SOME ASPECTS OF THE MESON

THEORY OF NUCLEAR FORCES

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I. General Discussion and Purpose

The main purpose of this paper is to illustrate some of the elementary, general principles of field theory and its application to nuclear problems. This object will be realized by selecting a specific field postulate, by applying the general principles to this particular case to obtain the field equations, and by applying these equations to the solution of a particular problem for which experimental data are available.

The field postulate that will be selected will be that of the charged meson theory in which the field quantities will be specified by two complex conjugate 4-vectors, U and U*. After having obtained the equations of motion for these fields (the electromagnetic field will, of course, also play an important part in the following development), these equations will be applied to the problem of the calculation of the exchange magnetic moment of the deuteron. The equations will also be used to obtain an expression for the potential function employed to describe the nature of the interaction between two nuclear particles.

These last two objectives are to some extent mutually exclusive, as the charged-meson-field-postulate does not allow, to the order of approximation employed in this paper, for interactions between

nucleons of the same charge; whereas charged mesons must be assumed if the phenomenon of exchange currents between nuclear particles is to be predicted by the theory. A more general field postulate, the pseudoscalar field, is available to remedy this particular defect but will not be considered in this work.

There are three reasons for the choice of the exchange magnetic moment problem as a test case for a particular field theory in its application to nuclear physics. 1) There is considerable reliable experimental data on nuclear magnetic moments with which to compare the quantitative predictions of the theory. 2) Some of these data are apparently not explainable on the basis of "classical" assumptions, that is, new assumptions concerning the nature of intra-nuclear forces must be made--perhaps even new assumptions concerning the nature of the nucleus itself. 3) The problem of carrying through the calculations necessary for a quantitative prediction is a comparatively straightforward process.

II. <u>The Field Concept and Its Mathematical Formalism</u>

A field can be defined as a region in space in which at every point there is associated, in a unique manner, one or more quantities. The field equations are then the analytical representations of the dependence of these quantities upon their position in space and time. These equations are therefore the field analogues of the classical equations of motion of particle mechanics. In fact, as pointed out by Schiff[1], the concept of wave fields as employed in quantum field theory is actually a sort of generalization-to-the-continuum of classical many-body mechanics.

As in particle mechanics, the task of setting up the equations of motion of the system may proceed from several starting points. For the purposes of this development, we shall start from the basic assumption that the correct equations of motion can be derived from the variational principle when applied to a proper Lagrangian

$$\delta \int_t L \, dt = 0. \tag{2.1}$$

(For a more general discussion of field theories and their properties, see Pauli[2].)

The first problem is therefore to construct the proper Lagrangian function to describe the system. To ensure Lorentz invariance of the resultant equations of motion, the Lagrangian shall consist only of quantities that are relativistically invariant. Since we shall be dealing with nucleons (protons and neutrons) and their interactions with an electromagnetic and a meson field, the Lagrangian will, in general, consist of five parts.

$$L = L^P \; L^N \; L^M \; L^{EM} \; L^I$$

where

L^P = proton Lagrangian

L^N = neutron Lagrangian

L^M = meson Lagrangian

L^{EM} = electromagnetic field Lagrangian

L^I = interaction Lagrangian.

It should be noted here that fundamental to the field theory is the postulate that there is no particle-particle interaction, only interactions between particles and fields. The specific form of the component parts of the above function will depend, of course, upon the nature of the wave equation assumed for the nucleons and the type of meson field chosen for study. The general procedure for construction of the Lagrangian for the nucleons and the electromagnetic field are well known and will not be entered into here. However, the interacting terms will first be written into the particle Lagrangians to illustrate the manner in which the field assumptions affect the nuclear particles.

III. The Equations for the Vector Meson Field

Yukawa[3] was the first to recognize that the range properties of nuclear forces could be at least qualitatively accounted for by the assumption that there is associated with nucleons a field which, if described by the scalar quantity, ϕ, satisfies the equation

$$\Box \phi - \kappa^2 \phi = 0 \tag{3.1}$$

where $$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}, \quad \phi = \frac{e^{-\kappa |r_k - r_i|}}{|r_k - r_i|}$$

$|r_k - r_i|$ being the distance between the i^{th} and k^{th} nucleons. This may be immediately generalized in analogy with the electromagnetic system of equations (the above corresponding to Laplace's equation for the electrostatic field) by designating $i\phi = U_4$, the fourth, or time, component of a 4-vector, and U_1, U_2, U_3 as the space components of the same vector and obtain the set of four equations

$$\Box \vec{U} - K^2 \vec{U} = 0. \tag{3.2}$$

If we compare this with the Klein-Gordon equation for a free particle

$$E^2 = \vec{p}^2 + M^2 c^4 \tag{3.3}$$

which becomes

$$\Box \psi \; \frac{-M^2 c^2}{h^2} \psi = 0 \tag{3.4}$$

after making the operator substitutions

$$E \rightarrow i\hbar \frac{\partial}{\partial t}, \qquad \vec{p} \rightarrow -i\hbar \text{ grad},$$

we see that we may identify a heavy "quantum" associated with the U field whose rest mass is related to the range constant, K by

$$K = \frac{MC}{\hbar}, \qquad M = \frac{K\hbar}{C}.$$

The relationship between K and the rest mass of the particle associated with the field equation

$$\Box \psi - K^2 \psi = 0$$

may be interestingly brought out in another manner.

Separate the variables and designate the separation constant by ω^2

$$\frac{\nabla^2 \psi(r)}{\psi(r)} = \omega^2 \tag{3.5}$$

$$\frac{-1}{\psi(t)\, c^2} \frac{\partial^2 \psi(t)}{\partial t^2} - K^2 = \omega^2 \tag{3.6}$$

Solve the time equation, obtaining

$$\psi(t) = A e^{\pm i c \sqrt{K^2 + \omega^2}\, t}$$

thus indicating the nature of the time dependence of the solutions. To this point we have regarded the equation as a field equation.

Now let us regard it as the equation of motion of a particle and solve for the energy eigenvalues of this system.

$$E\psi = i\hbar \frac{\partial \psi}{\partial t}$$

$$i\hbar\, \psi^{-1}(t) \frac{\partial \psi(t)}{\partial t} = E\, \psi(t)\, \psi(t)$$

Choosing the negative exponential, we have

$$E = \hbar c \sqrt{K^2 + \omega^2}$$

But if $\nabla^2 \psi = 0$, the eigenvalues of the momentum operator are zero; the particle is at rest. Since

$$\omega = 0,\ E = K\hbar c = MC^2$$

we have $$M = \frac{K\hbar}{C}.$$

This quantum, or meson, plays an entirely analogous role in nuclear interactions to that of the photon in electromagnetic quantum theory of radiation. In the former case, the nucleons emit and absorb mesons in the same fashion that charged particles emit and absorb photons. Hence in formulating the problem mathematically it would seem reasonable to be guided to some extent by the formalism already developed for electromagnetic radiation theory. For instance, a charged particle in an electrostatic field experiences potential energy due to its position in that field given by

$$PE = e\,\phi .$$

Similarly a nuclear particle in a meson field will experience potential energy given by

$$PE = gU_4$$

where g is a strength constant characteristic of the particle, and has the same dimensions as charge, $cm^{3/2} sec^{-1} gm^{1/2}$. Since in the electromagnetic case the Lagrangian for the field is known to be

$$L^{EM} = \int_\tau \frac{F_{\kappa\beta}^2}{4}\, d\tau \qquad (3.7)$$

where

$$F_{\alpha\beta} = \frac{\partial \varphi_\beta}{\partial X_\alpha} - \frac{\partial \varphi_\alpha}{\partial X_\beta} \qquad (3.8)$$

the φ_κ 's being the components of the electromagnetic 4-potential, the proper Lagrangian for the meson field alone is assumed to be

$$L^{m} = \int_\tau \left\{ \frac{|G_{\rho\nu}|^2}{2} + K^2 |U_\rho|^2 \right\} d\tau \qquad (3.9)$$

where

$$G_{\rho\nu} = \frac{\partial U_\nu}{\partial X_\rho} - \frac{\partial U_\rho}{\partial X_\nu} \qquad (3.10)$$

However, the mesons in this theory are assumed to be charged and hence will be acted upon by electromagnetic fields. In general, the momentum operator, $\frac{\partial}{\partial X_\rho}$, for a particle in the absence of an external field transforms in the presence of a field as follows:

$$\frac{\partial}{\partial X_\rho} \rightarrow \frac{\partial}{\partial X_\rho} - \frac{ie}{c\hbar}\varphi_\rho$$

Hence the tensor, $G_{\rho\nu}$, will be transformed in the presence of an electromagnetic field

$$G_{\rho\nu} \rightarrow \left(\frac{\partial}{\partial X_\rho} - \frac{ie}{c\hbar}\varphi_\rho\right)U_\nu - \left(\frac{\partial}{\partial X_\nu} - \frac{ie}{c\hbar}\varphi_\nu\right)U_\rho \qquad (3.11)$$

and its complex conjugate

$$G_{\rho\nu}^* \rightarrow \left(\frac{\partial}{\partial X_\rho} + \frac{ie}{c\hbar}\varphi_\rho\right)U_\nu^* - \left(\frac{\partial}{\partial X_\nu} + \frac{ie}{c\hbar}\varphi_\nu\right)U_\rho^*$$

Two sets of field quantities are necessary: one set to be associated with positive mesons and the other with negative mesons. These equations were first set up by Proca[4] as a device for linearization of the wave equations for the electron.

Hence in this theory the complete Lagrangian for the mesons, including their interaction with the electromagnetic field, is assumed to be

$$L^M = \int_\tau \left(\frac{G_{\rho\nu} G_{\rho\nu}^*}{2} + \kappa^2 U_\rho U_\rho^* \right) d\tau \qquad (3.12)$$

(For a discussion of the mathematical formulation of the many other types of meson theory, see Frohlich, Heitler, and Kemmer[5], and Kemmer[6]; and for a detailed development of the theory of charged mesons, see Bhabha[7].)

IV. The Nucleon Equations

For lack of a better one, Dirac's equation will be chosen to describe the motions of the nucleons. To put it in 4-vector notation we proceed as follows: The proton equation in the electromagnetic field is

$$E - e\phi = c(\alpha \cdot p) - e(\alpha \cdot A) + \beta mc^2 \qquad (4.1)$$

where

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The quantities α and β are Dirac's 4 x 4 matrices, and the σ 's are Pauli's spin matrices. Rearranging and going over to the operator equation where

$$E \rightarrow i\hbar \frac{\partial}{\partial t}, \quad p \rightarrow -i\hbar \nabla$$

we have

$$c\hbar\left(\frac{\partial}{\partial X_4} - \frac{ie}{c\hbar}\varphi_4\right) - c\hbar(i\alpha)\cdot\left(\nabla - \frac{ie}{c\hbar}A\right) + \beta mc^2\Big]\psi = 0 \qquad (4.2)$$

Multiply through by β, and define a new set of matrices as follows:

$$\gamma^\rho = -i\beta\alpha^\rho \qquad \alpha^4 = \epsilon \qquad \gamma^4 = -i\beta\epsilon = \beta$$

ϵ being $+\sqrt{-1}$. It is to be noted that throughout these calculations ϵ is not regarded as a complex number, that is, if

$$A = a + \epsilon b, \quad A^* \neq a - \epsilon b,$$

A, a, and b being real quantities. Employing the above definitions, we obtain

$$\gamma^4\left[\gamma^\rho\left(\frac{\partial}{\partial X_\rho} - \Pi\,\varphi_\rho\right) + \frac{M_P c}{\hbar}\right]\psi_P = 0 \qquad (4.3)$$

And for the neutron

$$\gamma^4\left[\gamma^\rho\frac{\partial}{\partial X_\rho} + \frac{M_N c}{\hbar}\right]\psi_N = 0 \qquad (4.4)$$

where

$$\Pi = \frac{ie}{c\hbar}.$$

If now we assume that a nucleon experiences potential energy due to its position in the meson field generated by nearby nucleons, then additional terms of the form gU and gU* must be added to the above equations. And this brings us to what is perhaps the most important concept in the whole field theory: the concept of a mechanism whereby forces of interaction between particles are generated.

We designate the meson fields due to the presence of nucleons by U and U*. Let us assume that the proton interacts with the U*-field, the energy accruing to the proton due to the interaction being gU*. Furthermore the proton is assumed to absorb a negative meson and become a neutron. Likewise neutrons in the U-field would have potential energy equal to gU and would absorb a positive meson and become a proton. Symbolically this exchange process that is postulated as the mechanism of nucleon interactions may be represented as follows:

$$P + M^- \rightarrow N$$

$$N + M^+ \rightarrow P.$$

Or conversely

$$P \rightarrow M^+ + N$$

$$N \rightarrow M^- + P.$$

To incorporate the new energy terms and the exchange process into the nucleon equations, we first postulate that the proton and neutron states of a nucleon are merely the different states of the same particle. We introduce a new coordinate into the description of the system. This coordinate is usually referred to as the isotopic spin variable. Now define an eight-rowed, one column eigenfunction, Ψ , which we may refer to as the nucleon eigenfunction.

$$\Psi(x, y, z, \sigma, \text{charge}) = \begin{pmatrix} \psi_N \\ \hline \psi_P \end{pmatrix}$$

Define the following 8 x 8 matrices

$$\tau_{PN} = \frac{\tau_1 + i\tau_2}{2} = \frac{1}{2}\left[\left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right) + i\left(\begin{array}{c|c} 0 & -i \\ \hline i & 0 \end{array}\right)\right] = \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right)$$

$$\tau_{NP} = \frac{\tau_1 - i\tau_2}{2} = \frac{1}{2}\left[\left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right) - i\left(\begin{array}{c|c} 0 & -i \\ \hline i & 0 \end{array}\right)\right] = \left(\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right)$$

The following equations are readily verifiable

$$\tau_{PN}\,\Psi = \begin{pmatrix} \psi_N \\ 0 \end{pmatrix} \qquad \tau_{NP}\,\Psi = \begin{pmatrix} 0 \\ \psi_P \end{pmatrix}$$

The τ 's are the charge exchange operators.

