133
635
THS

THE PROPAGATION OF ELECTROMAGNETIC WAVES IN PARABOLIC PIPES

Thesis for the Degree of M. S.

MICHIGAN STATE COLLEGE

Robert Dean Spence

1942

MICHIGAN STATE LIBRARIES
3 1293 01774 9643

[redacted]

LIBRARY
Michigan State
University

PLACE IN RETURN BOX to remove this checkout from your record.
TO AVOID FINES return on or before date due.
MAY BE RECALLED with earlier due date if requested.

DATE DUE	DATE DUE	DATE DUE

1/98 c:/CIRC/DateDue.p65-p.14

THE PROPAGATION OF ELECTROMAGNETIC WAVES IN PARABOLIC PIPES

by

Robert Dean Spence

A THESIS

Submitted to the Graduate School of Michigan
State College of Agriculture and Applied
Science in partial fulfilment of the
requirements for the degree of

MASTER OF SCIENCE

Department of Physics

1942

ACKNOWLEDGMENT

The writer wishes to express his sincere thanks to Dr. C. P. Wells for his aid and counsel on this problem and to those members of the staff of the Department of Physics who contributed numerous helpful discussions.

CONTENTS

142678

I. INTRODUCTION

The purpose of this work is to discuss and formulate mathematically the physical properties of a particular system for the transmission of guided electromagnetic waves. The system consists of a straight, hollow, conducting sheath or pipe of parabolic cross section as shown in figure 1. The electromagnetic waves are assumed to be confined to the interior of the pipe and are propagated along its axis. The discussion is limited to the problem of determining the waves that may exist in such a system and to the attenuation that they experience as they move down the pipe. The question of reflection and radiation that may occur at discontinuities in the interior dielectric or the conducting sheath will not be discussed here.

The first discussion of such systems for guiding electromagnetic waves appears in a paper by Lord Rayleigh, published in the "Philosophical Magazine"[1] in 1897. In this paper he showed that for perfectly conducting pipes of rectangular and circular cross section all waves greater than a certain critical wave length were completely attenuated, while wave lengths shorter than the critical wave length were freely transmitted. Futhermore, he was able to separate the types of waves that occur in such

[1] See references 1-5 in bibliography at the end of this paper.

systems into two types which he called "waves of the first and second kind" which are identical with the E and H waves to be defined later in this paper. In 1931 Southworth[1] began an experimental investigation of wave guides of circular cross section. He reported the results of his work in 1936 and at the same time Carson, Mead, and Schelkunoff[2] published the first theoretical study of wave guides whose sheath was of finite conductivity. Barrow and Chu[3] reported theoretical work on pipes of rectangular and elliptical cross section in two papers that appeared in 1938. In the case of the elliptical pipe Chu found that there existed two kinds of E and H waves which he designated as odd and even and both of which degenerated into waves of the circular pipe for the case of zero eccentricity. The work on rectangular pipes by Barrow and Chu is interesting because it includes a discussion of the radiation from the open end of the pipe; The radiation from a circular pipe was discussed by Chu in a paper published in 1940.

[1] Ibid.
[2] Ibid.
[3] Ibid.

II. SOLUTION OF THE WAVE EQUATION

According to Eisenhart the scalar wave equation is separable in only four distinct systems of cylindrical coordinates. On the basis of Eisenhart's criterion[1] the only types of pipes for which the scalar wave equation may be separated are those whose cross sections are either rectangular, circular, elliptical, or parabolic. The first three have been well investigated theoretically, but the fourth has received little if any attention.[2]

The parabolic coordinate system used to describe the parabolic pipe is defined by the transformation:

$$x = \tfrac{1}{2}(\eta^2 - \xi^2)$$
$$y = \eta\xi$$
$$z = z$$

The Z axis is taken to be the axis of the pipe and the surface of the pipe itself is formed by the intersection of the two parabolic cylinders $\xi = \eta =$ constant.

Plane waves, nonhomogeneous in ξ and η are assumed propagated down the pipe in a positive direction. This and the assumption of a simple periodic time variation is equivalent to postulating an electromagnetic field of the form:

$$\left.\begin{matrix}\bar{E}\\ \bar{H}\end{matrix}\right\} = \bar{F}(\eta,\xi)\, e^{-hz + i\omega t} \qquad (1)$$

1 Eisenhart, L. P., Annals of Mathematics 35, pp. 284-304. 1934
2 See references 1-5 in bibliography at end of this paper.

