ON THE STABILITY/SENSITIVITY OF RECOVERING VELOCITY FIELDS FROM BOUNDARY MEASUREMENTS

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ABSTRACT

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The thesis investigates the stability/sensitivity of the inverse problem of recovering velocity fields in a bounded domain from the boundary measurements. The problem has important applications in geophysics where people are interested in finding the inner structure (the velocity field in the elastic wave models) of earth from measurements on the surface. Two types of measurements are considered. One is the boundary dynamic Dirichlet-to-Neumann map (DDtN) for the wave equation. The other is the restricted Hamiltonian flow induced by the corresponding velocity field at a sufficiently large time and with domain the cosphere bundle of the boundary, or its equivalent form the scattering relation. Relations between these two type of data are explored. Three main results on the stability/sensitivity of the associated inverse problems are obtained: (1). The sensitivity of recovering scattering relations from their associated DDtN maps. (2). The sensitivity of recovering velocity fields from their induced boundary DDtN maps. (3). The stability of recovering velocity fields from their induced Hamiltonian flows. In addition, a stability estimate for the X-ray transform in the presence of caustics is established. The X-ray transform is introduced by linearizing the operator which maps a velocity field to its corresponding Hamiltonian flow. Micro-local analysis are used to study the X-ray transform and conditions on the background velocity field are found to ensure the stability of the inverse transform. The main results suggest that the DDtN map is very insensitive

to small perturbations of the velocity field, namely, small perturbations of velocity field can result changes to the DDtN map at the same level of large perturbations. This differs from existing Hölder type stability results for the inverse problem in the case when the velocity fields are simple. It gives hint that the methodology of velocity field inversion by DDtN map is inefficient in some sense. On the other hands, the main results recommend the methodology of inversion by Hamiltonian flow (or its equivalence the scattering relation), where the associated inverse problem has Lipschitz type stability.

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Chapter 1

Introduction

1.1 The main inverse problem

The work in the thesis mainly comes from [9], where the sensitivity of the inverse problem of recovering velocity fields from boundary DDtN maps is investigated. The result on the stability of recovering velocity fields from Hamiltonian flows are obtained as a byproduct. For this reason, we refer the inverse problem of recovering velocity fields from boundary DDtN maps as our main problem, which we formulate now. Let Ω be a bounded strictly convex smooth domain in $\mathbb{R}^d$, $d \geq 2$, with boundary Γ. Let $c(x)$ be a velocity field in Ω which characterizes the wave speed in the medium and let T be a sufficiently large positive number. Consider the following wave equation system:

$$\begin{aligned} \frac{1}{c^2(x)}u_{tt} - \Delta u &= 0, \quad (x,t) \in \mathbb{R}^d \times (0,T) &(1.1)\\ u(0,x) = u_t(0,x) &= 0, \quad x \in \Omega, &(1.2)\\ u(x,t) &= f(x,t), \quad (x,t) \in \Gamma \times (0,T). &(1.3) \end{aligned}$$

For each $f \in H_0^1([0,T] \times \Gamma)$, it is known that (see for instance [31]) there exists an unique solution $u \in C^1(0,T;L^2(\Omega)) \bigcap C(0,T;H^1(\Omega))$, and furthermore $\frac{\partial u}{\partial \nu} \in L^2([0,T] \times \Gamma)$, where ν is the unit outward normal to the boundary. The DDtN map Λ_c is defined by

$$\Lambda_c(f) := \frac{\partial u}{\partial \nu}|_{[0,T]\times\Gamma}.$$

The inverse problem is to recover the velocity function c from the DDtN map Λ_c.

The uniqueness of the inverse problem is solved by the boundary control method first introduced by Belishev in [10]. The method can also be used to solve the uniqueness for more general problems, for instance, the anisotropic medium case. We refer to [11], [13], [12], [30] and the references therein for more discussions.

We are interested in the sensitivity of the above inverse problem. Namely, we want to investigate how sensitive or stable is it to recover the velocity field from the DDtN map and characterize how small changes in the DDtN map affect the recovered velocity field.

The inverse problem of recovering velocity field is closely related to the inverse kinematic problem in geophysics, which we shall briefly review in chapter and we refer to [43] for more discussions. It also can be viewed as a special case of the inverse problem of recovering a Riemannian metric on a Riemannian manifold. Indeed, it corresponds to the case when the metrics are restricted to the class of those which are conformal to the Euclidean one. The inverse problem of recovering a Riemannian metric has been extensively studied in the literature. The uniqueness is proved by Belishev and Kurylev in [13] by using the boundary control method. However, it is still unclear that if their approach can give a stability estimate since it uses in an essential way an unique continuation property of the wave equation.

The first stability result on the determination of the metric from the DDtN map was given by Stefanov and Uhlmann in [46], where they proved conditional stability of Hölder type for metrics close enough to the Euclidean one in C^k for $k \gg 1$ in three dimensions. Later, they extended the stability result to generic simple metrics, [49]. Here we recall the definition of simple manifold.

Definition 1.1.1. *A compact Riemannian manifold (M, g) with boundary ∂M is called simple if ∂M is strictly convex with respect to g, and for any $x \in M$, the exponential map $exp_x : exp_x^{-1}(M) \to M$ is a diffeomorphism.*

An important feature of their approach is to first derive a stability estimate of recovering the boundary distance function from the DDtN map and then apply existing results from the boundary rigidity problem in geometry. Their approach was extended by Montalto in [33] to study the more general problem of determine a metric, a co-vector and a potential simultaneously from the DDtN map, and a similar Hölder type conditional stability result was obtained. The stability of the inverse problem of determining the conformal factor to a fixed simple metric was studied by Bellassoued and Ferreira in [14]. They proved the Hölder type conditional stability result for the case when the conformal factors are close to one. We comment that the result in [14] holds for all simple metrics. For other stability results on the related problems, we refer to the references in [33].

We emphasize that all of the above stability results deal with the case when the metrics are simple. To our best knowledge, no stability result is available for the general case when the metrics are not simple.

The thesis is devoted to the study of the general case when the metrics induced by the velocity fields are not simple. To avoid technical complications due to the boundary, we restrict our study to situations when the velocity fields are equal to one near the boundary. From this point of view, our results can be regarded as interior estimates. We refer to [49], [51] and the references therein for useful boundary estimates. In contrast to existing results mentioned above, where Hölder type stability estimate are suggested for the case when the velocity fields are simple, our result shows that the inverse problem of recovering velocity fields from their induced DDtN maps is insensitivity to small perturbations of the data. In fact, we showed that for a quite general velocity field, which we call "fold-regular" (see definition 6.2.3, 6.2.4, 8.0.1), if another velocity field is sufficiently close to it and satisfies a certain orthogonality condition, then the two must be equal if the two corresponding DDtN maps are sufficiently close. On the other hand, we showed that the inverse problem of recovering velocity fields from their induced Hamiltonian flows at a sufficiently large time is well-posed, in the sense that a local Lipschitz stability estimate for the inverse problem can be established.

1.2 Overview of the approach of solving the main problem

We now briefly review the approach we used to solve the main problem introduced in the previous section. We first derive a sensitivity result of recovering the scattering relation from the DDtN map. Our result shows that two scattering relations must be identical if the two corresponding DDtN maps are sufficiently close in some suitable norm. Equivalently, any arbitrarily small change in the scattering relation can imply a certain change in the DDtN map. To our best knowledge, this is the first sensitivity result for the problem in the non-simple metric case. Moreover, our result is fundamentally different from those in the literature where Lipschitz, Hölder or logarithmic estimates are derived , see for examples [8], [2], [46], [47], [48], [49], [33] and [28]. This is the reason the term "sensitivity" is used instead of "stability" whenever it is more proper. We remark that when the geometry induced by the velocity field is simple, the scattering relation is equivalent to the boundary distance function. In that case, a Hölder type interior stability estimate for recovering the boundary distance function from DDtN map has been established in [49]. Compared to the Hölder type result, our result is much stronger. Our approach is based on Gaussian beam solutions to the wave equation, which are capable of dealing with caustics, major obstacles to the construction of classic geometric-optics solutions. We refer to [37] and [30] for more discussions on Gaussian beams and its applications.

We observe that for any velocity filed c, the induced Hamiltonian flow $\mathcal{H}_c^t$ when restricted to the unit cosphere bundle $S^*\mathbb{R}^d$ determines the scattering relation $\mathfrak{S}_c$. We linearize the operator which maps c to $\mathcal{H}_c^t|_{S^*\mathbb{R}^d}$ and obtain a geodesic X-ray transform operator $\mathfrak{I}_c$ with matrix-valued weight. Note that the scattering relation (or Hamiltonian flow) is the natural object to study when the metric is not simple. It is related to the lens rigidity problem in geometry. We refer to [53] for more discussions on the topic. The boundary distance function (global or local) has received extensive attention in the literature for the problem where the metrics are "simple" or "regular", see for instance [48], [53]. However, it is unlikely to work for the case of general non-simple metrics. In the thesis, we attempt to overcome the difficulty by analyzing the scattering relation (or Hamiltonian flow).

We study the inverse problem of recovering a vector-valued function f from its weighted geodesic transform $\mathfrak{I}_c f$. For a fixed interior point x, we use a carefully selected set of geodesics whose conormal bundle can cover the cotangent space $T_x^*\mathbb{R}^d$ to recover the singularity of f at x. We allow fold caustics along these geodesics, but require that these caustics contribute to a smoother term in the transform than x itself. It is still an open problem to show that such a set of geodesics exists generically for a general velocity field with caustics. But we draw evidence from the classification result on caustics and regularity theory of Fourier Integral Operators (FIOs) to show that it is the case under some natural assumptions in the dimensions equal to or greater than three. We call the interior point which has the above set of geodesics "fold-regular". A local stability estimate is derived near fold-regular points.

We define a velocity field to be fold-regular if every interior point is fold-regular with respect to the Hamiltonian flow induced by it. As a consequence of the above local stability result, we obtain a Lipschitz stability result, up to a finite dimensional space, for the X-ray transform in a fold-regular background velocity field, or the linearized inverse problem of recovering velocity fields from their induced Hamiltonian flows at a fold-regular background velocity field. By standard arguments of linearization, it yields a similar stability estimate for the nonlinear inverse problem. We remark that it is still an open problem to show whether the finite dimensional space is empty or not. This is closely related to the injectivity of the X-ray transform $\mathfrak{I}_c f$.

Finally, We combine the stability result on the X-ray transform and the sensitivity result on recovering scattering relations from DDtN maps to deduce a sensitivity result for the inverse problem of recovering velocity fields from their induced DDtN maps.

1.3 Outline of the thesis

The thesis is organized as follows. In the first two chapters, chapter 2 and 3, we present some background material for the problem we contributed in the thesis. The results shown therein are standard and well-known in the field. In chapter 2, we give a brief introduction to the inverse kinematic problems of seismic (or travel time tomography in seismic). It serves as a major motivation for our investigation of the stability of recovering velocity field from Hamiltonian flow, which can be viewed as a generalization of the travel time tomography to the more general case when the velocity fields are not simple (see chapter 7). In chapter 3, we introduce the X-ray transform for scalar functions in the case when the background metric is simple. It prepares necessary preliminaries for the more general results in chapter 6, which works for the case when the background metric is not simple. Starting from chapter 4, we investigate the main problem in the thesis. This is where new ideas and results are developed. We first give some preliminaries in chapter 4, fixing notations and conventions. We then derive a sensitivity result for recovering scattering relations from their corresponding DDtN maps in chapter 5. In chapter 6, we show the equivalence of the scattering relation and the Hamiltonian flow. We linearize the Hamiltonian flow with respect to the velocity field. This leads to a X-ray transform. We study properties of the X-ray transform and establish some stability estimate for the transform. In chapter 7, we apply the stability estimate to the nonlinear inverse problem of recovering velocity fields from their induced Hamiltonian flows, a Lipschitz type stability result is obtained. Finally, in chapter 8, we combining the results from chapter 5 and 6 to study the sensitivity of recovering velocity fields from their induced DDtN maps.

Chapter 2

The inverse kinematic problem of seismic

2.1 Introduction

In geophysics, a basic problem is to study the earth's inner structure by making observations of various wave fields on the surface. In the elastic model, one assumes that the earth is an elastic body and the inner structure of earth is characterized by the velocity field of the elastic waves. Geophysicists are interested in finding the velocity field by measuring the travel time of seismic waves between points on the surface of the earth. The problem is referred to the inverse kinematic problem of seismic or travel time tomograph in seismic. The first result on the problem was obtained by G.Herglotz, E.Wiechert and K.Zeoppritz in 1905. They modeled the earth by a ball with spherically symmetric metric and studied the following mathematical problem: Let $M = \{x \in \mathbb{R}^3 : |x| \leq R\}$. Assume that the metric of M is given by

$$d\tau^2 = n^2(r)dx^2 \tag{2.1}$$

where $n(r)$ is a positive function on $[0, R]$. The mathematical problem then becomes to determine the function $n(r)$ from the induced boundary distance function.

They solved the problem and constructed solution to the inverse problem under the following "non-trapping" assumption

$$(rn'(r))' > 0.$$

The general problem is to recover a metric of the following form

$$d\tau^2 = n^2(x_1, x_2, x_3)dx^2 \tag{2.2}$$

from the boundary distance function induced by it. Linearization of the problem leads to a linear integral geometry problem. Both the linear and the nonlinear problem in two dimensions were solved by Mukhometov [20, 21] under assumptions which can be interpreted as the "simple metric" assumption. In addition Romanov studied the linearized problem near a spherically symmetric metric in 1967 [41].

Generalizations of the above two-dimensional case to multinational case were studied by Mukhometov [22, 23], Anikonov and Romanov [4], Bernstein and Gerver [15], Romanov [42], Pestov and Sharafutdinov [34].

We remark that all the results above require that the velocity fields are "simple", in the sense that the metrics induced by them are simple. In the general case when the velocity fields are not simple, the inverse problem of travel time tomography is no longer well-posed. In fact, in that case, the geodesics may have conjugate points (or caustics) and hence are not necessarily length minimizing. Therefore, linearization of the travel time tomography problem may not be well-defined. For the general case without "simple metric" assumption, we need more data than the travel time in order to "regularize" the problem. This leads to the lens rigidity problem where additional information about directions of geodesics (the scattering relation) are used to recover the associated metric (or the velocity field in our case). We shall briefly introduce the lens rigidity problem in Section 2.3 and its connection with our result in chapter 6.

2.2 Mukhometov's solution in 2D

In this section, we present Mukhometov's solution to the inverse kinematic problem in the two dimensional case. We choose his solutions for two main reasons: one is the simplicity of the results; the other is its importance in the development of the field.

We first introduce the concept "regular family of curves" which is a key assumption in his results. Let M be a bounded simply connected domain on the plane with C^1 smooth boundary ∂M. Parameterize ∂M by

$$x_1 = \tau_1(t),\ x_2 = \tau_2(t), \quad 0 \le t \le T,$$

where t is the arc length parameter and T is the length of ∂M. We call that a family of curves Γ in $\overline{M}$ are "regular" if the following four conditions are satisfied:

(1). For any two different points on ∂M, there exists a unique curve $\gamma \in \Gamma$ joining them.

(2). The end points and inner points of any $\gamma \in \Gamma$ belong to ∂M and M respectively.

Moreover, the lengths of all curves are uniformly bounded.

(3). For each point $z \in M$, direction θ, there exists a unique curve $\gamma \in \Gamma$ passing through z with direction θ. We parameterize the curve by $\gamma = \gamma(z, \theta, s)$ where s is the standard length parameter.

(4). The vector-valued function γ in (3) is C^3 in M and satisfies

$$\frac{\partial \gamma}{\partial(\theta, s)} \geq sC$$

for some $C > 0$.

We remark that in the case when the family of curves are the geodesics (with respect to some metric equipped to M) passing through M, the "regular" condition can be shown to be equivalent to the "simple metric" assumption on the underlining metric.

Now, we formulate and present Mokhometov's solution to the 2D inverse kinematic problem of seismic. Let $n(x, y) > 0$ be a function defined in M. Consider the Riemannian metric

$$d\tau^2 = n^2(x, y)(dx^2 + dy^2)$$

and the corresponding functional

$$J(\gamma) = \int_\gamma d\tau = \int_\gamma n(x, y)\sqrt{dx^2 + dy^2}.$$

Assume that the family of extremals of the functional $J(\gamma)$ is regular in the sense just defined. We can thus define the boundary distance function

$$\Gamma(t_1,t_2)=\int_{\gamma(t_1,t_2)} nds,$$

where $\gamma(t_1,t_2)$ is the unique extremal of the functional $J(\gamma)$ joining the boundary points $(\tau_1(t_1),\tau_2(t_1))$ and $(\tau_1(t_2),\tau_2(t_2))$.

Mukhometov established the following result.

Theorem 2.2.1. *Let $n\in C^4(\bar{M})$ be such that the corresponding family of extremals for J is regular. Then $n(x,y)$ can be uniquely determined from $\Gamma(t_1,t_2)$, moreover, the following stability estimate holds:*

$$\|n_1-n_2\|_{L^2(M)}\leq \frac{1}{\sqrt{2\pi}}\|\frac{\partial(\Gamma_1-\Gamma_2)}{\partial t_1}\|_{L^2([0,T]\times[0,T])}.$$

Proof. See [21].

In addition to the nonlinear inverse problem considered above, Mukhometov investigated the linearized problem, which is a X-ray transform, and derived a similar Lipschitz stability estimate.

Theorem 2.2.2. *Let $f\in C^2(\bar{M})$, define*

$$I(f)(t_1,t_2)=\int_{\gamma(t_1,t_2)} f(x_1,x_2)\,ds,\quad 0\leq t_1,t_2\leq T,$$

where $\gamma(t_1,t_2)$ is the unique curve in Γ joining the boundary points $(\tau_1(t_1),\tau_2(t_1))$ and $(\tau_1(t_2),\tau_2(t_2))$. If the family of curves $\{\gamma(t_1,t_2)\}_{0\leq t_1,t_2,\leq T}$ are regular, then the following *estimate holds*

$$\|f\|_{L^2(D)} \leq \frac{1}{\sqrt{2\pi}} \|\frac{\partial g(t_1, t_2)}{\partial t_1}\|_{L^2([0,T]\times[0,T])}.$$

Proof. See [20].

2.3 The lens rigidity problem

Let (M, g) be a compact Riemannian manifold with boundary ∂M. Let $\mathcal{H}^t$ be the geodesic flow on the tangent bundle TM and let SM be the unit tangent bundle. Denote

$$\begin{aligned} S_+\partial M &= \{(x, \xi) : x \in \partial M, |\xi|_g = 1, \langle \xi, \nu(x) \rangle > 0\}; \\ S_-\partial M &= \{(x, \xi) : x \in \partial M, |\xi|_g = 1, \langle \xi, \nu(x) \rangle < 0\}. \end{aligned}$$

where ν is the unit out-normal and $\langle \cdot, \cdot \rangle$ stands for the inner product. We now define the scattering relation. There are several definitions. Here, we only introduce the simplest one. We assume that the metric g is known for all points on the boundary. For each $(x, \xi) \in S_-\partial M$, define $L(g)(x, \xi) > 0$ to be the first positive moment at which the unit speed geodesic passing through (x, ξ) hits the boundary ∂M. If $L(g)(x, \xi)$ does not exist, we define formally $L_g(x, \xi) = \infty$ and call the corresponding geodesic trapped. We define $\Sigma(g) : S_-\partial M \to \overline{S_+\partial M}$ by

$$\Sigma(g)(x, \xi) = \Phi^{L(g)}(x, \xi).$$

We call the manifold (M, g) non-trapping if there exists $T > 0$ such that $L(g)(x, \xi) \leq T$ for all $(x, \xi) \in S_-\partial M$. We call the pair $(\Sigma(g), L(g))$ the full scattering relation induced by the metric g.

The lens rigidity problem can be formulated as recovering the metric g from its induced full scattering relation $(\Sigma(g), g)$. It is known that there is no uniqueness to this problem. The first obstruction comes from the diffeomorphisms which leave the boundary ∂M fixed. In addition to this, trapping of geodesics also prevents the uniqueness of the problem. See the counterexamples constructed in [17]. Therefore, a more natural formulation for the lens rigidity problem is as follows:

Conjecture 1. *Given a compact non-trapping Riemannian manifold (M, g) with boundary ∂M, can we recover the metric g by its induced full scattering relation $(\Sigma(g), L(g))$, up to an action of diffeomorphism which leaves the boundary fixed.*

It was observed by Michel that the lens rigidity and boundary rigidity problem are equivalent for simple metrics. We refer to [48] and the reference therein for the uniqueness and stability results for the boundary rigidity problem. With regarding to the lens rigidity problem, there are only two uniqueness results available:

1. Croke (2005) [16]: The finite quotient space of a lens rigid manifold is lens rigid.

2. Stefanov and Uhlmann (2009) [53]: Uniqueness up to diffeomorphism fixing the boundary for metrics a priori close to a generic regular metrics.

