SOME APPLICATIONS OF THE FUNCTIONS OF A COMPLEX VARIABLE

THESIS FOR THE DEGREE OF MASTER OF ART

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1935

Functions

SOME APPLICATIONS OF THE FUNCTIONS OF A COMPLEX VARIABLE

A Thesis

Submitted to the Faculty

of

MICHIGAN STATE COLLEGE

of

AGRICULTURE AND APPLIED SCIENCE

In Partial Fulfillment of the

Requirements for the Degree

of

Master of Arts

by

Dalon Hal Ely

1935

ACKNOWLEDGMENT

To Doctor James Ellis Powell
whose suggestions and help are
responsible for this thesis.

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INTRODUCTION

A student is first introduced to complex variable when he studies elementary algebra. At that time he feels that the word "imaginary" expressed very well the usefulness of such numbers. This feeling is usually retained until he takes a course in the functions of a complex variable, and even after a first course he seldom appreciate the applications of the theory to other branches of mathematics and to physics. It is the purpose of this paper to point out some of these applications. We hope that it will add interest to the study of complex functions and that it will induce more students of mathematics, physics, and engineering to make such a study.

No attempt has been made to give an extensive treatment of the applications; it has been our purpose, instead, to give enough applications in the different fields to suggest others. In some of the problems discussed the only method of solution is that of complex variable. While in other problems the solution is arrived at more readily by this method. We propose to so arrange the material that it will be readily available to the teacher for presentation to a class, or to the student who feels that time spent on such a subject would be wasted as far as usefulness is concerned.

We will discuss problems arising in algebra where the theory of poles is applied; problems from statistics are discussed where Euler's formula and the principle of inversion are used; we will show how to evaluate certain definite integrals by the use of the theory of residues; we will give a proof of the existence of a solution of a linear differential equation, using some of the properties of complex series; the existence of a relationship between trigonometric functions and exponential functions will be shown and illustrations as to how this relationship is used will be given; and for the use of the teacher of elementary algebra, we will show how certain geometrical theories can be proved in a simple manner by introducing complex numbers.

We will also discuss problems from the physical field. Thus, we will apply Laplace's equation and conformal representation to problems in hydrodynamics; Euler's formula and the principle of inversion to electrical problems, Laplace's equation and the theory of inversion to problems arising in the theory of the potential, the principle of inversion to problems in the theory of heat flow, and Euler's formula to the theory of light.

We also will show how Euler's formula and conformal representation are used by cartographers and map makers in mapping the earth on a plane.

HYDRODYNAMICS

The first problem we shall discuss is that of a two-dimensional, non-rotational motion that often appears in the theory of hydrodynamics. The straight forward solution of this problem requires that we find a solution of Laplace's equation that will also satisfy the boundary conditions.* When we have done this we determine the pressure by means of the equation

(1) $$\frac{\mathcal{P}}{\rho} = -\omega - \frac{1}{2} g^2 + c,$$

where

$\mathcal{P}$ = pressure,
g = velocity $= \sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}$,
ω = force potential,
ρ = cross section area

and $\phi(x, y)$ is a solution of Laplace's equation.

This process may be tedious or even practically impossible. Another method, which, while it is not direct, is much more fruitful, is to take a particular class of solutions of Laplace's equation and see to what class of problems they may be applied.

If we consider the plane of motion as the complex plane, then the complex analytic function,

$$w = u(x,y) + i v(x,y),$$

which satisfies Laplace's equation is such that if we plot from the z-plane to the w-plane with

$$w = u + i v,$$

* R. A. Houstoun, Mathematical Physics

we will have a conformal map.* Hence, since the velocity potential is perpendicular to the stream line, if we consider the real part of the function w to be the velocity potential, the imaginary part will give the stream function. On the other hand, if we consider the imaginary part of w to be the velocity potential, the real part will be the stream function.

As an example, let us consider the analytic function

$$w = \cosh(x+iy). \tag{2}$$

Since

$$\begin{aligned} u+iv &= \cosh(x+iy) \\ &= \cosh x \cosh iy + \sinh x \sinh iy \\ &= \cosh x \cos y + i \sinh x \sin y, \end{aligned}$$

we have

$$\begin{aligned} u &= \cosh x \cos y, \\ v &= \sinh x \sin y, \end{aligned} \tag{3}$$

and

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1, \tag{4}$$

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1. \tag{5}$$

Plotting (4) and (5) on the u,v-plane we have a family of ellipses and a family of hyperbolas so arranged that the ellipses and hyperbolas

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* For a discussion of conformal maps see section X of this paper.

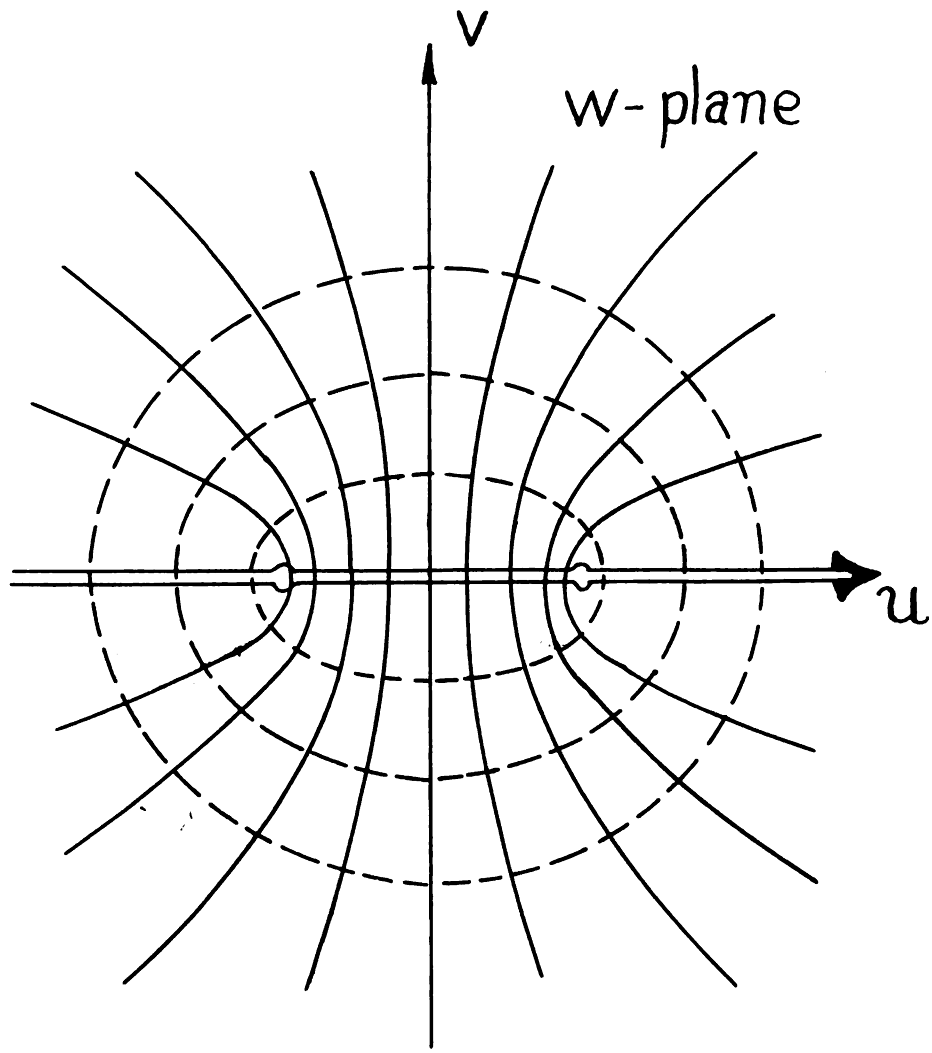

Fig. 1

are perpendicular.

Now it is known that if a liquid is in motion the lines of force are perpendicular to the stream lines. Considering the figure we see that the vertices of the family of hyperbolas all lie along the u-axis between the points (-1,0) and (1,0). Hence we see that if the hyperbolas are taken as the stream lines the figure represents the flow of a liquid through a slit and the lines of force set up by such a system are elliptical. On the other hand we might consider the family of ellipses as representing the stream lines. Again considering the figure we see that the ellipses approach the straight line segment from (-1,0) to (1,0). In this case we have the motion of a liquid rotating about a thin plate and we see that as we get away from the plate the stream lines are elliptical and the lines of force hyperbolic. It is readily seen that by taking $y > 0$ we have the motion about an elliptic cylinder.

Another problem in this field of an entirely different type is of interest in that we employ the same function $w = a \cosh z$.

It is known that when a liquid seeps through a porous soil, the component of the velocity, in any direction, is proportional to the negative pressure gradient in that same direction. Thus, if

P = pressure gradient,

K = a constant of proportionality,

(6)
$$u = -K\frac{\partial P}{\partial x},$$
$$v = -K\frac{\partial P}{\partial y}.$$

If we insert these values in the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

we find that,

(7)
$$\nabla^2 p = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0.$$

Now suppose we consider the problem of the seepage flow under a gravity dam resting on a material that permits seepage.* Here we seek a function, P, which satisfies Laplace's equation and also certain boundary conditions which depend on the nature of the surface of the ground. That is, the pressure must be uniform on the surface of the ground upstream from the heel of the dam, and zero on the surface of the ground down stream from the toe of the dam.