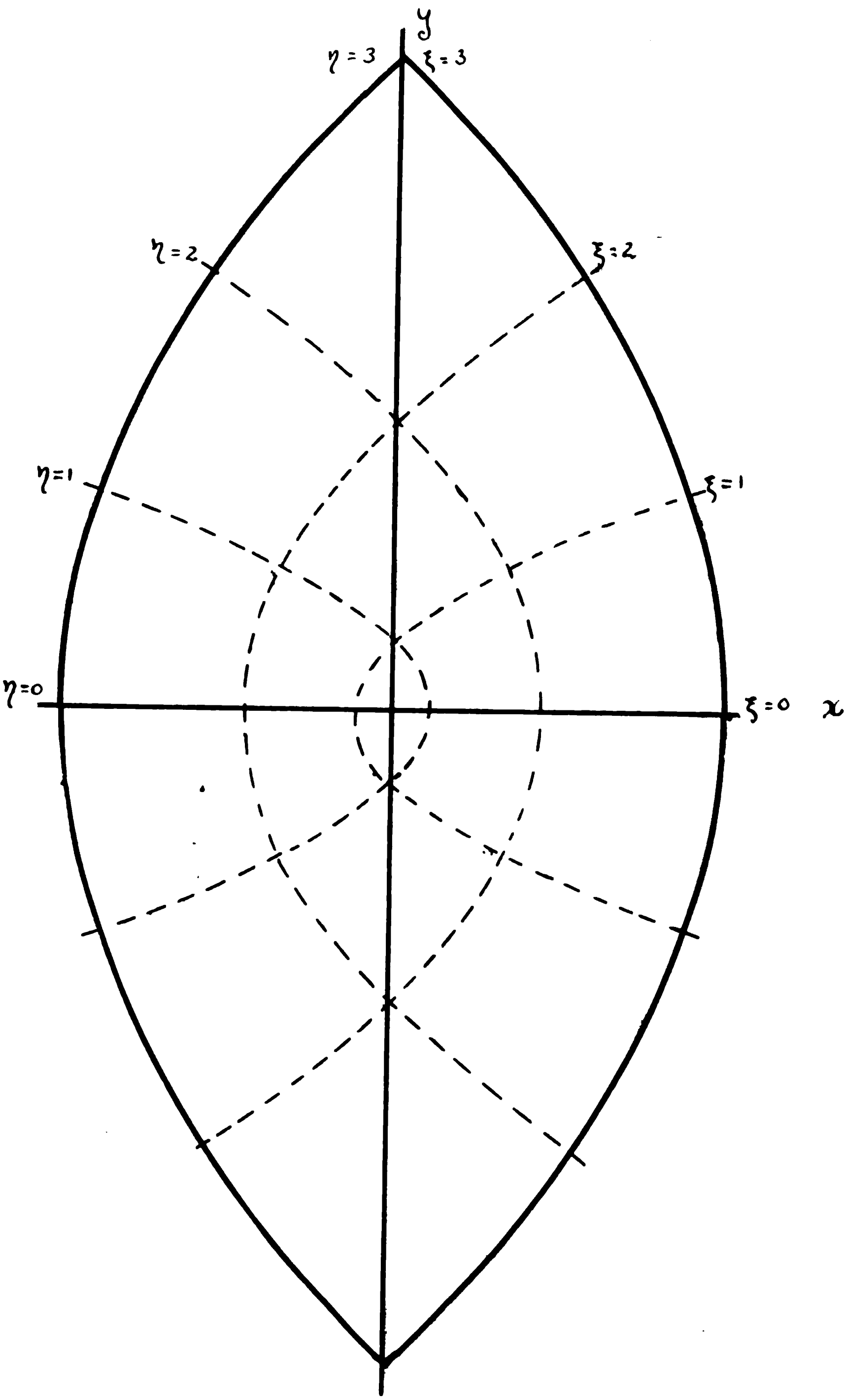

Figure 1. Parabolic cross-section with parabolic coordinates inserted.

in which ω is equal to 2π times the frequency of the source, and h is the propagation constant in the positive z direction. The propagation constant is itself complex and equal to $\alpha + \beta i$ where α is the attenuation constant and β is the phase constant.

The scalar components of the electromagnetic field are obtained from the wave equation and from Maxwell's equations. In parabolic coordinates these are:

$$\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\xi^2} + (k^2 + h^2)(\eta^2 + \xi^2)\begin{Bmatrix} E_z \\ H_z \end{Bmatrix} = 0 \tag{2}$$

$$r^2\sqrt{\eta^2+\xi^2}\, H_\xi = -h\frac{\partial H_z}{\partial \xi} + (i\omega\epsilon + \sigma)\frac{\partial E_z}{\partial \eta} \tag{3a}$$

$$r^2\sqrt{\eta^2+\xi^2}\, H_\eta = -h\frac{\partial H_z}{\partial \eta} - (i\omega\epsilon + \sigma)\frac{\partial E_z}{\partial \xi} \tag{3b}$$

$$r^2\sqrt{\eta^2+\xi^2}\, E_\xi = -h\frac{\partial E_z}{\partial \xi} - i\omega\mu\frac{\partial H_z}{\partial \eta} \tag{3c}$$

$$r^2\sqrt{\eta^2+\xi^2}\, E_\eta = -h\frac{\partial E_z}{\partial \eta} + i\omega\mu\frac{\partial H_z}{\partial \xi} \tag{3d}$$

where $k^2 = \omega^2\epsilon\mu - i\omega\mu\sigma$ and $r^2 = k^2 + h^2$. We shall call r the wave guide constant. It will depend only on the physical size of the pipe and the mode of the wave excited.

The practical system of units is used, where:

ϵ = dielectric constant $\frac{10^{-11}}{36\pi}$ farads per cm. (in air)

μ = permeability $4\pi \times 10^{-9}$ henrys per cm. (in air)

σ = conductivity in mhos per cm.

H = magnetic field in amperes per sq. cm.

E = electric field in volts per cm.