To our best knowledge, there is no stability result on the lens rigidity problem. In Chapter 7, we obtained a stability estimate for an equivalent problem in the case when the metrics are conformal to the Euclidean metric, i.e. the inverse problem of recovering velocity fields from their induced Hamiltonian flows. By exploring the relation between the scattering relation and the information we used in the Hamiltonian flow, we can establish a Lipschitz stability estimate for the corresponding lens rigidity problem.

Chapter 3

Weighted X-ray transform for scale functions

3.1 Introduction

The X-ray transform is an integral transform first introduced by Fritz John in 1938. In the simplest form, it can be stated as follows. Let f be a compactly supported function in $\mathbb{R}^d$. for each straight line passing through point z with direction ω, define

$$I(f)(z, \omega) = \int f(z + t\omega)dt, \quad z \in \mathbb{R}^d, \omega \in \mathbb{S}^{d-1}.$$

$I(f)$ is called the X-ray transform of the function f. There are two nature questions for the transform. One is the uniqueness, namely, given $I(f)(z, \omega)$ for all or some (z, ω), can we recover f? The other is the stability, namely, how perturbations in the data $I(f)$ affect the reconstructed f.

The X-ray transform has important applications in the field of medical imaging. X-ray computed tomography, also called computed tomography (CT), utilizes computer-processed X-rays to produce tomographic images or slices of specific areas of human body. In that case, f represents the density of an inhomogeneous medium and $I(f)$ represents the scattered data of a tomographic scan. Besides medical imaging, the X-ray transform also finds applications in the inverse kinematic problems of seismic and nondestructive materials testing.

The X-ray transform coincides with the Radon transform in the two dimensional case. The Radon transform was first introduced by Johann Radon in 1917, who also derived a formula for the inverse transform. We remark that both the X-ray transform and the Radon transform belong to the broad field of integral geometry. We refer to [44] for more detail.

We shall present some basics about the X-ray transform in this chapter. Section 3.2 and 3.3 studies the case in the Euclidean space with constant and variable weight respectively, while the case in simple Riemannian manifold is studied in Section 3.4. These results gives necessary preliminaries for us to understand the stability results we derived for an X-ray transform in the case when the Riemannian manifold is not simple in Chapter 6.

3.2 X-ray transform in the Euclidean space

We study the X-ray transform in the Euclidean space with constant weight in this section. We first derive a formula for the inverse of the transform. We start by parameterizing the set of straight lines by

$$\mathbb{S}^{d-1} \times \mathbb{R}^{d-1} = \{(\theta, x) : \theta \in \mathbb{S}^{d-1}, \langle x, \theta \rangle = 0\}.$$

We then define the X-ray transform in the following form

$$I(f)(\theta, x) = \int f(x + t\theta)dt.$$

We have the following important theorem which is also called the Projection-Slice Theorem.

Theorem 3.2.1. *The following identity holds*

$$I(f)^\wedge(\theta, \xi) = (2\pi)^{1/2} f^\wedge(\xi), \xi \perp \theta,$$

where the fourier transform on the left hand side is the (d-1)-dimensional Fourier transform in $x \in \theta^\perp$*, namely,*

$$I(f)^\wedge(\theta, \xi) = \int_{x \in \theta^\perp} e^{-ix\cdot\xi} I(f)(\theta, x)dx, \quad \textit{for} \;\; \xi \perp \theta.$$

while the Fourier transform on the right hand side is the usual d-dimensional Fourier transform in $x \in \mathbb{R}^d$*.*

We now introduce the adjoint of the X-ray transform. We define

$$I^\dagger g(x) = \int_{\mathbb{S}^{d-1}} g(\theta, E_\theta x)d\theta,$$

where E_θ is the orthogonal projection onto $\theta^\perp$, i.e. $E_\theta x = x - \langle x, \theta \rangle \theta$.

With the help of the adjoint operator $I^\dagger$, we obtain the following recovering formula.

Theorem 3.2.2.

$$f = \frac{1}{2\pi|\mathbb{S}^{d-1}|} I^{\dagger} R(I(f)),$$

where the operator R is defined by $(Rh)^{\wedge}(\theta,\xi) = |\xi| h^{\wedge}(\theta,\xi)$.

As a consequence of the recovering formula above, we have the following uniqueness result on the inverse of the X-ray transform.

Theorem 3.2.3. *Let $d = 3$, and assume that each great circle on $\mathbb{S}^2$ meets S_0. Then, f is uniquely determined by $I(f)(\theta, \cdot)$ for $\theta \in S_0$.*

The condition on S_0 above is called Orlov's completeness condition. An obvious example of S_0 satisfying Orlov's completeness condition is the great circle.

3.3 X-ray transform with weight in the Euclidean space

We now study the X-ray transform with weight in $\mathbb{R}^d$. Let M be a convex domain in $\mathbb{R}^d$ and let $w = w(x,\theta)$ be a smooth function defined in $M \times \mathbb{S}^{d-1}$. Assume that supp$f \subset M$. Define the weighted X-ray transform by

$$I_w(f)(x,\theta) = \int w(x+t\theta,\theta) f(x+t\theta) dt, \quad , x \in \mathbb{R}^d, \theta \in \mathbb{S}^{d-1}.$$

We are interested in the uniqueness and stability of recovering f from $I_w f$. Unfortunately, for some weight, the uniqueness fails, as is shown by the following counterexample first constructed by Boman in 1993.

Theorem 3.3.1. *Let B be the unit disk in the plane. There exist functions $w \in C^\infty(B)$ and $f \in C^\infty(B)$ with the property that $w > c_0 > 0$, $f \neq 0$, but*

$$I_w(f) \equiv 0.$$

Although the uniqueness does not hold in the case of a general weight. We still can study the stability of the inverse transform up to a finite dimensional space. This is because of the fact that the kernel of the X-ray transform can be shown to be of finite dimensional for general smooth weight.

We now focus on the stability of the X-ray transform. Define

$$\begin{aligned} S_-\partial M &= \{(x,\omega) : x \in \partial M, |\omega| = 1, \langle \omega, \nu(x) \rangle < 0\}, \\ SM &= \{(x,\theta) : x \in M, |\theta| = 1\}. \end{aligned}$$

The set $S_-\partial M$ parameterizes all straight lines passing through M. We define a measure μ on $S_-\partial M$ by

$$d\mu(x,\omega) = |\langle \omega, \nu(x) \rangle| dS(x) d\sigma(\omega),$$

where dS and $d\sigma$ are the Lebesgue measures in ∂M and $\mathbb{S}^{d-1}$. We have the following elementary results about the X-ray transform and its adjoint.

Lemma 3.3.1. *I_w is a bounded linear operator from $L^2(M)$ to $L^2(S_-\partial M, d\mu)$.*

Lemma 3.3.2. *The adjoint operator $I_w^\dagger$ has the following representation*

$$I^\dagger g(x) = \int_{\mathbb{S}^{d-1}} g^\sharp(x,\theta) w(x,\theta) d\sigma(\theta),$$

where $g^\sharp(x,\theta)$ is defined as the function that is constant along the ray and is equal to $g(x,\theta)$ on $S_-\partial M$.

We define the normal operator N_w to be $I_w^\dagger I_w$. We can show that N_w has the following expression

$$N_w f(x) = c_n \int \frac{W(x,y)f(y)}{|x-y|^{n-1}} dy,$$

where

$$W(x,y) = \bar{w}(x, -\frac{x-y}{|x-y|})w(y, -\frac{x-y}{|x-y|}) + \bar{w}(x, \frac{x-y}{|x-y|})w(y, \frac{x-y}{|x-y|}).$$

We now present some properties of the normal operator N_w.

Theorem 3.3.2. *N_w is a elliptic ΨDO of order -1, provided the following elliptic condition on w is satisfied:*

$$\forall\, (x,\xi) \in M \times \mathbb{S}^{d-1}, \quad \exists\, \theta \in \mathbb{S}^{d-1} \textit{such that } \ \theta \perp \xi \ \textit{and } w(x,\theta) \neq 0.$$

Moreover, in that case there exists a constant $C > 0$ such that

$$\|f\|_{L^2(M)} \leq C(\|N_w f\|_{H^1(M)} + \|f\|_{H^{-s}(M)}), \quad \forall f \in L^2(M)$$

for any $s > 0$.

Proof. See [52].

As a consequence of the above theorem, we have the following corollary.

Theorem 3.3.3. *Let $w \in C^\infty(M \times \mathbb{S}^{d-1})$ satisfy the elliptic condition. Assume that I_w is injective on $C^\infty(M)$. Then $N_w : L^2(M) \to H^1(M)$ is onto, and there exists a constant $C > 0$ such that*

$$\|f\|_{L^2(M)} \leq C\|N_w f\|_{H^1(M)}, \quad \forall f \in L^2(M).$$

Moreover, the above estimation remains true under small C^1 perturbations of w.

3.4 X-ray transform with weight on Riemannian Manifold

We consider the X-ray transform with weight defined on a Riemannian manifold in this section. Let (M, g) be a Riemannian Manifold with boundary ∂M. Assume that ∂M is strictly convex. Define

$$\begin{aligned} S_-\partial M &= \{(x, \omega) \in TM : x \in \partial M, |\omega| = 1, \langle w, \nu(x)\rangle < 0\}, \\ SM &= \{(x, \omega) \in TM : x \in M, |\omega| = 1\}. \end{aligned}$$

Define measure μ on $S_-\partial M$ by

$$d\mu(x, \omega) = |\langle \omega, \nu(x)\rangle| dS_x(x) d_x\sigma(\omega),$$

where $d_xS(x)$ and $d_x\sigma(\omega)$ are in surface measure in ∂M and S_xM in the metric, respectively. $d\sigma_x(\omega) = (\det g)^{1/2} d\sigma_0(\omega)$, where $d\sigma_0(\omega)$ is the Lebesgue measure in $\mathbb{S}^{d-1}$.

For each $(x, \omega) \in S_-\partial M$, denote $\gamma_{x,\omega}(t)$ the unit speed geodesic through (x, ω). We call M non-trapping if $\gamma_{x,\omega}(t)$ hit ∂M at finite time for all $(x,\omega) \in S_{-}\partial M$.

We define the X-ray transform on M with weight w by

$$I_w f(z,\omega) = \int w(\gamma_{x,\omega}(t), \dot{\gamma}_{x,\omega}(t)) f(\gamma_{x,\omega}(t)) dt.$$

We now present some basic results on the uniqueness and stability of the inverse of the above X-ray transform.

Theorem 3.4.1. *Let (M,g) be a simple Riemannian manifold, then $I_w^{\dagger} I_w$ is an elliptic ΨDO of order -1, provided the following elliptic condition on w is satisfied:*

$$\forall\, (x,\zeta) \in T^*M, \quad \exists\, \theta \in T_x M \text{ such that } \theta \perp \zeta \text{ and } w(x,\theta) \neq 0.$$

Theorem 3.4.2. *Let (M,g) be a simple Riemannian manifold, and let $w \equiv 1$, then*

1. *I is injective;*

2. $\|f\|_{L^2(M)} \leq C \|N_w f\|_{H^1(M)}$.

Proof. See [22, 23, 15] and [47].

Theorem 3.4.3. *Let (M,g) be a simple Riemannian manifold. Assume the ellipticity condition on the weight w, then*

1. *I_w has a finitely dimensional smooth kernel;*

2. $\|f\|_{L^2(M)} \leq C \|N_w f\|_{H^1(M)}, \quad \forall f \in (Ker I)^{\perp}$*;*

3. *If I is injective, then* $\|f\|_{L^2(M)} \leq C \|N_w f\|_{H^1(M)} \quad \forall f \in L^2(M)$.

Proof. See [47].

Chapter 4

Preliminaries

Starting from this chapter, we study the main inverse problem introduced in section 1.1 in the following chapters: 5, 6, 7 and 8. For the convenience of reading, we first fix notations and definitions in this chapter. They will be used throughout the rest of the thesis. Let Ω be a strictly convex smooth domain in $\mathbb{R}^d$ with boundary Γ. Let c be a smooth velocity field defined in Ω which is equal to one near the boundary. Then c has natural extension to $\mathbb{R}^d$. Throughout the paper, we always use the natural coordinate system of the cotangent bundle $T^*\mathbb{R}^d$ in which we write (x, ξ) for the co-vector $\xi_j dx^j$ in $T^*_x\mathbb{R}^d$. For ease of notation, we also use ξ for the co-vector $\xi_j dx^j$. The meaning of ξ should be clear from the context. The velocity field c introduces a Hamiltonian function $H_c(x, \xi) = \frac{1}{2}c^2(x)|\xi|^2$ to $T^*\mathbb{R}^d$. It also defines a norm to each cotangent space $T^*_x\mathbb{R}^d$ by

$$|\xi|_c = c(x)|\xi|, \quad \text{for} \;\; \xi \in T^*_x\mathbb{R}^d.$$

Here, $|\cdot|$ stands for the usual Euclidean norm in $\mathbb{R}^d$, while $|\cdot|_c$ stands for the norm in $T^*\mathbb{R}^d$ induced by the function c. When there is no other velocity field in the context, we drop the subscript c and write $\|\cdot\|$ instead.

Denote the corresponding Hamiltonian flow by $\mathcal{H}^t_c$, i.e. for each $(x_0, \xi_0) \in T^*\mathbb{R}^d$,

$\mathcal{H}_c^t(x_0,\xi_0) = (x(t,x_0,\xi_0),\xi(t,x_0,\xi_0))$ solves the following equations:

$$
\begin{aligned}
\dot{x} &= \frac{\partial H_c}{\partial \xi} = c^2 \cdot \xi, \quad x(0) = x_0, && (4.1)\\
\dot{\xi} &= -\frac{\partial H_c}{\partial x} = -\frac{1}{2}\nabla c^2 \cdot |\xi|^2, \quad \xi(0) = \xi_0. && (4.2)
\end{aligned}
$$

We call $(x(\cdot,x_0,\xi_0),\xi(\cdot,x_0,\xi_0))$ the bicharateristic curve emanating from (x_0,ξ_0) and $x(\cdot,x_0,\xi_0)$ the geodesic. By the assumptions on c, the flow $\mathcal{H}_c^t$ is defined for all $t \in \mathbb{R}$. Note that the flow $\mathcal{H}_c^t$ is also well-defined on the cosphere bundle $S^*\mathbb{R}^d = \{(x,\xi) : x \in \mathbb{R}, |\xi|_c = 1\}$.

We say that a velocity field c is non-trapping in Ω for time $T > 0$ if the following condition is satisfied:

$$\mathcal{H}_c^T(S^*\Omega) \bigcap S^*\Omega = \emptyset. \tag{4.3}$$

Denote

$$
\begin{aligned}
S_+^*\Gamma &= \{(x,\xi) : x \in \Gamma, |\xi|_c = 1, \langle \xi, \nu(x)\rangle > 0\};\\
S_-^*\Gamma &= \{(x,\xi) : x \in \Gamma, |\xi|_c = 1, \langle \xi, \nu(x)\rangle < 0\}.
\end{aligned}
$$

Assume that the velocity field c is non-trapping in Ω for time T; we now define the scattering relation $\mathfrak{S}_c : S_-^*\Gamma \to S_+^*\Gamma$. For each $(x_0,\xi_0) \in S_-^*\Gamma$, let $l(x_0,\xi_0)$ be the first moment that the geodesic $x(\cdot,x_0,\xi_0)$ hits the boundary Γ. Define

$$\mathfrak{S}_c(x_0,\xi_0) = \mathcal{H}_c^{l(x_0,\xi_0)}(x_0,\xi_0).$$

For future reference, we define $l_- : S^*\Omega \to (-\infty, 0]$ by letting $l_-(x,\xi)$ be the first negative moment that the bicharacteristic curve $\mathcal{H}^t(x,\xi)$ hits the boundary $S^*_-\Gamma$ and $\tau : S^*\Omega \to S^*_-\Gamma$ by

$$\tau(x,\xi) = \mathcal{H}^{l_-(x,\xi)}(x,\xi).$$

We remark that $l_-(\cdot)$ and $\tau(\cdot)$ are well-defined by the assumption (4.3).

We now introduce the class of admissible velocity fields that are considered in the paper.

Definition 4.0.1. *Let M_0, ϵ_0 and T be positive numbers. A velocity field c is said to belong to the admissible class $\mathfrak{A}(M_0, \epsilon_0, \Omega, T)$ if and only if the following three conditions are satisfied:*

1. *$c \in C^3(\mathbb{R}^d)$, $0 < \frac{1}{M_0} \le c \le M_0$, and $\|c\|_{C^3(\mathbb{R}^d)} \le M_0$;*
2. *the support of $c - 1$ is contained in the set $\Omega_0 =: \{x \in \Omega : dist(x, \Gamma) > \epsilon_0\}$;*
3. *the Hamiltonian H_c is non-trapping in Ω for time T.*

By Condition 2 above, we can find two small positive constants ϵ^* and ϵ_1, both depending on ϵ_0, such that for any $(x_0, \xi_0) \in S^*_-\Gamma$, if

$$\{\mathcal{H}^t(x_0,\xi_0) : t \in (0, l(x_0,\xi_0))\} \bigcap S^*\Omega_0 \neq \emptyset,$$

then

$$\langle \xi_0, \nu(x_0) \rangle \leq -\epsilon^*, \tag{4.4}$$

$$\langle \xi_1, \nu(x_1) \rangle \geq \epsilon^*, \tag{4.5}$$

$$l(x_0, \xi_0) \geq \epsilon_1, \tag{4.6}$$

where $(x_1, \xi_1) = \mathfrak{S}_c(x_0, \xi_0)$.

Finally, we remark that we set up the discussion in the paper in the cotangent space $T^*\mathbb{R}^d$. But one can also set up the discussion in the tangent space $T\mathbb{R}^d$, see for instance [45], [53]. The equivalence of the two setups can be seen from the procedure of "raising and lowing indices" in Riemannian geometry. We choose the cotangent setup mainly because the following three reasons. First, it is more natural to the construction of Gaussian beams. Second, the classification result of singular Lagrangian maps is more complete than that of singular exponential maps in the literature, though these two problems are equivalent in Riemannian manifold. Finally, it is more natural to study caustics in the cotangent space.

Chapter 5

Sensitivity of recovering scattering relations from their induced DDtN maps

We study the sensitivity of recovering scattering relations from their induced Boundary DDtN maps in this chapter.

5.1 Gaussian beam solutions to the wave equation

We first construct Gaussian beam solutions to the wave equation system (1.1)-(1.3) in this section. Let c be a velocity field in the class $\mathfrak{A}(\epsilon_0, \Omega, M_0, T)$. We now construct a Gaussian beam in $\mathbb{R}^d$. Following [36], we define $G(x, \xi) = c(x)|\xi|$. For a given $(x_0, \xi_0) \in S^*_-\Gamma$, let

$(x(t), \xi(t), M(t), a(t))$ be the solution to the following ODE system:

$$
\begin{aligned}
\dot{x} &= G_p, \qquad x(t_0) = x_0, \\
\dot{\xi} &= -G_x, \qquad \xi(t_0) = \xi_0, \\
\dot{M} &= -G_{x\xi}^{\dagger} M - M G_{\xi x} - M G_{\xi\xi} M - G_{xx}, \quad M(t_0) = \sqrt{-1} \cdot Id, && (5.1) \\
\dot{a} &= -\frac{a}{2G}(c^2 \text{trace}(M) - G_x^{\dagger} G_{\xi} - G_{\xi}^{\dagger} M G_{\xi}), \quad a(t_0) = \lambda^{\frac{d}{4}}. && (5.2)
\end{aligned}
$$

The corresponding Gaussian beam with frequency λ ($\lambda \gg 1$) is given as follows

$$
g(t, x, \lambda) = a(t) e^{i\lambda\tau(t,x)}
$$

where $\tau(t, x) = \xi(t) \cdot (x - x(t)) + \frac{1}{2}(x - x(t))^{\dagger} M(t)(x - x(t))$.