We may now write the two nucleon equations as a single equation

$$\gamma^4\Big[\gamma^\rho\Big(\frac{\partial}{\partial x_\rho} - \mathrm{II}\,\tau_{NP}\,\varphi_\rho \quad - \tau_{NP}\, g\, U_\rho^* - \tau_{PN}\, g\, U_\rho\Big) + \tau_{NP}\,\frac{M_P c}{\hbar} + \tau_{PN}\,\frac{M_N c}{\hbar}\Big]\Psi = 0 \quad (4.5)$$

All matrices in the equation are, of course, 8 x 8

$$\gamma^\rho = \left(\begin{array}{c|c} \gamma^\rho & 0 \\ \hline 0 & \gamma^\rho \end{array}\right) \qquad \text{,etc.}$$

It has been shown experimentally by Rabi[8] and his co-workers that the deuteron behaves in an applied electromagnetic field as though it possesses a non-vanishing electric quadrupole moment. That this experimental fact suggests the necessity of postulating a tensor interaction term in the potential function of the Lagrangian may be illustrated as follows: We consider the interaction between elements of charge $\rho_e\, d\tau_e$ in the electron cloud distribution about the nucleus and the elements of the charge $\rho_N\, d\tau_N$ in the distributed nuclear charge.

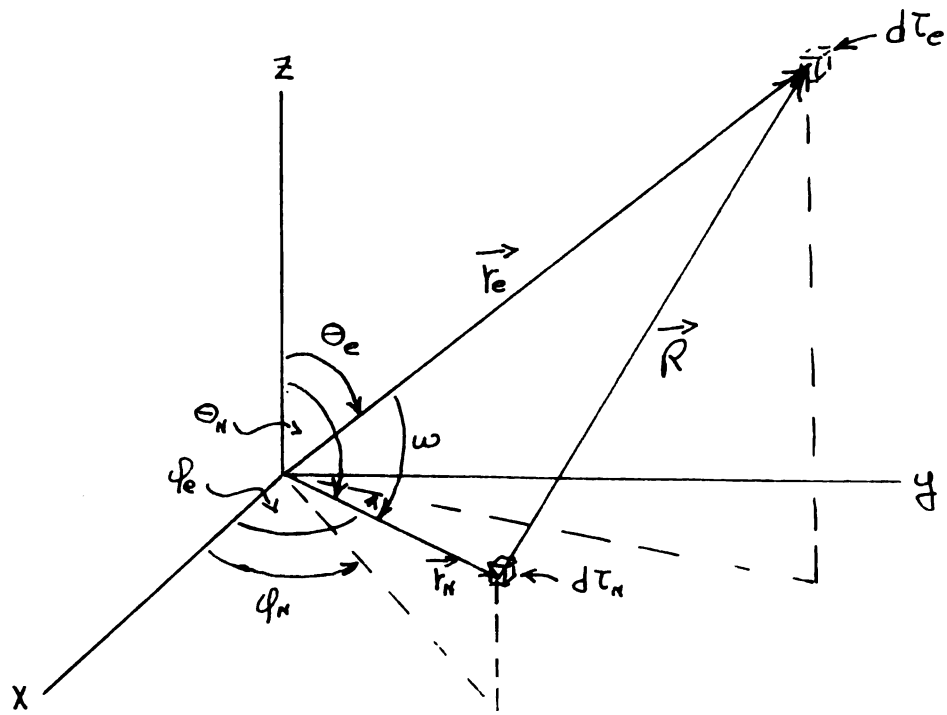


Figure 4.1

The interaction energy is given by

$$V = \int_{\tau_N}\int_{\tau_e} \frac{\rho_N d\tau_N \rho_e d\tau_e}{|R|} \tag{4.6}$$

where

$$\frac{1}{R} = \sum_{l=0}^{\infty} \left(\frac{r_N}{r_e}\right)^l \frac{P_l(\cos\omega)}{r_e} \qquad r_N < r_e \tag{4.7}$$

From the addition formula for Legendre polynomials[9], we have

$$P_l(\cos\omega) = \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!}(2-\delta_{o,m}) P_l^m(\cos\theta_e) P_l^m(\cos\theta_N) \cos m(\varphi_e - \varphi_N)$$

Inserting this in the equation for V and examining the terms for

l = 2, we have

$$V_{l=2} = \sum_{m=0}^{2} (2-\delta_{o,m}) \frac{(2-m)!}{(2+m)!} \int_{\tau_e} \frac{\rho_e d\tau_e}{r_e^3} P_2^m(\cos\theta_e) \times$$

$$\int_{\tau_N} \rho_N r_N^2 P_2^m(\cos\theta_N) \cos m(\varphi_e - \varphi_N)\, d\tau_N \tag{4.8}$$

If now we assume that the nuclear charge distribution is axially symmetric (that ρ_N does not depend upon ϕ_N), then the only non-vanishing contribution will be for M = 0. Thus

$$V_2 = \int_{\tau_e} \frac{\rho_e \, d\tau_e}{r_e^3} P_2(\cos\theta_e) \int_{\tau_N} \rho_N r_N^2 P_2(\cos\theta_N) \, d\tau_N . \quad (4.9)$$

It is the quantity

$$eQ = \int_{\tau_N} \rho_N r_N^2 (3\cos^2\theta_N - 1) \, d\tau_N$$

that Rabi detected experimentally. This integral indicates that the charge distribution in the nucleus exhibits angle dependence. But the charge distribution is a function of the forces between particles, hence it is reasonable to assume that these forces must also be angle dependent, that is, non-central. In view of these considerations, we add to the potential function of the nuclear equation the tensor interaction terms

$$\frac{g_2}{2K} \tau_{PN} \gamma^{\rho\nu} G_{\rho\nu} \qquad \frac{g_2}{2K} \tau_{NP} \gamma^{\rho\nu} G_{\rho\nu}^*$$

The nucleon equation is now

$$\gamma^4 \Big[\gamma^\rho \Big(\frac{\partial}{\partial x_\rho} - \frac{\Pi}{} \tau_{NP} \phi_\rho - g_1 \tau_{NP} U_\rho^* - g_1 \tau_{PN} U_\rho$$

$$- \frac{g_2}{2K} \gamma^\nu \{ \tau_{PN} G_{\rho\nu} + \tau_{NP} G_{\rho\nu}^* \} \Big) + \tau_{NP} \frac{M_P c}{\hbar} + \tau_{PN} \frac{M_N c}{\hbar} \Big] \Psi \quad (4.10)$$

This equation is still objectionable as it predicts an integral dipole moment (in units of nuclear magnetons) for the proton and no dipole moment for the neutron. This situation could be somewhat improved by adding additional terms of the form

$$\tau_{PN}\,\mu_N \cdot H + \tau_{NP}\left(\mu_P - \mu_B\right) \cdot H$$

where μ_N and μ_P are the observed dipole moments of the neutron and proton respectively. This paper, however, is concerned primarily with the exchange properties of the charged meson theory, and consequently the incorporation of these terms would be merely an unnecessary complication.

To put the nucleon equation into the form of a Lagrangian we note that if H is any Hermitian operator, the requirement that

$$\delta \int_t \int_\tau \breve{\psi}\, H\, \psi \, d\tau\, dt = 0$$

leads to

$$\int_t \int_\tau \left[(\delta \breve{\psi})\, H\, \psi + \breve{\psi}\, H\, (\delta\psi) \right] d\tau\, dt = 0$$

$$\int_t \int_\tau \left[(\delta \breve{\psi})\, H\, \psi + \overbrace{(\delta \breve{\psi})\, H\, \psi} \right] d\tau\, dt = 0$$

$$H\psi = 0 \quad , \quad \overbrace{H\psi} = 0$$

Thus we obtain two equations, one the Hermitian conjugate of the other. Hence we obtain the desired form for the nucleon equation by defining

$$\epsilon\, \breve{\Psi}\, \gamma^4 = \overline{\Psi}$$

and writing

$$\bar{\Psi}\Big[\gamma^{\rho}\Big(\frac{\partial}{\partial x_{\rho}} - \Pi\,\tau_{NP}\,\varphi_{\rho} - g_{1}\,\tau_{NP}\,U_{\rho}^{*} - g_{1}\,\tau_{PN}\,U_{\rho}$$

$$- \frac{g_{2}}{2K}\,\gamma^{\nu}\Big\{\tau_{PN}\,G_{\rho\nu} + \tau_{NP}\,G_{\rho\nu}^{*}\Big\}\Big) + \tau_{NP}\,\frac{M_{P}c}{\hbar} + \tau_{PN}\,\frac{M_{N}c}{\hbar}\Big]\Psi \quad (4.11)$$

for the final form of the Lagrangian density of the nucleons in the electromagnetic and meson fields.

For convenience in calculation, we break it down into three parts:

$$\bar{L}^{P} = \sum_{i} \bar{\psi}_{P}^{i}\Big[\gamma^{\rho}\Big(\frac{\partial}{\partial x_{\rho}} - \Pi\,\varphi_{\rho}\Big) + \frac{M_{P}c}{\hbar}\Big]\psi_{P}^{i} \quad (4.12)$$

$$\bar{L}^{N} = \sum_{j} \bar{\psi}_{N}^{j}\Big[\gamma^{\rho}\frac{\partial}{\partial x_{\rho}} + \frac{M_{N}c}{\hbar}\Big]\psi_{N}^{j} \quad (4.13)$$

$$\bar{L}^{I} = \sum_{i,j} \bar{\Psi}\Big[-g_{1}\,\gamma^{\rho}\Big(\tau_{NP}^{i}\,U_{\rho}^{*} + \tau_{PN}^{j}\,U_{\rho}\Big)$$

$$- \frac{g_{2}}{2K}\,\gamma^{\rho\nu}\Big(\tau_{NP}^{i}\,G_{\rho\nu}^{*} + \tau_{PN}^{j}\,G_{\rho\nu}\Big)\Big]\Psi \quad (4.14)$$

The bar indicates that these are Lagrangian densities. To summarize, we have also

$$\bar{L}^{M} = \frac{G_{\rho\nu}\,G_{\rho\nu}^{*}}{2} + K^{2}\,U_{\rho}\,U_{\rho}^{*} \quad (4.15)$$

and

$$\bar{L}^{EM} = \frac{F_{\rho\nu}\,F_{\rho\nu}}{4} \quad (4.16)$$

for the meson and electromagnetic fields. The summations are over i protons and j neutrons, thus theoretically extending the validity of the results to cases of many particles.

V. The Derivation of the Equations of Motion

Now that the Lagrangians for the system under consideration have been set up, the process of deriving the equations of motion is, in principle at least, a simple one. We proceed as in classical particle mechanics and require that

$$\delta \int_t (L^P + L^N + L^M + L^I + L^{EM}) \, dt = 0,$$

or, as in the previous notation,

$$\delta \int_t \left[\int_\tau (\bar{L}^P + \bar{L}^N + \bar{L}^M + \bar{L}^I + \bar{L}^{EM}) \, d\tau \right] dt = 0 \qquad (5.1)$$

To illustrate the procedure, we will carry through the variation with respect to U_ρ in detail. The other equations resulting from variations with respect to U_ρ^* and φ_ρ will then be simply written down.

As the Lagrangians have been defined, it is seen that the coordinate U_ρ does not appear in $\bar{L}^P$, $\bar{L}^N$, or $\bar{L}^{EM}$. Hence

$$\delta \int_\Omega L \, d\Omega = \int_\Omega (\delta L^I + \delta L^M) \, d\Omega = 0 \qquad (5.2)$$

Consider first

$$\delta \int_\Omega L^M d\Omega = \int_\Omega \delta \left[\frac{G_{\rho\nu} G_{\rho\nu}^*}{2} + K^2 U_\rho U_\rho^* \right] d\Omega \qquad (5.3)$$

$$= \int_\Omega \frac{G_{\rho\nu}^*}{2} \left[\delta\left(\frac{\partial U_\nu}{\partial x_\rho} - \Pi \varphi_\rho U_\nu \right) - \delta\left(\frac{\partial U_\rho}{\partial x_\nu} - \Pi \varphi_\nu U_\rho \right) + K^2 U_\rho^* \delta U_\rho \right] -$$

$$= \int_{\Omega}\left\{\frac{G_{\rho\nu}^{*}}{2}\left[\frac{\partial(\delta U_{\nu})}{\partial X_{\rho}} - \frac{\partial(\delta U_{\rho})}{\partial X_{\nu}} - \Pi \varphi_{\rho}\,\delta U_{\nu} + \Pi \varphi_{\nu}\,\delta U_{\rho}\right]\right.$$

$$\left. + K^{2} U_{\rho}^{*}\,\delta U_{\rho}\right\} d\Omega$$

$$= \int_{S_1} \frac{G_{\rho\nu}^{*}}{2}(\delta U_{\nu})\Big]_{(X_\rho)_1}^{(X_\rho)_2} dS_1 - \int_{S_2} \frac{G_{\rho\nu}^{*}}{2}(\delta U_{\rho})\Big]_{(X_\nu)_1}^{(X_\nu)_2} dS_2$$

$$+ \frac{1}{2}\int_{\Omega}\left[-\delta U_{\nu}\frac{\partial G_{\rho\nu}^{*}}{\partial X_{\rho}} + \delta U_{\rho}\frac{\partial G_{\rho\nu}^{*}}{\partial X_{\nu}} + G_{\rho\nu}^{*}\left(\Pi \varphi_{\nu}\,\delta U_{\rho}\right.\right.$$

$$\left.\left. - \Pi \varphi_{\rho}\,\delta U_{\nu}\right) + 2K^{2} U_{\rho}^{*}\,\delta U_{\rho}\right] d\Omega$$

The integrals over S_1 and S_2 vanish because of the requirement that $\delta U_{\beta}(X_{\alpha})_1 = \delta U_{\beta}(X_{\alpha})_2 = 0$ at the boundaries. Remembering that repeated indices are to be summed and that

$$G_{\rho\nu} = - G_{\nu\rho}$$

Eq. (5.3) becomes

$$\frac{1}{2}\int_{\Omega}\left[\delta U_{\rho}\frac{\partial G_{\rho\nu}^{*}}{\partial X_{\nu}} + \delta U_{\rho}\frac{\partial G_{\rho\nu}^{*}}{\partial X_{\nu}} + 2\Pi G_{\rho\nu}^{*}\varphi_{\nu}\,\delta U_{\rho}\right.$$

$$\left. + K^{2} U_{\rho}^{*}\,\delta U_{\rho}\right] d\Omega$$

$$= \int_{\Omega}\left[\frac{\partial G_{\rho\nu}^{*}}{\partial X_{\nu}} + \Pi G_{\rho\nu}^{*}\varphi_{\nu} + K^{2} U_{\rho}^{*}\right]\delta U_{\rho}\, d\Omega \tag{5.4}$$

Consider now

$$\delta \int_\Omega L^I d\Omega = \int_\Omega \sum_{i,j} \bar{\Psi} \Big[-g_1 \gamma^\rho \tau_{PN}^j \, \delta U_\rho$$

$$- \frac{g_2}{\partial K} \gamma^{\rho\nu} \tau_{PN}^j \, \delta G_{\rho\nu} \Big] d\Omega \tag{5.5}$$

