A SOLUTION OF THE HELMHOLTZ EQUATION IN CONICAL COORDINATES

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A SOLUTION OF THE HELMHOLTZ EQUATION IN CONICAL COORDINATES

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I. Definition of Coordinates

The conical coordinates r, u, and v are defined by

$$r = r \tag{1a}$$

$$u = k\,\mathrm{cn}(\xi, k) \tag{1b}$$

$$v = k'\,\mathrm{cn}(\eta, k') \tag{1c}$$

where $k^2 + k'^2 = 1$.

II. Some Geometrical Considerations

The transformations from the cartesian x, y, z coordinates to the conical r, u, v are given by the following relations:

$$x^1 = x = \pm \frac{r}{k}\sqrt{k^2 - u^2}\sqrt{k^2 + v^2} \tag{2a}$$

or

$$x = r\,\mathrm{sn}\,\xi\,\mathrm{dn}\,\eta.$$

$$x^2 = y = \pm \frac{r}{k'}\sqrt{k'^2 + u^2}\sqrt{k'^2 - v^2} \tag{2b}$$

or

$$y = r\,\mathrm{sn}\,\eta\,\mathrm{dn}\,\xi.$$

$$x^3 = z = \frac{r}{kk'}\,u v \tag{2c}$$

or

$$z = r\,\mathrm{cn}\,\xi\,\mathrm{cn}\,\eta.$$

The cartesian equations for the coordinate surfaces may be obtained from (2a), (2b), and (2c) by eliminating two of the conical variables at a time from the three equations. In this discussion we shall consider a particular case and let $k^2 = k'^2 = \frac{1}{2}$.

. If we eliminate u and v , we find

$$x^2 + y^2 + z^2 = r^2. \tag{3}$$

Hence, the surfaces $r =$ constant are spheres with centers at the origin. Elimination of r and v yields

$$x^2\left(\tfrac{1}{2} + u^2\right) - y^2\left(\tfrac{1}{2} - u^2\right) - z^2\left(\frac{1}{4u^2} - u^2\right) = 0. \tag{4}$$

Since $u^2 \leq \frac{1}{2}$, the surfaces $u =$ constant are cones with axes along the x-axis. The cones have elliptical cross-sections. Finally, by eliminating r and u we find the equation

$$y^2\left(\tfrac{1}{2} + v^2\right) - x^2\left(\tfrac{1}{2} - v^2\right) - z^2\left(\frac{1}{4v^2} - v^2\right) = 0. \tag{5}$$

We also have $v^2 \leq \frac{1}{2}$; hence, the surfaces $v =$ constant are cones with axes along the y-axis. These cones also have elliptical cross-sections. Fig. 1 illustrates the coordinate-surfaces.

In order to determine the signs of the coordinate-surfaces, we must consider the ξ and η of equations (1b) and (1c) as the variables. Since the functions $cn\,\xi$ and $cn\,\eta$ are periodic, it is necessary to take ξ and η as the variables if we are to be able to determine a point uniquely in the conical coordinates.

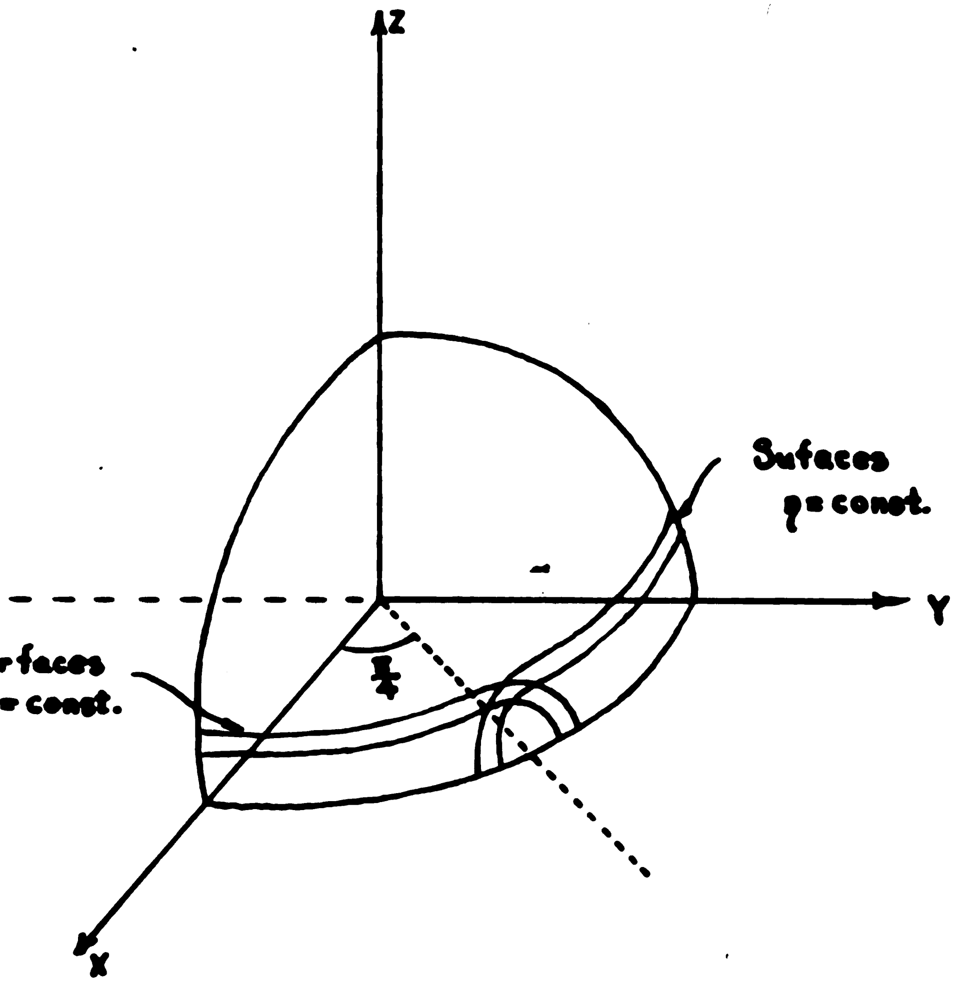

Fig.1 Conical Coordinates

We are led to a proper choice for the range of ξ and η through the following considerations:

1. The surface area of a sphere calculated from an element of area which was expressed in conical coordinates.
2. The values of ξ and η which are necessary to make the cartesian coordinates x and y from (2a) and (2b) change signs properly.
3. The orthogonality of certain surface harmonics when these are transformed into conical coordinates.

We shall choose for ξ the range $0 \leq \xi \leq 4K$ and for η the range $-K \leq \eta \leq K$. Here the quantity K is the complete elliptic integral of the first kind:

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}} .$$

This should not be confused with the parameter K of equation (1b). Fig. 1 and Fig. 2 illustrate the signs of the conical surfaces; however, a description may prove helpful for the interpretation of Fig. 2.

The cones whose axes coincide with the positive y-axis are positive. These are the surfaces for which $\mathrm{cn}\,\eta$ is constant, and the range of η is $0 \leq \eta \leq K$. The surfaces for which $\mathrm{cn}\,\eta$ is constant also represent cones whose axes coincide with

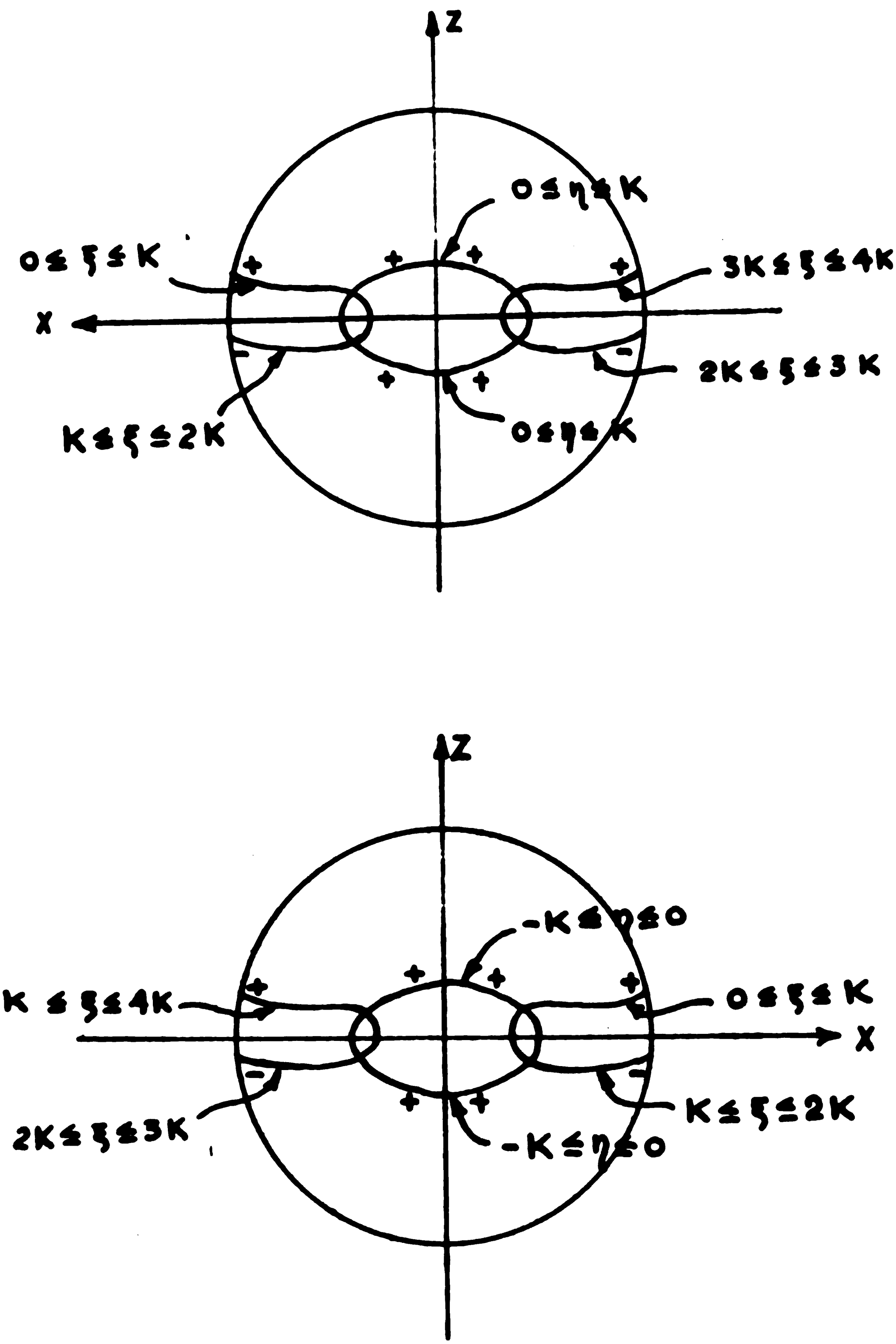

Fig. 2. Signs of the coordinate surfaces

the negative y-axis. These surfaces are positive, and we have $-K \leq \eta \leq 0$ for them.

The cones whose axes coincide with the positive and negative x-axes are the surfaces for which $\cos\zeta$ is constant. These surfaces change sign at the xy-plane. That is, the portions of the cones which lie above the xy-plane are positive; those portions which lie below the xy-plane negative. The following outline indicates the distribution of the values for the variable ζ.

1. Cones whose axes coincide with the positive x-axis:

   (a) Portions above the xy-plane,
   $$0 \leq \zeta \leq K.$$

   (b) Portions below the xy-plane,
   $$K \leq \zeta \leq 2K.$$

2. Cones whose axes coincide with the negative x-axis:

   (a) Portions above the xy-plane,
   $$3K \leq \zeta \leq 4K.$$

   (b) Portions below the xy-plane,
   $$2K \leq \zeta \leq 3K.$$

From the transformations (2a), (2b), and (2c) we may calculate the metric tensor $\bar{g}_{mn}$. For convenience we now change the notation slightly and write $r = \bar{x}^1$; $u = \bar{x}^2$; $v = \bar{x}^3$.

For the metric tensor we have the expression

$$\bar{g}_{mn} = \frac{\partial x^i}{\partial \bar{x}^m} \frac{\partial x^i}{\partial \bar{x}^n}. \tag{6}$$

The barred x's refer to conical coordinates, and the unbarred x's refer to cartesian coordinates. In the expression for $\bar{g}_{mn}$ summation is understood for the repeated index i . We find

$$\bar{g}_{11} = 1 \tag{7a}$$

$$\bar{g}_{22} = \frac{(\bar{x}^1)^2\left[(\bar{x}^2)^2 + (\bar{x}^3)^2\right]}{\left[k^2 - (\bar{x}^2)^2\right]\left[k'^2 + (\bar{x}^2)^2\right]} \tag{7b}$$

$$\bar{g}_{33} = \frac{(\bar{x}^1)^2\left[(\bar{x}^2)^2 + (\bar{x}^3)^2\right]}{\left[k^2 + (\bar{x}^3)^2\right]\left[k'^2 - (\bar{x}^3)^2\right]}. \tag{7c}$$

Also $\bar{g}_{mn} = 0$ for $m \neq n$. Hence, we see that the conical coordinates are orthogonal. We define the following quantities

$$\bar{g} = \begin{vmatrix} \bar{g}_{11} & 0 & 0 \\ 0 & \bar{g}_{22} & 0 \\ 0 & 0 & \bar{g}_{33} \end{vmatrix} = \bar{g}_{11}\bar{g}_{22}\bar{g}_{33}. \tag{8}$$

$$h_1 = \sqrt{\bar{g}_{11}} \tag{9a}$$

$$h_2 = \sqrt{\bar{g}_{22}} \tag{9b}$$

$$h_3 = \sqrt{\bar{g}_{33}}. \tag{9c}$$

III. The Helmholtz Equation $\nabla^2\psi + k^2\psi = 0$ in Conical Coordinates

For any scalar ψ and generalized coordinates $\bar{x}^i$, the Laplacian becomes

$$\nabla^2\psi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \bar{x}^i}\left(\sqrt{g}\,\bar{g}^{ij}\frac{\partial\psi}{\partial\bar{x}^j}\right)$$

, or

$$\nabla^2\psi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \bar{x}^1}\left(\sqrt{g}\,\bar{g}^{11}\frac{\partial\psi}{\partial\bar{x}^1}\right)$$

$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial \bar{x}^2}\left(\sqrt{g}\,\bar{g}^{22}\frac{\partial\psi}{\partial\bar{x}^2}\right)+\frac{1}{\sqrt{g}}\frac{\partial}{\partial \bar{x}^3}\left(\sqrt{g}\,\bar{g}^{33}\frac{\partial\psi}{\partial\bar{x}^3}\right). \qquad (10)$$

The $\bar{g}^{ij}$ may be found from the $\bar{g}_{jk}$ by means of the following relation:

$$\bar{g}^{ij}\bar{g}_{jk} = \delta^i_k. \qquad (11)$$

Then from equations (9a,b,c) and equation (11) we find

$$\bar{g}^{ii} = \frac{1}{\bar{g}_{ii}} = \frac{1}{\bar{h}_i^2}. \qquad (12)$$

If we substitute from equations (7a,b,c), employ equations (9a,b,c) and equation (12), and carry out the indicated differentiations in equation (10), we find the Helmholtz equation in conical coordinates

$$\nabla^2\psi + k^2\psi =$$

$$\frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\partial\psi}{\partial r}\right) + \frac{-2u^3}{r^2(u^2+v^2)}\frac{\partial\psi}{\partial u}$$

$$+\frac{(\frac{1}{2}-u^2)(\frac{1}{2}+u^2)}{r^2(u^2+v^2)}\frac{\partial}{\partial u}\left(\frac{\partial\psi}{\partial u}\right)+\frac{-2v^3}{r^2(u^2+v^2)}\frac{\partial\psi}{\partial v}$$

$$+ \frac{(\frac{1}{2}-v^2)(\frac{1}{2}+v^2)}{r^2(u^2+v^2)} \frac{\partial}{\partial v}\left(\frac{\partial \psi}{\partial v}\right) + k^2\psi = 0 . \tag{13}$$

In this equation we have returned to the original notation and written the conical coordinates as r , u , v. We shall use this notation in the remainder of the discussion.

