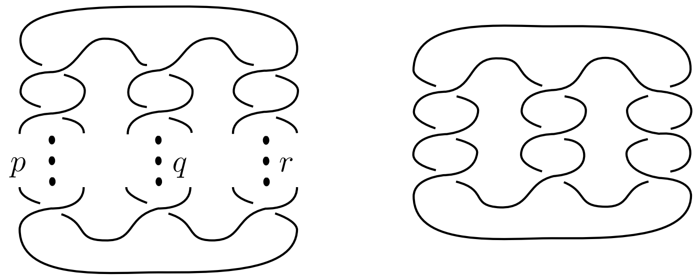


Figure 3.3: On the left is a diagram for $P(p, q, r)$ with p, q and r positive. Note that each of p, q and r denote the total number of crossings in the corresponding twist region. On the right is the pretzel knot $P(3, 3, -3)$.

3.5 Pretzel knots

Let K be a three string pretzel knot $P(p, q, r)$ with p, q and r odd as defined in Figure 3.3. The knot determinant of K is given by $\det(K) = |pq + qr + pr|$ (see [23]), and if K is non-trivial, then it has genus one. It is known that K is algebraically slice if and only if $pq + qr + pr = -m^2$, for some odd $m \in \mathbb{Z}$ [22].

Corollary 3.5.1. *The pretzel knot $P(p, q, r)$ with p, q and r odd does not admit a cosmetic generalized crossing change of any order if $pq + qr + pr \neq -m^2$, for every $m \in \mathbb{Z}$.*

Proof. This follows immedaitely from Theorem 3.2.1 and the discussion above. □

Corollary 3.5.2. *The knot $P(p, q, r)$ with p, q and r odd does not admit a cosmetic crossing change (i.e. a cosmetic generalized crossing change of order ± 1) if either of the following is true.*

1. $q + r = 0$ *and* $\gcd(p,\ q) \neq 1$

2. $p + q = 0$ *and* $\gcd(p,\ r) \neq 1$

Proof. There is a genus-one surface for $P(p,q,r)$ for which a Seifert matrix is $V_{(p,q,r)} := \frac{1}{2}\begin{pmatrix} p+q & q+1 \\ q-1 & q+r \end{pmatrix}$. (See Example 6.9 of [23].) Suppose that $q+r=0$. If $\gcd(p,\ q) \neq 1$, then $\gcd(p+q,\ q) \neq 1$ and the conclusion in Case 1 follows by Corollary 3.3.3. The proof for Case 2 is identical. □

3.6 Genus-one knots with low crossing number

The purpose of this section is to study potential cosmetic generalized crossing changes in genus-one knots with up to 12 crossings. We will need the following theorem of Trotter.

Theorem 3.6.1 (Trotter, Corollary 4.7 of [44]). *Let V be a Seifert matrix with $|\det(V)|$ a prime or* 1*. Then any matrix which is S-equivalent to V is congruent to V over $\mathbb{Z}$.*

With this in mind, we can prove the following.

Theorem 3.6.2. *Let K be a genus-one knot that has a diagram with at most 12 crossings. Then K admits no cosmetic crossing changes.*

Proof. Table 1, obtained from KnotInfo [9], gives the 23 knots of genus one with at most 12 crossings and the values of their determinants. We observe that there are four knots with square determinants. These are 6_1, $9_{46}, 10_3$ and $11n_{139}$, which are all known to be algebraically slice. Thus, Corollary 3.2.2 excludes cosmetic crossings for all but these four knots. Now 6_1 and 10_3 are 2-bridge knots, so by [43] they do not admit cosmetic crossing changes. The knot $K = 9_{46}$ is isotopic to the pretzel knot $P(3,3,-3)$ of Figure 3.3. By Corollary 3.5.2, this knot admits no cosmetic crossing changes.

The only remaining knot from Table 1 is the knot $K = 11n_{139}$. This knot is isotopic to the pretzel knot $P(-5,3,-3)$. There is therefore a genus-one Seifert surface for K for which

K	$\det(K)$	K	$\det(K)$	K	$\det(K)$
3_1	3	9_2	15	$11a_{362}$	39
4_1	5	9_5	23	$11a_{363}$	35
5_2	7	9_{35}	27	$\mathbf{11n_{139}}$	**9**
$\mathbf{6_1}$	**9**	$\mathbf{9_{46}}$	**9**	$11n_{141}$	21
7_2	11	10_1	17	$12a_{803}$	21
7_4	15	$\mathbf{10_3}$	**25**	$12a_{1287}$	37
8_1	13	$11a_{247}$	19	$12a_{1166}$	33
8_3	17	$11a_{343}$	31	-	-

Table 3.1: Genus-one knots with at most 12 crossings.

a Seifert matrix is $V := \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$ as described in Section 3.5. Since $|\det(V)| = 2$ is prime, by Corollary 3.2.3 and Theorem 3.6.1 it suffices to show that V is not integrally congruent to two matrices of the forms

$$V_1 := \begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix} \text{ and } V_2 := \begin{pmatrix} a+1 & b \\ b+1 & 0 \end{pmatrix}.$$

If this were true, V_1 and V_2 would be integrally congruent to each other, which cannot happen by Proposition 3.4.2 unless $b = 0$ or -1. In this case, $\Delta_K(t) \doteq 1$, but the Alexander polynomial of $11n_{139}$ is $\Delta_{11n_{139}}(t) \doteq 2 - 5t + 2t^2$. □

We also have the following corollary regarding cosmetic generalized crossing changes.

Corollary 3.6.3. *Let K be a genus-one knot that has a diagram with at most 12 crossings. If K admits an order-q cosmetic generalized crossing change, then one of the following must be true.*

1. *$K = 9_{46}$ and $q = 3n$ for some $n \in \mathbb{Z}$*
2. *$K = 11n_{139}$*

Proof. By the proof of Theorem 3.6.2, the only such knots which could admit a cosmetic generalized crossing change of any order are 9_{46} and $11n_{139}$. If $K = 9_{46} = P(3, 3, -3)$, then K has a Seifert matrix $\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$. Hence $H_1(Y_K)$ has the presentation matrix $\begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}$. Thus by Lemma 3.3.2, $H_1(Y_K) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$, and by Theorem 3.3.1, 3 divides q. □

Chapter 4

Crossing changes and embedded tori

4.1 Essential tori

In this chapter we will continue to investigate potential cosmetic generalized crossing changes by studying the interaction of the crossing circle L with embedded tori in the complement of K. We will continue using the notation of Chapter 3. The results of this chapter can also be found in [3, 5].

Fix a knot K and let L be a crossing circle for K. Let $M(q)$ denote the 3-manifold obtained from $M_{K \cup L}$ via a Dehn filling of slope $(-1/q)$ along $\partial\eta(L)$. (Recall that $M_{K \cup L} := \overline{S^3 - \eta(K \cup L)}$.) So, for $q \in \mathbb{Z} - \{0\}$, $M(q) = M_{K(q)}$, and $M(0) = M_K$. We will sometimes use $K(0)$ to denote $K \subset S^3$ when we want to be clear that we are considering $K \subset S^3$ rather than $K \subset M_L$.

Suppose there is some $q \in \mathbb{Z}$ for which $K_L(q)$ is a satellite knot. Then there is a companion torus T for $K_L(q)$ and, by definition, T is essential in $M(q)$. This essential torus T must occur in one of the following two ways.

Definition 4.1.1. Let $T \subset M(q)$ be an essential torus. We say T is *Type 1* if T can be isotoped into $M_{K \cup L} \subset M(q)$. Otherwise, we say T is *Type 2*. If T is Type 2, then we may isotope T so that it is the image of a punctured torus $(P, \partial P) \subset (M_{K \cup L}, \partial\eta(L))$ and each component of ∂P has slope $(-1/q)$ on $\partial\eta(L)$.

In general, let $\mathcal{L}$ be any knot or link in S^3 and let Σ be a boundary component of $M_{\mathcal{L}}$. If $(P, \partial P) \subset (M_{\mathcal{L}}, \Sigma)$ is a punctured torus and each component of ∂P is homotopically essential on Σ, then every component of ∂P has the same slope on Σ, which we call the *boundary slope* of P.

Suppose C_1 and C_2 are two non-separating simple closed curves (or boundary slopes) on a torus Σ. Let s_i be the slope of C_i on Σ, and let $[C_i]$ denote the isotopy class of C_i for $i = 1, 2$. Then $\Delta(s_1, s_2)$ is the minimal geometric intersection number of $[C_1]$ and $[C_2]$. If s_i is the rational slope $(1/q_i)$ for some $q_i \in \mathbb{Z}$ for $i = 1, 2$, then $\Delta(s_1, s_2) = |q_1 - q_2|$. (See [15] for more details.) Note that we consider $\infty = (1/0)$ to be a rational slope.

Gordon [15] proved the following theorem relating the boundary slopes of punctured tori in link complements. In fact, Gordon proved a more general result, but we state the theorem here only for the case which we will need later in Section 4.3.

Theorem 4.1.2 (Gordon, Theorem 1.1 of [15])**.** *Let $\mathcal{L}$ be a knot or link in S^3 and let Σ be a boundary component of $M_{\mathcal{L}}$. Suppose $(P_1, \partial P_1)$ and $(P_2, \partial P_2)$ are punctured tori in $(M_{\mathcal{L}}, \Sigma)$ such that the boundary slope of P_i on Σ is s_i for $i = 1, 2$. Then $\Delta(s_1, s_2) \leq 5$.*

4.2 Preliminary results

The goal of Section 4.3 is to prove Theorem 4.3.1, which says that any satellite knot K with an order-q cosmetic generalized crossing change with $|q| \geq 6$ admits a pattern knot K' with a cosmetic generalized crossing change of the same order. To do this we will need a few preliminary results, which we present in this section.

Suppose K is a knot contained in a solid torus $V \subset S^3$. We call K *geometrically essential* (or simply *essential*) in V if every meridional disk of V meets K at least once. With this in mind, we have the following lemma of Kalfagianni and Lin.

Lemma 4.2.1 (Kalfagianni and Lin, Lemma 4.6 of [19])**.** *Let $V \subset S^3$ be a knotted solid torus such that $K \subset \text{int(V)}$ is a knot which is geometrically essential in V, so* $\text{wrap}(V, K) \neq 0$. *Suppose K has a crossing disk D with $D \subset \text{int(V)}$. If K is isotopic to $K(q)$ in S^3, then $K(q)$ is also geometrically essential in V. Further, if K is not the core of V, then $K(q)$ is also not the core of V.*

Proof. Suppose, by way of contradiction, that $K(q)$ is not essential in V. Then there is a 3-ball $B \subset V$ such that $K(q) \subset B$. This means that the winding number $w(K(q), V) = 0$.

Let S_1 be a Seifert surface for K which is of minimal genus in M_L, where $L = \partial D$. We may isotope S_1 so that $S_1 \cap D$ consists of a single curve α as in Section 3.1. Then twisting S_1 q times at α gives rise to a Seifert surface S_2 for $K(q)$. By Proposition 3.1.1, S_1 and S_2 are minimal-genus Seifert surfaces in S^3 for K and $K(q)$, respectively.

Since $w(K(q), V) = 0$, $S_1 \cap \partial V = S_2 \cap \partial V$ is homologically trivial in ∂V. For $i = 1, 2$, we can surger S_i along disks and annuli in ∂V which are bounded by curves in $S_i \cap \partial V$ to get new minimal-genus Seifert surfaces $S_i' \subset \text{int(V)}$. Then S_2' is incompressible and V is irreducible, so we can isotope S_2' into int(B). Hence α and therefore D can also be isotoped into int(B). But then K must not be essential in V, which is a contradiction.

Finally, if K is not the core of V, then ∂V is a companion torus for the satellite knot K since V is knotted by assumption. Since a satellite knot cannot be isotopic to the core of its companion torus, $K(q)$ cannot be the core of V. □

The following lemma and its proof are of the same flavor as Lemma 4.2.1 and will be used in Section 4.4.

Lemma 4.2.2. *Let K be a satellite knot, T be a companion torus for K, and V be the solid torus bounded by T in S^3. Suppose that $w(K,V) = 0$ and that there are no essential annuli in $\overline{S^3 - V}$. Finally, suppose that K admits a cosmetic generalized crossing change of order-q, and let L be the corresponding crossing circle. Then we can isotope L so that L and a crossing disk bounded by L both lie in V.*

Proof. Let K and L be as in the statement of the lemma. Let S be a minimal-genus Seifert surface for K in M_L, and let D be the crossing disk for K which is bounded by L. We may isotope S so that $S \cap D$ is a single embedded arc α. Then performing $(-1/q)$-surgery at L twists both K and S at α, producing a surface $S(q) \subset M(q)$ which is a Seifert surface for $K(q)$. Again, by Proposition 3.1.1, S and $S(q)$ are minimal-genus Seifert surfaces in S^3 for K and $K(q)$, respectively.

We may isotope D to be a 2-dimensional neighborhood of α which is orthogonal to S. So if $S \subset \text{int}(V)$, then $D \subset V$ and there is nothing more to show. Assume that $S \not\subset V$, and let $C := S \cap T$. We may isotope S so that C is a collection of simple closed curves which are essential in both S and T. Since $w(K,T) = 0$, C must be homologically trivial in T, where each component of C is given the orientation induced by S. Hence C bounds a collection of annuli in T which we will denote by A_0.

Let $S_0 := S - (S \cap V)$. Suppose that $\chi(S_0) < 0$, where $\chi(\cdot)$ denotes the Euler characteristic. We may create S^* from S by replacing S_0 by A_0, isotoped slightly if necessary so that each component of A_0 is disjoint. Then S^* is a Seifert surface for K, and $\chi(S^*) > \chi(S)$ since $\chi(A_0) = 0$. This contradicts the fact that S is a minimal-genus Seifert surface for K, so it must be that $\chi(S_0) \geq 0$. Since S_0 contains no closed component, and no component of C bounds a disk in S, we conclude that S_0 consists of annuli.

By assumption, there are no essential annuli in $\overline{S^3 - V}$, so each component of S_0 must

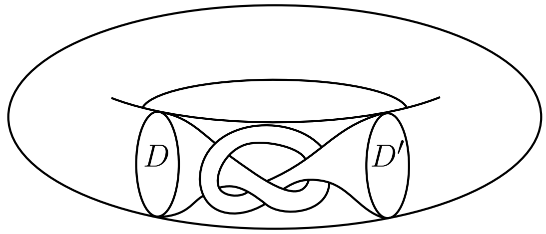


Figure 4.1: A knotted 3-ball B inside of a solid torus with disks $D, D' \subset \partial B$.

be boundary parallel in $\overline{S^3 - V}$. Thus we can isotope S_0 so that $S \subset V$, and therefore D can be isotoped into V as well. □

Before moving on to Theorem 4.3.1 and its corollaries, we state the following results of Motegi [30] (see also [39]) and McCullough [28] which we will need in Section 4.3.

Lemma 4.2.3 (Motegi, Lemma 2.3 of [30]). *Let K be a knot embedded in S^3 and let V_1 and V_2 be knotted solid tori in S^3 such that the embedding of K is essential in V_i for $i = 1, 2$. Then there is an ambient isotopy $\phi : S^3 \to S^3$ leaving K fixed such that one of the following holds.*

1. $\partial V_1 \cap \phi(\partial V_2) = \emptyset$.

2. *There exist meridian disks D and D' for both V_1 and V_2 such that some component of V_1 cut along $(D \sqcup D')$ is a knotted 3-ball in some component of V_2 cut along $(D \sqcup D')$.*

By a *knotted 3-ball*, we mean a ball B for which there is no isotopy which takes B to the standardly embedded 3-ball while leaving D and D' fixed. (See Figure 4.1.)

Theorem 4.2.4 (McCullough, Theorem 1 of [28]). *Suppose M is a compact, orientable 3-manifold that admits a homeomorphism which restricts to Dehn twists on the boundary of M along a simple closed curve in $C \subset \partial M$. Then C bounds a disk in M.*

4.3 Obstrucing cosmetic crossings in satellite knots

The goal of this section is to prove the following theorem and its corollaries.

Theorem 4.3.1. *Suppose K is a satellite knot which admits a cosmetic generalized crossing change of order q with $|q| \geq 6$. Then K admits a pattern knot K' which also has an order-q cosmetic generalized crossing change.*

We begin with the following lemma.