If we now choose our coordinate system with the origin at the midpoint of the base of the dam (Fig.2),

(8)
$$P(u,v) = \frac{P_0\, y(u,v)}{\pi},$$

---

* Warren Weaver, Conformal Representation, with Applications to Problems of Applied Mathematics. American Mathematical Monthly, October 1932, p. 465.

where $y(u,v)$ is determined by

$$w = u + iv = a\cosh(x+iy).$$

One may now easily find the distribution of uplift pressure across the base of the dam. In fact, the base of the dam is the representation, in the (u,v) plane, of the line

$$u = 0,$$
$$0 \leq y \leq \pi,$$

of the (x,y) plane. Hence on the base of the dam the equations

$$u = a\cosh x \cos y,$$
$$v = a\sinh x \sin y,$$

reduce to

$$(9)\qquad u = a\cos y, \quad v = 0,$$

so that

$$(10)\qquad p(u,0) = \frac{p_0}{\pi}\cos^{-1}\frac{u}{a}.$$

The total uplift pressure per unit length of dam is found to be

$$(11)\qquad P = \frac{p_0}{\pi}\int_{-a}^{a}\cos^{-1}\frac{u}{a}\,du = p_0 a,$$

which is the pressure that would be produced if the entire base of the dam were subjected to a head of water just one-half the head above the dam, or if the pressure decreased linearly from the static head P. at the head of the dam to a value zero at the toe.

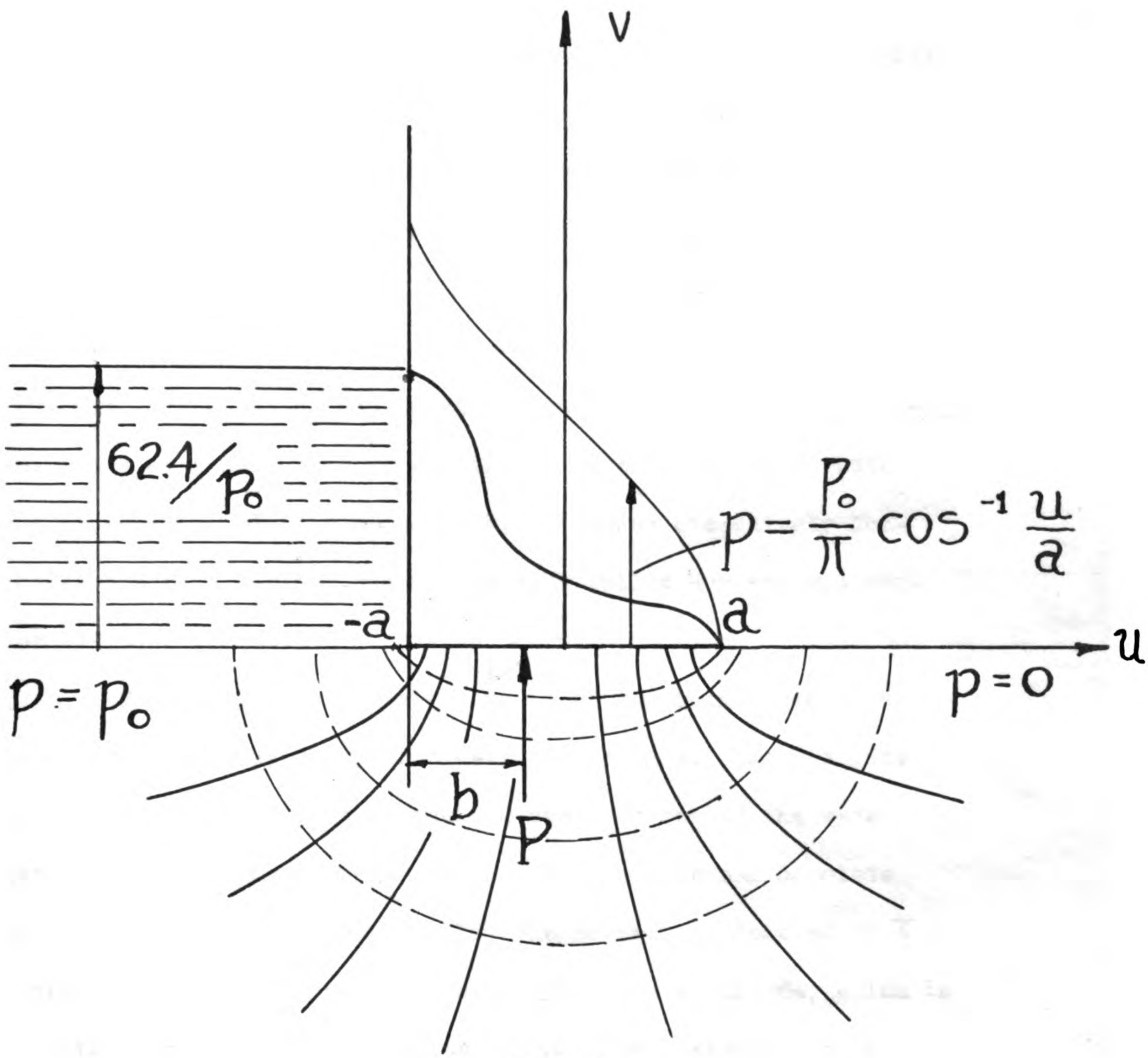

Fig. 2

LIGHT

One of the most interesting physical phenomenon that we encounter in daily life is that of light. Man's ideas of the nature of light have changed from time to time through the centuries but little was done in a scientific way to determine just what light was until the time of Sir Isaac Newton. Newton performed experiments along this line and arrived at the conclusion that light was in the nature of bullets hurtling through space at an exceedingly high velocity. At the same time a rival school, headed by Huygens, arose on the continent, which held that light was energy transmitted as a wave. Today physicists have incorporated both ideas along with modern quantum physics into an exceedingly complicated theory. For our purpose we will follow Huygens and consider light in the nature of a wave motion.

This motion is conceived to be of the transverse type, as a water wave, consisting of crests and troughs. The distance from one crest to the next succeeding crest is called the wave length and is designated by λ ; the time taken for one complete wave to pass a given point is called the period and denoted by τ; the velocity, of the wave, is equal to $\frac{\lambda}{\tau}$; the amplitude, which is the height of the crest above the normal, is designated by b ; the intensity of light is equal to b^2.

It is possible to show experimentally that if two waves of equal lengths and equal amplitudes are traveling in the same direction through a medium, such that the crest of one falls on the crest of the other, the resultant wave has the same length and an amplitude of $2b$. If, however, the crest of one falls on the trough of the other, the resultant amplitude is zero. In the latter case the waves are said to destructively interfere.

To produce interference between light waves it is necessary to take light from the same source, split it into two parts, lead the parts over different paths and reunite them at a small angle. If the difference in paths is an odd number of half wave lengths the waves will interfere and darkness will result.

A wave, whose amplitude is one, may be expressed mathematically by the function

$$\sin \frac{2\pi}{\tau}\left(t - \frac{x}{v}\right).$$

The problem we are considering here is that of interference produced by a medium whose sides are perfectly plane and parallel.* Let B be such a medium and consider a plane wave front, of amplitude one, incident upon it in the direction AP. This will give rise to a reflected wave P_1R_1, and a refracted wave in the direction P_1C_1. This refracted wave will give rise to a reflected wave in the direction C_1P_2 and a transmitted wave in the direction C_1T_1. The reflected wave, C_1P_2, gives rise to a reflected wave in the direction P_2C_2 and a refracted wave in the direction P_2R_2, and so on; the original incident wave gives rise to a series of reflected waves and a series of transmitted waves of rapidly decreasing intensity. What we are after is an expression for the intensities of the resultant reflected and transmitted waves.

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* R. A. Houstoun, A Treatise on Light, P. 140

Let ϕ be the angle of incidence, θ the angle of refraction, e the thickness of the plate. Let R, R' be amplitudes of succeeding reflected waves, C, C' be amplitude of succeeding transmitted waves, and μ the index of refraction.

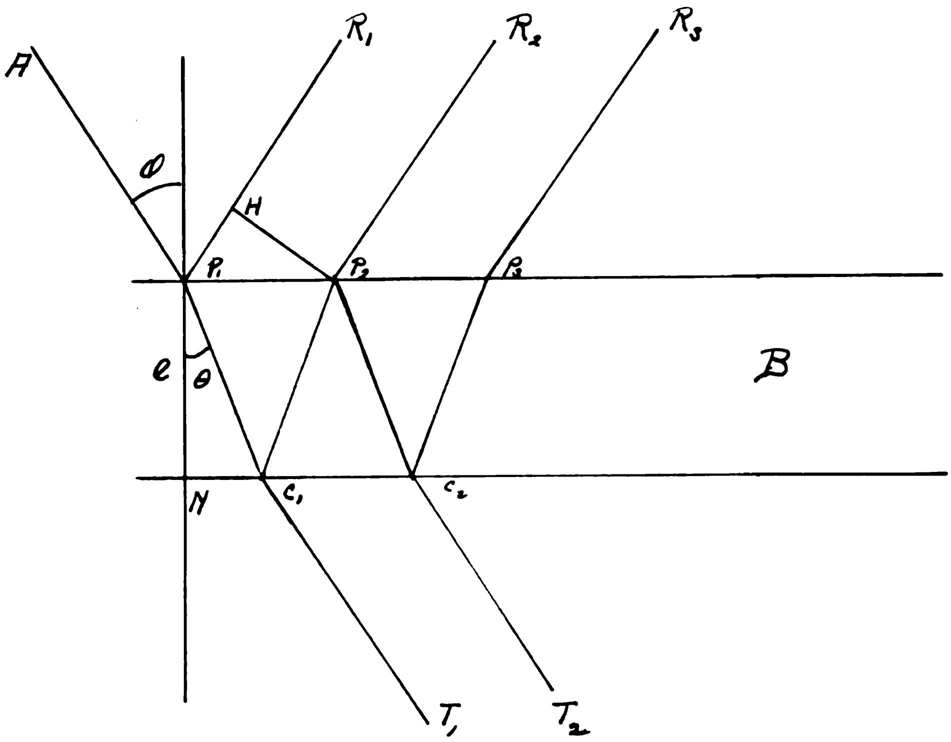

Fig. 3

Draw P_1N perpendicular to the surfaces and P_2H perpendicular to P_1R_1.

