INVARIANTS OF TOPOLOGICAL AND LEGENDRIAN LINKS IN LENS SPACES WITH A UNIVERSALLY TIGHT CONTACT STRUCTURE

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ABSTRACT

INVARIANTS OF TOPOLOGICAL AND LEGENDRIAN LINKS IN LENS SPACES WITH A UNIVERSALLY TIGHT CONTACT STRUCTURE

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In this thesis a HOMFLY polynomial is found for knots and links in a lens space $L(p, q)$. Further study of this polynomial invariant finds a relationship with the classical invariants of Legendrian and transverse links, when $L(p, q)$ is endowed with a universally tight contact structure. In fact certain criteria are found which, if satisfied by any numerical invariant of links in $L(p, q)$, guarantee that the invariant fits into a Bennequin type inequality. A linear function of the degree of the HOMFLY polynomial is then shown to satisfy these criteria. A corollary is that certain "simple" Legendrian and transverse realizations of knots admitting grid number one diagrams maximize the classical invariants in their knot type.

In order to obtain the above results, formulae are found for computing the classical invariants of Legendrian and transverse links from a toroidal front projection. Having these formulae, and known results about fibered links that support a given contact structure, it is found whether the duals of some families of Berge knots support the universally tight contact structure.

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

In this thesis we study knots and links in lens spaces through the tool of toroidal grid diagrams. Grid diagrams of knots and links in S^3 have garnered recent attention, due to applications in Legendrian knot theory and knot Floer homology. Their counterparts that describe links in a lens space are of similar interest in this setting. In [5] toroidal grid diagrams were considered in order to construct combinatorial knot Floer homology in lens spaces. Later, in [4], the correspondence between Legendrian links in universally tight contact lens spaces and toroidal grid diagrams was developed.

The construction in [5] is of particular relevance to an open problem known as the Berge Conjecture, which posits a complete description of knots in S^3 that admit Dehn surgery resulting in a lens space. In fact, the conjecture is only still open for integer surgeries, and thus can be reformulated in terms of which knots in a given lens space yield an integer surgery that gives S^3. In this setting, the Berge Conjecture states that such a knot must admit a toroidal grid diagram with grid number 1.

In the work of Baker and Grigsby [4], that indicates the correspondence between Legendrian links in universally tight contact lens spaces and toroidal grid diagrams, a relationship is found between the Thurston–Bennequin number of a knot, that of its mirror, and the grid number of the knot. This relationship suggests that finding bounds for the Thurston–Bennequin number of a knot might be useful in controlling the grid number.

Our work here uses skein polynomials (specifically the HOMFLY polynomial) to find such a bound on the Thurston–Bennequin number. In fact, previous to our work, the existence of a HOMFLY polynomial for links in lens spaces was an open problem. In the work below we resolve this problem, and toroidal grid diagrams are fundamental to the proof of the existence of the HOMFLY polynomial. In addition, we develop a skein theory of toroidal grid diagrams. We then give sufficient criteria for an invariant of links in lens spaces to bound from above the self-linking number of a transverse link in the universally tight contact structure. Equivalently, this means that the invariant bounds the sum of the Thurston-Bennequin number and absolute value of the rotation number, of a Legendrian representative of the link. We then derive an invariant from the HOMFLY polynomial that satisfies these criteria.

1.1 Summary of results

Throughout what follows we let p, q be coprime integers with $0 \leq |q| < p$. The lens space which results as $-\frac{p}{q}$ surgery on the unknot in S^3 is denoted by $L(p, q)$ (note that this excludes the case $S^1 \times S^2$). For each $L(p, q)$ we construct a collection of links called *trivial links*. There is exactly one trivial link in each homotopy class of links. We prove the following.

Theorem 1.1.1. *Let $\mathscr{L}$ be the set of isotopy classes of oriented links in $L(p, q)$ and let $\mathscr{TL} \subset \mathscr{L}$ denote the set of isotopy classes of trivial links. Define $\mathscr{TL}^* \subset \mathscr{TL}$ to be* *those trivial links with no null-homotopic components. Let* U *be the isotopy class of the standard unknot, a local knot in* $L(p,q)$ *that bounds an embedded disk. Suppose we are given a value* $J_{p,q}(\tau) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ *for every* $\tau \in \mathscr{T}\mathscr{L}^*$*. Then there is a unique map* $J_{p,q} : \mathscr{L} \to \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ *such that*

(i) $J_{p,q}$ *satisfies the skein relation*

$$a^{-p} J_{p,q}(L_+) - a^p J_{p,q}(L_-) = z J_{p,q}(L_0).$$

(ii) $J_{p,q}(U) = a^{-p+1}$.

(iii) $J_{p,q}\left(U \coprod L\right) = \frac{a^{-p} - a^p}{z} J_{p,q}(L)$.

As usual, the links L_+, L_-, and L_0 differ only in a small neighborhood. The exact construction of these links in $L(p,q)$ is made clear in the subsequent text.

Remark 1.1.2. In a large class of $\mathbb{Q}$-homology spheres, Kalfagianni has found a power series valued invariant of framed links that satisfies the Kauffman skein relation [32]. The ideas in the proof of Theorem 1.1.1 should also be capable of showing that this invariant provides a Kauffman polynomial for links in $L(p,q)$.

We also give criteria for a $\mathbb{Q}$-valued invariant of links in $L(p,q)$ to bound the classical invariants in $(L(p,q), \xi_{UT})$, where ξ_{UT} is a universally tight contact structure on $L(p,q)$ defined by the pushforward of the standard contact structure. Our result is a lens space analogue of a theorem of Lenny Ng [41]. To a given trivial link τ let $T(\tau)$ be a particular transverse representative of τ defined below, in Remark 4.1.5.

Theorem 1.1.3. *Let* i *be a* $\mathbb{Q}$*-valued invariant of oriented links in* $L(p,q)$ *such that*

$$i(L_+) + 1 \leq \max\left(i(L_-) - 1, i(L_0)\right) \qquad \text{and} \qquad i(L_-) - 1 \leq \max\left(i(L_+) + 1, i(L_0)\right),$$

where L_+, L_-, and L_0 are oriented links that differ as in the skein relation. If $\mathrm{sl}_{\mathbb{Q}}(T(\tau)) \leq -i(\tau)$ *for every trivial link τ in $L(p,q)$, then for every link L in $L(p,q)$,*

$$\mathrm{sl}_{\mathbb{Q}}(L_t) \leq -i(L),$$

where L_t is any transverse representative in $(L(p,q), \xi_{UT})$ of L. Moreover, if L_l is a Legendrian representative of L then

$$\mathrm{tb}_{\mathbb{Q}}(L_l) + \left|\mathrm{rot}_{\mathbb{Q}}(L_l)\right| \leq -i(L).$$

In order to prove Theorem 1.1.3 we provide explicit formulae to calculate the invariants $\mathrm{tb}_{\mathbb{Q}}$, $\mathrm{rot}_{\mathbb{Q}}$, and $\mathrm{sl}_{\mathbb{Q}}$ from a projection of the link to a Heegaard torus. These formulas are in the spirit of those used to compute the classical invariants in (S^3, ξ_{st}) from a front projection (see [16]). Moreover, combining the formula for $\mathrm{tb}_{\mathbb{Q}}$ with a result of Baker and Grigsby [4], we find a very short proof of a result of Fintushel and Stern [18], that if integral surgery on a knot in $L(p,q)$ yields S^3, then $\pm q$ is a quadratic residue mod p.

Given a link K in $L(p,q)$, the polynomial $J_{p,q}(K)$ is a two-variable polynomial in variables a and z, and its definition depends on a normalization on the set of trivial links. The following is a corollary of Theorem 1.1.3.

Corollary 1.1.4. *Let $J_{p,q}$ denote the HOMFLY polynomial invariant in $L(p,q)$, normalized so that if τ is a trivial link with no nullhomotopic components, or is the unknot, then $J_{p,q}(\tau) = a^{p \cdot \mathrm{sl}_{\mathbb{Q}}(T(\tau))+1}$. Given an oriented link L in $L(p,q)$, set $e(L)$ to be the lowest* *degree in a of $J_{p,q}(L)$. If L_t is a transverse representative of L in $(L(p,q), \xi_{UT})$, then*

$$\mathrm{sl}_{\mathbb{Q}}(L_t) \leq \frac{e(L)-1}{p}.$$

The Franks-Williams-Morton (FWM) inequality is a celebrated result that relates the braid index and algebraic crossing number of a braid to the HOMFLY polynomial of its closure (which is a link in S^3). We point out that Corollary 1.1.4 is a generalization of this result to lens spaces. For more details, see the discussion of the FWM inequality below. Finally, we pinpoint the maximum self-linking and maximum Thurston-Bennequin numbers of trivial knots as a corollary of Corollary 1.1.4.

Corollary 1.1.5. *If τ is a trivial link in $L(p,q)$, then $T(\tau)$ has maximal self-linking number among all transverse representatives of τ. If τ is a trivial knot, then the Legendrian knot associated to its grid number one diagram has maximal Thurston-Bennequin number.*

The thesis is organized as follows: In Chapter 2 we give definitions and background that are fundamental to the study of polynomial invariants, and of Legendrian and transverse links. We also review the work of Kalfagianni and Lin that paves the way for Theorem 1.1.1. We end the chapter with a discussion of toroidal grid diagrams and their correspondence to Legendrian links. In Chapter 3 we consider homotopies of links in $L(p,q)$ and develop a skein theory of toroidal grid diagrams. Using these tools, we prove the existence of the HOMFLY-PT link polynomial for links in $L(p,q)$. In Chapter 4 we review constructions of [7],[4] that extend classical invariants of Legendrian and transverse knots to the rationally null-homologous setting. We then give formulae for the Thurston-Bennequin, rotation, and self-linking numbers from data given by a projection adapted to a grid diagram. In that same chapter we prove Theorem 1.1.3 and Corollaries 1.1.4 and 1.1.5. We end the chapter with a computation, giving a sequence of Legendrian knots and links in $(L(5,1), \xi_{UT})$ on which the FWM inequality is sharp and arbitrarily negative. Finally, in Chapter 5 we discuss fibered knots in certain lens spaces $L(p,q)$ that are the cores of surgery on two families of Berge knots. In particular, we show that many of these knots do not support the universally tight contact structure on $L(p,q)$.

Chapter 2

Definitions and Background

2.1 Skein Polynomials

For our study of oriented knots and links in a lens space it is requisite that we consider n-singular links. Let M be an oriented 3-dimensional manifold, and let S be a disjoint union of oriented circles. A piecewise-linear (or alternatively, smooth) map $K : S \to M$ is an *n-singular link* if it has exactly n transverse double points, and is an embedding away from these points. We often write K for the image $K(S)$. Two n-singular links K_1, K_2 are equivalent if there is an ambient isotopy $M \times I \to M$, taking K_1 to K_2, such that the double points remain transverse throughout the isotopy. We sometimes say that K_1 and K_2 are "smoothly" isotopic, to distinguish such isotopy classes from more restrictive notions of equivalence, such as Legendrian and transverse isotopy (defined below). Note that a 0-singular link is a link in M, and is a knot if the domain S is connected.

Much of classical knot theory (the study of knots and links in S^3) benefits from the ability to discretize isotopies. That is, two links K_1, K_2 in S^3 are isotopic if and only if any regular projection of K_1 to the equatorial plane can be taken to a regular projection of K_2 via a finite sequence of three types of local changes to the projection, known as the three Reidemeister moves.

Given any n-singular link K, give S the structure of a simplicial complex so that, for each double point x in K, there are distinct 1-simplexes $\sigma_1, \sigma_2 \subset S$ with the interior of σ_i containing one of the points in $K^{-1}(x)$. We may assume that $K(\sigma_1) \cup K(\sigma_2)$ is contained in an embedded disk $D \subset M$ with $K(\partial\sigma_1) \cup K(\partial\sigma_2) \subset \partial D$. Moreover, since σ_i inherits an orientation from that of S, we may refer to the initial and terminal points of σ_i.

Let $x \in M$ be a double point of K. Subdividing the simplicial structure on S if necessary, there is a small ball neighborhood $B \subset M$ of x such that D is properly embedded in B and $K \cap B = K(\sigma_1) \cup K(\sigma_2)$. Define a_1 and a_2 to be two simple arcs in distinct components of $\partial B \setminus \partial D$, going from the initial point to the terminal point of $K(\sigma_1)$. Define b_1 and b_2 to be simple arcs on ∂D with b_1 going from the initial point of $K(\sigma_1)$ to the terminal point of $K(\sigma_2)$ and b_2 going from the initial point of $K(\sigma_2)$ to the terminal point of $K(\sigma_1)$.

We define three $(n-1)$-singular links from K in the following manner:

$$\begin{aligned} K_+ &= \overline{K(S \setminus \sigma_1)} \cup a_1; \\ K_- &= \overline{K(S \setminus \sigma_1)} \cup a_2; \\ K_0 &= \overline{K(S \setminus (\sigma_1 \cup \sigma_2))} \cup (b_1 \cup b_2). \end{aligned} \tag{2.1.1}$$

Remark 2.1.1. Note that a different choice of a_1 and a_2 would interchange K_+ and K_-. If M is oriented, this ambiguity can be dealt with. Since x is a transverse double point, consider the ordered pair of tangent vectors $K'(\sigma_1), K'(\sigma_2)$ at x. There is a unique vector normal to D at x that completes this ordered pair to an oriented frame that agrees with the orientation on B at x. Such a vector points into one of the components of $B \setminus D$. Take a_1 to be in the same component.

The first motivation for discussing n-singular links is to define a skein triple of links.

Definition 2.1.2. Let K_1, K_2, K_3 be any three links in M that are isotopic to K_+, K_-, and K_0 respectively, for some 1-singular link K. Then (K_1, K_2, K_3) is called an *oriented skein triple* in M. Given a 3-manifold M, let $\mathscr{L}$ denote the set of isotopy classes of (oriented) links in M. A *(Laurent) polynomial invariant of oriented links* in the variables $x_1, \ldots, x_m$ is a well-defined map $J : \mathscr{L} \to \mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. A polynomial invariant of oriented links J is called a *skein polynomial* if there is a homogeneous degree 1 polynomial $F(X, Y, Z)$ with coefficients in $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ such that for any skein triple (K_+, K_-, K_0), we have $F(J(K_+), J(K_-), J(K_0)) = 0$.

Skein polynomials in the setting of S^3 include the Alexander, Jones, and HOMFLY-PT polynomials. We note that the Alexander and Jones polynomials are one-variable specializations of the two-variable HOMFLY-PT polynomial. Furthermore, all mentioned skein polynomials can be computed from regular projections.

The linear polynomial $F(X, Y, Z)$ in the definition of a skein polynomial is often called the *skein relation* for that polynomial invariant. As our focus is on the HOMFLY-PT polynomial, we will use this skein relation as an example. Note that the HOMFLY polynomial was introduced in [20] (see also [50]), and is determined by its skein relation, along with a choice of normalization on the unknot. As observed by Ocneanu, the HOMFLY polynomial can also be expressed using a trace of representations into Hecke algebras (see also [28]).

Let K_+, K_-, K_0 be three links in S^3 which admit regular projections that are identical outside a neighborhood of some crossing, and that differ as in Figure 2.1 near the crossing.

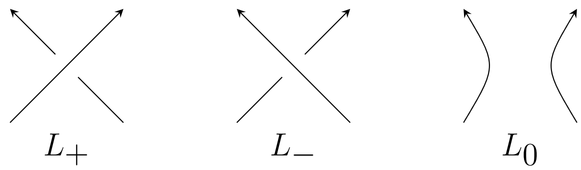


Figure 2.1: Projections in a skein relation

Then (K_+, K_-, K_0) is a skein triple. The HOMFLY-PT oriented link polynomial $J : \mathscr{L} \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ satisfies the skein relation

$$v^{-1}J(K_+) - vJ(K_-) = zJ(K_0), \tag{2.1.2}$$

(that is, $F(X, Y, Z) = v^{-1}X - vY - zZ$) and is usually normalized so that $J(\text{unknot}) = 1$.

In fact, using this normalization on the unknot, we can use induction to see that any oriented invariant of links satisfying (2.1.2) must be Laurent polynomial-valued. Indeed, consider a regular projection D of a link K in S^3 and let $c(D)$ be the set of crossings of D. There is a subset of $c(D)$ such that if we change which strand is over-crossing at each crossing in this subset, we obtain a projection of the standard unlink. Let $u(D)$ be the number of crossings in the smallest such subset. Suppose D has a positive crossing in this subset, as on the left in Figure 2.1 (the case where the subset has only negative crossings being similar). Then we may consider K as K_+ in a skein triple (K_+, K_-, K_0). Further, $c(D_0) < c(D_+)$ and $u(D_-) < u(D_+)$, where $D = D_+$ and D_-, D_0 are diagrams of K_-, K_0 as shown in Figure 2.1. Thus we may inductively assume that $J(K_-), J(K_0)$ are Laurent polynomials, noting that (2.1.2) implies

$$J\left(L \coprod \circlearrowleft\right) = \frac{v^{-1} - v}{z} J\left(L\right).$$

Our proof that there exists a HOMFLY-PT polynomial for links in a lens space uses a complexity function argument, as we did in the previous paragraph. We define a complexity function that behaves nicely with respect to skein triples. Then we deduce that an invariant of links J_M found by Kalfagiannin and Lin [33], which satisfies the HOMFLY-PT skein relation, takes Laurent polynomial values in the setting of a lens space. By construction, J_M takes values in a ring of power series, but it was not known to converge to a polynomial invariant. It is still an open problem whether J_M is Laurent polynomial-valued when M is not a lens space (see [31]).

The construction of J_M, which we will return to in Section 2.3, uses Vassiliev (or finite-type) link invariants. Note that, given any invariant of $(n-1)$-singular links (in particular, beginning with an honest link invariant), we can define an invariant of n-singular links by setting $F(L_\times) = F(L_+) - F(L_-)$. Thus any invariant of n-singular links gives an invariant of m-singular links for $m > n$. A link invariant is called a *Vassiliev invariant of order* $\leq n$ if the derived invariant vanishes on any $n+1$-singular link, but does not on some n-singular link.

2.2 Contact structures on 3-dimensional manifolds

Contact structures are a natural geometric structure that may be studied on any odd-dimensional manifold. We focus on the setting of a 3-dimensional manifold Y. In this case, a *contact structure* ξ on Y is a rank 2 subbundle of TY that does not agree locally (as subbundles) with the tangent bundle of any embedded surface in Y. If ξ is oriented then there is a global 1-form α satisfying $\alpha \wedge d\alpha > 0$ such that $\xi = \ker \alpha$. The pair (Y, ξ) is called a *contact 3-manifold.* Two contact 3-manifolds $(Y_1, \xi_1), (Y_2, \xi_2)$ are *contactomorphic* if there is a diffeomorphism $Y_1 \to Y_2$ which carries ξ_1 to ξ_2.

An embedded disk in (Y, ξ) is *overtwisted* if its tangent bundle restricted to its boundary agrees with ξ. A contact structure ξ is *overtwisted* if (Y, ξ) contains an overtwisted disk and ξ is *tight* otherwise. The property of ξ being tight or overtwisted is clearly invariant under contactomorphism. Finally given any map of 3-manifolds, $\varphi : \tilde{Y} \to Y$, and a contact structure ξ on Y, one can take the pullback bundle $\varphi^*\xi$, and this will be a contact structure on $\tilde{Y}$. If $\tilde{Y}$ is the universal cover of Y and $\varphi^*\xi$ is tight, then ξ is called *universally tight.*

If $K \subset Y$ is a link and, for some contact structure ξ on Y, K is tangent to ξ at each point, then K is called a *Legendrian link* in (Y, ξ). Two Legendrian links are *Legendrian isotopic* if there is an isotopy from one to the other, through Legendrian links. A link K in Y is called *transverse* if TK is transverse to ξ at each point of the image of K, and two such links are *transversely isotopic* if there is an isotopy through transverse embeddings from one to the other. Note that (assuming ξ is oriented) a transverse link comes with a natural orientation.

A fundamental example of a contact 3-manifold is $(\mathbb{R}^3, \xi_{st})$, where $\mathbb{R}^3$ has coordinates (x, y, z) and $\xi_{st} = \ker(dz - ydx)$. Projection to the xz-plane in $(\mathbb{R}^3, \xi_{st})$ is called *front projection*, and the front projection of a Legendrian link has a particularly nice form since the y-coordinate can be recovered from the slope in the xz-plane of the front projection. As a consequence, a diagram of a link in the xz-plane is the front projection of some Legendrian link in $(\mathbb{R}^3, \xi_{st})$ if and only if the only singular points of the projection are semi-cubical cusps and transverse double points, the projection has no vertical tangencies, and at a double point the strand with more negative slope is the over-crossing strand. An example is given in Figure 2.2.

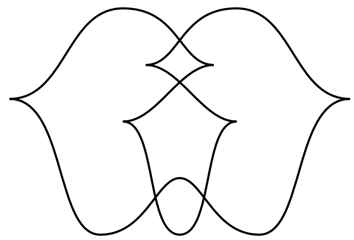


Figure 2.2: The front projection of a Legendrian figure 8 knot

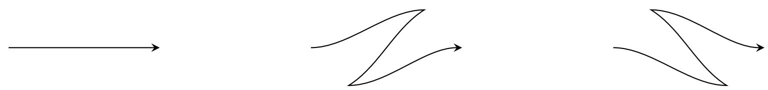
In addition to the underlying smooth isotopy class of a Legendrian link K, there are two invariants of Legendrian links called the *classical invariants of* K. These invariants are traditionally defined when K is nullhomologous in Y. The first is the *Thurston-Bennequin number* $\mathrm{tb}(K)$, which measures the framing induced from the contact planes, relative to the framing given by a Seifert surface for K (a connected oriented surface, properly embedded in the link complement, with boundary equal to K). That is, consider the simple closed curve K' on the boundary of a regular neighborhood of K which is determined by a non-zero section in the (real) line bundle $\xi_K \cap \nu$, where ν is the normal bundle of K. Choose a Seifert surface Σ for K. Then $\mathrm{tb}(K)$ is the algebraic intersection $\Sigma \cdot K'$. The second classical invariant is the *rotation number* $\mathrm{rot}(K)$, which is defined as the winding number of TK after a trivialization of ξ_Σ, given a Seifert surface Σ for K. We note that $\mathrm{rot}(K)$ actually depends on the homology class of Σ, but not on the chosen trivialization. However, as this thesis will only work in the setting of lens spaces, $\mathrm{rot}(K)$ will only depend on K.

