A STUDY OF CERTAIN LOCI
ASSOCIATED WITH ANALYTIC FUNCTIONS
OF A COMPLEX VARIABLE

Thesis for the Degree of M. S.

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INTRODUCTION

The aim of this thesis is to study nets whose curves in the z plane are the maps by means of an analytic function $z=f(w)$ of the lines u=constant, v=constant in the plane of a complex variable w. We find all the nets whose Laplace transforms also enjoy the property of being maps of the lines u=constant, v=constant by means of an analytic function of w. We shall discuss the curves corresponding by duality to the curves of the net under the same restrictions.

1. AN ADMISSIBLE SET OF DEFINING DIFFERENTIAL EQUATIONS FOR $z=f(w)$ INTEGRABILITY CONDITIONS

Let $z=f(w)$ be an analytic function of a complex variable. If $z=x+iy$, and $w=u+iv$, we have $x_u=y_v$ and $x_v=-y_u$. The homogeneous coordinates[1] of a point in the z plane

$$x = x(u,v),\quad y = y(u,v),\quad z = 1, \tag{1}$$

are solutions of differential equations of the form[2]

$$\begin{aligned} \theta_{uu} &= a\theta_u + b\theta_v + c\theta, \\ \theta_{uv} &= a'\theta_u + b'\theta_v + c'\theta, \\ \theta_{vv} &= a''\theta_u + b''\theta_v + c''\theta. \end{aligned} \tag{2}$$

1. Define a Net. E. J. Wilczynski, One Parameter Families and Nets of Plane Curves, Transactions of American Mathematical Society, Vol. 12 (1911), p.473.

2. E. J. Wilczynski, One Parameter Families and Nets of Plane Curves, Transactions of American Mathematical Society, Vol. 12 (1911), p. 474.

Substituting the functions (1) into the first of equations (2) we find that

(3)
$$x_{uu} = a x_u + b x_v + c x,$$
$$y_{uu} = a y_u + b y_v + c y,$$
$$z_{uu} = a z_u + b z_v + c z.$$

Solving for a, b and c, we find:

(4)
$$a = \frac{x_{uu} y_v - y_{uu} x_v}{x_u^2 + x_v^2},$$

$$b = \frac{x_u y_{uu} - x_{uu} y_u}{x_u^2 + x_v^2},$$

$$c = 0.$$

Similarly we find the following:

(5)
$$a' = \frac{x_{uv} y_v - y_{uv} x_v}{x_u^2 + x_v^2}, \qquad a'' = \frac{x_{vv} y_v - y_{vv} x_v}{x_u^2 + x_v^2},$$

$$b' = \frac{x_u y_{uv} - x_{uv} y_u}{x_u^2 + x_v^2}, \qquad b'' = \frac{x_u y_{vv} - x_{vv} y_u}{x_u^2 + x_v^2},$$

$$c' = 0, \qquad c'' = 0,$$

by using the second and third equations in(2).

Therefore we can say that the homogeneous coordinates in (1) are solutions of differential equations of the form

$$\theta_{uu} = a\theta_u + b\theta_v + c\theta,$$
$$\theta_{uv} = a'\theta_u + b'\theta_v + c'\theta,$$
$$\theta_{vv} = a''\theta_u + b''\theta_v + c''\theta,$$

for the values of a, b, c, a', b', c', a", b" and c" given in (4) and (5) when x and y are not constants.

Let us consider the known relations:

(6)
$$x_u = y_v,$$
$$x_v = -y_u,$$

Differentiating each of the equations in (6) with respect u, then with respect to v, we obtain the equations:

$$(7) \qquad x_{uu} = y_{uv}, \qquad x_{uv} = y_{vv},$$
$$x_{uv} = -y_{uu}, \qquad x_{vv} = -y_{uv}.$$

Using these values of these derivatives as defined by the differential equations

$$(8) \qquad \theta_{uu} = a\theta_u + b\theta_v,$$
$$\theta_{uv} = a'\theta_u + b'\theta_v,$$
$$\theta_{vv} = a''\theta_u + b''\theta_v,$$

we find that

$$a x_u + b x_v = a' y_u + b' x_v,$$

whence from (6) we obtain

$$a x_u + b x_v = -a' x_v + b' x_u,$$

or

$$(a-b')x_u + (b+a')x_v \equiv 0. \qquad (x_u \neq 0, \; x_v \neq 0)$$

therefore

$$a = b', \quad b = -a'.$$

In a similar manner we find

$$a' = b'' \text{ and } b' = -a''.$$

Hence

$$(9) \qquad a = b' = -a'', \quad b = -a' = -b''.$$

We shall call a set of equations of the form (8) an <u>admissible set</u> of defining differential equations if they admit x, y, 1, such that $x_u = y_v$, and $x_v = -y_u$.

In order that the system of differential equations (8) have a solution, it is necessary and sufficient that the following integrability conditions[3] be satisfied:

$$(10) \qquad a''b + a_v = a''b' + a'_u,$$
$$ab' + bb'' + b_v = a'b + b'^2 + b'_u,$$
$$a'^2 + a''b' + a'_v = aa'' + a'b'' + a''_u,$$
$$a'b' + b'_v = a''b + b''_u.$$

3. E. J. Wilczynski, <u>One Parameter Families</u> and <u>Nets</u> of <u>Plane Curves</u>, Transactions of American Mathematical Society, Vol. 12 (1911), p. 474.

Since a=b'=-a" and b=-a'=-b", these conditions become

$$-ab+a_v=-ab-b_u,$$
$$a^2-b^2+b_v=-b^2+a^2+a_u,$$
$$b^2-a^2-b_v=-a^2+b^2-a_u,$$
$$-ab+a_v=-ab-b_u.$$

These four conditions may be reduced to the following two conditions:

(11)
$$a_v+b_u=0,$$
$$a_u-b_v=0.$$

These conditions restrict a and b to be functions satisfying the Cauchy-Riemann differential equations.