When the first and second reflected waves pass the plane P_2H they are in different phases, for they were in the same phase at P_1 and have traveled different optical distances in coming from P_1. Let their relative phase difference be denoted by d; then d denotes the phase difference between any two successive waves.

We have

(12) $$P_1C_1 = \frac{e}{\cos\theta},$$

and

$$P_1H = P_1P_2 \cos HP_1P_2$$

$$= 2NC_1 \sin\varphi$$

(13) $$= 2e \tan\theta \sin\varphi$$

$$= 2\mu e \frac{\sin^2\theta}{\cos\theta}.$$

Hence

$$d = \frac{2\pi}{\lambda}(\mu P_1C_1 + \mu C_1P_2 - P_1H)$$

(14) $$= \frac{2\pi}{\lambda}\left(\frac{2\mu e}{\cos\theta} - 2\mu e \frac{\sin^2\theta}{\cos\theta}\right)$$

$$= \frac{2\pi}{\lambda} 2\mu e \cos\theta.$$

If the incident wave is denoted by

$$\sin\frac{2\pi}{\tau}\left(t - \frac{x}{v}\right)$$

we can represent the resultant wave by the function

$$R \sin\left[\frac{2\pi}{\tau}\left(t - \frac{x}{v}\right) - \delta\right],$$

in which the amplitude, R, and the phase, δ, are to be determined. The first reflected wave is represented by

$$r \sin\frac{2\pi}{\tau}\left(t - \frac{x}{v}\right),$$

the second by

$$r'cc'\sin\left\{\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)-d\right\},$$

the third by

$$r'^{3}cc'\sin\left\{\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)-2d\right\},$$

and so on. The resultant is the sum of these waves. Hence

$$\begin{aligned}R\sin\left\{\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)-\delta\right\} &= r\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)+r'cc'\sin\left\{\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)-d\right\}\\ &\quad+r'^{3}cc'\sin\left\{\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)-2d\right\}+\cdots\\ &\quad+r'^{2n-3}cc'\sin\left\{\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)-(n-1)d\right\}+\cdots\end{aligned}$$

To evaluate R and δ we must sum this series. To do this as the series stands we would have to apply the principles of Fourier; if, however, we introduce complex terms by the use of Euler's formula we will have a series of complex terms which can be easily summed. Using the expansion for $\sin(a-b)$, we have

$$\begin{aligned}&R\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\cos\delta-R\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\sin\delta\\ &\quad= r\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\\ &\quad+r'cc'\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\cos d-r'cc'\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\sin d\\ &\quad+r'^{3}cc'\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\cos 2d-r'^{3}cc'\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\sin 2d\\ &\quad+\cdots\cdots\\ &\quad+r'^{2n-3}cc'\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\cos((n-1)d)-r'^{2n-3}cc'\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)\sin(n-1)d\\ &\quad+\cdots\end{aligned}\tag{15}$$

Formally equating the coefficients of $\sin\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)$ and $\cos\frac{2\pi}{\tau}\left(t-\frac{x}{v}\right)$ we get

$$\begin{aligned}R\cos\delta &= r+r'cc'\left(\cos d+r'^{2}\cos 2d+r'^{4}\cos 3d+\cdots+r'^{2n-2}\cos nd+\cdots\right),\\ R\sin\delta &= r'cc'\left(\sin d+r'^{2}\sin 2d+r'^{4}\sin 3d+\cdots+r'^{2n-2}\sin nd+\cdots\right).\end{aligned}\tag{16}$$

The series of (15) and (16) are power series in r' with the coefficient of each term less then one and hence are convergent. Now by the use of Euler's formula

$$e^{ix} = \cos x + i \sin x$$

we can combine these equations into the following form

$$(17)\qquad \begin{aligned} Re^{i\delta} &= r + r'cc'(e^{id} + r'^2 e^{2id} + r'^4 e^{3id} + \dots\dots + r'^{2n-2} e^{nid} + \dots) \\ &= r + r'cc' \frac{e^{id}}{1 - r'^2 e^{id}} . \end{aligned}$$

This can be written in the form

$$Re^{i\delta} = r + \frac{r'cc'e^{id}(1 - r'^2 e^{-id})}{(1 - r'^2 e^{id})(1 - r'^2 e^{-id})}$$

$$(18)\qquad = r + \frac{r'cc'(\cos d + i \sin d - r'^2)}{1 - 2r'^2 \cos d + r'^4}$$

$$= r + \frac{r'cc'(\cos d - r'^2)}{1 - 2r'^2 \cos d + r'^4} + i \frac{r'cc' \sin d}{1 - 2r'^2 \cos d + r'^4} .$$

But if two complex numbers are equal their amplitudes are equal. Hence

$$R^2 = \left\{ r + \frac{r'cc'(\cos d - r'^2)}{1 - 2r'^2 \cos d + r'^4} \right\}^2 + \left\{ \frac{r'cc' \sin d}{1 - 2r'^2 \cos d + r'^4} \right\}^2 .$$

This may be simplified to

$$(19)\qquad \begin{aligned} R^2 &= \frac{4r^2 \sin^2 \frac{d}{2}}{1 - 2r'^2 \cos d + r'^4} \\ &= \frac{4r^2 \sin^2 \frac{d}{2}}{(1 - r'^2)^2 + 4r'^2 \sin^2 \frac{d}{2}} . \end{aligned}$$

If T is the amplitude of the resultant transmitted wave, we have by the principle of energy

$$R^2 + T^2 = 1.$$

Hence

$$(20) \qquad T^2 = \frac{(1-r^2)^2}{1-2r'^2\cos\alpha+r'^4}\,.$$

We are interested in the maximum and minimum values of R^2 and T^2. We have

$$R^2 = 0$$

when

$$\sin^2\frac{\alpha}{2} = 0\,.$$

Therefore

$$(21) \qquad \frac{\alpha}{2} = \frac{2\pi\mu e\cos\theta}{\lambda} = n\pi$$

or

$$2\mu e\cos\alpha = n\lambda$$

where N is a positive integer. Thus when the path difference is an integral number of wave lengths we have no reflected light.

Also

$$(22) \qquad R^2 = \frac{4r^2}{\dfrac{(1-r^2)^2}{\sin^2\frac{\alpha}{2}}+4r^2}$$

is a maximum when

$$\sin^2\frac{\alpha}{2} = 1.$$

Therefore

$$\frac{\alpha}{2} = \frac{2\pi\mu e\cos\theta}{\lambda} = (2n+1)\frac{\pi}{2}$$

or

$$2\mu e\cos\alpha = (2n+1)\frac{\lambda}{2}$$

where N is a positive interger. Thus R^2 is a maximum when the path difference is an odd number of half wave lengths. The minimum value of R is zero and the maximum value is

$$\frac{4r^2}{(1+r^2)^2}\,.$$

When R^2 is a minimum, T^2 has a maximum valve equal to one, and when R^2 is maximum, T^2 has a minimum valve equal to

$$\left(\frac{1-r^2}{1+r^2}\right)^2.$$

III ELECTRICITY

The applications of the complex variable to problems in electricity are divided into two parts. In electrical potential theory we consider problems of the same type as those discussed under hydrodynamics. For example, the transformation

$$w = \cosh z,$$

which has been discussed before in this paper, might be interpreted as representing the electrostatic field due to a source in the shape of an elliptic cylinder, or the electrostatic field due to a charged plate from which a strip has been removed.

The use of complex variable in the problem of an electrical circuit has been explained by T. C. Fry of the Bell Telephone Laboratories in the following manner.*

"From the physical standpoint it (electrical circuit theory) is naturally a study in real quantities, since the physical forces and currents are themselves real, and mathematical methods originally used in connection with it dealt almost exclusively with real variables. During the present century, however, a progressive transition has taken place, until now the use of complex quantities is by all odds the usual thing, and real numbers are rather rare.

* T. C. Fry, Differential Equations as a Foundation for Electrical Circuit Theory, American Mathematical Monthly, 1929, P. 499.