Let the beam g impinge on the surface Γ, we want to construct the reflected beam g^-. Without loss of generality, we may assume that the ray $x(t)$ hits Γ at the point $x(t_1) = x_1$. Write $\xi(t_1) = \xi_1$. We parameterize Γ in a neighborhood of x_1, say $V(x_1)$, by a smooth diffeomorphism $F : U(x_1) \to V(x_1)$, where $U(x_1)$ is a neighborhood of the origin in $\mathbb{R}^{d-1}$. We require that $F(0) = x_1$. With the coordinate $x = F(y)$, we can rewrite functions restricted to the boundary Γ. For example, we rewrite

$$
g(t, x) = g(t, F(y)) = \hat{g}(t, y), \quad \tau(t, x) = \tau(t, F(y)) = \hat{\tau}(t, y), \quad \text{for } x \in V(x_1).
$$

We next derive formulas for $\hat{\tau}(t, y)$ and $\hat{g}(t, y)$. For this, we need to calculate $\hat{\tau}(t_1, 0)$,

$\frac{\partial\hat\tau}{\partial t}(t_1,0)$, $\frac{\partial\hat\tau}{\partial y}(t_1,0)$ and $\hat M(t_1) =: \frac{\partial^2\hat\tau}{\partial(t,y)^2}(t_1,0)$. In fact, by direct calculation, we have

$$\frac{\partial\hat\tau}{\partial t}(t_1,0) = -1, \quad \frac{\partial\hat\tau}{\partial y}(t_1,0) = (\frac{\partial F}{\partial y}(0))^\dagger\xi_1.$$

Moreover, the imaginary part and real part of the matrix $\frac{\partial^2\hat\tau}{\partial(t,y)^2}(t_1,0)$ are given below

$$\begin{aligned}
\Im\hat M(t_1) &= \begin{pmatrix} c^4(x_1)\xi_1^\dagger\Im M(t_1)\xi_1 & -c^2(x_1)\xi_1^\dagger\Im M(t_1)\cdot\frac{\partial F}{\partial y}(0) \\ -c^2(x_1)(\frac{\partial F}{\partial y}(0))^\dagger\Im M(t_1)\xi_1 & (\frac{\partial F}{\partial y}(0))^\dagger\Im M(t_1)\frac{\partial F}{\partial y}(0) \end{pmatrix} \\
&= R(t_1,x_1)^\dagger\Im M(t_1)R(t_1,x_1), \\
\Re\hat M(t_1) &= R(t_1,x_1)^\dagger\Re M(t_1)R(t_1,x_1) + \begin{pmatrix} \frac{1}{2}\nabla c^2(x)\xi_1 & -(\nabla\ln c(x_1))^\dagger\frac{\partial F}{\partial y}(0) \\ -(\frac{\partial F}{\partial y}(0))^\dagger\nabla\ln c(x_1) & \frac{\partial^2 F}{\partial y^2}(0)\xi_1 \end{pmatrix},
\end{aligned}$$

where $R(t_1,x_1) = (c^2(x_1)\xi_1, \frac{\partial F}{\partial y}(0))$.

We claim that

$$\Im\hat M(t_1) > 0.$$

Indeed, note that the column vectors in the matrix $\frac{\partial F}{\partial y}(0)$ are linearly independent and hence span the tangent space of the surface Γ at the point x_1. By (4.5), ξ_1 forms a nonzero angle with the tangent space and thus is linearly independent with all the column vectors in the matrix $\frac{\partial F}{\partial y}(0)$. Therefore the matrix $R(t_1,x_1)$ is invertible, and our claim follows. Using (4.4) and (4.5), we further have

$$\Im\hat M(t_1) > C \tag{5.3}$$

for some $C > 0$ depending on ϵ_0 and M_0.

Now, we have calculated $\hat{\tau}(t_1, 0)$, $\frac{\partial \hat{\tau}}{\partial t}(t_1, 0)$, $\frac{\partial \hat{\tau}}{\partial y}(t_1, 0)$ and $\frac{\partial^2 \hat{\tau}}{\partial (t,y)^2}(t_1, 0)$. It follows that

$$\begin{aligned}
\hat{\tau}(t_1, y) &= \hat{\tau}(t_1, 0) + (\frac{\partial \hat{\tau}}{\partial (t,y)}(t_1, 0))^\dagger (t - t_1, y) + (t - t_1, y) \frac{\partial^2 \hat{\tau}}{\partial (t,y)^2}(t_1, 0)(t - t_1, y)^\dagger \\
&\quad + O(|(t - t_1, y)|^3) \\
&= \langle (-1, \frac{\partial F}{\partial y}(0)^\dagger \xi_1), (t - t_1, y) \rangle + (t - t_1, y) \hat{M}(t_1)(t - t_1, y)^\dagger + O(|(t - t_1, y)|^3).
\end{aligned}$$

We proceed to construct the reflected beam g^-. Write

$$g^-(t, x, \lambda) = a^-(t) e^{i\lambda \tau^-(t,x)}$$

with

$$\tau^-(t, x) = \xi^-(t) \cdot (x - x^-(t)) + \frac{1}{2}(x - x^-(t))^\dagger M^-(t)(x - x^-(t)).$$

We need to find $(x^-(t_1), \xi^-(t_1), a^-(t_1), M^-(t_1))$ such that the $g^- + g \approx 0$ on the boundary. Following [3], we impose the following condition

$$\partial_{t,y}^\alpha \hat{\tau}(t_1, 0) = \partial_{t,y}^\alpha \hat{\tau}^-(t_1, 0), \quad \text{for all } |\alpha| \leq 2. \tag{5.4}$$

The above condition with $|\alpha| = 0$ gives that $\hat{\tau}^-(t_1, 0) = \hat{\tau}(t_1, 0) = 0$; with $|\alpha| = 1$ gives that

$$(\frac{\partial F}{\partial y}(0))^\dagger \xi_1 = (\frac{\partial F}{\partial y}(0))^\dagger \xi_1^-, \tag{5.5}$$

where $\xi_1^- = \xi^-(t_1)$. Since the column vectors in $(\frac{\partial F}{\partial y}(0))^\dagger$ spans the tangent space $T_{x_1}\Gamma$, we see that the tangential component of ξ_1^- and ξ_1 are equal. Besides, note that $|\xi_1| = |\xi_1^-| = 1$. Thus,

$$\xi^-(t_1) = \xi_1^- = \xi_1 - 2\langle \xi_1, \nu(x_1)\rangle \nu(x_1).$$

Condition (5.4) with $|\alpha| = 2$ gives that $\hat{M}^-(t_1) = \hat{M}(t_1)$. Recall the relation between $\Im M^-(t_1)$ and $\Im M(t_1)$, $\Re M^-(t_1)$ and $\Re M(t_1)$, we have the following two identities:

$$\begin{aligned}
R(t_1,x_1)^\dagger \Im M(t_1) R(t_1,x_1) &= R(t_1,x_1)^\dagger \Im M^-(t_1) R(t_1,x_1), \\
R(t_1,x_1)^\dagger \Re M(t_1) R(t_1,x_1) &= R(t_1,x_1)^\dagger \Im M^-(t_1) R(t_1,x_1) \\
&\quad + \begin{pmatrix} -\frac{1}{2}\nabla c^2(x)(\xi_1^- - \xi_1) & 0 \\ 0 & \frac{\partial^2 F}{\partial y^2}(0)(\xi_1^- - \xi_1) \end{pmatrix}.
\end{aligned}$$

Solving the above equations, we obtain $\Im M^-(t_1)$ and $\Re M^-(t_1,)$ and hence $M^-(t_1)$. Finally, set $a^-(t_1) = -a(t_1)$. Then all of the four components of $(x^-(t_1), \xi^-(t_1), a^-(t_1), M^-(t_1))$ are constructed. We then solve an ODE system to get $(x^-(t), \xi^-(t), M^-(t), a^-(t))$ as we did for the beam g. This completes the construction for the reflected beam g^-.

We now present some properties about the constructed beam. The following lemma is crucial in the subsequent estimates. We refer to [37] for the proof.

Lemma 5.1.1. *Both the matrices $M(t)$ and $M^-(t)$ are uniformly bounded for $t \in [0, T+\epsilon_1]$. Moreover, there exists $C > 0$, depending on M_0 and ϵ_0, such that $\Im M(t) > C$ and $\Im M^-(t) > C$ for all $t \in [0, T+\epsilon_1]$.*

We next introduce two auxiliary beams below

$$\hat{g}_*(t,y,\lambda) = a(t_1)e^{i\lambda\hat{\tau}_*}, \quad \hat{g}_*^-(t,y,\lambda) = a^-(t_1)e^{i\lambda\hat{\tau}_*^-},$$

where

$$\hat{\tau}_* = \left\langle (-1, \frac{\partial F}{\partial y}(0)^\dagger \xi_1), (t-t_1, y) \right\rangle + (t-t_1, y)\hat{M}(t_1)(t-t_1, y)^\dagger,$$
$$\hat{\tau}_*^- = \left\langle (-1, \frac{\partial F}{\partial y}(0)^\dagger \xi_1^-), (t-t_1, y) \right\rangle + (t-t_1, y)\hat{M}^-(t_1)(t-t_1, y)^\dagger.$$

It is clear that $\hat{\tau}_* = \hat{\tau}_*^-$ and $\hat{g}_* = -\hat{g}_*^-$.

Lemma 5.1.2.

$$\hat{g}(t, y, \lambda) = \hat{g}_*(t, y, \lambda) + O(\sqrt{\lambda}) \quad in \quad H^1((3\epsilon_1/4, t_1 + \epsilon_1/2) \times U(x_1)), \tag{5.6}$$
$$\hat{g}^-(t, y, \lambda) = \hat{g}_*^-(t, y, \lambda) + O(\sqrt{\lambda}) \quad in \quad H^1((3\epsilon_1/4, t_1 + \epsilon_1/2) \times U(x_1)). \tag{5.7}$$

Proof:. We only show (5.6), since (5.7) follows in a similar way. For simplicity, denote $D = (3\epsilon_1/4, t_1 + \epsilon_1/2) \times U(x_1)$. We first show that

$$\hat{g}(t, y, \lambda) = \hat{g}_*(t, y, \lambda) + O(\frac{1}{\sqrt{\lambda}}) \quad \text{in} \quad L^2(D). \tag{5.8}$$

Indeed, by direct calculation,

$$\hat{g}(t, y, \lambda) - \hat{g}_*(t, y, \lambda) = (a(t) - a(t_1))e^{i\lambda\hat{\tau}_*} + a(t)(e^{i\lambda\hat{\tau}} - e^{i\lambda\hat{\tau}_*}). \tag{5.9}$$

It suffices to show that

$$R_1 := \|(a(t) - a(t_1))e^{i\lambda\hat{\tau}_*}\|_{L^2(D)} \lesssim \frac{1}{\sqrt{\lambda}},$$
$$R_2 := \|a(t)(e^{i\lambda\hat{\tau}} - e^{i\lambda\hat{\tau}_*})\|_{L^2(D)} \lesssim \frac{1}{\sqrt{\lambda}}.$$

We first estimate R_1. By Lemma 3.1 in [7], we have $|a(t)| \approx \lambda^{\frac{d}{4}}$. By equation (5.2), we further derive that $|\dot{a}(t)| \approx \lambda^{\frac{d}{4}}$, thus

$$a(t) - a(t_1) = \int_0^1 \dot{a}(t_1 + s(t - t_1))\, ds(t - t_1) = O(\lambda^{\frac{d}{4}})|t - t_1|.$$

Therefore,

$$\|(a(t) - a(t_1))e^{i\lambda\hat{\tau}_*}\|^2_{L^2(D)} \lesssim \int_D \lambda^{\frac{d}{2}}(t - t_1)^2 e^{-\lambda(t-t_1,y)\Im\hat{M}(t_1)(t-t_1,y)^\dagger}\, dtdy \lesssim \frac{1}{\lambda}.$$

This proves $R_1 \lesssim \frac{1}{\sqrt{\lambda}}$.

We next estimate R_2. Write $\hat{\tau} = \hat{\tau}_* + \delta\hat{\tau}$, then $\delta\hat{\tau} = O(|(t - t_1, y)|^3)$ and hence $|1 - e^{i\lambda\delta\hat{\tau}}| \lesssim \lambda \cdot O(|(t - t_1, y)|^3)$. It follows that

$$\begin{aligned} R_2 &\leq \int_D |a(t)e^{i\lambda\hat{\tau}_*}|^2 \cdot |1 - e^{i\lambda\delta\hat{\tau}}|\, dtdy \\ &\lesssim \int_D \lambda^{\frac{d}{2}} \cdot \lambda \cdot |(t - t_1, y)|^3 e^{-2\lambda(t-t_1,y)\Im\hat{M}(t_1)(t-t_1,y)^\dagger}\, dtdy \\ &\lesssim \frac{1}{\lambda}. \end{aligned}$$

This completes the proof of (5.8).

We now proceed to show (5.6). By direct calculation,

$$\begin{aligned} \frac{\partial\hat{g}}{\partial y} - \frac{\partial\hat{g}_*}{\partial y} &= i\lambda\frac{\partial\hat{\tau}}{\partial y} \cdot \hat{g} - i\lambda\frac{\partial\hat{\tau}_*}{\partial y} \cdot \hat{g}_* \\ &= i\lambda(\frac{\partial\hat{\tau}}{\partial y} - \frac{\partial\hat{\tau}_*}{\partial y}) \cdot \hat{g} + i\lambda\frac{\partial\hat{\tau}_*}{\partial y} \cdot (\hat{g} - \hat{g}_*) \end{aligned}$$

One can check that $\frac{\partial\hat{\tau}}{\partial y} - \frac{\partial\hat{\tau}_*}{\partial y} = O|(t - t_1, y)|^2$, then a similar argument as used in the estimate of R_1 above shows that

$$\|\lambda(\frac{\partial\hat{\tau}}{\partial y}-\frac{\partial\hat{\tau}_*}{\partial y})\cdot\hat{g}\|^2_{L^2(D)}\lesssim 1.$$

Besides, (5.8) implies that

$$\|\lambda\frac{\partial\hat{\tau}_*}{\partial y}\cdot(\hat{g}-\hat{g}_*)\|^2_{L^2(D)}\lesssim\lambda.$$

Combining these two estimates together, we conclude that

$$\|\frac{\partial\hat{g}}{\partial y}-\frac{\partial\hat{g}_*}{\partial y}\|^2_{L^2(D)}\lesssim\lambda.$$

Similarly, we can show that

$$\|\frac{\partial\hat{g}}{\partial t}-\frac{\partial\hat{g}_*}{\partial t}\|^2_{L^2(D)}\lesssim\lambda.$$

This completes the proof of (5.6) and hence the lemma.

Note that $\|\hat{g}(t,y,\lambda)\|_{L^2((t_1-\epsilon_1/2,t_1+\epsilon_1/2)\times U(x_1))}\approx 1$. As a direct consequence of Lemma 5.1.2, we obtain the following norm estimate for the beam g restricted to the boundary Γ.

Lemma 5.1.3.

$$\|g(\cdot,\cdot,\lambda)\|_{L^2((t_1-\epsilon_1/2,t_1+\epsilon_1/2)\times V(x_1))}\approx 1. \tag{5.10}$$

We now present an H^1-norm estimate for g^-+g and an approximation for the Neumann data $\frac{\partial g^-}{\partial\nu}+\frac{\partial g}{\partial\nu}$ on the boundary.

Lemma 5.1.4.

$$\begin{aligned} g^{-}(t,x,\lambda)+g(t,x,\lambda) &= O(\sqrt{\lambda}) \quad in \ \ H^1((3\epsilon_1/4, t_1+\epsilon_1/2)\times V(x_1)); &(5.11)\\ \frac{\partial g^-}{\partial \nu}+\frac{\partial g}{\partial \nu} &= 2i\lambda g\cdot\langle \xi_1, \nu(x_1)\rangle + O(\sqrt{\lambda}) &(5.12)\\ & \quad in \ \ L^2((3\epsilon_1/4, t_1+\epsilon_1/2)\times V(x_1)). \end{aligned}$$

Proof:. Denote $D=(3\epsilon_1/4, t_1+\epsilon_1/2)\times U(x_1)$ again. We first show (5.11). Since x is restricted to $V(x_1)\subset\Gamma$, it suffices to show that

$$\hat{g}^{-}(t,y,\lambda)+\hat{g}(t,y,\lambda)=O(\sqrt{\lambda}) \quad \text{in} \ \ H^1(D).$$

But this is a direct consequence of Lemma 5.1.2 and the fact that $\hat{g}_*^- = -\hat{g}_*$.

We now prove (5.12). By direct calculate

$$\begin{aligned} \frac{\partial g}{\partial \nu}(t,x) &= \frac{\partial}{\partial \nu}(a(t)e^{i\lambda\tau(t,x)}) = i\lambda g\cdot\frac{\partial \tau}{\partial \nu}\\ &= i\lambda g\cdot\langle \xi(t)+M(t)(x-x(t),\nu(x)\rangle\\ &= i\lambda g\cdot\langle \xi(t_1),\nu(x_1)\rangle + i\lambda g\cdot(\langle \xi(t),\nu(x)\rangle - \langle \xi(t_1),\nu(x_1)\rangle)\\ &\quad +i\lambda g\cdot\langle M(t)(x-x(t),\nu(x)\rangle. \end{aligned}$$

Note that in the coordinate $x=F(y)$,

$$\begin{aligned} |\langle M(t)(x-x(t)),\nu(x)\rangle| &= O(|(t-t_1,y)|),\\ |\langle \xi(t),\nu(x)\rangle - \langle \xi(t_1),\nu(x_1)\rangle| &= O(|(t-t_1,y)|). \end{aligned}$$

It follows that

$$\begin{aligned}\|g\cdot(\langle\xi(t),\nu(x)\rangle-\langle\xi(t_1),\nu(x_1)\rangle)\|_{L^2(D)}^2 &\lesssim \frac{1}{\lambda},\\ \|g\cdot\langle M(t)(x-x(t),\nu(x)\rangle\|_{L^2(D)}^2 &\lesssim \frac{1}{\lambda}.\end{aligned}$$

Thus

$$\frac{\partial g}{\partial\nu}(t,x)=i\lambda g\cdot\langle\xi(t_1),\nu(x_1)\rangle+O(\sqrt{\lambda}).$$

Similarly,

$$\frac{\partial g^-}{\partial\nu}(t,x)=i\lambda g^-\cdot\langle\xi^-(t_1),\nu(x_1)\rangle+O(\sqrt{\lambda}).$$

Finally, using (5.11) and the fact that $\langle\xi^-(t_1),\nu(x_1)\rangle=-\langle\xi(t_1),\nu(x_1)\rangle$, we conclude that (5.12) holds. This completes the proof of the lemma.

Now, we are ready to construct Gaussian beam solutions to the initial boundary value problem of the wave system (1.1)-(1.3). We first choose $\chi_{\epsilon_1}(t)\in C_0^\infty(\mathbb{R})$ such that $\chi_{\epsilon_1}(t)=1$ for $t\in(\epsilon_1/4,\epsilon_1/2)$ and $\chi_{\epsilon_1}(t)=0$ for $t\in(-\infty,0)\bigcup(3\epsilon_1/4,\infty)$. Let $(x_0,\xi_0)\in S^*_-\Gamma$ and $(x_0^*,\xi_0^*)=\mathcal{H}^{-\frac{\epsilon_1}{4}}(x_0,\xi_0)=(x_0-\frac{\epsilon_1\cdot\xi_1}{4},\xi_0)$. Let g be the Gaussian beam constructed with the initial data $x(0)=x_0^*,\xi(0)=\xi_0^*,M(0)=i\cdot Id$ and $a(0)=\lambda^{\frac{d}{4}}$. The beam g is reflected by Γ at $(x_1,\xi_1)=\mathfrak{S}_c(x_0,\xi_0)=\mathcal{H}_c^{l(x_0,\xi_0)}(x_0,\xi_0)$ at $t_1=l(x_0,\xi_0)+\frac{\epsilon_1}{4}$. We construct the reflected beam g^- by the preceding procedure. Let u be the exact solution to the wave system (1.1)-(1.3) with

$$f(t,x,\lambda)=g(t,x,\lambda)\cdot\chi_{\epsilon_1}(t).$$

Then $u = g + g^- + R$, where the remaining term R satisfies the following equation system

$$
\begin{aligned}
\mathcal{P}R &= -\mathcal{P}(g+g^-), \quad (t,x) \in \Omega \times (0, t_1 + \epsilon_1/2), \\
R(0,x,\lambda) &= -(g+g^-)(0,x,\lambda), \quad x \in \Omega, \\
R_t(0,x,\lambda) &= -(g_t+g_t^-)(0,x,\lambda), \quad x \in \Omega, \\
R(t,x,\lambda) &= -g(t,x,\lambda)(1-\chi_{\epsilon_1}(t)) - g^-(t,x,\lambda), \quad (t,x) \in (0, t_1+\epsilon_1/2) \times \Gamma.
\end{aligned}
$$

Here $\mathcal{P}$ stands for the wave operator $\frac{1}{c^2(x)}\partial_{tt} - \Delta$.

Lemma 5.1.5.

$$
\|\frac{\partial R}{\partial \nu}\|_{L^2([0,t_1+\epsilon_1/2]\times\Gamma)} \le C\sqrt{\lambda}
$$

for some constant $C > 0$ depending on ϵ_0 and M_0.