Lemma 4.3.2. *Let K be a prime satellite knot with a cosmetic crossing circle L of order q. Then at least one of the following must be true.*

1. *$M(q)$ contains no Type 2 tori*

2. *$|q| \leq 5$*

Proof. Suppose $M(q)$ contains a Type 2 torus. We claim that $M(0)$ must also contain a Type 2 torus. Assuming this is true, $M(0)$ and $M(q)$ each contain a Type 2 torus and hence there are punctured tori $(P_0, \partial P_0)$ and $(P_q, \partial P_q)$ in $(M, \partial\eta(L))$ such that P_0 has boundary slope $\infty = (1/0)$ and P_q has boundary slope $(-1/q)$ on $\partial\eta(L)$. Then, by Theorem 4.1.2, $\Delta(\infty, -1/q) = |q| \leq 5$, as desired. Thus it remains to show that there is a Type 2 torus in $M(0)$.

Let $M := M_{K\cup L}$. Since L is not nugatory, Lemma 3.0.4 implies that M is irreducible and hence the Haken number $\tau(M)$ is well-defined. First assume that $\tau(M) = 0$. Since K is a satellite knot, $M(0)$ must contain an essential torus, and it cannot be Type 1. Hence $M(0)$ contains a Type 2 torus.

Now suppose that $\tau(M) > 0$ and let T be an essential torus in M. Then T bounds a solid torus $V \subset S^3$. Let ext(V) denote $S^3 - V$. If $K \subset$ ext(V), then L must be essential in V. If

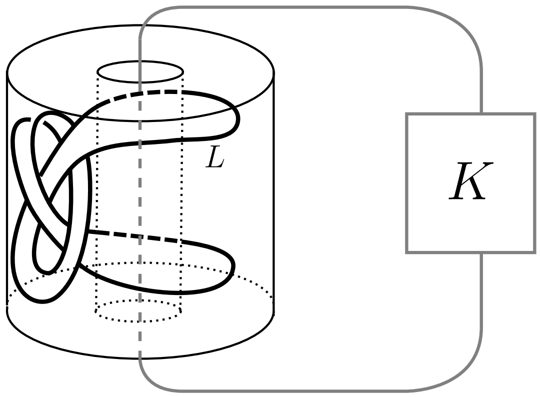


Figure 4.2: An example of an unknotted torus containing a crossing circle L which bounds a crossing disk for the knot $K \cong K\#U$.

V is knotted, then either L is the core of V or L is a satellite knot with companion torus ∂V. This contradicts the fact that L is unknotted. Hence T is an unknotted torus. By definition, L bounds a crossing disk D. Since D meets K twice, $D \cap \text{ext}(V) \neq \emptyset$. We may assume that D has been isotoped (rel boundary) to minimize the number of components in $D \cap T$. Since an innermost component of $D - (T \cap D)$ is a disk and L is essential in the unknotted solid torus V, $D \cap T$ consists of standard longitudes on the unknotted torus T. Hence $D \cap \text{ext}(V)$ consists of either one disk which meets K twice, or two disks which each meet K once. In the first case, L is isotopic to the core of V, which contradicts T being essential in M. In the latter case, the linking number $\text{lk}(K, V) = \pm 1$. So K can be considered as the trivial connect sum $K\#U$, where U is the unknot, and the crossing change at L takes place in the unknotted summand U. (See Figure 4.2.) The unknot does not admit cosmetic crossing changes of any order by [37], so $K_L(q) \cong K\#K'$ where $K' \ncong U$. This contradicts the fact that $K_L(q) \cong K$. Hence, we may assume that T is knotted and K is contained in the solid torus V bounded by T.

If $L \subset \text{ext(V)}$ and cannot be isotoped into V, then $D \cap T$ has a component C that is both homotopically non-trivial in $\overline{D - \eta(K)}$ and not parallel to ∂D. So C must encircle exactly one of the two points of $K \cap D$. This means that $\text{wrap}(K, V) = \pm 1$. Since T cannot be boundary parallel in M, K is not the core of T, and hence T is a follow-swallow torus for K and K is composite. But this contradicts the assumption that K is prime. Hence we may assume that L and, in fact, D are contained in int(V).

Since V is knotted and $D \subset \text{int(V)}$, Lemma 4.2.1 implies that if T is a companion torus for $K(0)$, then T is also a companion torus for $K(q)$. This means every Type 1 torus in $M(0)$ is also a Type 1 torus in $M(q)$. Since $K(0)$ and $K(q)$ are isotopic, $\tau(M(0)) = \tau(M(q))$. By assumption, $M(q)$ contains a Type 2 torus, which must give rise to a Type 2 torus in $M(0)$, as desired. □

The following is an immediate corollary of Lemma 4.3.2.

Corollary 4.3.3. *Let K be a prime satellite knot with a cosmetic crossing circle L of order q with $|q| \geq 6$. Then $\tau(M_{K \cup L}) > 0$.*

We are now ready to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Let K be a satellite knot as in the statement of the theorem. Let L be a crossing circle bounding a crossing disk D which corresponds to a cosmetic generalized crossing change of order q. Let $M := M_{K \cup L}$.

If K is a composite knot, then Torisu [43] showed that the crossing change in question must occur within one of the summands of $K = K_1 \# K_2$, say K_1. We may assign to K the follow-swallow companion torus T, where the core of T is isotopic to K_2. Then the patten knot corresponding to T is K_1 and the theorem holds.

Now assume K is prime. By Corollary 4.3.3, $\tau(M) > 0$. Let T be an essential torus in M and let $V \subset S^3$ be the solid torus bounded by T in S^3. As shown in the proof of

Lemma 4.3.2, V is knotted in M and D can be isotoped to lie in int(V). This means T is a companion torus for the satellite link $K \cup L$. Let $K' \cup L'$ be a pattern link for $K \cup L$ corresponding to T. So there is an unknotted solid torus $V' \subset S^3$ such that $(K' \cup L') \subset V'$ and there is a homeomorphism $f : (V', K', L') \to (V, K, L)$.

Let $\mathcal{T}$ be a Haken system for M such that $T \in \mathcal{T}$. We will call a torus $J \in \mathcal{T}$ *innermost with respect to* K if M cut along $\mathcal{T}$ has a component C such that ∂C contains $\partial\eta(K)$ and a copy of J. In other words, $J \in \mathcal{T}$ is innermost with respect to K if there are no other tori in $\mathcal{T}$ separating J from $\eta(K)$. Choose T to be innermost with respect to K.

Let $W := \overline{V - \eta(K \cup L)}$. We first wish to show that W is atoroidal. By way of contradiction, suppose that there is an essential torus $F \subset W$. Then F bounds a solid torus in V which we will denote by $\widehat{F}$. Since T is innermost with respect to K, either F is parallel to T in M, or $K \subset V - \widehat{F}$. By assumption, F is essential in W and hence not parallel to $T \subset \partial W$. So $K \subset V - \widehat{F}$ and, since F is incompressible, $L \subset \widehat{F}$. By the arguments of the proof of Lemma 4.3.2, $\widehat{F}$ is unknotted and we may assume that D has been isotoped to minimize $|D \cap F|$. Then $D \cap \overline{(S^3 - \widehat{F})}$ consists of either one disk which meets K twice or two disks each meeting K once. The first case contradicts the fact that F is essential in M. In the second case, we may consider K as $K \# U$ as in Figure 4.2, where U is the unknot, and arrive at a contradiction as in the proof of Lemma 4.3.2 since U admits no cosmetic generalized crossing changes. Hence W is indeed atoroidal, and $W' := \overline{V' - \eta(K' \cup L')}$ must be atoroidal as well.

To finish the proof, we must consider two cases, depending on whether T is compressible in $V - \eta(K(q))$.

Case 1: $K(q)$ is essential in V.

We wish to show that there is an isotopy $\Phi : S^3 \to S^3$ such that $\Phi(K(q)) = K(0)$ and

$\Phi(V) = V$. First, suppose $K(q)$ is the core of T. By Lemma 4.2.1, K is also the core of T. Since L is cosmetic, there is an ambient isotopy $\psi : S^3 \to S^3$ taking $K(q)$ to $K(0)$. Since $K(q)$ and $K(0)$ are both the core of V, we may choose ψ so that $\psi(V) = V$ and let $\Phi := \psi$.

If $K(q)$ is not the core of T, then T is a companion torus for $K(q)$. Since $K(0) = (K(q))_L(-q)$, we may apply Lemma 4.2.1 to $K(q)$ to see that T is also a companion torus for $K(0)$. Again, there is an ambient isotopy $\psi : S^3 \to S^3$ taking $K(q)$ to $K(0)$ such that V and $\psi(V)$ are both solid tori containing $K(0) = \psi(K(q)) \subset S^3$. If $\psi(V) = V$, we once more let $\Phi := \psi$. If $\psi(V) \neq V$, we may apply Lemma 4.2.3 to V and $\psi(V)$. If part (2) of Lemma 4.2.3 were satisified, then $\psi(V) \cap V$ would give rise to a knotted 3-ball contained in either V or $\psi(V)$. This contradicts the fact that W, and hence $\psi(W)$, are atoroidal. Hence part (1) of Lemma 4.2.3 holds, and there is an isotopy $\phi : S^3 \to S^3$ fixing $K(0)$ such that $(\phi \circ \psi)(T) \cap T = \emptyset$. Let $\Phi := (\phi \circ \psi) : S^3 \to S^3$. Recall that by Lemma 4.3.2, $M(q)$ contains no Type 2 tori. Hence T remains innermost with respect to $K(q)$ in S^3 and therefore $\Phi(T)$ is also innermost with respect to $K(0)$. Either $T \subset S^3 - \Phi(V)$ or $\Phi(T) \subset S^3 - V$. In either situation, the fact that T and $\Phi(T)$ are innermost implies that T and $\Phi(T)$ are in fact parallel in M_K. So, after an isotopy which fixes $K(0) \subset S^3$, we may assume that $\Phi(V) = V$.

Now let $h := (f^{-1} \circ \Phi \circ f) : V' \to V'$. Note that h preserves the canonical longitude of $\partial V'$ (up to sign). Since h maps $K'(q)$ to $K'(0)$, $K'(q)$ and $K'(0)$ are isotopic in S^3. So either L' gives an order-q cosmetic generalized crossing change for the pattern knot K', or L' is a nugatory crossing circle for K'.

Suppose L' is nugatory. Then L' bounds a crossing disk D' and another disk $D'' \subset M_{K'}$. We may assume $D' \cap D'' = L'$. Let $A := D' \cup (D'' \cap V')$. Since $\partial V'$ is incompressible in $V' - \eta(K')$, by surgering along components of of $D'' \cap \partial V'$ which bound disks or cobound annuli in $\partial V'$, we may assume A is a properly embedded annulus in V' and each component

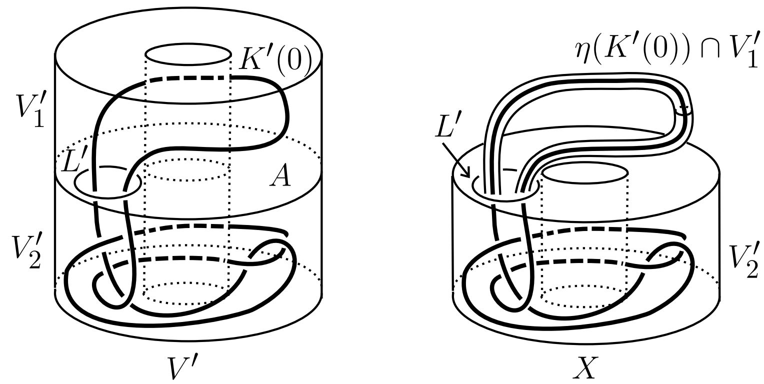


Figure 4.3: On the left is the solid torus V', cut into two solid tori by the annulus A. On the right is a diagram depicting the construction of X from the proof of Theorem 4.3.1.

of $A \cap \partial V'$ is a longitude of V'. Since $L' \subset V'$, we can extend the homeomorphism h on $V' \subset S^3$ to a homeomorphism H on all of S^3. Since V' is unknotted, let C be the core of the solid torus $S^3 - \mathrm{int(V')}$. We may assume that H fixes C. Since $D' \cup D''$ gives the same (trivial) connect sum decomposition of $K'(0) \cong K'(q)$ and H preserves canonical longitudes on $\partial V'$, we may assume $H(D')$ is isotopic to D' and $H(D'')$ is isotopic to D''. In fact, this isotopy may be chosen so that $H(C)$ and $H(V')$ remain disjoint throughout the isotopy and $H(V') = V'$ still holds after the isotopy. Thus, we may assume $h(A) = A$ and A cuts V' into two solid tori V_1' and V_2', as shown in Figure 4.3.

We now consider two subcases, depending on how h acts on V_1' and V_2'.

Subcase 1.1: $h : V_i' \to V_i'$ for $i = 1, 2$.

Up to ambient isotopy, we may assume the following.

1. $K'(q) \cap V_1' = K'(0) \cap V_1'$
2. $K'(q) \cap V_2'$ is obtained from $K'(0) \cap V_2'$ via q full twists at L'

Let X be the 3-manifold obtained from $V_2' - \eta(V_2' \cap K'(0))$ by attaching to $A \subset \partial V_2'$ a thickened neighborhood of $\partial\eta(K'(0)) \cap V_1'$. (See Figure 4.3.) Then $h|_X$ fixes X away from V_2' and acts on $X \cap V_2'$ by twisting $\partial\eta(K'(0)) \subset \partial X$ q times at L'. Hence there is a homeomorphism from X to $h(X)$ given by q Dehn twists at $L' \subset \partial X$. So, by Theorem 4.2.4, L' bounds a disk in $X \subset (V' - \eta(K'))$. But this means L bounds a disk in $(V - \eta(K)) \subset M_K$ and hence L is nugatory, contradicting our initial assumptions.

Subcase 1.2: h maps $V_1' \to V_2'$ and $V_2' \to V_1'$.

Again, we may assume the following.

1. $K'(q) \cap V_1' = K'(0) \cap V_2'$
2. $K'(q) \cap V_2'$ is obtained from $K'(0) \cap V_1'$ via q full twists at L'

This time we construct X from $V_1' - \eta(V_1' \cap K'(0))$ by attaching a thickened neighborhood of $\partial\eta(K'(0)) \cap V_2'$ to $A \subset \partial V_1'$. Then the arguments of Subcase 1.1 once again show that L must have been nugatory, giving a contradiction.

Hence, in Case 1, we have a pattern knot K' for K admitting an order-q cosmetic generalized crossing change, as desired.

Case 2: T is compressible in $V - \eta(K(q))$.

In this case, $K(q)$ is contained in a 3-ball $B \subset V$. Since $K(q)$ is not essential in V, by Lemma 4.2.1, $K(0)$ is also not essential in V, and $K(0) = f(K'(0))$ can be isotoped to $K(q) = f(K'(q))$ via an isotopy contained in the 3-ball $B \subset V$. This means that, once again, $K'(0)$ is isotopic to $K'(q)$ in S^3, and either L' gives an order-q cosmetic generalized crossing change for the pattern knot K' or L' is a nugatory crossing circle for K'. Applying the arguments of each of the subcases in Case 1, we see that L' cannot be nugatory, and hence K' is a pattern knot for K admitting an order-q cosmetic generalized crossing change. □

Theorem 4.3.1 gives obstructions to when cosmetic generalized crossing changes can occur in satellite knots. This leads us to several useful corollaries, including Corollaries 4.3.4 and 4.3.7, which concern hyperbolic knots, and Corollary 4.3.5, which addresses torus knots.