Considering the second term in the integrand by itself

$$\sum_{i,j} - \frac{g_2}{\partial K} \int_\Omega \bar{\Psi} \gamma^{\rho\nu} \tau_{PN}^j \Big[\Big(\frac{\partial(\delta U_\nu)}{\partial X_\rho} - \Pi \varphi_\rho \, \delta U_\nu \Big)$$

$$- \Big(\frac{\partial(\delta U_\rho)}{\partial X_\nu} - \Pi \varphi_\nu \, \delta U_\rho \Big) \Big] \Psi \, d\Omega \tag{5.6}$$

$$= \sum_{i,j} - \frac{g_2}{\partial K} \int_{S_1} \bar{\Psi} \gamma^{\rho\nu} \tau_{PN}^j \Psi \, \delta U_\nu \Big]_{(X_\rho)_1}^{(X_\rho)_2} dS_1$$

$$+ \sum_{i,j} - \frac{g_2}{\partial K} \int_{S_2} \bar{\Psi} \gamma^{\rho\nu} \tau_{PN}^j \Psi \, \delta U_\rho \Big]_{(X_\nu)_1}^{(X_\nu)_2} dS_2$$

$$- \sum_{i,j} - \frac{g_2}{\partial K} \int_\Omega \Big[\delta U_\nu \frac{\partial}{\partial X_\rho} \left(\bar{\Psi} \gamma^{\rho\nu} \tau_{PN}^j \Psi \right)$$

$$- \delta U_\rho \frac{\partial}{\partial X_\nu} \left(\bar{\Psi} \gamma^{\rho\nu} \tau_{PN}^j \Psi \right) + \Pi \varphi_\rho \, \delta U_\nu \bar{\Psi} \gamma^{\rho\nu} \tau_{PN}^j \Psi$$

$$- \Pi \varphi_\nu \, \delta U_\rho \bar{\Psi} \gamma^{\rho\nu} \tau_{PN}^j \Psi \Big] d\Omega$$

Noting that $\gamma^{\rho\nu} = -\gamma^{\nu\rho}$ and recalling the remainder of the integrand in Eq. (5.5), we obtain

$$\int_\Omega \sum_{i,j} \left\{ -\frac{g_2}{K}\left[\frac{\partial}{\partial X_\nu}\left(\bar{\Psi}\gamma^{\rho\nu}\tau_{PN}^{j}\Psi\right) + \Pi\varphi_\nu \bar{\Psi}\gamma^{\rho\nu}\tau_{PN}^{j}\Psi\right] - g_1\bar{\Psi}\gamma^{\rho}\tau_{PN}^{j}\Psi\right\}\delta U_\rho \, d\Omega \tag{5.7}$$

Hence

$$\delta\int_\Omega L\,d\Omega = \int_\Omega \left[\left(\frac{\partial}{\partial X_\nu} + \Pi\varphi_\nu\right)\left(G_{\rho\nu}^{*} - \sum_j \frac{g_2}{K}\bar{\Psi}\gamma^{\rho\nu}\tau_{PN}^{j}\Psi\right) - \sum_j g_1\bar{\Psi}\gamma^{\rho}\tau_{PN}^{j}\Psi + K^2 U_\rho^{*}\right]\delta U_\rho \, d\Omega = 0$$

Thus the equations of motion resulting from the variation with respect to U are

$$\left(\frac{\partial}{\partial X_\nu} + \Pi\varphi_\nu\right)\left(G_{\rho\nu}^{*} - \sum_j \frac{g_2}{K}\bar{\Psi}\gamma^{\rho\nu}\tau_{PN}^{j}\Psi\right) - g_1\sum_j \bar{\Psi}\gamma^{\rho}\tau_{PN}^{j}\Psi + K^2 U_\rho^{*} = 0 \tag{5.8}$$

The conjugate equations are

$$\left(\frac{\partial}{\partial X_\nu} - \Pi\varphi_\nu\right)\left(G_{\rho\nu} - \sum_i \frac{g_2}{K}\bar{\Psi}\gamma^{\rho\nu}\tau_{NP}^{i}\Psi\right) - g_1\sum_i \bar{\Psi}\gamma^{\rho}\tau_{NP}^{i}\Psi + K^2 U_\rho = 0 \tag{5.9}$$

The equations that result from the variation with respect to φ_ρ are

$$\frac{\partial F_{\rho\nu}}{\partial x_\nu} - \sum_{\iota} \overline{\psi}_P^{\,\iota} \, \Pi \, \gamma^\rho \psi_P^{\,\iota} + \Pi \left(U_\nu^* G_{\rho\nu} - U_\nu G_{\rho\nu}^* \right)$$

$$+ \sum_{\iota,\lambda} \frac{g_2}{K} \overline{\Psi} \, \Pi \, \gamma^{\rho\nu} \left(U_\nu \tau_{PN}^{\lambda} - U_\nu^* \tau_{NP}^{\iota} \right) \Psi = 0. \qquad (5.10)$$

These last equations are of particular importance, as they yield the expression for current and charge density due to protons and charged mesons. We rewrite it as follows

$$\frac{\partial F_{\rho\nu}}{\partial x_\nu} = \frac{J}{c} = \frac{J^P}{c} + \frac{J^M}{c}$$

where

$$\frac{J^P}{c} = \sum_{\iota} \overline{\psi}_P^{\,\iota} \, \Pi \, \gamma^\rho \psi_P^{\,\iota} \qquad (5.11)$$

and

$$\frac{J^M}{c} = \Pi \left(U_\nu G_{\rho\nu}^* - U_\nu^* G_{\rho\nu} \right)$$

$$+ \sum_{\iota,\lambda} \frac{g_2}{K} \overline{\Psi} \, \Pi \, \gamma^{\rho\nu} \left(U_\nu^* \tau_{NP}^{\iota} - U_\nu \tau_{PN}^{\lambda} \right) \Psi. \qquad (5.12)$$

VI. The Hamiltonian

Since we will not be concerned with the specific form of the nucleon Hamiltonians, we will indicate them by H^N and H^P for neutrons and protons respectively. The Hamiltonian density for the remainder of the system may be obtained in the usual way

$$\bar{H} = \sum {}_u\Pi_\rho \mathring{U}_\rho + {}_{u^*}\Pi_\rho \mathring{U}_\rho^* + {}_\varphi\Pi_\rho \mathring{\varphi}_\rho - \bar{L}^M - \bar{L}^{EM} - \bar{L}^I$$

The quantities ${}_u\Pi_\rho$ are the momenta canonically conjugate to the field coordinates U_ρ. They may be obtained from the defining relations

$${}_u\Pi_\rho = -\frac{i}{c}\frac{\partial \bar{L}}{\partial U_{\rho,4}} \quad , \quad {}_{u^*}\Pi_\rho = -\frac{i}{c}\frac{\partial \bar{L}}{\partial U^*_{\rho,4}} \quad , \quad {}_\varphi\Pi_\rho = -\frac{i}{c}\frac{\partial \bar{L}}{\partial \varphi_{\rho,4}}$$

where the notation $\varphi_{\rho,4}$ indicates $\frac{\partial \varphi_\rho}{\partial x_4}$. The momenta are

$${}_u\Pi_\rho = \frac{i}{\mathcal{X}c}\left[G^*_{\rho 4} - \frac{g_2}{K}\bar{\Psi}\gamma^{\rho 4}\tau_{PN}\Psi\right] \tag{6.1}$$

$${}_{u^*}\Pi_\rho = \frac{i}{\mathcal{X}c}\left[G_{\rho 4} - \frac{g_2}{K}\bar{\Psi}\gamma^{\rho 4}\tau_{NP}\Psi\right] \tag{6.2}$$

and

$${}_\varphi\Pi_\rho = \frac{i}{\mathcal{X}c}F_{\rho 4}. \tag{6.3}$$

The summation over nucleons has been neglected since we shall be concerned only with the potential energy terms contributed to the Hamiltonian by the interaction between two nucleons. Hence we have

$$\bar{H} = \bar{H}^P + \bar{H}^N + \frac{i}{\mathcal{X}c}\left[G^*_{\rho 4} - \frac{g_2}{K}\bar{\Psi}\gamma^{\rho 4}\tau_{PN}\Psi\right]\frac{\partial U_\rho}{\partial t}$$

$$+ \frac{i}{\mathcal{X}c}\left[G_{\rho 4} - \frac{g_2}{K}\bar{\Psi}\gamma^{\rho 4}\tau_{NP}\bar{\Psi}\right]\frac{\partial U^*_\rho}{\partial t} + \frac{i}{\mathcal{X}c}F_{\rho 4}\frac{\partial \varphi_\rho}{\partial t}$$

$$+\bar{\Psi}\Big[g_1\gamma^\rho\big(U_\rho\tau_{PN}+U_\rho^*\tau_{NP}\big)+\frac{g_2}{2K}\gamma^{\rho\nu}\big(G_{\rho\nu}\tau_{PN}$$

$$+G_{\rho\nu}^*\tau_{NP}\big)\Big]\Psi-\left(\frac{G_{\rho\nu}G_{\rho\nu}^*}{2}+K^2U_\rho U_\rho^*\right)-\left(\frac{F_{\rho\nu}^2}{4}\right). \tag{6.4}$$

VII. Reduction to 3-Vector Notation

For purposes of further calculation it is desirable to rewrite the equations of motion in ordinary 3-vector notation. We make use of the following defining relations between the tensors and 4-vectors employed previously, and the 3-vector that will be employed hereafter.

$$G_{\rho\nu} = \begin{pmatrix} 0 & G_3 & -G_2 & -iF_1 \\ -G_3 & 0 & G_1 & -iF_2 \\ G_2 & -G_1 & 0 & -iF_3 \\ iF_1 & iF_2 & iF_3 & 0 \end{pmatrix}$$

plus conjugate relations.

$$F_{\rho\nu} = \begin{pmatrix} 0 & H_3 & -H_2 & -iE_1 \\ -H_3 & 0 & H_1 & -iE_2 \\ H_2 & -H_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}$$

$$\bar{\Psi}\gamma^{\rho\nu}\Psi = \breve{\Psi}\begin{pmatrix} 0 & -\beta\sigma_3 & \beta\sigma_2 & -\beta\alpha_1 \\ \beta\sigma_3 & 0 & -\beta\sigma_1 & -\beta\alpha_2 \\ -\beta\sigma_2 & \beta\sigma_1 & 0 & -\beta\alpha_3 \\ \beta\alpha_1 & \beta\alpha_2 & \beta\alpha_3 & 0 \end{pmatrix}\Psi$$

$$(S, T) = \left(\frac{g_2}{K} \breve{\Psi} \beta \sigma \tau_{PN} \Psi, \frac{g_2}{K} \breve{\Psi} \beta \alpha \tau_{PN} \Psi \right)$$

$$(M, N) = \left(g_1 \breve{\Psi} \alpha \tau_{PN} \Psi, g_1 \breve{\Psi} \tau_{PN} \Psi \right)$$

and the conjugate vectors S^*, T^*, M^* and the scalar n^* involving the converse exchange operators, τ_{NP}.

Employing the above definitions, it is easily verifiable that the equations of motion become

$$(\nabla - \Pi A) \times (G + S^*) - \left(\frac{1}{c}\frac{\partial}{\partial t} - \Pi A_4\right)(F + T^*) - M^* + K^2 U = 0 \tag{7.1}$$

$$(\nabla - \Pi A) \cdot (F + T^*) - N^* + K^2 U_4 = 0 \tag{7.2}$$

plus conjugate equations. Equation (5.12) becomes

$$\frac{J^m}{c} = \Pi [U \times (G^* + S)] - \Pi [U^* \times (G + S^*)] - \Pi [U_4^* F - U_4 F^*]$$
$$- \Pi [U_4^* T^* - U_4 F] \tag{7.3}$$

for the current density and

$$J_4^m = \epsilon \Pi [U \cdot F^* - U^* \cdot F] - \epsilon \Pi [U \cdot T - U^* \cdot T^*] \tag{7.4}$$

for the charge density.

Again it is to be noted that the summation over the nucleons has been neglected. It would be entirely feasible to bring the summation through to the above equations, and in fact necessary, if one were to apply these equations to the problem of the Triton for example. This paper, however, is in the main restricted to the two particle problem, so we shall dispense with the many particle notation henceforth.

The Hamiltonian in vector notation is

$$\overline{H} = \overline{H}^{P} + \overline{H}^{N} + \frac{1}{2c}\Big[(F^* \cdot \frac{\partial U}{\partial t}) + (F \cdot \frac{\partial U^*}{\partial t}) + (T \cdot \frac{\partial U}{\partial t}) +$$

$$(T^* \cdot \frac{\partial U^*}{\partial t}) + (E \cdot \frac{\partial A}{\partial t})\Big] + (M \cdot U) + (M^* \cdot U^*)$$

$$(NU_4 + N^*U_4^*) + (T \cdot F) + (T^* \cdot F^*) - (G \cdot S) - (G^* \cdot S^*)$$

$$- K^2(U \cdot U^* + U_4U_4^*) - (G \cdot G^*) + (F \cdot F^*) + \tfrac{1}{2}(E^2 - H^2). \qquad (7.5)$$

The defining relations for G and F are

$$G = (\nabla \times U) + \Pi\,(U \times A) \qquad (7.6a)$$

$$G^* = (\nabla \times U^*) - \Pi\,(U^* \times A) \qquad (7.6b)$$

$$F = -\frac{1}{c}\frac{\partial U}{\partial t} - \nabla U_4 - \Pi A_4 U + \Pi U_4 A \qquad (7.7a)$$

$$F^* = -\frac{1}{c}\frac{\partial U^*}{\partial t} - \nabla U_4^* + \Pi A_4 U^* - \Pi U_4^* A \qquad (7.7b)$$

VIII. Integral Representation of the Field Quantities

It is desirable at this point to express the field quantities in an analytical form that lends itself to the application to specific problems. The method employed for this purpose is identical in principle to that developed by Møller and Rosenfield[10] and by Ma and Yu[11]. The actual technioue differs slightly.

We restrict ourselves at the outset to problems that depend only on the stationary states of the system, hence all time derivatives may be neglected. We further assume that the motion of the nucleons in a given system is such that non-relativistic wave equations will adequately describe them. Thus all quantities involving the operator α will be small and may be neglected. The latter approximation depends upon the interesting significance of α

as a velocity operator.