The results thus far obtained may be stated in the theorem --<u>An admissible set of defining differential equations for</u> z=f(w), <u>where</u> $x_u=y_v$ <u>and</u> $x_v=-y_u$ <u>is</u>

(12)
$$\theta_{uu}=a\theta_u+b\theta_v,$$
$$\theta_{uv}=-b\theta_u+a\theta_v,$$
$$\theta_{vv}=-a\theta_u-b\theta_v,$$

<u>where</u> a <u>and</u> b <u>satisfy</u> **<u>the Cauchy-Riemann</u>** <u>conditions</u>.

The transformations $\bar{u}=\phi(u)$, $\bar{v}=\psi(v)$, leave the curves u= constant, v=constant invariant but change the function z=f(w) into $\bar{z}=\bar{f}(\bar{w})$. Let us investigate the nature of the function ϕ and ψ, such that $\bar{f}(\bar{u})$ is analytic. By partial differentiation we find:

(13)
$$\frac{\partial x}{\partial \bar{u}}=\frac{\partial x}{\partial u}\frac{du}{d\bar{u}}=\frac{x_u}{\phi_u},\qquad \frac{\partial y}{\partial \bar{v}}=\frac{\partial y}{\partial v}\frac{dv}{d\bar{v}}=\frac{y_v}{\psi_v},$$

$$\frac{\partial x}{\partial \bar{v}}=\frac{\partial x}{\partial v}\frac{dv}{d\bar{v}}=\frac{x_v}{\psi_v},\qquad \frac{\partial y}{\partial \bar{u}}=\frac{\partial y}{\partial u}\frac{du}{d\bar{u}}=\frac{y_u}{\phi_u}.$$

Since by assumption $x_{\bar{u}}=y_{\bar{v}}$, $x_{\bar{v}}=-y_{\bar{u}}$; $x_u=y_v$ and $x_v=-y_u$ we find from equations in (13) that

(14)
$$\phi_u=\psi_v.$$

But ϕ is a function of u alone and ψ is a function of v alone, whence from (14)

$$(15)\qquad \phi = \bar{u} = ku + k_1,$$
$$\psi = \bar{v} = kv + k_2.$$

Let us consider now the transformation

$$(16)\qquad \bar{x} = a_{11}x + a_{12}y,$$
$$\bar{y} = a_{21}x + a_{22}y.$$

Differentiating equations (16) and using conditions (6) we find:

$$a_{11}y_v + a_{12}y_u = -a_{21}y_u + a_{22}y_v,$$
$$-a_{11}y_u + a_{12}y_v = -a_{21}y_v - a_{22}y_u,$$

therefore

$$(17)\qquad a_{11} = a_{22}, \quad a_{12} = -a_{21}.$$

By substituting the values for a_{22} and a_{21} found in (17), in (16) and differentiating, it can readily be verified that the Cauchy-Riemann conditions are satisfied for $\bar{x}$ and $\bar{y}$.

Therefore *the most general affine transformation transforming the analytic function* $z=f(w)$ *into the analytic function* $\bar{z}=\bar{f}(w)$ *is an affine similarity transformation.*

2. DEFINING DIFFERENTIAL EQUATIONS OF THE LAPLACEAN TRANSFORMS

The first and minus first Laplacean transforms of the given net are defined in homogeneous coordinates[4] by

$$(18)\qquad \bar{X}_2 = x_v - a'x, \qquad \bar{X}_1 = x_u - b'x,$$
$$X_2 = y_v - a'y, \qquad X_1 = y_u - b'y,$$
$$\bar{\bar{X}}_2 = z_v - a'z, \qquad \bar{\bar{X}}_1 = z_u - b'z,$$

respectively. In non-homogeneous coordinates they are

$$(19)\qquad \bar{x}_2 = \frac{x_v - a'x}{-a'}, \qquad \bar{x}_1 = \frac{x_u - b'x}{-b'},$$
$$x_2 = \frac{y_v - a'y}{-a'}, \qquad x_1 = \frac{y_u - b'y}{-b'},$$

4. E. J. Wilczynski, *One Parameter Families and Nets of Plane Curves*, Transactions of American Mathematical Society, Vol. 12 (1911). p. 486.

since in our problem z=1.

Substituting the values of $x_{\bar{u}}$, $x_{\bar{v}}$, $y_{\bar{u}}$, $y_{\bar{v}}$ in (13) into the second equation of (2) we have

$$K^2 \theta_{\bar{u}\bar{v}} = a' K \theta_{\bar{u}} + b' K \theta_{\bar{v}} + c' \theta,$$

or

$$\theta_{\bar{u}\bar{v}} = \frac{a'}{K} \theta_{\bar{u}} + \frac{b'}{K} \theta_{\bar{v}} + \frac{c'}{K^2} \theta.$$

Let $\bar{a}' = \frac{a'}{K}$, $\bar{b}' = \frac{b'}{K}$, $\bar{c}' = \frac{c'}{K^2} = 0,$

The expressions (18) become

$$x_u - b'x = K x_{\bar{u}} - K \bar{b}' x = K(x_{\bar{u}} - \bar{b}' x),$$
$$y_u - b'y = K y_{\bar{u}} - K \bar{b}' y = K(y_{\bar{u}} - \bar{b}' y),$$
$$z_u - b'z = K z_{\bar{u}} - K \bar{b}' z = K(z_{\bar{u}} - \bar{b}' z),$$

and

$$x_v - a'x = K x_{\bar{v}} - \bar{a}' K x = K(x_{\bar{v}} - \bar{a}' x),$$
$$y_v - a'y = K y_{\bar{v}} - \bar{a}' K y = K(y_{\bar{v}} - \bar{a}' y),$$
$$z_v - a'z = K z_{\bar{v}} - \bar{a}' K z = K(z_{\bar{v}} - \bar{a}' z).$$

We have shown that the transformation (15) leaves the expressions (18) the same except for a factor.