Corollary 4.3.4. *Let K be a satellite knot admitting a cosmetic generalized crossing change of order q with $|q| \geq 6$. Then K admits a pattern knot K' which is hyperbolic.*

Proof. Applying Theorem 4.3.1, repeatedly if necessary, we know K admits a pattern knot K' which is not a satellite knot and which also admits an order-q cosmetic generalized crossing change. Kalfagianni has shown that fibered knots do not admit cosmetic generalized crossing changes of any order [18], and it is well-known that all torus knots are fibered. Hence, by Thurston's classification of knots [42], K' must be hyperbolic. □

Corollary 4.3.5. *Suppose K' is a torus knot. Then no prime satellite knot with pattern K' admits an order-q cosmetic generalized crossing change with $|q| \geq 6$.*

Proof. Let K be a prime satellite knot which admits a cosmetic generalized crossing change of order q with $|q| \geq 6$. By way of contradiction, suppose T is a companion torus for K corresponding to a pattern torus knot K'. Since K is prime, Lemma 4.3.2 implies that T is Type 1 and hence corresponds to a torus in $M := M_{K \cup L}$, which we will also denote by T. If T is not essential in M, then T must to be parallel in M to $\partial\eta(L)$. But then T would be compressible in M_K, which cannot happen since T is a companion torus for $K \subset S^3$ and is thus essential in M_K. So T is essential in M and there is a Haken system $\mathcal{T}$ for M with $T \in \mathcal{T}$. Since the pattern knot K' is a torus knot and hence not a satellite knot, T must be innermost with respect to K. Then the arguments in the proof of Theorem 4.3.1 show that K' admits an order-q cosmetic generalized crossing change. However, torus knots are fibered and hence admit no cosmetic generalized crossing changes of any order, giving us our desired contradiction. □

Note that if K' is a torus knot which lies on the surface of the unknotted solid torus V', then (K', V') is a pattern for a satellite knot which is, by definition, a cable knot. Since any cable of a fibered knot is fibered, it was already known by [18] that these knots do not admit cosmetic generalized crossing changes. However, Corollary 4.3.5 applies not only to cables of non-fibered knots, but also to pattern torus knots embedded in *any* unknotted solid torus V' and hence gives us a new class of knots which do not admit a cosmetic generalized crossing change of order q with $|q| \geq 6$.

The proof of Corollary 4.3.5 leads us to the following.

Corollary 4.3.6. *Let K' be a knot such that $g(K') = 1$ and K' is not a satellite knot. If there is a prime satellite knot K such that K' is a pattern knot for K and K admits a cosmetic generalized crossing change of order q with $|q| \geq 6$, then K' is hyperbolic and algebraically slice.*

Proof. By Corollary 4.3.5 and its proof, K' is hyperbolic and admits a cosmetic generalized crossing change of order q. Then by Lemma 3.2.1, K' is algebraically slice. □

Finally, the following corollary summarizes the progress we have made in this section towards answering Problem 1.1.4.

Corollary 4.3.7. *If there exists a knot admitting a cosmetic generalized crossing change of order q with $|q| \geq 6$, then there must be such a knot which is hyperbolic.*

Proof. Suppose there is a knot K with a cosmetic generalized crossing change of order q with $|q| \geq 6$. Since K cannot be a fibered knot, K is not a torus knot, and either K itself is hyperbolic, or K is a satellite knot. If K is a satellite knot, then by Corollary 4.3.4 and its proof, K admits a pattern knot K' which is hyperbolic and has an order-q cosmetic generalized crossing change. □

4.4 Cosmetic crossings and twisting operations

Throughout this section, let $\mathbb{K}$ denote the class of knots which are known not to admit cosmetic generalized crossing changes. As noted in Section 1.1, $\mathbb{K}$ contains all fibered knots, 2-bridge knots and genus-one, algebraically non-slice knots. Torisu shows in [43] that the connect sum of two or more knots in $\mathbb{K}$ is also in $\mathbb{K}$.

Let V be a solid torus standardly embedded in S^3, and let K be a knot that is geometrically essential in V. Recall from Definition 2.5.1 that given $n \in \mathbb{Z}$, the knot $K_{n,V}$ is the image of K under the n^{th} power of a meridional Dehn twist of V and is called a *twist knot of* K. When there is no danger of confusion, we will simply write K_n instead of $K_{n,V}$.

Our first lemma regarding twist knots is similar to Lemmas 4.2.1 and 4.2.2.

Lemma 4.4.1. *Let V be a solid torus standardly embedded in S^3, let K' be a knot that is geometrically essential in the interior of V, and let $K := K'_n$ be a twist knot of K' for some $n \in \mathbb{Z}$. Suppose that K admits a cosmetic generalized crossing change of order q, and let L be the corresponding crossing circle. Then we can isotope L so that L and a crossing disk bounded by L lie in V.*

Proof. Let S be a minimal genus Seifert surface for K in M_L, and let D be the crossing disk for K which is bounded by L. As in Section 3.1, we may isotope S so that $S \cap D$ is a single embedded arc α. Then performing $(-1/q)$-surgery at L twists both K and S at α, producing a surface $S(q) \subset M(q)$ which is a Seifert surface for $K(q)$. By Proposition 3.1.1, S and $S(q)$ are minimal-genus Seifert surfaces in S^3 for $K(0)$ and $K(q)$, respectively.

Let W be the solid torus $\overline{S^3 - V}$. Since S is minimal-genus, each component of $S \cap W$ is incompressible in W and therefore is either a disk or an annulus which is parallel to an annulus in $T := \partial V$. We can isotope S to remove the annular components of $S \cap W$, so we may assume that each component $C \in T \cap S$ bounds a disk $D_C \subset S$ in the complement of V. For each such disk, the intersection $\alpha \cap D_C$ is a collection of properly embedded arcs in D_C. Each of these arcs can be isotoped onto $\partial(D_C) \subset T$ by an isotopy on D_C rel $\partial\alpha$ and then isotoped slightly further into $\text{int}(V)$. By starting with an outermost arc of $\alpha \cap D_C$, we can successively isotope each arc of $\alpha \cap D_C$ in such a way and such that α remains an embedded arc on S throughout the isotopies until $\alpha \subset \text{int}(V)$. Since we may assume that L lies in small neighborhood of α, this process brings L and D into V, as desired. □

The following lemma follows from the proof of Theorem 4.3.1 and discusses the interplay of nugatory generalized crossing changes and twisting operations.

Lemma 4.4.2. *Let K' be a prime knot that is essential in a standardly embedded solid torus V', and let K be a twist knot of K'. Consider the twisting homeomorphism $f : (V', K') \longrightarrow (V, K)$, where $V := f(V')$. Let L' be a crossing circle for K' that lies in V', and let $L := f(L')$ be the corresponding crossing circle for K in V. Suppose that there is a diffeomorphism $h : V' \to V'$ that takes the canonical longitude of V' to itself and such that $h(K'(0)) = K'(q)$. Then, if L' is nugatory for K' in S^3, L is also nugatory for K in S^3.*

Proof. Suppose L' is nugatory. Then L' bounds a crossing disk D' and another disk D'' in the complement of K'. As in the proof of Theorem 4.3.1, we may assume $D' \cap D'' = L'$ and $D' \cup (D'' \cap V')$ contains a properly embedded annulus $(A, \partial A) \subset (V', \partial V')$ such that each component of ∂A is a standard longitude of V'. By the arguments in the proof of Theorem 4.3.1, we may assume $h(A) = A$ and A cuts V' into two solid tori, V_1' and V_2', as shown in Figure 4.3. Then the conclusion of the lemma holds by considering Cases 1 and 2 in the proof of Theorem 4.3.1. □

Before getting to the main theorem of this section, we state a result of Shibuya which will be needed later. See [40] for a proof.

Theorem 4.4.3 (Shibuya, Theorem 3.10 of [21])**.** *Let K be a knot in S^3 and let V be a standardly embedded solid torus containing K such that* $\text{wrap}(K, V) = w(K, V) \geq 3$*. Then $K_{n,V} \cong K_{m,V}$ implies $m = n$.*

We are now ready to prove the following.

Theorem 4.4.4. *Let $K' \in \mathbb{K}$ be contained in a standardly embedded solid torus V' with* $w(K', V') = \text{wrap}(K', V') \geq 3$*. Then, for every $n \in \mathbb{Z}$, the twist knot $K'_{n,V'}$ does not admit a cosmetic generalized crossing change of any order.*

Proof. Suppose that for some $K' \in \mathbb{K}$ there is an embedding of K' into a standardly embedded solid torus V' as in the statement of the theorem, and that for some $n \in \mathbb{Z}$, the twist knot $K := K'_{n,V'}$ admits an order-q cosmetic crossing change corresponding to a crossing circle L. That is, $K(0)$ and $K(q)$ are isotopic in S^3. Let $f : V' \longrightarrow V := f(V')$ denote the twisting homeomorphism bringing K' to K.

By Lemma 4.4.1, we may isotope L into V. Now L pulls back, via f, to a crossing circle L' of K' in V', and the generalized crossing change on K pulls back to a generalized crossing change on K'. Let $K'(q) := K'_{L'}(q)$ denote the result of this crossing change on K'.

Since K is essential in V, by Lemma 4.2.1, $K(q)$ is also essential in V. As in the proof of Theorem 4.3.1, there is an orientation-preserving diffeomorphism $\phi : S^3 \longrightarrow S^3$ that brings $K = K(0)$ to $K(q)$. Since V is an unknotted solid torus in S^3, $\phi|_V$ is given by a meridional twist on V of some order $m \in \mathbb{Z}$. By Theorem 4.4.3, the hypotheses of the theorem imply that $m = 0$. Thus $\phi|_V$ must take canonical longitudes to canonical longitudes, preserving orientation. Hence, we may assume that ϕ fixes V.

Let $h := (f^{-1} \circ \phi \circ f) : V' \to V'$. Then h maps K' to $K'(q)$, and hence K' and $K'(q)$ are isotopic in S^3. So either L' gives an order-q cosmetic generalized crossing change for K', or L' is a nugatory crossing circle for K'. Since $K' \in \mathbb{K}$, L' has to be nugatory. By Lemma 4.4.2 and the fact that h maps the canonical longitude of V' to itself, L is also nugatory for $K = K'_{n,V'}$, which contradicts our assumption that L is cosmetic. □

This gives rise to the following corollary.

Corollary 4.4.5. *Let K be a fibered m-braid with $m \geq 3$. Then for every $3 \leq p \leq m$ and $q \in \mathbb{Z}$, there is no twisted fibered braid $K_{p,q}$ which admits a cosmetic generalized crossing change of any order.*

Proof. Consider a fibered braid K and let $K_{p,q}$ be a twisted fibered braid obtained from K by inserting Δ_p^{2q}, where $3 \leq p \leq m$. Consider a disk D that intersects K only in the strings to be twisted, exactly once for each string. Let $B := D \times [0, 1]$ be a neighborhood of D that meets K in exactly p unknotted arcs corresponding to the p points of $D \cap K$. Let B' be a 3-ball that engulfs the part of K outside B and such that $B \cap B' = D \times \{0, 1\}$. Then $V := B \cup B'$ is a solid torus containing K, and D is a meridional disk of V. (See, for example, Figure 2.6.)

Clearly, $K_{p,q}$ is the result of K under an order-q Dehn twist of V along ∂D. Thus $K_{p,q}$ is a twist knot of K, and the conclusion follows from Theorem 4.4.4. □

Knots that are closures of braids on three strands fit into the setting of Corollary 4.4.5 to give us the following corollary. Recall that the braid group B_3 has generators σ_1 and σ_2 as defined in Figure 2.5, and $\Delta^2 := (\sigma_1\sigma_2)^3$ generates the center of B_3.

Corollary 4.4.6. *Let K be a knot that can be represented as the closure of a 3-braid. Then K admits no cosmetic crossing changes of any order.*

Proof. Suppose that knot $K = \hat{w}$, the closure of some braid $w \in B_3$. By Schreier [38], w is conjugate to a braid in exactly one of the following forms.

1. $\Delta^{2k}\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s}$, where $k \in \mathbb{Z}$ and p_i, q_i, and s are all positive integers

2. $\Delta^{2k}\sigma_1^{p}$ for some $k, p \in \mathbb{Z}$

3. $\Delta^{2k}\sigma_1\sigma_2$ for some $k \in \mathbb{Z}$

4. $\Delta^{2k}\sigma_1\sigma_2\sigma_1$ for some $k \in \mathbb{Z}$

5. $\Delta^{2k}\sigma_1\sigma_2\sigma_1\sigma_2$ for some $k \in \mathbb{Z}$

This form is unique up to cyclic permutation of the word following Δ^{2k}. Since we are concerned with knots only and cases (2) and (4) will produce links for all $k, p \in Z$, we need not consider these cases.

We will call braids of the form (1) *generic*. In this case, we first notice that an alternating braid of the form $\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s}$ is homogeneous and hence its closure is fibered by Stallings [41]. Thus, if the closure of a generic braid is a knot, then it is a twisted fibered braid. In case (3), $\hat{w}$ is also a twisted fibered braid since the closure of $\sigma_1\sigma_2$ is the unknot and hence fibered. Finally, the closure of the braid $\sigma_1\sigma_2\sigma_1\sigma_2$ represents the trefoil, which is also a fibered knot. Thus the closure of a braid of type (5) is also a twisted fibered braid. Since in all cases the twisting occurs on 3 strings, Corollary 4.4.5 applies to give the desired conclusion. □

Remark 4.4.7. A weaker form of Corollary 4.4.6 was derived by Wiley in [46] where he shows that a closed 3-braid *diagram* cannot admit a cosmetic crossing change.

4.5 Another obstruction in satellite knots

In Section 4.3 we saw that any prime satellite knot with pattern a non-satellite in $\mathbb{K}$ does not admit a cosmetic generalized crossing change of order greater than 5. In this section, we restrict ourselves to satellite knots with winding number zero and obtain the following result.

Theorem 4.5.1. *Let C be a prime knot that is not a cable knot and let V' be a standardly embedded solid torus in S^3. Let K' be a non-satellite knot in $\mathbb{K}$ such that K' is geometrically essential in the interior of V' with $w(K', V') = 0$. Then any knot that is a satellite of C with pattern (K', V') admits no cosmetic generalized crossing change of any order.*

Proof. Let (K', V') be as in the statement of the theorem, and consider the satellite map $f : (V', K') \to (V, K)$ with $C = \text{core}(V)$ and $T := \partial V$. Suppose that K admits an order-q cosmetic crossing change, and let D be the corresponding crossing disk with $L = \partial D$. By Lemma 2 of [25] and the assumption that C is not a cable knot, there are no essential annuli in $\overline{S^3 - V}$. Hence, by Lemma 4.2.2, we may assume $D \subset V$, so T is also a companion torus for the satellite link $K \cup L$.

Now $K' \cup L'$ is a pattern link for $K \cup L$ with the satellite map $f : (V', K', L') \to (V, K, L)$ as above. We will show that L' is an order-q cosmetic crossing circle for K', which is a contradiction since $K' \in \mathbb{K}$. From here, the proof is similar to those of Theorem 4.3.1 and Lemma 4.4.2.

Since L is cosmetic, $M := M_{K \cup L}$ is irreducible. Consider a Haken system $\mathcal{T}$ for M with $T \in \mathcal{T}$. Since K' is not a satellite knot, T is innermost in $\mathcal{T}$ with respect to K. As shown in the proof of Theorem 4.3.1, $W := \overline{V - \eta(K \cup L)}$ and $W' := \overline{V' - \eta(K' \cup L')}$ are atoroidal.

If $K(q)$ is not geometrically essential in V, then, by Lemma 4.2.1, $K(0)$ is also not essential in V. But this contradicts V being a companion for K, so $K(q)$ must be essential in V. Hence T is a companion torus for both $K(q)$ and $K(0)$. Since L is cosmetic, there is an ambient isotopy $\psi : S^3 \to S^3$ taking $K(q)$ to $K(0)$ such that V and $\psi(V)$ are both solid tori containing $K(0) = \psi(K(q)) \subset S^3$. By Lemma 4.2.3 and the fact that W is atoroidal, there is another isotopy $\phi : S^3 \to S^3$ fixing $K(0)$ such that $(\phi \circ \psi)(T) \cap T = \emptyset$. Let $\Phi := (\phi \circ \psi) : S^3 \to S^3$. Since T is innermost with respect to K, $\Phi(T) \cap T = \emptyset$ implies T and $\Phi(T)$ are parallel in M_K. So, after an isotopy which fixes $K(0) \subset S^3$, we may assume that $\Phi(V) = V$.

Let $h := (f^{-1} \circ \Phi \circ f) : V' \to V'$. Then h maps $K'(q)$ to $K'(0)$, and hence $K'(q)$ and $K'(0)$ are isotopic in S^3. So either L' gives an order-q cosmetic generalized crossing change for the pattern knot K', or L' is a nugatory crossing circle for K'. Since $K' \in \mathbb{K}$, L' has to be nugatory. By Lemma 4.4.2, L is nugatory for K, which contradicts our assumption that L is cosmetic. □

4.6 Whitehead doubles

Given a knot K recall that $D_+(K, n)$ is the n-twisted Whitehead double of K with a positive clasp and $D_-(K, n)$ is the n-twisted Whitehead double of K with a negative clasp as shown in Figure 2.4. Since non-trivial Whitehead doubles have genus one and the Whitehead double $D_\pm(K, n)$ is a satellite knot for all $K \ncong U$, where U is the unknot, we may apply the results of Chapters 3 and 4 to find obstructions to Whitehead doubles admitting cosmetic generalized crossing changes.