IV. Separation of Variables

$$\text{Let} \quad \psi = R(r)\,S(u)\,T(v). \tag{14}$$

If we substitute this value for ψ in (13) and divide by (RST), we find

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2u^3}{S(u^2+v^2)}\frac{dS}{du}$$
$$+ \frac{(\frac{1}{2}-u^2)(\frac{1}{2}+u^2)}{S(u^2+v^2)}\frac{d^2S}{du^2} - \frac{2v^3}{T(u^2+v^2)}\frac{dT}{dv}$$
$$+ \frac{(\frac{1}{2}-v^2)(\frac{1}{2}+v^2)}{T(u^2+v^2)}\frac{d^2T}{dv^2} + r^2K^2 = 0. \tag{15}$$

We may separate from (15) the part which depends upon r alone and find

$$\frac{2r}{R}\frac{dR}{dr} + \frac{r^2}{R}\frac{d}{dr}\left(\frac{dR}{dr}\right) + r^2K^2 = a$$

, where a is a separation constant. Or

$$\frac{d}{dr}\left(\frac{dR}{dr}\right) + \frac{2}{r}\frac{dR}{dr} + \left(K^2 - \frac{a}{r^2}\right)R = 0. \tag{16}$$

We find for the part which depends upon u alone

$$\frac{d}{du}\left(\frac{dS}{du}\right)+\frac{-2u^3}{(\frac{1}{2}-u^2)(\frac{1}{2}+u^2)}\frac{dS}{du}+\frac{(au^2-b)S}{(\frac{1}{2}-u^2)(\frac{1}{2}+u^2)}=0, \tag{17}$$

where b is also a separation constant. And finally, for the part which depends upon v alone we find

$$\frac{d}{dv}\left(\frac{dT}{dv}\right)+\frac{-2v^3}{(\frac{1}{2}-v^2)(\frac{1}{2}+v^2)}\frac{dT}{dv}+\frac{(av^2+b)T}{(\frac{1}{2}-v^2)(\frac{1}{2}+v^2)}=0. \tag{18}$$

In the finite complex plane, equations (16), (17), and (18) have the following regular singular points:

Equation (16) $r=0$

Equation (17) $u=\pm\frac{1}{\sqrt{2}}$ and $u=\pm i\frac{1}{\sqrt{2}}$

Equation (18) $v=\pm\frac{1}{\sqrt{2}}$ and $v=\pm i\frac{1}{\sqrt{2}}$

Equation (16) has an irregular singular point at infinity, and equations (17) and (18) each have a regular singular point at infinity.

V. Solutions of the Ordinary Differential Equations

Consider the radial equation (16)

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(k^2-\frac{a}{r^2}\right)R=0.$$

If we let the separation constant a equal $l(l+1)$ where l is an integer, we find the spherical Bessel functions for solutions of the radial equation. Hence, we shall tentatively assign the value $a = l(l+1)$.

Next we shall consider the "angular" equations (17) and (18). With $a = l(l+1)$ in (17), let us assume a formal solution

$$S = \sum_m a_m \left(\tfrac{1}{2} - u^2\right)^{m+\alpha}. \tag{19}$$

This solution is designed to converge in the neighborhood of the singular points $u = \pm \frac{1}{\sqrt{2}}$. Substituting (19) in (17) and equating the lowest power of $(\frac{1}{2} - u^2)$ to zero, we find the indicial equation $\alpha(2\alpha - 1) = 0$

Hence,

$$\left.\begin{aligned} \alpha &= 0 \\ \text{or} \quad \alpha &= \tfrac{1}{2} \end{aligned}\right\} \tag{20}$$

In order to obtain a solution which converges in the neighborhood of the singular points $u = \pm i \frac{1}{\sqrt{2}}$, we assume a formal solution

$$S = \sum_m a_m \left(\tfrac{1}{2} + u^2\right)^{m+\beta}. \tag{21}$$

The indicial equation which one obtains after the substitution of (21) in (17) has the two roots

$$\left.\begin{aligned} \beta &= 0 \\ \text{and} \quad \beta &= \tfrac{1}{2} \end{aligned}\right\} \tag{22}$$

Since equation (18) with $a = l(l+1)$ possesses the singular points $v = \pm\frac{1}{\sqrt{2}}$ and $v = \pm i\frac{1}{\sqrt{2}}$, we assume formal solutions of the forms

$$T = \sum_m a_m \left(\tfrac{1}{2} - v^2\right)^{m+\alpha} \tag{23}$$

and

$$T = \sum_m a_m \left(\tfrac{1}{2} + v^2\right)^{m+\beta} \tag{24}$$

From the respective indicial equations we find for α of (23) the values

$$\left.\begin{array}{l} \alpha = 0 \\ \alpha = \frac{1}{2} \end{array}\right\} \tag{25}$$

and for β of (24) the values

$$\left.\begin{array}{l} \beta = 0 \\ \beta = \frac{1}{2} \end{array}\right\} \tag{26}$$

First, consider equation (17). Suppose we make the following substitution for S.

$$S = \left(\tfrac{1}{2} - u^2\right)^{\frac{1}{2}} P, \tag{27}$$

where P is a function of u not yet determined. Substitution of (27) in (17) yields

$$\left(\tfrac{1}{4} - u^4\right)\frac{d^2P}{du^2} - (4u^3 + u)\frac{dP}{du} + \left[-b - \tfrac{1}{2} + (l-1)(l+2)u^2\right]P = 0 \tag{28}$$

from which we shall find the function P. Let us assume a power series about $u = 0$ for P,

$$P = \sum_{n=0}^{\infty} h_n u^n. \tag{29}$$

By substituting (29) in (28), we find a relationship between the h_n's :

$$\left[(n-2)(n-3) + 4(n-2) - (l-1)(l+2)\right] h_{n-2} - \frac{1}{4}(n+2)(n+1)h_{n+2} + (n+b+\tfrac{1}{2})h_n = 0. \tag{30}$$

We wish to express n in terms of l ; and hence, obtain a series whose powers of u depend upon l .

If in (30) we let $n = l + g$, we may determine the h_{l+g} . Then for all h_{l+g} with $(l+g) \geq l$ we shall obtain a series in which the lowest power of u is u^l . On the other hand, for all h_{l+g} with $(l+g) < l$ we obtain a polynomial in u , since we allow no negative powers of u .

Let us make the following supposition:
We may express the solution P either as a series whose lowest power of u is l or as a polynomial in u of degree $(l-1)$. We shall consider the series solution first. If we may express our solution in this form, then all h_{l+g} with $(l+g) < l$ are zero. With $n = l + g$ equation (30) becomes

$$\left[(l+g-2)(l+g-3) + 4(l+g-2) - (l-1)(l+2)\right] h_{l+g-2} - \frac{1}{4}(l+g+2)(l+g+1)h_{l+g+2} + (l+g+b+\tfrac{1}{2})h_{l+g} = 0. \tag{31}$$

For $g=-2$, we find from (31)

$$\frac{1}{4}(l)(l-1)h_l=0$$

. This is to be true for $l\geq 2$; hence, $h_l=0$.

For $g=-1$,

$$\frac{1}{4}(l+1)(l)h_{l+1}=0,$$

or $h_{l+1}=0$.

For $g=0$,

$$\frac{1}{4}(l+2)(l+1)h_{l+2}-(l+b+\tfrac{1}{2})h_l=0.$$

Since $h_l=0$, $h_{l+2}=0$.

For $g=1$,

$$\frac{1}{4}(l+3)(l+2)h_{l+3}-(l+b+\tfrac{3}{2})h_{l+1}=0.$$

Therefore,

$$h_{l+3}=0.$$

For $g=2$

$$(2l+2)h_l-\frac{1}{4}(l+4)(l+3)h_{l+4}+(l+b+\tfrac{5}{2})h_{l+2}=0,$$

or $h_{l+4}=0$.

In general all the h_{l+g} with $(l+g)\geq 0$ are zero. Since we have assumed the h_{l+g} with $0\leq(l+g)<l$ are zero, our solution is $P=0$. Either the h_{l+g} with $0\leq(l+g)<l$ are not zero or our supposition is not true.

Next we consider the polynomial solution. If P is to be a polynomial of degree $(l-1)$, then all h_{l+g} with $(l+g)\geq l$

must be zero. If we let $g=0$ in the recurrence relation (31), we find

$$(-2l)h_{l-2} - \frac{1}{4}(l+2)(l+1)h_{l+2} + (l+b+\tfrac{1}{2})h_l = 0.$$

Therefore,

$$h_{l-2} = 0.$$

For $g=1$

$$[(l-1)(l-2)+4(l-1)-(l-1)(l+2)]h_{l-1}$$
$$-\frac{1}{4}(l+3)(l+2)h_{l+3} + (l+b+\tfrac{3}{2})h_{l+1} = 0.$$

However,

$$[(l-1)(l-2)+4(l-1)-(l-1)(l+2)] \equiv 0.$$

Consequently, h_{l-1} is arbitrary; and we assume that it is not zero.

For $g=-1$

$$(2-4l)h_{l-3} - \frac{1}{4}(l+1)(l)h_{l+1} + [(l-1)+b+\tfrac{1}{2}]h_{l-1} = 0$$

or

$$(2-4l)h_{l-3} + (l+b-\tfrac{1}{2})h_{l-1} = 0.$$

For $g=-2$

$$(6-6l)h_{l-4} + (l+b-\tfrac{3}{2})h_{l-2} = 0.$$

Since $h_{l-2}=0$, $h_{l-4}=0$. We assume that l is such that $(l-g)\geq 0$. For $g=-3$

$$(12-8l)h_{l-5} - \frac{1}{4}(l-1)(l-2)h_{l-1} + (l+b-\tfrac{5}{2})h_{l-3} = 0.$$

From this we can determine $h_{\ell-5}$ in terms of $h_{\ell-1}$ which is arbitrary. In general we are able to find non-vanishing $h_{\ell-\gamma}$ with the assumption that all $h_{\ell+\gamma}$ with $(\ell+\gamma) \geq \ell$ are zero.

If in the recurrence relation (31) we consider only even γ with $\gamma < 0$, we have a relationship between three consecutive h's of the form $h_{\ell-p}$ with p even. We have shown that $h_{\ell-2} = h_{\ell-4} = 0$. Consequently, all $h_{\ell-\gamma}$ with γ even are zero.

Since we need consider only odd γ with $\gamma < 0$, equation (31) is a relationship between three consecutive h's of the form $h_{\ell-p}$ with p odd. We have shown that $h_{\ell-1}$ is arbitrary, and we can express $h_{\ell-3}$ in terms of $h_{\ell-1}$. Hence, from the recurrence relation (31) it follows that we may express any $h_{\ell-\gamma}$ with odd γ in terms of $h_{\ell-1}$.

The solution is now of the form

$$P = h_{\ell-1} u^{\ell-1} + h_{\ell-3} u^{\ell-3} + h_{\ell-5} u^{\ell-5} + \cdots \tag{32}$$

If according to our supposition, equation (32) is to be a polynomial of degree $(\ell-1)$, then we must be able to make all $h_{\ell+\gamma}$ with $(\ell+\gamma) < 0$ vanish. We consider separately the cases for ℓ odd or ℓ even.

Suppose ℓ is odd. Then $h_{\ell-(\gamma+2)}$ with $\gamma = \ell$ must be zero, and all subsequent h's must vanish. Let $\gamma = -\ell$ in (31). Then we have

$$\ell(\ell+1) h_{-2} + \tfrac{1}{2} h_2 - (b + \tfrac{1}{2}) h_0 = 0.$$

If h_{-2} is zero, this becomes

$$\frac{1}{2} h_2 - (b+\tfrac{1}{2}) h_0 = 0.$$

For a given ℓ we express h_2 and h_0 in terms of $h_{\ell-1}$. This yields an equation for the separation constant b. The roots of this equation are values of b which make h_{-2} zero.

We keep in mind that we are considering only odd γ and odd ℓ. Then let $\gamma = -(\ell+2)$ in (31). The result is

$$-(\ell+3)(\ell-2) h_{-4} + \tfrac{1}{4}(0)(1) h_0 + (b-\tfrac{3}{2}) h_{-2} = 0.$$

The coefficient of h_0 vanishes; and if $h_{-2}=0$, then h_{-4} is zero. Next let $\gamma = -(\ell+r)$ with r even and $r > 2$. The relation (31) yields

$$\left[(r+2)(r+3) - 4(r+2) - (\ell-1)(\ell+2)\right] h_{-(r+2)}$$

$$-\tfrac{1}{4}(2-r)(1-r) h_{2-r} + (b+\tfrac{1}{2}-r) h_{-r} = 0. \tag{33}$$

We have shown that $h_{-2} = h_{-4} = 0$; hence, from (33) it follows that all $h_{\ell+\gamma}$ with $(\ell+\gamma) < 0$ are zero.

If ℓ is even, $h_{\ell-\gamma}$ with $\gamma = \ell+1$ must be zero. Let $\gamma = 1-\ell$ in the recurrence relation (31). We have

$$-\ell(\ell+1) h_{-1} - \tfrac{3}{2} h_3 + (b+\tfrac{3}{2}) h_1 = 0.$$

If h_{-1} is zero, then we have

$$\tfrac{3}{2} h_3 - (b+\tfrac{3}{2}) h_1 = 0.$$

In this equation we express h_3 and h_1 in terms of the arbitrary $h_{\ell-1}$. The result is an equation for the separation constant b. The roots of this equation are values of b which make h_{-1} zero. In (31) let $g=-(\ell+1)$:

$$(\ell-1)(\ell+2)h_{-3}+\tfrac{1}{4}(1)(0)h_1-(b-\tfrac{1}{2})h_{-1}=0.$$

The coefficient of h_1 is zero. If h_{-1} equals zero, then h_{-3} is also zero. Finally, let $g=-(\ell+r)$ where r is odd and $r>1$. Then (31) becomes

$$[(r+2)(r+3)-4(r+2)-(\ell-1)(\ell+2)]h_{-(r+2)}$$

$$-\tfrac{1}{4}(2-r)(1-r)h_{2-r}+(b+\tfrac{1}{2}-r)h_{-r}=0.$$

This is a relationship between three consecutive h's. We have shown that $h_{-1}=h_{-3}=0$. From this equation it follows that all $h_{\ell+g}$ with $(\ell+g)<0$ are zero.

We have seen that in order to obtain the polynomial solution for P we must determine properly the separation constant b. The proper values of b for a given ℓ arise from the solutions of an equation in b which we obtain by equating the proper coefficient h_m to zero. Next we shall consider the degree of this equation for a general value of ℓ. Again we treat separately the cases for ℓ odd and ℓ even.