In $(\mathbb{R}^3, \xi_{st})$, one can compute the Thurston-Bennequin number and rotation number of an oriented Legendrian link from its front projection in the following way. Let P be a front projection of a Legendrian link K, let $w(P)$ be the writhe of P, and $c(P)$ the number of cusps in the projection. Then, using the fact that the vector field $\partial/\partial z$ is transverse to ξ_{st} everywhere, one can show that $\mathrm{tb}(K) = w(P) - \frac{1}{2}c(P)$. Also, given an orientation, let $c_d(P)$ (resp. $c_u(P)$) be the downward (resp. upward) oriented cusps, then $\text{rot}(K) = \frac{1}{2}(c_d(P) - c_u(P))$. If K is the Legendrian knot with front projection shown in Figure 2.2, then $\text{tb}(K) = -3$ and, given either orientation, $\text{rot}(K) = 0$.

Transverse links have one *classical invariant* other than the underlying smooth link type. If K is a transverse link, this invariant is called the *self-linking number* of K and written $\text{sl}(K)$. The self-linking number is defined as follows: given a Seifert surface Σ for K, since one can trivialize ξ_Σ, we can choose a non-vanishing vector field v in ξ_Σ. Normalizing v determines a simple closed curve K' on the boundary of a regular neighborhood of K, since K is transverse to ξ. Define $\text{sl}(K)$ to be the linking of K with K', i.e. $\Sigma \cdot K'$.

Our study is that of links in a lens space, which of course need not be nullhomologous. However, the definitions given above can be extended to the setting of a rationally nullhomologous link with the help of *rational Seifert surfaces* (defined below), which are a generalization of a Seifert surface and are embedded in the exterior of the link.

There is an operation one can perform on a Legendrian knot K in any contact 3-manifold called *positive (resp. negative) stabilization* on K that gives a different Legendrian knot denoted $S^+(K)$ (resp. $S^-(K)$). To describe the operation, we point out that a theorem of Darboux states that given a point p in a contact 3-manifold (Y, ξ), there exists a neighborhood U of p, such that there is a contactomorphism $\varphi : (U, \xi|_U, p) \to (\mathbb{R}^3, \xi_{st}, 0)$. If K is Legendrian in (Y, ξ) then suppose p is a point on K. Suppose that Figure 2.3(a) is the front projection of $\varphi(U \cap K)$. Then $S^+(K)$ is defined by replacing $U \cap K$ with the image of Figure 2.3(b) under φ^{-1}, and $S^-(K)$ is defined by replacing $U \cap K$ with the image of Figure 2.3(c) under φ^{-1}. The Legendrian isotopy type of $S^\pm(K)$ does not depend on the choice of point p or contactomorphism φ.



(a) (b) (c)

Figure 2.3: Positive and negative stabilization of K

2.3 Review of known results

In this section we review what is known about the problem of finding skein polynomials in manifolds other than S^3, in addition to results that relate invariants of smooth isotopy of knots to the classical invariants of their Legendrian and transverse realizations. We will also introduce one of the main tools for our study of links in $L(p,q)$, toroidal grid diagrams, and their correspondence to Legendrian links in $L(p,q)$ with a universally tight contact structure.

2.3.1 Power series invariants in $\mathbb{Q}$-homology 3-spheres

Original constructions of the HOMFLY polynomial have been difficult to reproduce in manifolds other than S^3, as they rely on planar link projections. However, it was shown in [12] that the HOMFLY and Kauffman polynomials are generating functions for particular sequences of Vassiliev (or finite-type) link invariants (see also [8],[9], and [34]). In [30] and [33], Kalfagianni and Lin considered Vassiliev invariants of links in (certain) $\mathbb{Q}$-homology spheres in their effort to find a HOMFLY polynomial. They succeeded in finding an invariant that takes values in a ring of power series.

Kalfagianni and Lin begin by considering homotopies of links in $\mathbb{Q}$-homology spheres. In particular, they show that any free homotopy of a link in M can be perturbed slightly to a particularly nice form called *almost general position.* Among other properties, a homotopy in almost general position only fails to be isotopy at finitely many moments, at which times one has a 1-singular link, and near these times one passes from one resolution of the 1-singular link to another (e.g. L_+ to L_-).

We described above how singular link invariants may be derived from a link invariant. Kalfagianni and Lin give an "integrability condition" in [33], determining when a given 1-singular link invariant is derived from a link invariant. Their theorem placed certain restrictions on the ambient manifold. Recently Kalfagianni [32] was able to remove some of these restrictions, and we cite her theorem here, adapted to the setting of unframed links. Recall that an *atoroidal* 3-manifold M is one such that $\pi_2(M) = 0$ and M has no essential tori. Let $\mathscr{L}^{(1)}$ denote the isotopy classes of 1-singular links in M.

Theorem 2.3.1 ([32])**.** *Suppose that M is a $\mathbb{Q}$-homology sphere with $\pi_2(M) = 0$, such that if $H_1(M) \neq 0$ then M is atoroidal. Also assume that R is a ring which is torsion free as an abelian group and let $f : \mathscr{L}^{(1)} \to R$ be an invariant of 1-singular links. Then there exists a link invariant $F : \mathscr{L} \to R$ so that*

$$f(L_\times) = F(L_+) - F(L_-)$$

for any 1-singular link $L_\times$ if and only if

$$f(\propto) = 0$$

$$f(L_{\times +}) - f(L_{\times -}) = f(L_{+\times}) - f(L_{-\times}),$$

where $L_{\times\pm}$ and $L_{\pm\times}$ denote the four 1-singular resolutions of any 2-singular link, and $\propto$ denotes a 1-singular link with a "small" loop near some double point.

Almost general position of homotopies was an essential ingredient in proving Theorem

2.3.1. One easily checks that some f derived from a link invariant F will satisfy the two conditions of the theorem. Given a singular link invariant satisfying these conditions, one takes a homotopy in almost general position from a trivial link in $\mathscr{TL}$ to the given link and sums the values of the singular invariant at the finite singular moments in the homotopy. To see this is well-defined one must show that the definition is independent of the homotopy used, or that a "global integrability" condition holds.

It is known [29] that the 2-variable HOMFLY polynomial for links in S^3 is equivalent to a sequence of 1-variable Laurent polynomials $\{J_n(t)\}$, and that replacing t by e^x gives a power series invariant $J_n(x)$ with coefficients that are Vassiliev invariants [8, 12]. Kalfagianni and Lin use these ideas in [33] to arrive at Theorem 2.3.2 below. They define their invariant J_M by defining inductively a sequence of Vassiliev invariants whose generating function is a power series satisfying the HOMFLY-PT skein relation. The i^{th} coefficient of this power series is defined from a singular link invariant, which they show "integrates" to a link invariant by showing it satisfies the conditions of Theorem 2.3.1.

The following theorem is then obtained. Let U be the isotopy class of a knot in M that is the standard unknot in a small ball neighborhood of a point of M. Fix $\mathscr{TL}$, a collection of links in M such that there is exactly one representative in $\mathscr{TL}$ for each homotopy class of links in M, and furthermore, if $TL \in \mathscr{TL}$ has k components that are homotopically trivial then $TL = L \coprod^k U$ for some link L that has no homotopically trivial components. Define $\widehat{R} := \mathbb{C}[[x, y]]$ to be the ring of formal power series in the variables x and y over $\mathbb{C}$. Define $v \in \widehat{R}$ by $v := e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \cdots$ and let $z \in \widehat{R}$ be defined by $z := e^x - e^{-x} = 2x + \frac{x^3}{3} + \frac{x^5}{60} + \cdots$.

Theorem 2.3.2 ([33])**.** *Let M be a $\mathbb{Q}$-homology 3-sphere with $\pi_2(M) = 0$, such that if*

$H_1(M) \neq 0$ then M is atoroidal. Let $\mathscr{L}$ be the set of isotopy classes of links in M. Then, given values $J_M(T)$ for each $T \in \mathscr{T}\mathscr{L}$ such that $J_M(T \coprod U) = \frac{v^{-1}-v}{z} J_M(T)$, there is a unique map $J_M : \mathscr{L} \to \widehat{R}$ such that

$$v^{-1} J_M(L_+) - v J_M(L_-) = z J_M(L_0).$$

Given M as above, it is unknown whether J_M can be made to take values that are Laurent polynomials. In this consideration, Kalfagianni asked the following question [31] which encapsulates the primary difficulty of the problem, and which we answer affirmatively when M is a lens space:

Question 1. Is there a choice of $\mathscr{T}\mathscr{L}^*$ such that every link $L \subset M$ can be reduced to disjoint unions of unlinks and elements in $\mathscr{T}\mathscr{L}^*$ by a series of finitely many skein moves?

2.3.2 Bennequin type inequalities

Much effort has gone into finding upper bounds for the classical invariants in (S^3, ξ_{st}), where ξ_{st} is the standard tight contact structure on the 3-sphere, e.g. [10], [19, 37], [51], [52], [48], [49], [55], [40], [58]. Much of this work benefits from the fact that the classical invariants of Legendrian/transverse knots in (S^3, ξ_{st}) can be computed easily from a front projection.

Less is known about these invariants of Legendrian knots in other contact manifolds. A theorem of Eliashberg [15] generalizes the Bennequin inequality for null-homologous Legendrian knots in any 3-manifold with a tight contact structure:

Theorem 2.3.3 (Eliashberg-Bennequin inequality)**.** *Let ξ be a tight contact structure on a*

3-manifold, Y. If K is a null-homologous knot in Y and F is a Seifert surface for K, then

$$\mathrm{tb}(K_l) + \left|\mathrm{rot}_F(K_l)\right| \leq 2g(F) - 1 \tag{2.3.1}$$

for any K_l, a Legendrian representative of K.

This bound can be improved in some settings. Lisca and Matic improved the bound in the case that the contact structure is Stein fillable [36], and this improvement was extended to the setting of a tight contact structure with non-vanishing Seiberg-Witten contact invariant by Mrowka and Rollin [39]. An analogous theorem was proved by Wu [57] for the Ozsváth-Szabó contact invariant.

These improvements involved replacing the Seifert genus in the Eliashberg-Bennequin inequality with the genus of a surface which is properly embedded in a 4-manifold bounded by Y. As such bounds involve the negative Euler characteristic of a surface with boundary K, they must be no less than -1. In [25], Hedden introduced an integer $\tau_\xi(K)$ that is defined via the filtration on knot Floer homology associated to $(Y, [F], K)$, where $[F]$ is the homology class of a Seifert surface for K. He showed that in the case that the Ozsváth-Szabó contact invariant is non-zero, the right side of (2.3.1) can be replaced by $2\tau_\xi(K) - 1$. With such a bound he showed that for any contact manifold with non-zero contact invariant, there exist prime Legendrian knots with arbitrarily negative classical invariants.

In another direction, one could consider rationally null-homologous knots in a contact manifold (Y, ξ). In such a setting there is a notion of rational Seifert surface and corresponding classical invariants $\mathrm{tb}_{\mathbb{Q}}$, $\mathrm{rot}_{\mathbb{Q}}$, and $\mathrm{sl}_{\mathbb{Q}}$ (see Definition 4.1.2 below). Baker and Etnyre [7] extend the Eliashberg-Bennequin inequality to this setting:

Theorem 2.3.4. *Let (Y, ξ) be a contact 3-manifold with ξ a tight contact structure. Let K be a knot in Y with order $r > 0$ in homology and let Σ be a rational Seifert surface for K. Then for K_t, a transverse representative of K,*

$$sl_{\mathbb{Q}}(K_t) \leq -\frac{1}{r}\chi(\Sigma).$$

Moreover, if K_l is a Legendrian representative of K then

$$tb_{\mathbb{Q}}(K_l) + \left| rot_{\mathbb{Q}}(K_l) \right| \leq -\frac{1}{r}\chi(\Sigma).$$

There is an inequality found by Franks and Williams [19], and independently by Morton [37], that relates the index and algebraic crossing number of a braid to a degree of the HOMFLY polynomial of its closure. Later, using the Bennequin's work, Fuchs and Tabachnikov reinterpreted the result in terms of the self-linking number of a transverse knot in (S^3, ξ_{st}) [21]. This inequality has come to be known as the Franks-Williams-Morton (FWM) inequality.

More precisely, we describe the FWM inequality as follows. The HOMFLY polynomial $J(K)$ is a polynomial invariant of links in the variables v, z such that if $U \subset S^3$ is the unknot then $J(U) = 1$, and J satisfies

$$v^{-1}J(K_+) - vJ(K_-) = zJ(K_0), \tag{2.3.2}$$

where K_+, K_-, and K_0 form a skein triple.

Theorem 2.3.5 (Franks-Williams-Morton (FWM) inequality)**.** *Let $e(K)$ denote the minimum degree of v in $J(K)$. Then for any transverse representative K_t of K,*

$$sl(K_t) \leq e(K) - 1.$$

Moreover, if K_l is a Legendrian representative of K then

$$tb(K_l) + \left|rot(K_l)\right| \leq e(K) - 1.$$

As discussed in the introduction, Theorem 1.1.1 gives a HOMFLY polynomial for links in lens spaces, and Corollary 1.1.4 extends the FWM inequality to universally tight contact lens spaces.

2.3.3 Links and grid diagrams in $L(p, q)$

In this section we review toroidal grid diagrams in $L(p, q)$, which are central to our discussion and view of links in lens spaces, and were fully studied in [4] (see also [5]). We will also (briefly) review the correspondence between Legendrian links in $L(p, q)$ with a universally tight contact structure and toroidal grid diagrams. Let us begin with the definition of that contact structure.

The manifold $L(p, q)$, which is the result of $-\frac{p}{q}$ surgery on the unknot, is also obtained as a quotient of $S^3 \subset \mathbb{C}^2$ by the equivalence relation $(u_1, u_2) \sim (\omega_p u_1, \omega_p^q u_2)$, where $\omega_p = e^{\frac{2\pi i}{p}}$. Let $\pi : S^3 \to L(p, q)$ be the quotient map.

Represent points (u_1, u_2) of S^3 in polar coordinates, letting $u_i = (r_i, \theta_i)$. The kernel ξ_{st}

of the 1-form $\alpha = r_1^2 d\theta_1 + r_2^2 d\theta_2$ is the unique (up to orientation) tight contact structure on S^3 [22]. The 1-form α is constant along any torus in S^3 determined by a fixed r_1. Since such a torus is fixed (not pointwise) under the action $(u_1, u_2) \mapsto (\omega_p u_1, \omega_p^q u_2)$, the pushforward $\xi_{UT} = \pi_*(\xi_{st})$ is a well-defined, and is clearly a contact structure on $L(p,q)$. Since ξ_{UT} pulls back to the standard contact structure on S^3, which is tight, ξ_{UT} is universally tight.

The points of $L(p,q)$ can be identified with points in a fundamental domain of the cyclic action on S^3. Thus, since r_2 is determined by r_1 in S^3, we can describe $L(p,q)$ by

$$L(p,q) = \left\{ (r_1, \theta_1, \theta_2) \mid r_1 \in [0,1],\ \theta_1 \in [0, 2\pi)\,,\ \theta_2 \in \left[0, \frac{2\pi}{p}\right) \right\}.$$

Analogous to the correspondence between planar grid diagrams and Legendrian links in (S^3, ξ_{st}) ([42],[53]), we can define toroidal grid diagrams in $L(p,q)$ to get a correspondence between grid diagrams and Legendrian links in $(L(p,q), \xi_{UT})$. To be precise we define grid diagrams in $L(p,q)$ as follows (cf. [4]).

Definition 2.3.6. A *grid diagram D* with grid number n in $L(p,q)$ is a set of data $(T, \vec{\alpha}, \vec{\beta}, \vec{\mathbb{O}}, \vec{\mathbb{X}})$, where:

- T is the oriented torus obtained via the quotient of $\mathbb{R}^2$ by the $\mathbb{Z}^2$ lattice generated by $(1,0)$ and $(0,1)$.

- $\vec{\alpha} = \{\alpha_0, \ldots, \alpha_{n-1}\}$, with α_i the image of the line $y = \frac{i}{n}$ in T. Call the n annular components of $T - \vec{\alpha}$ the *rows* of the grid diagram.

- $\vec{\beta} = \{\beta_0, \ldots, \beta_{n-1}\}$, with β_i the image of the line $y = -\frac{p}{q}(x - \frac{i}{pn})$ in T. Call the n annular components of $T - \vec{\beta}$ the *columns* of the grid diagram.

- $\vec{\mathbb{O}} = \{O_0, \ldots, O_{n-1}\}$ is a set of n points in $T - (\vec{\alpha} \cup \vec{\beta})$ such that no two O_i's lie in the same row or column.

- $\vec{\mathbb{X}} = \{X_0, \ldots, X_{n-1}\}$ is a set of n points in $T - (\vec{\alpha} \cup \vec{\beta})$ such that no two X_i's lie in the same row or column.

The components of $T - \vec{\alpha} - \vec{\beta}$ are called the *fundamental parallelograms* of D and the points $\vec{\mathbb{O}} \cup \vec{\mathbb{X}}$ are called the *markings* of D. Two grid diagrams with corresponding tori T_1, T_2 are considered equivalent if there exists an orientation-preserving diffeomorphism $T_1 \to T_2$ respecting the markings (up to cyclic permutation of their labels).

Such a grid diagram has "slanted" β curves. For considerations of both convenience and aesthetics, we alter the fundamental domain of T and "straighten" our pictures so that the β curves are vertical. Figure 2.4 shows how this "straightening" is accomplished.

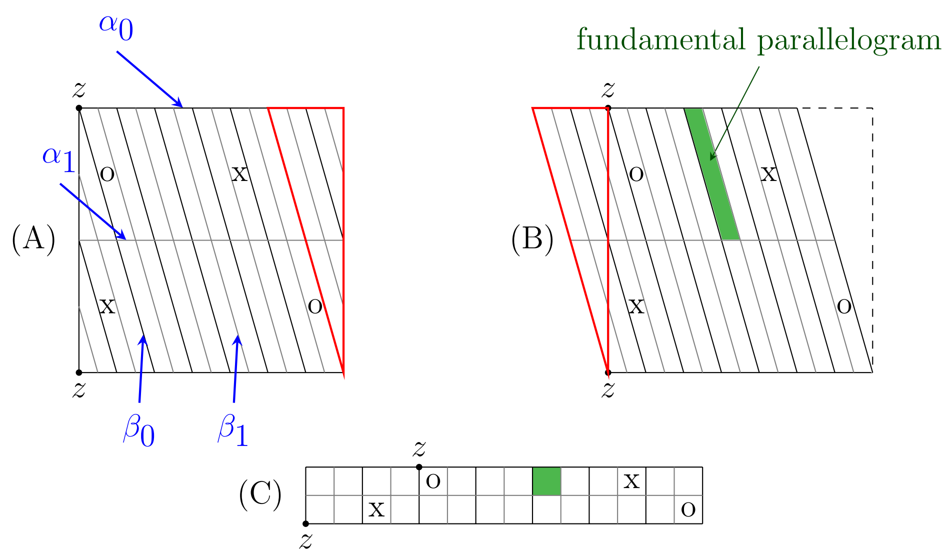


Figure 2.4: (A) shows a grid diagram (with grid number 2) in $L(7,2)$ on a fundamental domain of T. In (B) we alter the fundamental domain. (C) is the "straightening" of (B). For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this dissertation.

A link $K \subset L(p,q)$ is associated to a grid diagram D in $L(p,q)$ in the following manner.

Let Σ be the torus in $L(p,q)$ of constant radius $r_1 = 1/\sqrt{2}$ which splits $L(p,q)$ into two solid tori V^α and V^β. Identify T with $-\Sigma$ such that the α-curves of D are negatively-oriented meridians of V^α and the β-curves are meridians of V^β. Next connect each X to the O in its row by an "horizontal" oriented arc (from X to O) that is embedded in T and disjoint from $\vec{\alpha}$. Likewise, connect each O to the X in its column by a "vertical" oriented arc embedded in T and disjoint from $\vec{\beta}$. The union of the $2n$ arcs makes a multicurve γ. Remove self-intersections of γ by pushing the interiors of horizontal arcs up into V^α and the interiors of vertical arcs down into V^β.

We note that in the association of a link to a grid diagram in $L(p,q)$ the author used the opposite convention as that adopted in other places in the literature [4, 5, 45, 47]. The convention used in other places in the literature was adopted to fit conventions coming from knot Floer homology. However, for the purposes of this thesis, it is more clear to use the approach presented below (as in [13]) as there is no reference to Floer homology theories.

Definition 2.3.7. Let K be a link associated to a grid diagram D in $L(p,q)$ with grid number n. For some $0 < m < n$, suppose D' is a subcollection of m rows and m columns of D such that the $2m$ markings contained in the rows of D' are exactly the $2m$ markings contained in the columns of D'. Then D' is a grid diagram for some sublink of K. If this sublink has one component then D' is called a *component of* D.

Remark 2.3.8. No part of Definition 2.3.6 prohibits a marking in $\mathbb{X}$ and a marking in $\mathbb{O}$ from being in the same fundamental parallelogram. To a grid diagram that has grid number one (and so, only one marking in $\mathbb{X}$ and one marking in $\mathbb{O}$) and its two markings in the same fundamental parallelogram, we associate a knot in $L(p,q)$ that is contained in a small ball neighborhood and bounds an embedded disk.

Remark 2.3.9. Except for the case described in Remark 2.3.8, we assume that each marking of D is the center point of the fundamental parallelogram that contains it. Let the straightened fundamental domain of T have normalized coordinates $\{(\theta_1, \theta_2) \mid \theta_1 \in [0, p),\ \theta_2 \in [0, 1)\}$, so that each O and X sharing the same column have the same θ_1-coordinate mod 1.

Under the requirements of Remark 2.3.9, the projection of K to $-\Sigma$ (with vertical arcs crossing under horizontal arcs) is called a *grid projection* associated to D_K (the authors of [4] call this a rectilinear projection). Note that K has an orientation given by construction and so the grid projection is also oriented. Figure 2.5 shows an example of a grid diagram with a corresponding grid projection.

Given a grid diagram in a lens space $L(p, q)$ (with the identification of T to $-\Sigma$), the basis of vectors given by parallel translates of tangent vectors to $\{\alpha_0, \beta_0\}$ is coherently oriented with the global frame $\{d/d\theta_1, d/d\theta_2\}$.

The analogue to front projections in this setting, called *toroidal front projection*, projects radially through the tori V^α and V^β to Σ. A slight perturbation of a grid projection gives a toroidal front projection, which determines a Legendrian link in $(L(p, q), \xi_{UT})$ since the slope $d\theta_2/d\theta_1$ on the projection determines the radial coordinate r_1. The cusps of the front projection correspond to lower-left and upper-right corners of the grid projection, and we call these corners the *cusps* of a grid projection.

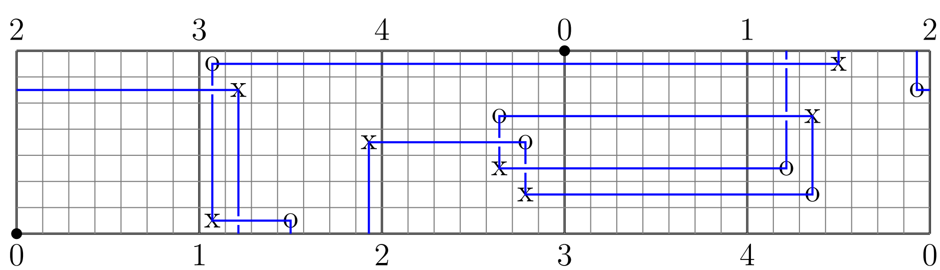


Figure 2.5: A grid diagram for $L(5, 3)$ with corresponding grid projection.

