# APPLICATION OF FOSTER'S REACTANCE THEOREM TO THE DESIGN OF ELECTRIC WAVE FILTERS

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Fuad Labib Abboud

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THESIS

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Major professor

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# APPLICATION OF FOSTER'S REACTANCE THEOREM TO THE DESIGN OF ELECTRIC WAVE FILTERS

By

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A THESIS

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# PREFACE

This thesis is intended to discuss and illustrate the extent of application of Foster's reactance theorem in the design of electric wave filters, and to investigate an abbreviated method for determining the analytic form of the reactance function for a given non-dissipative two-terminal network which is discussed in volumes I and II of the book " Communication Networks " by Mr. Ernst A. Guillemin.

The writer illustrates in this thesis that while this abbreviated method applies well to many networks, it fails to apply in its present form to many others. Therefore, the writer has set a rigid procedure for the application of the above method to any two-terminal network.

The writer wishes to extend his thanks and express his appreciation to Dr. Joseph A. Strelzoff for his instructions and suggestions during this year of advanced study which made the development of this thesis possible.

# INTRODUCTION

Filters in general are designed as non-dissipative structures. The presence of resistance in the actual physical filters only modifies the behavior predicted on a non-dissipative basis a little, hence the theory and design of such four terminal networks is based upon the behavior of structures with negligible or no ohmic resistance and therefore, this discussion will take up only purely reactive networks.

# CHAPTER I

## FOSTER'S REACTANCE THEOREM

### 1.1 A Statement of the Theorem

The most general driving-point impedance ( two-terminal impedance ) Z ($\omega$), obtainable by means of a finite resistanceless network is a pure reactance which is an odd rational function of the frequency and which is completely determined, except for a constant factor H, by assigning the resonant and anti-resonant frequencies, subject to the condition that they alternate and include both zero and infinity. Any such impedance may be physically constructed either by combining, in parallel, resonant circuits having impedances of the form $\left[ iLp + ( iCp )^{-1} \right]$, or by combining, in series, anti-resonant circuits having impedances of the form $\left[ iCp + ( iLp )^{-1} \right]$.

### 1.2 Discussion

For a network driven from the first mesh, the general form of the driving-point impedance function is

$$Z_{11}(\omega) = j\omega H \frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_3^2)\ldots\ldots(\omega^2 - \omega_{2n-1}^2)}{\omega^2(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)\ldots\ldots(\omega^2 - \omega_{2n-2}^2)} \qquad (1)$$

which can also be written as

$$Z_{11}(\omega) = \frac{D(\omega)}{B_{11}(\omega)} \tag{2}$$

in which $D(\omega)$ is the determinent of the network and $B_{11}(\omega)$ is its minor after eliminating the first row and the first column.

The zeros of the driving-point impedance which locate the resonant frequencies are determined by the roots of $D(\omega) = 0$, and the poles which locate the anti-resonant frequencies are determined by the roots of $B_{11}(\omega) = 0$.

For any possible combination of inductors and capacitors to be realized physically in the form of a driving-point impedance, it is necessary that the slope of the impedance function versus frequency be everywhere positive i.e.,

$$\frac{d\, Z_{11}(\omega)}{j\, d\, \omega} > 0 \qquad \text{for} \quad -\infty < \omega < \infty \tag{3}$$

which requires that the poles and zeros alternate as expressed by

$$0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{2n-2} < \omega_{2n-1} < \infty \tag{4}$$

## 1.3 Example

The following example will illustrate the reactance theorem.

Let us find the impedance function $Z_{11}(\omega)$ of Fig. 1-1 and determine the equivalent network according to Foster's theorem.

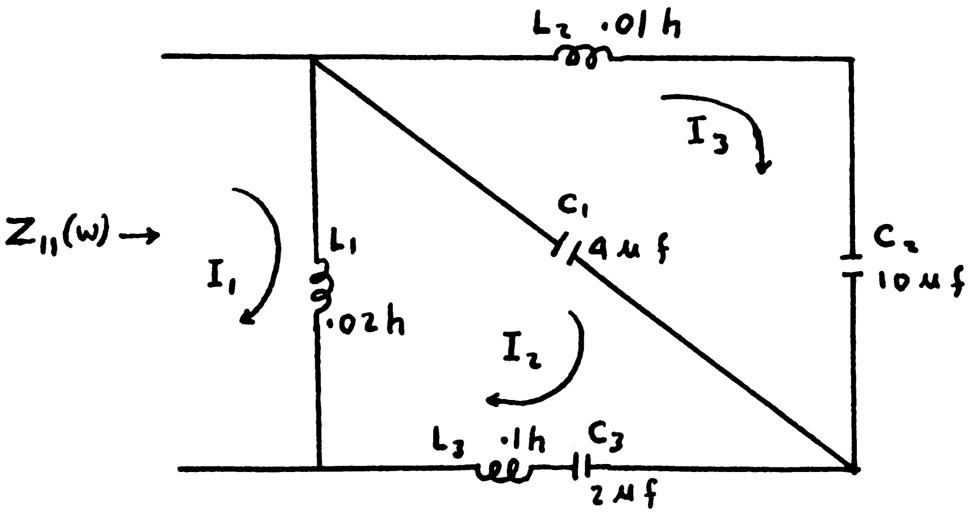



Fig. 1-1

The mesh equations for Fig. 1-1 are

$$E = j\omega L_1 I_1 - j\omega L_1 I_2 + 0$$

$$0 = -j\omega L_1 I_1 + \left[ j\omega(L_1 + L_3) + \frac{1}{j\omega \frac{(C_1 C_3)}{(C_1 + C_3)}} \right] I_2 - \frac{1}{j\omega C_1} I_3$$

$$0 = 0 - \frac{1}{j\omega C_1} I_2 + \left[ j\omega L_2 + \frac{1}{j\omega \frac{(C_1 C_2)}{(C_1 + C_2)}} \right] I_3$$

The driving-point impedance for Fig. 1-1 can now be written

$$Z_{11}(\omega) = \frac{D(\omega)}{B_{11}(\omega)} = \frac{\begin{vmatrix} j\omega L_1 & -j\omega L_1 & 0 \\ -j\omega L_1 & j\omega L_a + \frac{1}{j\omega C_a} & -\frac{1}{j\omega C_1} \\ 0 & -\frac{1}{j\omega C_1} & + j\omega L_2 + \frac{1}{j\omega C_b} \end{vmatrix}}{\begin{vmatrix} j\omega L_a + \frac{1}{j\omega C_a} & -\frac{1}{j\omega C_1} \\ -\frac{1}{j\omega C_1} & j\omega L_2 + \frac{1}{j\omega C_b} \end{vmatrix}}$$

where $L_a = L_1 + L_3$

$C_a = \frac{C_1 C_3}{C_1 + C_3}$ and

$C_b = \frac{C_1 C_2}{C_1 + C_2}$

Expanding the two determinents we have

$$Z_{11}(\omega) = j\omega L_1 \left[ \frac{\omega^4(L_2L_a - L_1L_2) - \omega^2\left(\frac{L_2}{C_a} + \frac{L_a}{C_b} - \frac{L_1}{C_b}\right) + \left(\frac{1}{C_aC_b} - \frac{1}{C_1^2}\right)}{\omega^4(L_3L_a) - \omega^2\left(\frac{L_2}{C_a} + \frac{L_a}{C_b}\right) + \left(\frac{1}{C_aC_b} - \frac{1}{C_1^2}\right)} \right]$$

Substituting the values from Fig. 1-1 and simplifying gives

$$Z_{11}(\omega) = j\omega \frac{1}{60} \left[ \frac{\omega^4 - \omega^2(4.25 \times 10^7) + (2 \times 10^{14})}{\omega^4 - \omega^2(4.125 \times 10^7) + (1.67 \times 10^{14})} \right]$$

The roots of the numerator are

$$\omega^2 = \frac{(4.25 \times 10^7) \pm \sqrt{(4.25 \times 10^7)^2 - 4(2 \times 10^{14})}}{2}$$

$$= 5.4 \times 10^6 \quad \text{and} \quad 3.71 \times 10^7$$

The roots of the denominator are

$$\omega^2 = \frac{(4.125 \times 10^7) \pm \sqrt{(4.125 \times 10^7)^2 - 4(1.67 \times 10^{14})}}{2}$$

$$= 4.54 \times 10^6 \quad \text{and} \quad 3.67 \times 10^7$$

and the final form of the driving-point impedance for Fig. 1-1 is

$$Z_{11}(\omega) = j\omega \frac{1}{60} \left[ \frac{(\omega^2 - 5.4 \times 10^6)(\omega^2 - 3.71 \times 10^7)}{(\omega^2 - 4.54 \times 10^6)(\omega^2 - 3.67 \times 10^7)} \right]$$

This shows that the resonant frequencies as characterized by the zeros of the system occur at

$$f_1 = 0 \quad ; \quad f_3 = 369 \text{ cps} \quad ; \quad f_5 = 969 \text{ cps}$$

while the anti-resonant frequencies as characterized by the poles of the system occur at

$$f_2 = 339 \text{ cps} \quad ; \quad f_4 = 961 \text{ cps} \quad ; \quad f_6 = \infty$$

The reactance function having these properties is illustrated in Fig. 1-2

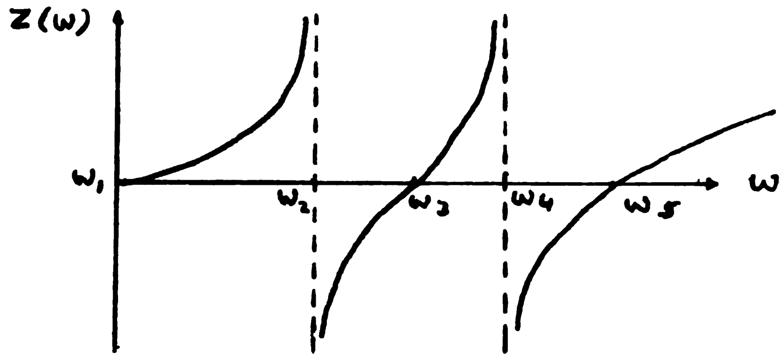



Fig. 1-2

Now let us determine the equivalent network that will have the same driving-point impedance.