"---- To reduce to mathematical terms, then, the problem of elementary circuit theory is that of solving the system of differential equations:

$$
(23) \qquad
\begin{aligned}
F_{11}(P)y_1 + F_{12}(P)y_2 + \cdots + F_{1n}(P)y_n &= f_1(t) \\
F_{21}(P)y_1 + F_{22}(P)y_2 + \cdots + F_{2n}(P)y_n &= f_2(t) \\
\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \quad & \quad \vdots \\
F_{n1}(P)y_1 + F_{n2}(P)y_2 + \cdots + F_{nn}(P)y_n &= f_n(t).
\end{aligned}
$$

Where the F's are linear differential operators with constant coefficients, the f's are simple harmonic functions or sums of such functions, and the boundary conditions require that the system shall be either at rest or in a steady state at the time zero.

" I need scarcely remark that the study of such systems is a recognized part of first courses in differential equations. The suggestion that the foundations for circuit theory be laid in such courses is, therefore, not a very revolutionary one; it requires at most a slight change of emphasis to accomplish it. Perhaps this will become even clearer as we mention the four facts upon which the use of complex quantities rests, none of which is unusual or non-mathematical in character.

"I The sum of any solution of (23) due to a set f_j, and any solution due to a set g_j, is a solution due to a set f_j+g_j.

"II This is an inverse to I. Of course, there is no exact converse to the principle of superposition, for even if we know a solution of (23) due to $f_j + g_j$ we cannot ordinarily separate the part which is due to f_j from that due to g_j. There is one important exceptional case however.

"If the functions f_j are real while the functions g_j are pure imaginaries, then the real part of the solution is due to f_j and the imaginary part of the solution is due to g_j.

"III. The derivatives of an exponential function are proportional to the function itself.

"IV. Euler's equation

$$e^{ix} = \cos x + i \sin x.$$

"Hence if (23) can be easily solved when $f(t)$ is an exponential, and if it is true that when $f(t)$ is a complex we can separate out the real solution from the imaginary solution, it follows at once that we can obtain the solution due to the simple harmonic function, $\cos(pt+\epsilon)$ by first finding the solution due to the complex exponential, $e^{i(pt+\epsilon)}$, and then discarding the imaginary part.

"This is the entire argument in favor of the use of complex imaginary numbers in electrical circuit theory. It is hard to imagine any explanation based on vectorial ideas, which would approach it either in conciseness, simplicity, or generality."

As an example of this method of solution, consider the differential equation *

* Page, Introduction to Mathematical Physics.

(24) $$a_1 \frac{d^2 u}{dt^2} + a_2 \frac{du}{dt} + a_3 u = b(t) + i\, c(t)$$

where a_1, a_2, a_3 are real constants, and b(t), c(t) real functions of the real variable t.

If we assume a solution of (24) to be

(25) $$u = x(t) + i y(t),$$

we have, after substituting (25) in (24),

(26) $$a_1 \frac{d^2 x}{dt^2} + i a_1 \frac{d^2 y}{dt^2} + a_2 \frac{dx}{dt} + i a_2 \frac{dy}{dt} + a_3 x + i a_3 y = b(t) + i c(t).$$

Equating the real and imaginary parts of this equation we get

(27) $$a_1 \frac{d^2 x}{dt^2} + a_2 \frac{dx}{dt} + a_3 x = b(t),$$
$$a_1 \frac{d^2 y}{dt^2} + a_2 \frac{dy}{dt} + a_3 y = c(t),$$

which are of the same form as (23).

Now since we know that the impressed electromotive force (EMF) is a harmonic function we may consider it to be the real part of the function

$$E_0 (\cos \omega t + i \sin \omega t) \equiv E_0 e^{i\omega t},$$

where E_0 is not necessarily real. We may now write

$$E_0 = E_0' e^{i\varphi}, \quad (E_0' \text{ real}).$$

The impressed E M F is then

(28) $$E = \text{real part of} \left\{ E_0' e^{i(\omega t + \varphi)} \right\} = E_0' \cos(\omega t + \varphi).$$

Now if we denote the current by I, the resistance by R, and the inductance of the circuit in henries by L, we may write the equation for Ohm's law in the following form

(29) $$L \frac{dI}{dt} + RI + \frac{1}{C} \int I\, dt = E_0 e^{i\omega t},$$

Comparing this with (24) we see that if we consider

$I = x$, $L = a_1$, $R = a_2$, and $\frac{1}{C} = a_3$ then (29) and (24) are identical.

To solve equation (29) let us assume a trial solution

$$I = I_0 e^{i\omega t} \quad (I_0 \text{ not necessarily real}).$$

Substituting this value in (24) we find it is a solution if

$$I_0 \left\{ R + i\left(\omega L - \frac{1}{\omega C}\right)\right\} = E_0,$$

$$E_0 = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \cdot I_0 e^{i\delta},$$

$$I_0 = \frac{E_0 e^{-i\delta}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}},$$

where $\tan\delta = \dfrac{\omega L - \frac{1}{\omega C}}{R}$.

Therefore E and I represent vectors rotating in the x,y-plane with an angular velocity ω, the latter lagging behind the former by an angle δ. The actual electromotive force and current at any instant are represented by the projections of these vectors on the x-axis.

If we put

$$Z = R + i\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, e^{i\delta},$$

$$A = \frac{1}{R + i\left(\omega L - \frac{1}{\omega C}\right)} = \frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\, e^{-i\delta},$$

it follows that

$$E = ZI,$$

$$I = AE.$$

IV HEAT FLOW

Many problems arise in the field of engineering to which the theories of heat flow can be applied. For example, it is of very practical importance to know the conductivity of a boiler plate, or a bearing packing. It is of theoretical interest to know that the length of time that the earth has sustained life may be determined by considering the conductivity of the earth's crust.

For the purpose of this discussion let us suppose that all space on the positive side of the y,z-plane is filled with a homogeneous solid of diffusivity K. Let the temperature on the y,z-plane be given as a harmonic function of the time, that is, the temperature at all points of the y,z-plane is the same but this temperature has a periodic variation. For example, if T = temperature, and t = time, then

$$T = a t^n \sin mt + b t^n \cos m't,$$

where a and b may be complex but n, m, m′ are real. Also let us suppose that this function is the same for all values of y and z. It is required to find the temperature throughout the solid when the periodic state is fixed.

Clearly T is independent of y and z. One of the conditions T must satisfy is

$$(30) \qquad \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}.$$

That is, the change of the temperature with respect to the time is proportional to the velocity of the change of the temperature with respect to the distance from the y,z-plane and the constant of proportionality is equal to the diffusivity of the solid.*

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* R. A. Houstoun, Introduction to Mathematical Physics.

That is to say, T must also satisfy the conditions

(31) $$T = a \sin mt, \text{ when } x = 0, \quad (m = \text{const.}),$$
$$T \neq \infty, \text{ when } x = \infty.$$

To solve equation (30) let us assume a trial solution

(32) $$T = e^{\alpha t + \beta x}.$$

When this is substituted in (30) we have

$$\alpha e^{\alpha t + \beta x} = K \beta^2 e^{\alpha t + \beta x},$$

or

(33) $$\alpha = K \beta^2.$$

It is obvious from (33) that α cannot be real and negative. Therefore let us assume α is imaginary and

$$\alpha = \pm i \gamma.$$

Substituting this value in (32) we have as a solution

(34) $$T = e^{\pm i\gamma t \pm \sqrt{\pm \frac{i\gamma}{K}}\, x}.$$

Now we know that

$$(1+i)^2 = 2i,$$

and

$$(1-i)^2 = -2i.$$

Hence

$$\sqrt{i} = \frac{1}{\sqrt{2}}(1+i),$$

and

$$\sqrt{-i} = \frac{1}{\sqrt{2}}(1-i).$$

Substituting these values in (34) we have

$$T = e^{\pm i\gamma t \pm \sqrt{\frac{\gamma}{2K}}(1 \pm i)x}$$
$$= e^{\pm\sqrt{\frac{\gamma}{2K}}\, x \pm i(\gamma t \pm \sqrt{\frac{\gamma}{2K}}\, x)}.$$

Now the condition for $x = \infty$ demands that the sign of the exponent be negative. We have, therefore

(35) $$T = e^{-\sqrt{\frac{\gamma}{2K}}\, x}\left\{A e^{+i(\gamma t - \sqrt{\frac{\gamma}{2K}}\, x)} + B e^{-i(\gamma t - \sqrt{\frac{\gamma}{2K}}\, x)}\right\},$$

where A and B are constants.

Substituting $x = o$ in this equation we find

$$T = Ae^{i\gamma t} + Be^{-i\gamma t}.$$

But to satisfy boundary conditions, when $x = o$ we must have

$$T = a \sin mt.$$

Therefore

$$Ae^{i(\gamma t - \sqrt{\frac{\gamma}{2K}}x)} + Be^{-i(\gamma t - \sqrt{\frac{\gamma}{2K}}x)}$$

must take the form

$$a \sin(\gamma t - \sqrt{\tfrac{\gamma}{2K}}\,x),$$

with

$$\gamma = m.$$

Hence

$$T = ae^{-\sqrt{\frac{m}{2K}}x} \sin(mt - \sqrt{\tfrac{m}{2K}}\,x)$$

is the solution to the problem.