If D has grid number n then there are 2^{2n} different grid projections as there are two choices of vertical arc for each column, and two choices of horizontal arc for each row. In a given row (resp. column), the difference in choice of horizontal (resp. vertical) arc corresponds to a Legendrian isotopy across a meridional disk of V^{α} (resp. V^{β}) (see [4]). So the Legendrian isotopy class of the link is independent of the choice of grid projection.

In view of the correspondence above, Legendrian links in $(L(p,q), \xi_{UT})$ can be discussed via grid diagrams. There is a set of grid moves such that two grid diagrams correspond to the same Legendrian link if and only if there is a sequence of such grid moves taking one grid diagram to the other [4]. These moves come in two flavors: grid (de)stabilizations and commutations.

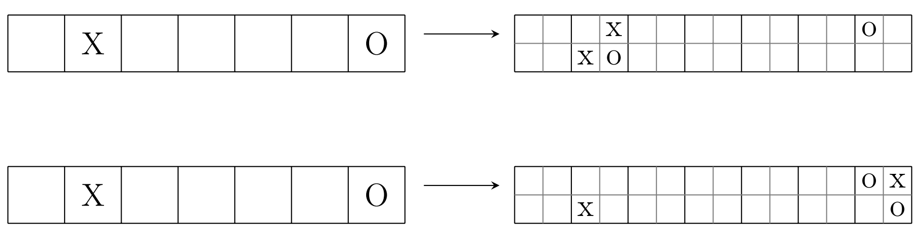


Figure 2.6: Stabilization of type X: NW (top) and of type O: SW (bottom)

Grid Stabilizations and Destabilizations: Grid stabilizations increase the grid number by one and should be thought of as adding a local kink to the knot. They are named with an X or O, depending on the type of marking at which stabilization occurs, and with NW, NE, SW, or SE, depending on the positioning of the new markings. Figure 2.6 shows an X:NW stabilization and an O:SW stabilization. Destabilizations are the inverse of a stabilization. Any (de)stabilization is a grid move that preserves the isotopy type.

However, the correspondence between our grid diagrams and toroidal front projections is such that cusps correspond to upper-right and lower-left corners of a grid projection. Only (de)stabilizations of types NW and SE preserve Legendrian isotopy type.

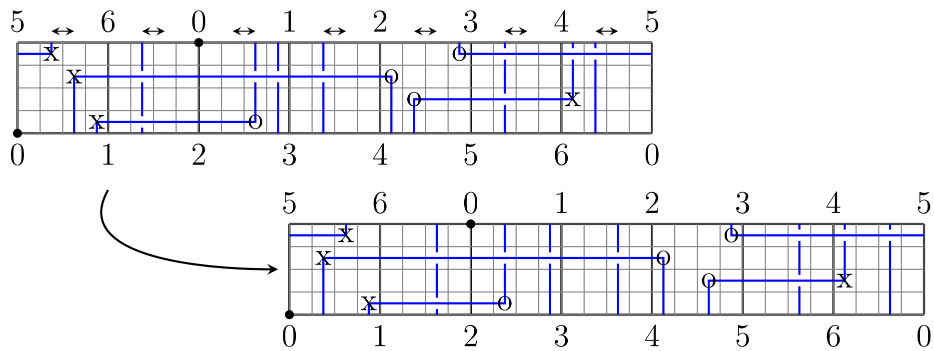


Figure 2.7: A non-interleaving commutation in $L(7,2)$.

Commutations: A commutation interchanges two adjacent columns (or rows) of the grid diagram. Let A be the annulus consisting of the two adjacent columns c_1, c_2 (resp. rows r_1, r_2) involved in the commutation. This annulus is sectioned into pn segments of the n rows (resp. columns) of the grid diagram. Let s_1, s_1' be the two segments in A containing the markings of c_1 (resp. r_1). If the markings of c_2 (resp. r_2) are contained in separate components of $A - s_1 - s_1'$, the commutation is called *interleaving*. If they are in the same component of $A - s_1 - s_1'$ the commutation is called *non-interleaving*. We note that in the literature a commutation typically refers only to what we call a non-interleaving commutation. We have extended the terminology to include the interleaving case. A non-interleaving commutation of columns (resp. rows) is a grid move that preserves Legendrian isotopy type [4]. An interleaving commutation corresponds to a crossing change (see Lemma 3.2.1). An example of non-interleaving commutation is shown in Figure 2.7.

Note that a commutation (interleaving or non-interleaving) does not include a column exchange of the type illustrated in Figure 2.8, where there is a row containing markings of both c_1 and c_2.

We end the section with a construction that plays an important role in what follows.

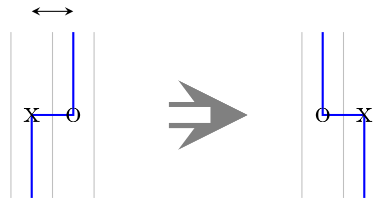


Figure 2.8: A move which is neither an interleaving nor non-interleaving commutation.

Given a grid diagram D in $L(p,q)$ define a grid diagram $\widetilde{D}$ of a link in S^3 as follows: cut T along α_0 to get an annulus A. The boundary of A is a disjoint union of two copies of α_0. Let one of these copies be α_0^+ and the other be α_0^-. Take p copies of A, say $A_0, \ldots, A_{p-1}$ and glue α_0^+ on A_i to α_0^- on $A_{i+1(\text{mod } p)}$ for $i = 0, \ldots, p-1$ by the identity map. Note that the torus constructed from $A_0, \ldots, A_{p-1}$ covers T under the cover $\pi : S^3 \to L(p,q)$ and so the link associated to $\widetilde{D}$ covers the link associated to D. Call $\widetilde{D}$ the *lift of* D *to* S^3. (An example is shown in Figure 2.9, where the grid diagram on the right is the lift of the grid diagram in $L(5,1)$ on the left. The link corresponding to the lift is a Hopf link.)

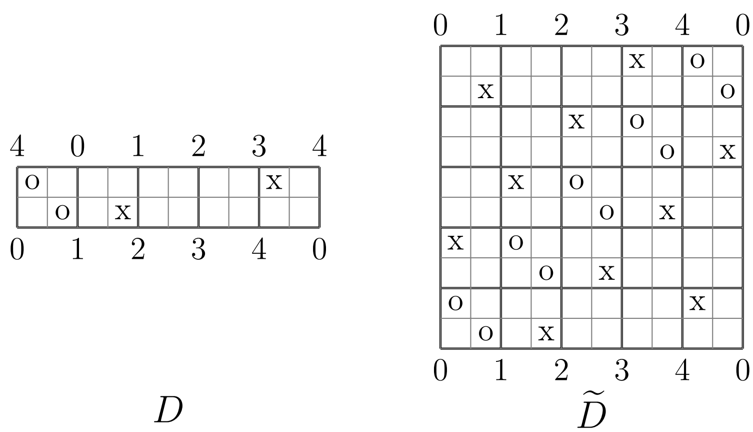


Figure 2.9: $\widetilde{D}$: the lift to S^3 of the grid diagram D in $L(5,1)$.

Chapter 3

Homotopies and the HOMFLY polynomial

3.1 Homology classes

Definition 3.1.1. For an oriented link K in $L(p,q)$, define $\mu(K)$ to be the homology class of K in $H_1(L(p,q))$.

Remark 3.1.2. Let K, K' be oriented links. We note that $\mu(K) = \mu(K')$ if and only if K and K' are freely homotopic, since the free homotopy class of an oriented knot in $L(p,q)$ is determined by its homology class.

Let C be the core of the handlebody V_α in $L(p,q)$ (dual to a meridional disk of V_α). Then $H_1(L(p,q)) \cong \mathbb{Z}/p$ is cyclically generated by $\gamma = [C] \in H_1(L(p,q))$. We identify $\mu(K)$ with the multiple of γ it represents.

Suppose $D_K = (T, \vec{\alpha}, \vec{\beta}, \mathbb{O}, \mathbb{X})$ is a grid diagram with associated link K in $L(p,q)$. Orient the α and β curves so that the algebraic intersection $\alpha_i \cdot \beta_j$ is positive on T, for all pairs

i, j. Let R_K be a grid projection for D_K with its given orientation. We can compute $\mu(K)$ from the grid diagram as follows.

Lemma 3.1.3. *Given D_K and R_K as above, choose an α-curve α_i in the grid diagram. Then $\mu(K)$ is equal (mod p) to $\alpha_i \cdot R_K$, the algebraic intersection of α_i with R_K.*

Proof. Push the interiors of horizontal arcs on R_K slightly into V_α to get a knot K' contained in V_α. The winding number of K' in V_α is clearly counted by $\alpha_i \cdot R_K$ and K' is isotopic to K, so $\mu(K') = \mu(K)$. □

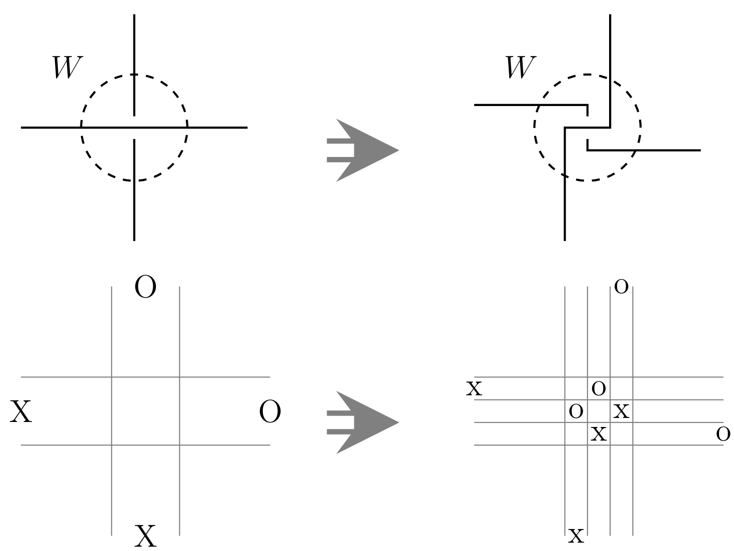


Figure 3.1: A crossing change on a grid projection.

3.2 Skein Theory

We now develop a skein theory of grid diagrams in $L(p, q)$. Note that grid diagrams have been discussed in the setting of singular links in S^3, and used to extend link Floer homology to singular links [2]. What we describe here as grid diagrams that differ by a "skein crossing change" are exactly the grid diagrams in [2] that correspond to the two resolutions of the double point of a 1-singular link (see, for example, Figure 10 in [2]).

If two links L_+ and L_- differ as described in (2.1.1), they are isotopic to links with associated grid diagrams differing by the grid move in Figure 3.1. The next lemma refines this. In a figure of a grid diagram, slanted gray bars indicate some number of columns (and their markings) which may be between the columns of interest.

Lemma 3.2.1. *Let L, L' be links corresponding to grid diagrams that differ by an interleaving commutation. Then L, L' differ by a crossing change.*

Proof. We prove the statement for row commutation. The case with columns is similar.

The proof is given by the sequence of grid moves detailed in Figure 3.2, where arrows indicate commutations to be performed, and a triple of markings that is to be destabilized is circled. Referring to Figure 3.2, there are 8 steps. From step 1 to 2 we perform a crossing change. From 2 to 5 we perform a number of non-interleaving commutations. Going from 5 to 6 is destabilization. From 6 to 7 involves several commutations. Each of these commutations is non-interleaving since the markings of the column being moved to the right are in adjacent rows. Finally we destabilize from 7 to 8.

Note that all grid moves above correspond to isotopy of the link, except the first which, as in Figure 3.1, corresponds to interchanging the two resolutions of a singular link defined by (2.1.1). □

Lemma 3.2.1 shows that if two grid diagrams differ only by the interleaving commutation of two adjacent columns (or rows), then the corresponding links are some pair L_+, L_-. Motivated by this result, we make the following definition.

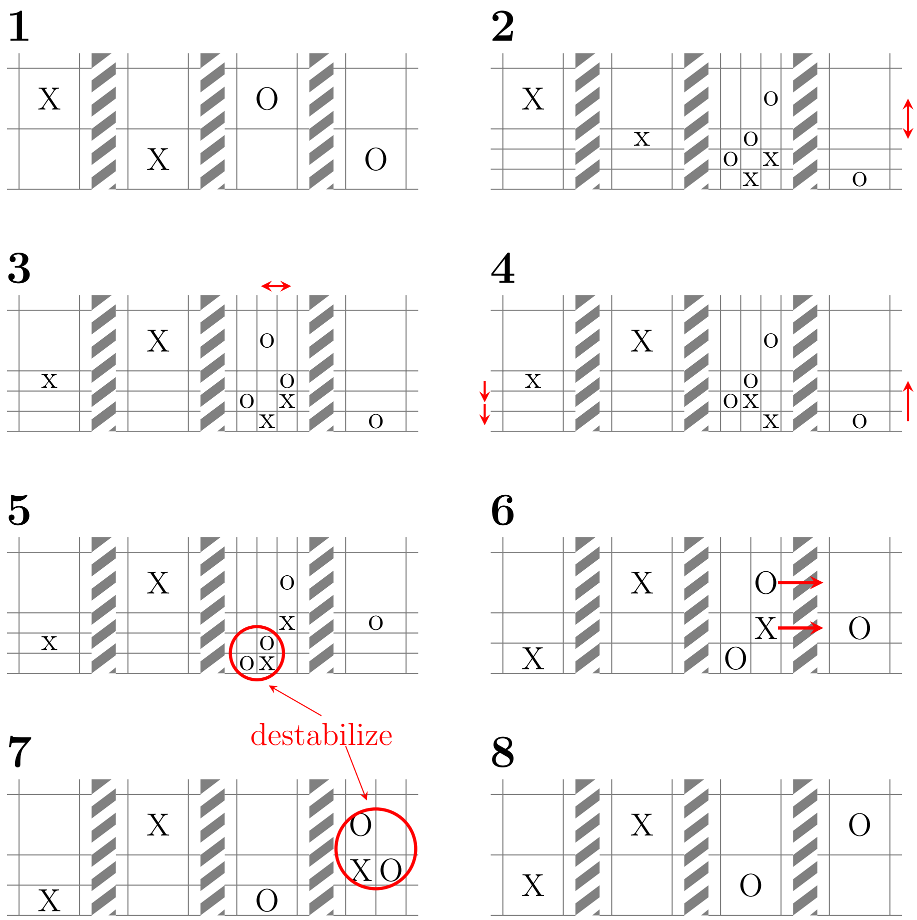


Figure 3.2: **1** → **2**: crossing change; **2** → **3**: row commutation; **3** → **4**: column commutation; **4** → **5**: two row commutations; **5** → **6**: destabilization; **6** → **7**: many column commutations; **7** → **8**: destabilization

Definition 3.2.2. A pair of adjacent columns in a grid diagram for $L(p, q)$ is called a *skein crossing* if their commutation is an interleaving commutation. Define a *skein crossing change* to be the commutation of a skein crossing pair. A skein crossing is called *positive* if it appears as in the left side of Figure 3.3 under some cyclic permutation of the rows and columns. It is *negative* if (after some cyclic permutation) it appears as in the right side of Figure 3.3. Given two grid diagrams that differ only by a skein crossing change, we call the diagram with the positive skein crossing D_+ and the diagram with the negative skein crossing D_-.

There is a function on grid diagrams that is related to the idea of skein crossings. We

Figure 3.3: Positive and negative skein crossings.

will use this function later, both to define trivial links and to produce a complexity function on grid diagrams that will prove Theorem 1.1.1.

Definition 3.2.3. Consider a (not necessarily adjacent) pair of columns c_1, c_2 in a grid diagram of $L(1,0) = S^3$. Let $R_i(\text{o})$ be the row in the diagram that contains the $\mathbb{O}$ marking of c_i and let $R_i(\text{x})$ be the row containing the $\mathbb{X}$ marking of c_i. Call the columns c_1, c_2 *interleaving* if $R_2(\text{o})$ and $R_2(\text{x})$ are in different annular components of $T - (R_1(\text{o}) \cup R_1(\text{x}))$. Note that if c_1, c_2 are adjacent and interleaving then they comprise a skein crossing.

Given a grid diagram D for a link in $L(p,q)$, define $scr(D)$ to be the number of interleaving pairs of columns in $\widetilde{D}$, the lift of D to S^3.

Note that, despite the name of the function, $scr(D)$ does not count the number of skein crossings of D. Instead, it counts all pairs of columns of $\widetilde{D}$ which, if they were adjacent, would make a skein crossing. Let us point out a few important properties of the function scr.

Proposition 3.2.4. *Let $\mathscr{D}(L(p,q))$ denote the set of all grid diagrams in $L(p,q)$. The function $scr : \mathscr{D}(L(p,q)) \to \mathbb{Z}$ satisfies the following:*

1. *(Orientation invariance) Given a grid diagram D, let rD denote the grid diagram obtained by exchanging every $\mathbb{O}$ marking for an $\mathbb{X}$ marking and vice versa. Then $scr(D) = scr(rD)$.*

2. *(Column commutation invariance) Let D, D' be two grid diagrams that only differ by a column commutation. Then $scr(D) = scr(D')$. Note that this is not true for row commutation.*

Proof. The fact that orientation invariance holds is obvious.

To prove the second property, first note that an adjacent pair of columns in D will correspond to p adjacent pairs of columns in $\widetilde{D}$. Thus, if D and D' differ by a column commutation, then $\widetilde{D}$ and $\widetilde{D'}$ differ by p column commutations. Given a column c of $\widetilde{D}$, if $R(\text{o})$ and $R(\text{x})$ are the rows that its markings occupy, then its markings are still in rows $R(\text{o})$ and $R(\text{x})$ after column commutation. It is clear then from Definition 3.2.3 that the number of columns in $\widetilde{D}$ that are interleaving with c is the same before and after a commutation with a column adjacent to c. □

We now consider how resolutions fit into the skein theory of grid diagrams. To begin, let us define what we mean by such a resolution.

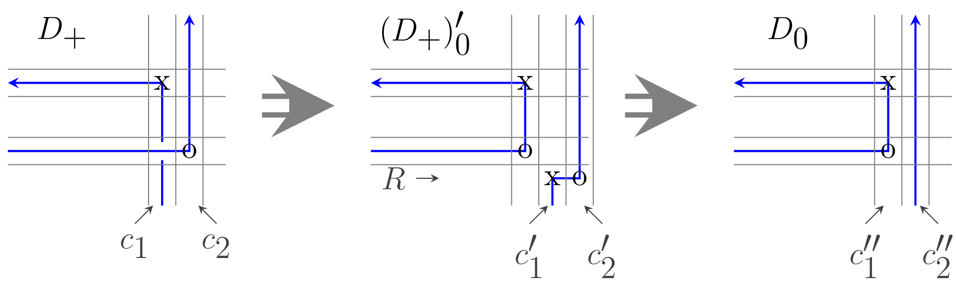


Figure 3.4: Resolution at a positive skein crossing

Given a grid diagram D_+ and a positive skein crossing of D_+, choose one of the $\mathbb{O}$ markings in the skein crossing. There exists some choice of grid projection that, near this marking, looks like the leftmost projection in Figure 3.4 (or a 180° rotation of this picture). Let $L_\times$ be the 1-singular link that has one double point at the crossing depicted in this projection, and is identical elsewhere to the link associated to D_+. We take the resolution L_0 of $L_\times$ defined in (2.1.1).

This operation is local and we see that the link associated to the grid diagram $(D_+)'_0$, pictured in the middle of Figure 3.4, is isotopic to L_0. Since columns c'_1 and c'_2 are adjacent in the diagram $(D_+)'_0$ any commutation of row R will be non-interleaving, and so an isotopy. Therefore, we may commute this row until we have a triple of markings that can be destabilized. After this destabilization we get the grid diagram D_0, as shown in Figure 3.4, with an associated link that is isotopic to L_0. Note that the grid diagram D_0 is obtained from D_+ by exchanging the θ_1 coordinates of the $\mathbb{O}$ markings in the two columns.

Likewise, given a grid diagram D_-, a negative skein crossing of D_-, and an $\mathbb{O}$ marking in the skein crossing, there is a grid projection of D_- that looks like the leftmost projection in Figure 3.5. We then define the resolution of this skein crossing similarly, getting the grid diagram D_0 depicted in Figure 3.5. In this case, we get the diagram D_0 by interchanging the $\mathbb{X}$ markings in the columns. Based on our observations, we make the following definition.

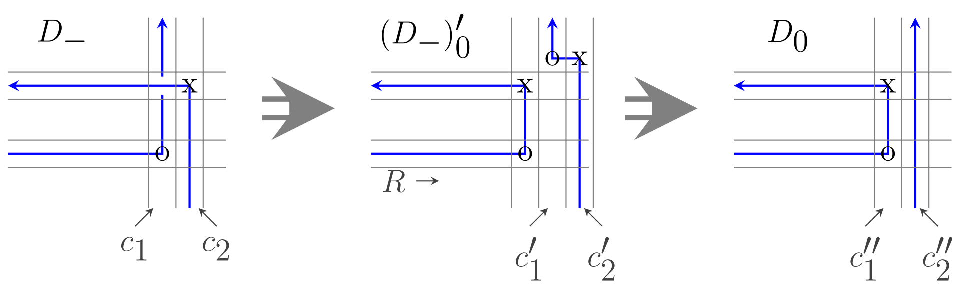


Figure 3.5: Resolution at a negative skein crossing

Definition 3.2.5. Given a grid diagram D and a positive (resp. negative) skein crossing of D, define D_0 to be the grid diagram that differs from D only in the columns of the skein crossing, where it differs only by interchanging the θ_1 coordinates of the $\mathbb{O}$ (resp. $\mathbb{X}$)

markings.

Remark 3.2.6. In Figure 3.4 (resp. 3.5), the grid diagram D_0 is obtained by interchanging the θ_1-coordinates of the $\mathbb{O}$ (resp. $\mathbb{X}$) markings in the skein crossing. Note that if we change the θ_1-coordinates of the $\mathbb{X}$ (resp. $\mathbb{O}$) markings instead, the resulting diagram differs from D_0 by a non-interleaving commutation of columns, and so its associated link is also isotopic to L_0. Finally, note that if D_+ and D_- differ only as shown, by a skein crossing change, then the grid diagram D_0 is obtained as a resolution of each, so D_+, D_-, and D_0 form a skein triple.

In the course of the next section we will prove Theorem 1.1.1, and in the proof we will need the function *scr* to decrease under a resolution $D_\pm \rightsquigarrow D_0$. To see that *scr* behaves this way, we use a map H from the columns of a grid diagram to the coordinate plane. The map H is defined below and we point out some of its properties.