The driving-point impedance for this example was found to have the following form

$$Z_{11}(\omega)=j\omega H\,\frac{(\omega^2-\omega_3^2)(\omega^2-\omega_5^2)}{(\omega^2-\omega_2^2)(\omega^2-\omega_4^2)} \qquad (5)$$

$$=j\omega H+\frac{j\omega^3 H\left[(\omega_2^2+\omega_4^2)-(\omega_3^2+\omega_5^2)\right]+j\omega H(\omega_3^2\omega_5^2-\omega_2^2\omega_4^2)}{(\omega^2-\omega_2^2)(\omega^2-\omega_4^2)}$$

$$=j\omega\frac{1}{60}\left[1+\frac{\omega^2(41.24-42.5)10^6+(20-16.67)10^{13}}{(\omega^2-4.54\times10^6)(\omega^2-3.67\times10^7)}\right]$$

By partial fraction expansion this becomes

$$Z_{11}(\omega)=j\omega\frac{1}{60}\left[1+\frac{A_1}{\omega^2-4.54\times10^6}+\frac{A_2}{\omega^2-3.67\times10^7}\right]$$

Equating the last two equations and solving for $A_1$ and $A_2$ gives

$$A_1 = -8.6 \times 10^5 \qquad ; \qquad A_2 = -3.9 \times 10^5$$

and finally

$$Z_{11}(\omega) = j\omega \frac{1}{60} \left[ 1 - \frac{8.6 \times 10^5}{\omega^2 - 4.54 \times 10^6} - \frac{3.9 \times 10^5}{\omega^2 - 3.67 \times 10^7} \right]$$

The last equation shows that Foster's network is composed of three branches in series, the first is an inductor whose reactance is

$$j\omega \frac{1}{60}$$

the second is a parallel branch of L and C whose reactance is

$$-j\omega \frac{1}{60} \cdot \frac{3.9 \times 10^5}{\omega^2 - 3.67 \times 10^7}$$

and the third again is an L C parallel branch whose reactance is

$$-j\omega \frac{1}{60} \cdot \frac{8.6 \times 10^5}{\omega^2 - 4.54 \times 10^6}$$

The form of Foster's equivalent network as determined by the partial fraction expansion is shewn in Fig. 1-3

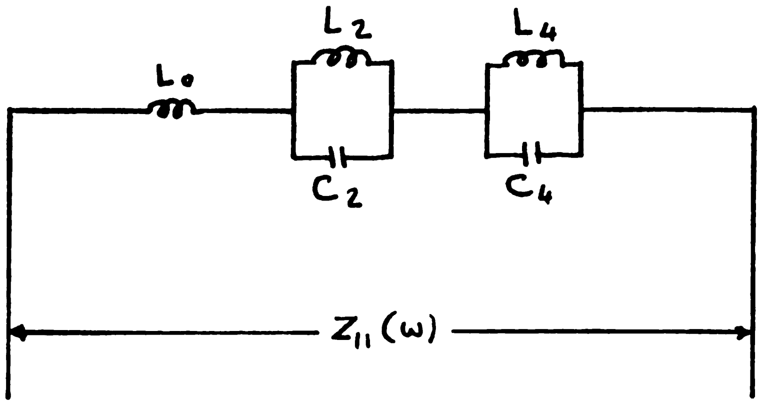



Fig. 1-3

The individual elements are determined from the following formulas which can be easily derived

$$L_0 = H$$

$$C_k = -\left[\frac{j\,\omega Y_{11}}{\omega^2 - \omega_k^2}\right]_{\omega=\omega_k} \qquad k = 0,2,4,\ldots\ldots$$

$$L_k = \frac{1}{\omega_k^2 \; C_k}$$

The following conclusions may be drawn from the above example, for proofs, the reader is referred to the references listed under bibliography :

a. From inspection of Fig. 1-1, it is seen to be constructed of six elements while the equivalent network, Fig. 1-3, involved five elements. Similar examples will show that the networks obtained by Foster's method always contain the least number of elements by means of which any driving-point impedance can be realized.

b. It is seen from equation (5) that the degree of the determinental equation is five, which is equal to the number of elements in the equivalent network of Fig. 1-3.

c. The least number of elements by means of which any two-terminal, linear, reactive network may be realized, and therefore the degree of the determinental equation of this equivalent network cannot exceed 2n where n is the number of meshes in the network.

d. The least number of elements by means of which a given driving-point impedance may be realized equals one more than the sum of its internal zeros and poles.

In the light of what paragraph (d) above suggests, it is interesting to point out the following :

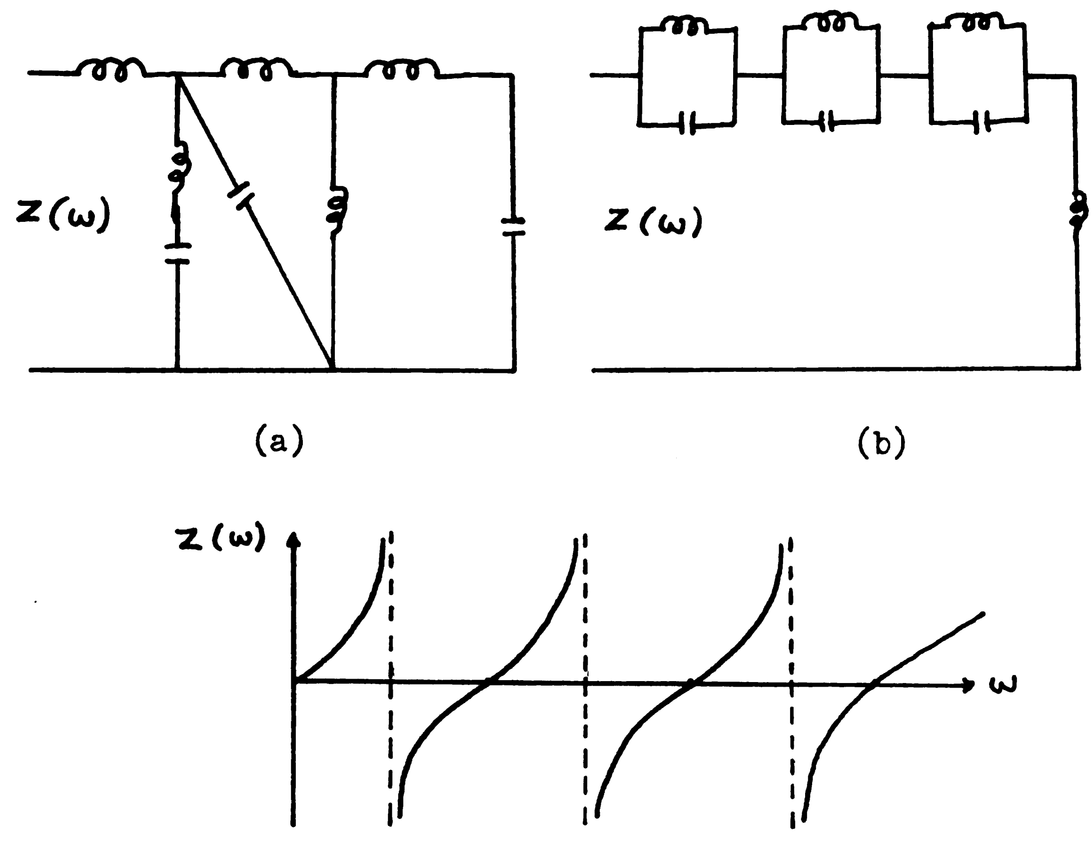
1. In chapter two, article eleven, page 66 of his book "Networks Lines and Fields", Prentice-Hall, Inc., 1949, Mr. John D. Ryder states :

   "In general, a network may have a total of resonant and anti-resonant points not exceeding its number of meshes plus one".

2. In section three, paragraph 21, page 201 of the Radio Engineers' Handbook, first edition, ninth impression, McGraw-Hill Book Company, Inc., 1943, Mr. Frederick Emmons Terman makes the following statement :

"The sum of the number of poles and number of zeros is one less than the number of independent meshes of the network".

We notice immediately that the above two statements are not in line with each other, and in addition, neither of them agrees with condition (d) above. That the statements (1) and (2) above are incorrect may be seen from the following illustration.




(c)

Fig. 1-4

Without resorting to mathematical analysis it can be seen that the network (a) above can be realized by network (b) according to the reactance theorem and that both have a reactance function as illustrated by (c), Fig. 1-4. While the network (a) is made of four meshes, we find from (c) that the reactance function has a total of eight zeros and poles or six internal zeros and poles, neither of which corresponds to either of the statements in (1) or (2) while condition (d) holds as stated.

## 1.4 An Abbreviated Process for Determining the Analytic Form of the Reactance Function for a Given Non-dissipative Two-terminal Network.

The determination of the equivalent network by Foster's method in the above example was relatively simple. Had the network involved five or more meshes the above method would have been rather lengthy and laborious.

Recalling that the least number of elements by means of which any given driving-point impedence function may be realized is equal to the degree of the determinental equation of the equivalent network, it seems that if we can find a simple method by means of which we can determine the degree of the determinental equation of any driving-point impedance, then we can determine Foster's equivalent network without much effort.

In section four, chapter five of his book "Communication Networks", volume two, Mr. Ernst A. Guillemin gives such an illustration in which Mr. Guillemin refers to a method outlined in section seven, chapter five of volume one for the determination of the degree of the determinental equation of any network. Mr. Guillemin then explains that the roots of the determinental equation are equal to the number of the internal zeros and poles plus one, and this number, he explains in section five, chapter five, volume two, is equal to the least number of elements by means of which any given driving-point reactance function may be realized. Finally, in section four, chapter five, volume two, Mr. Guillemin states that this method of setting up the reactance function is applicable to any case.

To show whether or not this method is applicable to any case, the example given in section seven, chapter five of volume one is repeated here for illustration.

Consider the network shewn in Fig. 1-5

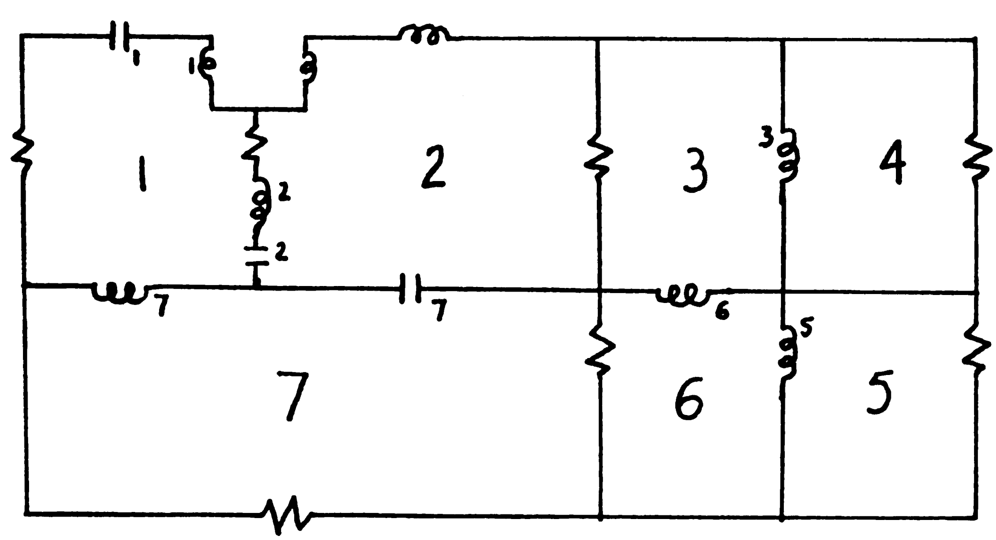



Fig. 1-5

We begin by following the contour of mesh number one. If it contains an inductance and a capacitance besides resistance we weight it two. If it contains only inductance or only capacitance besides resistance, we weight it one. If it contains only resistance, we weight it zero. If the contour contains several inductances or capacitances besides resistance the weight is also two except that in this case only one inductance and one capacitance need be checked off in order to give the mesh that weight.

In the same way we continue with the other meshes, bearing in mind that a coil or condenser that has already been checked off for a previous mesh does not count. Also, the presence of resistance is not essential unless the mesh contains only capacitance. Having thus weighted all meshes, the total number of modes ( the degree of the determinental equation ) equals the sum of all the weights.

The total weights for Fig. 1-5 add up to nine as shewn below :

| Mesh # | Weight |
|---|---|
| 1 | 2 |
| 2 | 2 |
| 3 | 1 |
| 4 | 0 |
| 5 | 1 |
| 6 | 1 |
| 7 | 2 |
| Total weights | 9 |

It would seem for an instant that we can apply this method indiscriminately to any two terminal network in order to obtain Foster's equivalent network. The following two examples will show that this is not the case.

Let us analyze Fig. 1-6

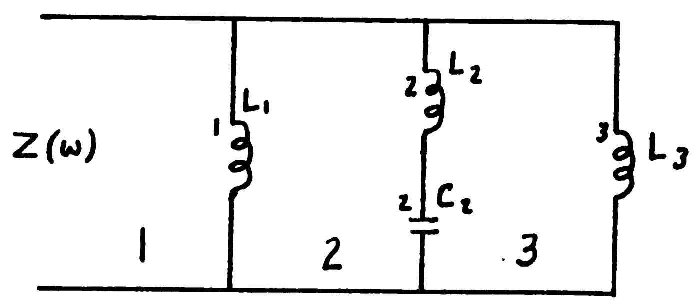



Fig. 1-6

It is readily seen that the total weights for the above network add up to four.