For odd ℓ we recall that $h_{\ell-(g+2)}$ with $g=\ell$ must be zero. From the recurrence relation (31) let us write explicitly the equations for the various h's. We obtain the following set of equations:

$$\left.\begin{array}{l}
(2-4\ell)h_{\ell-3}+(\ell+b-\frac{1}{2})h_{\ell-1}=0 \\
(12-8\ell)h_{\ell-5}-\frac{1}{4}(\ell-1)(\ell-2)h_{\ell-1}+(\ell+b-\frac{5}{2})h_{\ell-3}=0 \\
(30-12\ell)h_{\ell-7}-\frac{1}{4}(\ell-3)(\ell-4)h_{\ell-3}+(\ell+b-\frac{9}{2})h_{\ell-5}=0 \\
(56-16\ell)h_{\ell-9}-\frac{1}{4}(\ell-5)(\ell-6)h_{\ell-5}+(\ell+b-\frac{13}{2})h_{\ell-7}=0 \\
\quad\vdots \qquad\qquad\qquad\qquad \vdots \\
(\ell-1)(\ell+2)h_0+3h_4-(b+\frac{5}{2})h_2=0 \\
\frac{1}{2}h_2-(b+\frac{1}{2})h_0=0
\end{array}\right\} \qquad (34)$$

Consequently, for odd ℓ we have $\frac{\ell+1}{2}$ coefficients to determine and $\frac{\ell+1}{2}$ equations which involve these coefficients. We may consider these equations as a set of linear homogeneous equations for the h's. If we are to be able to solve (34) for the h's, then the determinant of the coefficients must be zero. Fig. 3 shows the determinant, and we notice that it has $\frac{\ell+1}{2}$ rows and $\frac{\ell+1}{2}$ columns. The separation constant b appears only on the diagonal; therefore, the equation for b must be of degree $\frac{\ell+1}{2}$.

From the determinant (Fig. 3) we may obtain a finite continued-fraction for b. Such a continued fraction may be convenient for the numerical calculation of b when ℓ is large. Suppose we denote the determinant of Fig. 3 by Δ and by Δ_j

$$
\begin{vmatrix}
\ell + b - \frac{1}{2} & 2 - 4\ell & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\frac{1}{4}(\ell-1)(\ell-2) & \ell + b - \frac{5}{2} & 12 - 8\ell & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -\frac{1}{4}(\ell-3)(\ell-4) & \ell + b - \frac{9}{2} & 30 - 12\ell & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{4}(\ell-5)(\ell-6) & \ell + b - \frac{13}{2} & 56 - 16\ell & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & -(b + \frac{5}{2}) & (\ell-1)(\ell+2) \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & -(b + \frac{1}{2})
\end{vmatrix}
$$

Fig. 3 Determinant formed from the coefficients of equation (34).

the determinant formed from Δ by deleting the first i rows and the first i columns. Then

$$\Delta = (l+b-\tfrac{1}{2})\Delta_1 + \tfrac{1}{4}(l-1)(l-2)(2-4l)\Delta_2 = 0.$$

Solving this equation for b, we have

$$b = (\tfrac{1}{2}-l) - \tfrac{1}{4}(l-1)(l-2)(2-4l)\frac{\Delta_2}{\Delta_1} \tag{35}$$

However, we may find Δ_1 in terms of Δ_2 and Δ_3:

$$\Delta_1 = (l+b-\tfrac{5}{2})\Delta_2 + \tfrac{1}{4}(l-3)(l-4)(12-8l)\Delta_3.$$

Then

$$\frac{\Delta_1}{\Delta_2} = (l+b-\tfrac{5}{2}) + \tfrac{1}{4}(l-3)(l-4)(12-8l)\frac{\Delta_3}{\Delta_2}$$

or

$$\frac{\Delta_2}{\Delta_1} = \frac{1}{(l+b-\tfrac{5}{2}) + \tfrac{1}{4}(l-3)(l-4)(12-8l)\frac{\Delta_3}{\Delta_2}} \tag{36}$$

Similarly,

$$\frac{\Delta_2}{\Delta_3} = (l+b-\tfrac{9}{2}) + \tfrac{1}{4}(l-5)(l-6)(30-12l)\frac{\Delta_4}{\Delta_3}$$

or

$$\frac{\Delta_3}{\Delta_2} = \frac{1}{(l+b-\tfrac{9}{2}) + \tfrac{1}{4}(l-5)(l-6)(30-12l)\frac{\Delta_4}{\Delta_3}}. \tag{37}$$

We may continue to solve for the ratios of the consecutive Δ_i's. If we substitute (37) in (36) and then this result in (35), we find for b the continued fraction

$$b=\left(\frac{1}{2}-\ell\right)-\cfrac{\frac{1}{4}(\ell-1)(\ell-2)(2-4\ell)}{\left(\ell+b-\frac{5}{2}\right)+\cfrac{\frac{1}{4}(\ell-3)(\ell-4)(12-8\ell)}{\left(\ell+b-\frac{9}{2}\right)+\frac{1}{4}(\ell-5)(\ell-6)(30-12\ell)\frac{a_4}{a_3}}}$$

This may be written as

$$b=\left(\frac{1}{2}-\ell\right)+\frac{-\frac{1}{4}(\ell-1)(\ell-2)(2-4\ell)}{\left(\ell+b-\frac{5}{2}\right)}\;\underset{+}{}\;\frac{\frac{1}{4}(\ell-3)(\ell-4)(12-8\ell)}{\left(\ell+b-\frac{9}{2}\right)}\;\underset{+}{}\;\cdots \tag{38}$$

If ℓ is even we find the following set of equations which corresponds to (34):

$$\left.\begin{array}{c}
(2-4\ell)h_{\ell-3}+\left(\ell+b-\frac{1}{2}\right)h_{\ell-1}=0\\
(12-8\ell)h_{\ell-5}-\frac{1}{4}(\ell-1)(\ell-2)h_{\ell-1}+\left(\ell+b-\frac{5}{2}\right)h_{\ell-3}=0\\
\vdots \qquad\qquad\qquad \vdots\\
(\ell+3)(\ell-2)h_1+5h_5-\left(\frac{7}{2}+b\right)h_3=0\\
\frac{3}{2}h_3-\left(b+\frac{3}{2}\right)h_1=0
\end{array}\right\} \tag{39}$$

The determinant of the coefficients is similar to that for odd ℓ; however, for even ℓ the determinant has $\frac{\ell}{2}$ rows and $\frac{\ell}{2}$ columns. Again the separation constant b appears only on the diagonal, and the equation for b is of degree $\frac{\ell}{2}$. We may also express b as a finite continued-fraction for even ℓ.

We may conclude that the function P is expressible as a polynomial of degree $(\ell-1)$ for either even or odd ℓ.

Consequently, we write this solution of (17) as

$$S^{(1)} = \left(\tfrac{1}{2}-u^2\right)^{\frac{1}{2}}\left(h_{\ell-1}u^{\ell-1} + h_{\ell-3}u^{\ell-3} + h_{\ell-5}u^{\ell-5} + \cdots\right). \tag{40}$$

If ℓ is odd P is an even function of u while if ℓ is even P is an odd function of u . The superscript (1) of (40) indicates that this is the first of four types of functions which we shall introduce. All four types satisfy the differential equation (17).

The second-type function is of the form

$$S^{(2)} = \left(\tfrac{1}{2}+u^2\right)^{\frac{1}{2}} P. \tag{41}$$

Substitution of (41) in the differential equation (17) yields

$$\left(\tfrac{1}{4}-u^4\right)\frac{d^2P}{du^2} + \left(u-4u^3\right)\frac{dP}{du} + \left[\tfrac{1}{2}-b+(\ell+2)(\ell-1)u^2\right]P = 0. \tag{42}$$

Once again we assume the power series

$$P = \sum_{m=0}^{\infty} h_m u^m. \tag{43}$$

for P . Substitution of (43) in (42) yields the recurrence relation

$$\tfrac{1}{4}(m+2)(m+1)h_{m+2}$$

$$\left[(\ell-1)(\ell+2)-4(m-2)-(m-2)(m-3)\right]h_{m-2} + \left(m+\tfrac{1}{2}-b\right)h_m = 0. \tag{44}$$

Let us compare (44) with (30). The same arguments which we used to find the function $S^{(1)}$ are valid for finding $S^{(2)}$. Hence, we shall present only the results.

In (44) replace m by $(\ell+g)$ and find

$$\frac{1}{4}(\ell+g+2)(\ell+g+1)h_{\ell+g+2}$$
$$+\left[(\ell-1)(\ell+2)-4(\ell+g-2)-(\ell+g-2)(\ell+g-3)\right]h_{\ell+g-2}$$
$$(\ell+g+\tfrac{1}{2}-b)h_{\ell+g}=0. \tag{45}$$

All $h_{\ell+g}$ with $(\ell+g)\geq\ell$ are zero, and we consider expressing P as a polynomial of degree $(\ell-1)$.

Proceeding with the recurrence relation (45) just as we did with (31) for the $S^{(1)}$, we find that $h_{\ell-1}$ is arbitrary; and we can express any $h_{\ell-g}$ with odd g in terms of the $h_{\ell-1}$. Furthermore, all $h_{\ell-g}$ with g even are zero; and all $h_{\ell+g}$ with g either even or odd and $(\ell+g)<0$ are zero.

From (45) we may write explicitly equations for the various h's:

$$\left.\begin{array}{c}
(4\ell-2)h_{\ell-3}+(\ell-\frac{1}{2}-b)h_{\ell-1}=0\\
(8\ell-12)h_{\ell-5}+\frac{1}{4}(\ell-1)(\ell-2)h_{\ell-1}+(\ell-\frac{5}{2}-b)h_{\ell-3}=0\\
(12\ell-30)h_{\ell-7}+\frac{1}{4}(\ell-3)(\ell-4)h_{\ell-3}+(\ell-\frac{9}{2}-b)h_{\ell-5}=0\\
\vdots \qquad\qquad\qquad \vdots\\
(\ell-1)(\ell+2)h_0+3h_4+(\frac{5}{2}-b)h_2=0\\
\frac{1}{2}h_2+(\frac{1}{2}-b)h_0=0
\end{array}\right\} \tag{46}$$

These equations are for odd l ; and hence, correspond to (34). If we form the determinant of the coefficients of the h's of (46), we find that the equation for the separation constant b is of degree $\frac{l+1}{2}$. We may write a similar set of equations for even l ; however, the equation for b is of degree $\frac{l}{2}$. As a result the second type function is of the form

$$S^{(2)} = \left(\tfrac{1}{2}+u^2\right)^{\frac{1}{2}}\left(h_{l-1}u^{l-1} + h_{l-3}u^{l-3} + h_{l-5}u^{l-5} + \cdots\right). \tag{47}$$

We have used the letter P for the polynomial of (47), since it is identical in form with the polynomial of $S^{(1)}$.

The third-type solution which we attempt to make satisfy (17) is

$$S^{(3)} = \left(\tfrac{1}{2}-u^2\right)^{\frac{1}{2}}\left(\tfrac{1}{2}+u^2\right) Q, \tag{48}$$

where Q is a function of u which must be determined. Substitution of (48) in (17) yields the differential equation

$$\left(\tfrac{1}{4}-u^4\right)\frac{d^2Q}{du^2} - 6u^3\frac{dQ}{du} + \left[-b + (l+3)(l-2)u^2\right] Q = 0. \tag{49}$$

Let us assume for Q the power series

$$Q = \sum_{m=0}^{\infty} h_m u^m \tag{50}$$

If we substitute (50) in (49), we find the following recurrence relation for the h's:

$$\tfrac{1}{4}(m+4)(m+3)h_{m+4} - \left[m(m-1) + 6m - (l+3)(l-2)\right] h_m - b\, h_{m+2} = 0. \tag{51}$$

We shall attempt to express m in terms of ℓ ; and hence, express the powers of u in terms of ℓ . For this purpose we let $m = \ell + g$ in (51) and find

$$\frac{1}{4}(\ell+g+4)(\ell+g+3)h_{\ell+g+4} - [(\ell+g)(\ell+g-1)+6(\ell+g)-(\ell+3)(\ell-2)]h_{\ell+g} - b h_{\ell+g+2} = 0. \tag{52}$$

We shall suppose that we may express Q either as a series whose lowest power is ℓ or as a polynomial of a degree less than ℓ . If Q is expressible as the series, then all $h_{\ell+g}$ with $(\ell+g)<\ell$ are zero. In (52) let $g=-4$ and find

$$\frac{1}{4}(\ell)(\ell-1)h_{\ell} - (2-4\ell)h_{\ell-4} - b h_{\ell-2} = 0.$$

According to our supposition, $h_{\ell-4}$ and $h_{\ell-2}$ are zero. Hence, $h_{\ell}=0$. For $g=-3$

$$\frac{1}{4}(\ell+1)(\ell)h_{\ell+1} + 2\ell h_{\ell-3} - b h_{\ell-1} = 0$$

or $h_{\ell+1}=0$.

With $g=-2$

$$\frac{1}{4}(\ell+2)(\ell+1)h_{\ell+2} - b h_{\ell} = 0$$

, and

$$h_{\ell+2}=0.$$

For $\gamma = -1$

$$\frac{1}{4}(\ell+3)(\ell+2)h_{\ell+3} - (2\ell+2)h_{\ell-1} - bh_{\ell+1} = 0.$$

Since $h_{\ell+1} = 0$, $h_{\ell+3} = 0$.

For $\gamma = 0$

$$\frac{1}{4}(\ell+4)(\ell+3)h_{\ell+4} - (4\ell+6)h_{\ell} - b\,h_{\ell+2} = 0.$$

Since $h_{\ell} = h_{\ell+2} = 0$, $h_{\ell+4} = 0$.

From the h's which we have shown to be zero and the relation (52) one sees that all $h_{\ell+\gamma}$ with $(\ell+\gamma) \geq \ell$ are zero. Hence, our attempt to express Q as a series whose lowest power is ℓ yields the result $Q = 0$. We turn next to the polynomial representation of Q.

If Q is to be a polynomial of a degree less than ℓ, then all $h_{\ell+\gamma}$ with $(\ell+\gamma) \geq \ell$ must be zero. Then if $\gamma = -1$, we find that $h_{\ell-1}$ is zero. For $\gamma = -2$ equation (52) yields

$$[(\ell-2)(\ell-3) + 6(\ell-2) - (\ell+3)(\ell-2)]h_{\ell-2} = 0.$$

However,

$$[(\ell-2)(\ell-3) + 6(\ell-2) - (\ell+3)(\ell-2)] \equiv 0.$$

Hence, $h_{\ell-2}$ is arbitrary; and we assume it is not zero. With $\gamma = -3$, we find from (52)

$$-2\ell h_{\ell-3} + bh_{\ell-1} = 0.$$

Since $h_{\ell-1} = 0$, $h_{\ell-3}$ is zero. For $\gamma = -4$,

$$(2-4\ell)h_{\ell-4} + bh_{\ell-2} = 0.$$

With this equation we are able to express $h_{\ell-4}$ in terms of $h_{\ell-2}$.

If in the recurrence relation (52) we consider only even γ with $(\gamma < -4)$ then it follows from (52) and the results which we have found for $\gamma = -2$ and $\gamma = -4$ that we can determine all $h_{\ell+\gamma}$ with $(\ell+\gamma) < \ell$ and γ even. All of the $h_{\ell+\gamma}$ so determined can be expressed in terms of $h_{\ell-2}$. On the other hand, if we allow only odd γ in (52) with $\gamma < -3$, then it follows from (52) and the fact that $h_{\ell-1} = h_{\ell-3} = 0$ that all $h_{\ell-\gamma}$ with γ odd are zero. Consequently, we need consider only even γ.