Separating variabiles in the wave equation leads to the two ordinary differential equations:

$$\frac{d^2U}{d\xi^2} + (r^2\xi^2 + m)U = 0 \qquad (4a)$$

$$\frac{d^2V}{d\eta^2} + (r^2\eta^2 - m)V = 0 \qquad (4b)$$

where m is the separation constant, restricted here to positive integral values. Both of these are forms of the confluent hypergeometric equation[1] or as they are sometimes called, parabolic cylinder equations. Solutions in the form of definite integrals were given by Weber[2] for the problem of the parabolic membrane. Later all solutions were classified by Epstein[3,4] in connection with the problem of diffraction of light by a parabolic cylinder.

Both even and odd solutions of order m exist for these equations. We shall designate the even solutions of (4a) and (4b) as ${}_eU_m$, ${}_eV_m$ and the odd solutions as ${}_oU_m$, ${}_oV_m$.

1 Whittaker and Watson, Modern Analysis, p. 341, 2nd ed. (1915)

2 Weber, Die Partiellen Differentialgleichugen der Math. Physik, Bd. a, Anfl. S. 258 (1912

3 Epstein, Diss. Munich.

4 See appendix for list of solutions.

The solution of the wave equation can be written:

$$\left.\begin{matrix} E_z \\ H_z \end{matrix}\right\} = \sum_{m=0}^{\infty} \left(b_m \, {}_eU_m(\xi) \, {}_eV_m(\eta) + c_m \, {}_oU_m(\xi) \, {}_oV_m(\eta) \right) e^{-hz + i\omega t} \qquad (5)$$

where b_m and c_m are complex constants and depend upon the strength of excitation.

III. PERFECTLY CONDUCTING PIPES

The boundary conditions depend upon the assumptions made regarding the conductivity σ. In this section we consider σ to be infinite in the walls of the pipes and zero in the dielectric inside the pipe. Under this condition the tangential components of the electric field vanish at the walls and the propagation constant h reduces to $i\beta$, since the attenuation constant must be zero. Boundary conditions of this type can be satisfied by considering two partial fields to exist in the pipe. The first, called an E wave is defined by $H_z = 0$ everywhere inside the pipe, and the second, called an H wave is defined by $E_z = 0$ everywhere inside the pipe. Thus for the two partial fields the boundary conditions are:

$$E \text{ wave} \quad \begin{cases} E_z = 0, & \xi = \eta = a \\ E_\eta = 0, & \xi = a \\ E_\xi = 0, & \eta = a \end{cases} \qquad (6a)$$

$$H \text{ wave} \begin{cases} E_\eta = 0\ ,\ \xi = a \\ E_\xi = 0\ ,\ \eta = a \end{cases} \tag{6b}$$

where $\eta = a$ or $\xi = a$ determines the boundary of the pipe. The boundary conditions are exceptional in that the walls of the pipes are formed by two cylindrical surfaces rather than just one as in the case of the circular or elliptical pipe.[1]

The above boundary conditions can be satisfied by using only one term of the series given by (5). For the E waves the components of the field are:

$$E_z = \left(b\ {}_eU_m(\xi)\,{}_eV_m(\eta) + c\ {}_oU_m(\xi)\,{}_oV_m(\eta)\right)e^{-hz+i\omega t} \tag{7a}$$

$$E_\xi = \frac{\beta}{\omega\epsilon} H_\eta = \frac{-i\beta}{r^2\sqrt{\eta^2+\xi^2}}\left(b\ {}_eU_m'(\xi)\,{}_eV_m(\eta) + c\ {}_oU_m'(\xi)\,{}_oV_m(\eta)\right)e^{-hz+i\omega t} \tag{7b}$$

$$E_\eta = \frac{-\beta}{\omega\epsilon} H_\xi = \frac{-\beta}{r^2\sqrt{\eta^2+\xi^2}}\left(b\ {}_eU_m(\xi)\,{}_eV_m'(\eta) + c\ {}_oU_m(\xi)\,{}_oV_m'(\eta)\right)e^{-hz+i\omega t} \tag{7c}$$

where the primes indicate the differentiation with respect to either ξ or η.

In order that the tangential components of the electric field vanish at the boundary, it is necessary that U and V satisfy the equations:

$${}_eU_m(a) = 0 \qquad {}_eV_m(a) = 0 \tag{8a}$$

[1] See appendix for list of solutions.

$$ {}_oU_m(a) = 0 \qquad {}_oV_m(a) = 0 \tag{8b} $$

The roots of these equations serve to determine the critical wave length. Corresponding to a root a_{nm} where n indicates the number of root and m the order, the critical wave length is: $\lambda_{n,m} = \frac{2\pi}{\gamma_{n,m}}$

For m = 0 the functions of U and V are identical. In this case it is not hard to verify that they can be written in terms of Bessel functions of order $\frac{1}{4}$ and $-\frac{1}{4}$. The even solution is:

$$ {}_eU_0(\xi) = c_1\sqrt{\xi}\,J_{-\frac{1}{4}}\left(\frac{\gamma\xi^2}{2}\right) \tag{9a} $$

and the odd solution is:

$$ {}_oU_0(\xi) = c_2\sqrt{\xi}\,J_{\frac{1}{4}}\left(\frac{\gamma\xi^2}{2}\right) \tag{9b} $$

where c_1 and c_2 are constants.