Differentiating the equations (19) as denoted by the subscript and making use of equations (2), we obtain the following results:

(20) (a) $\bar{x}_{1u} = \dfrac{(b'_u - ab' + b'^2)x_u - bb' x_v}{b'^2}$, (b) $\bar{x}_{1v} = \dfrac{(b'_v - a'b')x_u}{b'^2}$,

(c) $\bar{x}_{2u} = \dfrac{(a'_u - a'b')x_v}{a'^2}$, (d) $\bar{x}_{2v} = \dfrac{(a'_v + a'^2 - a'b'')x_v - a'a'' x_u}{a'^2}$,

(e) $\bar{x}_{1uu} = \dfrac{(b'b'_{uu} + 2ab'b'_u - a_u b'^2 - 2b'^2_u - a^2 b'^2 + ab'^3 - a'bb'^{2})x_u + (2bb'b'_u - b'^2 b_u - abb'^2)x_v}{b'^3}$

(f) $\bar{x}_{1uv} = \{(b'b'_{uv} - a_v b'^2 - 2b'_u b'_v + abb'_v + a'b'b'_u - aa'b'^2 + a'b'^3 - a''bb'^2)x_u$
$+ b'^2 b'_u - ab'^3 + b'^4 - bb'^2 b'' - b'^2 b_v + bab'_v)x_v\} \div b'^3$,

$$(g)\ \bar{x}_{1vv} = \frac{-a'^2 b'^2 + b b'_{vv} - b' a'_v - 2 b'^2_v + 2a' b b'_v}{b'^3}\, x_u + (b'^2 b'_v - a' b'^3)\, x_v,$$

$$(h)\ \bar{x}_{2uu} = \frac{a' a'_{uu} - a'^2 b'_u - 2 a'^2_u + 2 a' a'_u b' - a'^2 b'^2}{a'^3}\, x_v + (a'^2 a'_u - a'^3 b')\, x_u,$$

$$(i)\ \bar{x}_{2uv} = \{(a' a'_{uv} - a' b' a'_v - a'^2 b'_v - 2 a'_u a'_v + 2 a' a'_v b' + a' b'' a'_u - a' b' b'')\, x_v$$
$$+ (a' a'' a'_u - a'^2 a'' b')\, x_u\} \div a'^3,$$

$$(j)\ \bar{x}_{2vv} = \{(a' a'_{vv} + 2 a' a'_v b'' + a'^3 b'' - a'^2 b''_v - 2 a'^2_v - a'^2 b''^2 - a'^2 a'' b')\, x_v$$
$$+ (2 a' a'' a'_v + a'^3 a'' - a'^2 a'' b'' - a'^3 a'' - a'^2 a''_v)\, x_u\} \div a'^3.$$

Substituting the first derivatives obtained in(18-a,b,c,d) in equations (2) we find:

$$(21)\ (a)\ \bar{x}_{1uu} = a_1\left[\frac{(b'_u - a b' + b'^2)\, x_u - b b' x_v}{b'^2}\right] + b_1\left[\frac{(b'_v - a' b')\, x_u}{b'^2}\right],$$

$$(b)\ \bar{x}_{1uv} = a'_1\left[\frac{(b'_u - a b' + b'^2)\, x_u - b b' x_v}{b'^2}\right] + b'_1\left[\frac{(b'_v - a' b')\, x_u}{b'^2}\right],$$

$$(c)\ \bar{x}_{1vv} = a''_1\left[\frac{(b'_u - a b' + b'^2)\, x_u - b b' x_v}{b'^2}\right] + b''_1\left[\frac{(b'_v - a' b')\, x_u}{b'^2}\right],$$

$$(d)\ \bar{x}_{2uu} = a_2\left[\frac{(a'_u - a' b')\, x_v}{a'^2}\right] + b_2\left[\frac{(a'_v + a'^2 - a' b'')\, x_v - a' a'' x_u}{a'^2}\right],$$

$$(f)\ \bar{x}_{2uv} = a'_2\left[\frac{(a'_u - a' b')\, x_v}{a'^2}\right] + b'_2\left[\frac{(a'_v + a'^2 - a' b'')\, x_v - a' a'' x_u}{a'^2}\right],$$

$$(e)\ \bar{x}_{2vv} = a''_2\left[\frac{(a'_u - a' b')\, x_v}{a'^2}\right] + b''_2\left[\frac{(a'_v + a' - a' b'')\, x_v - a' a'' x_u}{a'^2}\right].$$

Eliminating x_u and x_v between the equations formed by setting the equivalents of corresponding second derivatives equal in **(20)** and (21) and using relations in (9), the coefficients of the defining differential equations of the Laplacean transforms are found to be:

(22) (a) $a_1 = \dfrac{ab_u + a^2 b - 2a_u b}{ab}$, (b) $b_1 = \dfrac{b a_{uu} + ab^3 - a_u b_u}{b(a_v + ab)}$,

(c) $a_1' = \dfrac{-a a_u - ab^2 + ab_v - a_v b}{ab}$, (d) $b_1' = \dfrac{a a_{uv} b + a^3 b^2 - a_u a_v b + a a_u^2 - a a_u b_v}{ab(a_v + ab)}$,

(e) $a_1'' = -\dfrac{ab + a_v}{b}$, (f) $b_1'' = \dfrac{ab a_{vv} + a^2 b b_v - 2a_v^2 b - 2a a_v b^2 - a^2 b^3 + a^2 a_u b + a a_u a_v}{ab(a_v + ab)}$

(g) $a_2 = \dfrac{-abb_{uu} + a a_u b^2 + 2ab_u^2 - 2a^2 b b_u + a^3 b^2 + ab^2 b_v - b b_u b_v}{-ab(b_u - ab)}$, (h) $b_2 = \dfrac{ab - b_u}{a}$,

(i) $a_2' = \dfrac{b b_{uv} - a_v b^2 - b_u b_v - b^2 b_u + ab^3}{b(b_u - ab)}$, (j) $b_2' = \dfrac{ab - b_u}{b}$,

(k) $a_2'' = \dfrac{a b_{vv} + a^3 b - a_v b_v}{a(b_u - ab)}$, (l) $b_2'' = \dfrac{-ab^2 - 2ab_v + a_v b}{ab}$.