Since Q is to be a polynomial, all $h_{\ell+\gamma}$ with $(\ell+\gamma) < 0$ must be zero. First, suppose ℓ is even. In (52) let $\gamma = -(\ell+2)$. We find

$$\frac{1}{2}h_2 + \ell(\ell+1)h_{-2} - bh_0 = 0.$$

If $h_{-2} = 0$,

$$\frac{1}{2}h_2 - bh_0 = 0.$$

We find h_2 and h_0 in terms of the arbitrary $h_{\ell-2}$. If these values are substituted in (52), the result is an equation for b. The roots of this equation are values of b which make $h_{-2} = 0$. Next, let $\gamma = -(\ell+4)$. Then (52) becomes

$$\frac{1}{4}(0)(-1)h_0 + (\ell+2)(\ell-1)h_{-4} - bh_{-2} = 0.$$

The coefficient of h_0 is zero; and if $h_{-2}=0$, then h_{-4} is also zero. Now consider all γ of the form $-(\ell+r)$ where r is even and $r>4$. The recurrence relation (52) becomes

$$\frac{1}{4}(4-r)(3-r)h_{-r+4}$$
$$-[r(r+1)-6r-(\ell+3)(\ell-2)]h_{-r}-bh_{2-r}=0. \tag{53}$$

From (53) and the fact that $h_{-2}=h_{-4}=0$, it follows that all $h_{\ell+\gamma}$ with $(\ell+\gamma)<0$ and even ℓ vanish.

If ℓ is odd, let $\gamma=-(\ell+1)$ in (52). The result is

$$\frac{3}{2}h_3+(\ell+2)(\ell-1)h_{-1}-bh_1=0.$$

If h_{-1} is zero,

$$\frac{3}{2}h_3-bh_1=0.$$

We express h_3 and h_1 in terms of the arbitrary $h_{\ell-2}$ and find an equation for b. By solving this equation, we find values of b which make h_{-1} zero. By setting $\gamma=-(\ell+3)$ in (52), we find

$$\frac{1}{4}(1)(0)h_1+\ell(\ell+1)h_{-3}-bh_{-1}=0.$$

The coefficient of h_1 vanishes; and if h_{-1} is zero, then h_{-3} is also zero. Finally, let γ be of the form $-(\ell+r)$ with r odd and $(r>3)$. Equation (52) gives

$$\frac{1}{4}(4-r)(3-r)h_{4-r}$$
$$-[r(r+1)-6r-(\ell+3)(\ell-2)]h_{-r}-bh_{2-r}=0.$$

This is equation (53). Since $h_{-1} = h_{-3} = 0$, it follows that all $h_{\ell+g}$ with odd ℓ and $(\ell+g) < 0$ are zero.

By means of (52) we may write explicitly equations for the various h_m's. With ℓ even we obtain the following set of equations:

$$\left.\begin{array}{c}(2-4\ell)h_{\ell-4}+bh_{\ell-2}=0\\ \frac{1}{4}(\ell-2)(\ell-3)h_{\ell-2}-(12-8\ell)h_{\ell-6}-bh_{\ell-4}=0\\ \frac{1}{4}(\ell-4)(\ell-5)h_{\ell-4}-(30-12\ell)h_{\ell-8}-bh_{\ell-6}=0\\ \frac{1}{4}(\ell-6)(\ell-7)h_{\ell-6}-(56-16\ell)h_{\ell-10}-bh_{\ell-8}=0\\ \vdots \qquad\qquad \vdots\\ 3h_4+(\ell+3)(\ell-2)h_0-bh_2=0\\ \frac{1}{2}h_2-bh_0=0\end{array}\right\} \qquad (54)$$

Equation (54) is a set of linear homogeneous equations for the h_m's. The determinant of the coefficients of the h_m's must be zero. Fig. 4 shows this determinant which has $\frac{\ell}{2}$ rows and $\frac{\ell}{2}$ columns. Since the separation constant b appears only on the diagonal, the equation for b, which we must solve in order to make the proper h_m zero, is of degree $\frac{\ell}{2}$.

From the determinant of Fig. 4 we may find a finite, continued fraction for b. Let Δ be the determinant and Δ_i the determinant formed from Δ by deleting the first i rows and columns. Then

$$\Delta = b\Delta_1 - \frac{1}{4}(\ell-2)(\ell-3)(2-4\ell)\Delta_2 = 0,$$

$$
\begin{vmatrix}
b & (2-4\ell) & 0 & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 \\
\frac{1}{4}(\ell-2)(\ell-3) & -b & -(12-8\ell) & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 \\
0 & \frac{1}{4}(\ell-4)(\ell-5) & -b & -(30-12\ell) & 0 & 0\ \cdots\ 0 & 0 & 0 \\
0 & 0 & \frac{1}{4}(\ell-6)(\ell-7) & -b & -(56-16\ell) & 0\ \cdots\ 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0\ \cdots\ 3 & -b & (\ell+3)(\ell-2) \\
0 & 0 & 0 & 0 & 0 & 0\ \cdots\ 0 & \frac{1}{2} & -b
\end{vmatrix}
$$

Fig. 4 Determinant formed from the coefficients of equation (54).

or

$$b = \frac{1}{4}(l-2)(l-3)(2-4l)\frac{\Delta_2}{\Delta_1}\ . \tag{55}$$

Solving for Δ_1 in terms of Δ_2 and Δ_3 , we find

$$\Delta_1 = -b\Delta_2 + \frac{1}{4}(l-4)(l-5)(12-8l)\Delta_3$$

$$\frac{\Delta_1}{\Delta_2} = -b + \frac{1}{4}(l-4)(l-5)(12-8l)\frac{\Delta_3}{\Delta_2}\ .$$

Then

$$\frac{\Delta_2}{\Delta_1} = \frac{-1}{b - \frac{1}{4}(l-4)(l-5)(12-8l)\frac{\Delta_3}{\Delta_2}}\ . \tag{56}$$

If (56) is substituted in (55), the result is

$$b = \frac{-\frac{1}{4}(l-2)(l-3)(2-4l)}{b - \frac{1}{4}(l-4)(l-5)(12-8l)\frac{\Delta_3}{\Delta_2}} \tag{57}$$

Similarly, we find

$$\frac{\Delta_3}{\Delta_2} = \frac{-1}{b - \frac{1}{4}(l-6)(l-7)(30-12l)\frac{\Delta_4}{\Delta_3}} \tag{58}$$

Substitution of (58) in (57) yields

$$b = \cfrac{-\frac{1}{4}(l-2)(l-3)(2-4l)}{b + \cfrac{\frac{1}{4}(l-4)(l-5)(12-8l)}{b - \frac{1}{4}(l-6)(l-7)(30-12l)\frac{\Delta_4}{\Delta_3}}} \tag{59}$$

We may continue to solve for the ratios of the consecutive Δ_i's.

The continued function (59) may be written as

$$b = \frac{-\frac{1}{4}(l-2)(l-3)(2-4l)}{b\ +}\ \frac{\frac{1}{4}(l-4)(l-5)(12-8l)}{b\ +}\cdots \tag{60}$$

For odd l the equations which correspond to (54) are:

$$\left.\begin{aligned}
(2-4l)h_{l-4}+bh_{l-2}&=0\\
\tfrac{1}{4}(l-2)(l-3)h_{l-2}-(12-8l)h_{l-6}-bh_{l-4}&=0\\
\tfrac{1}{4}(l-4)(l-5)h_{l-4}-(30-12l)h_{l-8}-bh_{l-6}&=0\\
\tfrac{1}{4}(l-6)(l-7)h_{l-6}-(56-16l)h_{l-10}-bh_{l-8}&=0\\
\\
5h_5+(l+4)(l-3)h_1-bh_3&=0\\
\tfrac{3}{2}h_3-bh_1&=0\,.
\end{aligned}\right\} \qquad (61)$$

In this case there are $\frac{l-1}{2}$ of the h_m's and $\frac{l-1}{2}$ equations from which to determine them. A determinant formed from the coefficients of the h's of (61) shows that the equation for the separation constant b is of degree $\frac{l-1}{2}$.

Consequently, we conclude that the function Q is a polynomial of degree $(l-2)$ and of the form

$$Q=h_{l-2}u^{l-2}+h_{l-4}u^{l-4}+h_{l-6}u^{l-6}+\cdots\,.$$

The third-type function becomes

$$S^{(3)}=(\tfrac{1}{2}-u^2)^{\frac{1}{2}}(\tfrac{1}{2}+u^2)^{\frac{1}{2}}(h_{l-2}u^{l-2}+h_{l-4}u^{l-4}+\cdots)\,. \qquad (62)$$

Finally, the fourth-type function which we shall introduce as a solution of (17) is a power series of the form

$$S=\sum_{n=0}^{\infty}h_n u^n.$$

Substitution of the power series in (17) the recurrence relation

$$\frac{1}{4}(m+4)(m+3)h_{m+4}-bh_{m+2}-[m(m+1)-l(l+1)]h_m=0. \tag{63}$$

We shall express the powers of the variable u in terms of l ; hence, we replace m in (63) by $(l+\gamma)$. Then (63) becomes

$$\frac{1}{4}(l+\gamma+4)(l+\gamma+3)h_{l+\gamma+4}-bh_{l+\gamma+2}$$

$$-[(l+\gamma)(l+\gamma+1)-l(l+1)]h_{l+\gamma}=0. \tag{64}$$

Suppose we write (64) for some specific values of γ :

$\gamma=0$

$$\frac{1}{4}(l+4)(l+3)h_{l+4}-bh_{l+2}-[l(l+1)-l(l+1)]h_l=0 \tag{65a}$$

$\gamma=-1$

$$\frac{1}{4}(l+3)(l+2)h_{l+3}-bh_{l+1}-[(l-1)(l)-l(l+1)]h_{l-1}=0 \tag{65b}$$

$\gamma=-2$

$$\frac{1}{4}(l+2)(l+1)h_{l+2}-bh_l-[(l-2)(l-1)-l(l+1)]h_{l-2}=0 \tag{65c}$$

$\gamma=-3$

$$\frac{1}{4}(l+1)(l)h_{l+1}-bh_{l-1}-[(l-3)(l-2)-l(l+1)]h_{l-3}=0 \tag{65d}$$

$\gamma=-4$

$$\frac{1}{4}(l)(l-1)h_l-bh_{l-2}-[(l-4)(l-3)-l(l+1)]h_{l-4}=0 \tag{65e}$$

As before, we shall attempt to express this solution either as a series whose lowest power we must determine or as a polynomial. A study of equations (65a,b,c,d,e) shows that a polynomial of a degree less than ℓ leads to the result that both the polynomial solution and the series solution are zero.

From (65a) we see that the coefficient of h_ℓ is identically zero; and hence, we shall try a polynomial of degree ℓ . Then according to (65a), h_ℓ is arbitrary; and we assume that it is not zero. From (65c) we may express $h_{\ell-2}$ in terms of h_ℓ ; and from (65b), $h_{\ell-1}$ is zero. With $h_{\ell-1}$ zero, (65d) shows that $h_{\ell-3}$ is zero. Then if we consider only odd g with $(g<-3)$ in (64), since $h_{\ell-1}=h_{\ell-3}=0$, all h's of the form $h_{\ell-p}$ with p odd are zero.

Since we can express $h_{\ell-2}$ in terms of h_ℓ , we see from (65e) that $h_{\ell-4}$ may be expressed in terms of h_ℓ . Now let g be even with $(g<-4)$ in (64). As a result we are able to find the h's of the form $h_{\ell-p}$ with p even in terms of the arbitrary, non-zero h_ℓ . Consequently, we consider only even g .

If the function is a polynomial, all $h_{\ell+g}$ with $(\ell+g)<0$ must be zero. For even ℓ let $g=-(\ell+2)$ and substitute for g in (64). The result is

$$\frac{1}{2}h_2-bh_0+(\ell+2)(\ell-1)h_{-2}=0.$$

If h_{-2} is zero,

$$\frac{1}{2}h_2-bh_0=0. \tag{66}$$

We find h_2 and h_0 in terms of the arbitrary h_l and substitute in (66). This yields an equation for the separation constant b, and the roots of this equation are values of b which make h_{-2} zero.

From (64) we can write explicitly a set of equations for the various h's with l even.

$$\left.\begin{aligned}
b h_l + (2-4l) h_{l-2} &= 0 \\
\tfrac{1}{4}(l)(l-1) h_l - b h_{l-2} - (12-8l) h_{l-4} &= 0 \\
\tfrac{1}{4}(l-2)(l-3) h_{l-2} - b h_{l-4} - (30-12l) h_{l-6} &= 0 \\
\tfrac{1}{4}(l-4)(l-5) h_{l-4} - b h_{l-6} - (56-16l) h_{l-8} &= 0 \\
&\;\vdots \\
3 h_4 - b h_2 + l(l+1) h_0 &= 0 \\
\tfrac{1}{2} h_2 - b h_0 &= 0
\end{aligned}\right\} \qquad (67)$$

Here are $\frac{l}{2}+1$ of the h_m's and $\frac{l}{2}+1$ linear, homogeneous equations which involve them. Fig. 5 shows the determinant formed from the coefficients of the h's in (67). We notice that the separation constant b appears only on the diagonal; and since the determinant has $\frac{l}{2}+1$ rows and $\frac{l}{2}+1$ columns, the equation for b is of degree $\frac{l}{2}+1$.

If l is odd, we obtain the following set of equations from (64):

$$\left.\begin{aligned}
b h_{\ell} + (2-4\ell) h_{\ell-2} &= 0 \\
\tfrac{1}{4}(\ell)(\ell-1) h_{\ell} - b h_{\ell-2} - (12-8\ell) h_{\ell-4} &= 0 \\
\tfrac{1}{4}(\ell-2)(\ell-3) h_{\ell-2} - b h_{\ell-4} - (30-12\ell) h_{\ell-6} &= 0 \\
\tfrac{1}{4}(\ell-4)(\ell-5) h_{\ell-4} - b h_{\ell-6} - (56-16\ell) h_{\ell-8} &= 0 \\
&\;\vdots \\
5 h_5 - b h_3 - (\ell+2)(\ell-1) h_1 &= 0 \\
\tfrac{3}{2} h_3 - b h_1 &= 0
\end{aligned}\right\} \qquad (68)$$

Since ℓ is odd, the last equation of this set is found by letting $\gamma = -(\ell+1)$ in (64) and equating h_{-1} to zero. In this case there are $\frac{\ell+1}{2}$ of the h_m's and the same number of linear homogeneous equations which relate the h_m's. A determinant formed from the coefficients of the h_m's of (68) shows that the separation constant b appears only on the diagonal. Hence, for odd ℓ the equation for b is of degree $\frac{\ell+1}{2}$.

From the determinant of Fig. 5 or the determinant formed from (68) we can express b through a continued fraction. We illustrate this fraction for the determinant of Fig. 5. We shall use the same notation as we have for the previous continued fractions.

$$\Delta = b\Delta_1 - \tfrac{1}{4}(\ell)(\ell-1)(2-4\ell)\Delta_2 = 0,$$

or

$$b = \tfrac{1}{4}(\ell)(\ell-1)(2-4\ell)\,\frac{\Delta_2}{\Delta_1}. \qquad (69)$$

$$
\begin{vmatrix}
b & (2-4\ell) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\frac{1}{4}(\ell)(\ell-1) & -b & -(12-8\ell) & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \frac{1}{4}(\ell-2)(\ell-3) & -b & -(30-12\ell) & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4}(\ell-4)(\ell-5) & -b & -(56-16\ell) & & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 3 & -b & \ell(\ell+1) \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & -b
\end{vmatrix}
$$

Fig. 5 Determinant formed from the coefficients of equation (67).