Proof. We apply Theorem 4.1 in [31] to derive the estimate. Note that the compatibility condition is satisfied on the boundary at time $t = 0$. It remains to show that the following four estimates hold:

$$
\begin{aligned}
\|\mathcal{P}(g+g^-)\|_{C([0,t_1+\epsilon_1/2];L^2(\Omega))} &\lesssim \sqrt{\lambda}, && (5.13)\\
\|(g+g^-)(0,\cdot,\lambda)\|_{H^1(\Omega)} &\lesssim \sqrt{\lambda}, && (5.14)\\
\|(g_t+g_t^-)(0,\cdot,\lambda)\|_{L^2(\Omega)} &\lesssim \sqrt{\lambda}, && (5.15)\\
\|g(t,x,\lambda)(1-\chi_{\epsilon_1}(t)) - g^-(t,x,\lambda)\|_{H^1([0,t_1+\epsilon_1/2]\times\Gamma)} &\lesssim \sqrt{\lambda}. && (5.16)
\end{aligned}
$$

First, (5.13) follows from the standard estimate for Gaussian beams, see for example [7].

We next show (5.14). By Lemma 5.1.1, there exists a constant $C > 0$ depending on M_0 and ϵ_0 such that the following two inequalities hold

$$
\begin{aligned}
|g(t,x,\lambda)| &\lesssim \lambda^{\frac{d}{4}} \cdot e^{-C\lambda\cdot|x-x^-(t)|^2}, \\
|g^-(t,x,\lambda)| &\lesssim \lambda^{\frac{d}{4}} \cdot e^{-C\lambda\cdot|x-x^-(t)|^2}.
\end{aligned}
$$

Thus the beam g and g^- are exponentially decaying away from the ray $x(t)$ and $x^-(t)$ respectively. Using this property, it is straightforward to show that $\|g(0,\cdot,\lambda)\|_{H^1(\Omega)} \lesssim 1$ and $\|g^-(0,\cdot,\lambda)\|_{H^1(\Omega)} \lesssim 1$, whence (5.14) and (5.15) follows.

Now, we show (5.16). We divide the domain $(0, t_1 + \epsilon_1/2) \times \Gamma$ into three parts:

$$
\Sigma_1 = (0, \epsilon_1/2) \times \Gamma, \quad \Sigma_2 = (\epsilon_1/2, t_1 - \epsilon_1/2) \times \Gamma, \quad \Sigma_3 = (t_1 - \epsilon_1/2, t_1 + \epsilon_1/2) \times \Gamma.
$$

We show that inequality (5.16) holds on each part.

For $(t,x) \in \Sigma_1$, we have $1 - \chi_{\epsilon_1}(t) = 0$. Consequently,

$$
g(t,x)(1 - \chi_{\epsilon_1}(t)) - g^-(t,x,\lambda) = g^-(t,x,\lambda).
$$

By the exponential decaying property of g^-, we obtain that

$$
\|g(t,x)(1 - \chi_{\epsilon_1}(t)) - g^-(t,x,\lambda)\|_{H^1(\Sigma_1)} \lesssim \sqrt{1}.
$$

For $(t,x) \in \Sigma_2$, by the exponential decaying property for both g and g^- again, we obtain

$$\|g(t,x,\lambda)(1-\chi_{\epsilon_1}(t)) - g^-(t,x,\lambda)\|_{H^1(\Sigma_2)} \lesssim 1.$$

Finally, for $(t,x) \in \Sigma_3$, note that $t_1 - \frac{\epsilon_1}{2} = l(x_0,\xi_0) + \frac{\epsilon_1}{4} - \frac{\epsilon_1}{2} \geq \frac{3\epsilon_1}{4}$. We can apply Lemma 5.1.4 to the part $x \in V(x_1)$ and the exponential decaying property for both g and g^- to the remaining part to conclude that

$$\|g(t,x,\lambda)(1-\chi_{\epsilon_1}(t)) - g^-(t,x,\lambda)\|_{H^1(\Sigma_3)} \lesssim \sqrt{\lambda}$$

This completes the proof of (5.16) and hence the lemma.

5.2 The sensitivity result

It is known that the DDtN map Λ_c determines the scattering relation $\mathfrak{S}_c$ uniquely [35]. We show that the following sensitivity result of recovering the scattering relation from the DDtN map holds.

Theorem 5.2.1. *Let c and $\tilde{c}$ be two velocity fields in the class $\mathfrak{A}(\epsilon_0,\Omega,M_0,T)$. Then there exists a constant $\delta > 0$ such that*

$$\mathfrak{S}_{\tilde{c}} = \mathfrak{S}_c$$

if $\|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1[0,3\epsilon_1/4]\times\Gamma \to L^2([0,T+\epsilon_1]\times\Gamma)} \leq \delta$.

Proof. For any $(x_0, \xi_0) \in S^*_-\Gamma$, let $(x_1, \xi_1) = \mathfrak{S}_c(x_0, \xi_0) = \mathcal{H}_c^{l(x_0,\xi_0)}(x_0, \xi_0)$ and $(\tilde{x}_1, \tilde{\xi}_1) = \mathfrak{S}_{\tilde{c}}(x_0, \xi_0) = \mathcal{H}_{\tilde{c}}^{\tilde{l}(x_0,\xi_0)}(x_0, \xi_0)$. We need to show that

$$(l(x_0, \xi_0), x_1, \xi_1) = (\tilde{l}(x_0, \xi_0), \tilde{x}_1, \tilde{\xi}_1)$$

if $\|\Lambda_{\tilde{c}} - \Lambda_c\|$ is sufficiently small. We do this in the following steps.

Step 1. Let $t_1 = l(x_0, \xi_0) + \frac{\epsilon_1}{4}$ and $\tilde{t}_1 = \tilde{l}(x_0, \xi_0) + \frac{\epsilon_1}{4}$. Without loss of generality, we may assume that $t_1 \leq \tilde{t}_1$. Let $V(x_1)$ be a neighborhood of x_1 in Γ which is parameterized by a smooth function $F : U(x_1) \to V(x_1)$ as before. We may assume that $\tilde{x}_1 \in V(x_1)$. Let $\tilde{x}_1 = F(\delta y)$. We construct the initial beam g, the reflected beam g^-, the boundary Dirichlet data f, the solution u to the wave equation with velocity field c and remanning term R as in the previous section. We similarly construct $\tilde{g}$, $\tilde{g}^-$, $\tilde{u}$ and $\tilde{R}$ to the system with velocity field $\tilde{c}$ and with boundary Dirichlet data $\tilde{f} = f$.

Step 2. Denote by $I(t_1, \epsilon_1/2)$ the interval $(t_1 - \epsilon_1/2, t_1 + \epsilon_1/2)$. Since $t_1 \leq \tilde{t}_1$ and $l(x_0, \xi_0) \geq \epsilon_1$, we have $I(t_1, \epsilon_1/2) \subset (3\epsilon_1/4, t_1 + \epsilon_1/2)$ and $I(t_1, \epsilon_1/2) \subset (3\epsilon_1/4, \tilde{t}_1 + \epsilon_1/2)$. Then we can apply (5.12) and Lemma 5.1.5 to obtain

$$\begin{aligned}
(\Lambda_{\tilde{c}} - \Lambda_c)f &= \frac{\partial u}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu} \\
&= \frac{\partial (g + g^-)}{\partial \nu} - \frac{\partial (\tilde{g} + \tilde{g}^-)}{\partial \nu} + \frac{\partial R}{\partial \nu} - \frac{\partial \tilde{R}}{\partial \nu} \\
&= 2i\lambda \cdot \left\{ \langle \xi_1, \nu(x_1) \rangle \cdot g - \langle \tilde{\xi}_1, \nu(\tilde{x}_1) \rangle \cdot \tilde{g} \right\} + O(\sqrt{\lambda})
\end{aligned}$$

in $L^2(I(t_1, \epsilon_1/2) \times V(x_1))$.

It follows that

$$\begin{aligned}\big\langle (\Lambda_{\tilde{c}}-\Lambda_c)f, g\big\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))} = 2i\lambda\cdot\Big[&\big\langle \xi_1, \nu(x_1)\big\rangle\cdot\big\langle g, g\big\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}\\ &-\big\langle \tilde{\xi}_1, \nu(\tilde{x}_1)\big\rangle\cdot\big\langle \tilde{g}, g\big\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}\Big]\\ &+O(\sqrt{\lambda}).\end{aligned}$$

Note that

$$\begin{aligned}&|\langle (\Lambda_{\tilde{c}}-\Lambda_c)f, g\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}|\\ \leq\;& \|(\Lambda_{\tilde{c}}-\Lambda_c)f\|_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}\cdot\|g\|_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}\\ \leq\;& \|(\Lambda_{\tilde{c}}-\Lambda_c)f\|_{L^2((0,T+\epsilon_1)\times\Gamma)}\cdot\|g\|_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}\\ \leq\;& \|\Lambda_{\tilde{c}}-\Lambda_c\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)\to L^2([0,T+\epsilon_1]\times\Gamma)}\cdot\|f\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)}\\ &\cdot\|g\|_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}\\ \lesssim\;& \lambda\cdot\|\Lambda_{\tilde{c}}-\Lambda_c\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)\to L^2([0,T+\epsilon_1]\times\Gamma)}.\end{aligned}$$

Thus the following inequality holds

$$\begin{aligned}&|\langle \xi_1, \nu(x_1)\rangle\cdot\langle g, g\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}| - |\langle \tilde{\xi}_1, \nu(\tilde{x}_1)\rangle\cdot\langle \tilde{g}, g\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}|\\ &\leq \frac{C}{\sqrt{\lambda}} + C\cdot\|\Lambda_{\tilde{c}}-\Lambda_c\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)\to L^2([0,T+\epsilon_1]\times\Gamma)}\end{aligned}\tag{5.17}$$

for some constant $C > 0$.

Step 3. We now estimate the two terms on the left hand side of the inequality (5.17).

First, by (4.5) and Lemma 5.1.3, we have

$$|\langle \xi_1, \nu(x_1)\rangle \cdot |\big\langle g, g\big\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}| \approx 1. \tag{5.18}$$

We next estimate $\langle \tilde{g}, g\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}$. In the coordinate $x = F(y)$, by Lemma 5.1.2, we have

$$\langle \hat{\tilde{g}}, \hat{g}\rangle_{L^2(I(t_1,\epsilon_1/2)\times U(x_1))} = \langle \hat{\tilde{g}}_*, \hat{g}_*\rangle_{L^2(I(t_1,\epsilon_1/2)\times U(x_1))} + O(\frac{1}{\sqrt{\lambda}}). \tag{5.19}$$

Here we recall that $\hat{g}_* = a(t_1)e^{i\lambda\hat{\tau}_*}$ and $\hat{\tilde{g}}_* = \tilde{a}(t_1)e^{i\lambda\hat{\tilde{\tau}}_*}$ with

$$\begin{aligned}
\hat{\tau}_* &= \langle(-1, \frac{\partial F}{\partial y}(0)^\dagger \xi_1), (t-t_1, y)\rangle + (t-t_1, y)\hat{M}(t_1)(t-t_1,y)^\dagger;\\
\hat{\tilde{\tau}}_* &= \langle(-1, \frac{\partial F}{\partial y}(\delta y)^\dagger \tilde{\xi}_1), (t-\tilde{t}_1, y-\delta y)\rangle + (t-\tilde{t}_1, y-\delta y)\hat{\tilde{M}}(t_1)(t-\tilde{t}_1, y-\delta y)^\dagger.
\end{aligned}$$

By Lemma 3.7 in [7], we have

$$|\langle \hat{\tilde{g}}_*, \hat{g}_*\rangle_{L^2(I(t_1,\epsilon_1/2)\times U(x_1))}| \lesssim e^{-c_0\lambda|\delta z|} \tag{5.20}$$

where c_0 is a positive constant depending only on $\|c\|_{C^3} + \|\tilde{c}\|_{C^3}$ and

$$|\delta z| = |t_1 - \tilde{t}_1|^2 + |\delta y|^2 + |\frac{\partial F}{\partial y}(\delta y)^\dagger\tilde{\xi}_1 - \frac{\partial F}{\partial y}(0)^\dagger \xi_1|^2.$$

It follows from (5.19) and (5.20) that

$$|\langle \tilde{g}, g\rangle_{L^2(I(t_1,\epsilon_1/2)\times V(x_1))}| \lesssim e^{-c_0\lambda|\delta z|} + O(\frac{1}{\sqrt{\lambda}}). \tag{5.21}$$

Step 4. Combining (5.17), (5.18) and (5.21), we see that

$$e^{-c_0\lambda|\delta z|} \gtrsim C_1 - C_2\|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)\to L^2([0,T+\epsilon_1]\times\Gamma)} - C_3\frac{1}{\sqrt{\lambda}}$$

for some positive constants C_1, C_2 and C_3 which are independent of (x_0, ξ_0). By letting $\lambda \to \infty$, we conclude that $\delta z = 0$ if

$$\|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1([0,3\epsilon_1/4]\times\Gamma)\to L^2([0,T+\epsilon_1]\times\Gamma)} < \frac{C_1}{C_2}.$$

Set $\delta = \frac{C_1}{C_2}$. From $\delta z = 0$ it follows that $t_1 = \tilde{t}_1$, $\delta y = 0$, and $\frac{\partial F}{\partial y}(0)^\dagger\tilde{\xi}_1 - \frac{\partial F}{\partial y}(0)^\dagger\xi_1 = 0$. It remains to show that $\tilde{\xi}_1 = \xi_1$. Indeed, $\frac{\partial F}{\partial y}(0)^\dagger\tilde{\xi}_1 - \frac{\partial F}{\partial y}(0)^\dagger\xi_1 = 0$ implies that the tangential component of $\tilde{\xi}_1$ and ξ_1 are equal. Besides, $\|\xi_1\| = \|\tilde{\xi}_1\|$. These together with (4.5) yield that $\tilde{\xi}_1 = \xi_1$. This completes the proof of the theorem.

Chapter 6

Stability of X-ray transform in the presence of caustics

6.1 The X-ray transform resulted from linearizing Hamiltonian flow with respect to velocity field

We begin with the following observation.

Lemma 6.1.1. *Let c and $\tilde{c}$ be two velocity fields in the class $\mathfrak{A}(\epsilon_0, \Omega, M_0, T)$, then $\mathfrak{S}_c = \mathfrak{S}_{\tilde{c}}$ if and only if $\mathcal{H}_c^T|_{S^*_-\Gamma} = \mathcal{H}_{\tilde{c}}^T|_{S^*_-\Gamma}$.*

The above lemma shows the equivalence of the Hamiltonian flow and the scattering relation. The next lemma shows that $\mathcal{H}_c^t$ satisfies an equivalent ordinary differential equation (ODE) system in $S^*\mathbb{R}^d$.

Lemma 6.1.2. *Let $(x_0, \xi_0) \in S^*\mathbb{R}^d = \{(x, \xi) \in \mathbb{R}^{2d} : c(x)|\xi| = 1\}$, and let $(x(t), \xi(t)) = \mathcal{H}_c^t(x_0, \xi_0)$, then $(x(t), \xi(t))$ satisfies the following ODE system*

$$\dot{x} = \frac{\xi}{|\xi|^2}, \tag{6.1}$$

$$\dot{\xi} = b(x). \tag{6.2}$$

where $b(x) = -\frac{1}{2}\nabla \ln c^2$. *Conversely, if* $(x(t), \xi(t)) \in S^*\mathbb{R}^d$ *satisfies the ODE system (6.1)-(6.2), then* $(x(t), \xi(t)) = \mathcal{H}_c^t(x_0, \xi_0)$.

We next linearize the operator which maps each velocity field to its induced Hamiltonian flow restricted to the cosphere bundle. Let c be a fixed smooth background velocity field. Denote the perturbed velocity field and Hamiltonian flow at time T as $\tilde{c}^2 = c^2 + \delta c^2$ and $\mathcal{H}_{\tilde{c}}^T = \mathcal{H}_c^T + \delta\mathcal{H}_c^T$ respectively. Denote also that $\delta b = -\frac{1}{2}\nabla(\ln \tilde{c}^2 - \ln c^2)$ and

$$A(x, \xi) = \begin{pmatrix} 0 & \frac{\partial}{\partial \xi}(\frac{\xi}{|\xi|^2}) \\ \frac{\partial b}{\partial x} & 0 \end{pmatrix}.$$

For each $(x_0, \xi_0) \in S_-^*\Gamma$, let $\Phi(t, x_0, \xi_0)$ be the solution of the following ODE system

$$\dot{\Phi}(t) = -\Phi(t)A(\mathcal{H}_c^t), \quad \Phi(0) = Id.$$

By the standard linearization argument, we have

$$\delta\mathcal{H}_c^T = \frac{\delta\mathcal{H}_c^T}{\delta b}(\delta b) + r(\delta b),$$

where

$$\frac{\delta\mathcal{H}_c^T}{\delta b}(\delta b)(x_0, \xi_0) = \int_0^T \Phi^{-1}(T, x_0, \xi_0) \cdot \Phi(s, x_0, \xi_0) \begin{pmatrix} 0 \\ \delta b(x(s, x_0, \xi_0)) \end{pmatrix} ds \tag{6.3}$$

and $\|r(\delta b)\|_{L^\infty} \leq C\|\delta b\|_{C^1}^2$ for some constant $C > 0$ depending only on $\|c\|_{C^3(\mathbb{R}^d)}$.

Formula (6.3) motivates us to define the following geodesic X-ray transform operator

$$\mathfrak{I}_c(f)(x_0,\xi_0) = \int_0^T \Phi(s,x_0,\xi_0) f(x(s,x_0,\xi_0))\, ds, \quad f \in \mathcal{E}'(\Omega,\mathbb{R}^{2d}). \tag{6.4}$$

Then $\frac{\delta \mathcal{H}_c^T}{\delta b}(\delta b)(x_0,\xi_0) = \Phi^{-1}(T,x_0,\xi_0)\cdot \mathfrak{I}_c(f)(x_0,\xi_0)$ with

$$f = \begin{pmatrix} 0 \\ \frac{1}{2}\nabla(\ln c^2 - \ln \tilde{c}^2) \end{pmatrix}. \tag{6.5}$$

We introduce a matrix $\Phi(x,\xi)$ for each $(x,\xi) \in S^*\Omega$. Let $(x_0,\xi_0) = \tau(x,\xi) = \mathcal{H}_c^{l_-(x,\xi)}(x,\xi)$. We then define

$$\Phi(x,\xi) = \Phi(-l_-(x,\xi),\tau(x,\xi)).$$

It is clear that the following identity holds

$$\Phi(\mathcal{H}_c^s(x_0,\xi_0)) = \Phi(s,x_0,\xi_0)$$

for all $s \in \mathbb{R}_+$ such that $\mathcal{H}_c^s(x_0,\xi_0) \in S^*\Omega$. We can rewrite the X-ray transform operator $\mathfrak{I}_c$ in the following standard form

$$\begin{aligned} \mathfrak{I}_c f(x_0,\xi_0) &= \int_0^T \Phi(\mathcal{H}_c^s(x_0,\xi_0)) f(\pi(\mathcal{H}_c^s(x_0,\xi_0)))\, ds \\ &= \int_0^{l(x_0,\xi_0)} \Phi(\mathcal{H}_c^s(x_0,\xi_0)) f(\pi(\mathcal{H}_c^s(x_0,\xi_0)))\, ds. \end{aligned} \tag{6.6}$$

Remark 6.1.1. *Formula (6.6) is derived in the coordinate of $T^*\mathbb{R}^d$. Hence it may not* *be geometrically invariant.*

Lemma 6.1.3. *Assume that $\mathfrak{S}_c = \mathfrak{S}_{\tilde{c}}$, let f be defined as in (6.5), then*

$$\|\mathfrak{I}_c f\|_{L^\infty} \lesssim \|f\|^2_{C^1(\Omega)}.$$

6.2 Statement of the main results for the stability of the geodesic X-ray transform $\mathfrak{I}_c$

We consider the stability estimate of the operator $\mathfrak{I}_c$. For simplicity, we drop the subscript c. Define $\beta : T^*\mathbb{R}^d \backslash \{(x, 0) : x \in \mathbb{R}^d\} \to S^*\mathbb{R}^d$ by

$$\beta(x, \xi) = \left(x, \frac{\xi}{\|\xi\|}\right).$$

Let $\pi : T^*\mathbb{R}^d \to \mathbb{R}^d$ be the natural projection onto the base space. We define $\phi : T^*\mathbb{R}^d \to \mathbb{R}^d$ by

$$\phi(x, \xi) = \pi \circ \mathcal{H}^{t=1}(x, \xi), \quad (x, \xi) \in T^*\mathbb{R}^d.$$

We remark that ϕ defined above is equivalent to the exponential map in Riemannian manifold.