The mesh equations for the above network are

$$E = j\omega L_1 I_1 - j\omega L_1 I_2 \qquad\qquad 0$$

$$0 = -j\omega L_1 I_1 + \left[j\omega(L_1+L_2) + \frac{1}{j\omega C_2}\right] I_2 - \left[j\omega L_2 + \frac{1}{j\omega C_2}\right] I_3$$

$$0 = 0 - \left[j\omega L_2 + \frac{1}{j\omega C_2}\right] I_2 + \left[j\omega(L_2+L_3) + \frac{1}{j\omega C_2}\right] I_3$$

The driving-point impedance can be written as follows

$$Z_{11}(\omega) = \frac{\begin{vmatrix} j\omega L_1 & -j\omega L_1 & 0 \\ -j\omega L_1 & j\omega(L_1+L_2) + \frac{1}{j\omega C_2} & -(j\omega L_2 + \frac{1}{j\omega C_2}) \\ 0 & -(j\omega L_2 + \frac{1}{j\omega C_2}) & j\omega(L_2+L_3) + \frac{1}{j\omega C_2} \end{vmatrix}}{\begin{vmatrix} j\omega(L_1+L_2) + \frac{1}{j\omega C_2} & -(j\omega L_2 + \frac{1}{j\omega C_2}) \\ -(j\omega L_2 + \frac{1}{j\omega C_2}) & j\omega(L_2+L_3) + \frac{1}{j\omega C_2} \end{vmatrix}}$$

Expanding the two determinents and taking into account all the initial conditions assigned to the network we have

$$D(\omega) = \omega^4 L_1L_2L_3 - \omega^2 \frac{L_1L_3}{C}$$

hence the determinental equation is of the fourth degree, which corresponds also to the total weights of the network, and therefore may deduce that the least number of elements by means of which the network in Fig. 1-6 may be realized is eqyal to four. If we attempt a partial fraction expansion of $Z_{11}(\omega)$ however, we obtain the following

$$Z_{11}(\omega) = j\omega L_1 \left\{ 1 + \frac{A}{\omega^2 - \frac{L_1+L_3}{C_2 L}} \right\}$$

where $L = L_1L_2 + L_1L_3 + L_2L_3$

The above form of the driving-point reactance function shows that Foster's equivalent network is composed of two branches in series, the first is an inductor, the second is an inductor and a capacitor in parallel making a total of only three elements and hence there is a discrepency.

As another example consider Fig. 1-7

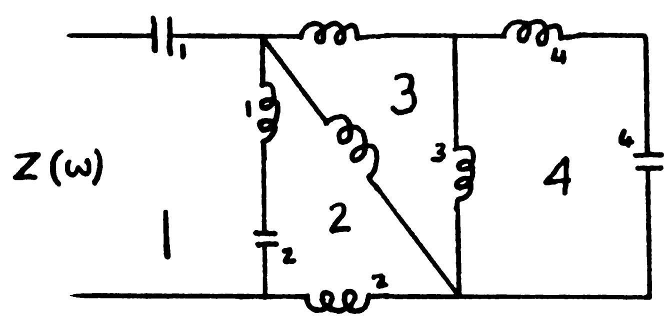



Fig. 1-7

In the above network the total weights add up to seven, but upon expanding the driving-point impedance function by partial fractions we will find that the equivalent network according to Foster is composed of only six elements.

Many other examples will show that while it is true that the number of modes (the degree of the determinental equation) of the equivalent network obtained by Foster's method is equal to the number of elements in that equivalent network, it is not always correct to assume that the least number of elements by means of which any two terminal network may be realized is equal to the degree of the determinental equation of the original network. Yet the method outlined earlier for determining the degree of the determinental equation could be applied to find the least number of elements by means of which any given two-terminal network may be realized according to Foster, if we follow this line of reasoning.

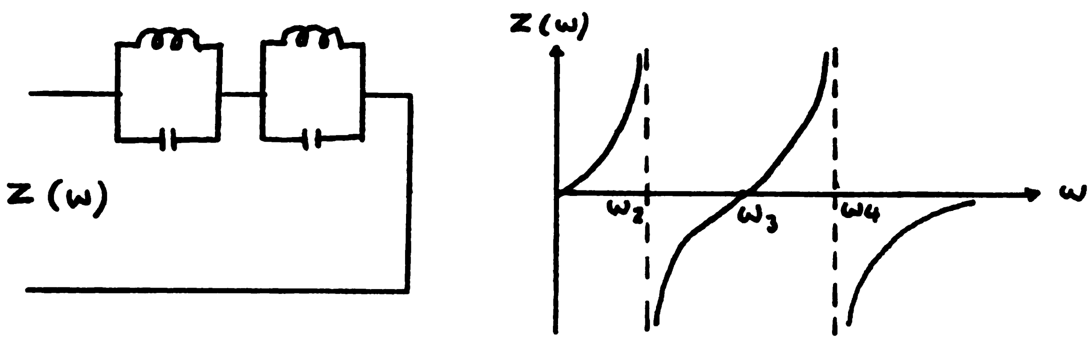
For a network having at least one independent inductor and one independent capacitor in each of its meshes, the total number of weights in that network is equal to the degree of the determinental equation and hence to the least number of elements in the equivalent network, and this number is equal to 2n, where n is the number of independent meshes. This condition is rather too narrow. In order to apply the method outlined earlier to any network, we should follow these steps:

1. Disregard any mesh that contains only inductors or only capacitors around its contour except the mesh from which the network is driven.
2. Number the remaining meshes each of which (except the first) should have at least one capacitor and one inductor (not necessarily independent) around its contour.
3. Count the total number of weights of the numbered meshes. If this number is even, then the elements are made of equal number of inductors and capacitors. If this number is odd, then the number of inductors should equal to one more or one less than the number of capacitors. Any discrepency is discarded.

## 1.5 Forms of the Possible Driving-Point Reactance Function According to Foster

The four different forms that the driving-point impedance can assume are given below :

a. If the network offers zero impedance at both zero and infinite frequencies, the total number of weights is an even number composed of equal number of inductors and capacitors as shewn in Fig. 1-8



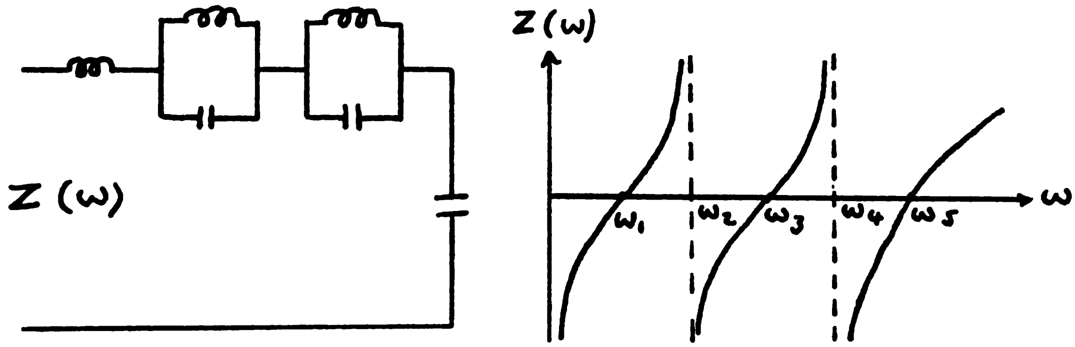

(a) (b)

Fig. 1-8

The mathematical expression for this function is

$$Z\ (\omega) = -j\omega H \frac{(\omega^2 - \omega_3^2)}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)} \qquad (6)$$

b. Should the network offer infinite impedance both at zero and infinite frequencies, the total number of weights is again even with the number of inductors equal to the number of capacitors. This is shewn in Fig. 1-9



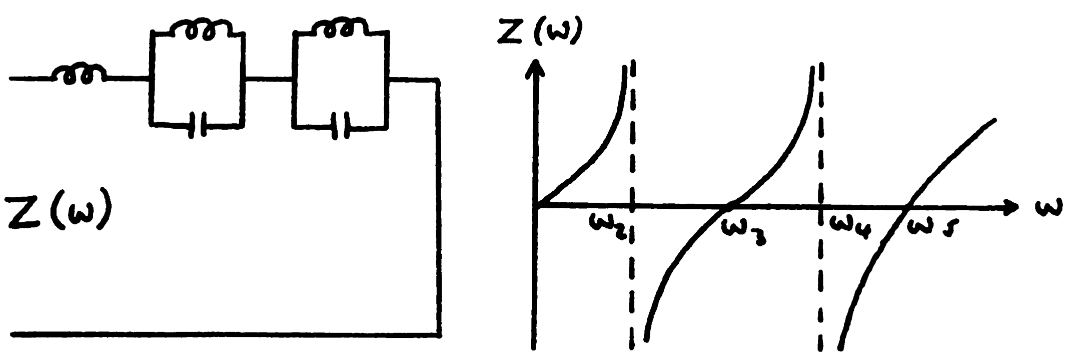

(a) (b)

Fig. 1-9

The mathematical expression for this condition is expressed by

$$Z\ (\omega) = j\omega H\ \frac{(\omega^2-\omega_1^2)(\omega^2-\omega_3^2)(\omega^2-\omega_5^2)}{(\omega^2-\omega_2^2)(\omega^2-\omega_4^2)} \qquad (7)$$

c. If a network offers zero impedance at zero frequency and infinite impedance at infinite frequency, we obtain the following



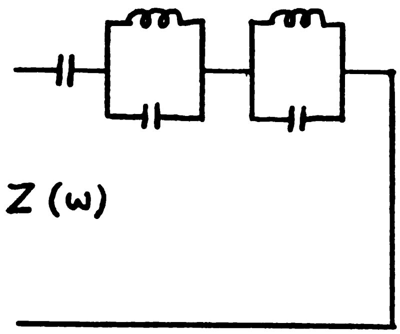

(a) (b)

Fig. 1-10

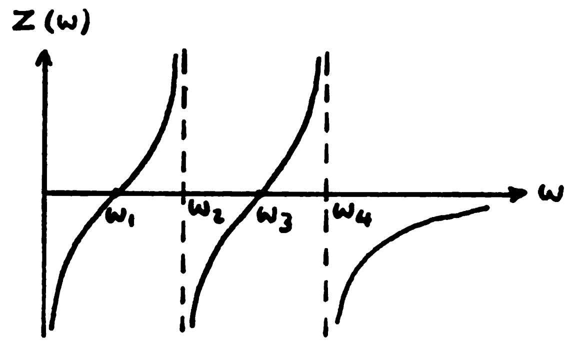
This condition is expressed by

$$Z\ (\omega) = j\omega H\ \frac{(\omega^2-\omega_3^2)(\omega^2-\omega_5^2)}{(\omega^2-\omega_2^2)(\omega^2-\omega_4^2)} \tag{8}$$

d. When the network offers infinite impedance at zero frequency and zero impedance at infinite frequency, the fourth possibility obtained is shewn below




(a)




(b)

Fig. 1-11

which is expressed by

$$Z\ (\omega) = -j\omega H\ \frac{(\omega^2-\omega_1^2)(\omega^2-\omega_3^2)}{\omega^2(\omega^2-\omega_2^2)(\omega^2-\omega_4^2)} \tag{9}$$

Before closing this discussion, it should be noted that the reactance functions discussed above may be realized physically in other fundamental forms than those given, but these other forms will not be presented here since this is outside the scope of this treatise.

# CHAPTER II

# ELECTRIC FILTERS

## 2.1 Fundamental Behavior of Filters

Electric wave filters are four terminal networks which discriminate between currents of different frequencies, transmitting those currents which lie within a certain range of frequencies and attenuating all others.

For our purposes here let us study the fundamental behavior of filters by presenting some examples.