The operator to be associated with the time-derivative of a coordinate x is defined in quantum mechanics by

$$\frac{dx}{dt} = v_x = \frac{i}{\hbar}[Hx - xH].$$

The Hamiltonian that appears in the commutator for this case is given by Eq. (4.1). Since the quantities depending upon the field coordinates commute with the quantities depending upon the space coordinates, it is only necessary to evaluate

$$[-c(\alpha\cdot\nabla)x + xc(\alpha\cdot\nabla)]\psi$$

$$= -cx(\alpha\cdot\nabla\psi) - c\psi(\alpha\cdot\nabla_x x) + xc(\alpha\cdot\nabla\psi)$$

$$= -c\psi\alpha_x$$

hence the operator to be associated with v_x is

$$-i\hbar v_x \rightarrow -ic\hbar\alpha_x$$

$$v_x \rightarrow c\alpha_x \quad , \text{ or } v \rightarrow c\alpha.$$

Consider

$$c\alpha\psi = \lambda\psi$$

where the λ 's are the eigenvalues of the velocity of the system;

$$\alpha\psi = \frac{\lambda}{c}\psi$$

$$\tilde{\psi}\alpha\psi = \frac{\lambda}{c}\tilde{\psi}\psi$$

so that for velocities appreciably less than the velocity of light, quantities involving α will be small and can reasonably be neglected. A non-relativistic approximation for the meson cannot be made.

A further simplification of the equations results from the fact that the electromagnetic field quantities need not be considered in the cases proposed herein. In fact, one of the reasons for restricting the actual computational problem to systems of at most one proton is to facilitate this particular simplification. Implicit in this approximation is the assumption that the electromagnetic interaction between a charged meson and a proton is small compared to proton-meson-field interaction. Of course this approximation also neglects the meson-meson interaction through the intermediation of the electromagnetic fields resulting from mesic charge.

As a consequence of these approximations, the field equations now become

$$\nabla \times (G + S^*) + \kappa^2 U = 0 \qquad (8.1)$$

$$\nabla \times (G^* + S) + \kappa^2 U^* = 0$$

$$\nabla \cdot F - N^* + \kappa^2 U_4 = 0 \qquad (8.2)$$

$$\nabla \cdot F^* - N + \kappa^2 U_4^* = 0$$

$$G = \nabla \times U, \quad G^* = \nabla \times U^* \qquad (8.3)$$

$$F = -\nabla U_4, \quad F^* = -\nabla U_4^*. \qquad (8.4)$$

And the equation for the current density due to exchange of charged mesons between nuclear particles becomes

$$\frac{J^m}{c} = \Pi \left[U \times (G^* + S) \right] - \Pi \left[U^* \times (G + S^*) \right] - \Pi \left[U_4^* F - U_4 F^* \right] \qquad (8.5)$$

whereas the charge density is

$$J_4^m = \epsilon\, \Pi \left[U \cdot F^* - U^* \cdot F \right] \qquad (8.6)$$

From the field equations we readily deduce that

$$\nabla^2 U - \kappa^2 U = \nabla \times S^* \qquad (8.7a)$$

$$\nabla^2 U^* - \kappa^2 U^* = \nabla \times S \qquad (8.7b)$$

$$\nabla^2 U_4 - K^2 U_4 = -N^* \tag{8.7c}$$

$$\nabla^2 U_4^* - K^2 U_4^* = -N. \tag{8.7d}$$

Now, in general

$$\int_\tau (\psi \nabla^2 \phi - \phi \nabla^2 \psi) d\tau = \int_\sigma (\psi \nabla \phi - \phi \nabla \psi) \cdot \vec{d\sigma} = 0$$

as the surface recedes to infinity. Also

$$\nabla^2 \psi = K^2 \psi - \delta(r - r')$$

so

$$\int_\tau \{ \psi \nabla^2 \phi - \phi [K^2 \psi - \delta(r-r')] \} d\tau = 0$$

$$4\pi \int_\tau \phi \, \delta(r - r') \, d\tau = - \int_\tau (\nabla^2 \phi - K^2 \phi) \psi \, d\tau$$

$$\phi(r') = - \frac{1}{4\pi} \int_\tau (\nabla^2 \phi - K^2 \phi) \psi(r - r') \, d\vec{r} \tag{8.8}$$

Having defined δ (r-r') by the relation

$$\nabla^2 \psi - K^2 \psi = - \delta(r-r')$$

for regions of no nucleons, we prove that

$$\int_\tau \delta(r - r') d\tau = 4\pi$$

as follows

$$- \int_\tau \delta(r - r') d\tau = \int_\tau (\nabla^2 \psi - K^2 \psi) d\tau = \int_\tau [\nabla \cdot (\nabla \psi) - K^2 \psi] d\tau$$

$$= \int_\sigma \nabla \psi \cdot d\sigma - K^2 \int_\tau \psi \, d\tau$$

Now

$$\nabla \frac{e^{-\kappa r}}{r} = \left(- \frac{e^{-\kappa r}}{r^3} - \frac{\kappa e^{-\kappa r}}{r^2} \right) \vec{r}$$

so we have

$$\int_\tau \delta(r - r')d\tau = \int_\sigma \left(\frac{e^{-\kappa r}}{r^3} + \frac{\kappa e^{-\kappa r}}{r^2} \right) \vec{r} \cdot \vec{d\sigma} - \kappa^2 \int_\tau \frac{e^{-\kappa r}}{r} d\tau$$

$$\text{and } \lim_{r \to r'} \int_\tau \delta(r - r')d\tau = \lim_{r \to 0} \left[4\pi e^{-\kappa r} + 4\pi r \kappa e^{-\kappa r} - \frac{4\pi r^2 \kappa^2}{3} \right]$$

$$= 4\pi .$$

It follows immediately from Eq. (8.7) and Eq. (8.8) that

$$U\,(r') = - \frac{1}{4\pi} \int_r (\nabla \times S^*)\, \varphi\,(r - r')\, \vec{dr}, \tag{8.9}$$

and

$$U_4\,(r') = \frac{1}{4\pi} \int_r N^* \varphi\,(r - r')\, \vec{dr}, \tag{8.10}$$

$$F\,(r') = - \frac{1}{4\pi} \int_r N^* \nabla' \varphi\,(r - r')\, \vec{dr}, \tag{8.11}$$

$$G\,(r') = - \frac{1}{4\pi} \int_r \nabla' \times [(\nabla \times S^*)\, \varphi\,(r - r')]\, \vec{dr}, \tag{8.12}$$

where

$$\varphi(r - r') = \frac{e^{-\kappa|r - r'|}}{|r - r'|} .$$

Equation (8.12) may be put in somewhat more convenient form as follows:

$$\nabla' \times [(\nabla \times S^*)\, \varphi\,(r - r')]$$

$$= \nabla' \varphi\,(r - r') \times (\nabla \times S^*) = - \nabla \varphi\,(r - r') \times (\nabla \times S^*)$$

$$= - \left\{ \nabla \times [\varphi(\nabla \times S^*)] - \varphi\, [\nabla \times (\nabla \times S^*)] \right\} .$$

So (8.12) becomes

$$G\,(r') = \frac{1}{4\pi} \int_r \nabla \times [\varphi(\nabla \times S^*)]\, \vec{dr} - \frac{1}{4\pi} \int_r \varphi\, [\nabla \times (\nabla \times S^*)]\, \vec{dr}. \tag{8.13}$$

The first integral may be transformed to a surface integral[12] which will vanish upon integration. Therefore

$$G\,(r') = -\frac{1}{4\pi} \int_r \varphi\, [\nabla \times (\nabla \times S^*)]\, \vec{dr}$$
$$= -\frac{1}{4\pi} \int_r \varphi\, [\nabla\, \nabla \cdot S^* - \nabla \cdot \nabla\, S^*]\, \vec{dr}$$
$$= \frac{1}{4\pi} \int_r \varphi\, \nabla \cdot \nabla\, S^*\, \vec{dr} - \frac{1}{4\pi} \int_r \varphi\, \nabla\, \nabla \cdot S^*\, \vec{dr}$$

The first integral may be partially integrated, the integrated part vanishing on the boundaries as the boundaries recede. We have finally

$$G\,(r') = \frac{1}{4\pi} \int_r S^*\, \nabla \cdot \nabla\, \varphi\, \vec{dr} - \frac{1}{4\pi} \int_r \varphi\, \nabla\, \nabla \cdot S^*\, \vec{dr}$$
$$= \frac{1}{4\pi} \int_r S^* [K^2 \varphi - \delta\,(r - r')]\, \vec{dr} - \frac{1}{4\pi} \int_r \varphi\, \nabla\, \nabla \cdot S^*\, \vec{dr}$$
$$= \frac{K^2}{4\pi} \int_r S^*\, \varphi\, \vec{dr} - \frac{1}{4\pi} \int_r \varphi\, \nabla\, \nabla \cdot S^*\, \vec{dr} - S^*. \tag{8.14}$$

IX. Calculation of the Exchange Dipole Moment of the Deuteron

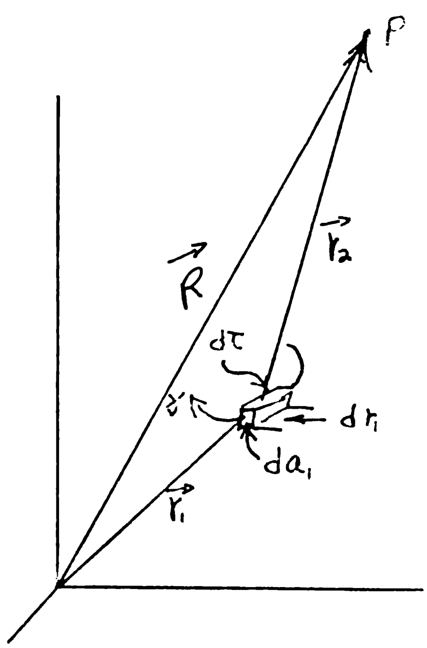


Figure 9.1

To calculate an exchange magnetic moment, it is necessary to obtain an expression for the magnetic moment of a volume distribution of current of density, J[13]. The components of the electromagnetic vector potential A (see Figure 9.1) are related to the volume distribution of current by

$$A_p = \frac{1}{c} \int_\tau \frac{J}{|r_2|} \, d\tau, \qquad (9.1)$$

where $\tau = \tau\,(x_1, y_1, z_1)$; x_1, y_1, z_1 being the coordinates of r_1-space, a region whose linear dimensions are small with respect to r_2 and R.

Now

$$f\,(\vec{r}_2) = f\,(\vec{R} - \vec{r}_1)$$

$$= \exp \left[-x_1 \frac{\partial}{\partial x} - y_1 \frac{\partial}{\partial y} - z_1 \frac{\partial}{\partial z} \right] f\,(R)$$

by Taylor's expansion. Hence

$$\frac{1}{r_2} = \exp \left[-x_1 \frac{\partial}{\partial x} - y_1 \frac{\partial}{\partial y} - z_1 \frac{\partial}{\partial z} \right] \frac{1}{(R)}.$$

So Eq. (9.1) becomes

$$A(x, y, z) = \frac{1}{c} \int_{\tau} J(x_1, y_1, z_1) \exp [- r_1 \cdot \nabla] \left(\frac{1}{R (x, y, z)}\right) d\tau . \tag{9.2}$$

The first term is

$$A = \frac{1}{c} \int_{\tau} \frac{J}{R} d\tau .$$

Now $I = |J|\, da_1$ is a current element along γ. So

$$I d\vec{r}_1 = |J|\, da_1 d\vec{r}_1 = \vec{J} d\tau .$$

Hence the volume integral reduces to a line integral along the path of a current filament. All filaments are presumed to close within the region, τ , hence

$$A = \frac{I}{cR} \int_{\gamma} d\vec{r}_1 = 0. \tag{9.3}$$

The second term in the expansion is

$$A_1 = \frac{1}{c} \int_{\tau} [- r_1 \cdot \nabla \left(\frac{1}{R}\right)] J d\tau = -\frac{I}{c} \int_{\gamma} [r_1 \cdot \nabla \left(\frac{1}{R}\right)] d\vec{r}_1. \tag{9.4}$$

We employ the identities

$$d \left\{ [r_1 \cdot \nabla \left(\frac{1}{R}\right)] r_1 \right\} = [r_1 \cdot \nabla \left(\frac{1}{R}\right)] dr_1 + [dr_1 \cdot \nabla \left(\frac{1}{R}\right)] r_1,$$

and

$$(r_1 \times dr_1) \times \nabla \left(\frac{1}{R}\right) = [r_1 \cdot \nabla \left(\frac{1}{R}\right)] dr_1 - [dr_1 \cdot \nabla \left(\frac{1}{R}\right)] r_1$$

to reduce (9.4) to

$$-\frac{I}{2c}\int_{\gamma} d\left\{\left[r_1 \cdot \nabla \frac{(1)}{(R)}\right] r_1\right\} \quad -\frac{I}{2c}\int_{\gamma} r_1 \times dr_1 \times \nabla \frac{(1)}{(R)}.$$

The first integral vanishes around a closed filament. Thus

$$A_1 = -\frac{I}{2c}\int_{\gamma} r_1 \times dr_1 \times \nabla \frac{(1)}{(R)}. \tag{9.5}$$

The dipole moment of the current filament is by definition

$$\mu_{\gamma} = \frac{I}{2c}\int_{\gamma} r_1 \times dr_1 = \frac{1}{2c}\int_{\gamma} r_1 \times Idr_1.$$

If we sum over all the filaments, that is, over the total current distribution, we obtain the dipole moment of the distribution

$$\mu = \frac{1}{2}\int_{\tau} (r_1 \times \frac{J}{c})\, d\tau , \tag{9.6}$$

since

$$Idr_1 = Jd\tau .$$

Thus the dipole moment of a system such as the deuteron may now be calculated from the following expression (see Eq. 8.5)

$$\mu \;\bar{=}\; \frac{\Pi}{2}\int_{r} r \times \left\{ \left[U \times (G^* + S)\right] - \left[U^* \times (G + S^*)\right] \right.$$

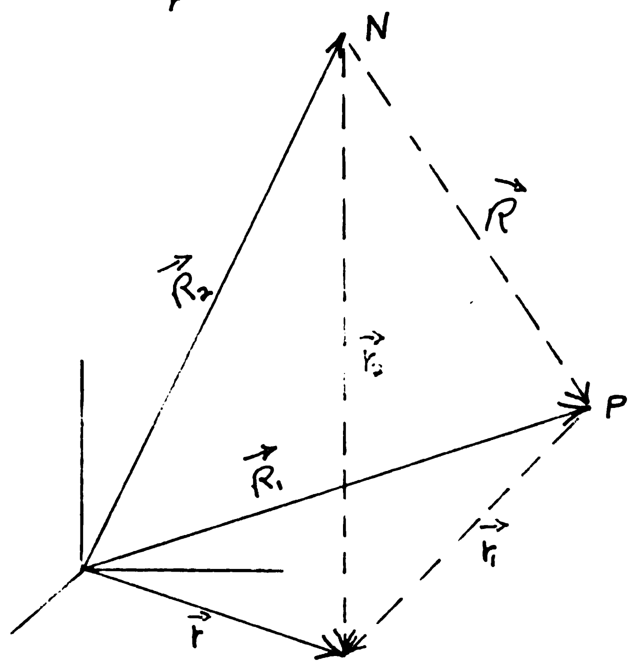
$$\left. - \left[U_4{}^*F - U_4F^*\right] \right\} \vec{dr}. \tag{9.7}$$