Solving for the ratio $\frac{\Delta_2}{\Delta_1}$, we find

$$\Delta_1 = -b\Delta_2 + \tfrac{1}{4}(\ell-2)(\ell-3)(12-8\ell)\Delta_3$$

$$\frac{\Delta_1}{\Delta_2} = -b + \tfrac{1}{4}(\ell-2)(\ell-3)(12-8\ell)\frac{\Delta_3}{\Delta_2}$$

or

$$\frac{\Delta_2}{\Delta_1} = \frac{1}{-b + \tfrac{1}{4}(\ell-2)(\ell-3)(12-8\ell)\frac{\Delta_3}{\Delta_2}} . \qquad (69a).$$

Substitute (69a) in (69) and find

$$b = \frac{\tfrac{1}{4}(\ell)(\ell-1)(2-4\ell)}{-b + \tfrac{1}{4}(\ell-2)(\ell-3)(12-8\ell)\frac{\Delta_3}{\Delta_2}} . \qquad (69b)$$

Also

$$\Delta_2 = -b\Delta_3 + \tfrac{1}{4}(\ell-4)(\ell-5)(30-12\ell)\Delta_4$$

$$\frac{\Delta_2}{\Delta_3} = -b + \tfrac{1}{4}(\ell-4)(\ell-5)(30-12\ell)\frac{\Delta_4}{\Delta_3}$$

$$\frac{\Delta_3}{\Delta_2} = \frac{1}{-b + \tfrac{1}{4}(\ell-4)(\ell-5)(30-12\ell)\frac{\Delta_4}{\Delta_3}} \qquad (69c)$$

With (69c), equation (69b) becomes

$$b = \cfrac{\tfrac{1}{4}(\ell)(\ell-1)(2-4\ell)}{-b + \cfrac{\tfrac{1}{4}(\ell-2)(\ell-3)(12-8\ell)}{-b + \tfrac{1}{4}(\ell-4)(\ell-5)(30-12\ell)\frac{\Delta_4}{\Delta_3}}} . \qquad (69d)$$

Finally, we write (69d) as

$$b = \frac{\tfrac{1}{4}(\ell)(\ell-1)(2-4\ell)}{-b} \,\underset{+}{}\, \frac{\tfrac{1}{4}(\ell-2)(\ell-3)(12-8\ell)}{-b} \,\underset{+}{}\, \cdots \qquad (70)$$

With $g = -(\ell+4)$, equation (64) yields

$$\tfrac{1}{4}(0)(-1)h_0 - bh_{-2} - [12 - \ell(\ell+1)]h_{-4} = 0.$$

If h_{-2} is zero, this equation shows that $h_{-4} = 0$. Now consider only even ℓ and let $g = -(\ell + r)$ where r is even and $r > 4$. Since $h_{-2} = h_{-4} = 0$, it follows from (64) that all $h_{\ell+g}$ with $(\ell+g) < 0$ and ℓ even are zero. For $g = -(\ell+3)$ we find from (64)

$$\tfrac{1}{4}(1)(0)h_1 - bh_{-1} - [6 - \ell(\ell+1)]h_{-3} = 0.$$

If h_{-1} is zero, then from this equation h_{-3} is also zero. For odd ℓ and $g = -(\ell+r)$ where r is odd and $r > 3$, we see from (64) that all $h_{\ell+g}$ with $(\ell+g) < 0$ and ℓ odd are zero. Therefore, the solution of the fourth type is a polynomial of degree ℓ and is of the form

$$S^{(4)} = h_\ell u^\ell + h_{\ell-2} u^{\ell-2} + h_{\ell-4} u^{\ell-4} + \cdots \quad (71)$$

A summary of our results will be convenient. We have shown the existence of four types of functions which satisfy the differential equation

$$\frac{d^2S}{du^2} - \frac{2u^3}{(\frac{1}{2}-u^2)(\frac{1}{2}+u^2)}\frac{dS}{du} + \frac{[\ell(\ell+1)u^2 - b]S}{(\frac{1}{2}-u^2)(\frac{1}{2}+u^2)} = 0.$$

The functions are:

$$S^{(1)} = (\tfrac{1}{2}-u^2)^{\frac{1}{2}}\left\{h_{\ell-1}u^{\ell-1} + h_{\ell-3}u^{\ell-3} + h_{\ell-5}u^{\ell-5} + \cdots\right\}$$

$$S^{(2)} = (\tfrac{1}{2}+u^2)^{\frac{1}{2}}\left\{h_{\ell-1}u^{\ell-1} + h_{\ell-3}u^{\ell-3} + h_{\ell-5}u^{\ell-5} + \cdots\right\}$$

$$S^{(3)} = (\tfrac{1}{2}-u^2)^{\frac{1}{2}}(\tfrac{1}{2}+u^2)^{\frac{1}{2}}\left\{h_{l-2}u^{l-2}+h_{l-4}u^{l-4}+h_{l-6}u^{l-6}+\cdots\right\}$$

$$S^{(4)} = h_l u^l + h_{l-2}u^{l-2} + h_{l-4}u^{l-4} + \cdots .$$

The expressions in the curly brackets of $S^{(1)}$ and $S^{(2)}$ are polynomials of degree $(l-1)$, and the expression in the curly brackets of $S^{(3)}$ is a polynomial of degree $(l-2)$. The function $S^{(4)}$ is a polynomial of degree l .

Our next consideration will be equation (18)

$$\frac{d^2T}{dv^2} - \frac{2v^3}{(\frac{1}{2}-v^2)(\frac{1}{2}+v^2)}\frac{dT}{dv} + \frac{[l(l+1)v^2+b]T}{(\frac{1}{2}-v^2)(\frac{1}{2}+v^2)} = 0.$$

We notice that except for the sign of the separation constant b equation (18) is identical with (17). We shall introduce four functions $T^{(i)}(v)$ which correspond to the functions $S^{(i)}(u)$. The development of the $T^{(i)}(v)$ is identical with that of the $S^{(i)}(u)$. Hence, we shall be brief and only indicate the results.

The first-type function is of the form

$$T = (\tfrac{1}{2}-v^2)^{\frac{1}{2}} R. \tag{72}$$

Substitution of (72) in (18) yields the equation

$$(\tfrac{1}{4}-v^4)\frac{d^2R}{dv^2} - (4v^3+v)\frac{dR}{dv} + [b-\tfrac{1}{2}+(l-1)(l+2)v^2]R = 0 \tag{73}$$

Let

$$R = \sum_{n=0}^{\infty} g_n v^n. \tag{74}$$

Substitute (74) in (73) and find the recurrence relation

$$\frac{1}{4}(n+4)(n+3)g_{n+4} + \left(b-n-\frac{5}{2}\right)g_{n+2}$$

$$-[n(n-1)+4n-(l-1)(l+2)]\,g_n = 0. \tag{75}$$

With $n = l+q$, (75) becomes

$$\frac{1}{4}(l+q+4)(l+q+3)g_{l+q+4} + \left(b-l-q-\frac{5}{2}\right)g_{l+q+2}$$

$$-[(l+q)(l+q-1)+4(l+q)-(l-1)(l+2)]\,g_{l+q} = 0. \tag{76}$$

For some particular values of q we find from (76) the equations:

$q=0$

$$\frac{1}{4}(l+4)(l+3)g_{l+4} + \left(b-l-\frac{5}{2}\right)g_{l+2} - (2l+2)g_l = 0 \tag{77a}$$

$q=-1$

$$\frac{1}{4}(l+3)(l+2)g_{l+3} + \left(b-l-\frac{3}{2}\right)g_{l+1} - [(l-1)(0)]g_{l-1} = 0. \tag{77b}$$

$q=-2$

$$\frac{1}{4}(l+2)(l+1)g_{l+2} + \left(b-l-\frac{1}{2}\right)g_l + (2l)g_{l-2} = 0. \tag{77c}$$

$q=-3$

$$\frac{1}{4}(l+1)(l)g_{l+1} + \left(b-l+\frac{1}{2}\right)g_{l-1} - (2-4l)g_{l-3} = 0. \tag{77d}$$

$q=-4$

$$\frac{1}{4}(l)(l-1)g_l + \left(b-l+\frac{3}{2}\right)g_{l-2} - (6-6l)g_{l-4} = 0. \tag{77e}$$

From a study of equations (77a, b, c, d, e) and by means of (76) one may establish the following results:

(1) The function R is expressible as a polynomial of degree $(\ell-1)$, and an attempt to find R as a series whose lowest power is ℓ yields the result $R=0$.

(2) All $g_{\ell+q}$ of the form $g_{\ell-p}$ with p even are zero.

(3) All $g_{\ell+q}$ of the form $g_{\ell-p}$ with p odd are expressible in terms of the arbitrary $g_{\ell-1}$.

(4) All $g_{\ell+q}$ with $(\ell+q)<0$ are zero if we make g_{-2} zero for odd ℓ and g_{-1} zero for even ℓ.

(5) The vanishing of g_{-2} and g_{-1} is accomplished through the determination of proper values of b.

(6) The equation for b is of degree $\frac{\ell+1}{2}$ if ℓ is odd and of degree $\frac{\ell}{2}$ if ℓ is even.

For particular values of q we may obtain from (76) the following set of equations for the g's if ℓ is odd:

$$\left.\begin{array}{c}
(b-\ell+\frac{1}{2})g_{\ell-1}-(2-4\ell)g_{\ell-3}=0 \\
\frac{1}{4}(\ell-1)(\ell-2)g_{\ell-1}+(b-\ell+\frac{5}{2})g_{\ell-3}-(12-8\ell)g_{\ell-5}=0 \\
\frac{1}{4}(\ell-3)(\ell-4)g_{\ell-3}+(b-\ell+\frac{9}{2})g_{\ell-5}-(30-12\ell)g_{\ell-7}=0 \\
\frac{1}{4}(\ell-5)(\ell-6)g_{\ell-5}+(b-\ell+\frac{13}{2})g_{\ell-7}-(56-16\ell)g_{\ell-9}=0 \\
\vdots \qquad\qquad\qquad \vdots \\
3g_4+(b-\frac{5}{2})g_2+(\ell-1)(\ell+2)g_0=0 \\
\frac{1}{2}g_2+(b-\frac{1}{2})g_0=0.
\end{array}\right\} \qquad (78)$$

$$
\begin{vmatrix}
b-\ell+\frac{1}{2} & -(2-4\ell) & 0 & 0 & 0 \;\cdots & 0 & 0 & 0 \\
\frac{1}{4}(\ell-1)(\ell-2) & (b-\ell+\frac{5}{2}) & -(12-8\ell) & 0 & 0 \;\cdots & 0 & 0 & 0 \\
0 & \frac{1}{4}(\ell-3)(\ell-4) & (b-\ell+\frac{9}{2}) & -(30-12\ell) & 0 \;\cdots & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4}(\ell-5)(\ell-6) & (b-\ell+\frac{13}{2}) & -(56-16\ell)\cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 \;\cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot & 3 & (b-\frac{5}{2}) & (\ell-1)(\ell+2) \\
0 & 0 & 0 & 0 & 0 \;\cdots & 0 & \frac{1}{2} & (b-\frac{1}{2})
\end{vmatrix}
$$

Fig. 6 Determinant formed from the coefficients of equation (78).

Fig. 6 shows the determinant of the coefficients of the g's of (78). We notice that the separation constant b appears only on the diagonal. The finite, continued fraction for b, which may be obtained from Fig. 5 is

$$b = (\ell - \tfrac{1}{2}) + \frac{-\frac{1}{4}(\ell-1)(\ell-2)(2-4\ell)}{(b-\ell+\frac{5}{2})} \; + \; \frac{\frac{1}{4}(\ell-3)(\ell-4)(12-8\ell)}{(b-\ell+\frac{9}{2})} \; + \cdots \tag{79}$$

We can find a similar result for even ℓ. Consequently, the first-type function which satisfies (18) is

$$T^{(1)} = (\tfrac{1}{2} - v^2)^{\frac{1}{2}} (g_{\ell-1} v^{\ell-1} + g_{\ell-3} v^{\ell-3} + g_{\ell-5} v^{\ell-5} + \cdots). \tag{80}$$

The second-type function is of the form

$$T = (\tfrac{1}{2} + v^2)^{\frac{1}{2}} R \tag{81}$$

By substituting (81) in (18), one finds the equation

$$(\tfrac{1}{4} - v^4)\frac{d^2R}{dv^2} + (v - 4v^3)\frac{dR}{dv} + \left[b + \tfrac{1}{2} + (\ell-1)(\ell+2)v^2\right]R = 0 \tag{82}$$

Let

$$R = \sum_{m=0}^{\infty} g_m v^m$$ and substitute for R in (82). This gives the recurrence relation

$$\tfrac{1}{4}(m+4)(m+3)g_{m+4} + (m+b+\tfrac{5}{2})g_{m+2} - \left[m(m-1) + 4m - (\ell-1)(\ell+2)\right]g_m = 0 \tag{83}$$

With $m = (l+g)$, (83) becomes

$$\frac{1}{4}(l+g+4)(l+g+3)g_{l+g+4} + (l+g+b+\tfrac{5}{2})g_{l+g+2}$$

$$-[(l+g)(l+g-1)+4(l+g)-(l-1)(l+2)]g_{l+g} = 0. \tag{84}$$

From equation (84) and a set of equations which corresponds to (77a, b, c, d, e) one may draw the same conclusions (1 to 6, p. 38) which we established for the function $T^{(1)}(v)$.

If l is odd, we may write the following equations:

$$\left.\begin{aligned}
(b+l-\tfrac{1}{2})g_{l-1} - (2-4l)g_{l-3} &= 0 \\
\tfrac{1}{4}(l-1)(l-2)g_{l-1} + (l+b-\tfrac{5}{2})g_{l-3} - (12-8l)g_{l-5} &= 0 \\
\tfrac{1}{4}(l-3)(l-4)g_{l-3} + (l+b-\tfrac{9}{2})g_{l-5} - (30-12l)g_{l-7} &= 0 \\
\tfrac{1}{4}(l-5)(l-6)g_{l-5} + (l+b-\tfrac{13}{2})g_{l-7} - (56-16l)g_{l-9} &= 0 \\
\vdots \qquad\qquad & \\
3g_4 + (b+\tfrac{5}{2})g_2 + (l-1)(l+2)g_0 &= 0 \\
\tfrac{1}{2}g_2 + (b+\tfrac{1}{2})g_0 &= 0.
\end{aligned}\right\} \tag{85}$$

From (85) one can write a determinant which is similar to that of Fig. 6. The determinant shows the degree of the equation for the separation constant b and provides a means of expressing b through a finite, continued fraction. Hence, the second-type function is

$$T^{(2)} = (\tfrac{1}{2}+v^2)^{\frac{1}{2}}(g_{l-1}v^{l-1} + g_{l-3}v^{l-3} + g_{l-5}v^{l-5} + \cdots) \tag{86}$$

The third-type function which we shall consider corresponds to $S^{(3)}_{\ell}(u)$ and is of the form

$$T = \left(\tfrac{1}{2}+v^2\right)^{\frac{1}{2}}\left(\tfrac{1}{2}-v^2\right)^{\frac{1}{2}} N. \tag{87}$$

Substitution of (87) in (18) yields the equation

$$\left(\tfrac{1}{4}-v^4\right)\frac{d^2N}{dv^2} - 6v^3\frac{dN}{dv} + \left[b + (\ell+3)(\ell-2)v^2\right]N = 0. \tag{88}$$

Let

$$N = \sum_{n=0}^{\infty} g_n v^n$$

and obtain from (88) the recurrence relation

$$\tfrac{1}{4}(n+4)(n+3)g_{n+4} + b\,g_{n+2}$$

$$-\left[n(n-1) + 6n - (\ell+3)(\ell-2)\right]g_n = 0. \tag{89}$$

With $n = \ell+\gamma$, we find for (89)

$$\tfrac{1}{4}(\ell+\gamma+4)(\ell+\gamma+3)g_{\ell+\gamma+4} + b\,g_{\ell+\gamma+2}$$

$$-\left[(\ell+\gamma)(\ell+\gamma-1) + 6(\ell+\gamma) - (\ell+3)(\ell-2)\right]g_{\ell+\gamma} = 0. \tag{90}$$

For some particular values of γ, we find from (90) the following equations:

$\gamma = 0$

$$\tfrac{1}{4}(\ell+4)(\ell+3)g_{\ell+4} + b\,g_{\ell+2} - (4\ell+6)g_\ell = 0 \tag{91a}$$

$\gamma = -1$

$$\tfrac{1}{4}(\ell+3)(\ell+2)g_{\ell+3} + b\,g_{\ell+1} - (2\ell+2)g_{\ell-1} = 0. \tag{91b}$$

$q = -2$

$$\frac{1}{4}(\ell+2)(\ell+1)g_{\ell+2} + b g_\ell - (0) g_{\ell-2} = 0 \tag{91c}$$

$q = -3$

$$\frac{1}{4}(\ell+1)(\ell)g_{\ell+1} + b g_{\ell-1} + (2\ell) g_{\ell-3} = 0 \tag{91d}$$

$q = -4$

$$\frac{1}{4}(\ell)(\ell-1)g_{\ell} + b g_{\ell-2} - (2-4\ell) g_{\ell-4} = 0 \tag{91e}$$

From equations (91a, b, c, d, e) and (90), one may draw the following conclusions:

(1) The function $N(v)$ is a polynomial of degree $(\ell-2)$. An attempt to express $N(v)$ as a series whose lowest power is ℓ yields the result that N is zero.