In terms of γa^2 a few of the roots of ${}_eU_0(a)=0$ are:

$$ {}_e\gamma_{1,0}\,a^2 = 4.01 \quad , \quad {}_e\gamma_{2,0}\,a^2 = 10.30 $$

For the odd solution ${}_oU_0(a) = 0$:

$$ {}_o\gamma_{1,0}\,a^2 = 5.50 \quad , \quad {}_o\gamma_{2,0}\,a^2 = 11.80 $$

The longest critical wave length for these modes is given by the even solution:

$$ {}_e\lambda_{1,0} = \frac{2\pi}{{}_e\gamma_{1,0}} = \frac{2\pi a^2}{4.01} = .78\,y_0 $$

where y_0 is the latus rectum of the parabola $\xi = a$.

For the values of $m \neq 0$, there appears to be no simple method of calculating the roots of the equations (8a), (8b).

The components of the field for the H wave are:

$$H_z = \left(b\ {}_eU_m(\xi)\,{}_eV_m(\eta) + c\ {}_oU_m(\xi)\,{}_oV_m(\eta)\right)e^{-hz+i\omega t} \qquad (10a)$$

$$E_\xi = \frac{\omega\mu}{\beta}H_\eta = \frac{-i\omega\mu}{r^2\sqrt{\eta^2+\xi^2}}\left(b\ {}_eU_m(\xi)\,{}_eV_m'(\eta) + c\ {}_oU_m(\xi)\,{}_oV_m'(\eta)\right)e^{-hz+i\omega t} \qquad (10b)$$

$$E_\eta = \frac{\omega\mu}{\beta}H_\xi = \frac{i\omega\mu}{r^2\sqrt{\eta^2+\xi^2}}\left(b\ {}_eU_m'(\xi)\,{}_eV_m(\eta) + c\ {}_oU_m'(\xi)\,{}_oV_m(\eta)\right)e^{-hz+i\omega t} \qquad (10c)$$

The boundary conditions here are:

$${}_eU'_m(a) = 0 \qquad {}_eV'_m(a) = 0 \qquad (11a)$$

$${}_oU'_m(a) = 0 \qquad {}_oV'_m(a) = 0 \qquad (11b)$$

Again for $m = 0$ we can reduce these to fairly simple expressions in terms of Bessel functions. For:

$${}_eU_0'(\xi) = \frac{d}{d\xi}\left(c_1\sqrt{\xi}\,J_{-\frac{1}{4}}\left(\frac{r\xi^2}{2}\right)\right) = c_3\,\xi^{\frac{3}{2}}J_{\frac{3}{4}}\left(\frac{r\xi^2}{2}\right) \qquad (12a)$$

Also:

$${}_oU_0'(\xi) = \frac{d}{d\xi}\left(c_2\sqrt{\xi}\,J_{\frac{1}{4}}\left(\frac{r\xi^2}{2}\right)\right) = c_4\,\xi^{\frac{3}{2}}J_{-\frac{3}{4}}\left(\frac{r\xi^2}{2}\right) \qquad (12b)$$

From the equation ${}_eU'_0(a) = 0$ we find:

$${}_er_{1,0}\,a^2 = 7.50\,, \qquad {}_er_{2,0}\,a^2 = 13.20$$

and from ${}_oU'_0(a) = 0$:

$${}_or_{1,0}\,a^2 = 2.12 \qquad {}_or_{2,0}\,a^2 = 4.27$$

The longest critical wave here is:

$${}_o\lambda_{1,0} = \frac{2\pi}{{}_or_{1,0}} = \frac{2\pi a^2}{2.12} = 1.48\,y_0$$