Let us impose the conditions that the defining differential equations of the minus first Laplace transformation be admissible. If $a_1 = -a_1''$, we have

$$\frac{ab_u + a^2 b - 2a_u b}{ab} = \frac{ab - b_u}{b},$$

or

$$b_u + a - \frac{2a_u}{a} = a - \frac{b_u}{b},$$

therefore

$$\frac{\partial}{\partial u} \log a/b = 0,$$

or

(23) $$a/b = \text{function of } v \text{ alone.}$$

If $b_1 = -b_1''$ or

$$\frac{a a_{vv} b + a^2 b b_v - a^2 b^3 - 2a_v^2 b - 2a a_v b^2 + a a_u a_v + a^2 a_u b}{ab(a_v + ab)} = \frac{a_u b_u - a_{uu} b - ab^3}{b(a_v + ab)},$$

then

$$a_{vv} + ab_v - \frac{2a_v^2}{a} - 2a_v b + \frac{a_u a_v}{b} + a a_u = \frac{a_u b_u}{b} - a_{uu},$$

or $a b_v - \dfrac{b_u^2}{a} + b b_u - \dfrac{b_u b_v}{b} = 0.$

Factoring we have

$$\left(\frac{b_v}{b} + \frac{b_u}{a}\right)\left(a - \frac{b_u}{b}\right) = 0.$$

Setting the first factor equal to zero and applying (23) we have

$$\bar{V} b_v + b_u = 0, \quad \text{or} \quad \bar{V} b_v = a_v.$$

But $a = b\bar{V}$.

This equation gives, on differentiating:

$$a_v = b_v \bar{V} + b\bar{V}_v,$$

therefore

(24) $\qquad b\bar{V}_v = 0, \quad$ or $\bar{V}$ is a constant.

Setting the second factor equal to zero and applying (23) we

have $\qquad \frac{b_u}{b} = a = bv, \quad or \; -b_u = -b^2\bar{V},$

integrating we find $\qquad 1/b = -\bar{V}u + C_{1(v)}.$

But $\qquad v = a/b = -a\bar{V}u + aC_{1(v)},$

therefore

$$a = \frac{\bar{V}}{-\bar{V}u + C_{1(v)}}, \quad 1/a = \frac{C_{1(v)} - \bar{V}u}{\bar{V}}.$$

Since $\quad b_u = b^2\bar{V} \quad$,then $-a_v = b^2\bar{V} = ab = \frac{a \cdot a}{\bar{V}},$

or

$$-\frac{a_v}{a^2} = \frac{1}{\bar{V}}$$

Integrate:

$$1/a = \frac{v + \bar{V}\,C_{2(u)}}{\bar{V}}$$

therefore

$$\frac{C_{1(v)} - \bar{V}\cdot u}{\bar{V}} = \frac{v + \bar{V}C_{2(u)}}{\bar{V}}$$

or

$$\bar{V} = \frac{C_{1(v)} - v}{C_{2(u)} - u}.$$

Since $\bar{V}$ is a function of v alone then $\quad C_{1(v)} = v \quad$ and $C_{2(u)} = u.$

Therefore

$$\frac{1}{a} = \frac{u\bar{V} + v}{\bar{V}} = u + \frac{v}{\bar{V}}.$$

But

$$\frac{b_u}{b} = a = \frac{a_u}{a}; \quad or \quad \frac{a_u}{a^2} = 1.$$

Integrate:

$$\frac{1}{a} = -u + k_{(v)}$$

Then $\qquad u + \frac{v}{\bar{V}} = -u + k_{(v)},$

which says that $u = -u$, which is impossible except for $u = 0$, thus making it necessary for $\frac{b_v}{b} + \frac{b_u}{a}$ to be zero. If $b_1' = a_1$, we have

$$\frac{a a_{uv} b + a^3 b^2 - a_u a_v b + a a_u^2 - a a_u b_v}{ab(a_v + ab)} = \frac{a b_u + a^2 b - 2 a_u b}{ab},$$

or

$$a_{uv} + a^2 b - \frac{a_u a_v}{a} + \frac{a_u^2}{b} - \frac{a_u b_v}{b} = \frac{a_v b_u}{b} + a a_v - \frac{2 a_u a_v}{a} + a b_u$$
$$+ a^2 b - 2 a_u b,$$

but we found in setting $a_1 = -a_1''$ that $\frac{b_u}{b} = \frac{a_u}{a}$, and applying this fact along with the integrability conditions (12), our equation reduces to $a_{uv} + 2 a_u b = 0$, or $b_{vv} + 2 b b_v = 0$, therefore

$$\frac{\partial}{\partial v}(b_v + b^2) = 0$$

or

(25)

$$b_v + b^2 = \text{ function of u alone.}$$

If $b_1' = -a_1''$, we have

$$\frac{a a_{uv} b + a^3 b^2 - a_u a_v b + a a_u^2 - a a_u b_v}{ab(a_v + ab)} = \frac{ab + a_v}{b},$$