(2) All $g_{\ell+q}$ of the form $g_{\ell-p}$ with odd p are zero.

(3) All $g_{\ell+q}$ of the form $g_{\ell-p}$ with p even can be expressed in terms of the arbitrary $g_{\ell-2}$.

(4) All $g_{\ell+q}$ with $(\ell+q) < 0$ vanish if g_{-2} is zero for even ℓ and if g_{-1} is zero for odd ℓ.

(5) The vanishing of g_{-2} and g_{-1} is accomplished through the determination of proper values of b.

(6) The equation for b is of degree $\frac{\ell}{2}$ if ℓ is even and of degree $\frac{\ell-1}{2}$ if ℓ is odd.

The recurrence relation (90) allows us to find the following set of equations for even ℓ :

$$
\begin{vmatrix}
b & -(2-4\ell) & 0 & 0 & 0 \cdots\cdots & 0 & 0 & 0 \\
\frac{1}{4}(\ell-2)(\ell-3) & b & -(12-8\ell) & 0 & 0 \cdots\cdots & 0 & 0 & 0 \\
0 & \frac{1}{4}(\ell-4)(\ell-5) & b & -(30-12\ell) & 0 \cdots\cdots & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4}(\ell-6)(\ell-7) & b & -(56-16\ell) \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 \cdots\cdots & 3 & b & (\ell+3)(\ell-2) \\
0 & 0 & 0 & 0 & 0 \cdots\cdots & 0 & \frac{1}{2} & b
\end{vmatrix}
$$

Fig. 7 Determinant formed from the coefficients of equation (92).

$$\left.\begin{array}{c} b g_{l-2} - (2-4l) g_{l-4} = 0 \\ \frac{1}{4}(l-2)(l-3) g_{l-2} + b g_{l-4} - (12-8l) g_{l-6} = 0 \\ \frac{1}{4}(l-4)(l-5) g_{l-4} + b g_{l-6} - (30-12l) g_{l-8} = 0 \\ \frac{1}{4}(l-6)(l-7) g_{l-6} + b g_{l-8} - (56-16l) g_{l-10} = 0 \\ \vdots \qquad\qquad \vdots \\ 3 g_4 + b g_2 + (l+3)(l-2) g_0 = 0 \\ \frac{1}{2} g_2 + b g_0 = 0 \end{array}\right\} \tag{92}$$

Fig. 7 is the determinant formed from the coefficients of the g's of (92). The separation constant b appears only on the diagonal. A finite, continued fraction for b, which may be found from (92) is

$$b = \frac{-\frac{1}{4}(l-2)(l-3)(2-4l)}{b} \underset{+}{} \frac{\frac{1}{4}(l-4)(l-5)(12-8l)}{b} \underset{+}{} \cdots \tag{93}$$

We conclude that the function of the third type is

$$T^{(3)} = (\tfrac{1}{2}+v^2)^{\frac{1}{2}}(\tfrac{1}{2}-v^2)^{\frac{1}{2}}(g_{l-2}v^{l-2} + g_{l-4}v^{l-4} + g_{l-6}v^{l-6} + \cdots). \tag{94}$$

We begin the development of the fourth function with the power series

$$T = \sum_{n=0}^{\infty} g_n v^n$$

. Substitution of the series in (18) gives the recurrence relation

$$\frac{1}{4}(n+4)(n+3) g_{n+4} - [n(n+1) - l(l+1)] g_n + b g_{n+2} = 0. \tag{95}$$

Once again setting $m = \ell + \gamma$, we find from (95) the equation

$$\frac{1}{4}(\ell+\gamma+4)(\ell+\gamma+3)\,g_{\ell+\gamma+4}$$

$$-\left[(\ell+\gamma)(\ell+\gamma+1)-\ell(\ell+1)\right]g_{\ell+\gamma}+b\,g_{\ell+\gamma+2}=0. \tag{96}$$

From (96) with particular values of γ we may write the following equations:

$\gamma = 0$

$$\frac{1}{4}(\ell+4)(\ell+3)\,g_{\ell+4}-\left[\ell(\ell+1)-\ell(\ell+1)\right]g_{\ell}+b\,g_{\ell+2}=0. \tag{97a}$$

$\gamma = -1$

$$\frac{1}{4}(\ell+3)(\ell+2)\,g_{\ell+3}-\left[(\ell-1)(\ell)-\ell(\ell+1)\right]g_{\ell-1}+b\,g_{\ell+1}=0. \tag{97b}$$

$\gamma = -2$

$$\frac{1}{4}(\ell+2)(\ell+1)\,g_{\ell+2}-\left[(\ell-2)(\ell-1)-\ell(\ell+1)\right]g_{\ell-2}+b\,g_{\ell}=0. \tag{97c}$$

$\gamma = -3$

$$\frac{1}{4}(\ell+1)(\ell)\,g_{\ell+1}-\left[(\ell-3)(\ell-2)-\ell(\ell+1)\right]g_{\ell-3}+b\,g_{\ell-1}=0. \tag{97d}$$

$\gamma = -4$

$$\frac{1}{4}(\ell)(\ell-1)\,g_{\ell}-\left[(\ell-4)(\ell-3)-\ell(\ell+1)\right]g_{\ell-4}+b\,g_{\ell-2}=0. \tag{97e}$$

Equations (97a, b, c, d, e) together with equation (96) allow us to conclude:

(1) The fourth-type function $T^{(4)}(v)$ can be expressed as a polynomial of degree ℓ .

(2) All $g_{\ell+\gamma}$ of the form $g_{\ell-p}$ with p odd are zero.

(3) All $g_{\ell+\gamma}$ of the form $g_{\ell-p}$ with p even are expressible in terms of the arbitrary g_{ℓ} .

$$
\begin{vmatrix}
b & -(2-4\ell) & 0 & 0 & 0 \cdots & 0 & 0 & 0 \\
\frac{1}{4}(\ell)(\ell-1) & b & -(12-8\ell) & 0 & 0 \cdots & 0 & 0 & 0 \\
0 & \frac{1}{4}(\ell-2)(\ell-3) & b & -(30-12\ell) & 0 \cdots & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4}(\ell-4)(\ell-5) & b & -(56-16\ell) \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 \cdots & 0 & 3 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot & 3 & b & \ell(\ell+1) \\
0 & 0 & 0 & 0 & 0 \cdots & 0 & \frac{1}{2} & b
\end{vmatrix}
$$

Fig. 8 Determinant formed from the coefficients of equation (98).

(4) All $g_{\ell+q}$ with $(\ell+q) < 0$ are zero if g_{-2} is zero for even ℓ and g_{-1} is zero for odd ℓ .

(5) Proper values of the separation constant b make g_{-2} and g_{-1} vanish.

(6) The equation for b is of degree $\frac{\ell}{2}+1$ if ℓ is even and of degree $\frac{\ell+1}{2}$ if ℓ is odd.

By assigning q different values in (96), one may obtain the following equations if ℓ is even:

$$\left.\begin{array}{c} b g_{\ell} - (2-4\ell) g_{\ell-2} = 0 \\ \frac{1}{4}(\ell)(\ell-1) g_{\ell} + b g_{\ell-2} - (12-8\ell) g_{\ell-4} = 0 \\ \frac{1}{4}(\ell-2)(\ell-3) g_{\ell-2} + b g_{\ell-4} - (30-12\ell) g_{\ell-6} = 0 \\ \vdots \qquad\qquad \vdots \\ 3 g_4 + b g_2 + \ell(\ell+1) g_0 = 0 \\ \frac{1}{2} g_2 + b g_0 = 0 \end{array}\right\} \tag{98}$$

A determinant formed from the coefficients of the g 's of (98) appears in Fig. 8. As we have seen before, the separation constant b appears only on the diagonal. A finite, continued fraction for b is given by the expression

$$b = \frac{-\frac{1}{4}(\ell)(\ell-1)(2-4\ell)}{b\ +}\ \frac{\frac{1}{4}(\ell-2)(\ell-3)(12-8\ell)}{b\ +}\ \cdots \tag{99}$$

Similar results are true for odd ℓ . Consequently, the fourth-type function is

$$T^{(4)} = g_{\ell} v^{\ell} + g_{\ell-2} v^{\ell-2} + g_{\ell-4} v^{\ell-4} + \cdots \tag{100}$$

Let us summarize the results. We have developed functions of four types which satisfy the differential equation (18)

$$\frac{d^2T}{dv^2} - \frac{2v^3}{(\frac{1}{2}-v^2)(\frac{1}{2}+v^2)}\frac{dT}{dv} + \frac{[\ell(\ell+1)v^2+b]}{(\frac{1}{2}-v^2)(\frac{1}{2}+v^2)}T = 0.$$

The functions are:

$$T^{(1)} = (\tfrac{1}{2}-v^2)^{\frac{1}{2}}\left\{g_{\ell-1}v^{\ell-1} + g_{\ell-3}v^{\ell-3} + g_{\ell-5}v^{\ell-5} + \cdots\right\}$$
$$T^{(2)} = (\tfrac{1}{2}+v^2)^{\frac{1}{2}}\left\{g_{\ell-1}v^{\ell-1} + g_{\ell-3}v^{\ell-3} + g_{\ell-5}v^{\ell-5} + \cdots\right\}$$
$$T^{(3)} = (\tfrac{1}{2}-v^2)^{\frac{1}{2}}(\tfrac{1}{2}+v^2)^{\frac{1}{2}}\left\{g_{\ell-2}v^{\ell-2} + g_{\ell-4}v^{\ell-4} + g_{\ell-6}v^{\ell-6} + \cdots\right\}$$
$$T^{(4)} = g_{\ell}v^{\ell} + g_{\ell-2}v^{\ell-2} + g_{\ell-4}v^{\ell-4} + \cdots .$$

The expressions in the curly brackets of $T^{(1)}$ and $T^{(2)}$ are polynomials of degree $(\ell-1)$, and the expression in the curly brackets of $T^{(3)}$ is a polynomial of degree $(\ell-2)$. The function $T^{(4)}$ is a polynomial of degree ℓ.

VI. The Functions $S^{(i)}$ and $T^{(i)}$ for $\ell=2$ and $\ell=3$.

We illustrate the general procedure for finding the functions $S^{(i)}$ and $T^{(i)}$ by calculating them for $\ell=2$ and $\ell=3$. In the work which follows we shall choose the value unity for all arbitrary coefficients. We treat the case $\ell=2$ first.

1. The functions $S^{(1)}$.

The coefficient h_1 of (40) is arbitrary.

From the first equation of (39) we have

$$b + \frac{3}{2} = 0$$

$$b = -\frac{3}{2}$$

Hence,

$$S^{(1)} = u\sqrt{\frac{1}{2} - u^2}$$

2. The functions $S^{(2)}$.

Again h_1 is arbitrary, and the first equation of (46) yields

$$\frac{3}{2} - b = 0$$

$$b = \frac{3}{2}$$

Hence,

$$S^{(2)} = u\sqrt{\frac{1}{2} + u^2}$$

3. The functions $S^{(3)}$.

The coefficient h_0 of (62) is arbitrary. From the first equation of (54) we have

$$b = 0$$

Hence,

$$S^{(3)} = \sqrt{\frac{1}{2} - u^2}\sqrt{\frac{1}{2} + u^2}$$

4. The functions $S^{(4)}$.

The coefficient h_2 of equation (71) is arbitrary.

From the first two equations of (67), we find

$$b h_2 - b h_0 = 0$$

$$\frac{1}{2} h_2 - b h_0 = 0$$

Since h_2 is arbitrary, we solve for h_0 in terms of h_2 :

$$h_0 = \frac{1}{2b} h_2 \quad . \text{ Then}$$

$$b h_2 - \frac{3}{b} h_2 = 0 .$$

We have

$$b_1 = \sqrt{3}$$

$$b_2 = -\sqrt{3}$$

$$h_0 = \pm \frac{\sqrt{3}}{6}$$

Hence,

$$S^{(4)}_{b_1} = u^2 + \frac{\sqrt{3}}{6}$$

$$S^{(4)}_{b_2} = u^2 - \frac{\sqrt{3}}{6} .$$

In order to determine the functions $T^{(i)}$ one proceeds in a similar manner. We shall only indicate the results.

$$T^{(1)} = v\sqrt{\tfrac{1}{2} - v^2} \qquad (b = \tfrac{3}{2})$$

$$T^{(2)} = v\sqrt{\tfrac{1}{2} + v^2} \qquad (b = -\tfrac{3}{2})$$

$$T^{(3)} = \sqrt{\tfrac{1}{2} + v^2}\sqrt{\tfrac{1}{2} - v^2} \qquad (b = 0)$$

$$T^{(4)}_{b_1} = v^2 - \frac{\sqrt{3}}{6} \qquad (b_1 = \sqrt{3})$$

$$T^{(4)}_{b_2} = v^2 + \frac{\sqrt{3}}{6} \qquad (b_2 = -\sqrt{3})$$

We may form the products of the proper S's and T's and obtain the "angular" parts of the wave functions for $\ell = 2$. If "X"$_i$ denote these "angular" solutions we find

$$X_1 = uv\sqrt{\tfrac{1}{2}-u^2}\sqrt{\tfrac{1}{2}+v^2} \qquad (b = -\tfrac{3}{2})$$

$$X_2 = uv\sqrt{\tfrac{1}{2}+u^2}\sqrt{\tfrac{1}{2}-v^2} \qquad (b = \tfrac{3}{2})$$

$$X_3 = \sqrt{\tfrac{1}{2}-u^2}\sqrt{\tfrac{1}{2}+u^2}\sqrt{\tfrac{1}{2}-v^2}\sqrt{\tfrac{1}{2}+v^2} \qquad (b = 0)$$

$$X_4 = \left(u^2 + \tfrac{\sqrt{3}}{6}\right)\left(v^2 - \tfrac{\sqrt{3}}{6}\right) \qquad (b = \sqrt{3})$$

$$X_5 = \left(u^2 - \tfrac{\sqrt{3}}{6}\right)\left(v^2 + \tfrac{\sqrt{3}}{6}\right) \qquad (b = -\sqrt{3})$$

Next, let $\ell = 3$.