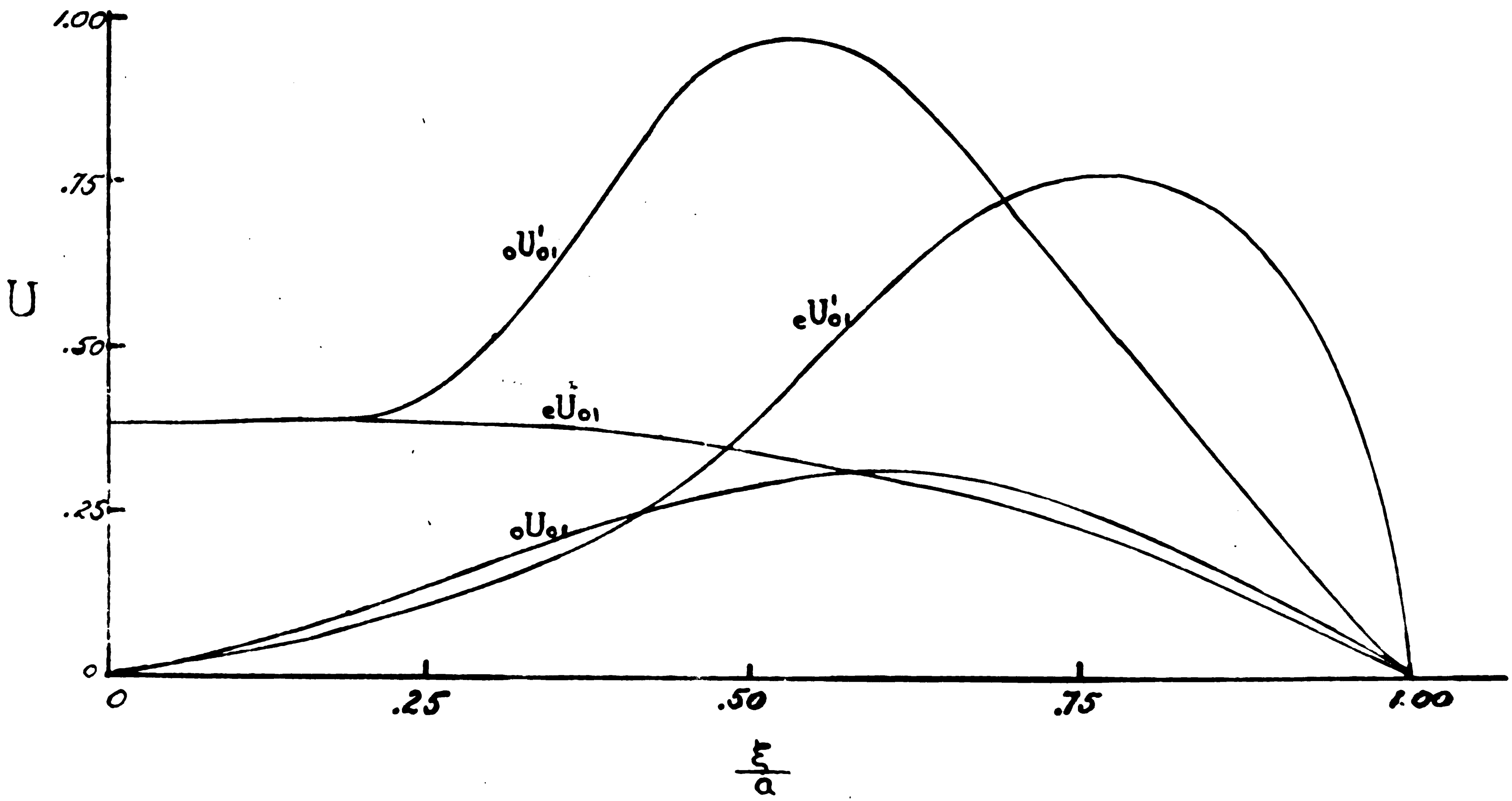

Figure 3. Wave functions of the ground state. Subscripts are in the order m,n.