Example 1

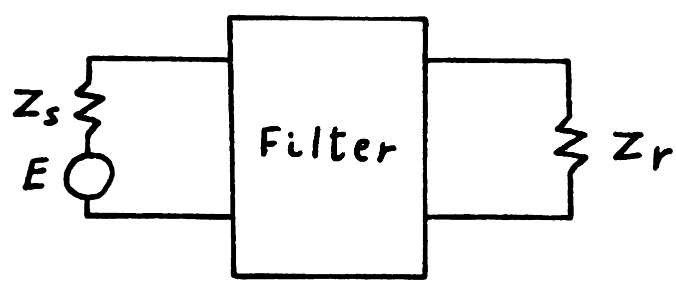
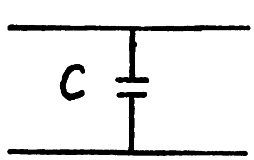
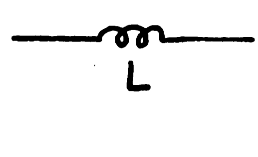



Fig. 2-1

Considering the terminating impedances $Z_s$ and $Z_r$ of Fig. 2-1 to be resistive, let us analyze the behavior of the filter when it takes the form of a shunt capacitor as shewn in Fig. 2-2(a).




(a)




(b)

Fig. 2-2

If $Z_s = 0$ and E is not changed by the current through the generator, it can be easily seen that the capacitor will have no effect on the voltage across $Z_r$. If $Z_s \neq 0$ and E is non-sinusoidal, we find that the same fraction of each harmonic of generated voltage appears across the load resistor $Z_r$ if the capacitor is not in the circuit. When the capacitor is in the circuit, we see that the voltage across $Z_r$ is the same as it was with the capacitor removed only for zero frequency while for all other frequencies the voltage across $Z_r$ is reduced, and as the harmonic order increases, it is increasingly effective in suppressing the reaction on the load of the generated voltage. Hence, this capacitor has less effect on lower freqyencies than on higher frequency harmonics and appears to pass the lower frequency effects of the generator more effectively.

Now let us replace the shunt capacitor by the series inductor appearing in Fig. 2-2(b). Here we find that at zero frequency the inductor offers no impedance and therefore does not affect the load action of a d-c component of the generated voltage, whereas at infinite frequency the series inductor offers infinite impedance and acts as an open circuit seperating the generator from the load as did the shunt circuit capacitor.

Thus, at zero and infinite frequencies the series inductor and shunt capacitor behave the same, both appear to pass the lower effects of the generator more effectively and therefore both are referred to as low-pass filters.

Example 2


(a) (b)

Fig. 2-3

By the same reasoning applied in example 1, we can see that a series capacitor and a shunt inductor as in Fig. 2-3 will have opposite effects to the shunt capacitor and series inductor.

At zero frequency the series capacitor offers infinite impedance, thus preventing any d-c component of generated voltage from appearing across the load while at infinite frequency it has no effect.

The shunt inductor acts as a short circuit at zero frequency, preventing any effect on the load while at infinite frequency it acts as an open circuit and has no effect on the load. Therefore both are referred to as high-pass filters.

Example 3

The four terminal network in Fig. 2-4(a) acts as a short circuit at the frequency of resonance and therefore when placed between the generator and the load in Fig. 2-1 will connect the generator directly to the load at that frequency while at zero and infinite frequencies the filter offers infinite impedance and the generator will be isolated from the load, or the circuit will be open.


(a) (b)

Fig. 2-3

At the frequency of resonance we also find that the network in Fig. 2-4(b) offers infinite impedance, thus connecting the generator directly to the load while at zero and infinite frequencies the inductor and the capacitor respectively short circuit the generator and therefore both networks are called band-pass filters.

Example 4


(a) (b)

Fig. 2-5

Finally consider the network in Fig. 2-5(a). Applying this four terminal network to Fig. 2-1, we will see that the generator will be disconnected from the load at the resonant frequency, and at both zero and infinite frequencies the generator will be connected directly to the load.

If the network in Fig. 2-5(b) was used instead, the generator will be shorted at the resonant frequency but will be across the load at zero and infinite frequencies and hence both networks of Fig. 2-5 are known as band-elemination filters..

## 2.2 Constant-K Filters

Most filters are designed as symmetrical T or symmetrical $\pi$ networks. Since T and $\pi$ networks can be made equivalent, it is immaterial which is used in this discussion.

Consider the following symmetrical T network

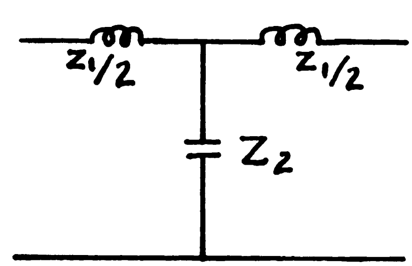



Fig. 2-6

This network is evidently a low-pass filter because it is constructed on a low-pass arrangement of inductors and capacitors.

From the discussion in the beginning of this chapter we recall that at zero frequency, the impedance of the series inductor was zero while the impedance of the shunt capacitor was infinite. Since in addition we find that at infinite frequency the series inductor offered infinite impedance but the shunt capacitor offered zero impedance, in other words, due to the fact that $Z_1$ had zeros when $Z_2$ had poles and vice versa, it follows from Foster's reactance theorem that the series inductor and the shunt capacitor are potentially reciprocal, and by proper choice of L and C, they can be made reciprocal with respect to any constant, say $k^2$ (where k is a real positive constant), hence the name "constant-k filters".

## 2.3 Determination of the Transmission and Attenuation Regions

Fig. 2-7 consists of a symmetrical T network connected between a generator and a load whose impedance $Z_r = Z_o$.

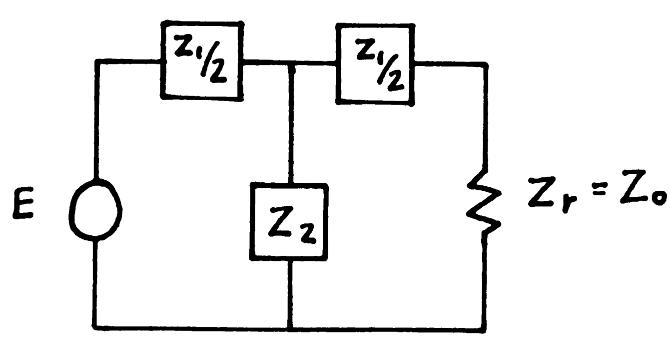



Fig. 2-7

The loop equations for the above network are

$$E = \left\{ \frac{Z_1}{2} + Z_2 \right\} I_1 - Z_2 I_2 \qquad (1)$$

$$0 = - Z_2 I_1 + \left\{ \frac{Z_1}{2} + Z_2 + Z_o \right\} I_2 \qquad (2)$$

Upon solving the above equations for $I_1$ a nd $I_2$, and defining $\frac{I_1}{I_2} = e^{\gamma}$ we obtain

$$\frac{I_1}{I_2} = \frac{\frac{Z_1}{2} + Z_2 + Z_o}{Z_2} = e^{\gamma} \qquad (3)$$

from which we get

$$\frac{Z_1}{2} = Z_2 \left( e^{\gamma} - 1 \right) - Z_o \qquad (4)$$

Substituting the value of $Z_o$ from equation (4) into the equation for the characteristic impedance of the T network

$$Z_o = \sqrt{Z_1 Z_2 + \frac{Z_1^{\,2}}{4}} \qquad (5)$$

gives

$$Z_2 \left(e^{\gamma} - 1\right)^2 - Z_1 e^{\gamma} = 0 \qquad (6)$$

Equation (6) can be simplified to

$$e^{2\gamma} - 2e^{\gamma} + 1 = \frac{Z_1}{Z_2} e^{\gamma} \qquad (7)$$

a nd after arranging terms we get

$$\frac{e^{\gamma} + e^{-\gamma}}{2} = 1 + \frac{Z_1}{2Z_2} \qquad (8)$$

or $$\cosh \gamma = 1 + \frac{Z_1}{2Z_2} \qquad (10)$$

where $$\gamma = \alpha + j\beta \qquad (10)$$

and therefore

$$\cosh \gamma = \cosh \alpha \cos \beta + j \sinh \alpha \sin \beta \qquad (11)$$

$\gamma$ is the propagation constant, $\alpha$ the attenuation constant and $\beta$ the phase constant.

Since $Z_1$ and $Z_2$ are pure reactances, their ratio is real and therefore equation (11) can be real only if the reactive term is equal to zero

$$\sinh \alpha \sin \beta = 0 \qquad (12)$$

and this relation is true if either $\sinh \alpha$ or $\sin \beta$ is equal to zero.

For $\sinh \alpha = 0$

$\alpha = 0$ and $\cosh \alpha = 1$

this condition gives

$$\cos \beta = 1 + \frac{Z_1}{2Z_2} \qquad (13)$$

and therefore

$$-1 < \left( 1 + \frac{Z_1}{2Z_2} \right) < 1$$

or

$$-1 < \cosh < 1 \qquad (14)$$

Since in this region the attenuation is zero, it must be a transmission region.

For $\sin\beta = 0$

$$\beta = \pm\pi \qquad \text{and } \cos\beta = \pm 1$$

When $\cos \;\; = + 1$ $\qquad \beta = 0$

and
$$\cosh\alpha = \left\{ 1 + \frac{Z_2}{2Z_2} \right\} > 1 \tag{15}$$

For $\cos\beta = -1$ $\qquad \beta = \pi$

and
$$\left\{ 1 + \frac{Z_1}{2Z_2} \right\} < -1 \tag{16}$$

The regions expressed by equations (15) and (16) are attenuation regions. Therefore transmission regions exist for

$$-1 < \cosh\gamma < 1$$

and outside this region attenuation exist.

The frequencies at which the network changes from a transmission to an attenuation region and vice versa are called cutoff frequencies. These frequencies occur such that

$$\cosh\gamma = \pm 1$$

Therefore
$$1 + \frac{Z_1}{2Z_2} = -1$$

gives
$$\frac{Z_1}{4Z_2} = -1 \quad \text{(as one cutoff condition)} \tag{17}$$

and
$$1 + \frac{Z_1}{2Z_2} = + 1$$

gives $\frac{Z_1}{4Z_2} = 0$ (as the second cutoff condition) (18)

Hence the transmission region is defined by

$$-1 \leq \frac{Z_1}{4Z_2} \leq 0 \qquad (19)$$

The transmission region can be expressed in a different form which will be useful in the following analysis.

From the relation

$$2 \sinh^2 \frac{X}{2} = \cosh X - 1$$

substitution in equation (9) yields

$$\sinh \frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} \qquad (20)$$

Let $\sqrt{\frac{Z_1}{4Z_2}} = \pm jX_k$ (21)

where $X_k$ may be complex.

For the ratio $\sqrt{\frac{Z_1}{4Z_2}}$ to be real, the expansion

$$\sinh \frac{\gamma}{2} = \cosh \frac{\alpha}{2} \sin \frac{\beta}{2} - j \sinh \frac{\alpha}{2} \cos \frac{\beta}{2}$$

requires that the imaginary part vanish. This may be so if either $\sinh \frac{\alpha}{2}$ or $\cos \frac{\beta}{2}$ is equal to zero.

For $\sinh \frac{\alpha}{2} = 0$

$\alpha = 0$ and $\cosh \frac{\alpha}{2} = 1$

hence $$\sin\frac{\beta}{2} = X_k \tag{22}$$

For $\cos\frac{\beta}{2} = 0$

$\beta = \pm\pi$ and $\sin\frac{\beta}{2} = \not\pm 1$

and $$\cosh\frac{\alpha}{2} = |X_k| \tag{23}$$

The transmission range can now be expressed

$$-1 \leq X_k \leq 1 \tag{24}$$

and the attenuation range by

$$|X_k| \geq 1 \tag{25}$$

In this last derivation, negative $\alpha$ was discarded because such values are impossible in a passive structure.