Corollary 4.6.1. *Given a knot K, the Whitehead double $D_+(K, n)$ does not admit a cosmetic generalized crossing change of any order if either $n < 0$ or $|n|$ is odd. Similarly*

$D_-(K,n)$ does not admit a cosmetic generalized crossing change if either $n > 0$ or $|n|$ is odd.

Proof. A Seifert surface of $D_+(K,n)$ is obtained by plumbing an n-twisted annulus with core K and a Hopf band, and gives rise to a Seifert matrix $V_n := \begin{pmatrix} -1 & 0 \\ -1 & n \end{pmatrix}$ as shown in Example 6.8 of [23]. Thus the Alexander polynomial is of the form

$$\Delta_n := \Delta_{D_+(K,n)}(t) \doteq -n(t^2+1) + (1+2n)t \tag{4.1}$$

Suppose that $D_+(K,n)$ admits an order-q cosmetic generalized crossing change. Then Δ_n should be of the form shown in equation (3.7). Comparing the leading coefficients in equations (3.7) and (4.1), we obtain $|n| = |b(b+1)|$, which implies that $|n|$ must be even. Thus we have shown that if $|n|$ is odd then $D_+(K,n)$ admits no cosmetic generalized crossing changes.

Suppose now that $n < 0$. Since the Seifert matrix V_n depends only on n and not on K, the knot $D_+(K,n)$ is S-equivalent to $D_+(U,n)$. This is a positive knot in the sense that all the crossings in the standard diagram of $D_+(U,n)$ are positive. Hence $D_+(K,n)$ has non-zero signature by [34]. Therefore $D_+(K,n)$ is not algebraically slice and, by Theorem 3.2.1, it cannot admit a cosmetic generalized crossing change.

A similar argument holds for $D_-(K,n)$. □

The following result applies only to cosmetic generalized crossing changes of order ± 1.

Corollary 4.6.2. *If K is not a cable knot, then $D_\pm(K,n)$ admits no cosmetic crossing changes for every $n \neq 0$.*

Proof. Suppose that K is not a cable knot. Then by [45], for every $n \neq 0$ the Whitehead doubles $D_{\pm}(K, n)$ have unique Seifert surfaces of minimal genus. By equation (4.1), $\Delta_n \neq 1$, and the conclusion follows by Theorem 3.4.3. □

By adding the assumption that K is prime, we can generalize Corollary 4.6.2 to include cosmetic generalized crossing changes of any order and the Whitehead doubles $D_{\pm}(K, 0)$.

Corollary 4.6.3. *Let K be a prime knot that is not a cable knot. Then no Whitehead double of K admits a cosmetic generalized crossing change of any order.*

Proof. Since all Whitehead doubles admit a pattern (U, V') where U is the unknot and $w(U, V') = 0$, the result follows immediately from Theorem 4.5.1. □

Chapter 5

On the additivity of the Turaev genus

5.1 Constructing the Turaev surface

Besides the definition given in Section 1.2, the Turaev genus of a knot can also be defined in a graph-theoretic way. To do this, consider the all-A smoothing of a projection P. This is the collection of simple closed curves in $\mathbb{R}^2$, which we will refer to as the *state circles* of the all-A smoothing. For each crossing of P, attach an edge to the all-A smoothing which goes from the state circle at one side of the crossing to the state circle at the other side of the crossing. This gives us a trivalent graph P_A. (See Figure 5.1.)

From P_A, we will construct a *ribbon graph* $\mathbb{G}_P$. A ribbon graph is a graph with a cyclic ordering on the adjacent edges at each vertex. To obtain $\mathbb{G}_P$ from P_A, we collapse each of the state circles to a vertex. If a state circle C is contained within an even number of other state circles in P_A, then the cyclic ordering at the vertex of $\mathbb{G}_P$ corresponding to C is the same as the ordering of the edges meeting C in P_A. If C is contained within an odd number of other state circles, then the cyclic ordering at the vertex corresponding to C is the opposite of the ordering of the edges meeting C in P_A.

We can thicken each of the vertices of $\mathbb{G}_P$ and think of them as disks. Likewise we can thicken the edges of $\mathbb{G}_P$ and think of them as untwisted bands or "ribbons". Then $\mathbb{G}_P$ becomes an embedded surface with boundary in S^3. It is known that every boundary component of this surface bounds an embedded disk $D^2 \subset S^3$ [7], so we can cap off each

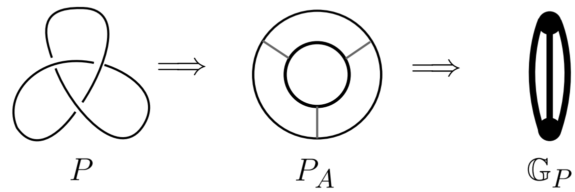


Figure 5.1: The construction of the ribbon graph $\mathbb{G}_P$ from a knot projection P.

boundary component to obtain a closed surface $\Sigma \subset S^3$. This surface is identical to the Turaev surface T_P. Further, $\mathbb{G}_P$ can be embedded in $\Sigma = T_P$, and this embedding can be used to recover the alternating projection $\mathbb{P} \subset T_P$ from Section 1.2. (See [13] for details.)

Note that $\mathbb{G}_P$ being planar, $T_P \cong S^2$, and P being alternating projection are all equivalent statements. It is also obvious from either construction of the Turaev surface that for any two knots K_1 and K_2, $g_T(K_1 \# K_2) \leq g_T(K_1) + g_T(K_2)$.

5.2 Turaev surfaces and decomposing spheres

Let K be a composite knot, let $T \subset S^3$ be a genus-n Turaev surface for K (not necessarily of minimum genus), and let $\mathbb{P}$ be the corresponding alternating projection of K on T. Suppose $A \subset S^3$ is a twice-punctured 2-sphere decomposing K into $K_1 \# K_2$. Following [29], we position K so that K lies on T, except near each crossing of $\mathbb{P}$, where K lies on the boundary of a bubble. (See Figure 5.2.) Let T_+ be T with each disk of T inside a bubble replaced by the upper hemisphere of that bubble. Likewise, we define T_- to be T with each disk of T inside a bubble replaced by the lower hemisphere of that bubble. Here we are choosing an orientation on T, so that upper and lower correspond to the direction of an outward or inward normal vector on T, respectively.

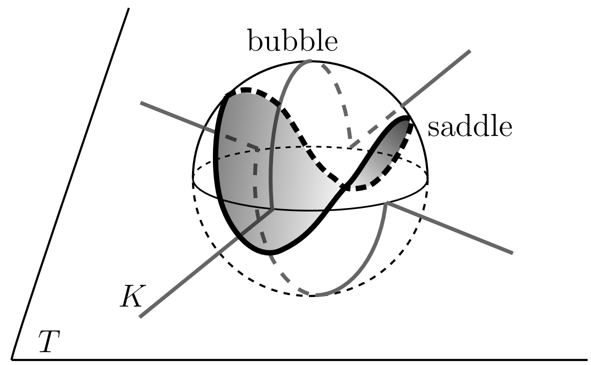


Figure 5.2: A bubble at a crossing of $\mathbb{P}$ containing a saddle-shaped disk where the decomposing sphere A meets the interior of the bubble.

Definition 5.2.1. A decomposing sphere A is *generic* if all of the following hold.

1. A meets T_+ and T_- transversely.

2. The punctures of A do not meet any bubbles.

3. Each time A meets the interior of a bubble, it does so in a saddle-shaped disk as shown in Figure 5.2.

Note that any decomposing sphere can be isotoped so that it is generic with respect to any given Turaev surface, as shown in [29].

Notation 5.2.2. Throughout this chapter, we will use the following notation.

1. M_+ will denote the genus-n handlebody bounded by T_+ lying outside of T and M_- will denote the genus-n handlebody bounded by T_- lying inside of T. So $M_\pm$ can be thought of as the exterior of $T_\pm$.

2. $\mathcal{C}_\pm := A \cap T_\pm$ and $\mathcal{C}'_\pm := A' \cap T_\pm$ for generic decomposing spheres A and A'.

3. $V_\pm := \overline{S^3 - M_\pm}$, so $V_\pm$ is the handlebody inside of $T_\pm$, whereas $M_\pm$ is the handlebody outside of $T_\pm$. Note that all bubbles are contained in both V_+ and V_-.

Let C be a component of $\mathcal{C}_\pm$ for some generic decomposing sphere A, so C is a simple closed curve on $T_\pm$. Choose an arbitrary orientation for C, and let $N := \eta(C) \cap T_\pm$. By traversing C along the chosen orientation, we can say an object in N is to the right or to the left of C. While the concepts of right and left are dependent upon the chosen orientation for C, we can discuss two objects being on the *same side* or *opposite sides* of C without having to specify an orientation.

Proposition 5.2.3 (**Alternating property**). *Let A be a generic decomposing sphere with respect to a Turaev surface T, and let C be a component of $\mathcal{C}_\pm$ such that C meets two bubbles B_1 and B_2 in succession. If the two arcs of $K \cap T_\pm$ in B_1 and B_2 lie on the same side of C, then C must cross K an odd number of times between meeting B_1 and B_2. Similarly, if the two arcs of $K \cap T_\pm$ in B_1 and B_2 lie on the opposite side of C, then C must cross K an even number of times between meeting B_1 and B_2.*

The alternating property was first stated in Section 2 of [29] in the case where T is a sphere. An example of the alternating property is shown in Figure 5.3(A). Note that this proposition is called the *alternating property* since it is only true because the projection $\mathbb{P}$ of K on T is alternating.

Proof of the alternating property. Let $\mathbb{P}$ be the alternating projection of K on T coming from the construction of T, and let G be the underlying four-valent graph. From the construction of T, we see that $\overline{T - \eta(G)}$ consists disks, and each edge of G is adjacent to two distinct disks of $\overline{T - \eta(G)}$. Since $\mathbb{P}$ is an alternating projection, we can orient each of the edges of G

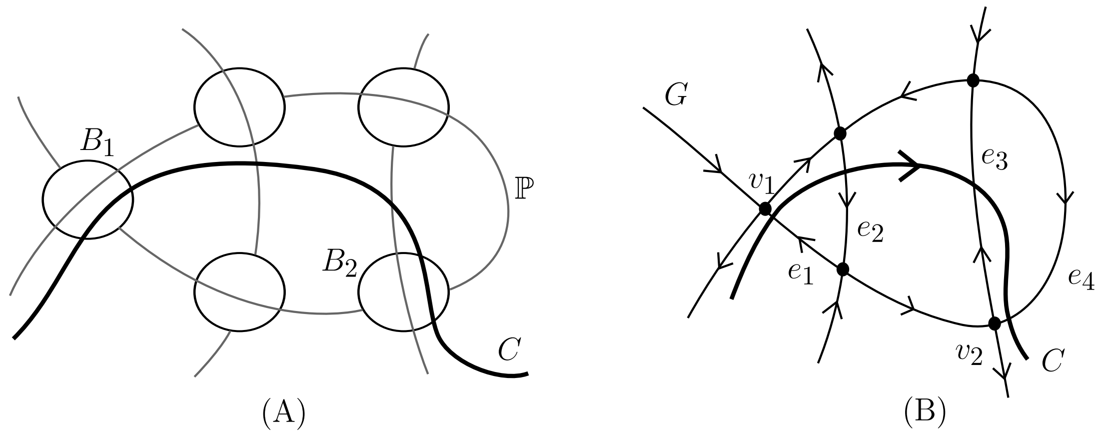


Figure 5.3: (A) A projection $\mathbb{P}$ illustrating the alternating property. (B) The corresponding 4-valent graph G.

from overcrossing to undercrossing, as in Figure 5.3(B). This gives a consistent orientation on the boundary of each disk of $\overline{T - \eta(G)}$.

Suppose C is a component of $\mathcal{C}_\pm$ and, for concreteness, orient C so that the arc of $K \cap T_\pm$ in B_1 is to the left of C. Let v_1 and v_2 be the vertices of G corresponding to B_1 and B_2, respectively. Project C onto T, so C crosses an edge $e_1 \subset G$ adjacent to v_1. Let $e_1, \ldots, e_n$ be the (not necessarily distinct) edges of G met by C between B_1 and B_2 in order, so e_1 is adjacent to v_1 and e_n is adjacent to v_2. Note that e_1 must be oriented towards v_1 and e_n must be oriented towards v_2.

Between two consecutive edges e_i and e_{i+1}, C crosses a single disk of $\overline{T - \eta(G)}$. Hence e_i and e_{i+1} must have opposite algebraic intersection number with C, and each consecutive edge of G met by C crosses C in the opposite direction of the edge before it. Suppose that the arc of $K \cap T_\pm$ in B_2 is also to the left of C. Since e_n is oriented towards v_2, this means both e_1 and e_n cross C from right to left. Hence n must be odd. Likewise, if the arc of $K \cap T_\pm$ in B_2 is to the right of C, then e_1 and crosses C from right to left and e_n crosses C from left to right and hence n is even. $\square$

5.3 Decomposing spheres in standard position

For each component $C \subset \mathcal{C}_\pm$, let $w_\pm(C)$ be a cyclic word in P (puncture) and S (saddle) which records, in order, the intersections of C with K and with bubbles, respectively. Note that $w_\pm(C)$ depends on a choice of orientation for C, but this will not matter for us. By the proof of the alternating property, $w_\pm(C)$ has even length for all $C \subset \mathcal{C}_\pm$. With this notation, we have the following lemma of Hayashi.

Lemma 5.3.1 (Hayashi, Lemma 2.1(i, ii) of [16]). *Let T be a Turaev surface for a composite knot K and let $M := M_+ \sqcup M_-$. Then any generic decomposing sphere A can be isotoped so that each of the following holds.*

1. *Every component $C \subset \mathcal{C}_\pm$ satisfies $w_\pm(C) \neq \emptyset$.*

2. *Each component of $A \cap M$ is incompressible in M.*

Proof. By construction of T, the corresponding projection $\mathbb{P}$ is such that $\overline{T - \eta(\mathbb{P})}$ consists of disks. Hence, if there exists some $C \subset \mathcal{C}_\pm$ with $w_\pm(C) = \emptyset$, C must bound a disk $D_1 \subset T$. Let C be an innermost such curve on T, so $D_1 \cap A = C$. Since A meets K exactly twice, C also bounds a disk $D_2 \subset A - \eta(K)$, and $D_1 \cup D_2$ bounds a ball in $S^3 - \eta(K)$. We can then isotope D_2 through this ball to D_1 and then off of T, removing C from $\mathcal{C}_\pm$.

Similarly, suppose R is a compressible component of $A \cap M$ and let $\Delta_1 \subset M$ be a compression disk for R. Since R is connected, R and Δ_1 are both contained in the same component of M, say M_+. Hence $\Delta_1 \cap K = \emptyset$ and $\partial \Delta_1$ also bounds a disk $\Delta_2 \subset A - \eta(K)$. Again, $\Delta_1 \cup \Delta_2$ bounds a ball in $S^3 - \eta(K)$, so we can isotope Δ_2 through this ball to Δ_1, replacing R with a disk, which is incompressible. This isotopy may eliminate some other components of $A \cap M$, but it will not introduce any new components, so we may iterate until all components of $A \cap M$ are incompressible in M. $\square$

Lemma 5.3.1 gives us the following corollary.

Corollary 5.3.2. *Let A be a generic decomposing sphere with respect to a Turaev surface T. Then A can be isotoped so that every component of $\mathcal{C}_\pm$ which bounds a disk on $T_\pm$ also bounds a disk in $A \cap M_\pm$ which is a peripheral disk in $M_\pm$.*

Proof. Let $C \subset \mathcal{C}_\pm$ be an innermost curve bounding a disk $D \subset T_\pm$, so $D \cap A = C$. By isotoping D slightly into $M_\pm$ we get a disk $D' \subset M_\pm$ with $D' \cap A = \partial D'$. By Lemma 5.3.1, D' cannot be a compression disk for the component of $A \cap M_\pm$ containing $\partial D'$. Hence this component of $A \cap M_\pm$ must be a disk peripheral to $T_\pm$.