or

$$a_{uv} + a^2 b - \frac{a_u a_v}{a} + \frac{a_u^2}{b} - \frac{a_u b_v}{b} = a^2 b + 2 a a_v + \frac{a_v^2}{b},$$

since $\frac{a_v}{b} = -\frac{a_u}{a}$, then

$$a_{uv} - 2 a a_v = 0$$

or

$$\frac{\partial}{\partial v}(a_u - a^2) = 0$$

(26) $$a_u - a^2 = \text{ function of u alone.}$$

If $a_1' = b_1''$, it follows that

$$-\frac{aa_u - ab^2 + ab_v - a_v b}{ab} = \frac{aa_{vv}b + a^2bb_v - 2a_v^2 b - 2aa_v b^2 - a^2b^3 + a^2a_u b + aa_u a_v}{ab(a_v + ab)},$$

then

$$-\frac{a_u a_v}{b} - a_v b + \frac{a_v b_v}{b} - \frac{a_v^2}{a} - aa_u - ab^2 + ab_v - a_v b = a_{vv} + ab_v$$
$$-\frac{2a_v^2}{a} - 2a_v b - ab^2 + aa_u + \frac{a_u a_v}{b},$$

simplifying we have

$$-\frac{a_u a_v}{b} - 2aa_u + \frac{a_v^2}{a} - a_{vv} = 0.$$

Regroup and factor and substitute:

$$a_v\left[-\frac{b_v}{b} + \frac{b_v \bar{v}}{b\bar{v}}\right] + a_{uu} - 2aa_u = 0.$$

Therefore

$$\frac{\partial}{\partial u}(a_u - a^2) = 0.$$

Therefore

(27) $\qquad a_u - a_z^2 \qquad$ is a function of v alone.

If $b_1 = -a_1'$, it follows that

$$\frac{a_{uu} b + ab^3 - a_u b_u}{b(a_v + ab)} = \frac{aa_u + ab^2 - ab_v + a_v b}{ab};$$

whence

$$a_{uu} + ab^2 - \frac{a_u b_u}{b} = \frac{a_u a_v}{b} + a_v b - \frac{a_v b_v}{b} + \frac{a_v^2}{a} + aa_u + ab^2 - ab_v$$
$$+ a_v b,$$

or

$$a_{uu} - 2a_v b + \frac{a_v b_v}{b} - \frac{a_v^2}{a} = 0.$$

But from (23) we have $a = b_v$, and since $\bar{v}$ is a constant we have

$$b_{uu}\bar{v} - 2bb_v\bar{v} + \frac{b_v^2\bar{v}}{b} - \frac{b_v^2\bar{v}^2}{b\bar{v}} = 0,$$

or

$$-b_{vv} = 2bb_v$$

Therefore

(28) $\qquad b_v + b^2 \qquad$ is a function of u alone.

From (26) and (27) we find $a_u - a^2$ =constant; differentiating this equation with respect to v we have $a_{uv} - 2aa_v = 0.$ Differentiating equation (25) we have

$$a_{uv} + \frac{2aa_v}{v^2} = 0,$$

since

$$b_v = a_u \;; \quad b = \frac{a}{v}.$$

Therefore

$$2aa_v\bar{v}^2 = -2aa_v,$$

or

$$\bar{v}^2 = -1, \quad \text{or } a_v = 0.$$

But $a = b\bar{v}$, and $a_v = b_v\bar{v}$; and since $a_v = 0$, it follows that $b_v = 0$, whence both a and b are constants. Since we are using real values for x and y, the only solution for our problem is for a and b to be constant.

In a similar manner, if we impose the conditions that the defining differential equations of the first Laplace transformation be admissible, we arrive at the same conclusion, namely, that a and b must be constants. We may state our results as follows: <u>In order that the minus first Laplace transform of a net have admissible defining differential equations it is necessary and sufficient that a and b be constants. It follows that the defining differential equations of the first Laplace transform is also admissible, the defining differential equations of the two Laplace transforms are identical with those of the original net.</u>

3. THE MOST GENERAL NET WHOSE DEFINING EQUATIONS ARE ADMISSIBLE THE DEFINING EQUATIONS OF WHOSE LAPLACE TRANSFORMS ARE ADMISSIBLE

Let us assume that a and b constants. Take $x = e^{\theta_1}$, $y = e^{\theta_2}$, wherein

$$\theta_1 = \alpha_1 u + \beta_1 v,$$
$$\theta_2 = \alpha_2 u + \beta_2 v, \qquad (\alpha_1, \alpha_2, \beta_1, \beta_2 \text{ being constants}).$$

From (12) we find:

$$\alpha_1^2 = a\alpha_1 + b\beta_1,$$
$$\alpha_1\beta_1 = -b\alpha_1 + a\beta_1,$$
$$\beta_1^2 = -a\alpha_1 - b\beta_1.$$

Solving these equations we find that $\alpha_1 = a - bi$, and $\beta_1 = b - ai$. In a similar manner we find that $\alpha_2 = a + bi$, and $\beta_2 = ai - b$. The solutions $x = e^{\theta_1}$, $y = e^{\theta_2}$ are complex. But

$$(29) \qquad \bar{x} = a_{11}x + a_{12}y, \quad \bar{y} = a_{21}x + a_{22}y,$$

are also solutions. We may determine $a_{11}, a_{12}, a_{21}, a_{22}$ so that $\bar{x}, \bar{y}$ will be real and satisfy the Cauchy-Riemann conditions as follows:

$$\bar{x}_u = a_{11}\alpha_1 e^{\theta_1} + a_{12}\alpha_2 e^{\theta_2},$$
$$\bar{y}_v = a_{21}\beta_1 e^{\theta_1} + a_{22}\beta_2 e^{\theta_2},$$
$$\bar{x}_v = a_{11}\beta_1 e^{\theta_1} + a_{12}\beta_2 e^{\theta_2},$$
$$\bar{y}_u = a_{21}\alpha_1 e^{\theta_1} + a_{22}\alpha_2 e^{\theta_2},$$

whence

$$(a_{11}\alpha_1 - a_{21}\beta_1)e^{\theta_1} + (a_{12}\alpha_2 - a_{22}\beta_2)e^{\theta_2} = 0,$$
$$(a_{21}\alpha_1 + a_{11}\beta_1)e^{\theta_1} + (a_{22}\alpha_2 + a_{12}\beta_2)e^{\theta_2} = 0.$$