Let $\mathscr{C}(D)$ be the set of columns of a grid diagram D in $L(1,0) = S^3$ and choose an α-curve α_0. Define a map $H_{\alpha_0} : \mathscr{C}(D) \to \mathbb{R}^2$ in the following way. Using the orientation on $\vec{\beta}$ we can define a height function θ_2 on the annulus $T - \alpha_0$. For $c \in \mathscr{C}(D)$, let m (resp. n) be the smaller (resp. larger) of the θ_2 coordinates of the markings of c. We define $H_{\alpha_0}(c) = (m, n)$. For convenience of notation, we will drop the subscript of H_{α_0} unless we wish to emphasize the dependence of the map on the choice of α_0.

By our definition, if D has grid number n, then $H(\mathscr{C}(D))$ is a set of n points in the plane between the y-axis and the line $y = x$. Further the map only depends on the positions of the markings in a column, so if rD is the grid diagram obtained from D by reversing orientation (interchanging the $\mathbb{X}$ and $\mathbb{O}$ markings), then the image set $H(rD)$ equals the image $H(D)$.

Finally, we point out that if c is a column of D and $p = H(c)$, it can be easily checked that another column c' is interleaving with c if and only if $H(c')$ is in one of the shaded regions of the plane shown in Figure 3.6. Call these regions the *interleaving regions* of p.

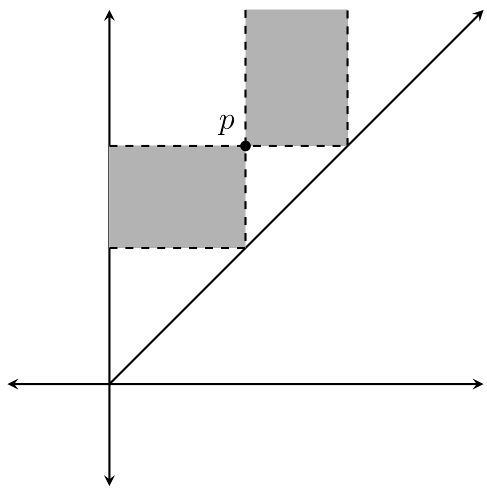


Figure 3.6: The interleaving regions of p, a point in the image of $H : \mathscr{C}(D) \to \mathbb{R}^2$

Remark 3.2.7. Observe that the property of two columns being interleaving in a grid diagram for a link in S^3 is completely independent of a preferred α-curve (Definition 3.2.3). This implies that, although the map H depends on the choice of α_0, the property of $H(c')$ being in one of the interleaving regions of $H(c)$ does not depend on such a choice.

3.3 The HOMFLY polynomial in $L(p, q)$

3.3.1 Trivial Links in $L(p, q)$.

Fix a lens space $L(p, q)$ with Heegaard torus Σ as described in Section 2.3. As mentioned in Remark 2.3.9, after a choice of α_0 and β_0 all grid diagrams are given coordinates $\{(\theta_1, \theta_2) \mid \theta_1 \in [0, p), \theta_2 \in [0, 1)\}$.

Given a grid diagram D, let $GN(D)$ denote the grid number of D. Given a link K, set

$GN(K)$ to be

$$GN(K) = \min \{GN(D) \mid D \text{ is a grid diagram for a link isotopic to } K\}.$$

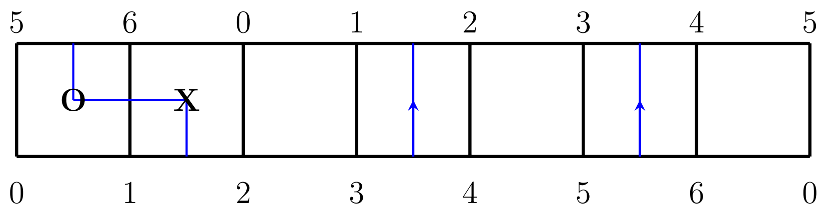


Figure 3.7: The grid number 1 knot K_3 in $L(7,2)$

Definition 3.3.1. A knot K in $L(p,q)$ is called a *trivial knot* if $GN(K) = 1$. If $\mu(K) = i$, we write $K = K_i$. Under this notation the unknot described in Remark 2.3.8 is a trivial knot and written K_0.

Remark 3.3.2. Note that μ is only defined up to congruence class mod p. Let D_{K_i} denote the grid diagram with grid number one of the trivial knot K_i. Since p, q are coprime, there exists some K_i for each $0 \leq i \leq p-1$. See Figure 3.7 for the diagram and a projection of K_3 in $L(7,2)$.

Having defined trivial knots we now define a collection of trivial links so that one component trivial links are trivial knots. Let $\mathcal{I} = (m_0, m_1, \ldots, m_{p-1})$ be a p-tuple of non-negative integers. We construct a grid diagram $D(\mathcal{I})$ with grid number $n = \sum m_i$ as follows. Place the markings $\mathbb{O} = \{O_1, O_2, \ldots, O_n\}$ so that $0 < \theta_1(O_i) < 1$ for each $1 \leq i \leq n$, and so that

$$\theta_1(O_i) + \frac{1}{n} = \theta_1(O_{i+1}) \qquad \text{and}$$
$$\theta_2(O_i) - \frac{1}{n} = \theta_2(O_{i+1}).$$

for each $1 \leq i \leq n-1$. Note that these conditions force O_1 to be in the left-uppermost fundamental parallelogram; i.e. O_1 has coordinates $\left(\frac{1}{2n}, 1 - \frac{1}{2n}\right)$.

In order to place the markings $\mathbb{X} = \{X_1, X_2, \ldots, X_n\}$, we define a bijection

$$\sigma : \{0, 1, \ldots, p-1\} \to \{0, 1, \ldots, p-1\}$$

by the congruence $\sigma(i) \cdot q \equiv i \mod p$. Now, for each i such that $\sum_{k=0}^{l-1} m_{\sigma(k)} + 1 \leq i \leq \sum_{k=0}^{l} m_{\sigma(k)}$, with $0 \leq l \leq p-1$, let v_i be the directed vertical arc (in the column of O_i) that satisfies $\alpha_0 \cdot v_i = \sigma(l)$ and has one endpoint at O_i and the other at the center of a fundamental parallelogram that is in the same row as O_i. Place X_i at the other endpoint of v_i. We call the resulting grid diagram $D(\mathcal{I})$. An example is shown in Figure 3.8 below.

Recall the definition of the components of a grid diagram (Definition 2.3.7). We first note that every component of $D(\mathcal{I})$ has grid number one, since for each $1 \leq i \leq n$, O_i and X_i are in the same row and the same column. Secondly, let v_i be the vertical arc from O_i to X_i as in the construction of $D(\mathcal{I})$. Denote the homology class of the component of $D(\mathcal{I})$ corresponding to O_i by $\mu(O_i) = \alpha_0 \cdot v_i$. Then our construction (by the definition of σ) guarantees that if

$$\mu(O_i)q \ (\mathrm{mod}\ p) < \mu(O_j)q \ (\mathrm{mod}\ p) \tag{3.3.1}$$

then $i < j$. Further, the coordinates of the $\mathbb{O}$ markings place these markings on a diagonal

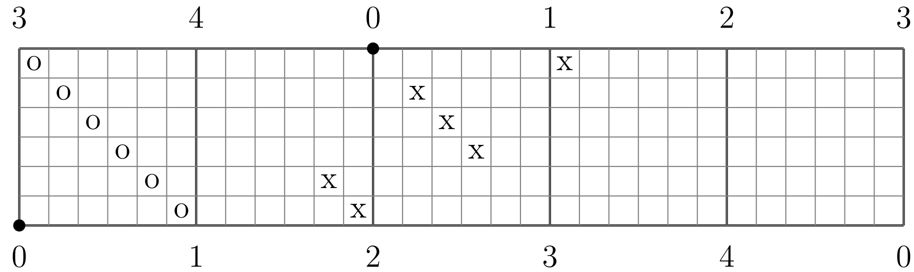


Figure 3.8: The trivial link diagram $D(\mathcal{I})$ in $L(5,2)$ with $\mathcal{I} = (0,1,2,0,3)$.

of a parallelogram in $T - \alpha_0 - \beta_0$.

Definition 3.3.3. For any p-tuple $\mathcal{I}$ of non-negative integers, define $K(\mathcal{I})$ to be the link associated to $D(\mathcal{I})$. We call a link K in $L(p,q)$ a *trivial link* if K is isotopic to $K(\mathcal{I})$ for some $\mathcal{I}$. The p-tuple $\mathcal{I}$ is called the *index set* associated to K and $D(\mathcal{I})$ is called a *trivial link diagram*. Note that each grid number one knot K_i corresponds to an index set with 1 in the i^{th} position and 0 otherwise.

In the course of proving Theorem 1.1.1 we will need that each trivial link K admits a grid diagram that minimizes the function *scr* in that free homotopy class of link, among diagrams with minimal grid number (just as n component unlinks minimize crossing number in their homotopy class). It may be that the diagram $D(\mathcal{I})$ for K does not achieve this, however in the following lemma we prove the existence of a grid diagram that does.

Let $\mathcal{I} = (m_0, m_1, \ldots, m_{p-1})$, with $m_i \geq 0$ for each i. Consider all grid diagrams D in $L(p,q)$ such that the components of D have grid number one (they are in the set $\{D_{K_0}, D_{K_1}, \ldots, D_{K_{p-1}}\}$), and so that D has m_i components of type D_{K_i} for each $0 \leq i \leq p-1$. Call this finite collection of grid diagrams $\mathscr{D}(\mathcal{I})$. Let $n = \sum m_i$. Then the grid number of every diagram in $\mathscr{D}(\mathcal{I})$ is n.

Lemma 3.3.4. *Consider the link $K(\mathcal{I})$ in $L(p,q)$ for an index set $\mathcal{I}$. There is a grid diagram*

$\hat{D}(\mathcal{I})$ *which has associated link isotopic to* $K(\mathcal{I})$*, such that* $scr(\hat{D}(\mathcal{I}))$ *is the minimum of the image* $scr(\mathscr{D}(\mathcal{I})) \subset \mathbb{Z}_{\geq 0}$.

Proof. Definition 3.2.3 gives a map $scr_{\mathcal{I}} : \mathscr{D}(\mathcal{I}) \to \mathbb{Z}_{\geq 0}$ by restriction. If z is the minimum of Im $scr_{\mathcal{I}}$, let $\mathscr{M}(\mathcal{I}) = scr_{\mathcal{I}}^{-1}(z)$. We must show that there is a diagram in $\mathscr{M}(\mathcal{I})$ whose associated link is isotopic to $K(\mathcal{I})$. To do so, we consider $\mathscr{M}(\mathcal{I})$ more thoroughly.

Recall from Proposition 3.2.4 that *scr* is invariant under column commutation, so if $D \in \mathscr{M}(\mathcal{I})$ and we get D' from D by a column commutation, then $D' \in \mathscr{M}(\mathcal{I})$. Begin with any grid diagram $D \in \mathscr{M}(\mathcal{I})$ and perform column commutations until the θ_1 coordinate of every $\mathbb{O}$ marking is between 0 and 1 (this is possible since each component has grid number one). The resulting grid diagram (which by abuse of notation we call D) is still in $\mathscr{M}(\mathcal{I})$.

Note that the $\mathbb{O}$ markings of D are in one-to-one correspondence with the components and with the columns. Therefore, perhaps after more commutations of columns, we may guarantee that if O, O' are two $\mathbb{O}$ markings and if

$$\mu(O)q \pmod{p} < \mu(O')q \pmod{p} \tag{3.3.2}$$

then $\theta_1(O) < \theta_1(O')$ (here $\mu(O)$ is the homology class of the component of D corresponding to O, as in (3.3.1)). As a result, all $\mathbb{O}$ markings with a fixed homology type are grouped into consecutive columns. Finally, among the $\mathbb{O}$ markings with homology $\mu(O) = i$, perform commutations of columns until the θ_2 coordinates are decreasing (among the markings with $\mu(O) = i$). Do this for each i with $m_i > 0$. Denote the resulting grid diagram, which is contained in $\mathscr{M}(\mathcal{I})$, by $\hat{D}(\mathcal{I})$.

The process above results in a grid diagram $\hat{D}(\mathcal{I})$ with an order on the $\mathbb{O}$ markings $\left\{\hat{O}_1, \hat{O}_2, \ldots, \hat{O}_n\right\}$ of $\hat{D}(\mathcal{I})$ so that:

(a) $\theta_1(\hat{O}_i) = \frac{2i-1}{2n}$;

(b) if $\mu(\hat{O}_i)q \pmod p < \mu(\hat{O}_j)q \pmod p$ then $i < j$;

(c) if $\mu(\hat{O}_i) = \mu(\hat{O}_{i+1})$ then $\theta_2(\hat{O}_i) - \frac{1}{n} = \theta_2(\hat{O}_{i+1})$.

Note that by condition (a), $\theta_1(\hat{O}_i) = \theta_1(O_i)$ for each i. This, in addition to condition (b), implies that for every $1 \leq i \leq n$ we have $\mu(\hat{O}_i) = \mu(O_i)$.

Now, suppose the p-tuple $\mathcal{I}$ is zero in every coordinate except one. Then (c) makes $\hat{D}(\mathcal{I}) = D(\mathcal{I})$, so $D(\mathcal{I})$ is in $\mathscr{M}(\mathcal{I})$ and we are finished. Suppose instead there is more than one homology class represented among the components. Property (c) implies that if $\theta_2(\hat{O}_i) - \theta_2(\hat{O}_{i+1}) \neq \theta_2(O_i) - \theta_2(O_{i+1})$ then $\mu(\hat{O}_i) \neq \mu(\hat{O}_{i+1})$. Thus, since $i+1$ is not less than i, it must be that

$$\mu(\hat{O}_i)q \pmod p < \mu(\hat{O}_{i+1})q \pmod p. \tag{3.3.3}$$

Let $k = \mu(O_{i+1})$ and for any $1 \leq j \leq n$, let r_j be the row of $\hat{D}(\mathcal{I})$ containing $\hat{O}_j$ (and thus also $\hat{X}_j$). Property (a) and inequality (3.3.3) then imply that r_i and r_{i+j} are non-interleaving rows for any $1 \leq j \leq m_k$.

So, for each $0 \leq k \leq p-1$, we may perform row commutations on the m_k rows of $\hat{D}(\mathcal{I})$ that have components in homology class k, one row at a time, until the corresponding $\mathbb{O}$ markings are on the diagonal (have coordinates $\left(\frac{2i-1}{2n}, 1 - \frac{2i-1}{2n}\right)$). Each of these row commutations is non-interleaving, using (a), (b) and the fact that the two rows involved in each commutation correspond to different homology classes. These row commutations describe an isotopy starting at the link associated to $\hat{D}(\mathcal{I})$ and ending at the link associated to $D(\mathcal{I})$. Since $\hat{D}(\mathcal{I})$ is in $\mathscr{M}(\mathcal{I})$, this finishes the proof. □

3.3.2 Every link is homotopic to a trivial link

In this section we show that every link in $L(p, q)$ is homotopic to a unique trivial link. More precisely, we show that this homotopy can be realized by a sequence of particular grid moves (see Proposition 3.3.6).

Note that a trivial link is determined by an index set $\mathcal{I}$. Any two different trivial links cannot be in the same homotopy class, since their index sets will differ and so at least one of their components differs in free homotopy class. Thus if a link is homotopic to some trivial link, it is homotopic to a unique trivial link. We will show that there is a particular type of homotopy from any link to a trivial link.

Lemma 3.3.5. *Let K be a Legendrian link in $L(p, q)$ associated to a grid diagram D_K. Suppose D_K has a component with grid number more than 1. Then there exists a sequence of commutations (both interleaving and non-interleaving) followed by a destabilization giving a new grid diagram D' such that $GN(D') < GN(D_K)$.*

Note that in Lemma 3.3.5 we arrive at the grid diagram D' without using the forbidden column exchange in Figure 2.8. We avoid the use of such moves for contact geometric considerations. Before proving Lemma 3.3.5 we use it to find a homotopy from any link to a trivial link through commutations and destabilizations.

Proposition 3.3.6. *Let D_K be a grid diagram for the link $K \subset L(p, q)$. There is a sequence of commutations and destabilizations taking D_K to a trivial link diagram. This provides a homotopy from K to a trivial link.*

Proof. We prove Lemma 3.3.5 below. By that lemma there is a grid diagram $D_{K'}$ corresponding to a link K' such that the grid number of each component of $D_{K'}$ equals one, and a sequence of commutations and destabilizations takes D_K to $D_{K'}$. Since $D_{K'} \in \mathscr{D}(\mathcal{I})$ for the index set $\mathcal{I}$ of K, we may successively interchange adjacent columns of $D_{K'}$ until we arrive at a diagram $\hat{D}$ with the θ_1-coordinates of its $\mathbb{O}$ markings lined up with $D(\mathcal{I})$, as in the proof of Lemma 3.3.4. By that proof, the link associated to $\hat{D}$ is a trivial link. This process is a sequence of column commutations, either interleaving or non-interleaving. Since non-interleaving commutations and destabilizations correspond to an isotopy, Lemma 3.2.1 implies that the resulting sequence of diagrams, from D_K to $\hat{D}$, corresponds to a homotopy from K to a trivial link. □

Proof of Lemma 3.3.5. We introduce the *length* of arcs in a grid projection. Every grid projection P is a union of $GN(P)$ horizontal arcs and $GN(P)$ vertical arcs. Define the *length* of an arc by the number of fundamental parallelograms it traverses. For example, the vertical arc in Figure 3.7 has length 3 and the horizontal arc has length 1.

Note that if two markings X_i and O_j share a row (resp. a column), there are two possible arcs that could join them in a grid projection. Define $len(X_iO_j)$ to be the minimum of the lengths of these two arcs. Note that if $len(X_iO_j) = 1$ then X_i and O_j are in adjacent columns (resp. rows). The converse, however, is not true since one column of the grid diagram (a component of $T - \vec{\beta}$) has p fundamental parallelograms in the same row.

For clarity, in the following figures we depict our argument by schematics rather than depicting the link projections on a grid diagram. The schematics show only the markings and horizontal and vertical arcs of the projection, but we remind the reader that – just as in a grid projection – a vertical arc in the schematic could be intersecting α_0 multiple times. The arrows in the schematics indicate how commutations will affect the columns, and a circled triple of markings indicates a destabilization.

Continuing with the proof, let $\hat{D}$ be a component of D_K with $GN(\hat{D}) > 1$. Then there are three markings of $\hat{D}$, say X_1, O_1, X_2, where X_1, X_2 are distinct, X_1, O_1 are in the same column, and X_2, O_1 are in the same row. Furthermore, among all such triples of markings of D_K, choose a triple X_1, O_1, X_2 such that $len(X_1O_1)$ is minimal.

Denote the column containing X_1 and O_1 by c_1 and the row containing O_1 and X_2 by r_1. Define O_2 to be the $\mathbb{O}$ marking in the column of X_2 and call this column c_2. Let X_3 be the $\mathbb{X}$ marking in the same row as O_2, which row we call r_2 (see Figure 3.9).

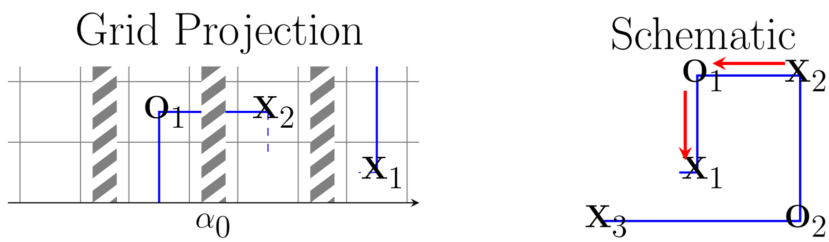


Figure 3.9: Decreasing Grid Number

Since X_1 and X_2 are distinct, we know the columns containing the pairs (X_1, O_1) and (X_2, O_2) are distinct. Thus there is a sequence of commutations of columns of D_K after which we have one of two cases. In case 1, $len(O_1X_2) = 1$. In case 2, which is depicted in Figures 3.10 and 3.11, $len(O_2X_3) = 1$. Each commutation (either interleaving or non-interleaving) exchanges column c_2 with an adjacent column, the result being that $len(O_1X_2)$ decreases. As previously shown, a commutation corresponds to either a skein crossing change or a Legendrian isotopy.

We now consider the two cases.

Case 1: By the minimality of our choice of $len(X_1O_1)$ we can commute row r_1 with an adjacent row, decreasing $len(X_1O_1)$ until $len(X_1O_1) = 1$. Since $len(O_1X_2) = 1$, c_1 and c_2 are adjacent columns, so each of the row commutations is non-interleaving. Performing a destabilization on (X_1, O_1, X_2), the grid number drops by one.

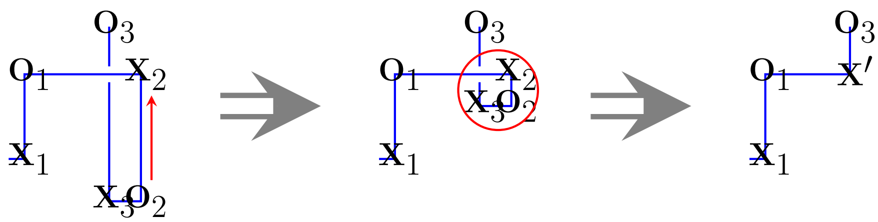


Figure 3.10: Decreasing Grid Number: Case 2a

Case 2: We are in one of the two situations of Figures 3.10 and 3.11. Let O_3 be the marking that shares a column with X_3.

In case 2a, $len(X_3O_3) \geq len(X_2O_2)$, so we can commute row r_2 until $len(X_2O_2) = 1$, with all commutations being non-interleaving (since $len(X_3O_2) = 1$). This is possible since X_3 is not the same marking as X_2, for this would force $GN(\hat{D}) = 1$. Note that X_3 may not be distinct from X_1. Performing a destabilization on the triple (X_2, O_2, X_3), the grid number decreases by one.

Otherwise, we are in case 2b where $len(X_3O_3) < len(X_2O_2)$. Commute row r_2, using only non-interleaving commutations, until $len(X_3O_3) = 1$, which again is possible since $GN(\hat{D}) \neq 1$ and $len(X_3O_2) = 1$. Perform a destabilization on the triple (O_2, X_3, O_3). Again the destabilization drops the grid number of the diagram by one.

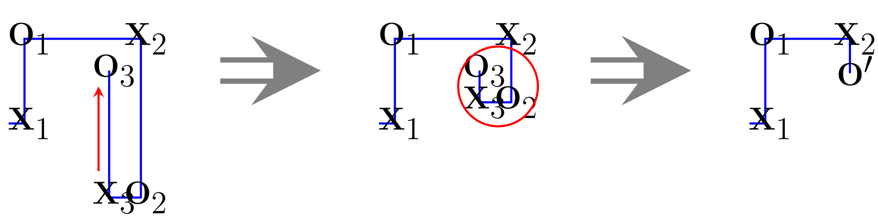


Figure 3.11: Decreasing Grid Number: Case 2b

□

Remark 3.3.7. Examining the proof of Lemma 3.3.5, we see that no row commutation performed in the reduction is interleaving.