We evaluate two integrals that will be of assistance in later calculations

$$\frac{1}{16\pi^2}\quad \frac{e^{-\kappa|r - R_1| \;-\; \kappa|r - R_2|}}{|r - R_1|\;|r - R_2|}\;\vec{dr}, \tag{9.8}$$

and

$$\frac{1}{16\pi^2}\int_r \vec{r}\,\frac{e^{-\kappa|r - R_1|}}{|r - R_1|}\;\frac{e^{-\kappa|r' - R_2|}}{|r - R_2|}\,dr. \qquad (9.9)$$



$R = R_1 - R_2$

$\rho = R_1 + R_2$

$r_1 = r - R_1$

$r_2 = r - R_2$

$r_1 = r_2 + R_2 - R_1 = r_2 - R$

$r_2 = r_1 + R_1 - R_2 = r_1 + R$

Figure 9.2

The second integral may then be written

$$\frac{2}{16\pi^2}\int_r \vec{r}\,\varphi(r_1)\,\varphi(r_2)\,dr = \frac{1}{16\pi^2}\int_{r_1}(\vec{r}_1 + \vec{R}_1)\,\frac{e^{-\kappa|r_1| - \kappa|r_1 + R|}}{|r_1|\;|r_1 + R|}\,dr_1$$

$$+\;\frac{1}{16\pi^2}\int_{r_2}(\vec{r}_2 + \vec{R}_2)\,\frac{e^{-\kappa|r_2 - R| - \kappa|r_2|}}{|r_2 - R|\;|r_2|}\,dr_2$$

$$= \frac{1}{16\pi^2}(\vec{R}_1 + \vec{R}_2)\int_r \varphi(r_1)\,\varphi(r_2)\,dr$$

$$+ \frac{1}{16\pi^2}\int_{r_1} \vec{r}_1 \,\frac{e^{-K|r_1|}}{|r_1|}\,\frac{e^{-K|r_1+R|}}{|r_1+R|}\,dr_1$$

$$+ \frac{1}{16\pi^2}\int_{r_2} \vec{r}_2 \,\frac{e^{-K|r_2|}}{|r_2|}\,\frac{e^{-K|r_2-R|}}{|r_2-R|}\,dr_2 .$$

Let the variable of integration in each of the last two integrals be X. They then may be combined as follows:

$$\frac{1}{16\pi^2}\int_X \vec{X}\,\frac{e^{-K|X|}}{|X|}\left\{\frac{e^{-K|X+R|}}{|X+R|} + \frac{e^{-K|X-R|}}{|X-R|}\right\}dX .$$

The integrand is clearly odd and hence will integrate to an even function which will vanish when evaluated over space. So Eq. (9.9) becomes

$$\frac{\vec{\rho}}{32\pi^2}\int_r \varphi(r_1)\,\varphi(r_2)\,dr .$$

Write the last integral as follows:

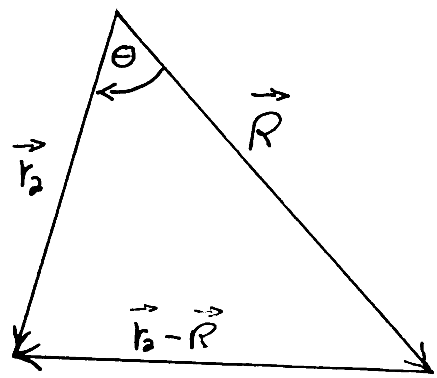


Figure 9.3

$$\frac{\rho}{32\pi^2}\int_{r_2} \frac{e^{-K|r_2 - R| - K|r_2|}}{|r_2|\,|r_2 - R|}\, \vec{dr_2}$$

$$|r_2 - R| = \left(r_2^2 + R^2 - 2r_2 R\cos\theta\right)^{1/2}$$

$$\vec{dr_2} = r_2^2 \sin\theta \, dr_2 \, d\theta \, d\varphi$$

When these substitutions are made, the integrations over φ and θ may be performed directly. We obtain

$$\frac{-\rho}{16\pi RK}\int_{r_2=0}^{\infty}\left[e^{-K\left(r_2^2+R^2-2r_2R\cos\theta\right)^{1/2}}\right]_{\theta=0}^{\pi} dr_2$$

$$= \frac{-\rho}{16\pi RK}\int_{r_2=0}^{\infty} e^{-K(r_2+R) - Kr_2}\, dr_2$$

$$+ \frac{\rho}{16\pi RK}\int_{r_2=0}^{R} e^{-K(R-r_2) - Kr_2}\, dr_2$$

$$+ \frac{\rho}{16\pi RK}\int_{r_2=R}^{\infty} e^{-K(r_2-R) - Kr_2}\, dr_2 .$$

When the integrations are performed, we have as a final result

$$\frac{1}{16\pi^2}\int_r \vec{r}\,\phi(r_1)\,\phi(r_2)\,dr = \frac{\vec{\rho}\,e^{-KR}}{16\pi K}\,. \tag{9.10}$$

Let us consider first the contribution to the exchange moment by the term

$$-\frac{\mathrm{II}}{2}\int_r \vec{r} \times \left[U_4^* F - U_4 F^*\right] dr$$

$$= -\frac{\mathrm{II}}{2}\int_r r \times \left[\left\{\frac{1}{4\pi}\int_{R_2} N_2\,\phi(r_2)\,dR_2\right\}\left\{-\frac{1}{4\pi}\int_{R_1} N_1^*\,\nabla\phi(r_1)\,dR_1\right\}\right] dr$$

$$+\frac{\mathrm{II}}{2}\int_r r \times \left[\left\{\frac{1}{4\pi}\int_{R_1} N_1^*\,\phi(r_1)\,dR_1\right\}\left\{-\frac{1}{4\pi}\int_{R_2} N_2\,\nabla\phi(r_2)\,dR_2\right\}\right] dr\,.$$

Making use of the relation $\nabla(\psi\phi) = \psi\nabla\phi + \phi\nabla\psi$

we have

$$= -\frac{\mathrm{II}}{32\pi^2}\int_r\int_{R_1}\int_{R_2} \phi(r_1)\,\phi(r_2)\,\vec{r} \times \left[N_1^*\,\nabla_2 N_2 - N_2\,\nabla_1 N_1^*\right] dR_1\,dR_2\,dr$$

$$= -\frac{\text{II}}{32\pi K}\int_{R_1}\int_{R_2} \vec{\rho} \times \left[N_1^* \nabla_2 N_2 - N_2 \nabla_1 N_1^*\right] e^{-\kappa R} dR_1 dR_2 .$$

Henceforward

$$e^{-\kappa R} = \psi , \quad \frac{e^{-\kappa R}}{R} = \phi(R).$$

Now

$$\int_{R_2} R_2 \times (\nabla_2 N_2)\psi \, dR_2 = \int_{R_2} N_2 [\nabla_2 \psi \times R_2] dR_2$$

by partial integration. In the same way

$$\int_{R_2} R_1 \times (\nabla_2 N_2)\psi \, dR_2 = -\int_{R_2} N_2 [R_1 \times \nabla_2 \psi] dR_2 .$$

Hence the integral becomes

$$\frac{-\text{II}}{32\pi K}\int_{R_1}\int_{R_2}\left\{-\left[N_1^* N_2 \vec{\rho} \times \nabla_2 \psi\right] + \left[N_1^* N_2 \vec{\rho} \times \nabla_1 \psi\right]\right\} dR_1 dR_2 .$$

Since
$$\nabla_1 \psi = -\kappa \vec{R}\, \phi(R)$$
$$\nabla_2 \psi = \kappa \vec{R}\, \phi(R)$$

so we have

$$= \frac{\text{II}}{16\pi K}\iint_{R_1 R_2} N_1^* N_2 \, (\rho \times R)\, \phi(R)\, dR_1 dR_2$$

$$= \frac{ie}{16\pi c k \hbar} \iint_{R_1 R_2} N_1^* N_2 \, (\rho \times R) \, \phi(R) \, dR_1 \, dR_2 . \tag{9.11}$$

Consider now the remainder of the expression for the exchange dipole moment.

$$\frac{i}{2}\int_r r \times \left\{ \left[U \times (G^* + S) \right] - \left[U^* \times (G + S^*) \right] \right\} dr$$

$$= \frac{-iK^2}{32\pi^2}\int_r \vec{r}\, \phi(r_1)\phi(r_2) \times \left\{ \int_{R_1} (\nabla_1 \times S_1^*) \times \int_{R_2} S_2 \right\} dR_1 dR_2 dr$$

$$+ \frac{i}{32\pi^2}\int_r \vec{r}\, \phi(r_1)\phi(r_2) \times \left\{ \int_{R_1} (\nabla_1 \times S_1^*) \times \int_{R_2} \nabla_2 \nabla_2 \circ S_2 \right\} dR_1 dR_2 dr$$

$$+ \frac{iK^2}{32\pi^2}\int_r \vec{r}\, \phi(r_1)\phi(r_2) \times \left\{ \int_{R_2} (\nabla_2 \times S_2) \times \int_{R_1} S_1^* \right\} dR_1 dR_2 dr$$

$$- \frac{i}{32\pi^2}\int_r \vec{r}\, \phi(r_1)\phi(r_2) \times \left\{ \int_{R_2} (\nabla_2 \times S_2) \times \int_{R_1} \nabla_1 \nabla_1 \circ S_1^* \right\} dR_1 dR_2 dr$$

$$= \frac{-iK}{32\pi}\int_{R_1}\int_{R_2} \vec{\rho} \times \left[(\nabla_1 \times S_1^*) \times S_2 \right] \psi \, dR_1 \, dR_2 \tag{9.12a}$$

$$+ \frac{iK}{32\pi}\int_{R_1}\int_{R_2} \vec{\rho} \times \left[(\nabla_2 \times S_2) \times S_1^* \right] \psi \, dR_1 \, dR_2 \tag{9.12b}$$

$$+\frac{\text{II}}{32\pi K}\int_{R_1}\int_{R_2}\vec{\rho}\times\left[(\nabla_1\times S_1^*)\times(\nabla_2\nabla_2\cdot S_2)\right]\psi\, dR_1\, dR_2 \tag{9.12c}$$

$$-\frac{\text{II}}{32\pi K}\int_{R_1}\int_{R_2}\vec{\rho}\times\left[(\nabla_2\times S_2)\times(\nabla_1\nabla_1\cdot S_1^*)\right]\psi\, dR_1\, dR_2 . \tag{9.12d}$$

To illustrate the method whereby these integrals were evaluated, we shall carry through the process in detail for Eq. (9.12c). The others may be evaluated similarly.

Consider first

$$\int_{R_1}\vec{R_1}\times\left[(\nabla_1\times S_1^*)\times(\nabla_2\nabla_2\cdot S_2)\right]\psi\, dR_1 .$$

Let $\vec{A} = \nabla_2\nabla_2\cdot S_2$. We have, after partial integration over R_1

$$= \int_{R_1}(S_1^*\times A)\psi\, dR_1 + \vec{i}\int_{R_1}\Big[R_{1y}A_y(S_1^*\times\nabla_1\psi)_x$$

$$+ R_{1z}A_z(S_1^*\times\nabla_1\psi)_x + S_{1x}A_x(R_1\times\nabla_1\psi)_x$$

$$+ A_x\nabla_{1x}\psi(S_1^*\times R_1)_x\Big]dR_1 + \vec{j}\int_{R_1}\Big[(R_{1x}A_x$$

$$+ R_{1z}A_z)(S_1^*\times\nabla_1\psi)_y + S_{1y}^*A_y(R_1\times\nabla_1\psi)_y$$

$$+ A_y \nabla_{1y} \psi (S_1^* \times R_1)_y \Big] dR_1 \; + \vec{k} \int_{R_1} \Big[(R_{1x} A_x$$

$$+ R_{1y} A_y)(S_1^* \times \nabla_1 \psi)_z \; + S_{1z} A_z (R_1 \times \nabla_1 \psi)_z$$

$$+ A_z \nabla_{1z} \psi (S_1^* \times R_1)_z \Big] dR_1 .$$

This becomes, upon rearranging and collecting terms,

$$\int_{R_1} (S_1^* \times A) \psi \, dR_1 \; + \int_{R_1} (R_1 \circ A)(S_1^* \times \nabla_1 \psi) \, dR_1$$

$$+ \int_{R_1} A \big[R_1 \circ (\nabla_1 \psi \times S_1^*) \big] dR_1$$

$$= \int_{R_1} (S_1^* \times A) \psi \, dR_1 \; + \int_{R_1} R_1 \times \big[A \times (\nabla_1 \psi \times S_1^*) \big] dR_1 .$$

Employing the identity

$$\nabla \times (\vec{u} \phi) = \nabla \phi \times u + \phi (\nabla \times u)$$

we have

$$\int_{R_1} (\nabla_1 \times S_1^*) \psi \, dR_1 = \int_{R_1} (S_1^* \times \nabla_1 \psi) \, dR_1$$

so that

$$\int_{R_1}\int_{R_2} R_2 \times [(\nabla_1 \times S_1^*) \times A]\, \psi \, dR_1 \, dR_2$$

$$= \int_{R_1}\int_{R_2} R_2 \times [A \times (\nabla_1 \psi \times S_1^*)]\, dR_1 \, dR_2 \, .$$

Hence

$$\int_{R_1}\int_{R_2} \vec{\rho} \times [(\nabla_1 \times S_1^*) \times (\nabla_2 \nabla_2 \circ S_2)]\, \psi \, dR_1 \, dR_2$$

$$= \int_{R_1}\int_{R_2} [S_1^* \times (\nabla_2 \nabla_2 \circ S_2)]\, \psi \, dR_1 \, dR_2$$

$$+ \int_{R_1}\int_{R_2} \vec{\rho} \times [(\nabla_2 \nabla_2 \circ S_2) \times (\nabla_1 \psi \times S_1^*)]\, dR_1 \, dR_2 \, .$$