Since e^{θ_1} and e^{θ_2} are functions of u and v, it follows that

$$a_{11}\alpha_1 - a_{21}\beta_1 = 0, \qquad a_{12}\alpha_2 - a_{22}\beta_2 = 0,$$
$$a_{21}\alpha_1 + a_{11}\beta_1 = 0, \qquad a_{22}\alpha_2 + a_{12}\beta_2 = 0,$$

therefore

$$a_{11}^2 + a_{21}^2 = 0, \qquad a_{22}^2 + a_{12}^2 = 0.$$

also

$$\frac{a_{11}}{a_{21}} = \frac{\beta_1}{\alpha_1} = -\frac{a_{21}}{a_{11}}, \qquad \frac{a_{22}}{a_{12}} = \frac{\alpha_2}{\beta_2} = -\frac{a_{12}}{a_{22}},$$

or

$$\frac{a_{11}}{a_{21}} = -\frac{(b+ai)}{a-bi} = -i, \qquad \frac{a_{22}}{a_{12}} = \frac{a+bi}{-(b-ai)} = -i.$$

Therefore

(30) $$a_{11} = -i\,a_{21}, \qquad a_{22} = -i\,a_{12}.$$

$\bar{x}$ may therefore be written in the form

$$\bar{x} = a_{11}\,e^{\alpha_1 u+\beta_1 v} + a_{12}\,e^{\alpha_2 u+\beta_2 v},$$

$$= e^{au-bv}\left[(a_{11}+a_{12})\cos(bu+av) + (a_{12}-a_{11})\,i\sin(bu+av)\right],$$

using equations (30)

$$\bar{x} = e^{au-bv}\left[(a_{12}-ia_{21})\cos(bu+av) + (a_{12}+ia_{21})\,i\sin(bu+av)\right],$$

then

$$\bar{x} = e^{au-bv}\left[A_1\cos(bu+av) + A_2\sin(bu+av)\right],$$

wherein

(31) $$A_1 = a_{12} - a_{21}\,i$$

and $A_2 = -a_{21} + a_{12}\,i$, are real numbers.

Solving equations (27) for a_{12} and a_{21}, we find

$$a_{12} = \frac{A_1 - A_2 i}{2}, \qquad a_{21} = \frac{A_2 - A_1 i}{2}.$$

In a similar manner we find

$$\bar{y} = e^{au-bv}\left[-A_2\cos(bu+av) + A_1\sin(bu+av)\right].$$

Multiplying and dividing the values of $\bar{x}$ and $\bar{y}$ by $\sqrt{A_1^2+A_2^2}$,

we obtain

$$\bar{x} = \sqrt{A_1^2+A_2^2}\,e^{au-bv}\left[\cos M\cos(bu+av) + \sin M\sin(bu+av)\right],$$

$$= \sqrt{A_1^2+A_2^2}\,e^{au-bv}\cos(bu+av-M),$$

$$\bar{y} = \sqrt{A_1^2+A_2^2}\,e^{au-bv}\sin(bu+av-M),$$

wherein

$$\cos M = \frac{A_1}{\sqrt{A_1^2+A_2^2}}, \qquad \sin M = \frac{A_2}{\sqrt{A_1^2+A_2^2}}.$$

It can readily be verified that these values of $\bar{x}$ and $\bar{y}$ satisfy the Cauchy-Riemann differential equations.

Hence <u>the functions</u>

(32) $$\bar{x} = K\,e^{au-bv}\cos(bu+av-M),$$
$$\bar{y} = K\,e^{au-bv}\sin(bu+av-M),$$

__where__ K __and__ M __are__ __arbitrary__ __constants__ __are__ __the__ __most__ __general__ __solutions__ __of__ __the__ __case__ __in__ __which__ __equations__ (2), (18) __and__ (19) __are__ __all__ __permissible__ __and__ __such__ __that__ __the__ __solutions__ __are__ __real__ __and__ __satisfy__ __the__ __Cauchy-Riemann__ __differential__ __equations__.

4. DEFINING DIFFERENTIAL EQUATIONS OF THE TANGENTS TO C_u AND C_v

The tangents to v=constant are

$$X(-y_u) - Y(-x_u) + 1(x\,y_u - x_u y) = 0.$$

Let μ, ν and 1 be the Plücker coordinates of this line. Then

$$(33)\qquad \mu = \frac{y_u}{x_u y - x y_u}, \qquad \nu = \frac{x_u}{x y_u - x_u y}, \qquad 1 = 1.$$