V. POTENTIAL THEORY

In discussing potential theory, we are interested, not only in the development by complex variable, but also in comparing this development with that in which complex variable is not used.*

In two dimensional problems, the equation to be satisfied by the potential is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

from which we get

$$(36) \qquad V = f(x+iy) + F(x+iy).^{**}$$

If V is to be real, then F must be the function obtained from f by changing i to $-i$. Now let

$$f(x+iy) = u + iv, \qquad \text{u and v both real.}$$

Then

$$F(x+iy) = u - iv,$$

and

$$V = 2u.$$

If we let

$$U = -2v,$$

we have

$$\begin{aligned} U + iV &= -2v + 2iu \\ (37) \qquad &= +2i(u+iv) \\ &= +2if(x+iy) = \omega(x+iy), \end{aligned}$$

where $\omega(x+iy)$ is a completely general function of the single

* For an example of the development without complex variable see MacMillan, Theory of the Potential.

** Jeans, Electricity and Magnetism. P. 261

complex variable $(x+iy)$.

Thus the most general function of the potential which is wholly real can be derived from the most general arbitrary function of the single complex variable $(x+iy)$ by taking the potential to be the imaginary part of this function.

Now, if

$$\Phi(x+iy) = U + iV$$

and

(38) $$z = x+iy, \quad w = U + iV$$

and if we let the element in the Z-plane be

(39) $$PP' = dz,$$

and the corresponding element in the w-plane

(40) $$QQ' = dw = \frac{dw}{dz}dz,$$

we can get any element QQ′ from PP′ by multiplying PP′ by $\frac{dw}{dz}$, that is, by

$$\frac{\partial}{\partial z}\Phi(z) \text{ or } \Phi'(x+iy).$$

If we express $\frac{dw}{dz}$ as

(41) $$\frac{dw}{dz} = \Phi'(x+iy) = \rho(\cos\theta + i\sin\theta)$$

we can find the element dw by multiplying dz by ρ or $\left|\frac{dw}{dz}\right|$ and turning it through the angle θ.

Now let us examine ρ, we have

(42) $$\rho(\cos\theta + i\sin\theta) = \frac{dw}{dz} = \Phi'(x+iy) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} + i\frac{\partial U}{\partial y},$$

or

(43) $$\rho = \left|\frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}\right| = \left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right\}^{\frac{1}{2}}.$$

Thus we see that if V is the potential, the modulus, ρ, measures the intensity R. And since

$$R = 4\pi\sigma$$

we have a simple means of finding σ, the surface density at any point of the surface.

VI ALGEBRA

Let us now turn to the realm of mathematics. It was here that the ideas of imaginary numbers and hence complex variable first arose. We first recognize the need of imaginary numbers when solving quadratic equations of the type

$$x^2+x+1=0.$$

Since the use of complex numbers in the solution of equations is familiar to anyone having had a course in college algebra, we will, instead, in this section, show how the fundamental theorem of algebra can be easily proved by the use of some ideas from functions of a complex variable.*

Theorem: An equation of degree n with complex coefficients

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_n=0,$$

has a complex (real or imaginary) root.

For, from our knowledge of singular points we know that $f(z)$ has no singular point in the finite plane and has a pole at infinity.

Putting

(44) $$z=\frac{1}{z'}$$

it follows that

(45) $$\phi(z')=f\left(\frac{1}{z}\right)$$

has a pole at

$$z'=0.$$

It can be shown that if

$$z=z_0$$

is a pole of order K of the analytic function $f(z)$, then $\frac{1}{f(z)}$ is analytic in the neighborhood of Z_0 and has a zero point of order

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* E. J. Townsend, Functions of a Complex Variable, P. 291

K at Z_0, and conversely.

Hence $\frac{1}{\phi(z')}$ is analytic in the neighborhood of the origin. Consequently, since

$$\frac{1}{f(\frac{1}{z'})} = \frac{1}{\phi(z')}$$

is analytic in the neighborhood of the origin, $\frac{1}{f(z)}$ is analytic in the neighborhood of $z=\infty$. But, as is easily proved, every analytic function, not a constant, must have at least one singular point either in the finite region or at infinity. Since $\frac{1}{f(z)}$ cannot have a singular point at infinity, there must be at least one point in the finite region, say Z_0, at which the function $\frac{1}{f(z_0)}$ has a singular point. This point cannot be as essential singular point, for in that case Z_0 would not be a regular point of $f(z)$. Hence Z_0 must be a pole of $\frac{1}{f(z)}$ and consequently a zero point of the given funtion $f(z)$. This establishes the theorm.*

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* For a proof without the use of the theories of a complex variable see Dickson, First Course in the Theory of Equations, p. 155.

Certain integrals that are difficult to evaluate by ordinary means can be evaluated by the use of the theory of residues.*

Consider for example,

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}.$$

To evaluate this, let

$$z = e^{i\theta},$$

(46) $$d\theta = \frac{dz}{iz},$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}.$$

Hence

(47) $$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \int_C \frac{\frac{1}{iz}}{\frac{1}{2}(z+4+\frac{1}{z})}\,dz = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1}$$

where the path, C, is determined by the substitution

$$z = e^{i\theta}.$$

As θ varies from 0 to 2π, z describes the unit circle about the origin. The integrand has poles at

$$z = -2 \pm \sqrt{3},$$

but the only one to fall within C is

$$z = -2+\sqrt{3}.$$

Hence if we expand $f(z)$ about

$$z = -2+\sqrt{3},$$

by considering

(48) $$\frac{1}{z^2+4z+1} = \frac{1}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$$

and then expanding the factor

$$\frac{1}{z+2+\sqrt{3}}$$

about the point by Taylor's series and multiplying each term

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* E. J. Townsend, Functions of a Complex Variable. P. 284

by $\frac{1}{z+2-\sqrt{3}}$, we get

(49) $$\frac{1}{z^2+4z+1} = \frac{1}{2\sqrt{3}}\left(z+2-\sqrt{3}\right)^{-1} + \cdots\cdots.$$

Since the theory requires only the coefficient of the term with the exponent (-1) this is all of the expansion we need.

The residue at

$$z = -2+\sqrt{3}$$

is $\frac{1}{2\sqrt{3}}$. Therefore we have

(50) $$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\cdot 2\pi i\cdot\frac{1}{2\sqrt{3}} = \frac{2\pi\sqrt{3}}{3}$$

as the value of the integral.

As a second example let us consider a function with infinite limits, for example

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3},$$

Substituting

$$x = z$$

we see that the function

$$\frac{1}{(z^2+1)^3}$$

has a pole at

$$z = i.$$

Expanding $f(z)$ in powers of $(z-i)$ we have

(51) $$f(z) = \frac{i}{8(z-i)^3} - \frac{3}{16(z-i)^2} - \frac{3i}{16(z-i)} + \cdots\cdots.$$

The residue at $z = i$ is

$$-\frac{3i}{16}.$$

Hence we have

(52) $$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = -\frac{3i}{16}\cdot 2\pi i = \frac{3\pi}{8}.$$

To discuss completely the many uses of the functions of a complex variable in the theory of differential equations is impossible in this paper. We will illustrate the use of some of the theory by giving a proof of an existence theorem of linear differential equations.* To simplify the proof we will consider an equation of the first order.

<u>Theorem. Given a linear differential equation of the first order</u>

(A) $$\frac{dy}{dx} = f(x,y)$$

<u>where x and y are complex and $f(x,y)$ is analytic in some finite region</u>

$$D: \quad |x-x_0| \leq a$$
$$|y-y_0| \leq b,$$

<u>for all x, y in the region, it follows that there exists a solution of (A) passing through the point (x_0, y_0).</u>

For, since $f(x,y)$ is analytic in D it follows that $f(x,y)$ has an upper bound, M, in D. **

Expanding $f(x,y)$ about the points x_0, y_0 we have

(53) $$f(x,y) = f(x_0 y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \frac{f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2}{2!} + \text{-------------}.$$

* Moulton, Differential Equations

** E. J. Townsend, Functions of a Complex Variable, P. 39.

Define the function $F(x,y)$ by

(54) $$F(x,y)=\frac{M}{\left(1-\frac{y-y_0}{b'}\right)\left(1-\frac{x-x_0}{a'}\right)},$$

where

$a'<a,$

$b'<b.$

Expanding

$\dfrac{M}{1-\frac{y-y_0}{b'}}$ and $\dfrac{1}{1-\frac{x-x_0}{a'}}$, we get

$$\frac{M}{1-\frac{y-y_0}{b'}}=M\left[1+\frac{y-y_0}{b'}+\left(\frac{y-y_0}{b'}\right)^2+\cdots+\left(\frac{y-y_0}{b'}\right)^n+\cdots\right],$$

and

$$\frac{1}{1-\frac{x-x_0}{a'}}=1+\frac{x-x_0}{a'}+\left(\frac{x-x_0}{a'}\right)^2+\cdots+\left(\frac{x-x_0}{a'}\right)^n+\cdots.$$

These series converge in a region

$D':\ |x-x_0|<a$

$\quad |y-y_0|<b$

since they are geometric progressions with the ratio less than one.

Then

(55) $$\frac{M}{\left(1-\frac{y-y_0}{b'}\right)\left(1-\frac{x-x_0}{a'}\right)}=M\left[1+\frac{y-y_0}{b'}+\cdots+\left(\frac{y-y_0}{b'}\right)^n+\cdots\right]\left[1+\frac{x-x_0}{a'}+\cdots+\left(\frac{x-x_0}{a'}\right)^n+\cdots\right]$$
$$=M\left[1+\frac{x-x_0}{a'}+\frac{y-y_0}{b'}+\left(\frac{x-x_0}{a'}\right)^2+\left(\frac{x-x_0}{a'}\right)\left(\frac{y-y_0}{b'}\right)+\left(\frac{y-y_0}{b'}\right)^2+\cdots\right]$$

is convergent in D' since it is the product of convergent power series.