3.3.3 The polynomial invariant

As a consequence of Proposition 3.3.6, there is a unique trivial link in each homotopy class of links. In this section we define a complexity function on grid diagrams so that trivial link diagrams minimize the complexity function. In classical knot theory one often considers the unlinking number of a link diagram as part of a complexity function. We consider an analogous number for grid diagrams.

Definition 3.3.8. Let D be a grid diagram for K. Define $u(D)$ to be the minimum number of skein crossing changes in a sequence of commutations and destabilizations that take D to a trivial link diagram $D(\mathcal{I})$, where the only interleaving commutations are column commutations. Proposition 3.3.6 , Remark 3.3.7, and the proof of Lemma 3.3.4 imply that there is such a number and that if $u(D) = 0$ then the link associated to D is isotopic to a trivial link.

Theorem 1.1.1. *Let $\mathscr{L}$ be the set of isotopy classes of links in $L(p, q)$ and let $\mathscr{T}\mathscr{L} \subset \mathscr{L}$ denote the set of isotopy classes of trivial links. Define $\mathscr{T}\mathscr{L}^* \subset \mathscr{T}\mathscr{L}$ to be those trivial links with no nullhomotopic components. Let U be the isotopy class of the standard unknot, a local knot in $L(p, q)$ that bounds an embedded disk. Suppose we are given a value $J_{p,q}(T) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ for every $T \in \mathscr{T}\mathscr{L}^*$. Then there is a unique map $J_{p,q} : \mathscr{L} \to \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ such that* *(i)* $J_{p,q}$ *satisfies the skein relation*

$$a^{-p}J_{p,q}(L_+) - a^p J_{p,q}(L_-) = zJ_{p,q}(L_0).$$

(ii) $J_{p,q}(U) = a^{-p+1}$.

(iii) $J_{p,q}(U \coprod L) = \frac{a^{-p}-a^p}{z} J_{p,q}(L)$.

Remark 3.3.9. The map $J_{p,q}$ is defined to be the map J_M, with $M = L(p,q)$, given by Theorem 2.3.2 except that we define a such that $a^p = v$. While not essential to the proof of the theorem, the change of variable is useful for Corollary 1.1.4. To prove the theorem at hand, by Theorem 2.3.2 we only need show that $J_{p,q}$ is Laurent-polynomial valued. Note that we have not specified what the fixed normalization on the trivial links is. Indeed, such a choice is not canonical.

Proof. Define a function ψ which assigns to every grid diagram D in $L(p,q)$ the triple of numbers

$$\psi(D) = (GN(D), scr(D), u(D)).$$

Ordering these triples lexicographically, we will use ψ as a complexity function.

There is a grid diagram $\hat{D}$ (the diagram $\hat{D}(\mathcal{I})$ of Lemma 3.3.4) associated to any trivial link $K(\mathcal{I})$ such that the following holds. If D_K is any grid diagram with associated link K, and K is in the same homotopy class as $K(\mathcal{I})$, then $\psi(\hat{D}) \leq \psi(D_K)$, equality holding only if $K(\mathcal{I})$ is isotopic to K. For in such a case, $GN(\hat{D}) \leq GN(D_K)$ since $\hat{D} \in \mathscr{D}(\mathcal{I})$. If $GN(\hat{D}) = GN(D_K)$ then $D_K \in \mathscr{D}(\mathcal{I})$, but by definition $\hat{D} \in \mathscr{M}$, implying $scr(\hat{D}) \leq$

$scr(D_K)$. As $u(\hat{D}) = 0 \leq u(D_K)$ by definition, with equality only if K is isotopic to $K(\mathcal{I})$, we see that $\psi(\hat{D}) < \psi(D_K)$ if K is not a trivial link.

Our argument inducts on the complexity function ψ. Let K be a non-trivial link in $L(p, q)$ and D a grid diagram associated to K. There is a sequence of $u(D)$ skein crossing changes that, with non-interleaving commutations and destabilizations, takes D to the trivial link diagram in its homotopy class. Suppose the first skein crossing in this sequence is positive and let $D = D_+$, D_-, and D_0 be diagrams as described in Section 3.2 (differing only at the two columns of the skein crossing) with associated links $K = K_+, K_-$, and K_0. By the skein relation,

$$a^{-p} J_{p,q}(K) - a^p J_{p,q}(K_-) = z J_{p,q}(K_0). \tag{3.3.4}$$

But $GN(D_-) = GN(D)$, and $scr(D_-) = scr(D)$ since the skein crossing change is a commutation of columns. As our choice of crossing change makes $u(D_-) < u(D)$, we have that $\psi(D_-) < \psi(D)$.

Recall the defintion of the grid diagram D_0 in Definition 3.2.5. D_0 has the same grid number as D. We prove below that

Lemma 3.3.10. *$scr(D_0) < scr(D)$.*

Provided Lemma 3.3.10 is true, D_- and D_0 have strictly lower complexity than D, so we may assume that $J_{p,q}(K_-)$ and $J_{p,q}(K_0)$ are Laurent polynomials. The skein relation (3.3.4) then implies $J_{p,q}(K) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ as well. The argument is similar if the first skein crossing change in the sequence is negative, only we use

$$a^{-p} J_{p,q}(K_+) - a^p J_{p,q}(K) = z J_{p,q}(K_0).$$

$\square$

Proof of Lemma 3.3.10. The proof will proceed as follows. Recall our definition of a map H from the columns of a grid diagram in S^3 to a subset of the coordinate plane (see the end of Section 3.2). Also recall that $scr(D)$ is defined to be the number of pairs of interleaving columns of $\widetilde{D}$, the lift of D to S^3, and these pairs need not be adjacent.

Given a skein crossing of D which is resolved to give D_0, the lifts $\widetilde{D}$ and $\widetilde{D_0}$ differ in exactly $2p$ columns, each of these being a lift of a column in the given skein crossing of D. Both $\widetilde{D}$ and $\widetilde{D_0}$ have grid number $p \cdot GN(D)$. Let c be the column of $\widetilde{D}$ between the β curves β_i, β_{i+1}. Then define c' to be the column in $\widetilde{D_0}$ between β_i and β_{i+1}. If c is not one of the $2p$ columns covering the skein crossing, we call c *unaltered* and otherwise c is called *altered*. Note that a column c in $\widetilde{D}$ is unaltered exactly when $H(c) = H(c')$. So a pair of columns $\{c, d\}$ that are both unaltered in $\widetilde{D}$ is interleaving if and only if $\left\{c', d'\right\}$ is interleaving in $\widetilde{D_0}$.

To prove the lemma then, we show that the number of interleaving pairs of columns in $\widetilde{D}$, such that one of the columns is altered, is larger than the number of interleaving pairs in $\widetilde{D_0}$. This is shown by proving inequalities (3.3.5) and (3.3.6) below. Now to the proof.

For one of the two columns in the given skein crossing of D, choose c_1 as one of the p lifts of that column in $\widetilde{D}$. Choose c_2 to be the lift of the other column in the skein crossing so that c_2 is adjacent to c_1. As in the outline of the proof, let c_1' and c_2' be the corresponding columns in $\widetilde{D_0}$.

Since $\widetilde{D}$ and $\widetilde{D_0}$ differ only in a specified way, we can refer to the same curve α_0 in both diagrams. For $H = H_{\alpha_0} : \mathscr{C}(\widetilde{D}) \to \mathbb{R}^2$, let $p_i = H(c_i)$ and for $H = H_{\alpha_0} : \mathscr{C}(\widetilde{D_0}) \to \mathbb{R}^2$, let $p_i' = H(c_i')$. If $q \in \text{Im } H$ and c is some column of $\widetilde{D}$ (or $\widetilde{D_0}$) then define $q(c) = 1$ if $H(c)$

is in an interleaving region of q (recall, the interleaving regions of a point in the image of H are as shown in Figure 3.6). Define $q(c) = 0$ otherwise. We will also abuse notation and sometimes write $q(H(c))$ to mean $q(c)$.

To prove Lemma 3.3.10 we first show that for any unaltered column c,

$$p_1(c) + p_2(c) \geq p_1'(c') + p_2'(c'). \tag{3.3.5}$$

Second we show that if d_1, d_2 is a pair of adjacent altered columns in $\widetilde{D}$, then

$$\sum_{i=1}^{2}(p_1(d_i) + p_2(d_i)) \geq \sum_{i=1}^{2}(p_1'(d_i') + p_2'(d_i')). \tag{3.3.6}$$

Since c_1 and c_2 interleave each other in $\widetilde{D}$ but c_1' and c_2' are non-interleaving in $\widetilde{D_0}$ by design, this would show that $scr(D_0) \leq scr(D) - p$.

To show that (3.3.5) holds, let c be an unaltered column of $\widetilde{D}$ and let $q = H(c) = H(c')$. Also let Z_1, Z_2 be the markings in c_1, c_2 that will swap columns to give c_1', c_2'. We can choose α_0 so that $\theta_2(Z_1)$ and $\theta_2(Z_2)$ are the x-coordinates of $H(c_1)$ and $H(c_2)$ respectively. Without loss of generality, we may also assume that $\theta_2(Z_1) < \theta_2(Z_2)$ and that α_0 is on the boundary of the row containing Z_1. Hence, no point in Im H has a smaller x-coordinate than $p_1 = H(c_1)$. In fact, with such a choice of α_0 the points p_1, p_2 appear as in Figure 3.12, and we get p_1', p_2' by interchanging the x-coordinates of p_1, p_2.

If q is in any region of the plane other than where it appears in Figure 3.12 then it is evident from the figure that $p_1(c) + p_2(c) = p_1'(c') + p_2'(c')$. On the other hand, if q is as in Figure 3.12 then $p_1(c) + p_2(c) = 2$ and $p_1'(c') + p_2'(c') = 0$. This shows that (3.3.5) holds.

We now show that (3.3.6) holds. As observed in Remark 3.2.7 our choice of α_0 does not

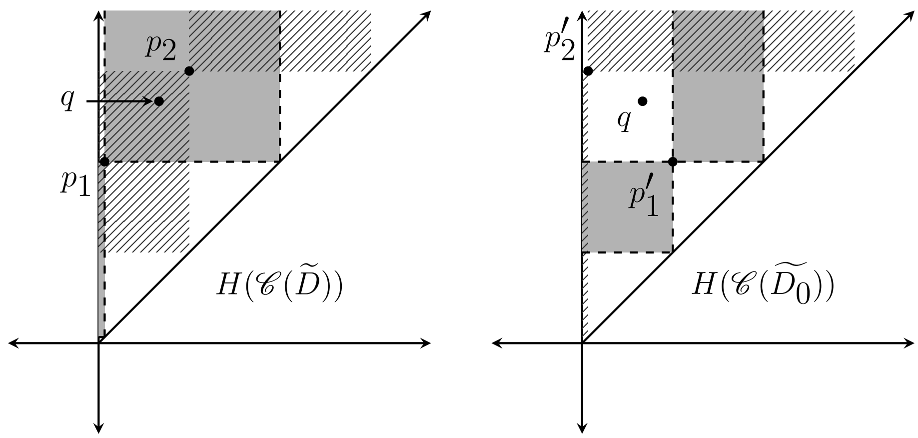


Figure 3.12: The interleaving of unaltered columns in $\widetilde{D}$

change the count in (3.3.6). Given a pair d_1, d_2 of interleaving columns in $\widetilde{D}$, we say that α_0 is *good* for the pair if the x-coordinates of $H_{\alpha_0}(d_1)$ and $H_{\alpha_0}(d_2)$ either both come from heights of $\mathbb{O}$ markings or both come from heights of $\mathbb{X}$ markings. Every pair of interleaving columns has at least 2 good α-curves.

Suppose d_1 and d_2 are a pair of ajacent, altered columns, d_1' and d_2' are the corresponding columns in $\widetilde{D_0}$, $q_i = H(d_i)$ and $q_i' = H(d_i')$. If α_0 is good for d_1, d_2 then the points q_1, q_1', q_2, q_2' are the corners of a rectangle in $\mathbb{R}^2$, with sides parallel to the axes and with q_1 (resp. q_2) being the lower left (resp. upper right) corner. In our proof of (3.3.5), the choice of α_0 was good for c_1, c_2.

Now suppose there is a choice of α_0 that is good for both pairs c_1, c_2 and d_1, d_2 simultaneously. With this choice of α_0 the points p_i, p_i' for $i = 1, 2$ appear as in Figure 3.13 with corresponding interleaving regions. Note that for an arbitrary point a in the second octant, we have that $p_1(a) + p_2(a) \geq p_1'(a) + p_2'(a)$. Let R be a rectangle in this octant (the sides of R parallel to the axes) with corner points in the image of H, labeled counter-clockwise by

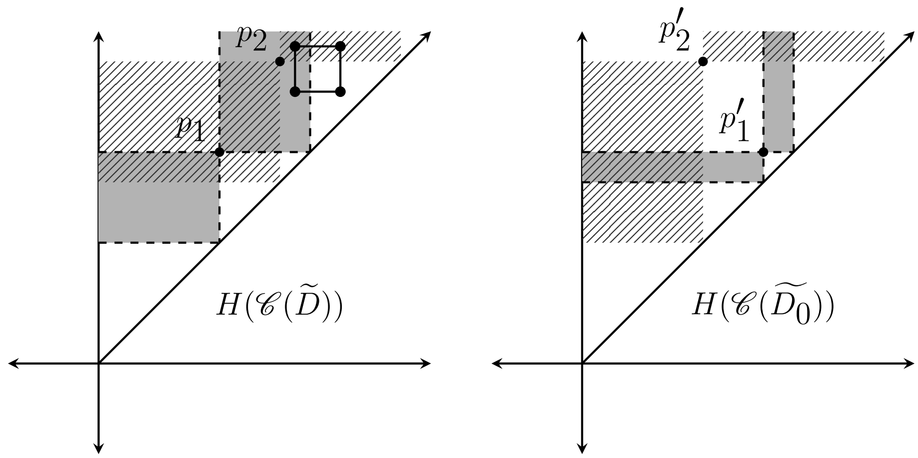


Figure 3.13: The interleaving of altered columns in $\widetilde{D}$ with a common good choice of α_0

r_1, s_1, r_2, s_2 with r_1 in the lower left corner. If

$$\sum_{i=1}^{2} p_1(r_i) + p_2(r_i) \geq \sum_{i=1}^{2} p_1(s_i) + p_2(s_i) \tag{3.3.7}$$

then since $p_1(s_j)+p_2(s_j) \geq p_1'(s_j)+p_2'(s_j)$ we see that (3.3.6) is satisfied. One easily checks that (3.3.7) is satisfied for any rectangle in the second octant. For the example rectangle shown on the left of Figure 3.13 we have $\sum p_1(r_i) + p_2(r_i) = 2$ and $\sum p_1(s_i) + p_2(s_i) = 2$.

We are left only to check the case when there is no choice of α_0 that is good for both c_1, c_2 and d_1, d_2. This can only occur when the set of good α-curves for d_1, d_2 is contained in the set of α-curves that are not good for c_1, c_2. In such a situation, we have two cases. Either d_i is interleaving with both c_1 and c_2 for $i = 1$ and $i = 2$, or d_i is non-interleaving with both c_1 and c_2, for $i = 1$ and $i = 2$. In the first case, since the right hand side of (3.3.6) can be at most 4, and this equals the left hand side, (3.3.6) is satisfied.

Consider the case when d_i is non-interleaving with both c_1 and c_2, for $i = 1, 2$. Then, since d_1 is interleaving with d_2, the heights of all the markings in d_1, d_2 are either between

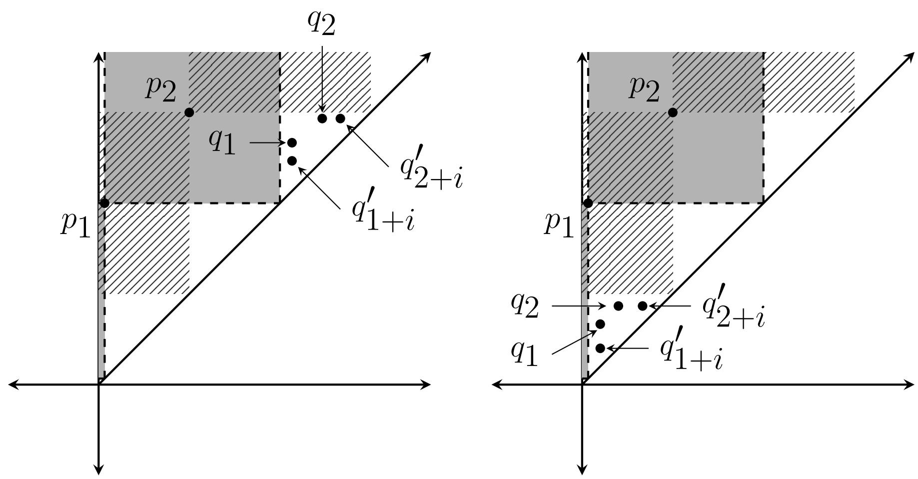


Figure 3.14: The interleaving of altered columns in $\widetilde{D}$ without a common good choice of α_0

the $\mathbb{O}$ markings of c_1, c_2 or between the $\mathbb{X}$ markings of c_1, c_2. Choose α_0 so that the x-coordinate of p_1 is minimal (as we did in Figure 3.12). With this choice we have the following: for $i \in \{1, 2\}$, both x- and y-coordinates of $q_i = H(d_i)$ are either numbers between the y-coordinates of p_1 and p_2 or are between the x-coordinates of p_1 and p_2. The first case is depicted on the left of Figure 3.14 and the second case on the right, and in the figure $i \in \{1, 2\}$, and indices are taken mod 2. In both cases, the left-hand and right-hand sides of (3.3.6) are both zero. □

Chapter 4

Classical invariants of $\mathbb{Q}$-nullhomologous Legendrian and transverse links

4.1 Definitions

We review the definitions given in [7] (cf. [4]) of the Thurston-Bennequin number and rotation number, and for transverse links the self-linking number. Throughout the section all knots and links are assumed to be $\mathbb{Q}$-nullhomologous.

To begin requires the notion of a *rational Seifert surface*. Let K be an oriented knot in a manifold M and let r be the order of the homology class of K in $H_1(M, \mathbb{Z})$. Then, writing $\nu(K)$ for a normal neighborhood of K, for some $s \in \mathbb{Z}$ there is a curve γ of slope (r, s) on $\partial\nu(K)$ that bounds an oriented surface $\Sigma^0 \subset M \setminus \nu(K)$. Let Σ be the union of Σ^0 and the cone in $\nu(K)$ of γ to K. Figure 4.1 depicts a meridional cross-section of $\nu(K)$ and the cone of γ to K.

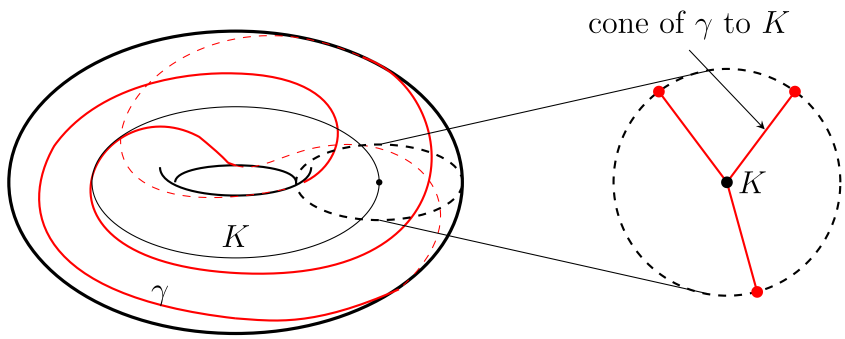


Figure 4.1: Constructing a rational Seifert surface

Note that the interior of Σ is an embedded surface in M and $\partial\Sigma$ is an r-fold cover of K.

Definition 4.1.1. Let K be an oriented knot in M with order r in $H_1(M, \mathbb{Z})$. A *rational Seifert surface for* K is an oriented surface Σ with a map $j : \Sigma \to M$ such that j is an embedding on the interior of Σ, $j(\partial\Sigma) = K$, and $j|_{\partial\Sigma}$ is an r-fold cover of K.

The previous discussion shows that every oriented ($\mathbb{Q}$-nullhomologous) knot has a rational Seifert surface. If we have a link, then we consider what Baker and Etnyre call a *uniform* rational Seifert surface of the link. Such a surface meets each boundary component of the link complement in an (r, s_i) sloped curve, where r is the order of the link, rather than in a curve with slope coming from the order of each link component. Often we will abuse terminology and call a rational Seifert surface simply a Seifert surface. We can now define the classical invariants for Legendrian and transverse knots.

Definition 4.1.2. Let K be an oriented ($\mathbb{Q}$-nullhomologous) link with order r as above and let $j : \Sigma \to M$ be a Seifert surface for K.

1. Given another oriented knot K', define

$$lk_{\mathbb{Q}}(K, K') = \frac{1}{r}\Sigma \cdot K'.$$

2. If in addition K is a Legendrian knot in (M, ξ) and K' is a pushoff of K in the direction of the contact framing given by $\xi|_K \cap \nu(K)$, then define the *(rational) Thurston-Bennequin number* of K by

$$\mathrm{tb}_{\mathbb{Q}}(K) = lk_{\mathbb{Q}}(K, K').$$

3. For $x \in K$ let dK_x denote the tangent vector to K at x. For K Legendrian as above, dK is a section of the bundle ξ_K. Take a trivialization $j^*\xi_\Sigma \cong \mathbb{R}^2 \times \Sigma$. Define the *(rational) rotation number* of K to be the winding number of j^*dK in $\mathbb{R}^2$ under this trivialization, divided by r:

$$\mathrm{rot}_{\mathbb{Q}}(K) = \frac{1}{r}\mathrm{winding}(j^*dK, \mathbb{R}^2).$$

4. If K is a transverse knot in (M, ξ), let v be a non-zero section of $j^*\xi$. Normalize v so that $v|_{\partial\Sigma}$ defines a curve K' in $\partial\nu(K)$. Define the *(rational) self-linking* of K to be

$$\mathrm{sl}_{\mathbb{Q}}(K) = lk_{\mathbb{Q}}(K, K').$$

Remark 4.1.3. In general, the rotation number of a Legendrian knot and the self-linking number of a transverse knot depend on the relative homology class of Σ. Yet once this class is fixed, they do not depend on other choices made – the trivialization of $j^*\xi_\Sigma$ in the case of $\mathrm{rot}_\mathbb{Q}$ and the section v in the case of $\mathrm{sl}_\mathbb{Q}$ (see [7]). The dependence on the homology of Σ does not apply to lens spaces, however, since $H_2(L(p,q)) = 0$.

In the case of null-homologous links (e.g. Legendrian links in (S^3, ξ_{st})) these numbers are known as the "classical invariants" of Legendrian and transverse links. In this case the classical invariants are always integers. However, in the case of rationally null-homologous links, these numbers are generally rational.