## 2.4 Design Procedure

We have already defined the constant-K filter to be a filter whose component impedances are reciprocal to a real positive constant. Let us write this statement in the form of an equation

$$Z_1 \; Z_2 = R^2 \tag{26}$$

If the characteristic impedance of the T network of Fig.2-1 is resistive and if the network is terminated with this characteristic impedance $Z_o$, it can be easily shewn that the input impedance will be equal to $Z_o$, and since the T network is purely reactive we can secure maximum power transfer. The nominal value of the characteristic impedance is R, defined by equation (26).

From equations (21) and (26) we obtain the following relation

$$X_k = \frac{Z_1}{2jR} \qquad (27)$$

and for reactive networks, it is seen that $X_k$ is a real function of frequency similar to $Z_1$.

The usual practice in the design of constant-K filters is to choose the value of the characteristic impedance R equal to the resistance the filter is to work into and out of. Next we need to design $Z_1$ only, for when this is determined, $Z_2$ can be evaluated by reciprocation. $Z_1$ is determined from equations (27) and (24). Equation (27) determines the cutoff frequencies which correspond to

$$Z_1 = \pm 2jR \qquad (28)$$

The point $X_k = 0$ corresponds to $Z_1 = 0$, i.e., to a zero for the reactance $Z_1$, therefore $Z_1$ must have as many zeros as the filter is to have transmission regions. Since a reactance function has either a zero or a pole at the origin and at infinity, that is, since $Z_1$ cannot have a finite non-zero value at either of the extremities of the frequency range, it follows that a pass-band starting at the origin or terminating at infinity requires that $Z_1$ have a zero at the origin or at infinity respectively.

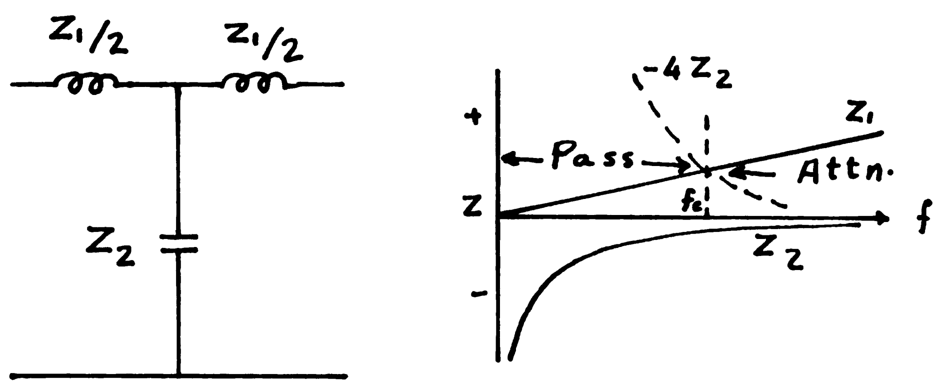
These regions are called external, to distinguish them from internal regions for which both boundaries lie at finite non-zero frequencies. Thus $Z_1$ must have one zero for each internal pass-band and a zero at the origin or at infinity for an external pass-band according to whether the latter is located at the origin or at infinity respectively. This determines the structure of $Z_1$.

Any reactance function is determined by specifying the locations of its internal poles and zeros plus one additional information which may be the value of the reactance at one other frequency. That is, the number of determining factors is equal to the number of internal zeros, plus the number of internal poles, plus one. This number equals the least number of elements by means of which the reactance function may be realized according to Foster's reactance theorem, and from the above discussion we see that this number coincides with the number of cutoff frequencies or boundaries between transmission and attenuatuon regions.

## 2.5 Examples on the Design of Constant-K Filters

### a. Low-pass Filters

In article 2.2 we saw that a symmetrical T network constructed on a low-pass arrangement of inductors and capacitors may be used as a low-pass filter, Fig. 2-8(a).




(a) (b)

Fig. 2-8

The equation expressing the transmission region

$$-1 \leq \frac{Z_1}{4Z_2} \leq 1 \qquad (18)$$

is demonstrated by Fig. 2-8(b).

The cutoff frequency is the point of intersection of $Z_1$ and $-4Z_2$ since at that point $Z_1 = -4Z_2$. Hence a transmission range starts at the frequency at which $Z_1 = 0$ and ends at the frequency at which $Z_1 = -4Z_2$, or the transmission range extends from $f = 0$ to $f = f_c$ and the frequencies above $f_c$ lie in an attenuation range.

It is seen however from Fig. 2-8(b) that the network (a) does not discriminate sharply between frequencies above and below $f_c$, the response changes gradually. In the ideal case it is desired that the transition between the transmission and attenuation regions be abrupt. In practice, this ideal condition can be approached, for a sharper frequency discrimination can be obtained by cascading two or more identical sections and connecting them between the generator and the load. This form of network is called a ladder-type filter. Still another approach to the ideal case may be had by making use of the m-derived type of filters the analysis of which will be given in article 2-6.

In Fig. 2-8(a) we have

$$Z_1 Z_2 = K^2 \tag{29}$$

or
$$Z_1 Z_2 = j\omega L . \frac{1}{j\omega C} = \frac{L}{C} = R^2 \tag{30}$$

where $R = \sqrt{\frac{L}{C}}$ was defined as the nominal value of the characteristic impedance.

The characteristic impedance of a T network is

$$Z_o = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right)}$$

or
$$Z_o = \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2 LC}{4}\right)} \tag{31}$$

Making use of the relation $Z_1 = -4Z_2$ we obtain

$$f_c = \frac{1}{\pi \sqrt{LC}} \tag{32}$$

therefore
$$Z_o = \sqrt{\frac{L}{C}\left\{1 - \left(\frac{f}{f_c}\right)^2\right\}} \tag{33}$$

or
$$Z_o = R\sqrt{1 - x^2} \tag{34}$$

where $$x = \frac{f}{f_c} \qquad (35)$$

Equation (34) shows that for frequencies below cutoff, the characteristic impedance is a pure resistance varying from $R = \frac{L}{C}$ for low frequencies to zero for $f = f_c$, and for frequencies above cutoff the characteristic impedance is a pure imaginary. The variations of the characteristic impedance with frequency are illustrated in Fig. 2-9.

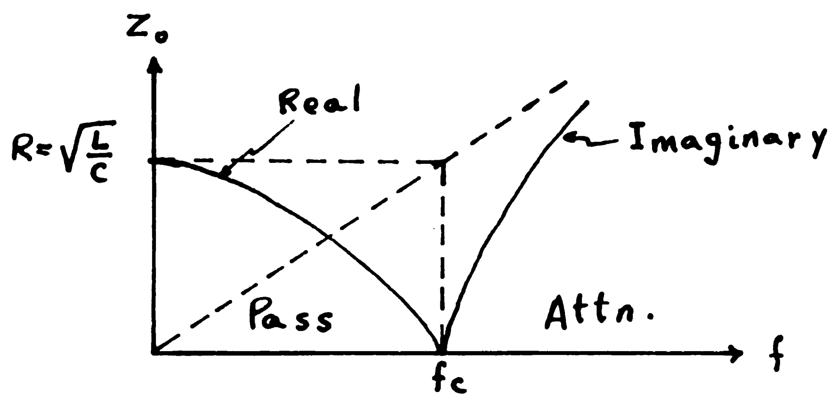



Fig. 2-9

Applying equations (22) and (23) to the network treated here, we obtain the following conditions

$$\left.\begin{aligned} \alpha &= 0 \\ \beta &= 2\sin^{-1}\left(\frac{f}{f_c}\right) \end{aligned}\right\} \quad (0 \leq f \leq f_c) \qquad (36)$$

and

$$\left.\begin{aligned} \alpha &= 2\cosh^{-1}\left(\frac{f}{f_c}\right) \\ \beta &= \pi \end{aligned}\right\} \quad (f_c \leq f < \infty) \qquad (37)$$

The above two conditions are demonstrated in Fig. 2-10

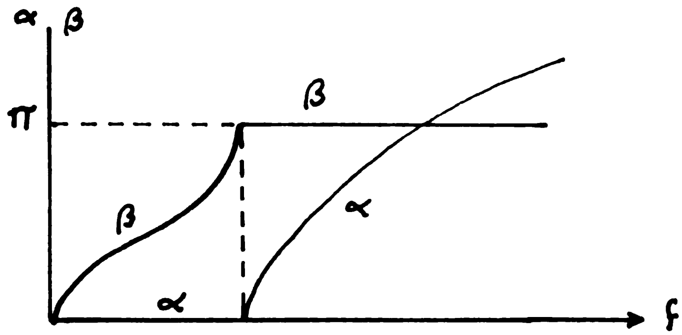
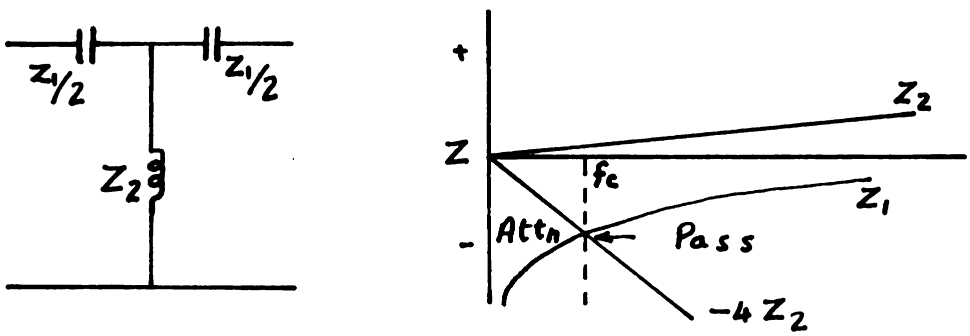



Fig. 2-10

b. High-Pass Filters




(a) (b)

Fig. 2-11

Fig. 2-11(a) is a high-pass filter because it is constructed on a high-pass arrangement of inductors and capacitors. This condition is illustrated by Fig. 2-11(b).

The characteristic impedance for this network is

$$Z_o = R\sqrt{1 - \frac{1}{x^2}} \qquad (38)$$

and the variations of this equation with frequency are illustrated in Fig. 2-12 below

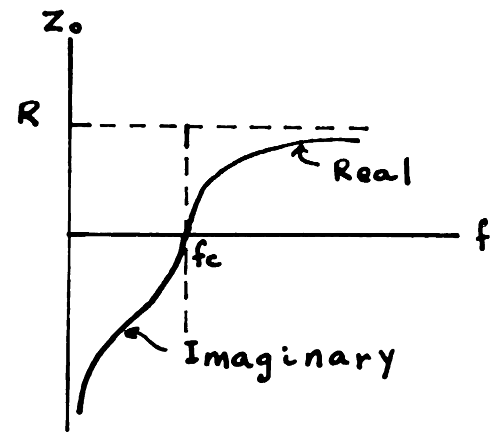



Fig. 2-12

The variations of the attenuation and phase characteristics are

$$\left.\begin{aligned} \alpha &= 2\cosh^{-1}\left(\frac{f_c}{f}\right) \\ \beta &= -\pi \end{aligned}\right\} \qquad (0 < f < f_c) \qquad (39)$$

$$\left.\begin{aligned} \alpha &= 0 \\ \beta &= -2\sin^{-1}\left(\frac{f_c}{f}\right) \end{aligned}\right\} \qquad (f_c < f < \infty) \qquad (40)$$