Next we will show that the series (55) dominates the series (53).*

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* For a discussion of dominant functions see F. S. Woods, <u>Advanced Calcalus</u>, p. 57.

That is, we will show that

$$M \geqq |f(x_0, y_0)|,$$

$$\frac{M}{a'} \geqq |f_x(x_0, y_0)|,$$

$$\frac{M}{b'} \geqq |f_y(x_0, y_0)|,$$

- - - - - - - - - - - -

$$\frac{M}{a'^n} \geqq \frac{\left|\frac{\partial^n}{\partial x^n} f(x_0, y_0)\right|}{n!},$$

$$\frac{M}{b'^n} \geqq \frac{\left|\frac{\partial^n}{\partial y^n} f(x_0, y_0)\right|}{n!},$$

- - - - - - - - - - - - - - - -,

$$\frac{M}{a'^m b'^n} \geqq \frac{\left|\frac{\partial^{m+n}}{\partial x^m \partial y^n} f(x_0, y_0)\right|}{(m+n)!},$$

- .

Let us consider first the two series

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots\cdots, \tag{56}$$

$$F(x) = \frac{M}{1 - \frac{x - x_0}{a'}} = M\left[1 + \frac{x - x_0}{a'} + \frac{(x - x_0)^2}{a'^2} + \cdots\cdots + \frac{(x - x_0)^n}{a'^n} + \cdots\right]. \tag{57}$$

By Cauchy's integral theorem we have

$$f(x_0) = \frac{1}{2\pi i}\int_c \frac{f(t)dt}{t - x_0} .$$

Substituting

$$(t - x_0) = re^{i\theta},$$

$$|(t - x_0)| = r,$$

$$dt = ire^{i\theta}d\theta,$$

we find that

$$|f(x_0)| \leqq \frac{1}{2\pi}\frac{M}{r}\cdot 2\pi r = M.$$

In a similar way

$$|f'(x_0)| = \frac{1}{2\pi i}\left|\int_c \frac{f(t)dt}{(t-x_0)^2}\right| \leq \frac{1}{2\pi}\frac{M}{a'^2}2\pi a' = \frac{M}{a'},$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$|f^{(n)}(x_0)| = \frac{1}{2\pi i}\left|\int_c \frac{f(t)dt}{(t-x_0)^{n+1}}\right| \leq \frac{1}{2\pi}\frac{M}{a'^{n+1}}\cdot 2\pi a' = \frac{M}{a'^n}.$$

This shows that (57) dominates (56). To show that a similar reasoning holds for two variables, let

(58)
$$f(x_0, y_0) = \frac{1}{2\pi i}\int_c \frac{f(t, y_0)dt}{(t-x_0)},$$
$$f(t, y_0) = \frac{1}{2\pi i}\int_c \frac{f(t,\tau)d\tau}{(\tau-y_0)},$$

$$|f(x_0,y_0)| = \left(\frac{1}{2\pi i}\right)^2\left|\int_{c_1}\int_{c_2}\frac{f(t,\tau)d\tau dt}{(t-x_0)(\tau-y_0)}\right| \leq \frac{1}{4\pi^2}\frac{M}{a'b'}\cdot 4\pi^2 a'b' = M,$$

$$\left|\frac{\partial f(x_0,y_0)}{\partial x}\right| = \left(\frac{1}{2\pi i}\right)^2\left|\int_{c_1}\int_{c_2}\frac{f(t,\tau)d\tau dt}{(t-x_0)^2(\tau-y_0)}\right| \leq \frac{M}{a'},$$

$$\left|\frac{\partial f(x_0,y_0)}{\partial y}\right| = \left(\frac{1}{2\pi i}\right)^2\left|\int_{c_1}\int_{c_2}\frac{f(t,\tau)d\tau dt}{(t-x_0)(\tau-y_0)^2}\right| \leq \frac{M}{b'},$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\left|\frac{\partial^m f(x_0,y_0)}{\partial x^m}\right| = \frac{1}{2\pi i}\left|\int_{c_1}\int_{c_2}\frac{f(t,\tau)d\tau dt}{(t-x_0)^{m+1}(\tau-y_0)}\right| \leq \frac{M}{a'^m},$$

$$\left|\frac{\partial^n f(x_0,y_0)}{\partial y^n}\right| = \left(\frac{1}{2\pi i}\right)^2 \left|\int_{c_1}\int_{c_2} \frac{f(t,\tau)\,d\tau\,dt}{(t-x_0)(\tau-y_0)^{n+1}}\right| \leq \frac{M}{b'^n},$$

$$\left|\frac{\partial^{m+n} f(x_0,y_0)}{\partial x^m \partial y^n}\right| = \left(\frac{1}{2\pi i}\right)^2 \left|\int_{c_1}\int_{c_2} \frac{f(t,\tau)\,d\tau\,dt}{(t-x_0)^{m+1}(\tau-y_0)^{n+1}}\right| \leq \frac{M}{a'^m b'^n}.$$

Now assume a solution of (53) in the form

(59) $$y-y_0 = \alpha_1(x-x_0)+\alpha_2(x-x_0)^2+\cdots+\alpha_n(x-x_0)^n+\cdots.$$

Substituting (59) in (53), we get

(60) $$\begin{aligned} &\alpha_1+2\alpha_2(x-x_0)+\cdots+n\alpha_n(x-x_0)^{n-1}+\cdots \\ &\quad = c_{00}+c_{10}(x-x_0) \\ &\qquad + c_{01}[\alpha_1(x-x_0)+\alpha_2(x-x_0)^2+\cdots+\alpha_n(x-x_0)^n+\cdots] \\ &\qquad + c_{20}[x-x_0]^2+c_{11}(x-x_0)[\alpha_1(x-x_0)+\alpha_2(x-x_0)^2+\cdots]+\cdots \end{aligned}$$

where $$c_{mn} = \frac{\dfrac{\partial^{m+n}}{\partial x^m \partial y^n} f(x_0,y_0)}{(m+n)!}.$$

Equating coefficients, we find that

$$\begin{aligned} \alpha_1 &= c_{00}, \\ 2\alpha_2 &= c_{10}+c_{01}\alpha_1 = c_{10}+c_{01}c_{00}, \\ 3\alpha_3 &= c_{01}\alpha_2+c_{02}+c_{11}\alpha_1+c_{02}\alpha_1^2, \\ &\vdots \end{aligned}$$

Hence it is possible to evaluate all the α's in terms of the C's, and hence we have a solution to (64) if we can prove the series (59) convergent in some region.

To do this consider

(61) $$\frac{dy}{dx} = F(x,y),$$

with

$$y(x_0) = y_0$$

By use of the same method as above we can solve this equation in a series, to get

(62) $$y-y_0 = A_1(x-x_0)+A_2(x-x_0)^2+\cdots+A_n(x-x_0)^n+\cdots.$$

The values of the A's here are of the same form as in (59). Furthermore, they are all positive. Hence, since the series (55) dominates series (53) it follows that the A's dominate the α's . That is

$$A_1 \geqslant |\alpha_1|, \ A_2 \geqslant |\alpha_2|, \ \cdots\cdots \ A_n \geqslant |\alpha_n|, \ \cdots\cdots .$$

However, we can solve (61) directly without the use of series, since

$$\frac{dy}{dx} = \frac{M}{\left(1 - \frac{y-y_0}{b'}\right)\left(1 - \frac{x-x_0}{a'}\right)},$$

and

$$dy\left(1 - \frac{y-y_0}{b'}\right) = \frac{M\,dx}{1 - \frac{x-x_0}{a'}} .$$

Therefore

$$y - \frac{(y-y_0)^2}{2b'} = -Ma'\log\left(1 - \frac{x-x_0}{a'}\right) + C.$$

Since

$$x = x_0, \ y = y_0,$$

we have

$$C = y_0 .$$

Hence

$$y - y_0 - \frac{(y-y_0)^2}{2b'} + Ma'\log\left(1 - \frac{x-x_0}{a'}\right) = 0 .$$

If

$$y - y_0 = Y,$$

then

$$Y - \frac{Y^2}{2b'} + Ma'\log\left(1 - \frac{x-x_0}{a'}\right) = 0,$$

$$Y^2 - 2b'Y - 2Ma'b'\log\left(1 - \frac{x-x_0}{a'}\right) = 0,$$

and

$$Y = b' \pm \sqrt{b'^2 + 2Ma'b'\log\left(1 - \frac{x-x_0}{a'}\right)} .$$

Now x is complex and therefore the series for the expansion of (53) is convergent in a circular region having x_0 as a center and a radius equal to the distance from x_0 to the nearest singular point.

There are singular points at

(63) $$\frac{x-x_0}{a'} = 1,$$

and

(64) $$b'^2 + 2a'b'M \log\left(1 - \frac{x-x_0}{a'}\right) = 0.$$

Solving (64) for $(x-x_0)$, we have

$$\log\left(1 - \frac{x-x_0}{a'}\right) = -\frac{b'}{2a'M},$$

$$1 - \frac{x-x_0}{a'} = e^{-\frac{b'}{2a'M}},$$

$$(x-x_0) = a' - a'e^{-\frac{b'}{2a'M}}.$$

Hence the radius of the circle of convergence is finite and different from zero. Since series (62) and (64) must be the same, series (62) must be convergent. Furthermore, since series (62) dominates (59), it follows that (59) must be convergent in some region.