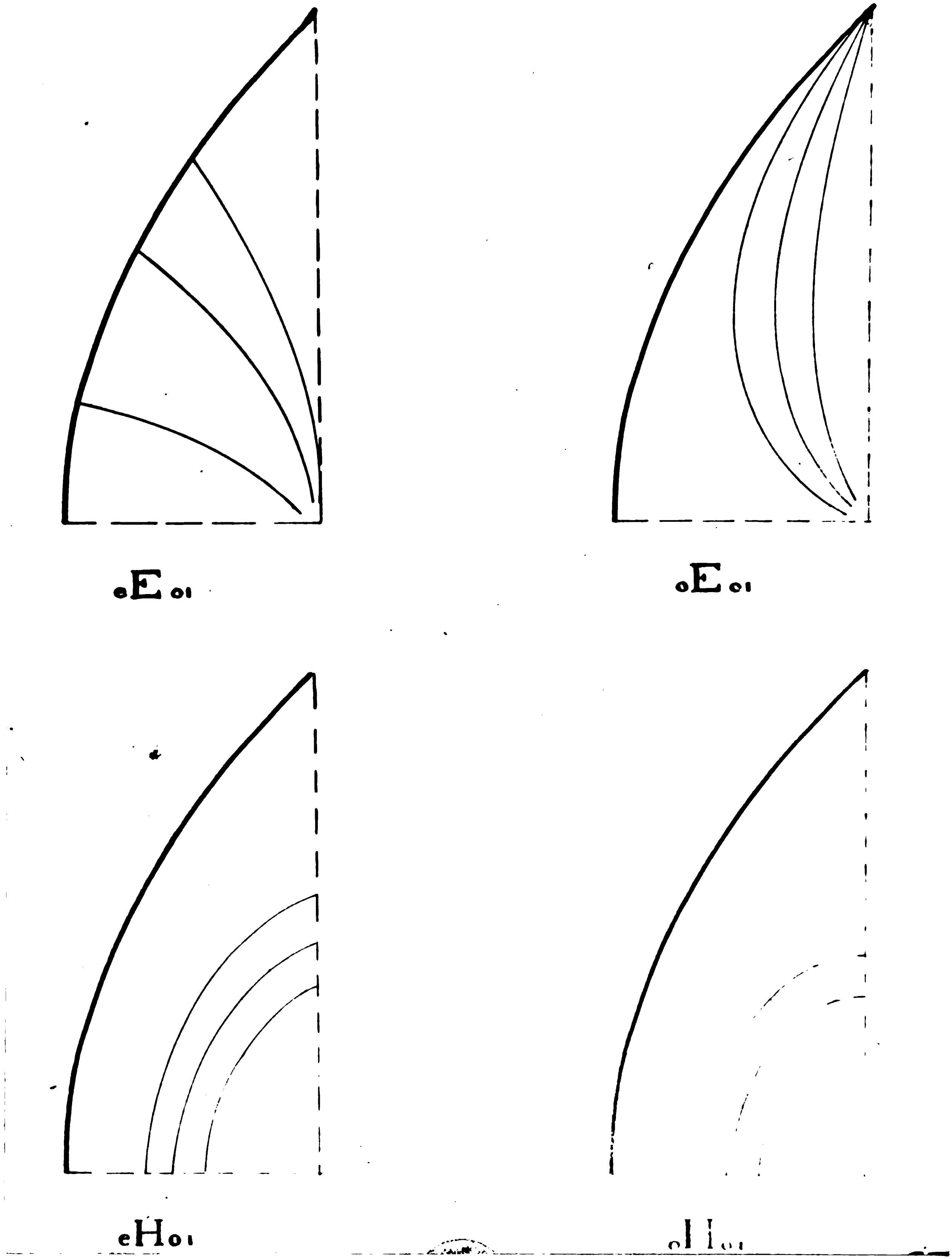

Figure 4. Electric field configurations for the ground state modes shown for quarter sections of the pipe. The subscripts are written in the order m,n.

As in the case of the E wave there appears to be no simple method for calculating roots of higher orders.

IV. IMPERFECTLY CONDUCTING PIPES

In this case we consider the conductivity to be finite though large. This required modification of the boundary conditions which are now the continuity of the tangential electric and tangential magnetic fields. The actual fields that exist will be a superposition of the two partial fields which we previously designated as E waves, or as H waves. However, if the conductivity is large, it will be assumed that one of these partial field will predominate and we shall still speak of E and H waves. For each case we shall define a set of associated impedances which will help clarify the physical statement of the problem[1]. A superscript (1) on the impedance will designate an E wave,

$$Z_z^{(1)} = \frac{E_\xi}{H_\eta} = -\frac{E_\eta}{H_\xi} = \frac{\omega\mu h}{k^2} = \frac{h Z_0}{k} \qquad (13a)$$

where $Z_0 = \frac{\omega\mu}{k}$ is the characteristic impedance. The impedance for the coordinates ξ and η are defined by:

$$E_z = -Z_\xi^{(1)} H_\eta, \qquad E_z = Z_\eta^{(1)} H_\xi \qquad (13b)$$

Likewise for the H wave:

$$Z_z^{(2)} = \frac{E_\xi}{H_\eta} = \frac{E_\eta}{H_\xi} = \frac{k Z_0}{h} \qquad (14a)$$

$$E_\eta = Z_\xi^{(2)} H_z, \qquad E_\xi = Z_\eta^{(2)} H_z \qquad (14b)$$

1 Stratton, Electromagnetic Theory, pp. 354-55. 1941.

Because of the similarity of the two boundaries $\xi = a$ and $\eta = a$ we shall limit the remainder of the discussion to the boundary $\xi = a$. Moreover we shall no longer distinguish between even and odd U and V as the details are the same for both.

For the field outside the boundary of the pipe $\xi = a$ we shall take

$$E'_z = \sum_{m=0}^{\infty} b'_m \, T'_m(\xi) \, V_m(\eta) \, e^{-hz + i\omega t} \tag{15}$$

where $T'_m(\xi)$ represents an asymptotic solution of the confluent hypergeometric equation. The primes will hereafter designate the value of the functions outside the pipe. As an approximation for $T'_m(\xi)$ we shall use:

$$T'_m(\xi) \sim \frac{e^{-\frac{i\gamma'\xi^2}{2}}}{2\gamma'\xi} \, e^{i(2m+1)\frac{\pi}{4}} \tag{16}$$

which is valid for values of $\xi \gg m$. Then (15) can be written:

$$E'_z = b' W(\eta) \frac{e^{-\frac{i\gamma'\xi^2}{2}}}{2\gamma'\xi} \, e^{-hz + i\omega t} \tag{17}$$

where

$$b' W(\eta) = \sum_{m=0}^{\infty} b'_m \, V_m(\eta) \, e^{i(2m+1)\frac{\pi}{4}}$$

and b' is a new constant.

The boundary conditions require $H_\eta = H'_\eta$ at $\xi = a$ where:

$$H_\eta = \frac{i\omega\epsilon}{r^2\sqrt{\eta^2+\xi^2}}\frac{\partial E_z}{\partial\xi}, \qquad H'_\eta = \frac{\sigma}{r'^2\sqrt{\eta^2+\xi^2}}\frac{\partial E'_z}{\partial\xi}$$

We have assumed for the second equation that $\sigma \gg i\omega\epsilon$ and $r' \sim k' \sim \sqrt{-i\omega\mu\sigma}$ Equating H_η and H'_η and setting $V(\eta)/W(\eta)$ equal to unity since it is independent of ξ, gives a relation for the constant b' in terms of b:

$$b' = -\frac{2b\omega\epsilon r'^2}{\sigma r^2} e^{\frac{i r' a^2}{2}} \frac{\partial U(a)}{\partial\xi} \tag{18}$$

In this expression we have assumed $i r' a^2 \gg 1$.

Calculation of the attenuation constant is the principle objective in the imperfectly conducting pipe. This can be done by means of the definition:

$$\alpha = \frac{1}{2}\frac{W_\xi}{W_z} \tag{19}$$

where w_ξ is the power lost through the wall, $\xi = a$ and w_z is the power transmitted down the pipe.