We now note that

$$\nabla_2 \nabla_2 \psi = \nabla_1 \nabla_1 \psi = \Phi$$

so the first integral becomes, after two partial integrations over R_2

$$- \int_{R_1}\int_{R_2} [(S_2 \circ \Phi) \times S_1^*]\, dR_1 \, dR_2$$

whereas the second one becomes

$$\iint_{R_1 R_2} \rho \times \left[(\nabla_2 \nabla_2 \circ S_2) \circ S_1^* \right] (\nabla_1 \psi) \, dR_1 dR_2$$

$$- \iint_{R_1 R_2} \rho \times \left[(\nabla_2 \nabla_2 \circ S_2) \circ \nabla_1 \psi \right] S_1^* \, dR_1 dR_2 .$$

The first of these vanishes because

$$\rho \times \left[(\nabla_2 \nabla_2 \circ S_2) \circ S_1^* \right] \nabla_1 \psi$$

$$= \left[(\nabla_2 \nabla_2 \circ S_2) \circ S_1^* \right] \left[(R_1 \times \nabla_1 \psi) + (R_2 \times \nabla_1 \psi) \right]$$

$$= \left[(\nabla_2 \nabla_2 \circ S_2) \circ S_1^* \right] \left[(R_1 \times \nabla_1 \psi) - (R_2 \times \nabla_2 \psi) \right]$$

and

$$R_1 \times \nabla_1 \psi = R_2 \times \nabla_2 \psi = \frac{K\psi}{R} (R_1 \times R_2) .$$

Hence the term (9.12c) becomes

$$- \iint_{R_1 R_2} \left[(S_2 \circ \Phi) \times S_1^* \right] dR_1 \, dR_2$$

$$- \iint_{R_1 R_2} \rho \times \left[(\nabla_2 \nabla_2 \circ S_2) \times \nabla_1 \psi \right] S_1^* \, dR_1 dR_2 . \tag{9.13}$$

In a similar manner (9.12d) becomes

$$- \iint_{R_1 R_2} \left[(S_1^* \circ \Phi) \times S_2 \right] dR_1 dR_2$$

$$-\iint_{R_1 R_2} \rho \times [(\nabla_1 \nabla_1 \circ S_1^*) \circ \nabla_2 \psi] S_2 \, dR_1 dR_2 . \tag{9.14}$$

Now

$$\iint_{R_1 R_2} \rho \times [(\nabla_2 \nabla_2 \circ S_2) \circ \nabla_1 \psi] S_1^* \, dR_1 dR_2$$

becomes, after two partial integrations,

$$-\iint_{R_1 R_2} (\rho \times S_1^*) [S_2 \circ \nabla_2 (\nabla_2 \circ \nabla_2 \psi)] \, dR_1 dR_2$$

$$+ \iint_{R_1 R_2} \{ [S_1^* \times (S_2 \circ \nabla_2 \nabla_2 \psi)] + [(S_1^* \times S_2)(\nabla_2 \circ \nabla_2 \psi)] \} dR_1 dR_2$$

and

$$\iint_{R_1 R_2} \rho \times [(\nabla_1 \nabla_1 \circ S_1^*) \circ \nabla_2 \psi] S_2 \, dR_1 dR_2$$

$$= -\iint_{R_1 R_2} (\rho \times S_2) [S_1^* \circ \nabla_1 (\nabla_1 \circ \nabla_1 \psi)] \, dR_1 dR_2$$

$$+ \iint_{R_1 R_2} \{ [S_2 \times (S_1^* \circ \nabla_1 \nabla_1 \psi)] + [(S_2 \times S_1^*)(\nabla_1 \circ \nabla_1 \psi)] \} dR_1 dR_2 .$$

We now have for the sum of the integrals (9.12c) and (9.12d)

$$-\iint_{R_1 R_2} [(S_2 \circ \Phi) \times S_1^*] \, dR_1 dR_2$$

$$+\iint_{R_1 R_2} (\rho \times S_1^*)\left[S_2 \circ \nabla_2 (\nabla_2 \circ \nabla_2 \psi)\right] dR_1 dR_2$$

$$-\iint_{R_1 R_2} \left\{\left[S_1^* \times (S_2 \circ \nabla_2 \nabla_2 \psi)\right] + \left[(S_1^* \times S_2)(\nabla_2 \circ \nabla_2 \psi)\right]\right\} dR_1 dR_2$$

$$+\iint_{R_1 R_2} \left[(S_1^* \circ \Phi) \times S_2\right] dR_1 dR_2$$

$$-\iint_{R_1 R_2} (\rho \times S_2)\left[S_1^* \circ \nabla_1 (\nabla_1 \circ \nabla_1 \psi)\right] dR_1 dR_2$$

$$+\iint_{R_1 R_2} \left\{\left[S_2 \times (S_1^* \circ \nabla_1 \nabla_1 \psi)\right] + \left[(S_2 \times S_1^*)(\nabla_1 \circ \nabla_1 \psi)\right]\right\} dR_1 dR_2$$

$$= \iint_{R_1 R_2} (\rho \times S_1^*)\left[S_2 \circ \nabla_2 (\nabla_2 \circ \nabla_2 \psi)\right] dR_1 dR_2$$

$$-\iint_{R_1 R_2} (\rho \times S_2)\left[S_1^* \circ \nabla_1 (\nabla_1 \circ \nabla_1 \psi)\right] dR_1 dR_2$$

$$-2\iint_{R_1 R_2} (S_1^* \times S_2)(\nabla \circ \nabla \psi)\, dR_1 dR_2 . \qquad (9.15)$$

In a similar manner it can be shown that the contribution from (9.12a) and (9.12b) is

$$-2\iint_{R_1 R_2} (S_1^* \times S_2)\, \psi \, dR_1 dR_2$$

$$- \iint_{R_1 R_2} \rho \times \left[(S_1^* \times \nabla_1 \psi) \times S_2 \right] dR_1 dR_2$$

$$+ \iint_{R_1 R_2} \rho \times \left[(S_2 \times \nabla_2 \psi) \times S_1^* \right] dR_1 dR_2 \quad . \tag{9.16}$$

Hence the total contribution to μ of terms involving ρ is

μ (terms involving ρ) =

$$\frac{ie}{16\pi c K \hbar} \iint_{R_1 R_2} (\rho \times R) N_1^* N_2 \Phi(R) \, dR_1 dR_2$$

$$+ \frac{ie}{32\pi c K \hbar} \iint_{R_1 R_2} (\rho \times S_1^*) \left[S_2 \circ \nabla_2 (\nabla_2 \circ \nabla_2 \psi) \right] dR_1 dR_2$$

$$- \frac{ie}{32\pi c K \hbar} \iint_{R_1 R_2} (\rho \times S_2) \left[S_1^* \circ \nabla_1 (\nabla_1 \circ \nabla_1 \psi) \right] dR_1 dR_2$$

$$- \frac{ieK}{32\pi c \hbar} \iint_{R_1 R_2} \rho \times \left[(S_1^* \times \nabla_1 \psi) \times S_2 \right] dR_1 dR_2$$

$$+ \frac{ieK}{32\pi c \hbar} \iint_{R_1 R_2} \rho \times \left[(S_2 \times \nabla_2 \psi) \times S_1^* \right] dR_1 dR_2 . \tag{9.17}$$

The terms involving ρ explicitly can all be made to vanish by locating the origin of our coordinate system midway between the two nucleons. The total contribution of terms independent of ρ is

$$\mu \text{ (independent of } \rho \text{)} = -\frac{\text{II}\, K}{16\pi c}\int_{R_1}\int_{R_2}(S_1^* \times S_2)\psi\, dR_1\, dR_2$$

$$\frac{-\text{II}}{16\pi K}\int_{R_1}\int_{R_2}(S_1^* \times S_2)(\nabla\cdot\nabla\psi)\, dR_1\, dR_2$$

$$= -\frac{ie}{16\pi c K \hbar}\int_{R_1}\int_{R_2}(S_1^* \times S_2)(\nabla\cdot\nabla + K^2)\psi\, dR_1\, dR_2.$$

Now

$$\nabla^2\psi = K^2 e^{-KR} - \frac{2Ke^{-KR}}{R}$$

and

$$m = \frac{\hbar K}{c}$$

so we have

$$\mu = -\frac{ie}{8\pi c^2 m}\int_{R_1}\int_{R_2}(S_1^* \times S_2)\left(K^2 - \frac{K}{R}\right)e^{-KR}\, dR_1\, dR_2. \tag{9.18}$$

This result agrees, except in sign, with that obtained by Ma and Yu[11]. It is obvious that for the deuteron $\mu = 0$, as the spin vectors S_1^* and S_2 must be either parallel or anti-parallel.

X. Calculation of the Potential Function for the Nuclear Particles in the Meson Field

To determine the potential function of the nuclear particles in the meson field, we must examine those terms contributed to the Hamiltonian by the particle-meson field interaction. Those terms are

$$\begin{aligned}\overline{V}_I = \; & \frac{i}{2c}\left(G^{*}_{\rho 4} - \frac{g_2}{K}\overline{\Psi}\,\gamma^{\rho 4}\,\tau_{PN}\,\Psi\right)\frac{\partial U_\rho}{\partial t} \\ & + \frac{i}{2c}\left(G_{\rho 4} - \frac{g_2}{K}\overline{\Psi}\,\gamma^{\rho 4}\,\tau_{NP}\,\Psi\right)\frac{\partial U^{*}_\rho}{\partial t} \\ & - g_1\,\overline{\Psi}\,\gamma^{\rho}\left(U_\rho \tau_{PN} + U^{*}_\rho \tau_{NP}\right)\Psi \\ & - \frac{g_2}{2K}\overline{\Psi}\,\gamma^{\rho\nu}\left(G_{\rho\nu}\tau_{PN} + G^{*}_{\rho\nu}\tau_{NP}\right)\Psi .\end{aligned} \tag{10.1}$$

If we consider only those terms independent of the time and make the usual non-relativistic approximation for the wave functions for the nucleons, Eq. (10.1) reduces to (in 3-vector notation)

$$\overline{V}_I = N^{*}U_4^{*} + NU_4 + S^{*}\cdot G^{*} + S\cdot G. \tag{10.2}$$

We may evaluate the potential energy from the integral

$$\int_\tau \overline{V}_I \, d\tau$$

We consider first the contribution from the terms

$$N^*U_4^* + NU_4$$

$$= \frac{1}{4\pi} N_1^* \int_{R_2} N_2 \,\phi(r_2)\, dR_2 + \frac{1}{4\pi} N_2 \int_{R_1} N_1^* \,\phi(r_1)\, dR_1$$

(Referring again to Figure 9.2)

$$= \frac{1}{4\pi}\int_{R_2} N_1^* N_2 \,\phi(r_2)\, dR_2 + \frac{1}{4\pi}\int_{R_1} N_1^* N_2 \,\phi(r_1)\, dR_1 . \tag{10.3}$$

Next, the terms

$$S_1^* \circ G^* + S_2 \circ G$$

$$= S_1^* \circ \left[\frac{K^2}{4\pi}\int_{R_2} S_2 \,\phi(r_2)\, dR_2 - \frac{1}{4\pi}\int_{R_2} \phi(r_2)\, \nabla_2 \nabla_2 \circ S_2 \, dR_2 - S_2\right]$$

$$+ S_2 \circ \left[\frac{K^2}{4\pi}\int_{R_1} S_1^* \,\phi(r_1)\, dR_1 - \frac{1}{4\pi}\int_{R_1} \phi(r_1)\, \nabla_1 \nabla_1 \circ S_1^* \, dR_1 - S_1^*\right]$$

$$= \frac{K^2}{4\pi}\int_{R_2} (S_1^* \circ S_2)\,\phi(r_2)\, dR_2 + \frac{K^2}{4\pi}\int_{R_1} (S_1^* \circ S_2)\,\phi(r_1)\, dR_1$$

$$- \frac{1}{4\pi}\int_{R_2} \phi(r_2)(S_1^* \circ \nabla_2 \nabla_2 \circ S_2)\, dR_2 - \frac{1}{4\pi}\int_{R_1} \phi(r_1)(S_2 \circ \nabla_1 \nabla_1 \circ S_1^*)\, dR_1$$

$$- 2(S_1^* \circ S_2). \tag{10.4}$$

Now

$$\int_\tau (\phi\,\nabla\nabla\cdot S)\,d\tau = \int_\tau \left[\nabla(\phi\,\nabla\cdot S) - (\nabla\cdot S)\nabla\phi\right]d\tau$$

$$= -\int_\tau (\nabla\cdot S)\,\nabla\phi\,d\tau = \int_\tau S\cdot\nabla\nabla\phi\,d\tau$$

so Eq. (10.4) becomes

$$\frac{\kappa^2}{4\pi}\int_{R_2}(S_1^*\cdot S_2)\,\phi(r_2)\,dR_2 + \frac{\kappa^2}{4\pi}\int_{R_1}(S_1^*\cdot S_2)\,\phi(r_1)\,dR_1$$

$$-\frac{1}{4\pi}\int_{R_2}\left[S_1^*\cdot S_2\cdot\nabla_2\nabla_2\phi(r_2)\right]dR_2 - \frac{1}{4\pi}\int_{R_1}\left[S_2\cdot S_1^*\cdot\nabla_1\nabla_1\phi(r_1)\right]dR_1$$

$$-2(S_1^*\cdot S_2). \tag{10.5}$$

If now we choose the origin at particle 1 (the proton) and evaluate the potential energy function due to meson interaction at the other particle, we have

$$\vec{R}_1 = -\vec{R}_2\,, \quad \phi(r_1) = \phi(r_2) = \phi(R)\,,$$

$$\nabla_1\nabla_1\phi(r_1) = \nabla_2\nabla_2\phi(r_2) = \nabla\nabla\phi(R)$$

so [Including Eq. (10.3)]

$$\overline{V}_I = \frac{1}{4\pi}\int_{R_2} N_1^* N_2\,\phi(R)\,dR_2 + \frac{1}{4\pi}\int_{R_1} N_1^* N_2\,\phi(R)\,dR_1$$

$$+\frac{\kappa^2}{4\pi}\int_{R_2}(S_1^*\circ S_2)\,\phi(R)\,dR_2 + \frac{\kappa^2}{4\pi}\int_{R_1}(S_1^*\circ S_2)\,\phi(R)\,dR_1$$

$$-\frac{1}{4\pi}\int_{R_2}\left[S_1^*\circ S_2\circ\nabla\nabla\phi(R)\right]dR_2 - \frac{1}{4\pi}\int_{R_1}\left[S_2\circ S_1\circ\nabla\nabla\phi(R)\right]dR_1$$

$$-2(S_1^*\circ S_2). \qquad (10.6)$$

Now

$$\nabla\nabla\phi(R) = \frac{1}{R^3}\left(\kappa^2 e^{-\kappa R} + \frac{3\kappa e^{-\kappa R}}{R} + \frac{3e^{-\kappa R}}{R^2}\right)\vec{R}\vec{R}$$