Since the corollary now holds when $\text{int}(D) \cap A = \emptyset$, we can complete the proof by inducting on $|\text{int}(D) \cap A|$. Let $C_n \subset \mathcal{C}_\pm$ bound a disk $D_n \subset T_\pm$ such that $|\text{int}(D_n) \cap A| = n$. For each curve $C \subset \text{int}(D_n) \cap A$ we may assume by induction that C bounds a peripheral disk in $A \cap M_\pm$ which we will call an *interior disk*. Isotope these interior disks (rel boundary) so that they lie just above $T_\pm$ but are all still disjoint from one another. Push D_n slightly into $M_\pm$, as before, until we get a disk $D'_n \subset M_\pm$ that is disjoint from each of the interior disks, with $D'_n \cap A = \partial D'_n$. As in the base case, D'_n cannot be a compression disk for the component of $A \cap M_\pm$ containing $\partial D'_n$, so this component of $A \cap M_\pm$ must also be a peripheral disk. □

The following lemma is similar to Lemma 1 of [29] and Lemma 2.1(iii) of [16].

Lemma 5.3.3. *Let A be a generic decomposing sphere with respect to a Turaev surface T. Then A can be replaced by a (non-trivial) generic decomposing sphere A' so that if some component $C \subset \mathcal{C}'_\pm$ meets the same bubble B at two different arcs γ_1 and γ_2, then for each component $\alpha \subset C - (\gamma_1 \cup \gamma_2)$ there is no disk $D \subset T_\pm$ such that ∂D consists of α and an arc in $\alpha' \subset (B \cap T)$.*

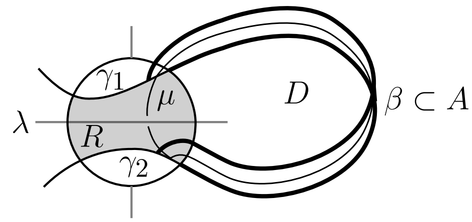


Figure 5.4: The rectangle R and the surgery curve μ from the proof of Lemma 5.3.3.

Proof. Suppose such a $D \subset T_+$ exists, noting that an identical proof holds for T_-. Let D be an innermost such disk on T_+. Let $H := B \cap T_+$ and $\lambda := K \cap H$. Then there is a rectangle $R \subset H$ whose boundary consists of γ_1, γ_2 and two arcs in ∂H. (See Figure 5.4.) Since D is innermost, $A \cap R \subset \partial R$. We now consider two cases, depending on whether or not $\lambda \subset R$.

Case 1: $\lambda \subset R$.

Since A does not meet the interior of R, γ_1 and γ_2 must belong to the same saddle σ of $A \cap B$. There is a band $\beta := \eta(\alpha) \cap M_+ \cap A$ connecting γ_1 to γ_2. By Corollary 5.3.2, we may assume that any component of $\mathcal{C}_+$ contained in D bounds a disk in $A \cap M_+$ which is boundary-parallel in M_+. Hence there is a simple closed curve $\mu \subset \beta \cup \sigma$ such that μ is isotopic in $S^3 - (A \cup \eta(K))$ to a meridian of $\eta(K)$. Hence, μ bounds a disk in $S^3 - A$ that meets K exactly once. Surgering A along this disk splits A into two spheres, at least one of which still decomposes K nontrivially. Let A' be this decomposing sphere. By isotoping A' slightly so that the new puncture no longer meets B, A' is generic we have removed γ_1 and γ_2 from $\mathcal{C}_+$.

Case 2: $\lambda \not\subset R$

There is still a band $\beta := \eta(\alpha) \cap M_+ \cap A$ connecting γ_1 to γ_2, but this time the γ_i each belong to different saddles σ_1 and σ_2 in B. Since D is innermost, σ_1 and σ_2 are adjacent in B. Hence there is a simple closed curve $\mu \subset \beta \cup R$ such that μ bounds a disk $\Delta' \subset M_+ - A$.

Thicken this disk slightly to $\Delta' \times [0,1]$ with $(\partial\Delta') \times [0,1]$ still contained in $\beta \cup R$, and let Δ'' denote this thickened disk. Then Δ'' together with the region inside B bounded by σ_1 and σ_2 is isotopic to a 3-ball in $S^3 - (A \cup \eta(K))$. We can isotope A through this 3-ball to eliminate σ_1 and σ_2. □

Note that while the sphere A' from Lemma 5.3.3 gives a nontrivial decomposition of K, it may no longer be the original decomposition given by A.

Definition 5.3.4. If A' is a generic decomposing sphere satisfying the conditions in Lemmas 5.3.1 and 5.3.3 and Corollary 5.3.2, we will say that A' is in *standard position* with respect to a Turaev surface T.

We now have the following lemma, which is similar in spirit to Lemma 2.4 of [16].

Lemma 5.3.5. *Let A' be a decomposing sphere in standard position with respect to a Turaev surface T. If $\mathcal{C}'_\pm$ contains a curve C which bounds a disk in $T_\pm$, then $w_\pm(C) = P^2$ or $PSPS$.*

Proof. Let C be a component of, say, $\mathcal{C}'_+$ which bounds a disk in T_+. (Again, an identical argument holds for T_-.) Choose C such that C is an innermost such curve on T_+, so C bounds a disk $D \subset T_+ - A'$. We know $w_+(C) \neq \emptyset$ by Lemma 5.3.1, and $w_+(C) \neq P$ by the construction of T. So either $w_+(C) = P^2$ or C meets some bubble B. Suppose the latter, and let $H := B \cap T_+$ and $\lambda := K \cap H$. If λ were contained in D, then C would have to meet B on both sides of λ and would hence violate Lemma 5.3.3. Hence D does not contain the overcrossing of any bubble met by C and, by the alternating property, C cannot meet two bubbles consecutively without first crossing a puncture. Thus $w_+(C) = PS$ or $PSPS$. If $w_+(C) = PS$, by passing to T_- we see that Lemma 5.3.3 is violated, as ilustrated in Figure 5.5. □

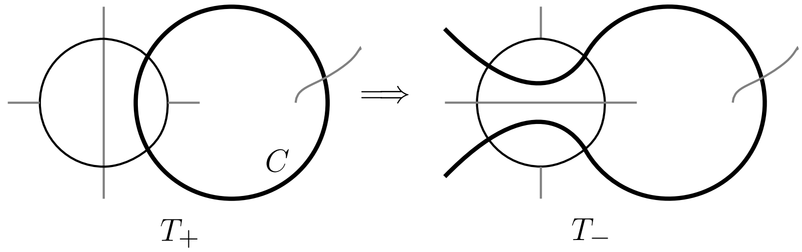


Figure 5.5: If $C \subset \mathcal{C}'_+$ bounds a disk on T_+ and has $w_+(C) = PS$, then C must contribute a curve to $\mathcal{C}'_-$, which violates Lemma 5.3.3.

5.4 Composite knots with Turaev genus one

Throughout this section we restrict our attention to the case where the Turaev surface is a torus and K is a composite knot with $g_T(K) = 1$. To start, fix a genus-one Turaev surface T and the corresponding alternating projection $\mathbb{P}$ of K on T. Since any generic decomposing sphere gives rise to a decomposing sphere which is in standard position with respect to T, let A' be such a sphere in standard position such that A' gives the decomposition $K = K_a \# K_b$.

Since we have chosen an orientation on $T \subset S^3$, T has a well-defined meridian and longitude. Namely, a *meridian* of T is a homologically nontrivial simple closed curve which bounds a disk in the solid torus inside of T and a *longitude* of T is such a curve which bounds a disk in the solid torus in the exterior of T. Define a *longitude* of $T_\pm$ to be a generator of $H_1(V_\pm)$, as defined in Notation 5.2.2. So a longitude of T_+ projects to a longitude of T and a longitude of T_- projects to a meridian of T. This also gives a well-defined *meridian* on $T_\pm$.

If there is a component $C \subset \mathcal{C}'_-$ which is separating on T_-, then Lemma 5.3.5 tells us that this curve is the unique such separating curve among the components of $\mathcal{C}'_-$ and $w_-(C) = P^2$ or $PSPS$. If $w_-(C) = P^2$, then C is also a component of $\mathcal{C}'_+$ and A' decomposes K along C into an alternating summand and a summand which has T as a Turaev surface. Hence, since $g_T(K) = 1$ by assumption, $g_T(K_a) + g_T(K_b) = 1$. If $w_-(C) = PSPS$, then $\mathcal{C}'_+$ contains no curves which are separating on T_+. Then $\mathcal{C}'_+$ consists of a collection of parallel, non-separating curves on T_+, and these curves are either meridians or longitudes of T_+.

Lemma 5.4.1. *Let A' be a decomposing sphere for K in standard position with respect to the genus-one Turaev surface T and suppose that $\mathcal{C}'_+$ contains no curves which are separating on T_+. Then $\mathcal{C}'_+$ must consist of longitudes of T_+.*

Proof. By way of contradiction, suppose $\mathcal{C}'_+$ consists of meridians of T_+. Since A' is a sphere, it cannot meet T_+ in a single meridian, so $|\mathcal{C}'_+| \geq 2$. Let C be a component of $\mathcal{C}'_+$ that is innermost in A', so C bounds a disk $D \subset A' - \mathcal{C}'_+$. Since C is a meridian of T_+, D must be a component of $A' \cap V_+$. There are at least two such innermost curves on A', so we may choose C such that C meets a bubble B. Let σ be a saddle of B met by C. Then $\sigma \subset D$ and C meets $H := B \cap T_+$ on both sides of σ. Let γ_1 and γ_2 be the arcs of $C \cap \sigma$ and let α be an arc in σ connecting γ_1 and γ_2.

The arc α cuts D into 2 disks, D_1 and D_2. Since $C = \partial D$ is a meridian of T_+, either ∂D_1 or ∂D_2 must bound a disk in T_+ (after isotoping α through B to H). But then this disk violates Lemma 5.3.3, so we have our desired contradiction. □

In light of Lemma 5.4.1, we have the following definition.

Definition 5.4.2. A decomposing sphere A' is a *preferred decomposing sphere* with respect to a Turaev surface T if A' is in standard position and $\mathcal{C}'_+$ consists of longitudes of T_+.

Note that if A' is a preferred decomposing sphere and $\mathcal{C}'_-$ contains a curve C which is separating on T, then $w_-(C) = PSPS$. Further, if $\mathcal{C}'_-$ contains any curves which are not separating on T_-, then the proof of Lemma 5.4.1 shows that these curves must be longitudes of T_-. This gives us the following corollary.

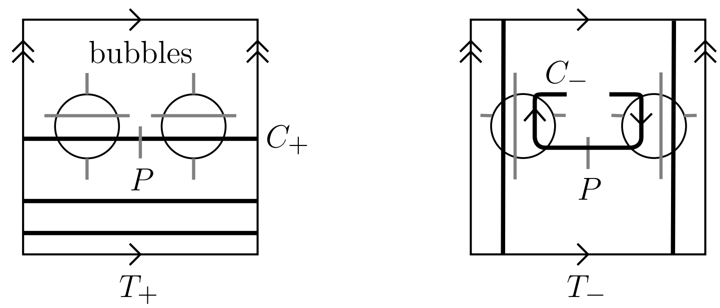


Figure 5.6: Square depictions of T_+ and T_- as seen from M_+. The thicker lines are components of $\mathcal{C}'_\pm$ and the cirlces are bubbles on $T_\pm$ with over- and undercrossings shown.

Corollary 5.4.3. *Let A' be a preferred decomposing sphere for K with respect to a genus-one Turaev surface T. Then each component of $A' \cap M_\pm$ is a disk.*

Proof. This follows from Lemmas 5.3.1 and 5.4.1 and Corollary 5.3.2. □

We now consider T_+ as a square with sides identified in the usual way to depict a torus as a moduli space so that the longitudes of $\mathcal{C}'_+$ appear as horizontal lines on T_+. (See Figure 5.6.) The curves of $\mathcal{C}'_-$ consist of a single separating curve and/or multiple longitudes on T_-. If we are viewing both T_+ and T_- from outside T (i.e. from M_+), then when we pass from the square depiction of T_+ to T_-, any longitudes of T_- in $\mathcal{C}'_-$ appear as (lines isotopic to) *vertical* lines. We will use this depiction to prove the following.

Lemma 5.4.4. *Let A' be a preferred decomposing sphere for K with respect to the genus-one Turaev surface T. Then there is a component $C_- \subset \mathcal{C}'_-$ which meets K twice. Further, if C_- is not separating on T_-, then $P^2 \subset w_-(C_-)$.*

Proof. Consider a component $C_+ \subset \mathcal{C}'_+$ which meets a puncture P and let C_- be the corresponding component of $\mathcal{C}'_-$ which meets the same puncture. If $w_+(C_+) = P^2S^i$ for some $i \geq 0$, then clearly $w_-(C_-)$ also contains P^2.

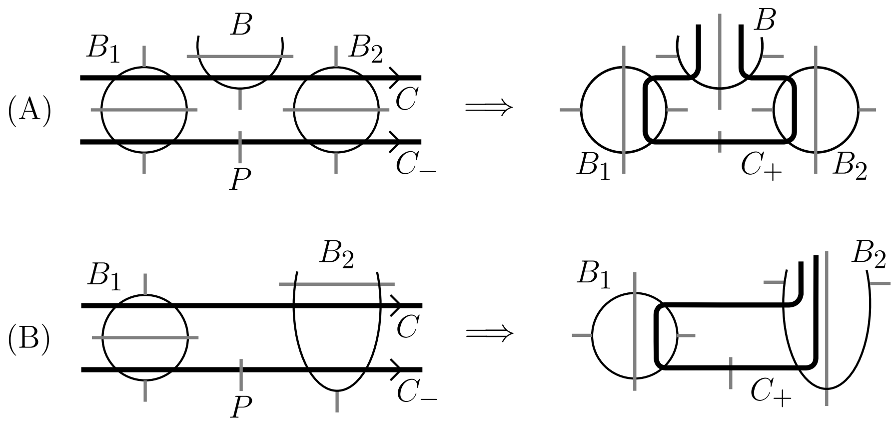


Figure 5.7: The curves C_-, C and C_+ from Cases 1 and 2 of the proof of Corollary 5.4.4.

Suppose $w_+(C_+) \neq P^2S^i$ and fix a square depiction of T_+. Since C_- will cross P horizontally on the induced square depiction of T_-, we may orient C_- so it is travelling upward just to the left of P. (Again, see Figure 5.6.) Applying the alternating property to C_+, we see that C_- must then be traveling downward just to the right of P. Since C_- is a simple closed curve on T_-, C_- must change direction again, from upward to downward or downward to upward. By the alternating property once more, this can only happen if C_- meets the other puncture of A'.

Since $P^2 \not\subset w_+(C_+)$, $w_-(C_-)$ contains the sequence SPS. Let B_1 and B_2 be the bubbles corresponding to the two occurrences of S in this sequence and let $\lambda_i := K \cap B_i \cap T_-$ for $i = 1, 2$. By the alternating property, λ_1 and λ_2 both lie on the same side of C_-. For concreteness, assume that the λ_i are both to the left of C_-, and C_- meets B_1 just before meeting B_2. (See Figure 5.7.) Let C be the component of $\mathcal{C}'_-$ immediately to the left of C_-, so C also meets both B_1 and B_2, and orient C in the same direction as C_-. We must now consider three cases, depending on which side of C λ_1 and λ_2 lie on, hoping to draw a contradiction in each case.

Case 1: Both λ_1 and λ_2 lie to the right of C, as shown in Figure 5.7(A).

By the alternating property and the fact that C_- meets K twice, C does not meet K and there must be some bubble B which meets C after B_1 and before B_2, with the overcrossing of B (with respect to T_-) to the left of C. Any bubble which meets C with its overcrossing to the right of C must also meet C_-, so B is the only bubble met by C after B_1 and before B_2. But then C_+ must meet B twice in a manner which contradicts Lemma 5.3.3.

Case 2: The λ_i are on opposite sides of C, as shown in Figure 5.7(B).

For concreteness, suppose λ_1 is to the right of C and λ_2 is to the left, noting that an identical argument holds if the roles are reversed. Since C meets no punctures, C must meet an even number of bubbles between B_1 and B_2. As before, any bubble which meets C with its overcrossing to the right of C also meets C_-. Hence C does not meet any other bubbles as it passes from B_1 to B_2. But then C_+ again contradicts Lemma 5.3.3.