In order to evaluate w_z it is necessary to integrate the real part of the Poynting vector in the z direction over the cross section of the pipe. For the E wave this yields:

$$W_z = \frac{|b|^2\beta\omega\epsilon}{2r^4}\int_0^a\int_0^a \left\{\left|V(\eta)\frac{\partial U(\xi)}{\partial\xi}\right|^2 + \left|U(\xi)\frac{\partial V(\eta)}{\partial\eta}\right|^2\right\} d\xi\, d\eta \tag{20}$$

The value of w_z for the H wave can be obtained by replacing ϵ by μ.

The power lost through the wall of the pipe is:

$$W_\zeta = \frac{1}{2} R \int_0^a (E_\eta' H_z^* - E_z' H_\eta'^*) \sqrt{\eta^2+\zeta^2}\, d\eta \qquad (21)$$

for a unit length of the pipe. Here R indicates the real part of the expression and * indicates the complex conjugate.

It is necessary to emphasize that for an imperfectly conducting pipe the so-called E wave actually has a small H_z component present. In computing the power lost through the pipe these cannot be neglected. Both a magnetic and an electric impedance will exist for either wave. For values of $\sigma \gg i\omega\epsilon$ the approximate values of the impedance are:

$$Z_\zeta^{(1)} = -\sqrt{\frac{\omega\mu'}{\sigma}} \frac{\sqrt{\eta^2+\zeta^2}}{\zeta} e^{\frac{\pi}{4}i} \qquad (22a)$$

$$Z_\zeta^{(2)} = -\sqrt{\frac{\omega\mu'}{\sigma}} \frac{\zeta}{\sqrt{\eta^2+\zeta^2}} e^{\frac{\pi}{4}i} \qquad (22b)$$

By means of equations (13) and (14) we find

$$E_z' = \sqrt{\frac{\omega\mu'}{\sigma}} \frac{\sqrt{\eta^2+\zeta^2}}{\zeta} H_\eta' e^{\frac{\pi}{4}i}, \quad E_\eta = \sqrt{\frac{\omega\mu'}{\sigma}} \frac{\zeta}{\sqrt{\eta^2+\zeta^2}} H_z' e^{\frac{\pi}{4}i} \qquad (23)$$

These give for the E wave:

$$W_\zeta = \frac{\sqrt{2}}{4} \sqrt{\frac{\omega\mu'}{\sigma}} \; \omega^2\epsilon^2 \frac{|b|^2}{a r^4} \left|\frac{dU(a)}{d\zeta}\right|^2 \int_0^a |V|^2 d\eta \qquad (24)$$

The attenuation for the E wave is:

$$\alpha = \frac{1}{2}\sqrt{\frac{\pi\mu\epsilon}{\mu\sigma}}\,\frac{f^{\frac{1}{2}}}{\sqrt{1-\frac{f_{n,m}^2}{f^2}}}\;\frac{\frac{1}{a}\left|\frac{dU(a)}{d\xi}\right|^2\int_0^a |V|^2\,d\eta}{\int_0^a\int_0^a\left\{\left|V\frac{dU}{d\xi}\right|^2+\left|U\frac{dV}{d\eta}\right|^2\right\}d\xi\,d\eta} \tag{25}$$

where $f = \frac{\omega}{2\pi}$ and $f_{n,m}$ is the frequency corresponding to the wave length $\lambda_{n,m}$.

Following the same method, the value of w_ξ for the H wave is:

$$W_\xi = \frac{\sqrt{2}}{4}\sqrt{\frac{\omega\mu'}{\sigma}}\,|b|^2|U(a)|^2\int_0^a\left\{|V|^2-\frac{h^2}{\gamma^2 a}\left|\frac{dV}{d\eta}\right|^2\right\}d\eta \tag{26}$$

The attenuation for the H wave is:

$$\alpha = \frac{1}{2}\sqrt{\frac{\pi\mu'\epsilon}{\sigma\mu}}\,\frac{|U(a)|^2}{a}\,\frac{f_{n,m}}{\sqrt{1-\frac{f_{n,m}^2}{f^2}}}\;\frac{\left(\frac{f_{n,m}}{f}\right)^{\frac{3}{2}}\gamma^2 a^2\int_0^a|V|^2\,d\eta+\left(\frac{f_{n,m}}{f}\right)^{-\frac{1}{2}}\left(1-\frac{f_{n,m}^2}{f^2}\right)\int_0^a\left|\frac{\partial V}{\partial\eta}\right|^2d\eta}{\int_0^a\int_0^a\left\{\left|V\frac{\partial U}{\partial\xi}\right|^2+\left|U\frac{\partial V}{\partial\eta}\right|^2\right\}d\xi\,d\eta} \tag{27}$$

The values of the attenuation for both waves are in agreement with those found by Chu for the pipe of elliptical cross section. Both expressions for the attenuation have one term that varies as $f^{\frac{1}{2}}$, thus increasing with frequency. Thus it of interest to note that the behavior of electromagnetic waves, insofar as the above theory applies, is the same for all cylindrical pipes whose

whose cross sections are defined by coordinates which separate the scalar wave equation.

Experimental work is now in progress to determine ways in which the various partial fields may be excited and to check the values of the critical frequencies and attenuation constants for various modes.