We recall that for any Legendrian knot K in a contact manifold (M, ξ), there is a related transverse knot $T_+(K)$ called the *positive transverse push-off of* K (see [16]). Using this construction on each component of a link, we can get the positive transverse push-off of a Legendrian link. To construct $T_+(K)$, find a tubular neighborhood of K that is contactomorphic, for sufficiently small ε, to $C_\varepsilon = \left\{[(x,y,z)] \mid y^2 + z^2 < \varepsilon^2, x = x+1\right\}$: the quotient of an ε-neighborhood of the x-axis in $(\mathbb{R}^3, \ker(dz - ydx))$ by the action $x \mapsto x+1$. Here the orientation of the image of K under the contactomorphism is in the direction of increasing x-values. The positive transverse push-off $T_+(K)$ is defined to be the image of $\{(x, \varepsilon/2, 0)\}$ in the neighborhood of K. We can also define the negative transverse push-off $T_-(K)$ to be the image of $\{(x, -\varepsilon/2, 0)\}$. Baker and Etnyre show the following [7]:

Lemma 4.1.4. *Given a Legendrian link K, let $T_\pm(K)$ be defined as above. Then*

$$\mathrm{sl}_\mathbb{Q}(T_\pm(K)) = \mathrm{tb}_\mathbb{Q}(K) \mp \mathrm{rot}_\mathbb{Q}(K).$$

Remark 4.1.5. We will define $T(K)$ to be the transverse push-off of K that satisfies

$\mathrm{sl}_{\mathbb{Q}}(T(K)) = \mathrm{tb}_{\mathbb{Q}}(K) + \left|\mathrm{rot}_{\mathbb{Q}}(K)\right|$. If $\mathrm{rot}_{\mathbb{Q}}(K) = 0$ then $T(K)$ is taken to be the positive transverse push-off.

4.2 Formulae for $\mathbf{tb}_{\mathbb{Q}}(K), \mathbf{rot}_{\mathbb{Q}}(K)$, and $\mathbf{sl}_{\mathbb{Q}}(K)$ from a grid projection in $L(p, q)$

A method for computing the (rational) Thurston-Bennequin number of a Legendrian link in $L(p, q)$ via the Maslov index of a corresponding grid diagram was given in [4]. The complexity of such computations increases quickly as the grid number of the diagram increases.

We recall that in (S^3, ξ_{st}) there are formulas for the classical invariants that can be computed from a front projection (see [16]). The formulas we give here for computing the (rational) Thurston-Bennequin, rotation, and self-linking numbers in $L(p, q)$ are in the same spirit. In fact, they are derived from the former.

Definition 4.2.1. Given a grid projection P for an oriented link in $L(p, q)$ denote the writhe of the projection by $w(P)$ and the number of cusps of the projection by $c(P)$. Also, let $\mu(P)$ denote the algebraic intersection number of α_0 with P and $\lambda(P)$ the algebraic intersection number of P with β_0.

Let K be the link associated to a given grid projection P on a grid diagram. We recall that for a given row (resp. column) in that diagram, P contains one of the two choices of horizontal (resp. vertical) arcs. A projection P' which is identical to P except that it contains the other arc in this row (resp. column) corresponds to a link K' that differs from K by a Legendrian isotopy across a meridian of V^α (resp. V^β). We call this isotopy a *disk slide*.

Recall that in (S^3, ξ_{st}) if P is the front projection of a Legendrian link K then $\mathrm{tb}(K) = w(P) - \frac{1}{2}c(P)$. Moreover, we note that $(S^3, \xi_{st}) = (L(1,0), \xi_{UT})$, and if a grid projection P in $L(1,0)$ is contained in a planar subset of T then there is a slight perturbation of P giving a front projection (see [4]).

Proposition 4.2.2. *Let K be an oriented Legendrian link in $(S^3, \xi_{st}) = (L(1,0), \xi_{UT})$ and let P be a grid projection of K. Let $l = \lambda(P)$ and $m = \mu(P)$. Then*

$$\mathrm{tb}(K) = w(P) - \frac{1}{2}c(P) - ml.$$

Proof. If P is a projection on a planar subset of Σ then we can consider it as a front projection. In this case $l = m = 0$ and the proposition follows immediately.

Suppose P' is any grid projection of K for which the proposition holds. We prove that if P differs from P' by a disk slide (either a disk with boundary parallel to α_0 or along one with boundary parallel to β_0) then the proposition holds for P as well. Since every grid projection of K is related to a planar grid projection by a sequence of disk slides, this will prove the proposition. Inherent in this proof is the fact that disk slides correspond to Legendrian isotopy [4].

We denote the algebraic intersection of two oriented curves γ, δ on Σ that meet transversely by $\langle \gamma, \delta \rangle$. Note that for any circle c on Σ, parallel to α_0 and transverse to P', we have $m' := \left\langle \alpha_0, P' \right\rangle = \left\langle c, P' \right\rangle$, since c separates $\Sigma \setminus \alpha_0$.

Suppose P differs from P' by a disk slide. Suppose further that the disk slide is along a disk Δ with boundary parallel to α_0. There is some orientation of Δ such that $\partial\Delta =$

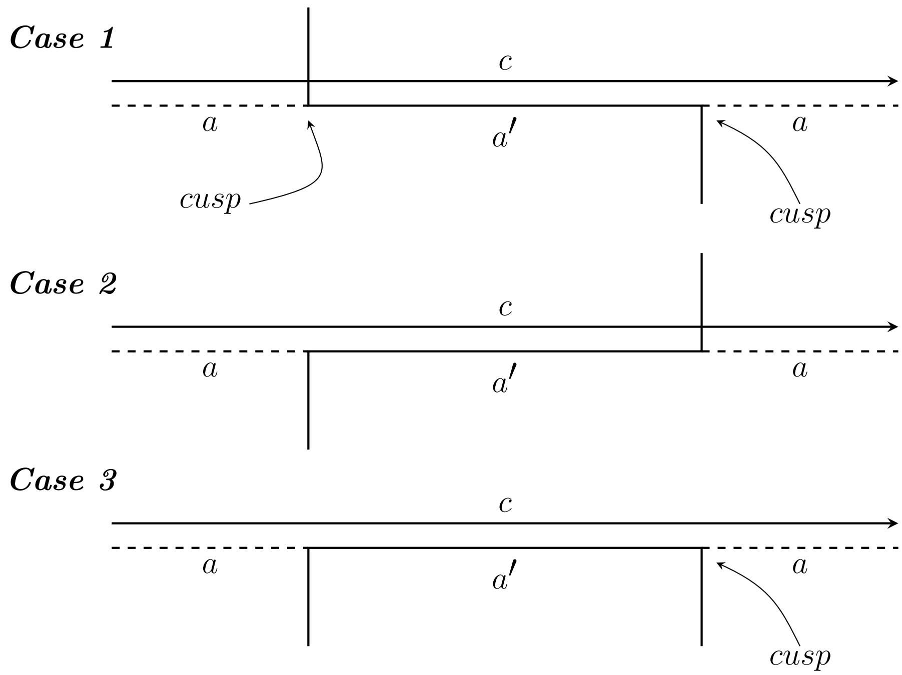


Figure 4.2: How tb changes with a disk slide

$(-a') \cup a$, where a' is an arc of P' and a is the arc that replaces it in the projection P (see Figure 4.2). Let c be a circle that is parallel and coherently oriented to α_0 and a small distance away from $(-a') \cup a$. Then c intersects P' and P tranversely. The writhe of P differs from that of P' only by the difference along double points of a' and double points of a. Thus if a' is an arc with two cusps as in case 1 of Figure 4.2, then

$$\pm(w(P) - w(P') + 1) = \left\langle c, P' \right\rangle = m',$$

where the sign on the left depends on the orientation of a with respect to c. If a' has no cusps as in case 2 of Figure 4.2 then

$$\pm(w(P) - w(P') - 1) = \left\langle c, P' \right\rangle = m'.$$

Finally, if a' is an arc with one cusp as in case 3 then

$$\pm(w(P) - w(P')) = \left\langle c, P' \right\rangle = m',$$

with the sign depending again on the orientation of a. Also, if a' is as in case 1 then $c(P) = c(P') - 2$, if we are in case 2 then $c(P) = c(P') + 2$, and if we are in case 3, then $c(P) = c(P')$.

Since $l = l' \pm 1$ (where $l' = \lambda(P')$) and $m = m'$ we can now check that

$$w(P) - \frac{1}{2}c(P) - ml = w(P') - \frac{1}{2}c(P') - m'l' = \mathrm{tb}(K)$$

for cases 1–3.

The argument is analogous if the disk slide is along a meridional disk. □

Corollary 4.2.3. *Let $K \subset (L(p,q), \xi_{UT})$ be an oriented Legendrian link with P a grid projection for K. Let $\lambda = \lambda(P)$ and $\mu = \mu(P)$. Then*

$$\mathrm{tb}_{\mathbb{Q}}(K) = w(P) - \frac{1}{2}c(P) - \frac{\mu\lambda}{p}.$$

Proof. Lift the projection P to a grid projection $\widetilde{P}$ in S^3 by lifting the Heegaard torus of $L(p,q)$ to Σ. Then clearly $pw(P) = w(\widetilde{P})$ and $pc(P) = c(\widetilde{P})$. Moreover, $\mu(P) = \mu(\widetilde{P})$ and $\lambda(P) = \lambda(\widetilde{P})$ by definition.

It was shown in [4] that $\mathrm{tb}_{\mathbb{Q}}(K) = \dfrac{\mathrm{tb}(\widetilde{K})}{p}$, and this completes the proof. □

Corollary 4.2.4. *Let $K \subset (L(p,q), \xi_{UT})$ be a Legendrian knot with its contact framing. If* *there is an integral surgery on K that gives a homology sphere S then $\mu(K)^2 \equiv \pm q' \mod p$, where $qq' \equiv 1 \mod p$. In particular, if there is a knot in $L(p,q)$ on which integer surgery yields S^3 then $\pm q$ is a quadratic residue mod p.*

Remark 4.2.5. We note that the last statement of the corollary is a well-known result of Fintushel and Stern [18].

Proof. In [4] it is shown that $p \cdot \mathrm{tb}_{\mathbb{Q}}(K) \equiv \pm 1 \mod p$. With Corollary 4.2.3 this implies that $\mu\lambda \equiv \pm 1 \mod p$. Since the projection of K is a closed curve, $\lambda \equiv \mu q \mod p$, so that $\mu^2 q \equiv \pm 1 \mod p$. □

The following proposition shows how to compute $\mathrm{rot}_{\mathbb{Q}}(K)$ given a projection P corresponding to a grid diagram for an oriented Legendrian link K in $(L(p,q), \xi_{UT})$. Its proof is similar to the proof of Proposition 4.2.2 and Corollary 4.2.3.

Proposition 4.2.6. *Let μ and λ be as in Corollary 4.2.3. Note that each cusp in a grid projection is comprised of a horizontal and a vertical arc. Define $c_u(P)$ to be the number of cusps whose horizontal arc is oriented against α_0 and $c_d(P)$ to be the number of cusps with horizontal arc oriented in the direction of α_0. Then*

$$\mathrm{rot}_{\mathbb{Q}}(K) = \frac{1}{2}\left(c_d(P) - c_u(P)\right) - \frac{(\lambda - \mu)}{p}.$$

Corollary 4.2.7. *Let $T_+(K)$ (resp. $T_-(K)$) be the positive (resp. negative) transverse* *pushoff of the Legendrian link K in $(L(p,q),\xi_{UT})$. Then*

$$\mathrm{sl}_{\mathbb{Q}}(T_+(K)) = w(P) - c_d(P) - \frac{\mu\lambda + (\mu-\lambda)}{p} \quad \textit{and}$$
$$\mathrm{sl}_{\mathbb{Q}}(T_-(K)) = w(P) - c_u(P) - \frac{\mu\lambda + (\lambda-\mu)}{p},$$

where P is the projection of K as above.

Proof. This results from Corollary 4.2.3, Proposition 4.2.6, and Lemma 4.1.4. □

4.3 Bennequin-type bounds in $L(p,q)$

In this section we prove the lens space analogue of the version of the FWM inequality observed by Fuchs and Tabachnikov [21]. To achieve this, we prove the analogue of a theorem of Lenny Ng (Theorem 1 in [41]) for oriented links in the contact lens space $(L(p,q),\xi_{UT})$. While the proof follows the ideas of Ng in [41], we note that it is independent of the work of Rutherford [54] that plays a key role in the proof given in [41]. It is an example of the power of the point of view of grid diagrams in understanding Legendrian links.

Theorem 1.1.3. *Let i be a $\mathbb{Q}$-valued invariant of oriented links in $L(p,q)$ such that*

$$i(L_+) + 1 \leq \max(i(L_-) - 1, i(L_0))$$

and

$$i(L_-) - 1 \leq \max(i(L_+) + 1, i(L_0))$$

where L_+, L_-, *and* L_0 *are oriented links that differ as in the skein relation. If* $\mathrm{sl}_{\mathbb{Q}}(T(\tau)) \leq -i(\tau)$ *for every trivial link* τ *in* $L(p,q)$, *then*

$$\overline{\mathrm{sl}}_{\mathbb{Q}}(L) \leq -i(L)$$

for every link L *in* $L(p,q)$. *Here* $\overline{\mathrm{sl}}_{\mathbb{Q}}$ *is the maximum self-linking number among transverse links in* $(L(p,q), \xi_{UT})$ *that are isotopic to* L.

Proof. In the proof we will abuse notation, often using P to refer to both a grid projection and the Legendrian link it specifies. Our strategy of proof is as follows. Assume our link is a positive transverse push-off $T_+(P)$. Define $\widetilde{i}(P) = i(P) + w(P)$, where $w(P)$ is the writhe of P. Then by the formula of Corollary 4.2.7, the inequality $\mathrm{sl}_{\mathbb{Q}}(T_+(P)) \leq -i(P)$ is equivalent to

$$-\mathrm{sl}_{\mathbb{Q}}(T_+(P)) - i(P) = c_d(P) + \frac{\mu\lambda + (\mu - \lambda)}{p} - \widetilde{i}(P) \geq 0. \qquad (4.3.1)$$

By Proposition 3.3.6, there is a minimal length sequence

$$P \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2}, \ldots, \xrightarrow{\alpha_n} P_n = P_\mathcal{T}$$

taking P to a grid projection $P_\mathcal{T}$ associated to a trivial link diagram, where each α_i is either a skein crossing change, Legendrian isotopy, or destabilization. We will show that if some α_i increases the left side of inequality (4.3.1) then it is a skein crossing change, and that in this case the resolution $P \rightsquigarrow P_0$ does not increase the left side of (4.3.1). If P, P_0 have underlying diagrams D, D_0, then $\psi(D) > \psi(D_0)$ (recall the complexity ψ from the proof of Theorem 1.1.1). As ψ is minimized on diagrams for trivial links, we may assume inductively that inequality (4.3.1) holds for P_0, since $\mathrm{sl}_{\mathbb{Q}}(T_+(\tau)) \leq \mathrm{sl}_{\mathbb{Q}}(T(\tau)) \leq -i(\tau)$ for any trivial link τ.

If instead our transverse link is $T_-(P)$ for some P then the same argument can be carried through, replacing inequality (4.3.1) with the corresponding formula. Since every transverse link is some transverse push-off $T_\pm(P)$ of a Legendrian link, this argument is sufficient to prove the theorem.

The expression on the left side of (4.3.1) is a Legendrian invariant, so if α_i is a Legendrian isotopy the expression is unchanged. In the case of a destabilization we make the following claim.

Claim 1: If α_i is a destabilization, then $-\mathrm{sl}_{\mathbb{Q}}(T_+(P_{i-1})) - i(P_{i-1}) \geq -\mathrm{sl}_{\mathbb{Q}}(T_+(P_i)) - i(P_i)$, so α_i does not increase the left side of (4.3.1).

Proof of Claim 1. First, μ and λ are each an algebraic intersection of P with some α-curve or β-curve respectively. Therefore they are unchanged by a destabilization.

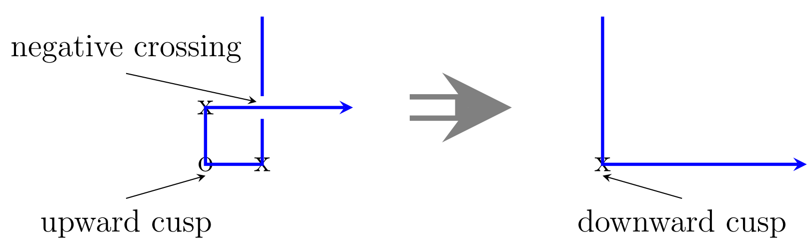


Figure 4.3: On an $X : NE$ destabilization $c_d(P)$ increases.

It is possible that for some destabilization that is not Legendrian isotopy, $c_d(P)$ changes. If $c_d(P)$ decreases, then $-\mathrm{sl}_{\mathbb{Q}}(T_+(P))$ decreases. The destabilization was an isotopy, so $i(P)$ remains unchanged and we see that the left side of (4.3.1) does not increase. If $c_d(P)$ increases the destabilization is of type X:NE or O:SW. Figure 4.3 depicts when the destabilization is type X:NE. In such a case, $c_d(P)$ increases by 1, but $w(P)$ does also. Thus $\mathrm{sl}_{\mathbb{Q}}(T_+(P))$ does not change, and so $-\mathrm{sl}_{\mathbb{Q}}(T_+(P)) - i(P)$ also does not change. $\square$

We remark that Claim 1 also holds for the case of a negative push-off $T_-(P)$. In the proof one only need consider $c_u(P)$ instead of $c_d(P)$.

Thus we only need concern ourselves with skein crossing changes. Suppose α_1 is a skein crossing change. If the pair of columns involved in α_1 are a positive (resp. negative) skein crossing in P, then use disk slides if necessary so that, at the skein crossing, P appears as in Figure 4.4 (resp. as in Figure 4.5). Since disk slides are Legendrian isotopy, this does not alter the left side of inequality (4.3.1).

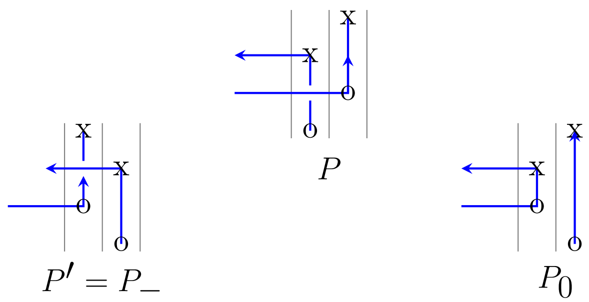


Figure 4.4: A skein crossing change and resolution at a positive skein crossing.

If α_1 does not increase the left hand side of (4.3.1), then we are finished by induction. Otherwise the left side of (4.3.1) does increase in passing from the projection P to P_- (resp. P_+). Lemma 4.3.1, which we prove below, shows that in this case $-\mathrm{sl}_{\mathbb{Q}}(T_+(P)) - i(P) \geq -\mathrm{sl}_{\mathbb{Q}}(T_+(P_0)) - i(P_0)$. As we remarked before, there is a complexity ψ which is minimized by diagrams for trivial links, such that if D_0 is the grid diagram for P_0 and D the diagram for P, then $\psi(D_0) < \psi(D)$. Thus we may assume that P_0 satisfies (4.3.1), so by Lemma 4.3.1, P does also. $\square$

Lemma 4.3.1. *Let P be a grid projection containing a skein crossing as in Figure 4.4 or as in Figure 4.5, and let P' be the grid projection obtained by a skein crossing change* *at that crossing. Further, suppose* $-\mathrm{sl}_{\mathbb{Q}}(T_+(P)) - i(P) < -\mathrm{sl}_{\mathbb{Q}}(T_+(P')) - i(P')$. *Then* $-\mathrm{sl}_{\mathbb{Q}}(T_+(P)) - i(P) \geq -\mathrm{sl}_{\mathbb{Q}}(T_+(P_0)) - i(P_0)$.

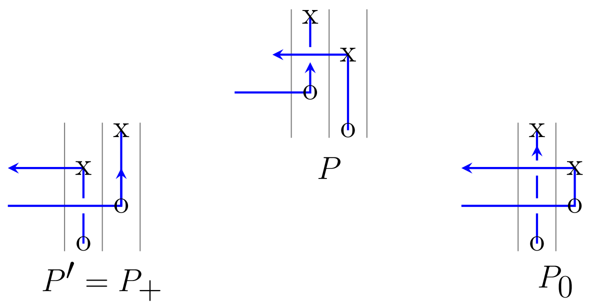


Figure 4.5: A skein crossing change and resolution at a negative skein crossing

Proof of Lemma 4.3.1. Firstly, λ is not changed in passing from P to either P' or P_0. The columns depicted in Figure 4.4 (resp. Figure 4.5) are adjacent, so there is a β-curve, say β_i, between the two columns. The intersection of the projection with some β-curve determines λ, but the value of λ is independent of the choice of β-curve for this intersection. Choose any β-curve other than β_i (at least one exists, because the existence of a skein crossing implies at least two columns in the grid diagram). Since the horizontal arcs of the projections P, P', and P_0 are identical outside of the two columns shown, $\lambda(P) = \lambda(P') = \lambda(P_0)$.

Also, consider the pair of vertical arcs depicted in each projection P, P', and P_0. The sum of the lengths of these two arcs is the same in all three projections. As P, P' and P_0 are identical elsewhere, $\mu(P) = \mu(P') = \mu(P_0)$.

Note that the number and nature of cusps in P are the same as those in P' and P_0, so $c_d(P) = c_d(P') = c_d(P_0)$. Therefore, the equality in (4.3.1) and the supposition that $-\mathrm{sl}_{\mathbb{Q}}(T_+(P)) - i(P) < -\mathrm{sl}_{\mathbb{Q}}(T_+(P')) - i(P')$ imply that $\widetilde{i}(P) > \widetilde{i}(P')$.

However, by our assumption on the invariant i, we see that

$$\widetilde{i}(L_+) \leq \max\left(\widetilde{i}(L_-), \widetilde{i}(L_0)\right)$$

and

$$\widetilde{i}(L_-) \leq \max\left(\widetilde{i}(L_+), \widetilde{i}(L_0)\right).$$

Since $P' = P_-$ or $P' = P_+$ (depending on the sign of the skein crossing in question), $\widetilde{i}(P) > \widetilde{i}(P')$ implies that $\widetilde{i}(P) \leq \widetilde{i}(P_0)$. But that implies that $-\mathrm{sl}_{\mathbb{Q}}(T_+(P)) - i(P) \geq -\mathrm{sl}_{\mathbb{Q}}(T_+(P_0)) - i(P_0)$, finishing the proof. □

Theorem 1.1.3 has the following applications.