The following result about the normal operator $\mathfrak{N} = \mathfrak{I}^\dagger \mathfrak{I}$ is well-known.

Lemma 6.2.1. *The normal operator $\mathfrak{N} : L^2(\Omega, \mathbb{R}^{2d}) \to L^2(\Omega, \mathbb{R}^{2d})$ is bounded and has* *the following representation*

$$\mathfrak{N}f(x) = \int_{T_x^*\Omega} W(x,\xi) f(\phi(x,\xi))\, d\sigma_x(\xi), \quad f \in L^2(\Omega, \mathbb{R}^{2d}) \tag{6.7}$$

where $d\sigma_x$ *denotes the measure in the space* $T_x^*\mathbb{R}^d$ *induced by the velocity field* c*, i.e.* $d\sigma_x(\xi) = c(x)^d d\xi$*, and* W *is defined as*

$$W(x,\xi) = \frac{1}{\|\xi\|^{d-1}}\{\Phi^\dagger \circ \beta(x,\xi)\cdot\Phi\circ\beta\circ\mathcal{H}(x,\xi) + \Phi^\dagger\circ\beta(x,-\xi)\cdot\Phi\circ\beta\circ\mathcal{H}^{-1}(x,-\xi)\}. \tag{6.8}$$

Proof. See [54] or [47].

We see from (6.7) that the local property of the normal operator $\mathfrak{N}$ restricted to a small neighborhood of $x \in \Omega$ is determined by the lagrangian map $\phi(x,\cdot) : T_x^*\mathbb{R}^d \to \mathbb{R}^d$. When the map is a diffeomorphism, it is known that the operator $\mathfrak{N}$ near x is a pseudo-differential operator (ΨDO). However, in general case, the map may not be a diffeomorphism and may have singular points which are called caustic vectors. The value of the map at caustic vectors are called caustics. When caustics occur, the Schwartz kernel of the operator $\mathfrak{N}$ has two singularities, one is from the diagonal which contributes to a ΨDO $\mathfrak{N}_1$, and the other is from the caustics which contributes to a singular integral operator $\mathfrak{N}_2$. The property of $\mathfrak{N}_2$ depends on the type of caustics. The case for fold caustics is investigated in [54], where it is shown that fold caustics contribute a Fourier Integral Operator (FIO) to $\mathfrak{N}_2$. Little is known for caustics of other type. Here we recall the following definition of fold caustics.

Definition 6.2.1. *Let* $f : \mathbb{R}^d \to \mathbb{R}^n$ *be a germ of* C^∞ *map at* x_0*, then* x_0 *is said to be a* *fold vector and $f(x_0)$ a fold caustic if the following two conditions are satisfied:*

1. *the rank of df at x_0 equals to $n-1$ and $\det df$ vanishes of order 1 at x_0;*

2. *the kernel of the matrix $df(x_0)$ is transversal to the manifold $\{x : \det df(x) = 0\}$ at x_0.*

We now introduce the following concept of "operator germ" to characterize the contribution of an infinitesimal neighborhood of a caustic or a regular point to the normal operator $\mathfrak{N}$.

Definition 6.2.2. *For each $\xi \in T_x^*\mathbb{R}^d \backslash 0$, the operator germ $\mathfrak{N}_\xi$ is defined to be the equivalent class of operators in the following form*

$$\mathfrak{N}_\xi f(y) = \int_{T_y^*\Omega} W(y,\eta) f(\phi(y,\eta)) \chi(y,\eta)\, d\sigma_y(\eta). \tag{6.9}$$

where χ is a smooth function supported in a small neighborhood of (x,ξ) in $\mathbb{R}^{2d}$. Two operators with χ_1 and χ_2 are said to be equivalent if there exists a neighborhood $B(x,\xi)$ of (x,ξ) such that $\chi_1 = \chi_2 \cdot \chi_3$ for some $\chi_3 \in C_0^\infty(B(x,\xi))$ with $\chi_3(x,\xi) \neq 0$.

The operator germ $\mathfrak{N}_\xi$ is said to has certain property if there exists a neighborhood $B(x,\xi)$ of (x,ξ) in $T^*\mathbb{R}^d$ such that the property holds for all operators of the form (6.9) with $\chi \in C_0^\infty(B(x,\xi))$.

Properties of the above defined operator germ will be given in Section 6.3.

We note from the preceding discussion that it is complicated to analyze the full operator $\mathfrak{N}$ which contains information from all geodesics. However, for a given interior point x, to recover f or the singularity of f at x from its geodesic transform, we need only to select a set of geodesics whose conormal bundle can cover the cotangent space $T_x^*\mathbb{R}^d$. Caustics may be allowed along these geodesics as long as they are of the simplest type, i.e fold type so that we can analyze their contributions. This idea can be carried out by introducing a cut-off function for the set of geodesics as we do now. We remark that this idea is motivated by the work [51]. For any $\alpha \in C_0^\infty(S_-^*\Gamma)$, we define

$$\mathfrak{I}_\alpha f(x_0,\xi_0) = \alpha(x_0,\xi_0)\int_0^{l(x_0,\xi_0)} \Phi(\mathcal{H}_c^s(x_0,\xi_0))f(\pi(\mathcal{H}_c^s(x_0,\xi_0)))\,ds \tag{6.10}$$

where $(x_0,\xi_0) \in S_-^*\Gamma$. Let $\alpha^\sharp$ be the unique lift of α to $S^*\Omega$ which is constant along bicharacteristic curves, i.e. $\alpha^\sharp(x,\xi) = \alpha\circ\tau(x,\xi)$ for $(x,\xi)\in S^*\Omega$. Then $\alpha^\sharp$ is smooth in $S^*\Omega$ and we have

$$\mathfrak{I}_\alpha f(x_0,\xi_0) = \int_0^{l(x_0,\xi_0)} (\alpha^\sharp\cdot\Phi)(\mathcal{H}_c^s(x_0,\xi_0))f(\pi(\mathcal{H}_c^s(x_0,\xi_0)))\,ds \tag{6.11}$$

With the original weight Φ being replaced by the new one $\alpha^\sharp\cdot\Phi$, we similarly can define $\mathfrak{N}_\alpha$. In fact, it is easy to check that $\mathfrak{N}_\alpha$ is defined as in (6.8) with W being replaced by

$$\begin{aligned} W_\alpha(x,\xi) \;\;=\;\; & \frac{1}{\|\xi\|^{d-1}}|\alpha\circ\tau\circ\beta(x,\xi)|^2\Phi^\dagger\circ\beta(x,\xi)\cdot\Phi\circ\beta\circ\mathcal{H}(x,\xi) \\ & +\frac{1}{\|\xi\|^{d-1}}|\alpha\circ\tau\circ\beta(x,-\xi)|^2\Phi^\dagger\circ\beta(x,-\xi)\cdot\Phi\circ\beta\circ\mathcal{H}^{-1}(x,-\xi). \end{aligned}$$

It can be shown that with properly chosen α, the analysis of the operator $\mathfrak{N}_\alpha$ becomes possible and we can recover the singularity of f from $\mathfrak{N}_\alpha f$.

We now give two definitions whose discussions are postponed to Section 6.

Definition 6.2.3. *A fold vector $\xi \in T_x^*\mathbb{R}^d$ is called fold-regular if there exists a neighbor-*

hood $U(x)$ of x such that the operator germ $\mathfrak{N}_\xi$ is compact from $L^2(\Omega_0, \mathbb{R}^{2d})$ to $H^1(U(x), \mathbb{R}^{2d})$ (or from $H^s(\Omega_0, \mathbb{R}^{2d})$ to $H^{s+1}(U(x), \mathbb{R}^{2d})$ for all $s \in \mathbb{R}$).

Definition 6.2.4. *A point x is called fold-regular if there exists a compact subset $\mathcal{Z}_2(x) \subset S^*_x\mathbb{R}^d$ such that the following two conditions are satisfied:*

1. *For each $\xi \in \mathcal{Z}_2(x)$, there exist only singular vectors of fold-regular type along the ray $\{t\xi : t \in \mathbb{R}\}$ for the map $\phi(x, \cdot)$;*
2. *$\forall \xi \in S^*_x\mathbb{R}^d$, $\exists \theta \in \mathcal{Z}_2(x)$, such that $\theta \perp \xi$.*

We remark that $\mathcal{Z}_2(x)$ parameterizes a subset of geodesics that pass through x and along which there exist only fold-regular caustics.

We now present the main results on the stability estimate for the geodesic X-ray transform operator. The proofs are given in Section ?.

Theorem 6.2.1. *Let x_* be a fold-regular point, then there exist a cut-off function $\alpha \in C_0^\infty(S^*_-\Gamma)$, a neighborhood $U(x_*)$ of x_*, a compact operator $\mathfrak{N}_{2,\alpha}$ from $L^2(\Omega_0, \mathbb{R}^{2d})$ to $H^1(U(x_*), \mathbb{R}^{2d})$ and a smoothing operator $\mathfrak{R}$ from $\mathcal{E}'(\Omega, \mathbb{R}^{2d})$ into $C^\infty(\overline{U(x_*)}, \mathbb{R}^{2d})$, such that for any $U_0(x_*) \Subset U(x_*)$ the following holds*

$$\|f\|_{H^s(U_0(x_*))} \lesssim \|\mathfrak{N}_\alpha f\|_{H^{s+1}(U(x_*))} + \|\mathfrak{N}_{2,\alpha} f\|_{H^{s+1}(U(x_*))} + \|\mathfrak{R} f\|_{H^s(U(x_*))} \quad (6.12)$$

for all $f \in \mathcal{D}'(\Omega_0, \mathbb{R}^{2d})$ and $s \in \mathbb{R}$.

Theorem 6.2.2. *Assume that the background velocity field c is strong fold-regular. Then there exist $U(x_j) \subset \Omega$, $\alpha_j \in C_0^\infty(S_-\partial\Omega)$, $j = 1, 2...N$, and a finite dimensional space*

$\mathfrak{L}_0 \in L^2(\Omega, \mathbb{R}^{2d})$ *such that the following estimate holds for any* $f \in L^2(\Omega, \mathbb{R}^{2d})$ *with support in* Ω_0*:*

$$\|f\|_{L^2(M)} \lesssim \sum_{j=1}^{N} \|\mathfrak{N}_{\alpha_j} f\|_{H^1(\tilde{U}(x_j))} + \|\mathfrak{N}_{2,\alpha_j} f\|_{H^1(U(x_j))} + \|\mathfrak{R}_{\mathrm{j}} f\|_{L^2(U(x_j))} \tag{6.13}$$

where each $\mathfrak{N}_{2,\alpha_j}$ *is a compact operator from* $L^2(\Omega_0, \mathbb{R}^{2d})$ *to* $H^1(U(x_j), \mathbb{R}^{2d})$*, and* $\mathfrak{R}_{\mathrm{j}}$ *smoothing from* $\mathcal{E}'(\Omega, \mathbb{R}^{2d})$ *into* $C^\infty(\overline{U(x_j)}, \mathbb{R}^{2d})$*. If we denote by* $\mathcal{P}_{\mathfrak{L}_0^\perp}$ *the orthogonal projection in* $L^2(\Omega, \mathbb{R}^{2d})$ *onto the subspace which is perpendicular to* $\mathfrak{L}_0$*, we have the following Lipschitz estimate*

$$\|\mathcal{P}_{\mathfrak{L}_0^\perp} f\|_{L^2(M)} \lesssim \sum_{j=1}^{N} \|\mathfrak{N}_{\alpha_j} f\|_{H^1(U(x_j))}. \tag{6.14}$$

6.3 Local properties of the normal operator $\mathfrak{N}$

In this subsection, we present some results about the local properties of the normal operator $\mathfrak{N}$ (see (6.7)).

From now on, we fix $x_* \in \Omega$. We first decompose $\mathfrak{N}$ locally into two parts based on the separation of singularities of its Schwartz kernel. Note that the map $\phi(x_*, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism in a neighborhood of the origin. In fact, we can check that $\frac{\partial \phi(x_*, \cdot)}{\partial \xi}(0) = c(x_*) \cdot Id$. Similar to the proof of existence of uniformly normal neighborhood in Riemannian manifold [32], we can find $\epsilon_2 > 0$ and a neighborhood of x_*, say $\tilde{U}(x_*) \subset \mathbb{R}^d$, such that $\phi(x, \cdot)$ are diffeomorphisms for each $x \in \tilde{U}(x_*)$ in the domain $\{\xi : \|\xi\| < 2\epsilon_2\}$.

Let $\chi_* \in C_0^\infty(\mathbb{R})$ be such that $\chi(t) = 1$ for $|t| < \epsilon_2$ and $\chi(t) = 0$ for $|t| > 2\epsilon_2$. We then define

$$\mathfrak{N}_1 f(x) = \int_{T_x^*\Omega} W(x,\xi) f(\phi(x,\xi))\chi_*(\|\xi\|)\, d\sigma_x(\xi), \tag{6.15}$$

$$\mathfrak{N}_2 f(x) = \int_{T_x^*\Omega} W(x,\xi) f(\phi(x,\xi))(1-\chi_*(\|\xi\|))\, d\sigma_x(\xi). \tag{6.16}$$

Note that for any f supported in Ω, $f(\phi(x,\xi)) = 0$ for all $\|\xi\| > T$. Thus we have

$$\mathfrak{N}_2 f(x) = \int_{\xi \in T_x^*\Omega,\ \epsilon_2 < \|\xi\| < T} W(x,\xi) f(\phi(x,\xi))(1-\chi_*(\|\xi\|))\, d\sigma_x(\xi).$$

It is clear that $\mathfrak{N}f = \mathfrak{N}_1 f + \mathfrak{N}_2 f$. This gives the promised decomposition of $\mathfrak{N}$. We next study $\mathfrak{N}_1$ and $\mathfrak{N}_2$ separately.

Lemma 6.3.1. $\mathfrak{N}_1 : C_0^\infty(\tilde{U}(x_*), \mathbb{R}^{2d}) \to \mathcal{D}'(\tilde{U}(x_*), \mathbb{R}^{2d})$ *is an elliptic* ΨDO *of order* -1 *and its principle symbol is*

$$\sigma_p(\mathfrak{N}_1)(x,\xi) = 2\pi \cdot \int_{S_x^*\Omega} \delta(\langle \xi, \theta\rangle)\Phi^\dagger(x,\theta)\cdot\Phi(x,\theta)\, d\sigma_x(\theta).$$

Proof. See [51] or [54].

We now proceed to study the operator $\mathfrak{N}_2$ whose property is determined by the Lagrangian map $\phi(x_*,\cdot)$. We shall study the operator germ $\mathfrak{N}_{2,\xi_*}$ for each $\xi \in T_{x_*}^*\mathbb{R}^d$. We first consider the case when ξ_* is not a caustic vector, i.e. ξ_* is a regular vector.

Lemma 6.3.2. *Let* $\xi_* \in S_{x_*}^*\mathbb{R}^d$ *be a regular vector, then there exists a neighborhood* $U(x_*)$ *of* x_* *and a neighborhood* $B(x_*,\xi_*)$ *of* (x_*,ξ_*) *such that for any* $\chi \in C_0^\infty(B(x_*,\xi_*))$ *the* *following operator*

$$\mathfrak{N}_{2,\xi_*} f(x) = \int_{T_x^*\Omega} W(x,\xi) f(\phi(x,\xi))(1-\chi_*(\|\xi\|)) \cdot \chi(x,\xi)\, d\sigma_x(\xi)$$

is a smoothing operator from $\mathcal{E}'(\Omega, \mathbb{R}^{2d})$ into $C^\infty(\overline{U(x_*)}, \mathbb{R}^{2d})$.

Proof. Since $\xi_* \in S^*_{x_*}\mathbb{R}^d$ is regular, there exist a neighborhood $V(x_*)$ of x_* in $\mathbb{R}^d$ and a neighborhood $B(x_*, \xi_*)$ of (x_*, ξ_*) in $\mathbb{R}^{2d}$ of the form $B(x_*, \xi_*) = V(x_*) \times B_0(\xi_*)$ for some open set $B_0(\xi_*)$ in $\mathbb{R}^d$ such that the map $\phi(x, \cdot)$ is a diffeomorphism between $B_0(\xi_*)$ and its image for all $x \in V(x_*)$. We denote the inverse of the map $\phi(x, \cdot)$ by $\phi^{-1}(x, \cdot)$. By a change of coordinate $\xi = \phi^{-1}(x, y)$ and use some cut-off function, we can write $\mathfrak{N}_{2,\xi_*}$ in the following form

$$\mathfrak{N}_{2,\xi_*} f(x) = \int_\Omega K(x,y) f(y)\, dy, f \in \mathcal{E}'(\Omega, \mathbb{R}^{2d})$$

for some smooth function K in $\Omega \times \Omega$. The Lemma follows immediately.

We next consider the case when ξ_* is a fold vector. We have the following slightly modified result from [54].

Lemma 6.3.3. *Let ξ_* be a fold vector of the map $\phi(x_*, \cdot)$. Then there exists a small neighborhood $U(x_*)$ of x_* and a small neighborhood $B(x_*, \xi_*)$ of (x_*, ξ_*) in $\mathbb{R}^{2d}$ such that for any $\chi \in C_0^\infty(B(x_*, \xi_*))$, the operator $\mathfrak{N}_{2,\xi_*} : \mathcal{E}'(\Omega, \mathbb{R}^{2d}) \to \mathcal{D}'(U(x_*), \mathbb{R}^{2d})$ defined by*

$$\mathfrak{N}_{2,\xi_*} f(x) = \int_{T_x^*\Omega} W(x,\xi) f(\phi(x,\xi))(1-\chi_*(\|\xi\|)) \cdot \chi(x,\xi)\, d\sigma_x(\xi), \quad f \in \mathcal{E}'(\Omega, \mathbb{R}^{2d}) \tag{6.17}$$

is an FIO of order $-\frac{d}{2}$ whose associated canonical relation is compactly supported in the following set

$$
\begin{aligned}
\Big\{(x,\xi,y,\eta); \qquad & x\in U(x_*), y=\phi(x,\omega), (x,\omega)\in B(x_*,\xi_*),\ \det d_\omega\phi(x,\omega)=0,\\
& \xi=-\eta_i\frac{\partial\phi^i(x,\omega)}{\partial x},\ \eta\in \mathit{Coker}(d_\omega\phi(x,\omega)).\Big\}
\end{aligned}
\tag{6.18}
$$

Proof. We sketch a proof here and refer to [54] for detail. We first note that by the fold condition, there exists a small neighborhood $B_1(x_*,\xi_*)$ of (x_*,ξ_*) in $\mathbb{R}^{2d}$ such that ξ_* is the only singular vector of the map $\phi(x_*,\cdot)$ along the ray $\{t\xi_*: t\in\mathbb{R}\}$ in $B_1(x_*,\xi_*)$. Define

$$
\begin{aligned}
S &= \{(x,w): \det d_\omega\phi(x,\omega)=0\}\subset\mathbb{R}^{2d},\\
\Sigma &= \{(x,y): y=\phi(x,\omega), (x,\omega)\in S\}\subset\mathbb{R}^{2d}.
\end{aligned}
$$

By shrinking $B_1(x_*,\xi_*)$ if necessary, we can show that $S\bigcap B_1(x_*,\xi_*)$ is a smooth $(2d-1)$-dimensional manifold in $\mathbb{R}^{2d}$, and ϕ is a diffeomorphism between $S\bigcap B_1(x_*,\xi_*)$ and its image. Denote

$$
S_1=S\bigcap B_1(x_*,\xi_*), \quad \Sigma_1=\phi(S_1).
$$

Note that Σ_1 is a smooth $(2d-1)$-dimensional manifold in a neighborhood of (x_*,y_*) in $\Omega\times\Omega$. Let π_2 be the projection from $\Omega\times\Omega$ to its second component. By the fold-condition for ξ_* and the fact that the matrix $d_{x,\xi}\phi(x_*,\xi_*)$ is surjective, we can show that $d\pi_2: T_{(x_*,\xi_*)}\Sigma_1\to T_{y_*}\mathbb{R}^d$ is surjective. Thus there exists a neighborhood $V_1(y_*)$ of y_*

in $\mathbb{R}^d$ such that $V_1(y_*) \subset \pi_2(\Sigma_1)$. We remark that the surjectivity of the linear map $d\pi_2$ implies that the conormal bundle of Σ_1 belongs to $(T^*\mathbb{R}^d \setminus 0) \times (T^*\mathbb{R}^d \setminus 0)$, where 0 stands for the zero section of the normal bundle $T^*\mathbb{R}^d$.