We therefore have the existence of a solution of (53).

It is often desirable to have a rather simple application of complex numbers to bring home to a group of college freshmen the idea that complex numbers have other uses than for solving algebraic equations of certain types. The applications that have so far been discussed are in the most part beyond the range of a freshman group and have little meaning to them. With this in mind, Lloyd L. Smail of Lehigh University has worked some theorems of plane geometry which he has proved with the aid of complex variable.* To do this he used only two theorems of complex variable which are easily proved to an elementary group. These are:

1. The distance between two points $P_1\{z_1\}$ and $P_2\{z_2\}$ is given by $P_1P_2 = |z_1 - z_2|$.

2. The point of division of a line segment joining points $P_1\{z_1\}$ and $P_2\{z_2\}$ which divides the segment into the ratio

(65) $$\frac{P_1P}{PP_2} = \lambda, \text{ is}$$
$$z = \frac{z_1 + \lambda z_2}{1+\lambda}.$$

For the special points where P' is the bisector and P'' is trisector, we have

(66) $$P' = \frac{1}{2}(z_1 + z_2), \quad \lambda = 1,$$
$$P'' = \frac{1}{3}(z_1 + 2z_2), \quad \lambda = 2.$$

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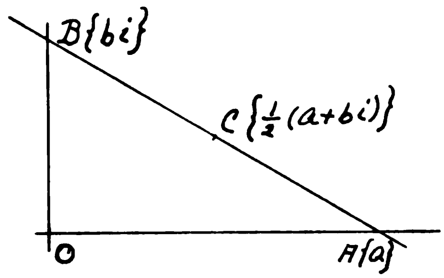
* L. L. Smail, Some Geometrical Applications of Complex Numbers, American Mathematical Monthly, 1929, P. 504.

One of the theorems that he proves is the following:

I. <u>The mid-point of the hypotenuse of a right triangle is equidistant from the vertices.</u>

For, consider the right triangle with vertices at $O\{0\}$, $A\{a\}$, $B\{bi\}$.



Then the point C which is the mid-point of the hypotenuse is given by $C\{\frac{1}{2}(a+bi)\}$. Now

$$OC = |\tfrac{1}{2}(a+bi) - 0| = \tfrac{1}{2}|a+bi| = \tfrac{1}{2}\sqrt{a^2+b^2},$$

$$AB = |a+bi| = \sqrt{a^2+b^2}.$$

Hence

$$OC = \tfrac{1}{2}AB = BC = CA.$$

By similar methods Professor Smail proves eight other theorems.

X. STATISTICS

One of the fields where one would least expect to find a use for complex variable is the field of statistics. However, in some of the developments of statistics, a knowledge of complex variable is very useful. Consider, for example, Laplace's development of Bernouillis' theorem.*

Let us consider the probability expansion

$$(p+q)^n,$$

where

$q =$ probability a thing will not happen,

$p =$ probability this same thing will happen, and n is a positive integer.

We have

$$(67)\qquad (p+q)^n = q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \cdots + \binom{n}{x} q^{n-x} p^x + \cdots + p^n,$$

where $$\binom{n}{x} = \frac{n(n-1)(n-2)\cdots[n-(x-1)]}{x!}.$$

The problem is to obtain a simple expression for the general term, which we will designate by y_x.

Let

$$(68)\qquad \begin{aligned} S' = (p+q)^n = {} & \binom{n}{0} q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \cdots \\ & + \binom{n}{x-1} q^{n-(x-1)} p^{x-1} + \binom{n}{x} q^{n-x} p^x + \binom{n}{x+1} q^{n-(x+1)} p^{x+1} + \cdots \\ & + p^n, \end{aligned}$$

* From notes taken from Lectures by Professor Carver of the University of Michigan, by Professor Crowe of Michigan State College.

and let

$$S = (q e^{\omega i} + p e^{-\omega i})^n = q^n e^{n\omega i} + \binom{n}{1} q^{n-1} p e^{(n-2)\omega i} + \cdots\cdots$$

$$(69) \qquad + \binom{n}{x-1} q^{n-x+1} p^{x-1} e^{(n-2x+2)\omega i} + \binom{n}{x} q^{n-x} p^x e^{(n-2x)\omega i}$$

$$+ \binom{n}{x+1} q^{n-x-1} p^{x+1} e^{(n-2x-2)\omega i} + \cdots\cdots + p^n e^{-n\omega i}.$$

Solving for the term containing $\binom{n}{x}$, we get

$$\binom{n}{x} q^{n-x} p^x = e^{-(n-2x)\omega i}\Big[(q e^{\omega i} + p e^{-\omega i})^n - q^n e^{n\omega i} - \cdots\cdots$$

$$- \binom{n}{x-1} q^{n-x+1} p^{x-1} e^{(n-2x+2)\omega i}$$

$$(70) \qquad - \binom{n}{x+1} q^{n-x-1} p^{n+1} e^{(n-2x-2)\omega i} - \cdots\cdots$$

$$- p^n e^{-n\omega i}\Big].$$

Integrating both members of the above equation with respect to ,

we have

$$(71) \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \binom{n}{x} q^{n-x} p^x \, d\omega = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-(n-2x)\omega i} (q e^{\omega i} + p e^{-\omega i})^n \, d\omega + 0.$$

To show that the other terms are zero consider any term as

$$\binom{n}{x-1} q^{n-x+1} p^{x-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2\omega i} \, d\omega$$

$$(72) \qquad = A \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2\omega + i \sin 2\omega) \, d\omega$$

$$= A\left(\frac{\sin 2\omega}{2} - \frac{i \cos 2\omega}{2}\right)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0,$$

$$A = \binom{n}{x-1} q^{n-x+1} p^{x-1}.$$

Hence

$$\binom{n}{x} q^{n-x} p^x = \frac{1}{\pi}\Big[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-(n-2x)\omega i} (q e^{\omega i} + p e^{-\omega i})^n \, d\omega\Big].$$

We now have the general term expressed as a definite integral. We now wish to evaluate this integral. It is easily seen that

(73) $$e^{-(n-2x)\omega i}\left(qe^{\omega i}+pe^{-\omega i}\right)^{n}=e^{-(n-2x)\omega i+n\log\left(qe^{\omega i}+pe^{-\omega i}\right)}.$$

Therefore

$$\binom{n}{x}q^{n-x}p^{x}=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-(n-2x)\omega i+n\log\left(qe^{\omega i}+pe^{-\omega i}\right)}d\omega.$$

Expanding $\log\left(qe^{\omega i}+pe^{-\omega i}\right)^{n}$ in terms of x, we get

(74) $$\log\left(qe^{x}+pe^{-x}\right)^{n}=c_0+c_1x+c_2\frac{x^2}{2}+\cdots\cdots,$$

where

$c_0=0,$

$c_1=n(p+q),$

$c_2=4npq,$

$c_3=-8npq(q-p),$

$c_4=16npq(1-6pq),$

$c_5=-32npq(q-p)(1-12pq),$

$c_6=64npq(1-30pq+120p^2q^2),$

$-----------------.$

Hence we have

(75) $$\binom{n}{x}q^{n-x}p^{x}=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{2(x-np)\omega i-2npq\omega^2+\cdots}d\omega.$$

Making the substitution

$$\omega=\frac{\omega'}{2},\quad d\omega=\frac{d\omega'}{2}$$

(75) becomes

$$\binom{n}{x} q^{n-x} p^{x} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(x-np)\omega i - \frac{npq\omega^2}{2} + \cdots} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(x-np)\omega i - \frac{npq\omega^2}{2}} \cdot e^{\frac{c_3(\omega i)^3}{3!\,2^3} + \frac{c_4(\omega i)^4}{4!\,2^4} + \cdots} d\omega$$

or,

$$(76) \qquad \binom{n}{x} q^{n-x} p^{x} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(x-np)\omega i - \frac{npq\omega^2}{2}} \left\{ 1 + \frac{c_3(\omega i)^3}{3!\,2^3} + \cdots \right\} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(x-np)\omega i - \frac{npq\omega^2}{2}} d\omega$$

$$(77) \qquad + \frac{c_3}{3!\,2^3} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(x-np)\omega i - \frac{npq\omega^2}{2}} \cdot (\omega i)^3 d\omega + \cdots .$$

Since the series in brackets in (76) converges to the value

$$e^{n \log(qe^{\omega i} + pe^{-\omega i})},$$

the integration performed in (77) is valid and the resultant series converges.

Letting

$$\varphi(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(x-np)\omega i - \frac{npq\omega^2}{2}} d\omega,$$

we have, on differentiating,

$$\varphi'(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(x-np)\omega i - \frac{npq\omega^2}{2}} (\omega i) d\omega,$$

$$(78) \qquad \varphi''(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(x-np)\omega i - \frac{npq\omega^2}{2}} (\omega i)^2 d\omega,$$

$$\vdots$$

$$\varphi^{(n)}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(x-np)\omega i - \frac{npq\omega^2}{2}} (\omega i)^n d\omega.$$

Comparing (77) and (78) we see that

$$\binom{r}{x} q^{r-x} p^{x} = \Phi(x) + \frac{c_3}{3!\,2^3}\,\Phi^{(3)}(x) + \frac{c_4}{4!\,2^4}\,\Phi^{(4)}(x) + \cdots.$$

This is analogous to MacLaurin's expansion.