Case 3: Both λ_1 and λ_2 lie to the left of C.

In this case, C must meet a bubble B after B_1 but before B_2 with the overstrand of B to the right of C. But then B also meets C_- after B_1 and before B_2, which cannot happen. □

For the next lemma, we will need the following definition.

Definition 5.4.5. Given a knot K with Turaev surface T and decomposing sphere A' which is in standard position with respect to T, let

$$a(A', T) := \min\{|w_\pm(C)| \mid C \text{ is a component of } \mathcal{C}'_\pm \text{ which is not separating on } T_\pm\}$$

Note that $a(A', T)$ is always a positive, even integer. With this in mind we have the following lemma, which is adapted from Theorem 1.3 of [16].

Lemma 5.4.6. *Let K be a composite knot with a genus-one Turaev surface T and let A' be* *a preferred decomposing sphere for K with respect to T. Then $a(A', T) = 2$.*

Proof. Fix a preferred decomposing sphere A' for K and let $a := a(A', T)$. We will use an Euler characteristic argument to show that $a = 2$.

Let $l := |\mathcal{C}'_+|$, $m := |\mathcal{C}'_-|$ and s be the number of saddles where A' meets a bubble of T. Let $M := M_+ \sqcup M_-$. By Corollary 5.4.3, $A' \cap M$ consists of disks. So A' is completely comprised of disks corresponding to the components of $A' \cap M_\pm$ and the saddles of A'. Hence A' has a cellular decomposition consisting of $4s$ 0-cells, $6s$ 1-cells and $(s+m+l)$ 2-cells, and $\chi(A') = m + l - s$.

First suppose that $\mathcal{C}'_-$ contains a curve C_- which is separating on T_-. Since A' is preferred, $w_-(C_-) = PSPS$. Since $C_- \subset \mathcal{C}'_-$, s is bounded below by both

$$s \geq \frac{a(m-1)+2}{2} \quad \text{and} \quad s \geq \frac{al-2}{2}. \tag{5.1}$$

This gives

$$-s \leq -\left(\frac{am}{2} - \frac{a-2}{2}\right) \quad \text{and} \quad l \leq \frac{2s}{a} + \frac{2}{a}. \tag{5.2}$$

Hence we have

$$\begin{aligned}
2 = \chi(A') &= m - s + l \\
&\leq m - s + \frac{2s}{a} + \frac{2}{a} && \text{by (5.2)} \\
&= m - s\left(1 - \frac{2}{a}\right) + \frac{2}{a} \;\leq\; m - \left(\frac{am}{2} - \frac{a-2}{2}\right)\left(1 - \frac{2}{a}\right) + \frac{2}{a} && \text{by (5.2)} \\
&= m\left(1 - \frac{a}{2}\left(1 - \frac{2}{a}\right)\right) + \frac{a}{2} - 2 + \frac{4}{a} = -m\left(\frac{a}{2} - 2\right) + \frac{a}{2} - 2 + \frac{4}{a} \\
&\leq -\left(\frac{a}{2} - 2\right) + \frac{a}{2} - 2 + \frac{4}{a} && \text{since } -m \leq -1 \\
&= \frac{4}{a}
\end{aligned}$$

So $a \leq 2$. Since a must be positive and even, $a = 2$ as desired.

If there is no component of $\mathcal{C}'_-$ which is separating on T_-, then m is even and at least 2. In this case, the inequality (5.1) becomes

$$s \geq \frac{am-2}{2} \quad \text{and} \quad s \geq \frac{al-2}{2}. \tag{5.3}$$

This gives

$$-s \leq -\left(\frac{am}{2} - 1\right) \quad \text{and} \quad l \leq \frac{2s}{a} + \frac{2}{a} \tag{5.4}$$

Hence we have

$$\begin{aligned}
2 = \chi(A') &= m - s + l \\
&\leq m - s + \frac{2s}{a} + \frac{2}{a} && \text{by (5.4)} \\
&= m - s\left(\frac{a-2}{a}\right) + \frac{2}{a} \;\leq\; m - \left(\frac{am}{2} - 1\right)\left(\frac{a-2}{a}\right) + \frac{2}{a} && \text{by (5.4)} \\
&= m\left(1 - \frac{a}{2}\left(\frac{a-2}{a}\right)\right) + \frac{a-2}{a} + \frac{2}{a} \\
&= -m\left(\frac{a}{2} - 2\right) + 1 \leq -2\left(\frac{a}{2} - 2\right) + 1 && \text{since } -m \leq -2 \\
&= -a + 5 && (5.5)
\end{aligned}$$

So $a \leq 3$ and again, since a is positive and even, we conclude $a = 2$. □

This gives us the following corollary.

Corollary 5.4.7. *Let K be a composite knot with a genus-one Turaev surface T and let A' be a preferred decomposing sphere for K with respect to T. If $\mathcal{C}'_-$ has no component which is separating on T_-, then there is some curve $C \subset \mathcal{C}'_\pm$ with $w_\pm(C) = S^2$.*

Proof. By Lemma 5.4.6, there is a $C \subset \mathcal{C}'_{\pm}$ such that $|w_{\pm}(C)| = 2$. First suppose $C \subset \mathcal{C}'_{-}$. If C meets K, then by Lemma 5.4.4, $P^2 \subset w_{-}(C)$ and hence $w_{-}(C) = P^2$. But then C is also a component of $\mathcal{C}_{+}$ and hence must bound a disk on T, which is a contradiction. Hence $w_{-}(C) = S^2$.

If $C \subset \mathcal{C}'_{+}$, by the above argument $w_{+}(C) \neq P^2$. Suppose $w_{+}(C) = PS$. and let C_{-} be the component of $\mathcal{C}'_{-}$ which meets the same puncture as C. Then $P^2 \not\subset w_{-}(C_{-})$ and, again by Lemma 5.4.4, C_{-} must bound a disk on T_{-}, which contradicts our initial assumptions. Hence $w_{+}(C) = S^2$, as desired. □

5.5 Turaev genus one and additivity

The goal of this section is to study the following question.

Problem 5.5.1. If $K = K_1 \# K_2$ and $g_T(K) = 1$, is it true that $g_T(K_1) + g_T(K_2) = 1$?

Let $K = K_1 \# K_2$. By Proposition 2.3.1, $g_T(K) = 0$ if and only if $g_T(K_1) = g_T(K_2) = 0$. This and the fact that the Turaev genus is sub-additive immediately gives us the converse of Problem 5.5.1. That is, if $g_T(K_1) + g_T(K_2) = 1$, then $g_T(K) = 1$. Hence, a positive answer to Problem 5.5.1 would give us additivity of the Turaev genus in the sense that $g_T(K_1 \# K_2) = g_T(K_1) + g_T(K_2)$ for composite knots of Turaev genus one.

The results of this chapter will primarily be stated in terms of preferred decomposing spheres. The following proposition justifies this choice.

Proposition 5.5.2. *Let K be a composite knot with a genus-one Turaev surface T. Assume that for any decomposing sphere A' which is preferred with respect to T, if A' gives the decomposition $K = K_a \# K_b$, then $g_T(K_a) + g_T(K_b) = 1$. Then $g_T(K) = 1$ and for any decomposition $K = K_1 \# K_2$, $g_T(K_1) + g_T(K_2) = 1$.*

Proof. The fact the $g_T(K) = 1$ follows immediately from the preceding discussion. Let A be a decomposing sphere for K which gives the decomposition $K = K_1 \# K_2$. We may assume that A is generic with respect to T. Surger A as in results of Section 5.3 to obtain a decomposing sphere A' which is in standard position with respect to T. Let $K = K_a \# K_b$ be the decomposition given by A'. If $\mathcal{C}'_\pm$ contains a curve C which is separating on $T_\pm$ with $w_\pm(C) = P^2$, then clearly $g_T(K_a) + g_T(K_b) = 1$. Otherwise, by choice of orientation, we may assume that A' is preferred, and $g_T(K_a) + g_T(K_b) = 1$ by assumption.

To show that $g_T(K_1) + g_T(K_2) = 1$, we induct on the number of summands n in the prime decomposition of K. If $n = 2$, then A' must give the same decomposition as A and there is nothing further to show.

For $n > 2$, if A' does not decompose K into $K_1 \# K_2$, then A' was obtained from A by surgering A (perhaps repeatedly) into two spheres, one of which was A'. Hence A' contains a summand of either K_1 or K_2. For concreteness, say K_a is a summand of K_1. So $K_b = K_{1b} \# K_{2b}$, where $K_1 = K_a \# K_{1b}$ and $K_2 = K_{2b}$. If $g_T(K_a) = 1$ and $g_T(K_b) = 0$, then $g_T(K_{1b}) = g_T(K_{2b}) = 0$ and hence $g_T(K_1) = 1$ and $g_T(K_2) = 0$. If $g_T(K_a) = 0$ and $g_T(K_b) = 1$, then, by induction, $g_T(K_{1b}) + g_T(K_{2b}) = 1$. If $g_T(K_{1b}) = 1$, then $g_T(K_1) = 1$ and $g_T(K_2) = 0$. Otherwise, $g_T(K_{2b}) = g_T(K_2) = 1$ and $g_T(K_1) = 0$. □

Throughout the rest of this section, K will denote a composite knot of Turaev genus one, and A' will denote a preferred decomposing sphere for K with respect to some genus-one Turaev surface T. We will first study the cases when $|\mathcal{C}'_-| \leq 3$. To state our first result, we need the following definitions.

Definition 5.5.3. The *alternating genus* of a knot K, denoted $g_{\text{alt}}(K)$, is the minimum genus of an unknotted surface on which K has an alternating projection. The *disk-alternating*

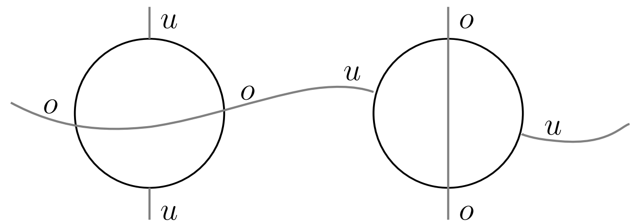


Figure 5.8: Overstrands and understrands of K on $T_\pm$, denoted by o and u, respectively.

genus of K, denoted $g_{DA}(K)$, is the minimum genus of an unknotted surface Σ on which K has an alternating projection $\mathbb{P}$ such that $\overline{\Sigma - \eta(\mathbb{P})}$ consists entirely of disks.

It is clear from this definition that $g_{\text{alt}}(K) \leq g_{DA}(K) \leq g_T(K)$ for any knot K. Also, K is alternating if and only if $g_{\text{alt}}(K) = g_{DA}(K) = g_T(K) = 0$.

Definition 5.5.4. Each segment of K between two crossings of the projection $\mathbb{P}$ on a Turaev surface T consists of two strands: one *overstrand* and one *understrand*, as illustrated in Figure 5.8 Hence each bubble of T corresponds to exactly two overstrands and two understrands of K on $T_\pm \cap T$.

Theorem 5.5.5. *Let K be a composite knot with $g_T(K) = 1$ and let A' be a preferred decomposing sphere for K with respect to the genus-one Turaev surface T such that A' decomposes K into $K_a \# K_b$. If $|\mathcal{C}'_-| = 1$, then, up to relabelling of the summands, K_a is alternating and $g_{alt}(K_b) = g_D(K) = 1$.*

Proof. Since $g_T(K) = 1$, K_a and K_b cannot both be alternating. Since $\mathcal{C}'_-$ consists of a single curve C_-, $F := A' \cap V_-$ must be a disk and $C_- = \partial F$ is a separating curve on T_-. Since A' is preferred, $w_-(C_-) = PSPS$. Since $\mathcal{C}'_- = C_-$, the two arcs of saddles met by C_- must actually belong to the same saddle, and hence C_- meets exactly one bubble B. Further, by applying Lemma 5.3.3 and the fact that $\mathcal{C}'_+$ consists of longitudes, we see that

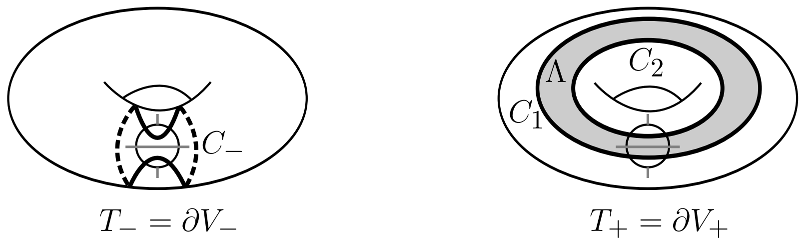


Figure 5.9: The components of $\mathcal{C}'_-$ and $\mathcal{C}'_+$ from Theorem 5.5.5.

each component of $C_- - B$ can be made into a meridian of T_- using an arc contained in $B \cap T$, as illustrated in Figure 5.9.

Hence $\mathcal{C}'_+$ consists of exactly two components, C_1 and C_2, with $w_+(C_1) = w_+(C_2) = SP$. Let $\Lambda \subset T_+$ be the annulus bounded by $C_1 \cup C_2$ with the overstrand of $H_+ := B \cap T_+$ contained in Λ. Since A' decomposes K into $K_a \# K_b$, one of these summands, say K_a, is contained in Λ. The other summand, K_b, is contained in $\Lambda_b := \overline{T_+ - \Lambda}$, other than at the undercrossing of K at B, which is part of K_b but passes under Λ. (See Figure 5.10.)

The two overstrands of B are contained in Λ and the two understrands of B are in Λ_b. Hence at each of the punctures where K meets $\partial\Lambda$, if we traverse K from Λ to Λ_b, we must be going from an overcrossing to an undercrossing, so we may decompose K into $K_a \# K_b$ as in Figure 5.10.

This gives an alternating projection of K_a, as desired. We also have a projection of K_b on a torus which is alternating, so $g_{\text{alt}}(K_b) = 1$. It remains to show that the complement of K_b on this torus consists of disks. To do this, recall that in the construction of the original Turaev surface T we glued disks to a thickening of a diagram of K in $\mathbb{R}^2$, and these disks came from the all-A and all-B smoothings of the diagram. So $\overline{T - \eta(\mathbb{P})}$ consists of disks which we will call A-disks and B-disks, depending on which smoothing type they originally came from. Project C_1 and C_2 onto T. Then each of the C_i pass through exactly one A-disk

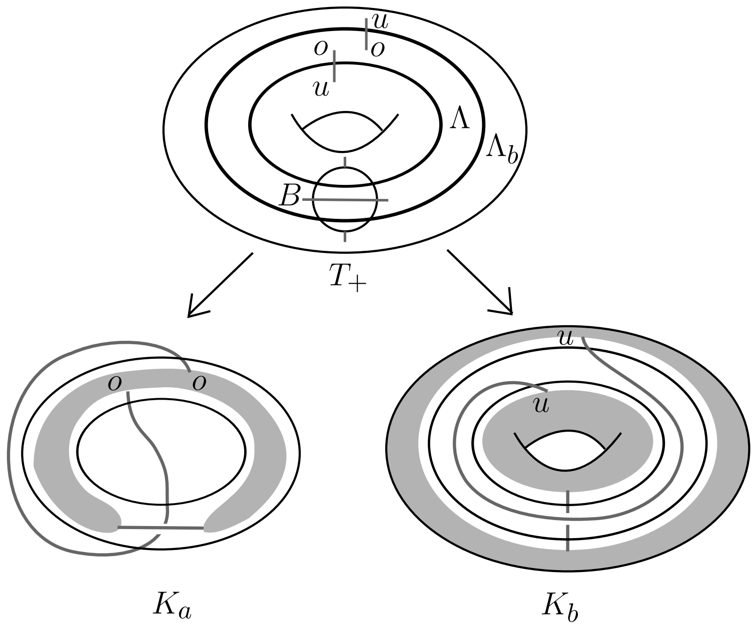


Figure 5.10: The decomposition of $K = K_a \# K_b$ from Theorem 5.5.5. The shaded regions contain alternating projections of subtangles of K_a and K_b.