Differentiating the equations (33) and making use of equations (2) we obtain the following:

$$(34)\ (a)\ \nu_u = \frac{-\lambda^{(u)} y_{uu} + \lambda^{(u)}_u y_u}{\lambda^{(u)2}}, \qquad (b)\ \nu_v = \frac{\lambda^{(u)} x_{uv} - x_u \lambda^{(u)}_v}{\lambda^{(u)2}},$$

$$(c)\ \mu_u = \frac{-\lambda^{(u)} y_{uu} + \lambda^{(u)}_u y_u}{\lambda^{(u)2}}, \qquad (d)\ \mu_v = \frac{-\lambda^{(u)} y_{uv} + \lambda^{(u)}_v y_u}{\lambda^{(u)2}},$$

$$\begin{aligned}(e)\ \nu_{uu} = \Big\{&\big(a_u \lambda^{(u)2} + a^2\lambda^{(u)2} - b^2\lambda^{(u)2} - \lambda^{(u)}\lambda^{(u)}_{uu} - 2a\lambda^{(u)}\lambda^{(u)}_u + 2\lambda^{(u)2}_u\big)x_u \\ &+ \big(ab\lambda^{(u)2} + b_u\lambda^{(u)2} + ab\lambda^{(u)2} - 2b\lambda^{(u)}_u\lambda^{(u)}\big)x_v\Big\} \div \lambda^{(u)3},\end{aligned}$$

$$\begin{aligned}(f)\ \mu_{uv} = \Big\{&\big(-a\lambda^{(u)}\lambda^{(u)}_v - a_v\lambda^{(u)2} + 2ab\lambda^{(u)2} + \lambda^{(u)}\lambda^{(u)}_{uv} - b\lambda^{(u)}\lambda^{(u)}_u - 2\lambda^{(u)}_u\lambda^{(u)}_v + 2a\lambda^{(u)}\lambda^{(u)}_v\big)y_u \\ &+ \big(b\lambda^{(u)}\lambda^{(u)}_v - a^2\lambda^{(u)2} - \lambda^{(u)2}b_v + b^2\lambda^{(u)2} + a\lambda^{(u)}\lambda^{(u)}_u\big)y_v\Big\} \div \lambda^{(u)3},\end{aligned}$$

$$\begin{aligned}(g)\ \mu_{vv} = \Big\{&\big(b_v\lambda^{(u)2} - b^2\lambda^{(u)2} + a^2\lambda^{(u)2} + \lambda^{(u)}\lambda^{(u)}_{vv} - 2b\lambda^{(u)}\lambda^{(u)}_v - 2\lambda^{(u)2}_v\big)y_u \\ &+ \big(2ab\lambda^{(u)2} - a_v\lambda^{(u)2} + 2a\lambda^{(u)}\lambda^{(u)}_v\big)y_v\Big\} \div \lambda^{(u)3},\end{aligned}$$

wherein $\lambda^{(u)} = \begin{vmatrix} x & x_u \\ y & y_u \end{vmatrix}$.

Substituting the first derivatives obtained in (34--a,b,c,d) in equations in (2), we find:

$$(35)\ (a)\ u_{uu} = \frac{(a\alpha_1\lambda^{(u)} - \alpha_1\lambda^{(u)}_u - b\beta_1\lambda^{(u)} - \beta_1\lambda^{(u)}_v)\chi_u + (\alpha_1 b\lambda^{(u)} + a\beta_1\lambda^{(u)})\chi_v}{\lambda^{(u)2}},$$

$$(b)\ u_{uv} = \frac{(-a\alpha_1'\lambda^{(u)} + \alpha_1'\lambda^{(u)}_u + b\beta_1'\lambda^{(u)} + \beta_1'\lambda^{(u)}_v)y_u + (-\alpha_1' b\lambda^{(u)} - a\beta_1'\lambda^{(u)})y_v}{\lambda^{(u)2}},$$

$$(c)\ u_{vv} = \frac{(-a\alpha_1''\lambda^{(u)} + \alpha_1''\lambda^{(u)}_u + b\beta_1''\lambda^{(u)} + \beta_1''\lambda^{(u)}_v)y_u + (-b\alpha_1''\lambda^{(u)} - a\beta_1''\lambda^{(u)})y_v}{\lambda^{(u)2}}.$$

Eliminating χ_u and λ_v between the equations formed by setting the equivalents of corresponding second derivatives equal in (34) and (35) the coefficients of the defining differential equations of the tangents to C_u are

$$(36)\quad (a)\ \alpha_1 = 3a + \frac{b_u}{b} - \frac{2\lambda^{(u)}_u}{\lambda^{(u)}}, \qquad (b)\ \beta_1 = -b,$$

$$(c)\ \alpha_1' = -b + \frac{b_v}{b} - \frac{\lambda^{(u)}_v}{\lambda^{(u)}}, \qquad (d)\ \beta_1' = a - \frac{\lambda^{(u)}_u}{\lambda^{(u)}},$$

$$(e)\ \alpha_1'' = 1/b\,(a_v + ab), \qquad (f)\ \beta_1'' = -3b - \frac{2\lambda^{(u)}_u}{\lambda^{(u)}}.$$

In a similar manner we find the defining differential equations of the tangents to C_v are

$$(37)\quad (a)\ \alpha_2 = 3a + \frac{b_u}{b} - \frac{2\lambda^{(v)}_v}{\lambda^{(v)}}, \qquad (b)\ \beta_2 = -b,$$

$$(c)\ \alpha_2' = -b + \frac{b_v}{b} - \frac{\lambda^{(v)}_u}{\lambda^{(v)}}, \qquad (d)\ \beta_2' = a - \frac{\lambda^{(v)}_v}{\lambda^{(v)}},$$

$$(e)\ \alpha_2'' = 1/b\,(a_v + ab), \qquad (f)\ \beta_2'' = -3b - \frac{2\lambda^{(v)}_v}{\lambda^{(v)}}.$$

Let us impose the condition that the defining differential equations of the tangents to C_u be **admissible.**