1. The functions $S^{(1)}$.

The coefficient h_2 is arbitrary, and equation (34) yields

$$-10h_0 + (b + \tfrac{5}{2})h_2 = 0$$

$$-\tfrac{1}{2}h_2 + (b + \tfrac{1}{2})h_0 = 0$$

Then $h_0 = \dfrac{1}{2(b+\frac{1}{2})}$, and $b^2 + 3b - \frac{15}{4} = 0$.

Therefore, $b_1 = -\frac{3}{2} + \sqrt{6}$ and $h_0 = \frac{1}{10}(1+\sqrt{6})$

while $b_2 = -\frac{3}{2} - \sqrt{6}$ and $h_0 = \frac{1}{10}(1-\sqrt{6})$.

Hence,
$$S^{(1)}_{b_1} = \sqrt{\tfrac{1}{2}-u^2}\left\{u^2 + \tfrac{1}{10}(1+\sqrt{6})\right\}$$

$$S^{(1)}_{b_2} = \sqrt{\tfrac{1}{2}-u^2}\left\{u^2 + \tfrac{1}{10}(1-\sqrt{6})\right\}.$$

2. The functions $S^{(2)}$.

The coefficient h_2 is arbitrary, and from equation (46) we have

$$10h_0 + \left(\tfrac{5}{2} - b\right) h_2 = 0$$

$$\tfrac{1}{2} h_2 + \left(\tfrac{1}{2} - b\right) h_0 = 0 .$$

We find

$$h_0 = -\tfrac{1}{10}\left(\tfrac{5}{2} - b\right) \qquad \text{, and}$$

$$b^2 - 3b - \tfrac{15}{4} = 0 \qquad .$$

Consequently,

$b_1 = \frac{3}{2} + \sqrt{6}$ and $h_0 = -\frac{1}{10}(1 - \sqrt{6})$ while
$b_2 = \frac{3}{2} - \sqrt{6}$ and $h_0 = -\frac{1}{10}(1 + \sqrt{6})$.

Then the functions $S^{(2)}$ become

$$S^{(2)}_{b_1} = \sqrt{\tfrac{1}{2} + u^2}\,\left\{u^2 - \tfrac{1}{10}(1 - \sqrt{6})\right\} \qquad \text{and}$$

$$S^{(2)}_{b_2} = \sqrt{\tfrac{1}{2} + u^2}\,\left\{u^2 - \tfrac{1}{10}(1 + \sqrt{6})\right\} \qquad .$$

3. The functions $S^{(3)}$.

The coefficient h_1 is arbitrary and from (54)

$$b = 0 .$$

Hence,

$$S^{(3)} = u\sqrt{\tfrac{1}{2} - u^2}\,\sqrt{\tfrac{1}{2} + u^2} .$$

4. The functions $S^{(4)}$.

The coefficient h_3 is arbitrary, and we find

$$b h_3 - 10 h_1 = 0$$

$$\frac{3}{2} h_3 - b h_1 = 0.$$

With $h_3 = 1$, these equations yield

$$h_1 = \frac{3}{2b} \qquad \text{and}$$

$$b^2 - 15 = 0 \qquad .$$

Therefore,

$$b_1 = \sqrt{15} \quad \text{and} \quad h_1 = \frac{\sqrt{15}}{10} \qquad \text{while}$$

$$b_2 = -\sqrt{15} \quad \text{and} \quad h_1 = -\frac{\sqrt{15}}{10} \qquad .$$

Consequently, we find for the functions $S^{(4)}$

$$S_{b_1}^{(4)} = u^3 + \frac{\sqrt{15}}{10} u$$

$$S_{b_2}^{(4)} = u^3 - \frac{\sqrt{15}}{10} u.$$

One can find the functions $T^{(i)}$ in a similar manner, and the results are:

$$T_{b_1}^{(1)} = \sqrt{\tfrac{1}{2} - v^2}\left\{v^2 + \tfrac{1}{10}(1 - \sqrt{6})\right\}, \qquad (b = \tfrac{3}{2} + \sqrt{6})$$

$$T_{b_2}^{(1)} = \sqrt{\tfrac{1}{2} - v^2}\left\{v^2 + \tfrac{1}{10}(1 + \sqrt{6})\right\}, \qquad (b = \tfrac{3}{2} - \sqrt{6})$$

$$T_{b_1}^{(2)} = \sqrt{\tfrac{1}{2} + v^2}\left\{v^2 - \tfrac{1}{10}(1 + \sqrt{6})\right\}, \qquad (b = -\tfrac{3}{2} + \sqrt{6})$$

$$T_{b_2}^{(2)} = \sqrt{\tfrac{1}{2} + v^2}\left\{v^2 - \tfrac{1}{10}(1 - \sqrt{6})\right\}, \qquad (b = -\tfrac{3}{2} - \sqrt{6})$$

$$T^{(3)} = v\sqrt{\tfrac{1}{2}+v^2}\sqrt{\tfrac{1}{2}-v^2} \quad , \quad (b=0)$$

$$T_{b_1}^{(4)} = v^3 - \frac{\sqrt{15}}{10}v \quad , \quad (b=\sqrt{15})$$

$$T_{b_2}^{(4)} = v^3 + \frac{\sqrt{15}}{10}v \quad , \quad (b=-\sqrt{15})$$

If we denote the "angular" parts of the wave functions by X_i, then the S's and T's which we have found for $\ell = 3$ yield:

$$X_1 = \sqrt{\tfrac{1}{2}-u^2}\sqrt{\tfrac{1}{2}+v^2}(u^2+c)(v^2-c) \qquad b = -\tfrac{3}{2}+\sqrt{6}$$

$$X_2 = \sqrt{\tfrac{1}{2}-u^2}\sqrt{\tfrac{1}{2}+v^2}(u^2+g)(v^2-g) \qquad b = -\tfrac{3}{2}-\sqrt{6}$$

$$X_3 = \sqrt{\tfrac{1}{2}+u^2}\sqrt{\tfrac{1}{2}-v^2}(u^2-g)(v^2+g) \qquad b = \tfrac{3}{2}+\sqrt{6}$$

$$X_4 = \sqrt{\tfrac{1}{2}+u^2}\sqrt{\tfrac{1}{2}-v^2}(u^2-c)(v^2+c) \qquad b = \tfrac{3}{2}-\sqrt{6}$$

$$X_5 = uv\sqrt{\tfrac{1}{2}-u^2}\sqrt{\tfrac{1}{2}+u^2}\sqrt{\tfrac{1}{2}+v^2}\sqrt{\tfrac{1}{2}-v^2} \qquad b = 0$$

$$X_6 = uv(u^2+a)(v^2-a) \qquad b = \sqrt{15}$$

$$X_7 = uv(u^2-a)(v^2+a) \qquad b = -\sqrt{15}$$

In the preceding equations,

$$c = \tfrac{1}{10}(1+\sqrt{6})\ ; \quad g = \tfrac{1}{10}(1-\sqrt{6})\ ; \quad a = \frac{\sqrt{15}}{10}.$$

VII. Orthogonality of the Functions $S^{(i)}(u)$ and $T^{(i)}(v)$.

In order to show the interval of orthogonality of the $S^{(i)}(u)$, we begin with equation (17) and write it as

$$\frac{d}{du}\left[\left(\tfrac{1}{4}-u^4\right)^{\frac{1}{2}}\frac{dS}{du}\right]+\left[\frac{au^2}{\left(\frac{1}{4}-u^4\right)^{\frac{1}{2}}}-\frac{b}{\left(\frac{1}{4}-u^4\right)^{\frac{1}{2}}}\right]S=0. \qquad (101)$$

with $a=l(l+1)$. According to the previous notation, $S^{(i)}(u)$ will represent any one of the four different types of functions which have been developed. We shall let $S_m^{(i)}$ be the function of the i th type determined by a particular value of the separation constant b, say b_m. Then (101) may be written as

$$\frac{d}{du}\left[\left(\tfrac{1}{4}-u^4\right)^{\frac{1}{2}}\frac{dS_m^{(i)}}{du}\right]+\left[\frac{au^2}{\left(\frac{1}{4}-u^4\right)^{\frac{1}{2}}}-\frac{b_m}{\left(\frac{1}{4}-u^4\right)^{\frac{1}{2}}}\right]S_m^{(i)}=0 \qquad (102a)$$

and

$$\frac{d}{du}\left[\left(\tfrac{1}{4}-u^4\right)^{\frac{1}{2}}\frac{dS_n^{(i)}}{du}\right]+\left[\frac{au^2}{\left(\frac{1}{4}-u^4\right)^{\frac{1}{2}}}-\frac{b_n}{\left(\frac{1}{4}-u^4\right)^{\frac{1}{2}}}\right]S_n^{(i)}=0 \qquad (102b)$$

Multiply (102a) by $S_n^{(i)}$ and (102b) by $S_m^{(i)}$ and subtract the results. We find

$$S_n^{(i)}\frac{d}{du}\left[\left(\tfrac{1}{4}-u^4\right)^{\frac{1}{2}}\frac{dS_m^{(i)}}{du}\right]-S_m^{(i)}\frac{d}{du}\left[\left(\tfrac{1}{4}-u^4\right)^{\frac{1}{2}}\frac{dS_n^{(i)}}{du}\right]$$

$$+(b_n-b_m)\frac{S_n^{(i)}S_m^{(i)}}{\left(\frac{1}{4}-u^4\right)^{\frac{1}{2}}}=0. \qquad (b_n\neq b_m) \qquad (103)$$

Integrate (103) between the limits $u = c$ and $u = d$:

$$\int_c^d S_n^{(i)} \frac{d}{du}\left[\left(\tfrac{1}{4}-u^4\right)^{\frac{1}{2}} \frac{dS_m^{(i)}}{du}\right]du - \int_c^d S_m^{(i)} \frac{d}{du}\left[\left(\tfrac{1}{4}-u^4\right)^{\frac{1}{2}} \frac{dS_n^{(i)}}{du}\right]du$$

$$+(b_n - b_m)\int_c^d \frac{S_n^{(i)} S_m^{(i)}}{\left(\frac{1}{4}-u^4\right)^{\frac{1}{2}}}\,du = 0. \tag{104}$$

Let us recall the range of the variables ξ and η . We have taken $0 \le \xi \le 4K$ and $-K \le \eta \le K$; and hence, $u(0) = \frac{1}{\sqrt{2}}\,\mathrm{cn}(0) = \frac{1}{\sqrt{2}}$ and $u(4K) = \frac{1}{\sqrt{2}}\,\mathrm{cn}(4K) = \frac{1}{\sqrt{2}}$. Therefore, if ξ is allowed to vary over its entire range, the integrals of (104) considered in the Riemann sense are identically zero. (We suppose the integrals to exist). Because of the periodicity of $u = \frac{1}{\sqrt{2}}\,\mathrm{cn}\,\xi$, we shall consider the variation of ξ .

If in the definition of the conical coordinates

$$u = \frac{1}{\sqrt{2}}\,\mathrm{cn}\,\xi$$
$$v = \frac{1}{\sqrt{2}}\,\mathrm{cn}\,\eta \quad ,$$

we treat ξ and η as the variables, the separated wave equation yields

$$\frac{d^2 S^{(i)}}{d\xi^2} + \frac{1}{2}\left(a^2\,\mathrm{cn}^2\xi - b\right) S^{(i)} = 0 \tag{105}$$

$$[a = l(l+1)]$$

$$\frac{d^2 T^{(i)}}{d\eta^2} + \frac{1}{2}\left(a^2\,\mathrm{cn}^2\eta + b\right) T^{(i)} = 0. \tag{106}$$

and the radial equation (16). Both equation (105) and (106) are forms of Lame's equation. By using $u = \frac{1}{\sqrt{2}}\, cn\, \xi$ we can transform (105) into (17), and with $v = \frac{1}{\sqrt{2}}\, cn\, \eta$ we can transform (106) into (18).

We write (105) for two different values of b, say b_m and b_n:

$$\frac{d^2 S_m^{(i)}}{d\xi^2} + \frac{1}{2}\left(a^2 cn^2 \xi - b_m\right) S_m^{(i)} = 0 \tag{107a}$$

$$\frac{d^2 S_n^{(i)}}{d\xi^2} + \frac{1}{2}\left(a^2 cn^2 \xi - b_n\right) S_n^{(i)} = 0 \tag{107b}$$

Here the superscript (i) refers to functions of the same type. If we multiply (107a) by $S_n^{(i)}$ and (107b) by $S_m^{(i)}$ and subtract, we obtain the equation

$$S_n^{(i)} \frac{d^2 S_m^{(i)}}{d\xi^2} - S_m^{(i)} \frac{d^2 S_n^{(i)}}{d\xi^2} - \frac{1}{2}\left(b_m - b_n\right) S_m^{(i)} S_n^{(i)} = 0. \tag{108}$$

Integrate the first two terms of (108) by parts between the limits α and β.

$$\left[S_n^{(i)} \frac{d S_m^{(i)}}{d\xi} - S_m^{(i)} \frac{d S_n^{(i)}}{d\xi} \right]_\alpha^\beta + \frac{1}{2}\left(b_n - b_m\right) \int_\alpha^\beta S_n^{(i)} S_m^{(i)}\, d\xi = 0 \tag{109}$$

We shall let $\alpha = 0$ and $\beta = 4K$ where as before

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}}\,.$$

Then $cn(\alpha) = cn(\beta) = 1$. By considering the forms of the functions $S^{(i)}$, we shall show that the $S^{(i)}$ are orthogonal for the range of ξ given by $0 \leq \xi \leq 4K$. We show the details for functions of the first and fourth types. Similar calculations apply to functions of the second and third types.

1. Functions of the first type.

From equation (40) and with the identity $sn^2\xi + cn^2\xi = 1$, we may write the $S^{(1)}$ as

$$S^{(1)} = sn\,\xi\, F(cn\,\xi) \tag{110}$$

where $F(cn\,\xi)$ is a polynomial. Then for different b's we have

$$S_m^{(1)} = sn\,\xi\, F(cn\,\xi) \tag{111a}$$

and

$$S_m^{(1)} = sn\,\xi\, G(cn\,\xi) \tag{111b}$$

Substitution of (111a) and (111b) and their derivatives into (109) yields for the expression in the square brackets

$$\Big[sn^2\xi\,(FG' - GF')\Big]_0^{4K} = 0 \qquad \text{, since}$$

$$sn\,(0) = sn(4K) = 0.$$

Here F' means $\frac{d}{d(cn\xi)} F$ and G' is $\frac{d}{d(cn\xi)} G$.

Consequently, we have

$$(b_n - b_m)\int_0^{4K} S_n^{(1)} S_m^{(1)} d\xi = 0. \qquad (b_n \neq b_m) \tag{112}$$

2. Functions of the fourth type.

According to equation (71), these functions are polynomials in $cn\xi$. Hence, we write for different values of b

$$S_m^{(4)} = F(cn\xi) \tag{113a}$$

$$S_n^{(4)} = G(cn\xi) \tag{113b}$$.