These conditions are illustrated in Fig. 2-13

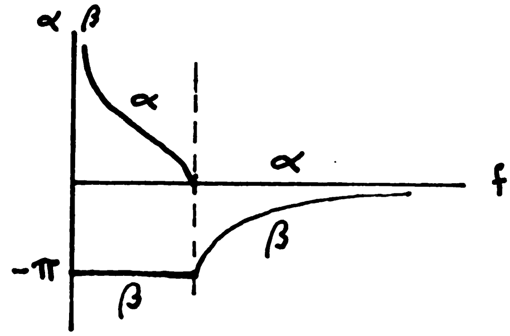



Fig. 2-13

c. Band-Pass Filters

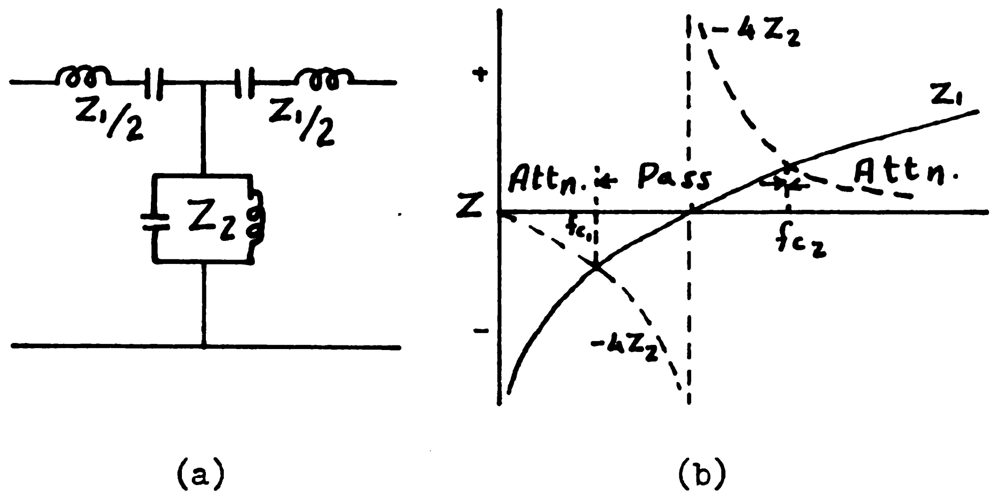



Fig. 2-14

In the above network

$$Z_1 = j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1}\right) \tag{41}$$

and

$$Z_2 = \frac{j\omega L_2}{1-\omega^2 L_2 C_2} \tag{42}$$

therefore

$$Z_1 Z_2 = R^2 = -\frac{L_2}{C_1}\frac{(\omega^2 L_1 C_1 - 1)}{(1-\omega^2 L_2 C_2)} \tag{43}$$

If the anti-resonant frequency of the shunt arm is made to correspond to the resonant frequency of the series arm, then

$$\omega_o^2 L_1 C_1 = \omega_o^2 L_2 C_2 \tag{44}$$

or

$$L_1 C_1 = L_2 C_2 \tag{45}$$

and hence

$$Z_1 Z_2 = \frac{L_2}{C_1} = \frac{L_1}{C_2} = R^2 \tag{46}$$

At the cutoff frequencies

$$Z_1 = -4Z_2$$

multiplying both sides of the above equation by $Z_1$ gives

$$Z_1{}^2 = -4Z_1Z_2 = -4R^2$$

or

$$Z_1 = \pm 2jR$$

so that

$Z_1$ at lower cutoff frequency $= -Z_1$ at upper cutoff frequency

The variations of the characteristic impedance with frequency are shewn in Fig. 2-15 and given by equation (47)

$$Z_0 = \frac{R\sqrt{(\omega_2^2-\omega^2)(\omega^2-\omega_1^2)}}{\omega\,(\omega_2-\omega_1)} \qquad (47)$$

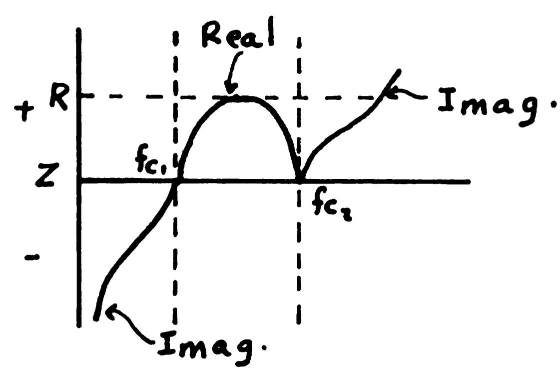



Fig. 2-15

The attenuation and phase characteristics are

$$\left.\begin{aligned} \alpha &= 2\cosh^{-1}\frac{\omega^2-\omega_0^2}{\omega(\omega_2-\omega_1)} \\ \beta &= \pi \end{aligned}\right\} \quad (f_{c2} < f < \infty) \qquad (48)$$

$$\left.\begin{aligned} \alpha &= 2\cosh^{-1}\frac{\omega_0^2-\omega^2}{\omega(\omega_2-\omega_1)} \\ \beta &= -\pi \end{aligned}\right\} \quad (0<f<f_{c1}) \qquad (49)$$

and

$$\left.\begin{aligned} \alpha &= 0 \\ \beta &= 2\sin^{-1}\frac{\omega^2-\omega_0^2}{\omega(\omega_2-\omega_1)} \end{aligned}\right\} \quad (f_{c1}<f<f_{c2}) \qquad (50)$$

The above variations are illustrated in Fig. 2-16

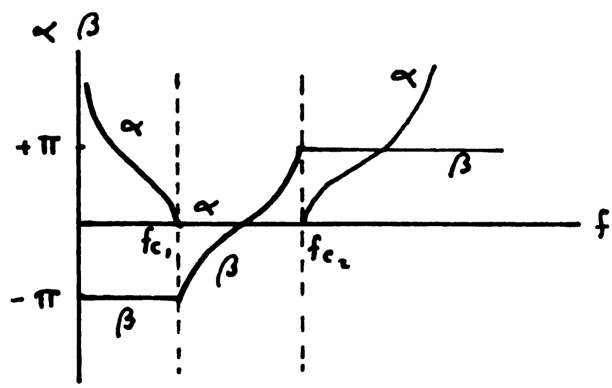
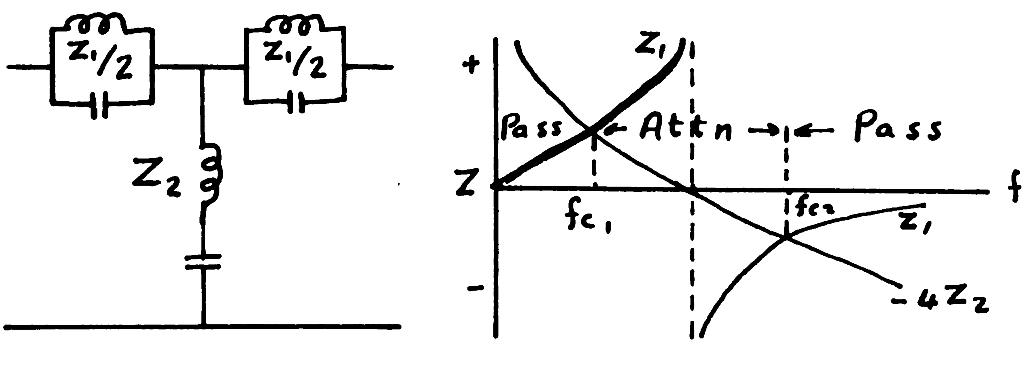



Fig. 2-16

## d. Band-Elimination Filters

This filter is obtained by interchanging the series and shunt arms of Fig. 2-14(a), as illustrated in Fig. 2-17




(a) (b)

Fig. 2-17

At the cutoff frequencies

$$Z_1 = -4Z_2$$

and

$$Z_2 = \pm \frac{jR}{2}$$

Equation (51) and Fig. 2-18 give the variations of the characteristic impedance with frequency

$$Z_o = \frac{R\sqrt{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)}}{(\omega_o^2-\omega^2)} \qquad (51)$$

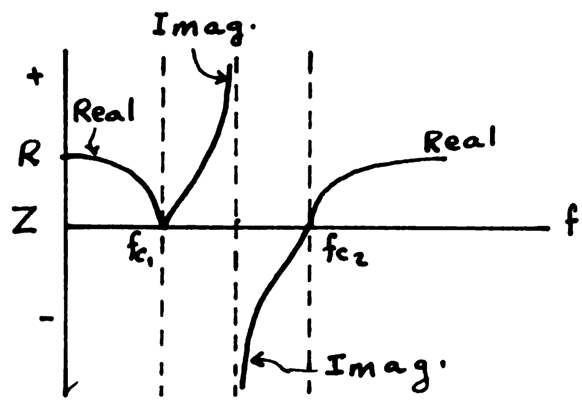



Fig. 2-18

The attenuation and phase characteristics are given below

$$\alpha = \infty \qquad f = f_o \qquad (52)$$

$$\left.\begin{aligned} \beta &= -\pi \\ \alpha &= 2\cosh^{-1}\frac{\omega(\omega_2-\omega_1)}{\omega_o^2-\omega^2} \end{aligned}\right\} \quad (f_o < f < f_{c2}) \qquad (53)$$

$$\left.\begin{aligned} \alpha &= 0 \\ \beta &= 2\sin^{-1}\frac{\omega(\omega_2-\omega_1)}{\omega_0^2-\omega^2} \end{aligned}\right\} \quad (f_{c2} < f < \infty) \qquad (54)$$

$$\left.\begin{aligned} \alpha &= 0 \\ \beta &= 2\sin^{-1}\frac{\omega(\omega_2-\omega_1)}{\omega_0^2-\omega^2} \end{aligned}\right\} \quad (0 < f < f_{c1}) \qquad (55)$$

$$\left.\begin{aligned} \beta &= \pi \\ \alpha &= 2\cosh^{-1}\frac{\omega(\omega_2-\omega_1)}{\omega_0^2-\omega^2} \end{aligned}\right\} \quad (f_{c1} < f < f_0) \qquad (56)$$

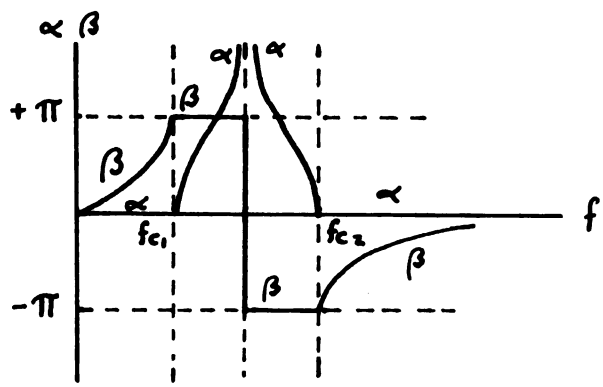



Fig. 2-19

## 2.6 The m-Derived T Section

We recall from the previous article that the constant-K filters do not discriminate sharply between frequencies above and below cutoff, and that the characteristic impedance is not constant over the pass-band so that a satisfying impedance match is not possible.

Where impedance matching is not important, the attenuation near cutoff may be built up by cascading several constant-K sections and in this case, if every section had the same characteristic impedance, the sections remain matched at all frequencies. The propagation constant will assume the value $n\gamma$, where n is the number of sections in cascade. The characterssstic impedance however still does not improve.

A more economical way than cascading several constant-K sections to improve the attenuation near cutoff is by use of the so-called m-derived filter which will now be discussed.

Referring to the constant-K low-pass filter shewn in Fig. 2-8(a), let us derive from this one an improved filter that will have the same allocation of attenuation and transmission regions, and that will have the same characteristic impedance.

Let us indicate the series and shunt arms of the derived filter by $Z_1'$ and $Z_2'$ respectively.