Corollary 1.1.4. *Let $J_{p,q}$ denote the HOMFLY polynomial invariant in $L(p,q)$, normalized so that if τ is a trivial link with no nullhomotopic components, or is the unknot, then $J_{p,q}(\tau) = a^{p\cdot \mathrm{sl}_{\mathbb{Q}}(T(\tau))+1}$. Given an oriented link L in $L(p,q)$, set $e(L)$ to be the lowest degree in a of $J_{p,q}(L)$. If L_t is a transverse representative of L in $(L(p,q), \xi_{UT})$, then*

$$\mathrm{sl}_{\mathbb{Q}}(L_t) \leq \frac{e(L)-1}{p}.$$

Proof. The defining skein relation of $J_{p,q}$ says that

$$e(L_+) \geq \min\left(e(L_-) + 2p, e(L_0) + p\right),$$

implying

$$-e(L_+) + 1 \leq \max\left(-e(L_-) + 1 - 2p, -e(L_0) + 1 - p\right).$$

Dividing by p and then adding 1 we have

$$\frac{-e(L_+)+1}{p} + 1 \leq \max\left(\frac{-e(L_-)+1}{p} - 1, \frac{-e(L_0)+1}{p}\right).$$

A similar computation shows

$$\frac{-e(L_-)+1}{p} - 1 \leq \max\left(\frac{-e(L_+)+1}{p} + 1, \frac{-e(L_0)+1}{p}\right).$$

Letting $i(L) = \frac{-e(L)+1}{p}$, we see that i satisfies the first hypothesis of Theorem 1.1.3. Moreover, by our choice of normalization $e(\tau) = p \cdot \mathrm{sl}_{\mathbb{Q}}(T(\tau)) + 1$ for any trivial link τ and so $-i(\tau) = \mathrm{sl}_{\mathbb{Q}}(T(\tau))$. Since all conditions of Theorem 1.1.3 are met, we are done. □

Corollary 1.1.5. *If τ is a trivial link in $L(p,q)$, then $T(\tau)$ has maximal self-linking number among all transverse representatives of τ. If τ is a trivial knot, then the Legendrian knot associated to its grid number one diagram has maximal Thurston-Bennequin number.*

Proof. Let τ_t be some transverse representative of τ. Corollary 1.1.4 says that $\mathrm{sl}_{\mathbb{Q}}(\tau_t) \leq \frac{e(\tau)-1}{p} = \mathrm{sl}_{\mathbb{Q}}(T(\tau))$.

Now consider a Legendrian knot K that is associated to a grid number one diagram and let K' be another Legendrian knot with the same knot type as K. Suppose that $\mathrm{tb}_{\mathbb{Q}}(K) < \mathrm{tb}_{\mathbb{Q}}(K')$. In [7] it is shown that K and K' are Legendrian isotopic after each has been positively and negatively stabilized some number of times, and therefore $\mathrm{tb}_{\mathbb{Q}}(K) - \mathrm{tb}_{\mathbb{Q}}(K')$ is an integer. So we have $\mathrm{tb}_{\mathbb{Q}}(K) + 1 \leq \mathrm{tb}_{\mathbb{Q}}(K')$.

By Corollary 4.2.6, if P is a grid projection for a grid diagram of K, then

$$\mathrm{rot}_{\mathbb{Q}}(K) = \frac{1}{2}(c_d(P) - c_u(P)) - \frac{\lambda - \mu}{p}.$$

Since we are in the grid number one case, P has one vertical and one horizontal arc. Also we can choose P so that $0 < \mu(P) < p$ and $0 < \lambda(P) < p$. This choice of P has no cusps at all, implying that $\left|\mathrm{rot}_{\mathbb{Q}}(K)\right| < 1$.

By Corollary 4.1.4 the transverse pushoff $T(K)$ of K has self-linking $\mathrm{sl}_{\mathbb{Q}}(T(K)) = \mathrm{tb}_{\mathbb{Q}}(K) + \left|\mathrm{rot}_{\mathbb{Q}}(K)\right|$. So since $\left|\mathrm{rot}_{\mathbb{Q}}(K)\right| < 1$, it must be that $\mathrm{sl}_{\mathbb{Q}}(T(K)) < \mathrm{tb}_{\mathbb{Q}}(K')$. This contradicts the fact that $\mathrm{sl}_{\mathbb{Q}}(T(K))$ is maximal, since one of the positive or negative transverse pushoffs of K' has self-linking number at least as large as $\mathrm{tb}_{\mathbb{Q}}(K')$. □

4.4 Examples and Computations

In this section we compute the polynomial $J_{p,q}(L)$ for two examples in $L(5,1)$. We normalize on the trivial links as in Corollary 1.1.4. The first example we compute is a nullhomologous knot B which is depicted via a grid diagram at the top of Figure 4.6. The second example L, also in $L(5,1)$ and shown in Figure 4.7, is homotopically nontrivial, having $\mu(L) = 2$ mod 5. The second example gives rise to a family of Legendrian links $\{L_n\}_{n=1}^{\infty}$. For every n, $T(L_n) = T_+(L_n)$, and we show that the family $T(L_n)$ has arbitrarily negative self-linking numbers that are maximal in their link type.

Example 1. In Figure 4.6 we show the skein tree of B. Note that the relevant skein crossing in B is positive. The skein crossing change (the left branch in the tree) takes B to the unknot K_0. Resolution of the skein crossing (the right branch) gives a link B_0.

To see that B_0 is isotopic to a trivial link, note that its index set is $(0,1,0,0,1)$ and that if we label the $\mathbb{O}$ markings O_1, O_2 so that $\theta_1(O_1) < \theta_1(O_2)$, then $\mu(O_1) = 1$ and $\mu(O_2) = 4$. Since $1 \cdot 1 < 4 \cdot 1(\bmod 5)$, we see that the grid diagram shown for B_0 is the exactly the trivial link diagram $D(0,1,0,0,1)$.

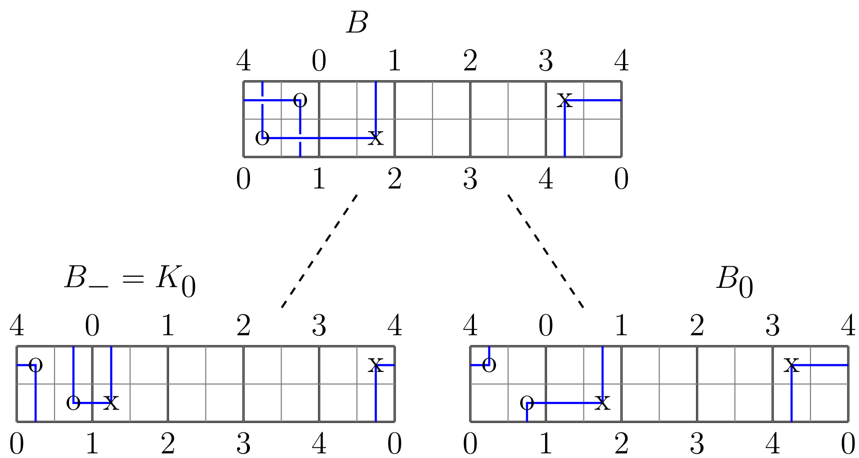


Figure 4.6: Example: A link B in $L(5,1)$ and its skein tree.

Recall the formulas for the classical invariants, in particular the result of Proposition 4.2.7. Letting B_0 also denote the corresponding Legendrian link in $(L(5,1), \xi_{UT})$, note that $\mathrm{tb}_{\mathbb{Q}}(B_0) = -\frac{4}{5}$ and $\mathrm{rot}_{\mathbb{Q}}(B_0) = 0$. So $T(B_0) = T_+(B_0)$ and $\mathrm{sl}_{\mathbb{Q}}(T(B_0)) = -\frac{4}{5}$. Since B_0 is a trivial link, we have that $J_{5,1}(B_0) = a^{-4+1} = a^{-3}$. And thus

$$\begin{aligned} J_{5,1}(B) &= a^{10} J_{5,1}(K_0) + a^5 z J_{5,1}(B_0) \\ &= a^{10}\left(a^{-4}\right) + a^5 z \left(a^{-3}\right) \\ &= a^6 + a^2 z. \end{aligned}$$

Example 2. In the second example we have a negative skein crossing associated to the first and second columns of the grid diagram for L. Via grid moves that correspond to isotopy (a non-interleaving row commutation, destabilization, a non-interleaving column commutation, then another destabilization) the diagram shown for L_+ in Figure 4.7 is taken to a grid number one diagram of the trivial knot K_2. So L_+ is isotopic to the trivial knot K_2.

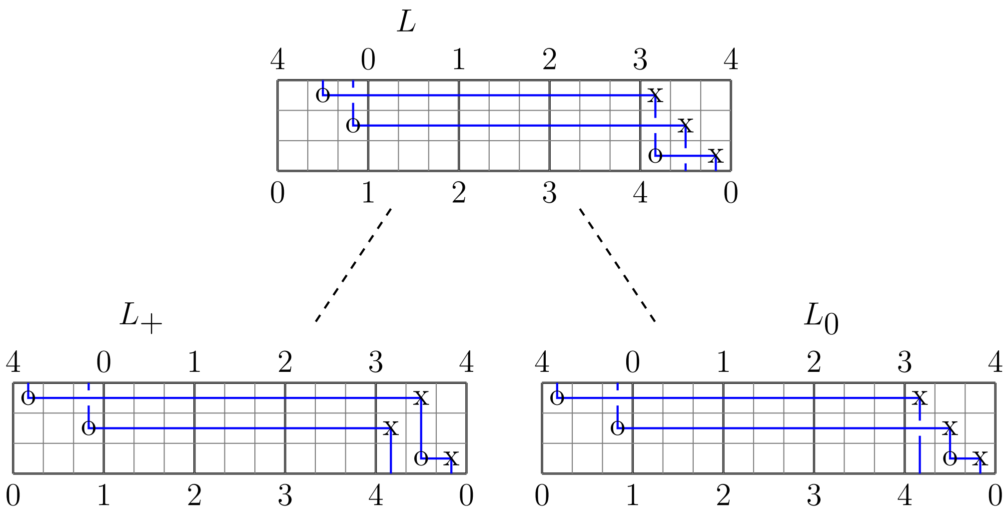


Figure 4.7: Example: A link L in $L(5,1)$ and its skein tree.

The diagram shown in the figure of L_0 (after a destabilization) has two grid number one components: both of type D_{K_1}. The $\mathbb{O}$ markings are adjacent and on a negative slope, so we have the diagram $D(0,2,0,0,0)$ and L_0 is trivial.

Since $\mathrm{sl}_{\mathbb{Q}}(T(K_2)) = \frac{1}{5}$ and $\mathrm{sl}_{\mathbb{Q}}(T(L_0)) = -\frac{4}{5}$, we compute

$$\begin{aligned} J_{5,1}(L) &= a^{-10} J_{5,1}(K_2) - a^{-5} z J_{5,1}(L_0) \\ &= a^{-10}(a^2) - a^{-5} z (a^{-3}) \\ &= a^{-8}(1 - z). \end{aligned}$$

Consider the family of links $\{L_n\}_{n \geq 0}$ where L_n is the link associated to the grid diagram in Figure 4.8 with grid number $n+2$. Then L_1 is the knot L in Example 2 above and L_0 is isotopic to the trivial 2-component link in the skein tree of L. The link L_n is a knot if n is odd and a 2-component link if n is even.

The first two columns of this grid diagram make a negative skein crossing. It is not difficult to see that for $n \geq 2$, commutation of these columns of L_n gives a link isotopic to L_{n-2} and the resolution of the same columns gives a link isotopic to L_{n-1}. Therefore

$$J_{5,1}(L_n) = a^{-10} J_{5,1}(L_{n-2}) - a^{-5} z J_{5,1}(L_{n-1}).$$

We know that $J_{5,1}(L_0) = a^{-3}$ and $J_{5,1}(L_1) = a^{-8}(1-z)$ by our computations above. Define f_n recursively: let

$$f_0 = 1, \qquad f_1 = 1 - z,$$

and define $f_n = f_{n-2} - z f_{n-1}$ for $n \geq 2$. Then the skein relation implies that $J_{5,1}(L_n) = a^{-5n-3} f_n$. Note that the recursive definition of f_n implies that it is not zero for any n. Using Corollary 1.1.4 we show that $T(L_n)$ has maximal self-linking number for each n. We will see below that $\mathrm{rot}_{\mathbb{Q}}(L_n) = -n$, and so $T(L_n) = T_+(L_n)$ for all n.

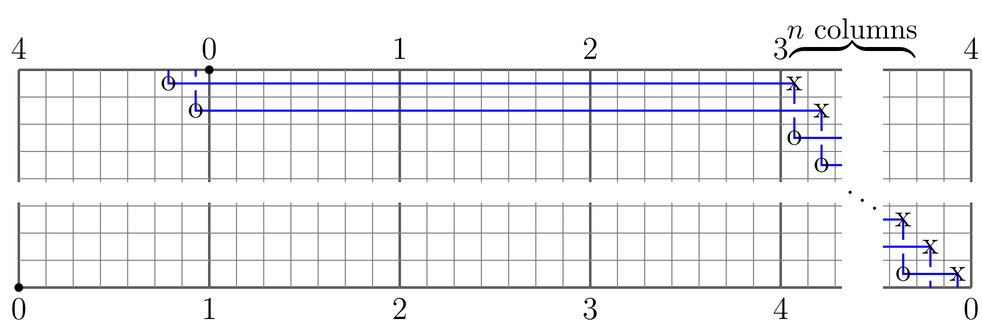


Figure 4.8: Grid diagram associated to the link L_n.

Let $T(L_n) = T_+(L_n)$ denote positive transverse pushoff of L_n. By Corollary 1.1.4

$$\mathrm{sl}_{\mathbb{Q}}(T(L_n)) \leq \frac{e(L_n) - 1}{5} = -n - \frac{4}{5}.$$

If we choose P_n to be a grid projection of L_n, then

$$\mathrm{sl}_{\mathbb{Q}}(T(L_n)) = w(P_n) - c_d(P_n) - \frac{\mu\lambda + (\mu - \lambda)}{5}.$$

Let P_n be the grid projection depicted in Figure 4.8. Then $w(P_n) = -n - 2$, $c_d(P_n) = 0$, and $c_u(P_n) = 2(n + 2)$. Moreover, $\mu(P_n) = 2$ and $\lambda(P_n) = -8$. Therefore, $\mathrm{tb}_{\mathbb{Q}}(L_n) = -n - 2 - (n + 2) + \frac{16}{5} = -2n - \frac{4}{5}$ and $\mathrm{rot}_{\mathbb{Q}}(L_n) = -n - 2 + \frac{10}{5} = -n$. As a consequence,

$$\mathrm{sl}_{\mathbb{Q}}(T(L_n)) = -n - \frac{4}{5},$$

showing that each $T(L_n)$ maximizes its self-linking number.

Chapter 5

Fibered knots and the universally tight contact structure

In this chapter we recall briefly an association between fibered links in a 3-manifold and contact structures on that manifold. We will review a bit of the history that explores which fibered links are associated to certain contact structures. Hedden and Plamenevskaya have a recent result [27], showing that, given a fibered knot associated to a contact structure with non-trivial Ozsváth-Szabó contact invariant, the dual knots to "sufficiently large" surgeries on that knot are also associated to a contact structure with non-zero contact invariant. As a consequence, the dual to any framed knot arising from Berge's double primitive construction is associated to a tight contact structure on the lens space obtained by the surgery. We consider certain families of these dual knots in $L(p, q)$, and we use results from Section 4.2 to determine whether these links are associated with the universally tight contact structure ξ_{UT} on $L(p, q)$.

5.1 Background

Given a rationally null-homologous link B in a closed, oriented 3-manifold Y, a *rational open book decomposition* (B, π, F) of Y is a fibration $\pi : Y \setminus \nu(B) \to S^1$ with compact fiber F, and $\nu(B)$ is a small tubular neighborhood of B in Y. Note that we do not require that F meet $\partial\nu(B)$ in a longitude of B, i.e. F is a rational Seifert surface for B. If B is not connected, we will only consider the case that F is a uniform Seifert surface. In the case that F is an honest Seifert surface, (B, π, F) is called an *integral open book decomposition.*

In what follows, we will often suppress the map π from the notation, denoting (B, π, F) as (B, F). Alexander showed [1] that every closed oriented Y^3 has an integral open book decomposition.

From any integral open book decomposition (B, F) of Y, with B oriented as the boundary of F, a construction of Thurston and Winkelnkemper [56] gives a contact 1-form α on Y with the properties:

(a) $\alpha(v) > 0$ for any v tangent to B (and coherently oriented with B);

(b) $d\alpha$ is a volume form on the interior of F.

We say that if $\xi = \ker \alpha$ for some contact 1-form α satisfying (a) and (b), then (B, F) *supports* ξ. By [56], every fibered knot (B, F) supports a contact structure, in fact it is unique. In [3], this was extended to rational open books, for if (B, F) is a rational open book decomposition, then we can still require that for a contact structure $\xi = \ker \alpha$ to be supported, the contact form α satisfy (a) and (b). It is shown in [3] that a rational open book (B, F) supports a unique contact structure, by determining an integral open book from (B, F) (called its integral resolution) and showing that (B, F) supports ξ if and only if its integral resolution does.

The exact correspondence between open book decompositions of Y and contact structures on Y was elucidated by Giroux: open book decompositions and contact structures are in bijective correspondence, if one considers open book decompositions of Y up to positive Hopf stabilization, and contact structures up to isotopy [23].

Given a fibered knot K in Y, one could ask which contact structure on Y is supported by K. When $Y = S^3$, an interesting result of Hedden (in [26]) shows that the answer is connected to strongly quasipositive knots and the Ozsváth-Szabó concordance invariant arising in knot Floer homology. We quickly recall the definitions of a strongly quasipositive knot and of τ in order to state Hedden's theorem.

Let $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ denote the standard generators of the braid group B_n on n strands. For any $i < j \leq n$, let $\sigma_{i,j}$ define the braid $\sigma_{i,j} = (\sigma_i \ldots \sigma_{j-2})\sigma_{j-1}(\sigma_i \ldots \sigma_{j-2})^{-1}$ in B_n. A knot K is called *strongly quasipositive* if K may be realized as the closure $\hat{\beta}$ of the braid

$$\beta = \prod_{k=1}^{m} \sigma_{i_k, j_k}.$$

We note that the maximum Euler characteristic of a strongly quasipositive knot is known. Suppose K is such a knot, realized as the closure of some $\beta \in B_n$ as above, where β is the product of m elements $\sigma_{i,j}$. Then K has a Seifert surface Σ constructed from n parallel disks with m half-twisted bands attached (the one corresponding to $\sigma_{i,j}$ being attached to the i^{th} and j^{th} parallel disk). Now, from the "Legendrianization" of the closure of β (see [26]) which we call L, one sees that

$$tb(L) + |rot(L)| = -n + m.$$

So the Bennequin inequality is sharp on K and the Seifert surface Σ has maximum Euler characteristic.

Let $\mathcal{F}(Y, K, i)$ denote the filtration that a null-homologous knot $K \subset Y$ induces on the Ozsváth-Szabó chain complex $\widehat{CF}(Y, \mathfrak{s})$ associated to Y and $\mathfrak{s}$, a spin^c-structure on Y. In the setting where $Y = S^3$, the concordance invariant is defined by

$$\tau(K) = \min\left\{ j \in \mathbb{Z} \mid i_* : H_*(\mathcal{F}(Y, K, j)) \to \widehat{HF}(S^3) \cong \mathbb{Z} \text{ is non-trivial.} \right\}$$

In [26] Hedden proves the following. Let $\xi(K, F)$ denote the contact structure corresponding to the fibered knot (K, F).

Theorem 5.1.1. *Let $K \subset S^3$ be a fibered knot with fiber F. Then*

$$(K, F) \textit{ is strongly quasipositive} \iff \xi(K, F) = \xi_{st} \iff \tau(K) = g(K),$$

where ξ_{st} is the standard tight contact structure on S^3 and $g(K)$ is the Seifert genus of K.

Remark 5.1.2. The method of Hedden's proof uses that $\xi(K, F) = \xi_{st} \iff c(\xi(K, F)) \neq 0$, where $c(\xi)$ is the Ozsváth-Szabó contact invariant of ξ, which equivalence uses the uniqueness of the tight contact structure on S^3.

In [46], Ozsváth and Szabó show that any knot K for which some positive integral surgery yields a lens space has the property that $\tau(K) = g(K)$. Thus, since the work of Ni shows that all such knots are also fibered [43], Theorem 5.1.1 above tells us that every knot with a positive integral lens space surgery supports the tight contact structure on S^3.

The discussion makes clear that if a fibered knot in S^3 supports the tight contact structure ξ_{st} then the Bennequin inequality is sharp for that knot. In fact, more generally one can see from the construction of Thurston and Winkelnkemper that if (K, F) is an integral open book decomposition of Y, then K is transverse to $\xi(K, F)$ and $sl_{\xi(K,F)}(K) = -\chi(F)$. Thus if $\xi(K, F)$ is tight then the Eliashberg-Bennequin inequality implies that F has maximum Euler characteristic among Seifert surfaces of K.

In [17], Etnyre and Van Horn-Morris considered whether, given a contact structure ξ on Y, and an integral open book (K, F) of Y, the equality $sl_\xi(K) = -\chi(F)$ is sufficient to say that K supports ξ. The question was generalized in [7] to the setting of a rational open book decompostion, with K being $\mathbb{Q}$-nullhomologous. The answer, stated below, gives a nice geometric meaning to the sharpness of the Eliashberg-Bennequin inequality in (Y, ξ).

Theorem 5.1.3. *Let (K, F) be a fibered, transverse rationally null-homologous link in a contact 3-manifold (Y, ξ) such that ξ restricted to the exterior of K is tight. Suppose that $[K]$ has order r in $H_1(Y)$. Then $r \cdot sl_\xi(K) = -\chi(F)$ if and only if either $\xi = \xi(K, F)$ or is obtained from $\xi(K, F)$ by adding Giroux torsion along incompressible tori in the complement of K.*

5.2 Surgeries on fibered knots, contact structures and Berge knots

One of the fundamental constructions in low-dimensional topology is that of Dehn surgery. If a knot $K \subset M$ has Dehn surgery that yields some 3-manifold N, then the core of the filling is a knot in N, which we call the *dual knot* to this surgery. Note that this dual knot has a surgery that yields M.

If the surgery coefficient on K was an integer p, then $[K']$ has order p in $H_1(N)$. Furthermore, the complement of a tubular neighborhood of K in M and of K' in N are homeomorphic. Thus, if K was fibered in M, then K' is fibered in N, with the boundary of a fiber being some (p, s)-sloped curve in the boundary of a neighborhood of K'. Thus K' supports a rational open book decomposition of N, and we may consider the associated contact structure.

Recently, Hedden and Plamenevskaya [27] considered this situation when K is fibered, with fiber surface F, and supports a contact structure ξ on M with $c(\xi) \neq 0$. They were able to prove the following.