$$-\frac{1}{R^2}\left(\kappa e^{-\kappa R} + \frac{e^{-\kappa R}}{R}\right)I - \delta(r-R_i), \quad i = 1, 2$$

and where I is the idem factor, so

$$(S_1^*\circ S_2)\circ(\nabla\nabla\phi(R)) = 3\,\frac{(S_1^*\circ R)(S_2\circ R)}{R^2}\left(\frac{\kappa^2}{3}\right.$$

$$\left.+\frac{\kappa}{R} + \frac{1}{R^2}\right)\frac{e^{-\kappa R}}{R} - (S_1^*\circ S_2)\left(\frac{\kappa}{R} + \frac{1}{R^2}\right)\frac{e^{-\kappa R}}{R}$$

$$-\delta(r-R_i).$$

Let $f(R) = \left(\frac{\kappa^2}{3} + \frac{\kappa}{R} + \frac{1}{R^2}\right)$, then

$$\overline{V}_I = \frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}^{2}\frac{N_i^*}{4\pi}\int_{R_2} N_j\,\phi(R)\,dR_2 + \frac{N_j}{4\pi}\int_{R_1} N_i^*\,\phi(R)\,dR_1$$

$$+ \frac{2/3\, K^2}{4\pi} S_i^* \cdot \int_{R_2} S_j\, \Phi(R)\, dR_2 + \frac{2/3\, K^2}{4\pi} S_j \cdot \int_{R_1} S_i^*\, \Phi(R)\, dR_1$$

$$- \frac{S_i^*}{4\pi} \cdot \int_{R_2} \left[\frac{3R\,(S_j \cdot R)}{R^2} - S_j \right] f(R)\, \Phi(R)\, dR_2$$

$$- \frac{S_j}{4\pi} \cdot \int_{R_1} \left[\frac{3R\,(S_i^* \cdot R)}{R^2} - S_i^* \right] f(R)\, \Phi(R)\, dR_1$$

$$+ \int_R \delta(r - R_i)\, dR - 2\,(S_1^* \cdot S_2)\,. \tag{10.7}$$

The above expression is written as shown in an effort to bring out more clearly the precise nature of the integration indicated. The first integral, for example, expresses the potential energy of the neutron in the field of the proton; the second integral expresses the potential energy of the same proton in the field of the same neutron. Thus the potential energy of the system is actually just one-half the sum of the two space integrals (ignoring the spin dependent terms, though the same argument applies for these also). But the system possesses a further degree of freedom--the isotopic spin variable that characterizes the charge state of the nucleons. The integration should therefore be extended over the spin coordinates also, as indicated by the summation over i and j. After integrating and summing, we have

$$\overline{V}_I = g_1^2 \left(\tau_{NP}^{(1)} \tau_{PN}^{(2)} + \tau_{PN}^{(1)} \tau_{NP}^{(2)}\right) \frac{e^{-\kappa R}}{R}$$

$$+ \frac{2}{3} g_2^2 \left(\tau_{NP}^{(1)} \tau_{PN}^{(2)} + \tau_{PN}^{(1)} \tau_{NP}^{(2)}\right)(\sigma_1 \cdot \sigma_2) \frac{e^{-\kappa R}}{R}$$

$$+ g_2^2 \left(\tau_{NP}^{(1)} \tau_{PN}^{(2)} + \tau_{PN}^{(1)} \tau_{NP}^{(2)}\right)\Big[(\sigma_1 \cdot \sigma_2)$$

$$- \frac{3(\sigma_1 \cdot R)(\sigma_2 \cdot R)}{R^2}\Big] f(R) \frac{e^{-\kappa R}}{R}. \tag{10.8}$$

To consider the significance of the operator

$$\frac{\tau_{NP}^{(1)} \tau_{PN}^{(2)} + \tau_{PN}^{(1)} \tau_{NP}^{(2)}}{2}$$

we recall the original definitions

$$\tau_{NP} = \frac{\tau_1 - i\tau_2}{2}$$

and

$$\tau_{PN} = \frac{\tau_1 + i\tau_2}{2}$$

and expand as follows:

$$\frac{\tau_{NP}^{(1)} \tau_{PN}^{(2)} + \tau_{PN}^{(1)} \tau_{NP}^{(2)}}{2} = \frac{\tau_1^{(1)} \tau_1^{(2)} + \tau_2^{(1)} \tau_2^{(2)}}{2} = P^H.$$

The eigenfunctions of the isotopic variables may be written explicitly

$$\chi^1 = (\xi_1)(\xi_2)$$

$$\chi^2 = \frac{1}{\sqrt{2}}\left[(\xi_1)(\eta_2) + (\xi_2)(\eta_1)\right]$$

$$\chi^3 = (\eta_1)(\eta_2)$$

$$\chi^4 = \frac{1}{\sqrt{2}}\left[(\xi_1)(\eta_2) - (\xi_2)(\eta_1)\right]$$

where the character operator τ_3 operating on the state functions ξ and η yields[14]

$$\tau_3 \xi = \xi \qquad \tau_3 \eta = -\eta .$$

Hence ξ designates the neutron state and η the proton state of the nucleon. Making use of the following relations[9],

$$\tau_1 \xi = \eta \qquad \tau_2 \xi = i\eta \qquad \tau_3 \xi = \xi$$

$$\tau_1 \eta = \xi \qquad \tau_2 \eta = -i\xi \qquad \tau_3 \eta = -\eta$$

it is easily verifiable that

$$P^{M} \chi^{1} = 0 \qquad P^{M} \chi^{2} = \chi^{2} \qquad P^{M} \chi^{3} = 0 \qquad P^{M} \chi^{4} = -\chi^{4}.$$

Since the functions χ^1 and χ^3 represent two-nucleon systems of two neutrons or two protons, it is seen that the charged meson theory indicates no interaction between like particles to the order of approximation employed herein. In the case of unlike particles there is an interchange of charge states between the particles due to the operator P^M. It is to be noted that the form of the potential function derived from the charged meson hypothesis is identical in form with that obtained in the neutral theory[15], with the exception of the operator P^M. The neutral theory indicates interactions only between unlike particles.

XI. The Contribution to the Electric Quadrupole Moment of the Deuteron by Mesic Space Charge

In Section IV we derived the expression for the electric quadrupole moment

$$eQ = \int_{\tau_N} \rho_N r_N^2 (3\cos^2\theta_N - 1)\, d\tau_N.$$

From Section VIII Eq. (8.6) we obtain

$$\rho_N = \frac{J_4^m}{\epsilon} = \mathrm{II}\,[U \cdot F^* - U^* \cdot F].$$

Hence we have

$$\rho_N = \mathrm{II}\left[-\frac{1}{4\pi}\int_{R_1} (\nabla_1 \times S_1^*)\,\phi(r_1)\,dR_1 \cdot \left(-\frac{1}{4\pi}\right)\int_{R_2} N_2 \nabla\phi(r_2)\,dR_2\right]$$

$$-\mathrm{II}\left[-\frac{1}{4\pi}\int_{R_2} (\nabla_2 \times S_2)\,\phi(r_2)\,dR_2 \cdot \left(-\frac{1}{4\pi}\right)\int_{R_1} N_1^* \nabla\phi(r_1)\,dR_1\right]. \quad (11.1)$$

$$\rho_N = \frac{\mathrm{II}}{16\pi^2}\left[\int_{R_1}\int_{R_2}\left\{\left[\nabla_2\phi(r_2) \times \nabla_1\phi(r_1)\right]\cdot S_1^*\right\} N_2\, dR_1\, dR_2\right.$$

$$\left.+\int_{R_1}\int_{R_2}\left\{\left[\nabla_2\phi(r_2) \times \nabla_1\phi(r_1)\right]\cdot S_2\right\} N_1^*\, dR_1\, dR_2\right]$$

$$= \frac{\mathrm{II}}{16\pi^2}\int_{R_1}\int_{R_2} (N_2 S_1^* + N_1^* S_2)\cdot\left[\nabla_2\phi(r_2) \times \nabla_1\phi(r_1)\right] dR_1\, dR_2.$$

(11.2)

Now
$$\nabla_1 \phi(r_1) = \left[K\phi(r_1) + \frac{\phi(r_1)}{r_1}\right](\vec{r} - \vec{R}_1)$$

$$\nabla_2 \phi(r_2) = \left[K\phi(r_2) + \frac{\phi(r_2)}{r_2}\right](\vec{r} - \vec{R}_2).$$

Let
$$\left[K\phi(r) + \frac{\phi(r)}{r}\right] = A.$$

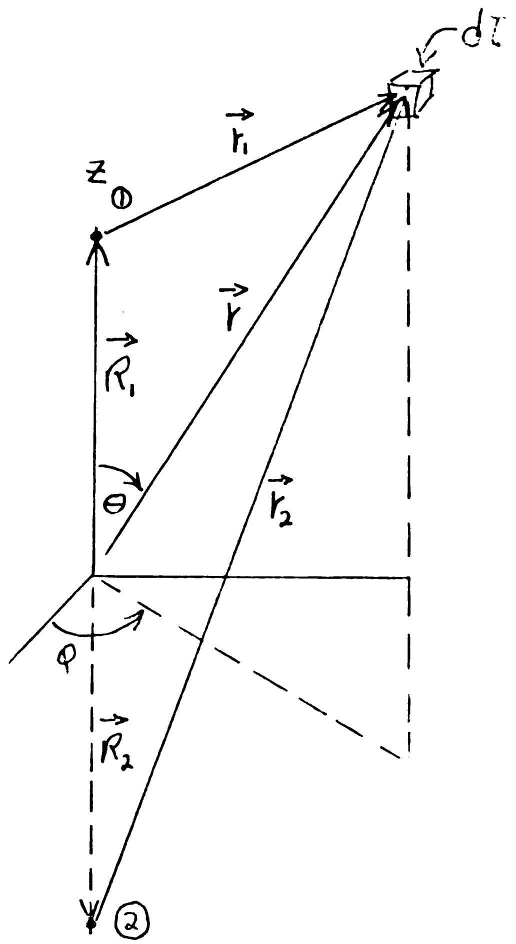


Figure 11.1

(See Figure 4.1)

$$\vec{R}_1 = -\vec{R}_2$$

$$\nabla_1 \phi(r_1) = A_1(\vec{r} - \vec{R}_1)$$

$$\nabla_2 \phi(r_2) = A_2(\vec{r} - \vec{R}_2)$$
$$= A_2(\vec{r} + \vec{R}_1)$$

Hence

$$\nabla_2 \phi(r_2) \times \nabla_1 \phi(r_1)$$
$$= A_1 A_2 \left[(r - R_1) \times (r + R_1)\right]$$
$$= A_1 A_2 \left[(r \times R_1) + (r \times R_1)\right]$$
$$= 2 A_1 A_2 (r \times R_1).$$

Also

$$\nabla_1 \phi(r_1) = A_1 (r + R_2)$$

so
$$\nabla_2 \phi(r_2) \times \nabla_1 \phi(r_1) = A_1 A_2 \left[(r + R_2) \times (r - R_2)\right]$$
$$= -2 A_1 A_2 (r \times R_2).$$

Thus

$$2\left[\nabla_2 \phi(r_2) \times \nabla_1 \phi(r_1)\right] = 2A_1 A_2\left[(r \times R_1) - (r \times R_2)\right]$$

$$= 2A_1 A_2 (r \times R)$$

where $R = R_1 - R_2$, (the internuclear distance vector).

Hence

$$\rho_N = \frac{\pi}{16\pi^2}\iint_{R_1 R_2}\left[(N_2 S_1^* + N_1^* S_2)\cdot A_1 A_2 (r \times R)\right] dR_1\, dR_2 . \tag{11.3}$$

Thus

$$(eQ)' = \frac{\pi}{16\pi^2}\iiint_{r R_1 R_2}\left[(N_2 S_1^* + N_1^* S_2)\cdot A_1 A_2 (r \times R)\, r^2 (3\cos^2\theta - 1)\right] dr\, dR_1\, dR_2 \tag{11.4}$$

where the prime indicates contribution from exchange processes only.

Making use of the fact that

$$\vec{r} = (r\cos\phi \sin\theta)\vec{i} + (r\sin\phi\sin\theta)\vec{j} + (r\cos\theta)\vec{k}$$

and $R = B\vec{k}$, B being the magnitude of the internuclear distance, we have

$$(r \times R) = (rB\sin\theta\sin\phi)\vec{i} - (rB\sin\theta\cos\phi)\vec{j}$$

so that we have now

$$(eQ)' = \frac{\pi}{16\pi^2}\iiint_{r R_1 R_2}\Big\{(N_2 S_1^* + N_1^* S_2)\cdot A_1 A_2\Big[(rB\sin\theta\sin\phi)\vec{i}$$

$$- (rB\sin\theta\cos\phi)\vec{j}\Big] r^2 (3\cos^2\theta - 1)\Big\}\, dr\, dR_1\, dR_2 . \tag{11.5}$$

$$(eQ)' = \frac{\text{II}}{16\pi^2}\iiint\limits_{r R_1 R_2}\Big\{A_1 A_2 \beta r^3(3\cos^2\theta - 1)\Big[(N_2 S_{1x}^{*}$$

$$+ N_1^{*} S_{2x}) \sin\theta \sin\varphi\, r^2 \sin\theta\, dr\, d\theta\, d\varphi\Big]dR_1\, dR_2$$

$$- \frac{\text{II}}{16\pi^2}\iiint\limits_{r R_1 R_2}\Big\{A_1 A_2 \beta r^3(3\cos^2\theta - 1)\Big[N_2 S_{1y}^{*}$$

$$+ N_1^{*} S_{2y}) \sin\theta \cos\varphi\, r^2 \sin\theta\, dr\, d\theta\, d\varphi\Big]dR_1\, dR_2 \qquad (11.6)$$

where $d\vec{r}$ has been replaced by

$$r^2 \sin\theta\, dr\, d\theta\, d\varphi.$$

It is clear that the integration over φ will vanish for both integrals, hence

$$(eQ)' = 0.$$

XII. Indication as to How the Calculation of the Exchange Moment of the Triton Could Be Accomplished

Making use of the general development carried out in Section V, and particularly equations (5.8), (5.9), and (5.11), it is entirely feasible to carry through the calculations for magnetic and electric quadrupole moments for systems of more than two particles. However, it is obvious that the problem would be extremely complicated if the system contained more than one charged particle, since then it would be necessary to consider the coulombian interactions that appear explicitly in the expression for the current density. Consequently the triton, a system of two neutrons and one proton, is the most reasonable test case for the extension of the development herein. There follows a brief development of the expression for the exchange magnetic moment, analogous to the previous development for the deuteron.