Since the derived filter in question is a low-pass filter, the series arm must remain to be an inductor, either a fraction of, equal to, or a multiple of $Z_1$ of the constant-K filter and hence let us assume

$$Z_1' = mZ_1 \tag{57}$$

m to be determined. In order that the characteristic impedances of the derived and prototype sections be equal, the following must be true

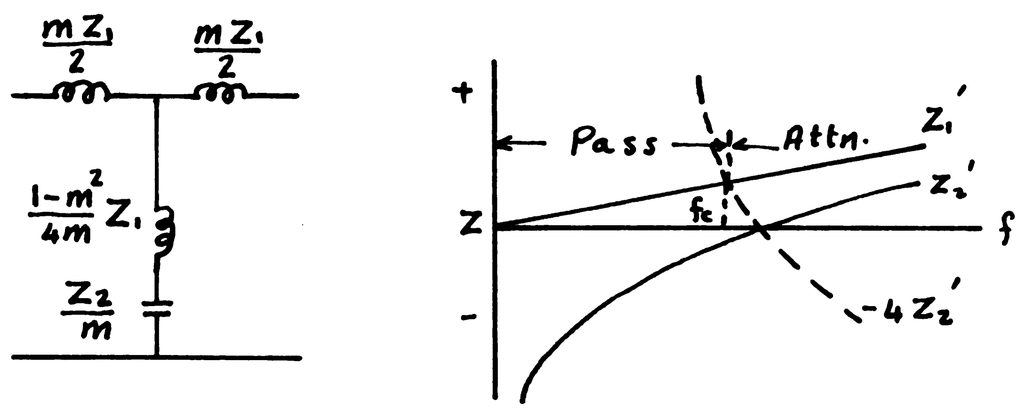
$$\frac{(mZ_1)^2}{4} + mZ_1Z_2' = \frac{Z_1^2}{4} + Z_1Z_2 \qquad (58)$$

from which

$$Z_2' = \frac{Z_2}{m} + \frac{1-m^2}{4m} Z_1 \qquad (59)$$

Equation (59) shows that the shunt arm of the derived network is made of an inductor and a capacitor in series having impedances equal to $\frac{1-m^2}{4m} Z_1$ and $\frac{Z_2}{m}$ respectively.

The m-derived low-pass filter and the reactance curve demonstrating its performance are shewn below




(a) (b)

Fig. 2-20

The shunt arm in Fig. 2-20(a) is seen to be a series resonant circuit with resonance above cutoff. At this frequency the shunt arm appears as a short circuit and the attenuation becomes infinite and by making the frequency at the point of infinite attenuation ($f_\infty$) close to the cutoff frequency ($f_c$), the attenuation near cutoff can be made high.

Since $Z_2$ in the constant-K filter is opposite in sign to $Z_1$, the same relation should remain in the two series impedances of the shunt arm of the m-derived filter. This can be true if $\frac{1-m^2}{4m}$ in equation (59) is positive, therefore $(1-m^2)$ and m must be positive, thus giving the limits on the value of m

$$0 < m < 1 \tag{60}$$

We have seen in Fig. 2-20 that the frequency of resonance occurs above cutoff, this follows from the requirement that below $f_c$ the shunt arm appears capacitive. Therefore at the resonant frequencu

$$\left| \frac{Z_2}{m} \right| = \left| \frac{1-m^2}{4m} \; Z_1 \right| \tag{61}$$

and for the low-pass filter we get

$$\frac{1}{2\pi f_\infty mC} = \frac{1-m^2}{4m} \; 2\pi f_\infty L \tag{62}$$

hence

$$f_\infty = \frac{1}{\pi \sqrt{(1-m^2)\ LC}} \tag{63}$$

Since the cutoff frequency for the low-pass filter is

$$f_c = \frac{1}{\pi \sqrt{LC}} \tag{64}$$

the frequency of infinite attenuation will be

$$f_\infty = \frac{f_c}{\sqrt{(1-m^2)}} \tag{65}$$

from which

$$m = \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \tag{66}$$

From equation (66) m may be found for any specified $f_\infty$.

The variation of attenuation over the stop band for a low-pass m-derived filter may be found from the relation

$$\cosh \gamma = 1 + \frac{Z_1}{2Z_2} \tag{9}$$

Following the same procedure applied in the derivation of the constant-K filter we find that in an attenuation region

$$\cos \beta = \pm 1$$

When $\cos \beta = + 1$, $\beta = 0$

and

$$\cosh \alpha = 1 + \frac{Z_1}{2Z_2}$$

$$= 1 - \frac{\omega m L}{2\left[\frac{1}{\omega m C} - \omega L \left(\frac{1-m^2}{4m}\right)\right]}$$

$$= 1 - \frac{\omega^2 m^2 LC}{2\left[1 - \omega^2 LC \left(\frac{1-m^2}{4}\right)\right]}$$

Applying the relations (61), (66), and (64) to the above we get

$$\cosh\alpha = 1 - \frac{\frac{2f^2}{f_c^2}\, m^2}{1 - \frac{f^2}{f_c^2}\,(1-m^2)} \qquad (67)$$

When $\cos\beta = -1$, $\beta = \pi$

and

$$\cosh\alpha = \frac{\frac{2f^2}{f_c^2}\, m^2}{1 - \frac{f^2}{f_c^2}\,(1-m^2)} - 1 \qquad (68)$$

and when $\alpha = 0$, $\cosh\alpha = 1$

and

$$\cos\beta = 1 + \frac{Z_1}{2Z_2}$$

$$= 1 - \frac{\frac{2f^2}{f_c^2}\, m^2}{1 - \frac{f^2}{f_c^2}\,(1-m^2)} \qquad (69)$$

The above conditions are demonstrated in Fig. 2-21 for $m = 0.6$

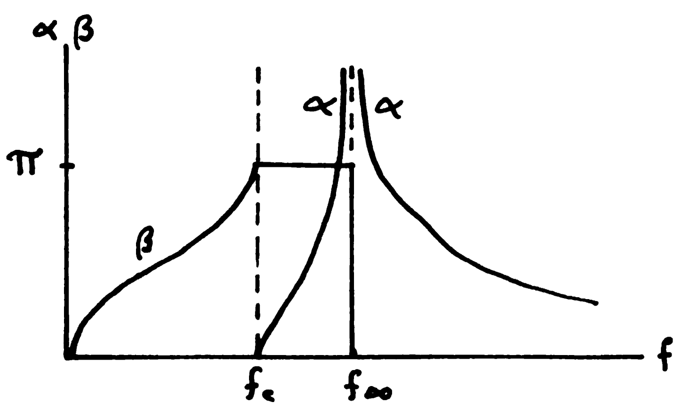



Fig. 2-21

Fig. 2-21 shows thatinfinite attenuation is achieved at $f_\infty$ but that the attenuation above $f_\infty$ falls to low values. If high attenuation is desired over the whole attenuation band it is necessary to use the above section in series with a prototype section to provide high attenuation at frequencies much higher than cutoff. Such a combination of networks is known as a composite filter.

A condition that was imposed on the m-derived filter was that its characteristic impedance be equal to that of the prototype. The proof is as follows

$$Z_{om} = \sqrt{Z_{oc}Z_{sc}}$$

$$= \sqrt{\left(\frac{mZ_1}{2} + \frac{1-m^2}{4m} Z_1 + \frac{Z_2}{m}\right)\left\{\frac{mZ_1}{2} + \frac{\frac{mZ_1}{2}\left(\frac{1-m^2}{4m} Z_1 + \frac{Z_2}{m}\right)}{\frac{mZ_1}{2} + \frac{1-m^2}{4m} Z_1 + \frac{Z_2}{m}}\right\}}$$

$$= \sqrt{Z_1Z_2 + \frac{Z_1^2}{4}} = Z_{OT} \qquad (70)$$

Applying the transformation relations developed above to the other three prototype networks lead to m-derived networks. Since the procedure is the same, these networks will not be derived here and the reader is referred to the references listed in the bibliography for immediate reference.

We recall that the characteristic impedance of the constant-K filter was not constant over the pass band, the same can also be said for the m-derived filter since $Z_{om} = Z_{OT}$ .

To improve the response of the characteristic impedance,i.e., to make it nearly constant over the transmission range, half sections of the m-derived filters are used. The characteristic impedance for one end of this network can be made to have the same value as that of the constant-K filter and the characteristic impedance of the other end will have nearly the desired properties.

## 2.7 Termination with m-Derived Half Sections

Here we will be concerned with the design of an L section (half section) such that it changes its characteristics with frequency in such a way that the filter will be approximately matched to its load at all frequencies over the transmission range.

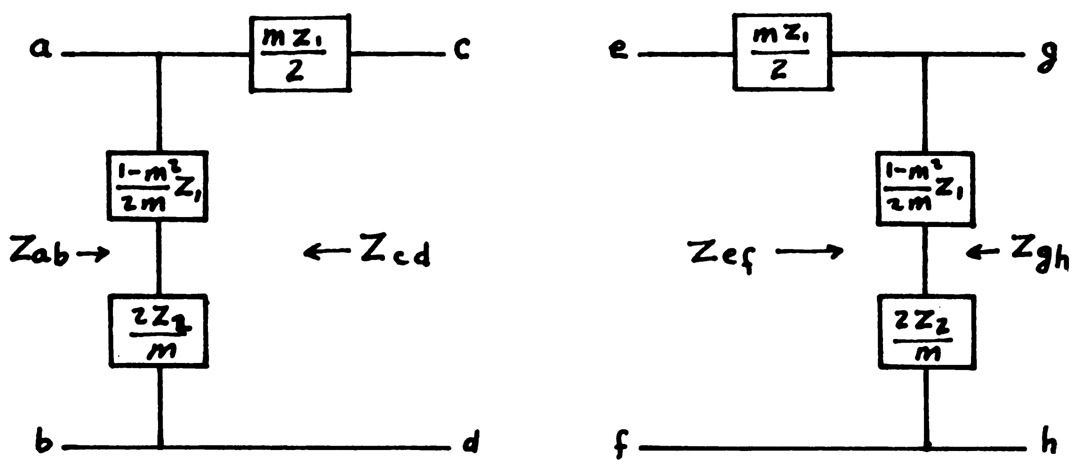



Fig. 2-22

Referring to the inverted L sections of an m-derived T section in Fig. 2-22 and solving for $Z_{ab}$ we get

$$Z_{ab} = \sqrt{Z_{aboc}\, Z_{absc}}$$

$$= \sqrt{\frac{\left(\frac{1-m^2}{2m} Z_1 + \frac{2Z_2}{m}\right)^2 \frac{mZ_1}{2}}{\frac{1-m^2}{2m} Z^1 + \frac{2Z_2}{m} + \frac{mZ_1}{2}}}$$

$$= \left[1 + (1-m^2)\frac{Z_1}{4Z_2}\right] \sqrt{\frac{Z_1 Z_2}{1 + \frac{Z_1}{4Z_2}}}$$

$$= \left[1 + (1-m^2)\frac{Z_1}{4Z_2}\right] Z_{o\pi} \qquad (71)$$

and hence $Z_{ab}$ is found to be a function of $Z_{o\pi}$ modified by a value which varies with m. For the low-pass filter equation (71) becomes

$$Z_{ab} = \frac{R\left[1- (1-m^2)\ \frac{f^2}{f_c^{\,2}}\right]}{\sqrt{1- \frac{f^2}{f_c^{\,2}}}} \qquad (72)$$

The values of $Z_{ab}$ are plotted below for several values of m

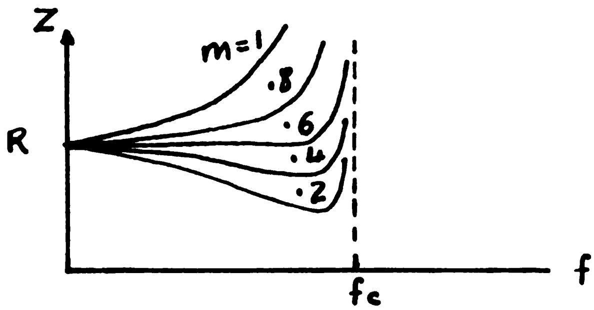



Fig. 2-23

Fig. 2-23 shows that by using the value m=0.6 for the L section, a nearly constant value of $Z_{ab}$ equal R is obtained for most of the pass band.