If $\alpha_1 = \beta_1' = -\alpha_1''$

we have $3a + \frac{b_u}{b} - \frac{2\lambda^{(u)}_u}{\lambda^{(u)}} = a - \frac{\lambda^{(u)}_u}{\lambda^{(u)}} = -\frac{a_v}{b} - a,$

or

$$3a + \frac{b_u}{b} - \frac{2\lambda_u^{(u)}}{\lambda^{(u)}} = a - \frac{\lambda_u^{(u)}}{\lambda^{(u)}},$$

then

$$\frac{\lambda_u^{(u)}}{\lambda^{(u)}} = 2a - \frac{b_u}{b},$$

also

$$a - \frac{\lambda_u^{(u)}}{\lambda^{(u)}} = -\frac{a_v}{b} - a,$$

Hence

$$\frac{\lambda_u^{(u)}}{\lambda^{(u)}} = 2a + \frac{a_v}{b};$$

therefore

$$2a - \frac{b_u}{b} = 2a + \frac{a_v}{b},$$

or

$$\frac{b_u}{b} + \frac{a_v}{b} = 0,$$

hence

$$b_u + a_v = 0.$$

If $\beta_1 = -\alpha_1' = \beta_1''$

we have

$$b - \frac{b_v}{b} + \frac{\lambda_v^{(u)}}{\lambda^{(u)}} = 3b + \frac{2\lambda_v^{(u)}}{\lambda^{(u)}} = -b,$$

whence

$$b - \frac{b_v}{b} + \frac{\lambda_v^{(u)}}{\lambda^{(u)}} = 3b + \frac{2\lambda_v^{(u)}}{\lambda^{(u)}},$$

or

$$\frac{\lambda_v^{(u)}}{\lambda^{(u)}} = -2b - \frac{b_v}{b},$$

also

$$3b + \frac{2\lambda_v^{(u)}}{\lambda^{(u)}} = -b,$$

or

$$\frac{\lambda_v^{(u)}}{\lambda^{(u)}} = -2b,$$

therefore

$$-2b - \frac{b_v}{b} = -2b,$$

and

$$b_v = 0,$$

$\qquad$ or b is a function of u alone.

But we know from theorem I that

$$a_u = b_v ,$$

Then $a_u = 0$, $\qquad$ or a is a function of v alone,

but

$a_v = -b_u$, therefore a and b are constants.

Under these conditions the coefficients (36) become

$$(38)\quad \alpha_1 = -a, \qquad \beta_1 = -b, \qquad \alpha_1'' = a,$$
$$\alpha_1' = b, \qquad \beta_1' = -a, \qquad \beta_1'' = b.$$

In a similar manner we find

$$(39)\quad \alpha_2 = -a, \qquad \beta_2 = -b, \qquad \alpha_2'' = a,$$
$$\alpha_2' = b, \qquad \beta_2' = -a, \qquad \beta_2'' = b.$$

By substituting the values for $\bar{x}$ and $\bar{y}$ of (37) in equivalents of μ and ν in (38) we have

$$\mu = \frac{aKe^{au-bv}\sin(bu+av-M) + bKe^{au-bv}\cos(bu+av-M)}{Kye^{au-bv}(a\cos(bu+av-M) - b\sin(bu+av-M)) - Kxe^{au-bv}(b\cos(bu+av-M) + a\sin(bu+av-M))},$$

$$\nu = \frac{K\{ae^{au-bv}\cos(bu+av-M) - be^{au-bv}\sin(bu+av-M)\}}{Kxe^{au-bv}(b\cos(bu+av-M) + a\sin(bu+av-M)) - yKe^{au-bv}(a\cos(bu+av-M) - b\sin(bu+av-M))},$$

or

$$\mu = \frac{Ke^{au-bv}\{a\sin(bu+av-M) + b\cos(bu+av-M)\}}{Ke^{au-bv}\{(ay-bx)\cos(bu+av-M) - (ax+by)\sin(bu+av-M)\}},$$

$$\nu = \frac{Ke^{au-bv}\{a\cos(bu+av-M) - b\sin(bu+av-M)\}}{Ke^{au-bv}\{(bx-ay)\cos(bu+av-M) + (ax+by)\sin(bu+av-M)\}},$$

Let $\phi = bu + av - M$ and $\theta = au - bv$,

then

(40)
$$\mu = \frac{e^{-\theta}}{-bK}(a \sin\phi + b \cos\phi),$$

$$\nu = \frac{e^{-\theta}}{-bK}(-a \cos\phi + b \sin\phi).$$

We may readily verify that $\nu_u = \mu_v$ and $\nu_v = -\mu_u$ and that μ and ν satisfy the system

$$\theta_{uu} = -a\theta_u - b\theta_v,$$

$$\theta_{uv} = b\theta_u - a\theta_v,$$

$$\theta_{vv} = a\theta_u + b\theta_v.$$

We may summarize our results as follows:

<u>The most general net whose defining differential equations are admissible and the defining differential equations of whose minus first Laplace transform is also admissible is defined by</u>

$$x = K e^{au-bv} \cos(bu + av - M),$$

$$y = K e^{au-bv} \sin(bu + av - M),$$

<u>wherein</u> K, M, a, b, <u>are arbitrary constants. Any other net with these properties is a similar affine transformation of any given one. The admissible defining differential equations are:</u>

$$\theta_{uu} = a\theta_u + b\theta_v,$$

$$\theta_{uv} = -b\theta_u + a\theta_v,$$

$$\theta_{vv} = -a\theta_u - b\theta_v.$$

It follows that the defining differential equations of the first Laplace transform is also admissible. The defining equations of both Laplace transforms coincide with the defining equations of the original net. Hence the Laplace transforms are similar affine transformations of the original net. Again the defining differential equations of the nets corresponding by duality to the curves of the original net are admissible if and only if a and b are constants. Each of the nets thus corresponding by duality to the curves of the net is a similar affine transformation of the other. The defining equations of the dual nets are:

$$\theta_{uu} = -a\theta_u - b\theta_v,$$
$$\theta_{uv} = b\theta_u - a\theta_v,$$
$$\theta_{vv} = a\theta_u + b\theta_v.$$

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