Let $U_1(x_*)$ be a small neighborhood of x_* such that $U_1(x_*) \subset \pi(S_1)$ and $\tilde{\chi} \in C_0^\infty(B_1(x_*, \xi_*))$. Consider the Schwartz kernel of operator

$$\tilde{\mathfrak{N}}_{2,\xi_*} : \mathcal{E}'(\Omega, \mathbb{R}^{2d}) \to \mathcal{D}'(U_1(x_*), \mathbb{R}^{2d})$$

defined by

$$\tilde{\mathfrak{N}}_{2,\xi_*} f(x) = \int_{T_x^*\Omega} W(x,\xi) f(\phi(x,\xi))(1 - \chi_*(\|\xi\|)) \cdot \tilde{\chi}(x,\xi)\, d\sigma_x(\xi), \quad f \in \mathcal{E}'(\Omega, \mathbb{R}^{2d}).$$

We can show that it has conormal singularity supported in the set Σ_1. Moreover, the conormal bundle $\mathcal{N}^*\Sigma_1$ is given by

$$\begin{aligned}\mathcal{N}^*\Sigma_1 = \Big\{(x,\xi,y,\eta); \qquad & x \in U(x_*), y = \phi(x,\omega), (x,\omega) \in B_1(x_*,\xi_*), \\ & \xi = -\eta_i \frac{\partial \phi^i(x,\omega)}{\partial x}, \eta \in \mathrm{Coker}\,(d_\omega \phi(x,\omega)), \det d_\omega \phi(x,\omega) = 0.\Big\}\end{aligned}$$

By analyzing the singularity of the Jacobian determinant of $d_\omega\phi(x,\omega))$, we can show that the Schwartz kernel of $\tilde{\mathfrak{N}}_{2,\xi_*}$ belongs to the conormal class $I^{-\frac{d}{2}}(\Omega \times \Omega, \Sigma_1, \mathcal{M}_{2d\times 2d})$, where $\mathcal{M}_{2d\times 2d}$ denotes the vector bundle of matrices from $\mathbb{R}^{2d}$ to $\mathbb{R}^{2d}$ over Ω. Especially, when the domain of $\tilde{\mathfrak{N}}_{2,\xi_*}$ is restricted to distribution sections supported in $V_1(y_*)$, the operator $\tilde{\mathfrak{N}}_{2,\xi_*}$ is a FIO of order $-\frac{d}{2}$ from $\mathcal{E}'(V_1(y_*), \mathbb{R}^{2d})$ to $\mathcal{D}'(U_1(x_*), \mathbb{R}^{2d})$.

The above $\tilde{\mathfrak{N}}_{2,\xi_*}$ with $\tilde{\chi} \in C_0^\infty(B_1(x_*,\xi_*))$ requires the domain to be $\mathcal{E}'(V_1(y_*), \mathbb{R}^{2d})$, we now show that this condition can be relaxed by further decreasing $B_1(x_*, \xi_*)$. Indeed, let $U(x_*)$ be a neighborhood of x_* such that $U(x_*) \Subset U_1(x_*)$ and $B(x_*, \xi_*)$ be a neighborhood of (x_*, ξ_*) such that $B(x_*, \xi_*) \Subset B_1(x_*, \xi_*)$. By choosing $U(x_*)$ and $B(x_*, \xi_*)$ to be sufficiently small, we can assume that the set $V(y_*) = \{\, \phi(x, \xi) : x \in U(x_*), \xi \in B(x_*, \xi_*)\}$ is compactly supported in $V_1(y_*)$. We then choose $\chi_V \in C_0^\infty(V_1(x_*))$ such that $\chi_V(x) = 1$ for $x \in V(x_*)$. One can check that for any $\chi \in C_0^\infty(B(x_*, \xi_*))$, the operator $\mathfrak{N}_{2,\xi_*}$ defined by

$$\mathfrak{N}_{2,\xi_*} f(x) = \int_{T_x^*\Omega} W(x, \xi) f(\phi(x, \xi))(1 - \chi_*(\|\xi\|)) \cdot \chi(x, \xi)\, d\sigma_x(\xi), \quad f \in \mathcal{E}'(\Omega, \mathbb{R}^{2d})$$

satisfies $\mathfrak{N}_{2,\xi_*} f(x) = \mathfrak{N}_{2,\xi_*}(\chi_V \cdot f)(x)$ for all $x \in U(x_*)$. Thus $\mathfrak{N}_{2,\xi_*}$ is well-defined from $\mathcal{E}'(\Omega, \mathbb{R}^{2d})$ to $\mathcal{D}'(U(x_*), \mathbb{R}^{2d})$, and is a FIO of order $-\frac{d}{2}$ with canonical relation compactly supported in the set (6.18). This completes the proof of the lemma.

6.4 Singularities of the map $\phi(x, \cdot)$

In this subsection, we present some properties about the map $\phi(x, \cdot)$ which is equivalent to the exponential map in Riemannian manifold.

By the classification result for Lagrangian maps (see [5] and [6] for detail), there are only a finite number of stable and simple singular Lagrangian map germs in dimensions between three and five and they are generic. In three dimensions, there are four types: fold, cusp, swallow-tail and D4. The others are unstable and can be removed by using arbitrarily small perturbations. We define

$$\begin{aligned}
\mathcal{K}(x) &= \{\xi \in T_x^*\mathbb{R}^d : \text{the map germ } \phi(x,\cdot) \text{ at } \xi \text{ is singular}\};\\
\mathcal{K}_1(x) &= \{\xi \in T_x^*\mathbb{R}^d : \text{the map germ } \phi(x,\cdot) \text{ at } \xi \text{ has singularity of fold type}\};\\
\mathcal{K}_2(x) &= \{\xi \in T_x^*\mathbb{R}^d : \text{the map germ } \phi(x,\cdot) \text{ at } \xi \text{ has singularity of cusp type}\};\\
\mathcal{K}_3(x) &= \{\xi \in T_x^*\mathbb{R}^d : \text{the map germ } \phi(x,\cdot) \text{ at } \xi \text{ has simple and stable singularities}\\
&\qquad \text{of types other than fold and cusp}\};\\
\mathcal{K}_4(x) &= \{\xi \in T_x^*\mathbb{R}^d : \text{the map germ } \phi(x,\cdot) \text{ at } \xi \text{ has singularity}\\
&\qquad \text{which are either not simple or stable}\}.
\end{aligned}$$

It is clear that $\mathcal{K}(x) = \bigcup_{j=1}^4 \mathcal{K}_j(x)$. Denote

$$\mathcal{S}(x) = \{\xi \in S_x^*\mathbb{R}^d : r\xi \in \mathcal{K} \text{ for some } \ r \in \mathbb{R}\},$$
$$\mathcal{S}_j(x) = \{\xi \in S_x^*\mathbb{R}^d : r\xi \in \mathcal{K}_j \text{ for some } \ r \in \mathbb{R}\}, \ \ j = 1,2,3,4.$$

We say that the map $\phi(x,\cdot)$ is in a general position(or generic) if the map germ $\phi(x,\cdot)$ is simple and stable at all caustic vectors in $\mathcal{K}(x)$, i.e. $\mathcal{K}_4(x) = \emptyset$. It is possible that the map $\phi(x,\cdot)$ can be brought to a general position by adding an arbitrarily small perturbation to the velocity field c. By the classification result of Lagrangian maps, see for instance [5], the following result holds for the set $\mathcal{K}(x)$.

Propsition 1. *Assume that the map $\phi(x,\cdot)$ is in a general position, then the sets $\mathcal{K}_1(x)$ and $\mathcal{K}_2(x)$ are smooth manifolds of dimensions $d-1$ and $d-2$, respectively. The set $\mathcal{K}_3(x)$ is a union of smooth manifolds of dimensions not greater than $d-3$. Especially,* *for $d = 3$, the sets $\mathcal{K}_1(x)$, $\mathcal{K}_2(x)$ and $\mathcal{K}_3(x)$ consists of smooth surfaces, smooth curves and isolated points, respectively.*

In the case when the map $\phi(x, \cdot)$ is not in a general position, it is known that $\mathcal{K}_1(x) \bigcup \mathcal{K}_2(x) \bigcup \mathcal{K}_3(x)$ is open and dense in $\mathcal{K}(x)$.

Note that $\mathcal{S}_j(x)$ are the images of $\mathcal{K}_j(x)$ under the map β which sends $\xi \in T_x^*\mathbb{R}^d$ to $\frac{\xi}{\|\xi\|} \in S_x^*\mathbb{R}^d$ for $\xi \neq 0$. We conclude that the following result holds.

Lemma 6.4.1. *Assume that the map $\phi(x, \cdot)$ is in general position, then the sets $\mathcal{S}_2(x)$ and $\mathcal{S}_3(x)$ are of finite $d-2$ and $d-3$ dimensional Hausdorff measures, respectively. Especially, for $d = 3$, the set $\mathcal{S}_2(x)$ is a curve (not necessarily smooth) of finite length in $S_x^*\mathbb{R}^3$ and $\mathcal{S}_3(x)$ consists of a finite number of points.*

6.5 Discussions on the concept of Fold-regular

In this subsection, we discuss the concept "fold-regular". We show that for a general velocity field in Ω whose induced metric is not simple, a given point in Ω is fold-regular under some natural assumptions.

We begin with the concept "fold-regular vector". It is still an open problem to find a complete characterization for it, i.e. what are the necessary and sufficient conditions for the map germ $\phi(x_*, \cdot)$ at ξ_* for ξ_* to be fold-regular. We have the following partial answer in the form of remarks.

Remark 6.5.1. *In dimension $d = 2$, the set of fold-regular vectors is generally empty. Indeed, for a fold vector ξ_*, the operator germ $\mathfrak{N}_{2,\xi_*}$ is a FIO of order -1, and hence the best* *estimate is that it is bounded from* $L^2(\Omega_0, \mathbb{R}^{2d})$ *to* $H^1(U(x_*), \mathbb{R}^{2d})$ *for some neighborhood* $U(x_*)$ *of* x_*.

Remark 6.5.2. *In dimension* $d \geq 3$, *a sufficient condition for a fold vector* ξ_* *to be fold-regular is that the following condition is satisfied*

$$d^2_\xi \phi(x_*, \xi_*)(N_{x_*}(\xi_*) \setminus 0 \times \cdot)|_{T_{\xi_*} S(x_*)} \quad \text{is of full rank.} \tag{6.19}$$

where $N_{x_*}(\xi_*)$ *denotes the kernel of* $d_\xi \phi(x_*, \xi_*)$ *and* $S(x_*)$ *the set of all vectors* $\xi \in T^*_{x_*}\mathbb{R}^d$ *such that* $\det d_\xi \phi(x_*, \xi) = 0$. *Indeed, in that case, it is shown in [54] that the canonical relation associated with the operator germ* $\mathfrak{N}_{2,\xi_*}$ *is locally a canonical graph and hence* $\mathfrak{N}_{2,\xi_*}$ *is bounded from* $L^2(\Omega_0, \mathbb{R}^{2d})$ *to* $H^{\frac{d}{2}}(U(x_*), \mathbb{R}^{2d})$ *for some neighborhood* $U(x_*)$ *of* x_*. *Note that for* $d \geq 3$, $H^{\frac{d}{2}}(U(x_*), \mathbb{R}^{2d})$ *is compactly embedded in* $H^1(U(x_*), \mathbb{R}^{2d})$, *so* $\mathfrak{N}_{2,\xi_*}$ *is compact from* $L^2(\Omega_0, \mathbb{R}^{2d})$ *to* $H^1(U(x_*), \mathbb{R}^{2d})$ *and we can conclude that* ξ_* *is fold-regular.*

The set of fold-regular vectors $\mathcal{Z}_1(x_*)$ *contains more elements than those which satisfy the graph condition (6.19). In fact, let* $\mathcal{C} \subset T^*\Omega \times T^*\Omega$ *be the canonical relation associated with the operator germ* $\mathfrak{N}_{2,\xi_*}$ *defined in Lemma 6.3.3. We have shown that* $\mathcal{C}$ *is homogeneous and* $\mathcal{C} \subset (T^*\Omega \backslash 0) \times (T^*\Omega \backslash 0)$. *By the main result in [24],* $\mathfrak{N}_{2,\xi_*}$ *is bounded from* $L^2(\Omega_0, \mathbb{R}^{2d})$ *to* $H^{\frac{d}{2}-\frac{1}{3}}(U(x_*), \mathbb{R}^{2d})$ *for some neighborhood* $U(x_*)$ *of* x_*, *if the only singularity of the projection of* $\mathcal{C}$ *to its first or second component at the point associated with* (x_*, ξ_*) *is fold or cusp. Since* $H^{\frac{d}{2}-\frac{1}{3}}(U(x_*), \mathbb{R}^{2d})$ *is compactly embedded in* $H^1(U(x_*), \mathbb{R}^{2d})$, *we see that* $\mathfrak{N}_{2,\xi_*}$ *is compact from* $L^2(\Omega_0, \mathbb{R}^{2d})$ *to* $H^1(U(x_*), \mathbb{R}^{2d})$ *and hence* ξ_* *is fold-regular.*

We now consider the concept "fold-regular point". We denote

$$\mathcal{Z}_1(x_*) = \{\xi \in S^*_{x_*}\mathbb{R}^d : \ \forall\, r \in \mathbb{R}, r\xi \text{ is either regular or fold-regular}\}.$$

It is clear that

$$\mathcal{Z}_1(x_*) \subset \mathcal{Z}(x_*) =: S^*_{x_*}\mathbb{R}^d \setminus \overline{(\mathcal{S}_2(x_*) \bigcup \mathcal{S}_3(x_*)) \bigcup \mathcal{S}_4(x_*)}.$$

We remark that $\mathcal{Z}(x_*)$ characterizes the set of geodesics that pass through x_* and along which the map $\phi(x_*, \cdot)$ only has singularities of fold type. By Morse's index theorem (a fold vector for the map $\phi(x, \cdot)$ corresponds to a fold conjugate vector for the exponential map $\exp_x(\cdot)$), for each $\xi \in \mathcal{S}(x_*)$, there are at most finitely many fold vectors along the geodesic $\pi \circ \mathcal{H}^t(x_*, \xi)$. Using the definition of "fold-regular", we can conclude that $\mathcal{Z}_1(x_*)$ is open in $S^*_{x_*}\mathbb{R}^d$.

Recall that x_* is fold-regular if there exists a compact subset $\mathcal{Z}_2(x_*) \subset \mathcal{Z}_1(x_*)$ such that the following completeness condition is satisfied

$$\forall\, \xi \in S^*_{x_*}\mathbb{R}^d, \exists\, \theta \in \mathcal{Z}_2(x_*), \quad \text{such that } \ \theta \perp \xi.$$

Remark 6.5.3. *A sufficient condition for the completeness of $\mathcal{Z}_1(x_*)$ is that $\mathcal{Z}_1(x_*)$ contains a set*

$$\theta^\perp = \{\xi \in S^*_{x_*}\mathbb{R}^d, \xi \perp \theta\}$$

for $\theta \in S^*_{x_*}\mathbb{R}^d$.

Remark 6.5.4. *If the completeness condition fails for $\mathcal{Z}_1(x_*)$, then there exists $\theta \in S^*_{x_*}\mathbb{R}^d$ such that*

$$\theta^\perp \subset \overline{\mathcal{S}_2(x_*) \bigcup \mathcal{S}_3(x_*) \bigcup \mathcal{S}_4(x_*)}. \tag{6.20}$$

Assume that the map $\phi(x_*, \cdot)$ is in a general position. By Lemma 6.4.1, the set on the right hand side of (6.20) is of finite $d-2$-dimensional Hausdorff measure, so is the set $\theta^\perp$ for each $\theta \in S^*_{x_*}\mathbb{R}^d$. Thus, we conclude that there exists at most a finite number of θ such that (6.20) holds.

6.6 Proof of Theorem 6.2.1

We prove Theorem 6.2.1 in this subsection. The proof can be divided into two major stages: in the first stage, we present some preliminaries and construct a cut-off function $\alpha \in C_0^\infty(S^*_-\Gamma)$ which selects a complete set of geodesics with only fold-regular caustics, see Lemma 6.6.1; in the second stage, we study the normal operator $\mathfrak{N}_\alpha = \mathfrak{I}_\alpha^\dagger \mathfrak{I}_\alpha$, see Lemma 6.6.2 and 6.6.3. Theorem 6.2.1 is then a direct consequence of Lemma 6.6.2 and 6.6.3.

We now present some preliminaries that are necessary for the construction of α. Let x_* be a fold-regular point with the compact subset $\mathcal{Z}_2(x_*) \subset S^*_{x_*}\mathbb{R}^d$ in Definition 6.2.4. Denote $\mathcal{C}\mathcal{Z}_2 = \{r\xi; \xi \in \mathcal{Z}_2(x_*), r \in \mathbb{R} \quad \text{and} \ \epsilon_2 \le |r| \le T\}$. For each $\xi_* \in \mathcal{C}\mathcal{Z}_2$, by Lemma 6.3.2 and Lemma 6.3.3, there exist a neighborhood $U(x_*, \xi_*)$ of x_* and a neighborhood $B(x_*, \xi_*)$ of (x_*, ξ_*) such that for any $\chi \in C_0^\infty(B(x_*, \xi_*))$ the following operator

$$\mathfrak{N}_{2,\xi_*} f(x) = \int_{T^*_x\Omega} W(x,\xi) f(\phi(x,\xi))(1 - \chi_*(\|\xi\|)) \cdot \chi(x,\xi)\, d\sigma_x(\xi)$$

is compact from $H^s(\Omega_0, \mathbb{R}^{2d})$ to $H^{s+1}(U(x_*, \xi_*), \mathbb{R}^{2d})$. Let $B_0(x_*, \xi_*)$ be another neighborhood of (x_*, ξ_*) in $\mathbb{R}^{2d}$ such that $B_0(x_*, \xi_*) \Subset B(x_*, \xi_*)$. Since $\mathcal{CZ}_2$ is compact, there exists a finite number of ξ_*'s in $\mathcal{CZ}_2$, say ξ_1, ξ_2,..., ξ_M, such that

$$\mathcal{CZ}_2 \subset \bigcup_{j=1}^{M} B_0(x_*, \xi_j).$$

We can then find smooth functions χ_1, χ_2,...,χ_M with $\mathrm{supp}\chi_j \subset B(x_*, \xi_j)$ for each j such that

$$\sum_{j=1}^{M} \chi_j(x, \xi) = 1 \quad \text{for all} \;\; (x, \xi) \in \bigcup_{j=1}^{M} B_0(x_*, \xi_j).$$

Denote by $\mathcal{A}_0$ be the greatest connected open symmetric subset in $\bigcup_{j=1}^{M} B_0(x_*, \xi_j)$ which contains $\mathcal{CZ}_2$. Here and after, we say that a set $\mathcal{B}$ in $\mathbb{R}^{2d}$ is symmetric if $(x, \xi) \in \mathcal{B}$ implies that $(x, -\xi) \in \mathcal{B}$. Define

$$\mathcal{A}_\epsilon = \{(x, \xi) \in \mathbb{R}^{2d} : |x - x_*| \le \epsilon, \;\epsilon_2 \le \|\xi\| \le T\}$$

for each $\epsilon > 0$. It is clear that $\mathcal{A}_\epsilon$ is compact in $\mathbb{R}^{2d}$, so is the set $\mathcal{A}_\epsilon \backslash \mathcal{A}_0$.