Differentiation of (113a) and (113b) and substitution into (109) yields for the quantity in the square brackets

$$\Big[sn\xi\, dn\xi\,(G'F - F'G)\Big]_0^{4K} = 0$$

Hence,
$$(b_n - b_m)\int_0^{4K} S_n^{(4)} S_m^{(4)} d\xi = 0. \tag{114}$$

In general, we may obtain from (109) the integral

$$(b_n - b_m)\int_0^{4K} S_n^{(i)} S_m^{(i)} d\xi = 0. \tag{115}$$

We may extend (115) to the case for which the functions are not of the same type. That is, the following integral is also valid:

$$(b_n - b_m)\int_0^{4K} S_n^{(i)} S_m^{(j)}\, d\xi = 0. \qquad (b_n \neq b_m) \tag{116}$$

Equation (116) can be proven by forming all the combinations $S_n^{(i)} S_m^{(j)}$ (for $m \neq n$) of our four types of functions. We illustrate the vanishing of (116) for two particular combinations.

1. Combination of functions of the first and fourth type.

   We have
   $$S_m^{(1)} = \mathrm{sn}\,\xi\, F(\mathrm{cn}\,\xi) \tag{117a}$$
   and
   $$S_n^{(4)} = G(\mathrm{cn}\,\xi) \tag{117b}$$

   Differentiation of (117a) and (117b) and substitution in (109) allows us to write for the quantity in the square brackets

   $$\Big[\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi\, FG - \mathrm{sn}^2\xi\,\mathrm{dn}\,\xi\,(F'G - FG')\Big]_0^{4K}$$

   The second term vanishes for $\xi = 0$ and $\xi = 4K$. For the first term we recall that F and G are polynomials in $\mathrm{cn}\,\xi$, and also $\mathrm{cn}(0) = \mathrm{cn}(4K) = \mathrm{dn}(0) = \mathrm{dn}(4K) = 1$. Consequently, evaluation of the square brackets gives zero; and we have the result

   $$(b_n - b_m)\int_0^{4K} S_m^{(1)} S_n^{(4)}\, d\xi = 0.$$

2. Combination of functions of the second and third type.

From (47) we write for the second-type function

$$S_m^{(2)} = dn\,\xi\, G(cn\,\xi) \tag{118a}$$

and from (62) we may express the third-type function as

$$S_m^{(3)} = sn\,\xi\, dn\,\xi\, F(cn\,\xi). \tag{118b}$$

We substitute (118a) and (118b) and their derivatives in (109) and obtain for the expression in the square brackets

$$\Big[cn\,\xi\, dn^2\xi\, FG - sn^2\xi\, dn^3\xi\,(F'G - FG')\Big]_0^{4K}.$$

The second term is zero for each limit of integration; and since $cn(0) = cn(4K) = dn(0) = dn(4K) = 1$, we see that the first term yields zero upon integration.

Similar arguments are valid for any combination $S_m^{(i)} S_n^{(j)}$ (with $m \neq n$) of the four types of functions.

Next, we investigate the orthogonality of the functions $T^{(i)}$. From equation (106) we can obtain

$$\left[T_m^{(i)} \frac{dT_n^{(i)}}{d\eta} - T_n^{(i)} \frac{dT_m^{(i)}}{d\eta}\right]_\alpha^\beta + \frac{1}{2}(b_n - b_m)\int_\alpha^\beta T_m^{(i)} T_n^{(i)}\, d\eta = 0. \tag{119}$$

We shall show that for functions of the same type and for the range of η given by $-K \leq \eta \leq K$, equation (119) becomes

$$(b_n - b_m)\int_{-K}^{K} T_m^{(i)} T_n^{(i)}\, d\eta = 0 \tag{120}$$

where K is the complete elliptic integral of the first kind. The proof of (120) depends upon the degrees of the polynomials $F(cn\,\eta)$ and $G(cn\,\eta)$ which appear in functions $T^{(i)}$. We shall treat each type of function individually.

For functions of the first type we may write

$$T_m^{(1)} = sn\,\eta\, G(cn\,\eta) \tag{121a}$$

$$T_m^{(1)} = sn\,\eta\, F(cn\,\eta)\,. \tag{121b}$$

With (121a) and (121b) we obtain from (119)

$$\left[T_m^{(i)}\frac{dT_m^{(i)}}{d\eta} - T_m^{(i)}\frac{dT_m^{(i)}}{d\eta}\right]_{-K}^{K} = \left[sn^3\eta\, dn\,\eta\,(G'F - GF')\right]_{-K}^{K}$$

For a given ℓ the polynomial F differs from the polynomial G only in its coefficients. Consequently, we see from (80) that if ℓ is even, the lowest power of $v = \frac{1}{\sqrt{2}}\, cn\,\eta$ in either G or F will be the first. Since $cn\,(K) = cn\,(-K) = 0$, the quantity in the brackets is zero for the limits of integration. If ℓ is odd, the lowest power of v in either F or G will be the zeroth; and the next higher power of $(cn\,\eta)$ will be the second. Hence, the lowest power of $(cn\,\eta)$ in F' or G' will be the first; and we have $G'(cn\,\eta) = F'(cn\,\eta) = 0$ for $\eta = \pm K$. Equation (120) with $i = 1$ follows.

For a given ℓ the degrees of the polynomials of the functions of types one and two are equal. Consequently, the

argument which we have used for the first-type function is also valid for functions of the second type. It follows that equation (120) is valid for $i = 2$.

According to equation (94), we may write the functions of the third type as

$$T_m^{(3)} = \mathrm{sn}\,\eta\,\mathrm{dn}\,\eta\,F(\mathrm{cn}\,\eta) \tag{122a}$$

$$T_n^{(3)} = \mathrm{sn}\,\eta\,\mathrm{dn}\,\eta\,G(\mathrm{cn}\,\eta) \tag{122b}$$

Substitution of (122a) and (122b) into (119) yields the expression

$$\left[T_m^{(3)}\frac{dT_n^{(3)}}{d\eta} - T_n^{(3)}\frac{dT_m^{(3)}}{d\eta}\right]_{-K}^{K} = \left[\mathrm{sn}^3\eta\,\mathrm{dn}^3\eta\,(F'G - FG')\right]_{-K}^{K}$$

For a given ℓ in equation (94), only the coefficients of the polynomials F and G will differ. If ℓ is even, the lowest power of ν in F or G will be the zeroth, and the next higher power will be the second. Hence, the lowest power of $(\mathrm{cn}\,\eta)$ in F' or G' will be the first. Since $\mathrm{cn}(K) = \mathrm{cn}(-K) = 0$, both F' and G' will vanish for $\eta = \pm K$. With odd ℓ the lowest power of $(\mathrm{cn}\,\eta)$ in the polynomials G and F is the first, and $G(\mathrm{cn}\,\eta) = F(\mathrm{cn}\,\eta) = 0$ for $\eta = \pm K$. As a result equation (120) is true for $i = 3$.

From equation (100) functions of the fourth type may be written

$$T_m^{(4)} = G(\mathrm{cn}\,\eta) \tag{123a}$$

$$T_m^{(4)} = F(\mathrm{cn}\,\eta) \tag{123b}$$

With (123a) and (123b) we find from (119) the expression

$$\left[T_m^{(4)}\frac{dT_m^{(4)}}{d\eta}-T_m^{(4)}\frac{dT_m^{(4)}}{d\eta}\right]_{-K}^{K}=\left[\operatorname{sn}\eta\operatorname{dn}\eta\,(GF'-FG')\right]_{-K}^{K}$$

If l is even, the lowest power of $(\operatorname{cn}\eta)$ in F or G is the zeroth; and the next higher power is the second. Differentiation of F and G will yield polynomials whose lowest power of v is the first. Consequently, $F'(\operatorname{cn}\eta)=G'(\operatorname{cn}\eta)=0$ for $\eta=\pm K$. When l is odd the lowest power of $(\operatorname{cn}\eta)$ in either F or G is the first. Therefore, $F(\operatorname{cn}\eta)=G(\operatorname{cn}\eta)=0$ for $\eta=\pm K$. Equation (120) for $i=4$ follows.

The extension of the integral of (120) to the type

$$(b_m-b_m)\int_{-K}^{K}T_m^{(i)}T_m^{(j)}\,d\eta=0. \qquad (i\neq j) \tag{124}$$

does not follow. Consider the case for $i=4$ and $j=2$. We write

$$T_m^{(2)}=\operatorname{dn}\eta\,F(\operatorname{cn}\eta) \tag{125a}$$

$$T_m^{(4)}=G(\operatorname{cn}\eta) \tag{125b}$$

With (125a) and (125b) we obtain for the square brackets of (119)

$$\left[T_m^{(4)}\frac{dT_m^{(2)}}{d\eta}-T_m^{(2)}\frac{dT_m^{(4)}}{d\eta}\right]_{-K}^{K}=$$

$$\left[-\tfrac{1}{2}\operatorname{sn}\eta\operatorname{cn}\eta\,FG+\operatorname{sn}\eta\operatorname{dn}^2\eta\,(FG'-F'G)\right]_{-K}^{K} \tag{126}$$

The first term vanishes at the limits of integration, since $\mathrm{cn}(K) = \mathrm{cn}(-K) = 0$. If ℓ is odd, then the lowest power of $(\mathrm{cn}\,\eta)$ in the polynomial F is the zeroth, and the lowest power of $(\mathrm{cn}\,\eta)$ in G is the first. For the derivatives F' and G', we have for the lowest power of $(\mathrm{cn}\,\eta)$ the first and zeroth respectively. Consequently, the second term in the square brackets need not vanish at the limits of integration.

As a specific example suppose we let ℓ equal three. Then we find for the polynomials

$$F = \frac{1}{2}\mathrm{cn}^2\eta - c, \qquad \left[c = \frac{1}{10}(1+\sqrt{6})\right]$$
$$\left[b = \frac{3}{2} + \sqrt{6}\right]$$

and

$$G = \frac{1}{2\sqrt{2}}\mathrm{cn}^3\eta - \frac{a}{\sqrt{2}}\mathrm{cn}\,\eta, \qquad \left[a = \frac{\sqrt{15}}{10}\right]$$
$$\left[b = \sqrt{15}\right]$$

The functions $T_m^{(4)}$ and $T_m^{(2)}$ are

$$T_m^{(4)} = \frac{1}{2\sqrt{2}}\mathrm{cn}^3\eta - \frac{a}{\sqrt{2}}\mathrm{cn}\,\eta \tag{127a}$$

$$T_m^{(2)} = \mathrm{dn}\,\eta\left(\frac{1}{2}\mathrm{cn}^2\eta - c\right) \tag{127b}$$

If we differentiate F and G, we have

$$F' = \mathrm{cn}\,\eta$$

$$G' = \frac{3}{2\sqrt{2}}\mathrm{cn}^2\eta - \frac{a}{\sqrt{2}}$$

With these expressions for F' and G', the term

$\operatorname{sn}\eta \operatorname{dn}^2\eta\,(F\,G' - F'\,G)$ of (126) becomes

$$\left[\operatorname{sn}\eta\operatorname{dn}^2\eta\left\{\frac{1}{4\sqrt{2}}\operatorname{cn}^4\eta + \frac{1}{2\sqrt{2}}(a-3c)\operatorname{cn}^2\eta + \frac{ac}{\sqrt{2}}\right\}\right]_{-K}^{K}$$

Evaluation yields

$$\frac{ac}{\sqrt{2}} \neq 0 \quad .$$

Hence, equation (124) is not true for this particular example, and we may say that (124) is not true in general.

VIII. Transformation of the Angular Momentum Operator.

We shall find the components in conical coordinates of the angular momentum operator

$$\vec{M} = -i\hbar\,(\vec{r} \times \nabla)\,. \tag{128}$$

Let $\vec{i}$, $\vec{j}$, and $\vec{k}$ be the unit vectors in the cartesian coordinates x, y, and z. In the conical coordinates we denote the unit coordinate-vectors as $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$. In the work which follows we shall treat the variables r, ξ, and η as the conical coordinates. The general procedure consists of expressing the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ in terms of the conical coordinates; finding (128) in conical coordinates; and computing from these results the scalar products $(\vec{i}\cdot\vec{M})$, $(\vec{j}\cdot\vec{M})$, and $(\vec{k}\cdot\vec{M})$. We shall begin by computing the component M_z.

In conical coordinates the operator ∇ may be written as

$$\nabla = \frac{\vec{e}_1}{h_1}\frac{\partial}{\partial r} + \frac{\vec{e}_2}{h_2}\frac{\partial}{\partial \xi} + \frac{\vec{e}_3}{h_3}\frac{\partial}{\partial \eta}$$

Then

$$(\nabla \vec{r}) = \left(\frac{\vec{e}_1}{h_1}\frac{\partial}{\partial r} + \frac{\vec{e}_2}{h_2}\frac{\partial}{\partial \xi} + \frac{\vec{e}_3}{h_3}\frac{\partial}{\partial \eta}\right)(\vec{i}x + \vec{j}y + \vec{k}z),$$

and

$$\vec{k} = (\nabla \vec{r})\cdot\vec{k} = \frac{\vec{e}_1}{h_1}\frac{\partial z}{\partial r} + \frac{\vec{e}_2}{h_2}\frac{\partial z}{\partial \xi} + \frac{\vec{e}_3}{h_3}\frac{\partial z}{\partial \eta}.$$

Since $z = r\,\mathrm{cn}\,\xi\,\mathrm{cn}\,\eta$,

$$\vec{k} = \frac{\vec{e}_1}{h_1}\mathrm{cn}\,\xi\,\mathrm{cn}\,\eta + \frac{\vec{e}_2}{h_2}(-r\,\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi\,\mathrm{cn}\,\eta) + \frac{\vec{e}_3}{h_3}(-r\,\mathrm{cn}\,\xi\,\mathrm{sn}\,\eta\,\mathrm{dn}\,\eta).$$

The operator $(\vec{r}\times\nabla)$ becomes

$$\vec{r}\times\nabla = -\vec{e}_2\frac{r}{h_3}\frac{\partial}{\partial \eta} + \vec{e}_3\frac{r}{h_2}\frac{\partial}{\partial \xi}$$

Consequently,

$$\vec{k}\cdot(\vec{r}\times\nabla) = \frac{r^2}{h_2h_3}\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi\,\mathrm{cn}\,\eta\frac{\partial}{\partial \eta} - \frac{r^2}{h_2h_3}\mathrm{cn}\,\xi\,\mathrm{sn}\,\eta\,\mathrm{dn}\,\eta\frac{\partial}{\partial \xi}.$$

With $h_2h_3 = \frac{r^2}{2}(\mathrm{cn}^2\xi + \mathrm{cn}^2\eta)$, we find

$$M_z = \frac{2i\hbar}{\mathrm{cn}^2\xi + \mathrm{cn}^2\eta}\left\{\mathrm{cn}\,\xi\,\mathrm{sn}\,\eta\,\mathrm{dn}\,\eta\frac{\partial}{\partial \xi} - \mathrm{sn}\,\xi\,\mathrm{dn}\,\xi\,\mathrm{cn}\,\eta\frac{\partial}{\partial \eta}\right\}. \tag{129}$$

One proceeds in a similar manner in order to find the components M_y and M_x. The results are as follows:

$$M_y = \frac{-2i\hbar}{cn^2\xi + cn^2\eta}\left\{ cn\eta\, dn\eta\, dn\xi \frac{\partial}{\partial\xi} + \frac{1}{2} sn\eta\, sn\xi\, cn\xi \frac{\partial}{\partial\eta} \right\} \tag{130}$$

$$M_x = \frac{2i\hbar}{cn^2\xi + cn^2\eta}\left\{ \frac{1}{2} sn\xi\, sn\eta\, cn\eta \frac{\partial}{\partial\xi} - cn\xi\, dn\xi\, dn\eta \frac{\partial}{\partial\eta} \right\} \tag{131}$$