The image impedance at the terminals c,d in Fig. 2-22 is found as follows

$$\begin{aligned} Z_{cd} &= \sqrt{Z_{cdoc}\, Z_{cdsc}} \\ &= \sqrt{\left(\frac{mZ_1}{2} + \frac{1-m^2}{2m} Z_1 + \frac{2Z_2}{m}\right) \frac{mZ_1}{2}} \\ &= \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right)} = Z_{0T} \qquad (73) \end{aligned}$$

Similarly we find that $Z_{ef} = Z_{0T}$ and $Z_{gh} = Z_{ab}$. Therefore a generator of internal impedance R may be connected to terminals a,b and a load of value R to terminals g,h and between the c,d and e,f terminals a constant-K and an m-derived T sections designed for a value R may be inserted and obtain a satisfactory match over the largest range of the transmission band and also obtain maximum power transfer. The characteristic impedance will be nearly constant and equal to the value R except near cutoff. These half sections are referred to as End or Terminating half-sections.

The preceding illustrates the advantages of the m-derived sections especially when cascaded with a prototype section. High attenuation at other frequencies in the attenuation band may still be obtained by cascading as many m-derived sections as required.

A further improvement however can be achieved by derivation of another m-section from the first m-section just as the latter was derived from the prototype. These sections are called "double m-derived" or "mm-derived" sections. The procedure is similar to that used in deriving the m-derived from the prototype section and therefore will not be repeated here since this is not the purpose of this discussion.

## 2.8 Example

The following example will illustrate the advantages of cascading a prototype section and an m-derived section terminated with half sections, the whole network being a composite low-pass filter.

Design a composite low-pass filter with a 2000 ohms resistance termination. A cutoff frequency of 3000 cycles and very high attenuation at 3840, 5000, and $\infty$ cycles are required.

First we proceed with the design of the prototype. The cutoff frequency for a constant-K filter is that at which $Z_1 = -4Z_2$ and for a low-pass filter this becomes

$$\omega_c L = \frac{4}{\omega_c C}$$

or

$$\pi^2 f_c^{\,2} LC = 1 \qquad (74)$$

but

$$R = \sqrt{\frac{L}{C}}$$

and hence $$L = R^2C$$

substituting this value of L in equation (74) gives for the value of the shunt capacitor

$$C = \frac{1}{\pi f_c R}$$

By a similar approach we find

$$L = \frac{R}{\pi f_c}$$

Therefore for the prototype section

$$L = 2\frac{R}{\pi f_c} = \frac{2000}{\pi \times 3000} = 0.213 \text{ henry}$$

and $$C = \frac{1}{\pi f_c R} = \frac{1}{\pi \times 3000 \times 2000} = 0.053 \text{ mfd}$$

This section is shewn in Fig. 2-24

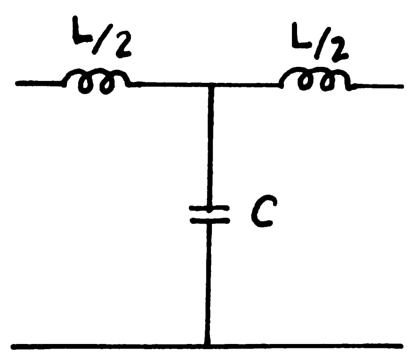



Fig. 2-24

The m-derived section providing high attenuation at 3840 cycles will have a value of m given by the equation

$$m = \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$= \sqrt{1-\left(\frac{3000}{3840}\right)^2}$$

$$= \sqrt{1-0.64} = 0.6$$

Therefore this section may be used for the terminating half sections, for we saw earlier that terminating half sections using the value m= 0.6 provide a nearly constant value of image impedance equal to R over about 85 per cent of the pass band. The component values for the half sections are then

$$\frac{mL}{2} = \frac{0.6 \times 0.213}{2} = 0.0639 \text{ henry}$$

$$\frac{1-m^2}{2m}L = \frac{1-0.36}{1.2} \times 0.213 = 0.114 \text{ henry}$$

and

$$\frac{mC}{2} = \frac{0.6 \times 0.053}{2} = 0.0159 \text{ mfd}$$

These end sections are shewn below

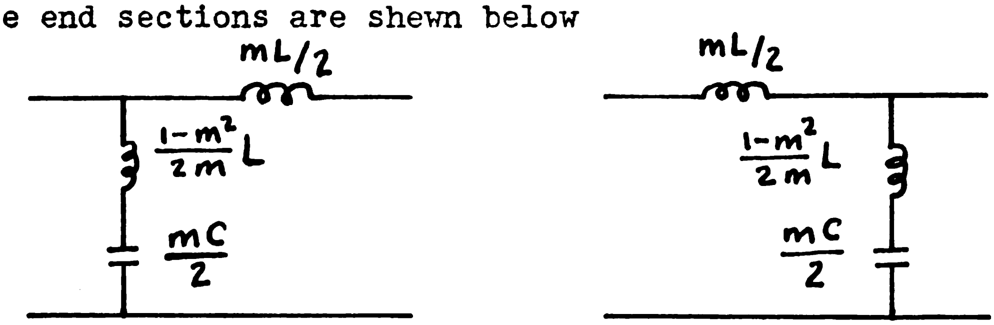



Fig. 2-25

To achieve high attenuation at 5000 cycles the m'-derived section requires an m' having the value

$$m' = \sqrt{1-\left(\frac{3000}{5000}\right)^2}$$

$$= \sqrt{1-0.36} = 0.8$$

The component values for the m'-derived section are

$$\frac{m'L}{2} = \frac{0.8 \times 0.213}{2} = 0.0852 \text{ henry}$$

$$\frac{1-m'^2}{4m'}L = \frac{1-0.36}{3.2} \times 0.213 = 0.0426 \text{ henry}$$

and $$m'C = 0.8 \times 0.053 = 0.0424 \text{ mfd}$$

The composite filter thus derived is shewn below

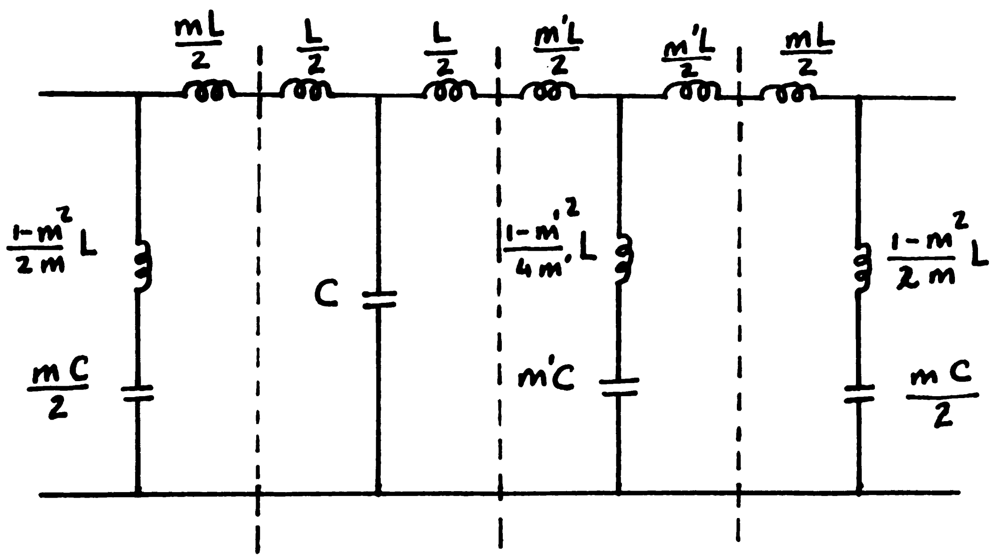



Fig. 2-26

Of course the series arms between any two sections in Fig. 2-26 may be combined to form one physical inductor.

The attenuation of each section and of the whole composite network of Fig.2-26 are shewn below

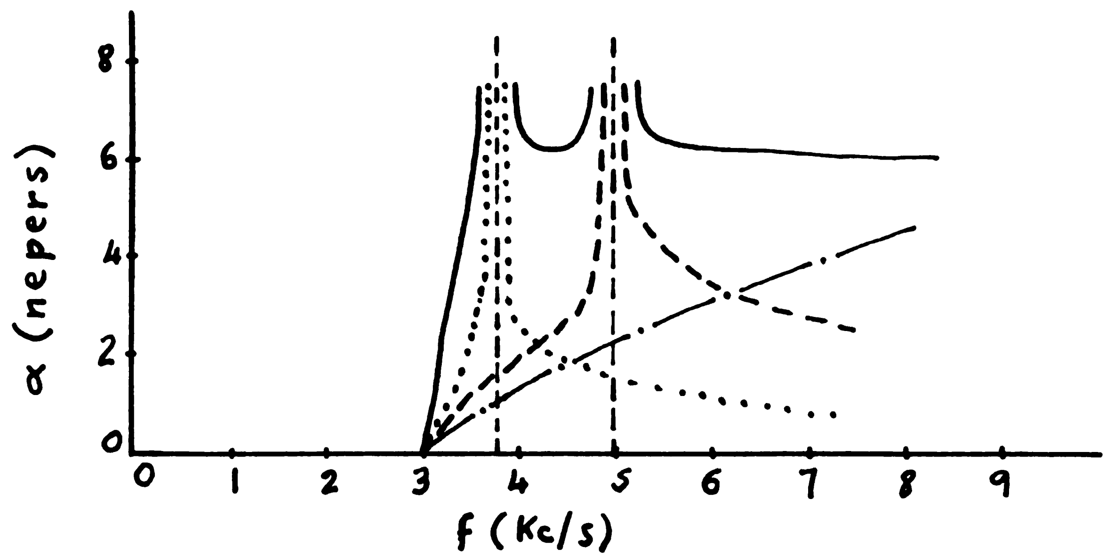



Fig. 2-27

In the above curves:

dash-dot curve = response of constant-K mid-section

dashed curve = response of m-derived mid-section

dotted curve = response of m-derived terminations

solid curve = response of composite filter.

## 2.9 Conclusion

In the examples given in a rticle 2-5 the application of Foster's reactance theorem in the design of the reactance arms of the constant-K filter networks was illustrated. In designing the series arm for example, $Z_1$ must have as many zeros as the filter is to have transmission regions, or one zero for each internal pass band and a zero at the origin or at infinity for an external pass band as the case may be. Thus a series arm of any form may be reduced by means of Foster's reactance theorem to an equivalent network containing the least number of elements as discussed in Chapter I. With $Z_1$ determined, $Z_2$ is its reciprocal with respect to $R^2$ .

The procedure in the design of the reactance arms of the m-derived filter follows the same as that of the constant-K filter in so far as the application of the reactance theorem is concerned. However, the shunt arm in the m-derived section is not the reciprocal of the series arm, it is rather derived from it as discussed earlier.

Therefore, the arms of a constant-K and an m-derived filter do contain the least number of elements according to Foster, for they were designed as two-terminal networks.

We should wonder at this point whether or not a composite filter may still be reduced to contain less number of elements. We have seen earlier that each section in the composite network contains least number of elements, and that we can use either the constant-K or the m-derived section to give us the required pass and stop bands, the cutoff point and the characteristic impedance. The reason for cascading these sections however was to build up the attenuation in the stop band at the cutoff and higher frequencies, for each m-derived section can be designed to boost the attenuation at some frequency above cutoff, and hence the composite network in the example of article 2-8 does contain the least number of elements designed for a cutoff frequency of 3000 cycles and to have high attenuation at 3840, 5000, and cycles, and we know that that network can be reduced to a single section, a prototype for example, and still have the same allocation of attenuation and transmission regions, cutoff frequency and characteristic impedance, but of course they will not be as high as desired at frequencies above cutoff in the stop band as offered by the composite network, neither will the characteristic impedance be nearly constant over the transmission range. Therefore, to achieve these improvements we cascade a prototype with as many m-derived sections as required, each of which contains least number of elements.

Now to try and apply Foster's and Cauer's theorems inorder to further reduce the number of elements in either the constant-K or the m-derived sections is not possible because these theorems are first limited to the treatment of networks involving only two kinds of elements namely R,C; R,L; and L,C networks and second, the question regarding the equivalence of networks with respect to more than one pair of terminals is completely outside the scope of the above theorems.

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