Theorem 5.2.1. *Suppose that $K \subset M$ is fibered, with fiber surface F, let g be the genus of F, and suppose that K supports a contact structure ξ on M such that $c(\xi) \neq 0$. If $K' \subset N$ is the dual to p-surgery on K, and $p \geq 2g$, then K' supports a contact structure ξ' on N with the property $c(\xi') \neq 0$.*

Theorem 5.2.1 has an application to a class of knots arising in the Berge Conjecture, which attempts to characterize all knots in S^3 which admit a surgery resulting in a lens space. This problem was first addressed by Moser [38], who showed that all torus knots admit such a surgery. By the cyclic surgery theorem of Culler, Gordon, Luecke, and Shalen [14], the remaining knots would need to admit integer surgeries giving a lens space. A conjectured solution was provided through the work of John Berge [11]. He describes knots that are doubly primitive with respect to a standard genus 2 Heegaard splitting of S^3 and shows that all such knots have a surgery that yields a lens space. It is conjectured that this completes the list.

Recent work on the Berge conjecture has indicated that it may be more tractable to consider the “inverse” problem: what knots in some given lens space $L(p, q)$ admit an integer surgery that yields S^3? When $K \subset S^3$ is a Berge knot, we understand K to carry information about its framing on the genus 2 surface. When we mention the dual to some Berge knot, this framing is part of the data. The surgery so described by this framing produces a dual to the Berge knot that is 1-bridge in a genus 1 Heegaard splitting of the lens space. Furthermore, this dual admits a grid number one diagram (see Lemma 1 and Theorem 2 in [11]).

As mentioned above, Berge knots are fibered. Furthermore, they each are the binding of an open book decomposition of S^3 that supports the standard (tight) contact structure, by Theorem 5.1.1. Using monopole Floer homology, it was shown by Kronheimer-Mrowka-Ozsváth-Szabó [35] that if p-surgery on a knot in S^3 yields a lens space, then $2g - 1 \leq p$, where g is the Seifert genus of the knot. Combining this with work of Greene that solves the lens space realization problem [24], we see that the surgery coefficient on a Berge knot must be at least twice the Seifert genus of the knot. Thus, we may apply Theorem 5.2.1, and conclude that the dual knot K' in $L(p, q)$ supports a contact structure ξ' that has $c(\xi') \neq 0$, and so ξ' is tight.

Below we use Theorem 5.1.3 and the formulae developed in Section 4.2 to see which duals of Berge knots support the *universally* tight contact structure on that lens space. The moral of our results is that one should not expect a Berge dual to support the universally tight contact structure. Berge's list separated double primitive knots into 3 types: knots in a solid torus, knots on a genus 1 fiber surface, and 4 "sporadic families" of knots. Our computations will deal only with the knots on a genus 1 fiber surface.

5.3 Computations

The only genus 1 fibered knots in S^3 are the right- and left-handed trefoils and the figure eight knot. We remark that every embedded closed curve on the Seifert surface of such knots may be represented either as the closure of a positive braid or a negative braid. This was shown to be the case in Ken Baker's thesis [6], and we illustrate the case of the right-handed trefoil in Figures 5.3 and 5.4. The maximum Euler characteristic of such closures, as they are strongly quasipositive knots (or mirrors thereof), can be found as the Euler characteristic of the quasipositive Seifert surface constructed above.

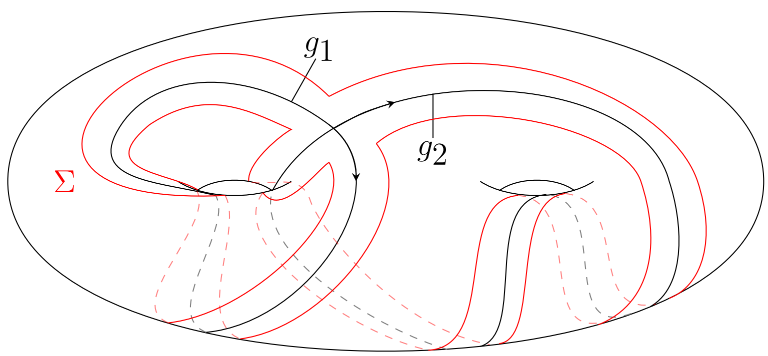


Figure 5.1: Berge knots on a Seifert surface of the trefoil

Consider a knot embedded on the Seifert surface of a right-handed trefoil, with fiber surface Σ shown in Figures 5.3 and 5.4. To see how Σ (and a Berge knot) sits on a genus 2 Heegaard surface, refer to Figure 5.1. Every Berge knot in this family is homologous to $a[g_1] + b[g_2]$, where g_1, and g_2 are the cores of the handles of Σ as shown in Figure 5.1, and a and b are relatively prime. If K is such a knot, with $[K] = a[g_1] + b[g_2]$ on Σ, then K is given a framing by Σ, and this is the framing for the surgery. It is shown in [11] that the prescribed surgery on K yields $L(p, q)$, where $p = a^2 + ab + b^2$ and $q \equiv a^2 b^{-2} (\text{mod } p)$.

Proposition 5.3.1. *Let $K \subset S^3$ be a double primitive knot on a Seifert surface Σ of the right-handed trefoil as described above with $[K] = a[g_1] + b[g_2]$. Also let $p = a^2 + ab + b^2$ and $q \equiv a^2 b^{-2} (\text{mod } p)$. If $K' \subset L(p, q)$ is the dual knot to K, then K' supports the universally tight contact structure on $L(p, q)$ if and only if $a = 1$ or $b = 1$.*

Remark 5.3.2. Note that if $a = 1$ or $b = 1$ then K is a torus knot.

Proof. The maximum Euler characteristic of a Seifert surface for K is $\chi(K) = 2a + 2b - a^2 - ab - b^2$, as Figures 5.3 and 5.4 indicate that K is the closure of a negative braid with index $a + b$ and $ab + a^2 - a + b^2 - b$ crossings.

We will show by computation that the maximum self-linking number of K' cannot make the Bennequin bound sharp for K'. For our computations, we use the following lemma.

Lemma 5.3.3. *Suppose r, s are integers with $0 < r < s$. Then $r^2 + r + 1 \equiv 0$ (mod s), if and only if there is a pair of integers a, b with $(a, b) = 1$ such that $s = a^2 + ab + b^2$ and $r \equiv \left(\frac{a^2}{b^2}\right)$ (mod s).*

Proof. First suppose that $s = a^2 + ab + b^2$ and $r \equiv \left(\frac{a^2}{b^2}\right)$ (mod s) for some relatively prime pair (a, b). Since $r^3 - 1 \equiv b^{-6}(a^6 - b^6) = b^{-6}(a - b)(a^3 + b^3)(a^2 + ab + b^2)$, we see that $r^3 \equiv 1$ (mod s). A similar calculation shows that $\left(\frac{a}{b}\right)^3 \equiv 1$, so $r^2 \equiv \frac{a}{b}$ (mod s). As a result,

$$
\begin{aligned}
r + r^2 + 1 &\equiv \left(\frac{a^2}{b^2}\right) + \left(\frac{a}{b}\right) + 1 \\
&\equiv \frac{1}{b^2}\left(a^2 + ab + b^2\right),
\end{aligned}
$$

which is a multiple of s.

Now, for any $0 < r < s$, suppose that $r^2+r+1 = ms$, for some multiple m. Then ms is a number properly represented by the positive definite quadratic form $f(x, y) = x^2 + xy + y^2$. A number is so represented if and only if its prime decomposition is of the form

$$ms = 3^{\varepsilon} \prod p_i^k,$$

where ε is 0 or 1 and each p_i is a prime congruent to 1 (mod 3) (see, for example, [44, Chapter 3 (p176)]). Since s must therefore have a prime decomposition of the same form, s is also properly represented by the quadratic form f. □

Let $\mu \equiv \mu(K')$. We know from Corollary 4.2.4 and the proof of Lemma 5.3.3 that

$$\mu^2 \equiv \pm q^{-1} \equiv \pm q^2 (\text{mod } p).$$

In fact, an examination of Berge's proof that $L(p, q)$ is the lens space obtained from this surgery on K reveals that $\mu \equiv \pm q (\text{mod } p)$ (see the last two paragraphs of the proof of Lemma 3 [11]). Let K'_l be the Legendrian representative of K' in the universally tight contact structure corresponding to a grid number one diagram. We get that

$$p \cdot \text{sl}_{\mathbb{Q}}(T_+(K'_l)) \equiv -\mu\lambda + \lambda - \mu \equiv -\mu^2 q + \mu q - \mu \qquad \text{and,}$$
$$p \cdot \text{sl}_{\mathbb{Q}}(T_-(K'_l)) \equiv -\mu\lambda - \lambda + \mu \equiv -\mu^2 q - \mu q + \mu$$

by Corollary 4.2.7. As a consequence, we have that

$$p \cdot \text{sl}_{\mathbb{Q}}(T_+(K'_l)) \equiv -1 \pm q^2 \mp q,$$

which is congruent to either $2q^2$ or $2q$ since, due to Lemma 5.3.3, $q^2+q+1 \equiv 0 (\text{mod } p)$. Similarly, we see that $p \cdot \text{sl}_{\mathbb{Q}}(T_-(K'_l))$ is congruent to either $2q$ or $2q^2$.

Suppose that $K' \subset L(p,q)$ is the binding of an open book decomposition that supports ξ_{UT}. Then by Theorem 5.1.3,

$$p \cdot \overline{\text{sl}_{\mathbb{Q}}(K')} = -\chi(K') = -\chi(K) = a^2 + ab + b^2 - 2a - 2b.$$

The maximum self-linking number of K' is realized by either the positive or negative transverse push-off of a Legendrian realization of K', and so either $2q^2 \equiv -2(a+b)(\text{mod } p)$ or $2q \equiv -2(a+b)(\text{mod } p)$. Recall that $q \equiv a^2b^{-2}$ and $q^2 \equiv ab^{-1}$. Since p cannot be even, 2 is invertible. Moreover, $q^3 \equiv 1(\text{mod } p)$, and so either of these cases implies

$$1 \equiv -(a+b)^3 = -(a^3 + 3a^2b + 3ab^2 + b^3) \equiv -(a^2b + ab^2)(\text{mod } p).$$

So $0 \equiv 1 + a^2b + ab^2 \equiv 1 - b^3(\text{mod } p)$, implying that either $b = 1$ or $b^2 + b + 1 \equiv 0(\text{mod } p)$, the latter only being possible if $a = 1$.

To see that we have the equality, $p \cdot \text{sl}_{\mathbb{Q}}(T(K'_l)) = -\chi(K')$, when $b = 1$, first recall that a grid number 1 diagram admits a projection with no cusps, and with $0 \leq \mu < p$ and $0 \leq \lambda < p$. Let w be the writhe of this grid projection for K'. Now since $b = 1$, we have $p = a^2 + a + 1$, $\mu = q = a^2$, and $\lambda \equiv \mu q = a^4$. But $0 \leq \lambda < p$, so $\lambda = a$. We now use Corollary 4.2.7 again and get that the equality holds if

$$(a^2 + a + 1)w - a^3 - a + a^2 = a^2 - a - 1, \tag{5.3.1}$$

since $b = 1$ implies $\chi(K') = -a^2 + a + 1$. Equation 5.3.1 holds if $w = a - 1$. To see that the writhe of our chosen projection is $a - 1$, we use that $q = a^2$, which implies that a grid number one diagram with $\mu = ja$ will have its $\mathbb{X}$ marking in the $(p - j)^{th}$ fundamental parallelogram (this follows from $-ka^2 \equiv ka + k (\text{mod } p)$ for $k = ja$). Thus, when $\mu = a^2$, and since we have chosen the projection with $\lambda > 0$, the horizontal arc intersects the vertical arc in every parallelogram it crosses, of which there are $a - 1$. □

Finally, we examine Berge knots that arise as simple closed curves on the Seifert surface Σ of a figure 8 knot. Similar to the previous setting, all such knots are homologous on Σ to $a[g_1] + b[g_2]$, where $(a, b) = 1$, and g_1 and g_2 are the cores of the handles of Σ depicted in Figure 5.2. If K is such a knot, the lens space obtained as Σ-framed surgery on K is $L(p, q)$, where $p = \left|b^2 - ab - a^2\right|$ and $q = a^2 b^{-2}$. We obtain the following result.

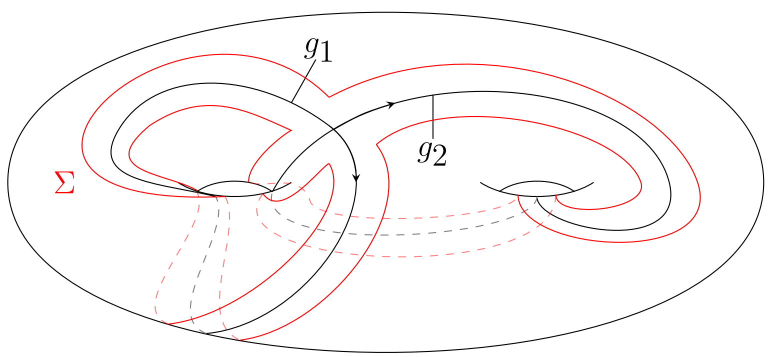


Figure 5.2: Berge knots on a Seifert surface of the figure 8 knot

Proposition 5.3.4. *Let $K \subset S^3$ be a double primitive knot on a Seifert surface of the figure 8 knot, as described above, with $[K] = a[g_1] + b[g_2]$. Also, suppose that $b \geq 13$ and $a > b^2$, and let $p = \left|b^2 - ab - a^2\right|$ and $q \equiv a^2 b^{-2} (\text{mod } p)$. If $K' \subset L(p, q)$ is the dual knot to K, then K' does not support the universally tight contact structure on $L(p, q)$.*

Remark 5.3.5. For lower values of a, b, it is, at best, quite rare that K' supports the universally tight contact structure. For example, among relatively prime pairs (a, b) with $a \leq 75$ and $b \leq 40$, there are at most 9 such pairs.

Proof. To calculate the maximum Euler characteristic of a surface (rationally) bounded by K', we appeal to [6, Appendix B], where we see that K is the closure of a positive braid with index $a+b$ and $ab+(a+b)(a+b-1)$ twists. Thus, the maximal Euler characteristic of a Seifert surface for K is

$$\chi(K) = \chi(K') = 2(a+b) - a^2 - 3ab - b^2 \equiv 2(a+b) - 2ab - 2b^2 (\text{mod } p).$$

Lemma 5.3.6. *Given relatively prime a, b, let $p = \left|b^2 - ab - a^2\right|$, $q \equiv a^2 b^{-2}$, and $r \equiv ab^{-1}$. Then $q^2 - 3q + 1 \equiv 0 (\text{mod } p)$ and $r^2 + r - 1 \equiv 0 (\text{mod } p)$.*

Proof. The proof is straightforward, noting that b is invertible mod p, since $(a, b) = 1$. □

As a consequence of Lemma 5.3.6, we discover that $r^{-1} \equiv r+1$, $q^{-1} \equiv r+2$, and $q^{-1} \equiv 3-q$. Some calculations then show that $-q \equiv r-1$, and so $r^3 - 1 \equiv -2q$ and $r^3 + 1 \equiv 2r$. As in the proof of the previous proposition, Corollary 4.2.7 implies that if K'_l is the Legendrian realization of K' in $(L(p,q), \xi_{UT})$ corresponding to the grid number one diagram, then

$$p \cdot \mathrm{sl}_{\mathbb{Q}}(T(K'_l)) \equiv -\mu^2 q \pm \mu q \mp \mu (\text{mod } p),$$

depending on whether $T(K'_l) = T_{\pm}(K'_l)$. Berge's work again implies that $\mu \equiv \pm q$, as before.

Combining this with the consequences of Lemma 5.3.6, we get

$$p \cdot \mathrm{sl}_{\mathbb{Q}}(T(K'_l)) \equiv \begin{cases} -2q^2, \text{ if } T(K'_l) = T_+(K'_l) \\ -2q^2 r, \text{ otherwise.} \end{cases}$$

Suppose that K' supports the contact structure ξ_{UT}. Then $p \cdot \mathrm{sl}_{\mathbb{Q}}(T(K'_l)) = -\chi(K')$. Thus, since p is odd, and so 2 is invertible mod p, we have

$$ab + b^2 - (a+b) \equiv \begin{cases} -q^2, \text{ if } T(K'_l) = T_+(K'_l) \\ -q^2 r, \text{ otherwise.} \end{cases}$$

But $a + b \equiv a(1 + r^{-1}) \equiv aq^{-1}$. So if $T(K'_l) = T_+(K'_l)$, we get that $-q^3 \equiv a(b-1)$. Using that $q^2 - 3q + 1 \equiv 0$ one then arrives at $8a - 5b \equiv a(b^2 - b) (\mathrm{mod}\ p)$. However, since $a > b^2$ and $b \geq 13$, we get that $p = a^2 + ab - b^2$ and

$$0 < 8a - 5b < 8a < (b^2 - b)a < ab^2 < a^2 < p,$$

a contradiction.

On the other hand, if $T(K'_l) = T_-(K'_l)$, then $-q^3 r \equiv a(b-1)$. This would require that $-13a + 8b \equiv a(b^2 - b)(\mathrm{mod}\ p)$. But now we have a contradiction, since

$$p > p - 13a + 8b > p - 13a > a(b^2 - b) > 0,$$

the second to last inequality resulting from the fact that $a > b^2$, and $b > 13$. □

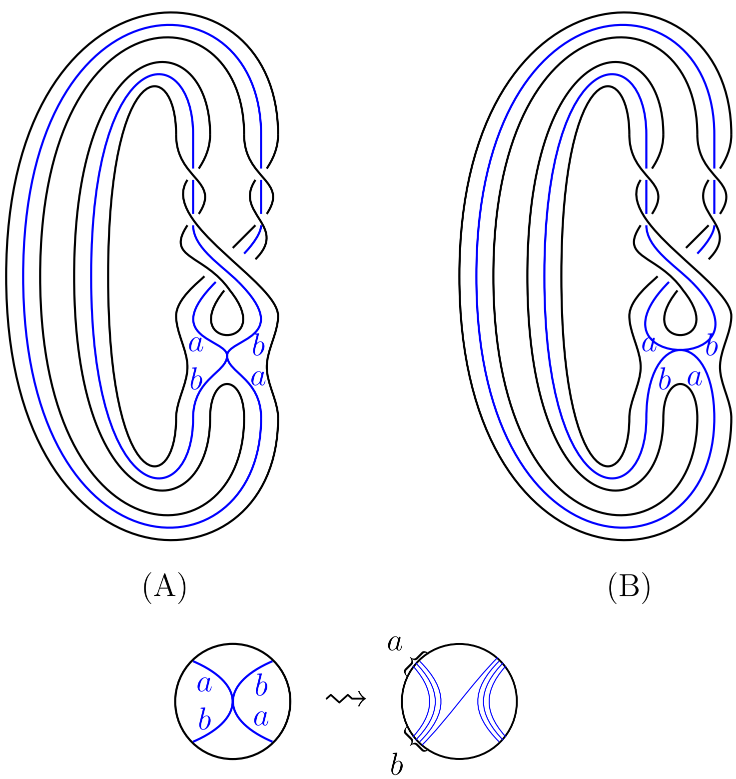


Figure 5.3: In surface (A) the embedded curve obtained from the weighted edges as indicated is already a negative braid. In surface (B), this is not the case. By isotopy of the surface, we get the top-left surface in Figure 5.4, where it shows how to get a negative braid closure.

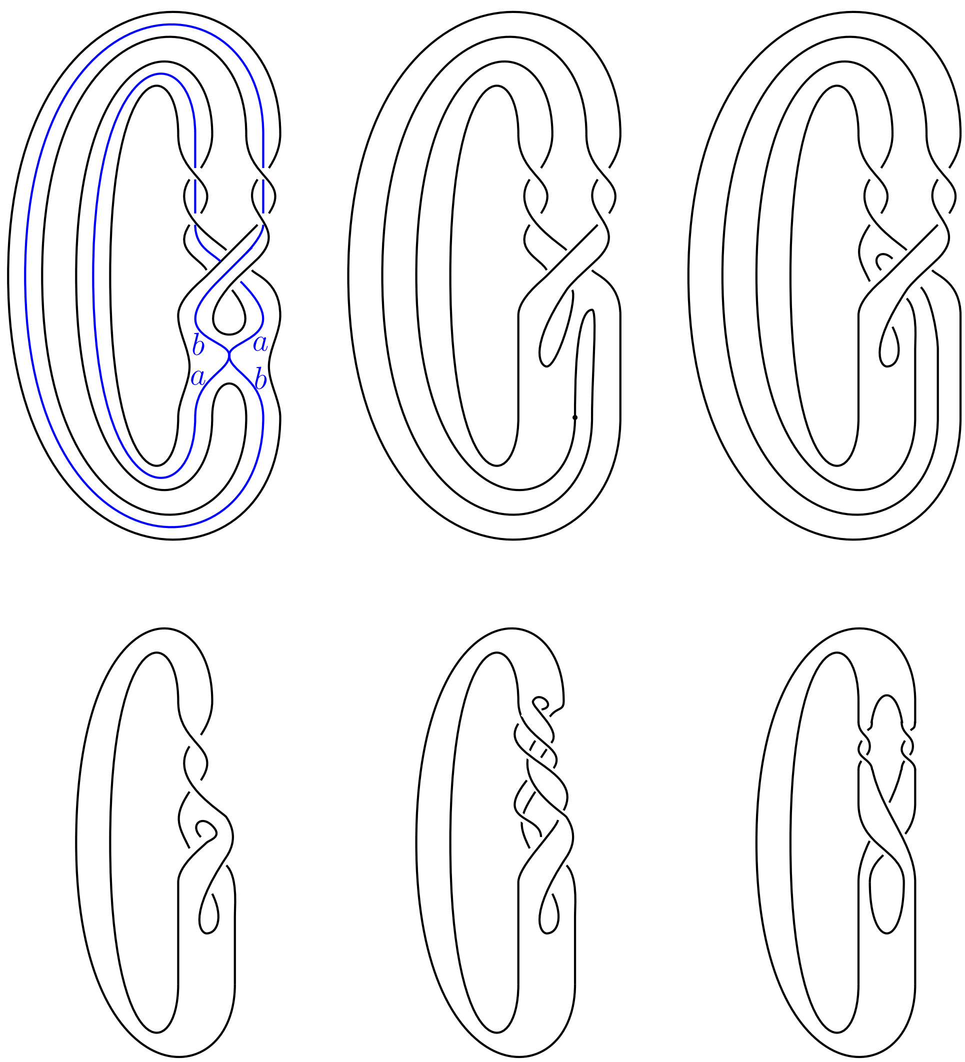


Figure 5.4: The isotopy of Σ above starts at the top-left figure and ends at the bottom-right figure. The isotopy from the top-right figure to the bottom-left figure is a "Reidemeister I move" of one of the bands in the closure, which move cancels the twisting. From the bottom-left figure to the bottom-middle figure, one performs a finger move through the full twist.

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