and one B-disk. Denote these disks by $\mathcal{A}_i$ and $\mathcal{B}_i$, respectively. (See Figure 5.11.) When we remove K_a from T, we have the effect of removing from T all of the A- and B-disks which meet Λ, other than the $\mathcal{A}_i$ and $\mathcal{B}_i$. We then merge $\mathcal{B}_1$ and $\mathcal{B}_2$ into one region of $T - \eta(K_b)$ which spans $\Lambda - (\mathcal{A}_1 \cup \mathcal{A}_2)$. Denote this region by $\mathcal{B}$. Now $\mathcal{B}$ is a disk unless $\mathcal{B}_1 = \mathcal{B}_2$, which cannot happen since, by construction, $\mathbb{P}$ has no nugatory crossings on T. Since nothing in Λ_b was changed when K_a was removed, $\overline{T - \eta(K_b)}$ consists of disks, as desired. □

Theorem 5.5.6. *Let K be a composite knot with $g_T(K) = 1$ and let A' be a preferred decomposing sphere for K with respect to the genus-one Turaev surface T such that A' decomposes K into $K_a \# K_b$. Then $|\mathcal{C}'_-| \neq 2$.*

Proof. Suppose the $|\mathcal{C}'_-| = 2$ and let C_1 and C_2 be the two components of $\mathcal{C}'_-$. Then C_1 and C_2 are longitudes of T_-. Without loss of generality, $P^2 \subset w_-(C_1)$ by Lemma 5.4.4. Note

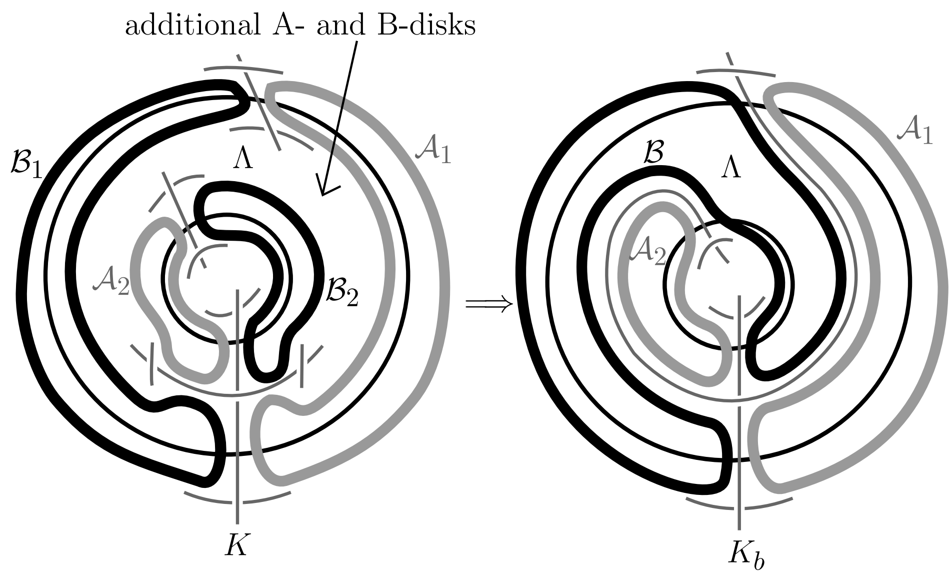


Figure 5.11: The A- and B-disks of T locally at the annulus Λ before and after the decomposition of K in Theorem 5.5.5.

that $w_{-}(C_2) \neq \emptyset$, and every saddle met by C_2 also meets C_1.

Now A' cuts T_{-} into exactly two annuli and, as before, K_a and K_b are each contained in one of these annuli except at the undercrossings corresponding to bubbles met by $\mathcal{C}'_{-}$. Let Λ_a be the annulus containing K_a away from these undercrossings and likewise define Λ_b with respect to K_b. We will argue that both K_a and K_b are, in fact, alternating knots, in which case we have the contradiction that K did not have Turaev genus one to begin with.

Focusing first on Λ_a, note that, by the alternating property, each component of $\mathcal{C}'_{-}$ meets an even number of bubbles, and every other bubble which meets $\mathcal{C}'_{-}$ has its overcrossing contained in Λ_a. (See Figure 5.12(A).) We will refer to these bubbles with their overcrossings in Λ_a as *a-bubbles* and to the other bubbles which meet $\mathcal{C}'_{-}$ (with their overcrossings contained in Λ_b) as *b-bubbles*. The a-bubbles cut Λ_a into rectangles, each of which contain a (possibly trivial) alternating subtangle of K_a. The overcrossings of the a-bubbles connect each of these rectangles together and hence will be called *connecting arcs*. (See Figure 5.12(B)).

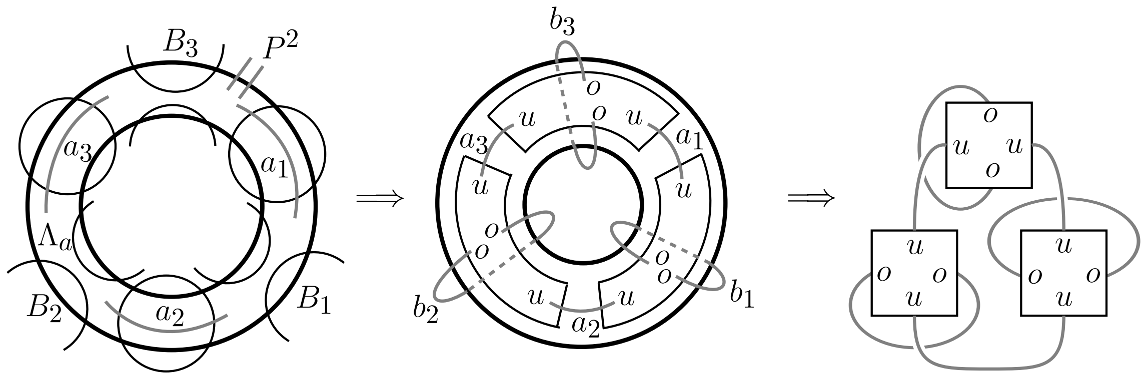


Figure 5.12: (A) An example of the annulus Λ_a from Theorem 5.5.6, which, in this example, meets six bubbles. The overcrossings of the a-bubbles are labeled as a_i. The b-bubbles are labeled as B_i. (B) A diagram of K_a which is contained in Λ_a away from the undercrossings of the b-bubbles, which are labeled as b_i. Each of the rectangular regions contains an alternating subtangle of K_a, and the a_i are now the *connecting arcs* of the diagram. (C) The induced alternating projection of K_a.

The undercrossings of the b-bubbles, which are part of K_a and pass behind the rectangles of Λ_a in the original projection of K, can each be isotoped to cross under a connecting arc, giving an alternating projection of K_a as shown in Figure 5.12(C).

By reversing the roles of the a- and b-bubbles, the same argument can be applied to Λ_b to get an alternating projection of K_b, as well. $\square$

Theorem 5.5.7. *Let K be a composite knot with $g_T(K) = 1$ and let A' be a preferred decomposing sphere for K with respect to the genus-one Turaev surface T such that A' decomposes K into $K_a \# K_b$. If $|\mathcal{C}'_-| = 3$, then $g_T(K_a) + g_T(K_b) = 1$.*

Proof. Let C_1 and C_2 be the components of $\mathcal{C}'_-$ which do not meet K. Note that the third component $C_- \subset \mathcal{C}'_-$ must be separating on T_- with $w_-(C_-) = PSPS$. The curves of $\mathcal{C}'_-$ cut T_- into three surfaces: a disk D bounded by C_-, a punctured annulus Λ_a with $\partial\Lambda_a = \mathcal{C}'_-$, and an annulus Λ_b with $\partial\Lambda_b = C_1 \cup C_2$. (See Figure 5.13.) Note that if C_- meets the same bubble B twice, then, when we pass to T_+, the components of $C_- - B$ create

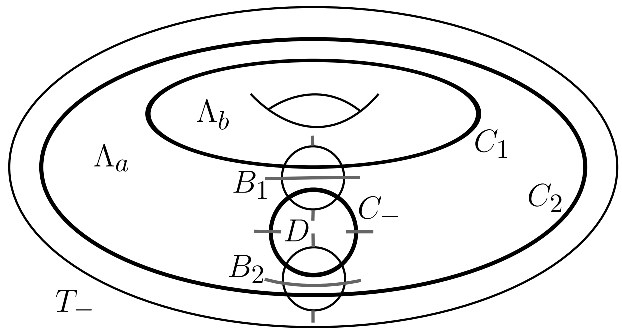


Figure 5.13: The components of $\mathcal{C}'_-$ and the regions of $\overline{T_- - \eta(\mathcal{C}'_-)}$ from Theorem 5.5.7.

either separating curves or meridians in T_+, which cannot happen. So C_- meets two distinct bubbles, B_1 and B_2.

By the alternating property and the fact that C_1 meets no punctures, $w_-(C_1) = S^{2n}$ for some $n \in \mathbb{Z}$. Further, by Lemma 5.3.3, C_1 cannot meet the same bubble more than once. So C_1 meets n distinct bubbles whose overcrossings are contained in Λ_a and another n distinct bubbles whose overcrossings are contained in Λ_b. If B_1 and B_2 were both to meet C_1, then C_2 would meet the same n bubbles as C_1 with overcrossings contained in Λ_b but only $(n-2)$ bubbles with overcrossings contained in Λ_a. This cannot happen, so, up to relabeling, we may assume B_1 meets C_1 and B_2 meets C_2. Note also that since no component of $\mathcal{C}'_-$ can meet any bubble more than once, each bubble which meets A' can contain only one saddle.

As in the proof of Theorem 5.5.6, after relabelling the summands of $K_a \# K_b$ if necessary, K_a is contained in Λ_a away from the undercrossings of the b-bubbles (which are part of K_a but pass under Λ_b). Similarly, K_b is contained in $\Lambda_b \cup D$ away from the undercrossings of the a-bubbles. Note that B_1 and B_2 are both a-bubbles.

The b-bubbles cut Λ_b into rectangles which contain alternating subtangles of K_b, as shown in Figure 5.14. The overcrossings of the b-bubbles are connecting arcs for these rectangles.

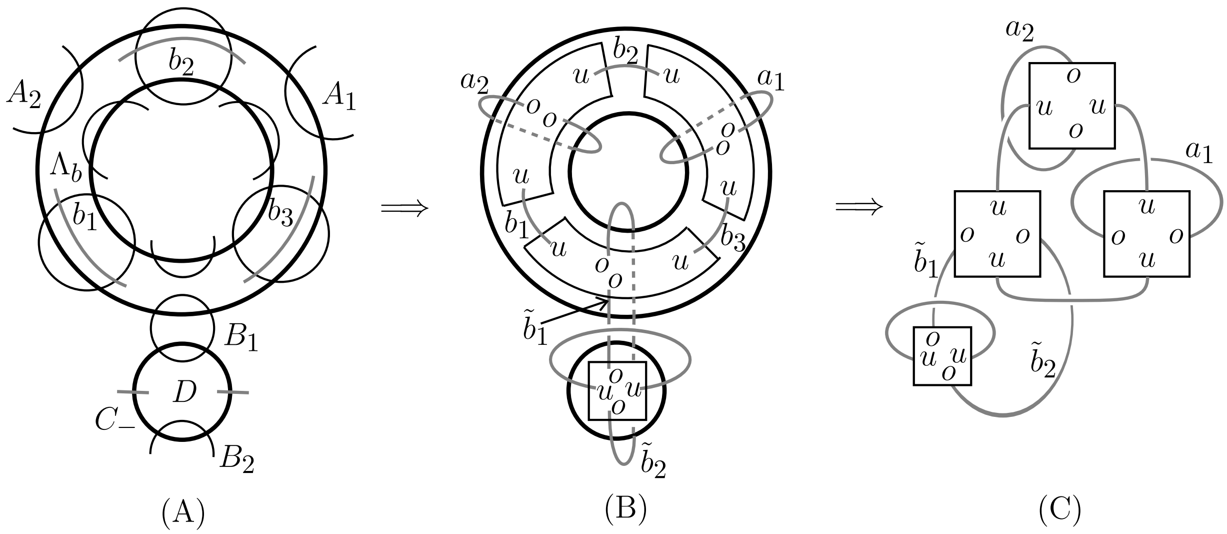


Figure 5.14: (A) An example of the annulus Λ_b and the disk D from Theorem 5.5.7. The a-bubbles, other than B_1 and B_2, are labeled as A_i, and the overcrossings of the b-bubbles are labeled as b_i. (B) A diagram of K_b contained (mostly) in $\Lambda_b \cup D$. The b_i are now connecting arcs of the diagram, and each of the rectangular regions contains an alternating subtangle. The a_i and $\tilde{b}_i$ correspond to the overcrossings of the A_i and B_i, respectively. (C) The induced alternating projection of K_b.

The disk D also contains an alternating subtangle of K_b, which is connected to one of the rectangles of Λ_b via the undercrossings of B_1 and B_2. As in the proof of Theorem 5.5.6, the undercrossings of the a-bubbles (including B_1 and B_2) can be isotoped to give an alternating diagram for K_b. Hence $g_T(K_b) = 0$.

Similarly, the a-bubbles (including B_1 and B_2) cut Λ_a into rectangles, two of which, R_1 and R_2, are adjacent to C_-. (See Figure 5.15.) Each of the rectangles of $\Lambda_a - \bigcup\{a\text{-bubbles}\}$ contains an alternating subtangle of K_a, and these rectangles are connected to each other via the overcrossings of the a-bubbles. Note that there are three connecting arcs between R_1 and R_2: two coming from the overcrossings of B_1 and B_2, and another coming from the unknotted arc $\alpha \subset D$ that was created in the decomposition of K. Each of the rectangles of $\Lambda_a - \bigcup\{a\text{-bubbles}\}$ is also connected to itself via the undercrossing of some b-bubble. These undercrossings can be isotoped under the connecting arcs to give a diagram for K_a as shown

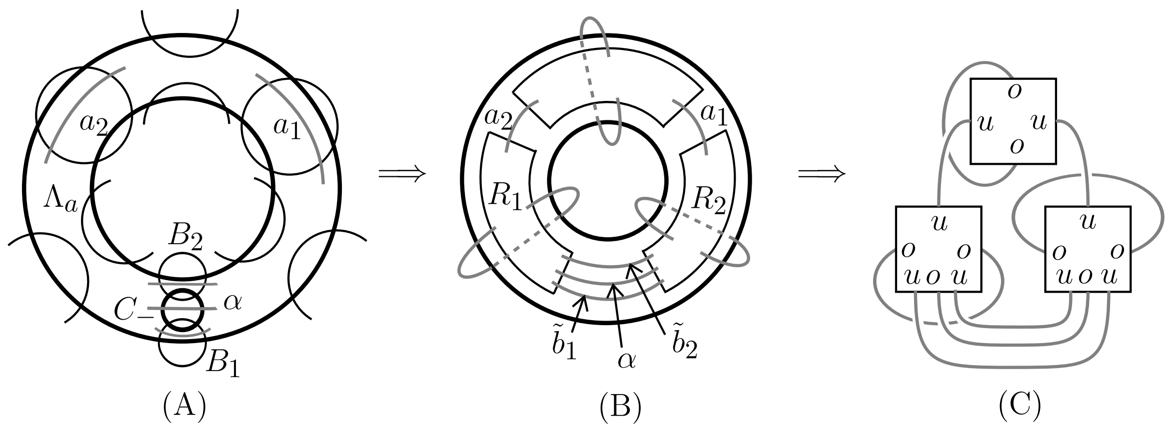


Figure 5.15: (A) An example of the punctured annulus Λ_a from Theorem 5.5.7. The overcrossings of the a-bubbles, other than B_1 and B_2, are labeled as a_i. (B) A projection of K_a contained (mostly) in Λ_a. The a_i are now connecting arcs in the diagram, as are α and the overcrossings of the B_i, which are labeled $\tilde{b}_i$. Each of the rectangular regions contains an alternating subtangle. (C) A diagram for K_a.

in Figures 5.15(C) and 5.16(A). Using this diagram, we get an associated ribbon graph of genus one, as shown in Figure 5.16(C). Hence $g_T(K_a) = 1$, as desired. $\square$

Before moving on to the cases when $|\mathcal{C}'_-| \geq 4$, we have the following corollary.