Using Euler's formula

$$e^{ix} = \cos x + i \sin x,$$

$\Phi(x)$ takes the form,

$$\Phi(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{\cos(x-rp)\omega + i\sin(x-rp)\omega\right\} e^{-\frac{rpq\omega^2}{2}}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\frac{rpq\omega^2}{2}} \cos(x-rp)\omega\,d\omega + \frac{i}{2\pi}\int_{-\pi}^{\pi} e^{-\frac{rpq\omega^2}{2}} \sin(x-rp)\omega\,d\omega.$$

The integral

$$\int_{-\pi}^{\pi} e^{-\frac{rpq\omega^2}{2}} \sin(x-rp)\,\omega\,d\omega$$

may be shown to be zero.

Using

$$\int_{-\infty}^{\infty} e^{-a^2x^2} \cos bx\,dx = 2\int_{0}^{\infty} e^{-a^2x^2} \cos bx\,dx = \frac{\sqrt{\pi}}{a} e^{-\frac{b^2}{4a^2}}$$

with

$$a^2 = \frac{rpq}{2}, \quad b = (x-rp),$$

we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\frac{rpq\omega^2}{2}} \cos(x-rp)\,\omega\,d\omega = \frac{1}{\pi}\int_{0}^{\pi} e^{-\frac{rpq\omega^2}{2}} \cos(x-rp)\,\omega\,d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} e^{-\frac{rpq\omega^2}{2}} \cos(x-rp)\,\omega\,d\omega - \frac{1}{\pi}\int_{\pi}^{\infty} e^{-\frac{rpq\omega^2}{2}} \cos(x-rp)\,\omega\,d\omega$$

$$= \frac{1}{\pi}\cdot\frac{\sqrt{\pi}}{\sqrt{2rpq}}\, e^{-\frac{(x-rp)^2}{2rpq}} - \int_{\pi}^{\infty} \text{remainder}.$$

It can be shown that when pqn is large, the remainder is small.

We find the value of the general term to be

(79) $$\binom{n}{x} q^{n-x} p^{x} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-np)^2}{2\sigma^2}} + \frac{8(q-p)(x-np)^3}{3!\,2^3\sigma^6\sqrt{2\pi}} e^{-\frac{(x-np)^2}{2\sigma^2}} + \cdots,$$

where

$$\sigma = \sqrt{npq}\,.$$

As an example of the use of this formula let us suppose that of 100,000 men of a certain age, 800 die within the year. It is of importance to insurance companies to know the probability that say 1000 men of the same age will die in another year.

In this problem

$$p = .008,$$
$$q = .992,$$
$$n = 100{,}000,$$
$$x = 1{,}000.$$

Substituting in (79)

$$\binom{n}{x} q^{n-x} p^{x} = \frac{1}{\sqrt{7936}\cdot\sqrt{2\pi}} e^{-\frac{(200)^2}{13872}} + \frac{8(.984)(200)^3}{6\cdot 8\cdot (7936)^{3}\sqrt{2\pi}} + \cdots$$

$$= .00039.$$

XI. CARTOGRAPHY

The problem of mapping is the problem of finding a one to one (1,1) correspondence between the surface to be mapped and the surface upon which the map is to be made. For the most part we are interested in mapping the earth on a plane. Now we know that it is impossible to apply a sphere to a plane; that is, a sphere cannot be mapped on a plane in such a way that corresponding arcs will have equal lengths on the sphere and the plane. Therefore, when a map of the earth is made there is necessarily some distortion.

The maps with which we are most familiar are conformal maps. These have the distinction that infinitesmal areas correspond and that the angle between two arcs on the earth is equal to the angle between corresponding arcs on the map. The most common projections which have this property are Lambert's conformal conic projection in which the earth is considered an ellipsoid of revolution and Mercator's projection in which the earth is considered a sphere.

If we have a surface,

$$F(u,v)=0,$$

on which the linear element is

$$ds^2 = V du^2 + U dv^2 \qquad (80)$$

in which V is a function of v alone and U a function of u alone, the element can be expressed in the form

$$ds^2 = m^2(du^2+dv^2),$$

where M is a function of u and v. To see this write ds^2 in the form

(31) $$ds^2 = UV\left(\frac{du^2}{U} + \frac{dv^2}{V}\right).$$

By changing the parameters so that

(32) $$\theta = \int \frac{du}{\sqrt{U}}, \qquad \eta = \int \frac{dv}{\sqrt{V}},$$

the linear element becomes

$$ds^2 = UV(d\theta^2 + d\eta^2).$$

When the linear element is in this form, the surface is said to be expressed in terms of an isothermal orthogonal system of parameters, and the net of u, v curves is said to form an isothermal conjugate net.

After the surface has been expressed in this manner in terms of isothermal orthogonal coordinates, the general conformal representation of the surface upon a plane can be determined. This general representation is given by the equation

$$x + iy = f(u+iv)$$

where $f(u+iv)$ is an analytic function. The element of length $ds^2 = m^2(du^2+dv^2)$ is represented in the plane by

$$ds^2 = dx^2 + dy^2.$$

If we let the curve C be represented by the curve C_1 in the plane and if θ represents the angle which the curve C makes with the curve

$$v = \text{const.},$$

we will have

$$(83)\qquad \begin{aligned} ds_v &= m\,dv, \\ \cos\theta &= \frac{ds_v}{ds} = \frac{du}{\sqrt{du^2+dv^2}}, \\ \sin\theta &= \frac{dv}{\sqrt{du^2+dv^2}}. \end{aligned}$$

Hence

$$e^{i\theta} = \cos\theta + i\sin\theta = \frac{du+idv}{\sqrt{du^2+dv^2}} = \sqrt{\frac{du+idv}{du-idv}},$$

or

$$e^{2i\theta} = \frac{du+idv}{du-idv}.$$

Therefore

$$e^{2i(\theta-\theta_1)} = \frac{du+idv}{du-idv}\cdot\frac{dx-idy}{dx+idy},$$

or

$$(84)\qquad e^{2i(\theta-\theta_1)} = \frac{du+idv}{dx+idy}\cdot\frac{dx-idy}{du-idv}.$$

Since

$$x+iy = f(u+iv)$$

so will

$$x-iy = f(u-iv).$$

If we differentiate with respect to the complex variable $x+iy$ we have

$$dx+idy = (du+idv)\,f'(u+iv),$$

or

$$(85)\qquad f'(u+iv) = \frac{dx+idy}{du+idv}.$$

In the same manner

$$(86)\qquad f'(u-iv) = \frac{dx-idy}{du-idv}.$$

Substituting (85) and (86) in equation (84) we obtain

$$e^{2i(\theta-\theta_1)} = \frac{f'(u-iv)}{f'(u+iv)}.$$

If B and B_1 are another pair of corresponding curves starting from the same point and if their angles are denoted by Φ and Φ_1 we will find by the method used above that

$$e^{2i(\Phi-\Phi_1)} = \frac{f'(u-iv)}{f'(u+iv)} .$$

Hence

$$e^{2i(\theta-\theta_1)} = e^{2i(\Phi-\Phi_1)},$$

or

$$\theta-\theta_1 = \Phi-\Phi_1,$$

and

$$\theta-\Phi = \theta_1-\Phi_1 .$$

This shows that the angles between curves on the surface is preserved in the representation on the plane and the map is conformal.

Now if we consider the earth to be a sphere its equations are

(87)
$$\begin{aligned} x &= \sin u \cos v, \\ y &= \sin u \sin v, \\ z &= \cos u, \end{aligned}$$

where v is the longitude and u the latitude. Here the element of length is

$$ds^2 = dx^2+dy^2+dz^2,$$

with

$$\begin{aligned} dx &= \cos u \cos v\, du - \sin u \sin v\, dv, \\ dy &= \cos u \sin v\, du + \sin u \cos v\, dv, \\ dz &= -\sin u\, du. \end{aligned}$$

Expressing ds^2 in terms of u and v we get

$$\begin{aligned} ds^2 = {} & (\cos^2 u \cos^2 v + \cos^2 u \sin^2 v)du^2 \\ & + (\sin^2 u \sin^2 v + \sin^2 u \cos^2 v)dv^2 \\ & + \sin^2 u\, du^2. \end{aligned}$$

The transformation (82) takes the form

(88)
$$\begin{aligned} \bar{u} &= \int \frac{du}{\sin u} = \int \csc u\, du = \log \tan \frac{u}{2}, \\ \bar{v} &= v. \end{aligned}$$

If we solve the first equation of (88) for u in terms of $\mathfrak{U}$ and then find the values of x, y, z in terms of $\mathfrak{U}$ and $\mathfrak{V}$ by substituting the values of u, v in (87), we can substitute these values in equations (84), (85), (86), and show that the transformation actually does give a conformal map.*

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* For a discussion of this transformation treating the earth an an ellipisoid of revolution and of the errors introduced by the transformation, see the phamphlet by O. S. Adams, <u>General Theory of the Lambert Conformal Conic Projection</u>, U. S. Coast and Geodetic Survey, Special Publication No. 53- 1918.

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