V. APPENDIX

We now consider the method of obtaining the solution of

$$\frac{d^2 U}{d\xi^2} + (r^2\xi^2 + m)U = 0 \tag{28}$$

Letting $U = e^{\frac{i r \xi^2}{2}} M(\xi)$ we obtain

$$\frac{d^2 M}{d\xi^2} + 2ir\xi\frac{dM}{d\xi} + (m + ir)M = 0 \quad . \tag{29}$$

Setting $x = ir\xi^2$ yields

$$x\frac{d^2 M}{dx^2} + \left(\frac{1}{2} - x\right)\frac{dM}{dx} - \left(\frac{1}{4} - \frac{im}{4r}\right)M = 0 \quad . \tag{30}$$

This is the same form as Krummer's first confluent hypergeometric equation

$$x\frac{d^2 M}{dx^2} + (c - x)\frac{dM}{dx} - aM \quad , \tag{31}$$

where $c = \frac{1}{2}$ and $a = \left(\frac{1}{4} - \frac{im}{4r}\right)$.

Now let

$$M = x^{\rho}\sum_n C_n x^n \quad . \tag{32}$$

P is determined from the indicial equation

$$\rho(\rho - 1) + c\rho = 0$$

and since $c = \frac{1}{2}$ $P = 0$ or $\frac{1}{2}$. The coefficients for the series are given by $C_{n+1}(n+1)(n+c) - C_n(n-a) = 0$

If $C_0 = 1$ the series is

$$\sum_n C_n x^n = 1 - \frac{a}{c}x + \frac{a(a-1)}{c(c+1)}\frac{x^2}{2!} - \frac{a(a-1)(a-2)}{c(c+1)(c+2)}\frac{x^3}{3!}\ldots \tag{33}$$

Since p is double valued M may be either odd or even in respect to ξ .

$$_eM = \sum_n C_n (i\nu\xi^2)^n \qquad (34a)$$

$$_oM = \sqrt{i\nu}\,\xi \sum_n C_n (i\nu\xi^2)^n \qquad (34b)$$

If a or (c-a) is an interger the series in finite and we obtain Hermetian or Sonine polynomials. However this condition can be satisfied only if the separation constant is permitted to take on imaginary values and this is impossible if we wish to satisfy real boundary conditions.

The final solution of our original differential equation must be real and so we take

$$_eU = \cos\frac{\nu\xi^2}{2}\ {}_eM_{real} + \sin\frac{\nu\xi^2}{2}\ {}_eM_{imag.} \qquad (35a)$$

$$_oU = \cos\frac{\nu\xi^2}{2}\ {}_oM_{real} + \sin\frac{\nu\xi^2}{2}\ {}_eM_{imag.} \qquad (35b)$$

These are given in the integral form by Weber:

$$_eU = \int_0^1 s^{\frac{1}{4}-1}(1-s)^{\frac{1}{4}-1} \cos\left(\nu\xi^2 (s-\tfrac{1}{2}) + \frac{m}{4\nu} \log\frac{s}{1-s}\right) ds \qquad (36a)$$

$$_oU = \xi \int_0^1 s^{\frac{3}{4}-1}(1-s)^{\frac{3}{4}-1} \cos\left(\nu\xi^2 (s-\tfrac{1}{2}) + \frac{m}{4\nu} \log\frac{s}{1-s}\right) ds \qquad (36b)$$

For the case where m = o the differential equation is a special form of the Bessel equation:

$$y'' + b x^2 y = 0$$

which has the solutions

$$y = \sqrt{x}\, J_{\frac{1}{4}}\left(\frac{\sqrt{b}}{2} x^2\right) \tag{37a}$$

$$y = \sqrt{x}\, J_{-\frac{1}{4}}\left(\frac{\sqrt{b}}{2} x^2\right) \tag{37b}$$

Thus

$$_e u_o = \sqrt{\xi}\, J_{-\frac{1}{4}}\left(\frac{r\xi^2}{2}\right) \tag{38a}$$

$$_o u_o = \sqrt{\xi}\, J_{\frac{1}{4}}\left(\frac{r\xi^2}{2}\right) \tag{38b}$$

BIBLIOGRAPHY

1. Lord Rayleigh, On Passage of Electric Waves Through Tubes or the Vibrations of Dielectric Cylinders. Philosophical Magazine, 43; 125. (1897)

2. Southworth, G.C., Hyper-Frequency Wave Guides. The Bell System Technical Journal. 15; 284. (1936)

3. Carson, J.R., Mead, S.P., and Schelkunoff, S.A., Hyper-Frequency Wave Guides-Mathematical Theory. The Bell System Technical Journal. 15; 310 (1936)

4. Chu, L.J., Electromagnetic Waves in Elliptic Hollow Pipes of Metal. Journal of Applied Physics, 9; 583, (1938)

5. Chu, L.J., and Barrow, W.L., Electromagnetic Waves in Hollow Pipes of Rectangular Cross Section. Proceedings of the Institute of Radio Engineers, 26; 1550 (1930)

6. Page, L.I., and Adams, N.I., Electromagnetic Waves in Conducting Tubes. Physical Review 52; 647 (1937)

7. Chu, L.J., Calculations of the Radiation Properties of Hollow Pipes and Horns. Journal of Applied Physics; 11, 603, (1940)

MICHIGAN STATE UNIV. LIBRARIES
31293017749643