Lemma 6.6.1. *There exist $\epsilon_3 > 0$ and $\alpha \in C_0^\infty(S_-^*\Gamma)$ such that the following two conditions are satisfied:*

$$\alpha(x_0, \xi_0) \;=\; 1 \quad \textit{for all} \;\; (x_0, \xi_0) \in \tau \circ \beta(\mathcal{CZ}_2(x_*)), \tag{6.21}$$

$$\alpha(x_0, \xi_0) \;=\; 0 \quad \textit{for all} \;\; (x_0, \xi_0) \in \tau \circ \beta(\mathcal{A}_{\epsilon_3} \backslash \mathcal{A}_0). \tag{6.22}$$

Proof. Note that both β and τ are continuous. Since $\mathcal{CZ}_2(x_*)$ and $\mathcal{A}_\epsilon \backslash \mathcal{A}_0$ are compact, so are the sets $\tau \circ \beta(\mathcal{CZ}_2(x_*))$ and $\tau \circ \beta(\mathcal{A}_\epsilon \backslash \mathcal{A}_0)$. We claim that there exists $\epsilon_3 > 0$ such that

$$\tau \circ \beta(\mathcal{CZ}_2(x_*)) \bigcap \tau \circ \beta(\mathcal{A}_\epsilon \backslash \mathcal{A}_0) = \emptyset$$

for all $\epsilon \leq \epsilon_3$. Indeed, assume the contrary, then

$$\tau \circ \beta(\mathcal{CZ}_2(x_*)) \bigcap \tau \circ \beta(\mathcal{A}_\epsilon \backslash \mathcal{A}_0) \neq \emptyset$$

for all $\epsilon > 0$. Note that the collection of compact subsets $\tau \circ \beta(\mathcal{CZ}_2(x_*)) \bigcap \tau \circ \beta(\mathcal{A}_\epsilon \backslash \mathcal{A}_0)$ is decreasing with respect to ϵ, so it satisfies the finite intersection property and we can thus conclude that

$$\tau \circ \beta(\mathcal{CZ}_2(x_*)) \bigcap_{\epsilon > 0} \tau \circ \beta(\mathcal{A}_\epsilon \backslash \mathcal{A}_0) \neq \emptyset.$$

But on the other hand, we can check that

$$\begin{aligned} \bigcap_{\epsilon > 0} \tau \circ \beta(\mathcal{A}_\epsilon \backslash \mathcal{A}_0) &= \tau((\mathcal{A}_\epsilon \backslash \mathcal{A}_0) \bigcap S^*_{x_*}\mathbb{R}^d) \\ \tau \circ \beta(\mathcal{CZ}_2(x_*)) &= \tau(\mathcal{CZ}_2(x_*) \bigcap S^*_{x_*}\mathbb{R}^d). \end{aligned}$$

Using the fact that τ is injective on $S^*_{x_*}\mathbb{R}^d$ and $\mathcal{CZ}_2 \subset \mathcal{A}_0$, we obtain

$$\tau((\mathcal{A}_\epsilon \backslash \mathcal{A}_0) \bigcap S^*_{x_*}\mathbb{R}^d) \bigcap \tau(\mathcal{CZ}_2(x_*) \bigcap S^*_{x_*}\mathbb{R}^d) = \emptyset.$$

Thus,

$$\tau \circ \beta(\mathcal{CZ}_2(x_*)) \bigcap_{\epsilon > 0} \tau \circ \beta(\mathcal{A}_\epsilon \backslash \mathcal{A}_0) = \emptyset.$$

This contradiction completes the proof of our claim.

Now, we have

$$\tau \circ \beta(\mathcal{CZ}_2(x_*)) \bigcap \tau \circ \beta(\mathcal{A}_{\epsilon_3} \backslash \mathcal{A}_0) = \emptyset.$$

By decreasing ϵ_3 if necessary, we may assume that

$$\{x : |x - x_*| \leq \epsilon_3\} \subset \pi(\mathcal{A}_0).$$

Since both the sets $\tau \circ \beta(\mathcal{CZ}_2(x_*))$ and $\tau \circ \beta(\mathcal{A}_{\epsilon_3} \backslash \mathcal{A}_0)$ are compact in $S^*_-\Gamma$, we can find $\alpha \in C_0^\infty(S^*_-\Gamma)$ as desired. This concludes the proof of the lemma.

The construction of α above completes the first stage of the proof of Theorem 6.2.1, we are now at the second stage. We define the truncated geodesic X-ray transform $\mathfrak{I}_\alpha f$ as in (6.10) or (6.11). By replacing the weight Φ with the new one $\alpha^\sharp \cdot \Phi$, we obtain $\mathfrak{N}_\alpha$, $\mathfrak{N}_{1,\alpha}$ and $\mathfrak{N}_{2,\alpha}$ from the corresponding formulas of $\mathfrak{N}$, $\mathfrak{N}_1$ and $\mathfrak{N}_2$. It is clear that Lemma 6.3.2, 6.3.3 still hold with the new weight.

Lemma 6.6.2. *There exist a neighborhood $U(x_*)$ of x_* and a smoothing operator $\mathfrak{R}$ from $\mathcal{E}'(\Omega, \mathbb{R}^{2d})$ into $C^\infty(\overline{U(x_*)}, \mathbb{R}^{2d})$, such that for for any $s \in \mathbb{R}$ and any neighborhood $U_0(x_*)$ of x_* with $U_0(x_*) \Subset U(x_*)$, the following estimate holds*

$$\|f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})} \lesssim \|\mathfrak{N}_{1,\alpha} f\|_{H^{s+1}(U(x_*),\mathbb{R}^{2d})} + \|\mathfrak{R} f\|_{H^s(\Omega,\mathbb{R}^{2d})}. \tag{6.23}$$

Proof. We first show that $\mathfrak{N}_{1,\alpha}$ is an elliptic ΨDO. Indeed, as in Lemma 6.3.1, $\mathfrak{N}_{1,\alpha}$

is a ΨDO of order -1 from $C_0^\infty(\tilde{U}(x_*), \mathbb{R}^{2d})$ to $\mathcal{D}'(\tilde{U}(x_*), \mathbb{R}^{2d})$ with principle symbol

$$\sigma_p(\mathfrak{N}_1)(x,\xi) = 2\pi \cdot \int_{S_x^*\Omega} \delta(\langle \xi, \theta \rangle) |\alpha^\sharp(x_*, \theta)|^2 \Phi^\dagger(x,\theta) \cdot \Phi(x,\theta)\, d\sigma_x(\theta).$$

By the construction of α, for any $\xi \in S_{x_*}^*\mathbb{R}^d$, we have $\alpha^\sharp(x_*, \theta) = 1$ for some $\theta \in S_{x_*}^*\mathbb{R}^d$ with $\theta \perp \xi$. Thus

$$\sigma_p(\mathfrak{N}_{1,\alpha})(x_*,\xi) = 2\pi \cdot \int_{\theta \in S_{x_*}^*\mathbb{R}^d,\, \theta\perp\xi} |\alpha^\sharp(x_*, \theta)|^2 \Phi^\dagger(x_*,\theta) \cdot \Phi(x_*,\theta)\, d\sigma_{x_*}(\theta) > 0$$

in the sense of symmetric positive definite matrix. By continuity, we can find a neighborhood $U(x_*) \subset \tilde{U}(x_*)$ of x_* such that $\sigma_p(\mathfrak{N}_{1,\alpha})(x,\xi) > 0$ for all $x \in U(x_*)$ and $\xi \in S_x^*\mathbb{R}^d$. Thus we can conclude that $\mathfrak{N}_{1,\alpha}$ is an elliptic ΨDO of order -1 from $C_0^\infty(U(x_*), \mathbb{R}^{2d})$ to $\mathcal{D}'(U(x_*), \mathbb{R}^{2d})$.

Now, let $\mathfrak{B}$ be the pseudo-inverse of $\mathfrak{N}_{1,\alpha}$ restricted to $U(x_*)$. Then $\mathfrak{B}$ is ΨDO of order 1 and there is a smoothing operator $\mathfrak{R}_1 : \mathcal{E}'(U(x_*), \mathbb{R}^{2d}) \to C^\infty(U(x_*), \mathbb{R}^{2d})$ such that

$$g = \mathfrak{B} \circ \mathfrak{N}_{1,\alpha} g + \mathfrak{R}_1 g \tag{6.24}$$

for all g supported in $U(x_*)$.

Next, let $U_0(x_*)$ be any given neighborhood of x_* with $U_0(x_*) \Subset U(x_*)$. For later convenience, we write $U_3(x_*)$ for $U(x_*)$. Then there exist two neighborhoods of x_*, say $U_1(x_*)$ and $U_2(x_*)$ such that $U_0(x_1) \Subset U_1(x_*) \Subset U_2(x_*) \Subset U_3(x_*)$. We choose three smooth cut-off functions χ_0, χ_1 and χ_2 such that $\mathrm{supp}\chi_j \subset U_{j+1}$ and $\chi_j|_{U_j} = 1$ for $j = 0, 1, 2$. Then both $\chi_0 \mathfrak{B}(1-\chi_1)$ and $\chi_1 \mathfrak{N}_{1,\alpha}(1-\chi_2)$ are smoothing operators.

Note that $\chi_2 f$ is compact supported in $U_3(x_*)$, so we have by (6.24) that

$$\chi_2 f = \mathfrak{B} \circ \mathfrak{N}_{1,\alpha}(\chi_2 f) + \mathfrak{R}_1(\chi_2 f).$$

Thus,

$$\begin{aligned}
\chi_0 f &= \chi_0 \cdot \chi_2 f = \chi_0 \cdot \mathfrak{B} \circ \mathfrak{N}_{1,\alpha}\chi_2 f + \chi_0 \mathfrak{R}_1(\chi_2 f) \\
&= \chi_0 \mathfrak{B} \chi_1 \mathfrak{N}_{1,\alpha}\chi_2 \cdot f + \big(\chi_0 \mathfrak{B}(1-\chi_1)\mathfrak{N}_{1,\alpha}\chi_2 + \chi_0 \mathfrak{R}_1 \chi_2\big) f \\
&= \chi_0 \mathfrak{B} \chi_1 \mathfrak{N}_{1,\alpha} f + \big(\chi_0 \mathfrak{B}\chi_1 \mathfrak{N}_{1,\alpha}(1-\chi_2) + \chi_0 \mathfrak{B}(1-\chi_1)\mathfrak{N}_{1,\alpha}\chi_2 + \chi_0 \mathfrak{R}_1 \chi_2\big) f \\
&= \chi_0 \mathfrak{B} \chi_1 \mathfrak{N}_{1,\alpha} f + \mathfrak{R} f
\end{aligned}$$

where $\mathfrak{R} = \chi_0 \mathfrak{B}\chi_1 \mathfrak{N}_{1,\alpha}(1-\chi_2) + \chi_0 \mathfrak{B}(1-\chi_1)\mathfrak{N}_{1,\alpha}\chi_2 + \chi_0 \mathfrak{R}_1 \chi_2$. We can check that $\mathfrak{R}$ is a smoothing operator from $\mathcal{E}'(\Omega, \mathbb{R}^{2d})$ to $C^\infty(\overline{U(x_*)}, \mathbb{R}^{2d})$.

Finally, we conclude that

$$\begin{aligned}
\|f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})} &\lesssim \|\chi_0 \mathfrak{B}\chi_1 \mathfrak{N}_{1,\alpha} f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})} + \|\mathfrak{R} f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})} \\
&\lesssim \|\mathfrak{B}\chi_1 \mathfrak{N}_{1,\alpha} f\|_{H^s(U_3(x_*),\mathbb{R}^{2d})} + \|\mathfrak{R} f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})} \\
&\lesssim \|\chi_1 \cdot \mathfrak{N}_{1,\alpha} f\|_{H^{s+1}(U_3(x_*),\mathbb{R}^{2d})} + \|\mathfrak{R} f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})} \\
&\lesssim \|\mathfrak{N}_{1,\alpha} f\|_{H^{s+1}(U_3(x_*),\mathbb{R}^{2d})} + \|\mathfrak{R} f\|_{H^s(U_0(x_*),\mathbb{R}^{2d})}.
\end{aligned}$$

This completes the proof of the Lemma.

We now study the operator $\mathfrak{N}_{2,\alpha}$.

Lemma 6.6.3. *There exists a small neighborhood of x_*, say $U(x_*)$, such that the following* *decomposition holds for the operator* $\mathfrak{N}_{2,\alpha} : \mathcal{E}'(\Omega, \mathbb{R}^{2d}) \to \mathcal{D}'(U(x_*), \mathbb{R}^{2d})$

$$\mathfrak{N}_{2,\alpha} = \sum_{j=1}^{M} \mathfrak{N}_{2,j} \tag{6.25}$$

where each $\mathfrak{N}_{2,j}$ *is compact from* $H^s(\Omega_0, \mathbb{R}^{2d})$ *to* $H^{s+1}(U(x_*), \mathbb{R}^{2d})$.

Proof. Recall that $\mathfrak{N}_{2,\alpha}$ has the following representation

$$\mathfrak{N}_{2,\alpha} f(x) = \int_{T_x^*\Omega} W_\alpha(x,\xi) f(\phi(x,\xi))(1 - \chi_*(\|\xi\|)) \cdot \, d\sigma_x(\xi),$$

where

$$\begin{aligned} W_\alpha(x,\xi) &= \frac{1}{\|\xi\|^{d-1}} |\alpha \circ \tau \circ \beta(x,\xi)|^2 \Phi^\dagger \circ \beta(x,\xi) \cdot \Phi \circ \beta \circ \mathcal{H}(x,\xi) \\ &\quad + \frac{1}{\|\xi\|^{d-1}} |\alpha \circ \tau \circ \beta(x,-\xi)|^2 \Phi^\dagger \circ \beta(x,-\xi) \cdot \Phi \circ \beta \circ \mathcal{H}^{-1}(x,-\xi). \end{aligned}$$

By (6.22) and the fact that $\mathcal{A}_0$ is symmetric, we see that supp $W_\alpha \subset \mathcal{A}_0$ for all x with $|x - x_*| \le \epsilon_3$.

Now, let χ_j's be as in the first stage. Define $\mathfrak{N}_{2,j} : \mathcal{E}'(\Omega, \mathbb{R}^{2d}) \to \mathcal{D}'(U(x_*, \xi_j), \mathbb{R}^{2d})$ by

$$\mathfrak{N}_{2,j} f(x) = \int_{T_x^*\Omega} W_\alpha(x,\xi) f(\phi(x,\xi))(1 - \chi_*(\|\xi\|)) \cdot \chi_j(x,\xi) \, d\sigma_x(\xi).$$

Let $U(x_*) = \bigcap_{j=1}^{M} (U(x_*, \xi_j)) \bigcap \{x : |x - x_*| < \epsilon_3\}$. Then $U(x_*)$ is a neighborhood of x_* and each $\mathfrak{N}_{2,j}$ is compact from $H^s(\Omega_0, \mathbb{R}^{2d})$ into $H^{s+1}(U(x_*), \mathbb{R}^{2d})$.

We claim that $\mathfrak{N}_{2,\alpha} = \sum_{j=1}^{M} \mathfrak{N}_{2,j}$ when both sides are viewed as operators from $\mathcal{E}'(\Omega, \mathbb{R}^{2d})$ to $\mathcal{D}'(U(x_*), \mathbb{R}^{2d})$. Indeed, for any $f \in C_0^\infty(\Omega, \mathbb{R}^{2d})$, since $\sum_{j=1}^{M} \chi_j = 1$ on

$\mathcal{A}_0$ and supp $W_\alpha \subset \mathcal{A}_0$, we have

$$W_\alpha(x,\xi)f(\phi(x,\xi))(1-\chi_*(\|\xi\|)) = W_\alpha(x,\xi)f(\phi(x,\xi))(1-\chi_*(\|\xi\|))\cdot\big(\sum_{j=1}^{M}\chi_j(x,\xi)\big)$$

for all $x \in U(x_*)$. Thus $\mathfrak{N}_{2,\alpha}f = \sum_{j=1}^{M}\mathfrak{N}_{2,j}f$ and the claim follows. This completes the proof of the lemma.

Finally, note that $\mathfrak{N}_\alpha = \mathfrak{N}_{1,\alpha} + \mathfrak{N}_{2,\alpha}$. Theorem 6.2.1 follows from Lemma 6.6.2 and 6.6.3.

6.7 Proof of Theorem 6.2.2

Proof of Theorem 6.2.2: Step 1. We show that (6.13) holds. For each $x \in \overline{\Omega_0}$, by Theorem 6.2.1, there exist neighborhoods $U(x) \Subset \tilde{U}(x_*)$ of x and a smooth function $\alpha \in C_0^\infty(S_-^*\Gamma)$ such that the estimate (5.13) holds. Since $\overline{\Omega_0}$ is compact, we can find finite number of points, say x_1, x_2, ... x_N, such that $\overline{\Omega_0} \subset \bigcup_{j=1}^{M} U(x_j)$ and the following estimate holds for each j:

$$\|f\|_{L^2(U(x_j))} \lesssim \sum_{j=1}^{N}\|\mathfrak{N}_{\alpha_j}f\|_{H^1(\tilde{U}(x_j))} + \|\mathfrak{N}_{2,\alpha_j}f\|_{H^1(U(x_j),\mathbb{R}^{2d})} + \|\mathfrak{R}_{\mathfrak{j}}f\|_{L^2(U(x_j),\mathbb{R}^{2d})}$$

where $f \in L^2(\Omega)$ has support in Ω_0. The estimate (6.13) follows by observing that $\|f\|_{L^2(\Omega)} \leq \sum_{j=1}^{N}\|f\|_{L^2(U(x_j))}$.

Step 2. From now on, we show that the second part of the theorem holds. Denote by

H the Hilbert space $\prod_{j=1}^{M} H^1(U(x_j), \mathbb{R}^{2d})$. We consider the following three operators

$$\begin{aligned} Tf &= (\mathfrak{N}_{\alpha_1} f, \mathfrak{N}_{\alpha_2} f, ..., \mathfrak{N}_{\alpha_M} f), \\ T_1 f &= (\mathfrak{N}_{2,\alpha_1} f, \mathfrak{N}_{2,\alpha_2} f, ..., \mathfrak{N}_{2,\alpha_M} f), \\ T_2 f &= (\mathfrak{R}_{\alpha_1} f, \mathfrak{R}_{\alpha_2} f, ..., \mathfrak{R}_{\alpha_M} f). \end{aligned}$$

It is clear that all three operators are bounded from $L^2(\Omega_0)$ to H. Moreover, T_1 and T_2 are also compact and the following estimate holds

$$\|f\|_{L^2(\Omega, \mathbb{R}^{2d})} \lesssim \|Tf\|_H + \|T_1 f\|_H + \|T_2 f\|_H. \tag{6.26}$$

Step 3. Let $\mathfrak{L}_0 \subset L^2(\Omega_0, \mathbb{R}^{2d}) \subset L^2(\Omega, \mathbb{R}^{2d})$ be the kernel of T. We claim that $\mathfrak{L}_0 \subset L^2(\Omega_0, \mathbb{R}^{2d})$ is of finite dimension. We prove by contradiction. Assume the contrary, then there exists an infinity number of orthogonal vectors in $\mathfrak{L}_0 \subset L^2(\Omega_0, \mathbb{R}^{2d})$, say, e_1, e_2, ..., such that $\|e_j\|_{L^2(\Omega_0, \mathbb{R}^{2d})} = 1$ and $Te_j = 0$ for all $j \in \mathbb{N}$. Since the sequence $\{e_j\}_{j=1}^{\infty}$ is bounded in $L^2(\Omega_0, \mathbb{R}^{2d})$ and the operators T_1 and T_2 are compact, we can find a subsequence, still denoted by $\{e_j\}_{j=1}^{\infty}$, such that both the sequences $\{T_1 e_j\}_{j=1}^{\infty}$ and $\{T_2 e_j\}_{j=1}^{\infty}$ are Cauchy in H. By applying Inequality (6.26) to the vectors $e_i - e_j$ and recall that $T(e_i - e_j) = 0$, we conclude that the sequence $\{e_j\}_{j=1}^{\infty}$ is also Cauchy in $L^2(\Omega_0, \mathbb{R}^{2d})$. This contradicts to the fact that $\|e_i - e_j\|_{L^2(\Omega_0, \mathbb{R}^{2d})} > 1$ for all $i \neq j$. This contradiction proves the claim.

Step 4. We claim that

$$\|f\|_{L^2(\Omega_0,\mathbb{R}^{2d})} \lesssim \|Tf\|_H \quad \text{for all } f \in \mathfrak{L}_0^{\perp}. \tag{6.27}$$

Indeed, assume the contrary, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathfrak{L}^{\perp}$ such that

$$\|f_n\|_{L^2(\Omega_0,\mathbb{R}^{2d})} = 1, \text{ and } \|Tf_n\|_H \leq \frac{1}{n} \text{ for all } n.$$

By the same argument as in Step 3, we can find a subsequence, still denoted by $\{e_j\}_{j=1}^{\infty}$, such that both the sequences $\{T_1 e_j\}_{j=1}^{\infty}$ and $\{T_2 e_j\}_{j=1}^{\infty}$ are Cauchy in H. By Inequality (6.26) and the fact that $\|Tf_n\|_H \leq \frac{1}{n}$, we can conclude that $\{f_n\}_{n=1}^{\infty}$ is also Cauchy in $L^2(\Omega_0, \mathbb{R}^{2d})$. Let $f_0 = \lim_{n\to\infty} f_n$, then $\|Tf_0\|_H = \lim_{n\to\infty} \|Tf_n\|_H = 0$. This implies that $f_0 \in \mathfrak{L}_0$. However, note that $\mathfrak{L}_0^{\perp}$ is closed, as the limit of a sequence of functions in $\mathfrak{L}_0^{\perp}$, f_0 must belong to $\mathfrak{L}_0^{\perp}$. Therefore, we see that $f_0 = 0$. But this contradicts to the fact that $\|f_0\|_{L^2(\Omega_0,\mathbb{R}^{2d})} = \lim_{n\to\infty} \|f_n\|_{L^2(\Omega_0,\mathbb{R}^{2d})} = 1$. The claim is proved, whence the estimate (6.14) follows.

Chapter 7

Stability of of recovering velocity fields from their induced Hamiltonian flows

We investigate the stability of the inverse problem of recovering velocity fields from their induced Hamiltonian flows.