Corollary 5.5.8. *Let K be a composite knot with $g_T(K) = 1$ and let A' be a preferred decomposing sphere for K with respect to the genus-one Turaev surface T such that A' decomposes K into $K_a \# K_b$. If no bubble of T meets A' more than once, then either $g_T(K_a)+g_T(K_b) = 1$ or $|\mathcal{C}'_-| = 1$ and, up to relabelling, K_a is alternating and $g_{alt}(K_b) = g_{DA}(K) = 1$.*

Proof. If $|\mathcal{C}'_-|$ is even, then $\mathcal{C}'_-$ has no curve which is separating on T_- and, by Corollary 5.4.7, there is a component $C^* \subset \mathcal{C}'_\pm$ with $w_\pm(C^*) = S^2$. Since $\mathcal{C}'_-$ has no component which is separating on T_-, we may choose the orientation on T so that $C^* \subset \mathcal{C}'_+$. Then, since each bubble contains at most one saddle, $|C'_-| = 2$, which cannot happen by Theorem 5.5.6.

Suppose $|\mathcal{C}'_-|$ is odd, so there is a curve $C_- \subset \mathcal{C}'_-$ which is separating on T_- with $w_-(C_-) = PSPS$. Since no bubble contains more than one saddle, each longitude $C \subset \mathcal{C}'_-$

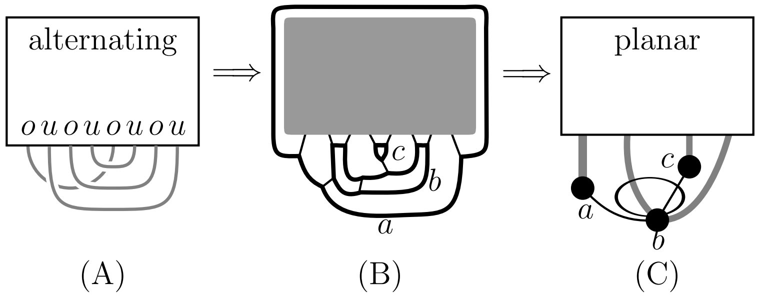


Figure 5.16: (A) A simplification of the diagram from Figure 5.15(C). (B) A diagram for the all-A smoothing of K_a. The thicker lines are each Seifert circles. The lighter line segments represent crossings from the knot diagram which will be edges in the ribbon graph. The shaded gray region contains additional Seifert circles and edges from the alternating portion of the knot diagram. (C) A genus-one ribbon graph for K_a. The rectangular region contains a planar sub-ribbon graph. The thicker gray lines each represent two or more planar edges connecting the vertices a, b and c to vertices in the planar portion of the graph.

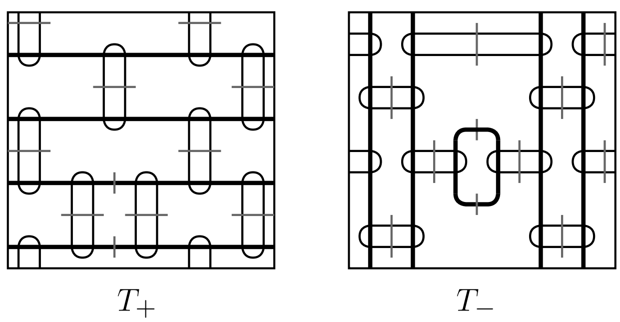


Figure 5.17: The curves of $\mathcal{C}'_\pm$ from Corollary 5.5.8 with the bubbles of $T_\pm$.

has $w_-(C) = S^{2n}$ where $|\mathcal{C}'_+| = 2n$. Likewise, $\mathcal{C}'_+$ has two components C_1 and C_2 with $w_+(C_i) = PS^{2m+1}$ and all other curves $C \subset \mathcal{C}'_+$ have $w_+(C) = S^{2m}$ where $|\mathcal{C}'_-| = 2m+1$. (See Figure 5.17.)

By Corollary 5.4.6, there is a curve $C^* \subset \mathcal{C}'_\pm$ with $|w_\pm(C^*)| = 2$. First suppose $C^* \subset \mathcal{C}'_+$. If $w_+(C^*) = PS$, then $m = 0$ and $|\mathcal{C}'_-| = 1$, so the result holds by Theorem 5.5.5. If $w_+(C^*) = S^2$, then $m = 1$ and $|\mathcal{C}'_-| = 3$, so the result holds by Theorem 5.5.7. Hence we may assume $C^* \subset \mathcal{C}'_-$ and $w_-(C^*) = S^2$, so $|\mathcal{C}'_+| = 2$. Then $\mathcal{C}'_+$ cuts T_+ into annuli Λ_a and Λ_b as in the proof of Theorem 5.5.6 and these annuli give rise to diagrams for the summands

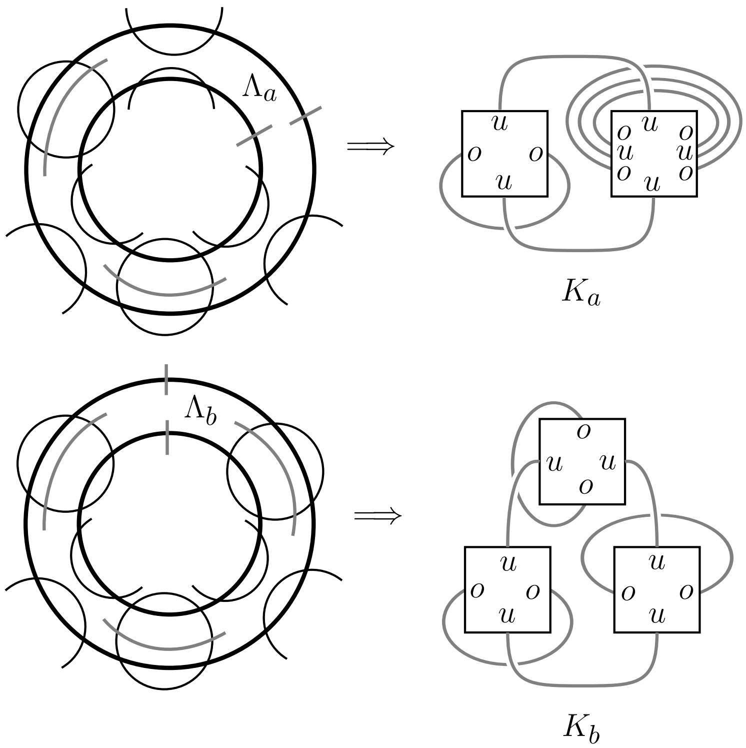


Figure 5.18: An example of the annuli of $\overline{T_+ - \eta(\mathcal{C}'_+)}$ from Corollary 5.5.8. The corresponding diagram of K_b is alternating, and the diagram of K_a has Turaev genus 1 by the arguments of Figure 5.16.

K_a and K_b of K with $g_T(K_a) + g_T(K_b) = 1$, as illustrated in Figure 5.18. □

In light of this corollary, what remains to be studied are decompositions given by preferred spheres such that $|\mathcal{C}'_-| \geq 4$ and there is some bubble on the Turaev surface meeting this decomposing sphere multiple times. The proof of Theorem 5.5.9, which restricts our attention to the case when $|\mathcal{C}'_-| = 4$, employs an Euler characteristic argument which may, in the future, be helpful in studying the general case of $|\mathcal{C}'_-| \geq 4$.

Let T be a genus-one Turaev surface for a knot K and let A' be a preferred decomposing sphere for K with respect to T such that $\mathcal{C}'_-$ contains no curve which bounds a disk on T_-. By Corollary 5.4.3, each component of $\mathcal{C}'_\pm$ contributes one disk to $A' \cap M_\pm$, and each of these disks are meridional disks in $M_\pm$. These disks cut $M_\pm$ into a collection of 3-balls. We will call the 3-balls of $\overline{M_- - \eta(A')}$ *drums* and those of $\overline{M_+ - (A')}$ *caps*.

Let B be some bubble of T which meets A' in $n \geq 1$ saddles. These saddles cut B into $(n+1)$ 3-balls which we will denote by b^i for $0 \leq i \leq n$. Recall that $V_+ = M_- \cup \{\text{bubbles}\}$. Let b^0 be the component of $B - \eta(A')$ which is adjacent to M_- in V_+ and let b^{i+1} be adjacent to b^i in B for $0 \leq i \leq n-1$.

Note that $\overline{V_+ - \eta(A')}$ has exactly two components, which we will denote by Y_1 and Y_2. Half of the drums of $\overline{M_- - \eta(A')}$ are contained in Y_1 and the other half are in Y_2, and we will regard these as the 0-cells of the Y_i. In $\overline{V_+ - \eta(A')}$, b^0 is part of some drum and the other b^i for $1 \leq i \leq n$ each contribute a 1-cell to some Y_i. Hence $Y_1 \sqcup Y_2$ has a cellular decomposition with 0-cells corresponding to drums of M_- and 1-cells corresponding to the saddles of A'. Then $\overline{S^3 - \eta(A')}$ consists of two 3-balls, X_1 and X_2, where each X_i is obtained from Y_i by attaching exactly half the caps of M_+, which are the 2-cells of X_i. Hence for $X := X_1 \sqcup X_2$,

$$2 = \chi(X) = |\mathcal{C}'_-| - s + |\mathcal{C}'_+| \tag{5.6}$$

where s is the total number of saddles in $A' \cap \{\text{bubbles}\}$.

For a bubble B which meets n saddles, b^n meets just one cap of M_+ and, for $i \neq 0$ or n, b^i meets exactly two caps. We will call b^n a *1-bridge* and the other b^i for $i \neq 0, n$ will be called *2-bridges*. For each bridge b^i, we will say that b^i has *length* i. Note that a bridge of length i connects two drums $\delta_1, \delta_2 \subset M_-$ by passing over $2i-1$ drums between δ_1 and δ_2.

Let Δ_i, β_i, Γ_i and κ_i denote the number of drums, 1-bridges, 2-bridges and caps in X_i, respectively, for $i = 1, 2$. Then $\Delta_i = \frac{1}{2}|\mathcal{C}'_-|$, $\kappa_i = \frac{1}{2}|\mathcal{C}'_+|$ and we have

$$1 = \chi(X_i) = \Delta_i + \kappa_i - \beta_i - \Gamma_i \tag{5.7}$$

for $i = 1, 2$.

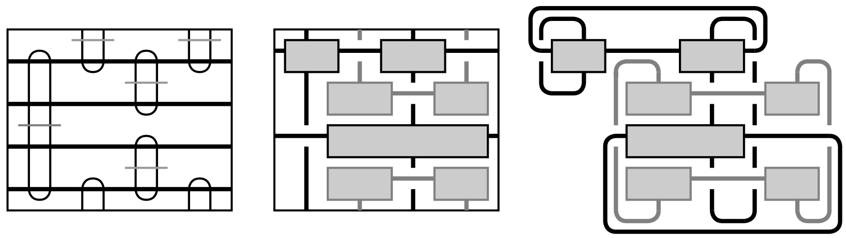
Theorem 5.5.9. *Let K be a composite knot with $g_T(K) = 1$ and let A' be a preferred decomposing sphere for K with respect to the genus-one Turaev surface T. If there exists $C \subset \mathcal{C}'_-$ with $w_-(C) = S^2$, then $|\mathcal{C}'_-| \neq 4$.*

Note that, by Lemma 5.4.6, there is some component $C \subset C'_\pm$ with $w_\pm(C) = S^2$, so we are simply choosing the orientation on T such that $C \subset \mathcal{C}'_-$.

Proof. Assume $|\mathcal{C}'_-| = 4$ and let $C \subset \mathcal{C}'_-$ have $w_-(C) = S^2$. Since $\mathcal{C}'_+$ consists of longitudes of T_+, C must meet two distinct bubbles. Hence, by Lemma 5.3.3, each bubble meets each component of $\mathcal{C}'_-$ at most once and therefore contains at most 2 saddles. Since every component of $\mathcal{C}'_+$ meets at least one of the bubbles meeting C, $|\mathcal{C}'_+| \leq 8$. Hence $\Delta_i = 2$ and $\kappa_i \leq 4$ for $i = 1, 2$. (In general, if we assume $w_-(C) = S^2$ for some $C \subset \mathcal{C}'_-$, then $\kappa_i \leq 2\Delta_i$.)

By Corollary 5.5.8, some bubble of T meets A' more than once, so, without loss of generality, we may assume that $\Gamma_2 > 1$. The number and length of the 1-bridges in X completely determines the number of caps and 2-bridges. To see this, notice that each cap of X_i is attached to Y_i along a gluing circle which meets either 1 bridge of length 2 or 2 bridges each of length 1. Hence each 1-bridge $b^1 \subset X_i$ of length 1 corresponds to half of a cap and no 2-bridges in X_i, and b^1 contributes $-\frac{1}{2}$ to $\chi(X_i)$. Likewise, each 1-bridge $b^2 \subset X_1$ of length 2 corresponds to 1 2-bridge (of length 1) in X_2 and 1 cap in each of X_1 and X_2. This means the 1-bridges of length 2 contribute nothing to $\chi(X_1)$ or $\chi(X_2)$. Letting β_i^j denote the number of 1-bridges of length j in X_i for $i = 1, 2$, we have

$$1 = \chi(X_i) = \Delta_i - \frac{1}{2}\beta_i^1. \qquad (5.8)$$



(A) (B) (C)

Figure 5.19: (A) The configuration of T_- from the first row of Table 5.1. (B) A diagram of $K_a \sqcup K_b$ on the torus given by this coniguration. The shaded boxes contain (possibly trivial) alternating sub-diagrams. (C) The resulting diagram of $K_a \sqcup K_b$ on $\mathbb{R}^2$, which shows that K_a and K_b are alternating knots.

Since $\Delta_i = 2$, we see that $\beta_i^1 = 2$ for $i = 1, 2$.

Note that $\beta_1^2 = \Gamma_2$. This gives $\beta_1 - \Gamma_2 = \beta_1^1 = 2$. An identical argument shows $\beta_2 - \Gamma_1 = 2$. Hence

$$\beta_1 + \beta_2 - \Gamma_1 - \Gamma_2 = \beta_1^1 + \beta_2^1 = 4. \tag{5.9}$$

Equation (5.7) applied to $\chi(X)$ gives

$$\beta_1 + \beta_2 + \Gamma_1 + \Gamma_2 = 2 + \kappa_1 + \kappa_2. \tag{5.10}$$

Combining equations (5.9) and (5.10) with the fact that $\kappa_1 = \kappa_2 \leq 4$, we have

$$2(\beta_1 + \beta_2) - 4 \leq 10 \tag{5.11}$$

This means $\beta_1 + \beta_2 \leq 7$ and hence $\beta_1^2 + \beta_2^2 \leq 3$ with $\beta_1^2 = \Gamma_2 > 0$ and $\beta_1^1 = \beta_2^1 = 2$. In light of the alternating property, this leaves exactly 6 possible configurations for the bubbles of T with respect to $\mathcal{C}'_-$, which are shown in Table 5.1. Since some curve $C_- \subset \mathcal{C}'_-$ has $P^2 \subset w_-(C_-)$ by Lemma 5.3.5, each of these configurations decomposes K into 2 disjoint,

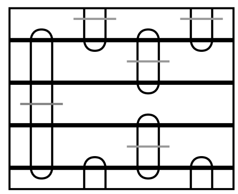
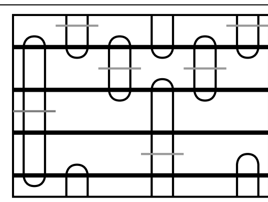
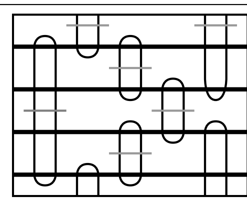
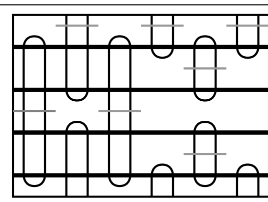
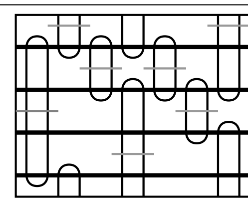
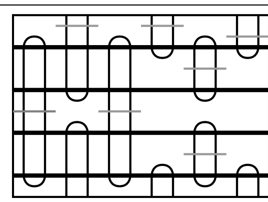
β_1^2	β_2^2	Possible configurations of T_-
1	0	
1	1	
2	0	
2	1	 
3	0	

Table 5.1: All possible configurations of $\mathcal{C}'_-$ and the bubbles of T_- from Theorem 5.5.9.

unlinked summands. See, for example, Figure 5.19. Hence K is alternating, contradicting the fact that $g_T(K) = 1$. □

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