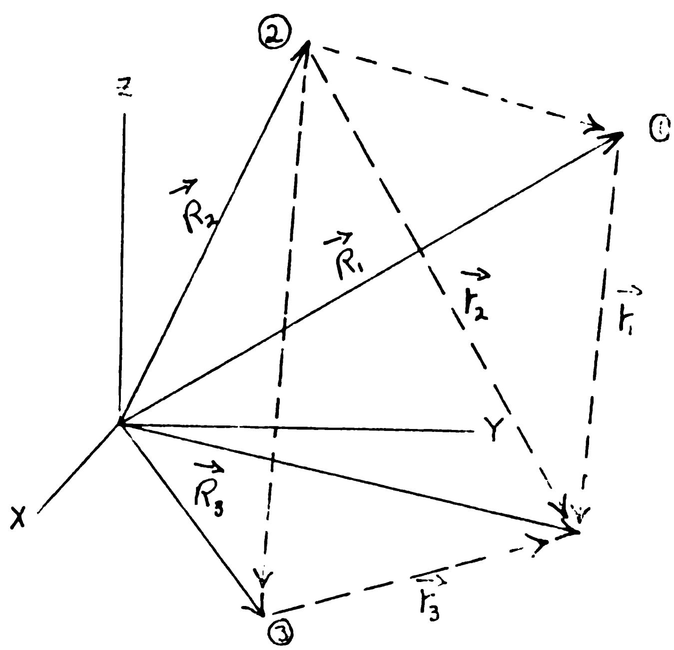


Figure 12.1

① Proton

② Neutron

③ Neutron

$$r_1 = r - R_1$$

$$r_2 = r - R_2$$

$$r_3 = r - R_3$$

$$\mu = \frac{1}{2c}\int_r (r \times J)\, d\vec{r}$$

$$J^m = \Pi\left[U \times \left(G^* + \sum_d S^d\right)\right] - \Pi\left[U^* \times \left(G + \sum_c S^{*c}\right)\right]$$

$$-\Pi\left[U_4^* F - U_4 F^*\right] \qquad (12.1)$$

$$u(r') = -\frac{1}{4\pi}\int_r \left(\nabla \times \sum_i S^{*i}\right) \phi(r_i - r')\, dr$$

$$u_4(r') = \frac{1}{4\pi}\int_r \left(\sum_i N^{*i}\right) \phi(r_i - r')\, dr \qquad (12.2)$$

$$F(r') = -\frac{1}{4\pi}\int_r \left(\sum_i N^{*i}\right) \nabla' \phi(r_i - r')\, dr$$

$$G(r') = \frac{\kappa^2}{4\pi}\int_r \left(\sum_i S^{*i}\right) \phi(r_i - r')\, dr$$

$$- \frac{1}{4\pi}\int_r \phi(r_i - r')\, \nabla\nabla \cdot \sum_i S^{*i}\, dr$$

$$- \sum_i S^{*i} \,.$$

We also have

$$\frac{1}{16\pi^2}\int_r \vec{r}\, \phi(r_i)\, \phi(r_j)\, dr = (R_i + R_j)\, \frac{e^{-\kappa|R_i - R_j|}}{16\pi\kappa} \qquad (12.3)$$

$$\nabla^2 U - K^2 U = \nabla \times \sum_i S^{*i}$$

$$\nabla^2 U^* - K^2 U^* = \nabla \times \sum_j S^j$$

$$\nabla^2 U_4 - K^2 U_4 = -\sum_i N^{*i}$$

$$\nabla^2 U_4^* - K^2 U_4^* = -\sum_j N^j \qquad (12.4)$$

For the triton

$$J^m = \text{II}\Big(U \times \Big[\frac{K^2}{4\pi}\int_{R_2} S_2 \varphi(r_2)\, dR_2 + \frac{K^2}{4\pi}\int_{R_3} S_3 \varphi(r_3)\, dR_3$$

$$- \frac{1}{4\pi}\int_{R_2} \varphi(r_2)(\nabla_2 \nabla_2 \circ S_2)\, dR_2 - \frac{1}{4\pi}\int_{R_3} \varphi(r_3)(\nabla_3 \nabla_3 \cdot S_3)\, dR_3\Big]\Big)$$

$$-\text{II}\Big(U^* \times \Big[\frac{K^2}{4\pi}\int_{R_1} S_1^* \varphi(r_1)\, dR_1 - \frac{1}{4\pi}\int_{R_1} \varphi(r_1)(\nabla_1 \nabla_1 \cdot S_1^*)\, dR_1\Big]\Big)$$

$$-\text{II}\Big(U_4^* \Big[\frac{1}{4\pi}\int_{R_1} N_1^* \nabla_1 \varphi(r_1)\, dR_1\Big] - U_4 \Big[\frac{1}{4\pi}\int_{R_2} N_2 \nabla_2 \varphi(r_2)\, dR_2$$

$$+ \frac{1}{4\pi}\int_{R_3} N_3 \nabla_3 \varphi(r_3)\, dR_3\Big]\Big). \qquad (12.5)$$

$$J^{M} = -\frac{\text{II}\, K^2}{16\pi^2}\Bigg[\int_{R_1}(\nabla_1 \times S_1^*)\,\phi(r_1)\,dR_1 \times \int_{R_2} S_2\,\phi(r_2)\,dR_2$$

$$+\int_{R_1}(\nabla_1 \times S_1^*)\,\phi(r_1)\,dR_1 \times \int_{R_3} S_3\,\phi(r_3)\,dR_3\Bigg]$$

$$+\frac{\text{II}}{16\pi^2}\Bigg[\int_{R_1}(\nabla_1 \times S_1^*)\,\phi(r_1)\,dR_1 \times \int_{R_2}(\nabla_2\nabla_2 \circ S_2)\,\phi(r_2)\,dR_2$$

$$+\int_{R_1}(\nabla_1 \times S_1^*)\,\phi(r_1)\,dR_1 \times \int_{R_3}(\nabla_3\nabla_3 \circ S_3)\,\phi(r_3)\,dR_3\Bigg]$$

$$+\frac{\text{II}\, K^2}{16\pi^2}\Bigg[\int_{R_2}(\nabla_2 \times S_2)\,\phi(r_2)\,dR_2 \times \int_{R_1} S_1^*\,\phi(r_1)\,dR_1$$

$$+\int_{R_3}(\nabla_3 \times S_3)\,\phi(r_3)\,dR_3 \times \int_{R_1} S_1^*\,\phi(r_1)\,dR_1\Bigg]$$

$$-\frac{\text{II}}{16\pi^2}\Bigg[\int_{R_2}(\nabla_2 \times S_2)\,\phi(r_2)\,dR_2 \times \int_{R_1}(\nabla_1\nabla_1 \circ S_1^*)\,\phi(r_1)\,dR_1$$

$$+\int_{R_3}(\nabla_3 \times S_3)\,\phi(r_3)\,dR_3 \times \int_{R_1}(\nabla_1\nabla_1 \circ S_1^*)\,\phi(r_1)\,dR_1\Bigg]$$

$$-\frac{\text{II}}{16\pi^2}\Bigg[\int_{R_2} N_2\,\phi(r_2)\,dR_2 + \int_{R_3} N_3\,\phi(r_3)\,dR_3\Bigg]\int_{R_1} N_1^*\,\nabla_1\phi(r_1)\,dR_1$$

$$+\frac{\text{II}}{16\pi^2}\Bigg[\int_{R_1} N_1^*\,\phi(r_1)\,dR_1\left(\int_{R_2} N_2\,\nabla_2\phi(r_2)\,dR_2 + \int_{R_3} N_3\,\nabla_3\phi(r_3)\,dR_3\right)\Bigg]. \tag{12.6}$$

Consider the contribution from the last two terms

$$
\begin{aligned}
& -\frac{\mathrm{II}}{16\pi^2}\int_{R_1}\int_{R_2} N_2\,\phi(r_2)\,N_1^*\,\nabla_1\phi(r_1)\,dR_1\,dR_2 \\
& -\frac{\mathrm{II}}{16\pi^2}\int_{R_1}\int_{R_3} N_3\,\phi(r_3)\,N_1^*\,\nabla_1\phi(r_1)\,dR_1\,dR_3 \\
& +\frac{\mathrm{II}}{16\pi^2}\int_{R_1}\int_{R_2} N_1^*\,\phi(r_1)\,N_2\,\nabla_2\phi(r_2)\,dR_1\,dR_2 \\
& +\frac{\mathrm{II}}{16\pi^2}\int_{R_1}\int_{R_3} N_1^*\,\phi(r_1)\,N_3\,\nabla_3\phi(r_3)\,dR_1\,dR_3 \\
= & \;\frac{\mathrm{II}}{16\pi^2}\int_{R_1}\int_{R_2} N_2\,(\nabla_1 N_1^*)\,\phi(r_1)\,\phi(r_2)\,dR_1\,dR_2 \\
& +\frac{\mathrm{II}}{16\pi^2}\int_{R_1}\int_{R_3} N_3\,(\nabla_1 N_1^*)\,\phi(r_1)\,\phi(r_3)\,dR_1\,dR_3 \\
& -\frac{\mathrm{II}}{16\pi^2}\int_{R_1}\int_{R_2} N_1^*\,(\nabla_2 N_2)\,\phi(r_1)\,\phi(r_2)\,dR_1\,dR_2 \\
& -\frac{\mathrm{II}}{16\pi^2}\int_{R_1}\int_{R_3} N_1^*\,(\nabla_3 N_3)\,\phi(r_1)\,\phi(r_3)\,dR_1\,dR_3
\end{aligned}
\tag{12.7}
$$

So the term

$$\frac{\text{II}}{2}\int_r \vec{r} \times \left[U_4 F^* - U_4^* F \right] dr$$

becomes

$$\frac{\text{II}}{32\pi^2}\int_r \phi(r_1)\,\phi(r_2)\,\vec{r} \times \iint_{R_1 R_2} \left[N_2 (\nabla_1 N_1^*) - N_1^* (\nabla_2 N_2) \right] dR_1 dR_2$$

$$+ \frac{\text{II}}{32\pi^2}\int_r \phi(r_1)\,\phi(r_3)\,\vec{r} \times \iint_{R_1 R_3} \left[N_3 (\nabla_1 N_1^*) - N_1^* (\nabla_3 N_3) \right] dR_1 dR_3 \qquad (12.8)$$

$$= \frac{\text{II}}{32\pi K}\iint_{R_1 R_2} (R_1 + R_2) \times \left[N_2 (\nabla_1 N_1^*) - N_1^* (\nabla_2 N_2) \right] dR_1 dR_2$$

$$+ \frac{\text{II}}{32\pi K}\iint_{R_1 R_3} (R_1 + R_3) \times \left[N_3 (\nabla_1 N_1^*) - N_1^* (\nabla_3 N_3) \right] dR_1 dR_3$$

$$= \frac{\text{II}}{16\pi K}\iint_{R_1 R_2} N_1^* N_2 \left[(R_1 + R_2) \times (R_1 - R_2) \right] e^{-K|R_1 - R_2|} dR_1 dR_2$$

$$+ \frac{\text{II}}{16\pi K}\iint_{R_1 R_3} N_1^* N_3 \left[(R_1 + R_3) \times (R_1 - R_3) \right] e^{-K|R_1 - R_3|} dR_1 dR_3$$

$$(R_1 + R_2) \times (R_1 - R_2) = -2(R_1 \times R_2)$$

so we have finally

$$\frac{\Pi}{8\pi K}\iint_{R_1 R_2} N_1^* N_2 \,(R_2 \times R_1)\, e^{-K|R_1 - R_2|}\, dR_1\, dR_2$$

$$+\frac{\Pi}{8\pi K}\iint_{R_1 R_3} N_1^* N_3 \,(R_3 \times R_1)\, e^{-K|R_1 - R_3|}\, dR_1\, dR_3 \,. \tag{12.9}$$

In exactly similar fashion the other terms can be shown to yield contributions to the exchange moment of precisely the same form as those obtained for the deuteron, with the addition of a complete set of identical terms in R_1, R_3. Given assumptions concerning the relationships between the vectors R_1, R_2, and R_3, it would be possible to carry through the indicated integrations. However, there would then remain the problem of calculating the expectation values for the dipole operator, taking into consideration all permissible combinations of space, spin, and charge wave functions that accord with the exclusion principle. Such a task appears feasible theoretically, but almost prohibitive from the computational standpoint, especially in view of the apparent inadequacy of a purely charged meson theory of nuclear forces.

XIII. Discussion

The purely charged meson theory is evidently unable to provide a mechanism to account for the anomalous magnetic moment of the deuteron or the electric quadrupole moment of the deuteron in terms of exchange pheonomena. Since the magnetic moment of the deuteron is not the sum of the moments of its constituent particles*, a contribution of the right order of magnitude and of the right sign to this moment by the charge exchange process would have been very encouraging. Also it might have been reasoned that the distributed mesic space charge due to exchange would contribute to the electric quadrupole moment**. The calculations based on the charged meson hypothesis, however, reveal that the exchange process contributes nothing to these quantities in that theory. It would appear that there is a non-vanishing contribution to the moment of the triton, but the calculations were not carried far enough to indicate whether or not the contribution is significant.

In connection with the problem of the triton, it should be noted that Villars[18] has carried through computations for the exchange moment predicted by the pseudoscalar theory. His results appear to be of the right order of magnitude and of the right sign for the dipole moment.

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*$\mu_P = 2.7896$ (Nuclear Magnetons)[16]

$\mu_D = 0.85647 \pm .0003$ [16]

$\mu_N = -1.9103 \pm .0012$ [16]

**$(eQ) = 2.73 \times 10^{-27}\ cm^2$ [17]

It is significant that a field theory of nuclear forces, postulating a mechanism for the origin of those forces, yields interaction terms in the Hamiltonian that agree qualitatively with experimental evidence. In fact, considerable attention has been given to a phenomenological approach to the problem of the deuteron in which the form of the interaction potential is taken to be that predicted by the neutral theory[19, 20, 21]

$$V = J_0 (r) + J_1 (r) \sigma_1 \cdot \sigma_2 + J_2 (r) S_{12}$$

where

$$S_{12} = 3 (\sigma_1 \cdot r \sigma_2 \cdot r)/r^2 .$$

Another approach to the whole problem of nuclear forces representing an extension of field theory to its logical extremes seems indicated in the various attempts to develop a Unitary Field Theory[22] in which fundamental quantities like mass and charge follow as consequences of mathematical consistency of proper field equations.

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