Let c be the smooth background velocity field and let $\tilde{c}$ be a perturbation. We require that $\tilde{c}$ is in the admissible class of velocity fields. Denote by $\mathfrak{X}$ the X-ray transform operator obtained from linearizing the map which associate velocity fields to their induced Hamiltonian flows, More precisely, $\mathfrak{X} = \frac{\delta \mathcal{H}_c^T}{\delta b}$, where $b(x) = -\frac{1}{2}\nabla \ln c^2$. It is clear that (see section 6.1)

$$\mathfrak{X}f(x_0, \xi_0) = \Phi^{-1}(T, x_0, \xi_0) \cdot \mathfrak{I}(f)(x_0, \xi_0). \tag{7.1}$$

We recall the following result on the linearization.

Lemma 7.0.1. *The following estimate holds*

$$\|\mathcal{H}^T(\tilde{c}) - \mathcal{H}^T(c) - \mathfrak{X}f\|_{C^1(S_-\partial M)} \lesssim \|f\|^2_{C^2(\Omega)}. \tag{7.2}$$

where f is defined as in (6.5).

Theorem 7.0.1. *Let $d = 3$. Let c be a fold-regular admissible velocity field. Then there exist $\epsilon > 0$, a finite dimensional subspace $\mathfrak{L} \subset L^2(\Omega, \mathbb{R}^3)$, and a finite number of smooth functions $\alpha_j \in C^\infty(S_-\partial\Omega)$, $j = 1, 2, ...N$, such that the following estimate holds*

$$\|\tilde{c}^2 - c^2\|_{H^1(\Omega)} \lesssim \sum_{j=1}^{N} \|\hat{\alpha}_j(\mathcal{H}^T(\tilde{c}) - \mathcal{H}^T(c))\|_{H^1(S_-\partial\Omega)}, \tag{7.3}$$

for all $\tilde{c}$ satisfying $\|\tilde{c} - c\|_{H^9(\Omega)} \leq \epsilon$ and $\nabla(\ln c^2 - \ln \tilde{c}^2) \perp \mathfrak{L}$.

Proof. Let f be as in (6.5). Denote by $\mathfrak{L}$ the projection of $\mathfrak{L}_0$ from $L^2(\Omega, \mathbb{R}^6)$ to the space $L^2(\Omega, \mathbb{R}^3)$ by taking the last three components. Note that the first three components of f are zero. Thus the condition $\nabla(\ln c^2 - \ln \tilde{c}^2) \perp \mathfrak{L}$ implies that $f \in \mathfrak{L}_0^\perp$. Therefore

$$\begin{aligned}
\|f\|_{L^2} &\lesssim \sum_{j=1}^{N} \|\mathfrak{N}_{\alpha_j} f\|_{H^1(U(x_j))} \qquad \text{(by Theorem 6.2.2)} \\
&\lesssim \sum_{j=1}^{N} \|\mathfrak{I}_{\alpha_j} f\|_{H^1(S_-\Gamma)} \qquad \text{(by the result that } \mathfrak{I}^\dagger \text{ is bounded from } H^1 \text{ to } H^1) \\
&\lesssim \sum_{j=1}^{N} \|\mathfrak{X}_{\alpha_j} f\|_{H^1(S_-\Gamma)} \qquad \text{(by (7.1))} \\
&\lesssim \sum_{j=1}^{N} \|\mathcal{H}^T(\tilde{c}) - \mathcal{H}^T(c)\|_{\alpha_j} f\|_{H^1(S_-\Gamma)} + \|f\|^2_{C^2(\Omega)} \qquad \text{(by lemma 7.0.1)} \\
&\lesssim \sum_{j=1}^{N} \|\alpha_j(\mathcal{H}^T(\tilde{c}) - \mathcal{H}^T(c))\|_{H^1(S_-\Gamma)} + \|f\|_{L^2(\Omega)} \cdot \|f\|_{H^8(\Omega)}. \\
&\quad \text{(by interpolation inequality)}
\end{aligned}$$

By choosing ϵ to be sufficiently small, we can show that

$$\|f\|_{L^2} \lesssim \sum_{j=1}^{N} \|\alpha_j(\mathcal{H}^T(\tilde{c}) - \mathcal{H}^T(c))\|_{H^1(S_-\Gamma)}.$$

Using the formula for f, we obtain

$$\|\ln c^2 - \ln \tilde{c}^2\|_{H^1} \lesssim \sum_{j=1}^{N} \|\alpha_j(\mathcal{H}^T(\tilde{c}) - \mathcal{H}^T(c))\|_{H^1(S_-\Gamma)}.$$

Finally, note that

$$\tilde{c}^2 - c^2 = c^2 \cdot (e^{\ln \tilde{c}^2 - \ln c^2} - 1).$$

The estimate (7.3) follows. This completes the proof of the theorem.

Remark 7.0.1. *Similar result also holds for $d \geq 3$.*

Chapter 8

Sensitivity of recovering velocity fields from their induced DDtN maps

In this chapter, we investigate the sensitivity of the inverse problem of recovering velocity fields from their induced DDtN maps. We first present a lemma which is a direct consequence of Theorem 5.2.1 and Lemma 6.1.3.

Lemma 8.0.2. *Let c and $\tilde{c}$ be two velocity field in $\mathfrak{A}(\epsilon_0, \Omega, M_0, T)$, and let f be as in (6.5). Then there exists $\delta > 0$ such that if $\|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1[0,3\epsilon_1/4]\times\Gamma \to L^2([0,T+\epsilon_1]\times\Gamma)} \leq \delta$, then*

$$\|\mathfrak{I}f\|_{L^\infty(S_-^*\Gamma, \mathbb{R}^{2d})} \leq C\|f\|^2_{C^1(\Omega, \mathbb{R}^{2d})} \tag{8.1}$$

for constant $C > 0$ depending M_0.

Definition 8.0.1. *An admissible velocity field c is called fold-regular if all points in Ω are fold-regular with respect to the Hamiltonian flow $\mathcal{H}_c^t$.*

We have established the following sensitivity result. For simplicity we only consider the case $d = 3$, similar results also hold for $d > 3$.

Theorem 8.0.2. *Let c and $\tilde{c}$ be two velocity fields in the class $\mathfrak{A}(\epsilon_0, \Omega, M_0, T)$. Assume that the velocity field c is smooth and is fold-regular. Then there exist a finite dimensional* *subspace $\mathfrak{L} \subset L^2(\Omega_0, \mathbb{R}^3)$, and a constant $\delta > 0$ such that for all $\tilde{c}$ sufficiently close to c in $H^{\frac{17}{2}}(\Omega)$ and satisfying $\nabla(\ln c^2 - \ln \tilde{c}^2) \perp \mathfrak{L}$, $\|\Lambda_{\tilde{c}} - \Lambda_c\|_{H_0^1[0,3\epsilon_1/4]\times\Gamma \to L^2([0,T+\epsilon_1]\times\Gamma)} \leq \delta$ implies that $c = \tilde{c}$.*

Proof. The proof is divided into the following three steps.

Step 1. Let $\mathfrak{L}_0$ be defined as in Let Theorem 6.2.2. Let f be as in (6.5). Assume that $f \perp \mathfrak{L}_0$. By Theorem 6.2.2, there exist $U(x_j) \subset \Omega$, $\alpha_j \in C_0^\infty(S_-\partial\Omega)$, $j = 1, 2...N$ such that the following estimate holds

$$\|f\|_{L^2(\Omega)} \lesssim \sum_{j=1}^{N} \|\mathfrak{N}_{\alpha_j} f\|_{H^1(U(x_j))}. \tag{8.2}$$

Step 2. We show that

$$\sum_{j=1}^{N} \|\mathfrak{N}_{\alpha_j} f\|_{H^1(U(x_j))} \lesssim \|f\|^{\frac{2}{5}}_{H^{\frac{15}{2}}(\Omega,\mathbb{R}^{2d})} \cdot \|f\|_{L^2(\Omega,\mathbb{R}^{2d})}. \tag{8.3}$$

Indeed, for each $\mathfrak{I}_{\alpha_j}$, by Lemma 8.0.2, we have $\|\mathfrak{I}_{\alpha_j} f\|_{L^\infty(S_-^*\Gamma)} \leq C_1 \|f\|^2_{C^1}$. Apply $\mathfrak{I}^\dagger_{\alpha_j}$ to both sides and use the fact that $\mathfrak{I}^\dagger_{\alpha_j}$ is bounded from L^2 to L^2(see [45]), we obtain

$$\|\mathfrak{N}_{\alpha_j} f\|_{L^2(\Omega,\mathbb{R}^{2d})} \lesssim \|f\|^2_{C^1(\Omega,\mathbb{R}^{2d})}. \tag{8.4}$$

Then,

$$\begin{aligned}
\|\mathfrak{N}_{\alpha_j} f\|_{H^1(U(x_j),\mathbb{R}^{2d})} &\lesssim \|\mathfrak{N}_{\alpha_j} f\|^{\frac{1}{3}}_{H^3(U(x_j),\mathbb{R}^{2d})} \cdot \|\mathfrak{N}_{\alpha} f\|^{\frac{2}{3}}_{L^2(\Omega,\mathbb{R}^{2d})} \\
&\quad \text{(by interpolation inequality)} \\
&\lesssim \|\mathfrak{N}_{\alpha_j} f\|^{\frac{1}{3}}_{H^3(U(x_j),\mathbb{R}^{2d})} \cdot \|f\|^{\frac{4}{3}}_{C^1(\Omega,\mathbb{R}^{2d})} \qquad \text{(by (8.4))} \\
&\lesssim \|f\|^{\frac{1}{3}}_{H^3(\Omega,\mathbb{R}^{2d})} \cdot \|f\|^{\frac{4}{3}}_{C^1(\Omega,\mathbb{R}^{2d})} \qquad \text{(by Lemma 6.6.2, 6.6.3)} \\
&\lesssim \|f\|^{\frac{1}{3}}_{H^3(\Omega,\mathbb{R}^{2d})} \cdot \|f\|^{\frac{4}{3}}_{H^3(\Omega,\mathbb{R}^{2d})} \quad \text{(by interpolation inequality)} \\
&= \|f\|^{\frac{5}{3}}_{H^3(\Omega,\mathbb{R}^{2d})} \\
&\lesssim \|f\|^{\frac{2}{5}}_{H^{\frac{15}{2}}(\Omega,\mathbb{R}^{2d})} \cdot \|f\|_{L^2(\Omega,\mathbb{R}^{2d})}. \quad \text{(by interpolation inequality)}
\end{aligned}$$

It follows that (8.3) holds.

Step 3. Denote by $\mathfrak{L}$ the projection of $\mathfrak{L}_0$ from $L^2(\Omega,\mathbb{R}^{2d})$ to the space $L^2(\Omega,\mathbb{R}^d)$ by taking the last three components. Note that the first three components of f are zero, see (6.5). Thus the condition $\nabla(\ln c^2 - \ln \tilde{c}^2) \perp \mathfrak{L}$ implies that $f \in \mathfrak{L}_0^{\perp}$. Consequently, Inequality (8.2) holds. Combining this with (8.3), we see that

$$\|f\|_{L^2(\Omega,\mathbb{R}^{2d})} \lesssim \|f\|^{\frac{2}{5}}_{H^{\frac{15}{2}}(\Omega,\mathbb{R}^{2d})} \cdot \|f\|_{L^2(\Omega,\mathbb{R}^{2d})}.$$

Therefore, we must have $f = 0$ for $\|f\|^{\frac{2}{5}}_{H^{\frac{15}{2}}(\Omega,\mathbb{R}^{2d})}$ sufficiently small. Finally, note that $\|f\|_{H^{\frac{15}{2}}(\Omega,\mathbb{R}^{2d})} \lesssim \|c-\tilde{c}\|_{H^{\frac{17}{2}}(\Omega)}$ and that both c and $\tilde{c}$ vanishes near the boundary, we conclude that $f = 0$ implies $c = \tilde{c}$. This completes the proof of the theorem.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] G. Alessandrini, J. Sylvester and Z. Sun. Stability for a multidimensional inverse spectral theorem, Comm.PDE, 15(1990), 711-736.

[2] G. Alessandrini and J. Sylvester. Stability for a multidimensional inverse spectral theorem, Comm.PDE, 15(1990), 711-736.

[3] R. Alexandre, J. Akian and S. Bougacha. Gaussian beams summation for the wave equation in a convex domain, Commun. Math. Sci, 7(4)(2009), 973-1008.

[4] Y. E. Anikonov and V. G. Romanov. On uniqueness of definition of first-order form by its integrals over geodesics, Ill-posed Math. and Geophys. Problems, Novosibirsk, in Russian, (1976), 22-27.

[5] V. I. Arnold. Singularity Theory, London Mathematical Society Lecture Note Series(53), Cambrige Univeristy Press, 1981.

[6] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko. Singularites of Differentiable Maps I, volume 82 of monographs in mathematics. Birkh*ä*user, Boston, 1985.

[7] G. Bao, J. Qian, L. Ying and H. Zhang. A convergent multiscale Gaussian-beam parametrix for wave equations, Comm.PDE, 38(2013), 92-134.

[8] G. Bao and K. Yun. On the stability of an inverse problem for the wave equation, Inverse Problems, 25(4)(2009), R1-R7.

[9] G. Bao and H. Zhang. Sensitivity analysis of an inverse problem for the wave equation with caustics, submitted.

[10] M. Belishev. An approach to multidimensional inverse problems for the wave equation, Dokl. Akad. Nauk. SSR, 297(1987), 524-527.

[11] M. Belishev. Boundary control in reconstruction of manifolds and metrics (the BC method), Inverse Problems, 13(1997), R1-R45.

[12] M. Belishev. Recent progress in the boundary control method, Inverse Problems, 23(2007), R1-R67.

[13] M. Belishev and Y. V. Kurylev. To the reconstruction of a Riemannian manifold via its spectral data (BC-method), Comm.P.D.E, 17(1992), 765-804.

[14] M. Bellassoued and D. D. S. Ferreira. Stability estimates for the anisotropic wave equation from the Dirichlet-to-Neumann map, Inverse Problems and Imaging, 5(4)(2011), 745-773.

[15] I. N. Bernstein and M. L. Gerver. Conditions of distinguishability of metrics by godographs, Methods and Algoritms of Interpretation of Seismological Information, Computerized Seismology, 13(1980), 50-73.

[16] C. Croke. Boundary and lens rigidity of finite quotients, Proc. AMS, 133(12) (2005), 3663-3668.

[17] C. Croke and B. Kleiner. Conjugacy and rigidity for manifolds with a parallel vecotor feild, J. Diff. Geom, 39(1994), 659-680.

[18] G. Eskin. A new approach to hyperbolic inverse problems, Inverse Problems, 22 (2006), 815-831.

[19] G. Eskin. A new approach to hyperbolic inverse problems II: global step, Inverse Problems, 23 (2007), 2343-2356.

[20] R. G. Mukhometov. On the problem of integral gometry, Math. Problems of Geophysics, Akad. Nauk. SSSR, Sibirsk. Otdel, Vychisl. Tsentr, Novosibirsk, 6(2) (1975), 212-242.

[21] R. G. Mukhometov. Inverse kinetic problem of seismic on the plane, Math. Problems of Geophysics, Akad. Nauk. SSSR, Sibirsk. Otdel, Vychisl. Tsentr, Novosibirsk, 6(2) (1975), 212-242.

[22] R. G. Mukhometov, The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry, Soviet Math. Dokl, 18(1), 27-37.

[23] R. G. Mukhometov. On a problem of reconstructing Riemannian metrics, Siberian Math. J, 22(3), 420-433.

[24] A. Greenleaf and A. Seeger. Fourier integral operators with cusp singularites, American Journal of Mathematics, 120(5)(1998), 1077-1119.

[25] L. Hömander. The Analysis of Linear Partial Differential Operators, III and IV, volume 274 and 275 of Grundelehren der Mathematischen Wissenschaften, Spring-Verlag, Berlin, 1985.

[26] V. Isakov. An inverse hyperbolic problem with many boundary measurements, Comm. PDE, 16 (1991), 1183-1195.

[27] V. Isakov and Z. Sun. Stability estimates for hyperbolic inverse problems with local boundary data, Inverse Problems, 8 (1992), 193-206.

[28] V. Isakov. Inverse Problems for Partial Differential Equations, volume 127 of Applied Mathematical Sciences, second edition, Springer Science and Business Media, Inc, New York, 2006.

[29] Y. V. Kurylev and M. Lassas, Hyperbolic inverse problem with data on a part of the boundary, in "Differential Equations and Mathematical Physics", AMS/IP Stud. Adv. Math. 16, Amer. Math. Soc., Providence, (2000), 259-272.

[30] A. Katchalov, Y. Kurylev and M. Lassas. Inverse Boundary Spectral Problems, Chapman Hall/CRC, Boca Raton, 2001.

[31] I. Lasiecka, J. L. Lions and R. Triggiani. Nonhomogeneous boundary value problems for second order hyperbolic operators, J. Math. Pures et Appl, 65(1986), 149-192.

[32] J. M. Lee. Riemannian manifolds, an introduction to curvature, volume 176 of Graduate Texts in Mathematics, Spring-Verlag, New York, 1997.

[33] C. Montalto. Stable determination of a simple metric, a covector field and a potential from the hyperbolic Dirichlet-to-Nuemann map, Arxiv, 2012.

[34] L. N. Pestov and V. A. Sharafutdinov. Integral geometry of tensor fields on a manifold of negative curvative, Siberian Math. J, 29(3), (1988), 427-441.

[35] L. Pestov and G. Uhlmann. The scattering relation and the Dirichlet-to-Neumann map, Contemporary Math, 412(2006), 249-262.

[36] J. Qian and L. Ying. Fast multiscale Gaussian wavepacket transforms and multiscale Gaussian beams for the wave equation, SIAM. MMS, 8 (2010).

[37] J. Ralston. Gassian beams and the propagation of singularities, Studies in Partial Differential Equations, MAA Studies in Mathematics, Edited by W. Littman, 23(1983), 206-248.

[38] Rakesh. Reconstruction for an inverse problem for the wave equation with constant velocity, Inverse Problems, 6 (1990), 91-98.

[39] Rakesh and W. Symes. Uniqueness for an inverse problems for the wave equation, Comm. PDE, 13 (1988), 87-96.

[40] A. Ramm and J. Sjostrand. An inverse problem of the wave equation, Math. Z, 206 (1991), 119-130.

[41] V. G. Romanov. Reconstructing a function by means of integral along a family of curves, Sibirsk. Mat. Z, 8(5) (1967), 1206-1208.

[42] V. G. Romanov. Integral geometry on geodesics of an isotropic Riemannian metric, Soviet Math. Dokl, 19(4) (1979), 847-851.

[43] V. G. Romanov. Invere Problems of Mathematical Physics, VNU Science Press, Utreht, the Netherlands, 1987.

[44] V. A. Sharafutdinov. Integral Geometry of Tensor Fields, VSP, Utrech, the Netherland, 1994.

[45] V. A. Sharafutdinov. Ray Transform on Riemannian Manifolds, Lecture notes, University of Oulu, 1999.

[46] P. Stefanov and G. Uhlmann. Stability estimates for the hyperbolic Dirichlet to Neumann map in anisotropic media, J. Funct. Anal, 154(2)(1998), 330-358.

[47] P. Stefanov and G. Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity, Duke Math. J, 123(2004), 445-467.

[48] P. Stefanov and G. Uhlmann. Boundary rigidity and stability for generic simple metrics, Journal Amer. Math. Soc, 18(2005), 975-1003.

[49] P. Stefanov and G. Uhlmann. Stable determination of generic simple metrics from the hyperbolic Dirichlet-to-Neumann map, IMRN, 17(2005), 1047-1061.

[50] P. Stefanov and G. Uhlmann. The X-Ray transform for a generic family of curves and weights, J. Geom. Anal, 18(1)(2008), 81-97.

[51] P. Stefanov and G. Uhlmann. Integral geometry of tensor fields on a class of non-simple Riemannian manifolds, American J. of Math, 130 (2008), 239-268.

[52] P. Stefanov and G. Uhlmann. Linearizing non-linear inverse problems and its applications to inverse backscattering, Journal Functional Analysis, 256 (2009), 2842-2866.

[53] P. Stefanov and G. Uhlmann. Local lens rigidity with incomplete data for a class of non-simple Riemannian manifolds, Journal Diff. Geometry, 82 (2009), 383-409.

[54] P. Stefanov and G. Uhlmann. The geodesic X-ray transform with fold caustics, Anal. and PDE, 5(2012), 219-260.

[55] Z. Sun. On continuous dependence for an inverse initial boundary value problem for the wave equation, J. Math. Anal. Appl, 150(1990), 188-204.

[56] G. Uhlmann. The cauchy data and the scattering relation, IMA Publications, Springer-Verlag, 137(2003), 263-288.