REPLACEMENT OF A DISTRIBUTED R. C. LINE BY LUMPED PARAMETERS FOR A THERMAL-ELECTRICAL ANALOGUE

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This is to certify that the

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REPLACEMENT OF A DISTRIBUTED R.C. LINE BY LUMPED PARAMETERS FOR A THERMAL-ELECTRICAL ANALOGUE

by

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INTRODUCTION

Temperature Control

In order to design a controller, it is necessary to know the dynamical characteristics of the system to be controlled. In a thermal system, this means to know the temperature-time response of the controlled medium, as the heat supply is varied.

The dynamical characteristics of a thermal system can be expressed mathematically in terms of a transfer function in Laplace transform. Such a transfer function - G is used in the theory of servomechanisms and regulating systems. (1, 3, 7). For a thermal system, it would contain a certain number of negative real roots.

If the transfer function of the system is known, a proper controller can be applied and the resulting equation investigated as to the expected error and stability under different load and reference input variations.

An alternate method, used in temperature control, is to obtain a graph which represents the temperature-time variation of the medium as heat is suddenly applied. This graph is called "process reaction curve" (9).

In general, all natural processes are capable of representation by an exponential curve. If the transfer function has more than one root, a so called "transfer lag" would exist as a result of superimposed exponential functions.

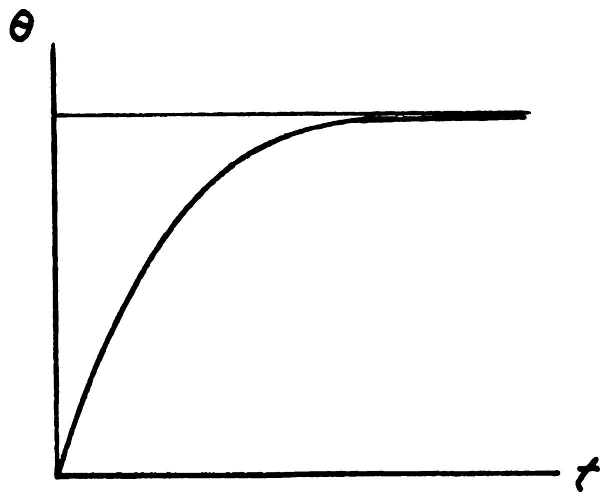


Fig. 1 Process Reaction Curve without Transfer Lag (1 root)

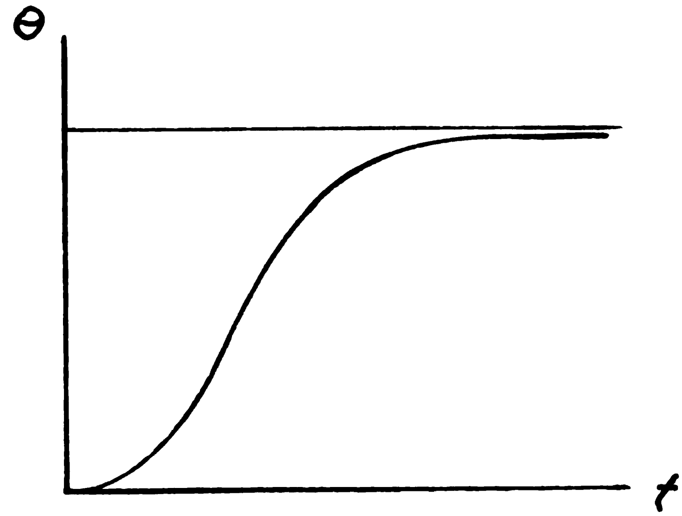


Fig. 2 Process Reaction Curve with Transfer Lag (more than one root)

A large transfer lag makes the control problem difficult, whereas it is quite simple if no transfer lag exists.

Therefore, to decide what type what type of controller should be used, the transfer function or process reaction curve should be determined. The latter may be plotted if the transfer function is known.

METHODS FOR OBTAINING TRANSFER FUNCTIONS

To obtain a transfer function or process reaction curve for a thermal process, an experiment may be performed on the system, thus determining the reaction curve by actual measurements of the temperature at certain time intervals.

Sometimes, however, this can not be done and other methods have to be employed.

If the thermal characteristics of the materials in the system as well as the dimensions and operating conditions are known, an analytical solution may be attempted. Thermal conductivities as well as specific heats would have to be considered since a transient solution is required. The calculation is extremely involved, however, and has been developed for only very simple cases.

Besides the mathematical-analytical method, three other methods of analysis are known.

1). A graphical method, first devised by Schmidt (24). It is suitable only for very simple cases and only after making a number of simplifying assumptions. Even then, it's use is rather cumbersome.

2). A numerical method, known as Southwell's relaxation method and applied to heat flow problems by Emmons (11).

The method is easy to learn and - within its range of applicability - is very useful. It becomes quite involved in case of changing thermal properties and if applied to cyclic heating.

3). An electric analogy method, which is very versatile

and subject to fewer limitations than the other methods. It's main drawback is its' rather expensive equipment (available at the Department of Mechanical Engineering, Columbia University).

THE ELECTRIC ANALOGY METHOD

The method has been first devised by Beuken (2), and introduced into the United States by Paschkis (20, 21, 22).

The method rests on the fundamental similarity between the flow of heat within a rigid body and that of a charge in a noninductive electric circuit. Conservation of the scalar quantity, charge, corresponds to conservation of heat. The scalar point function, electric potential, corresponds to the scalar point function, temperature. The concept "electric capacity of a conductor", corresponds to the concept "thermal capacity of a portion of mass".

There is a direct identity in form between the defining equations for thermal and electrical resistance and thermal and electrical capacity.

The temperature distribution in a body, at any time, is given by Fourier's general law of heat conduction, which is a partial differential equation derived by the usual methods considering an infinitely small cube, Schlack (23, p. 29). It is:

$$\frac{\partial \theta}{\partial t} = \frac{K}{\rho c}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2}\right)$$

where

θ = temperature at any point given by the coordinates x, y, z

t = time

K = coefficient of thermal conductivity

c = specific heat of unit mass

ρ = mass per unit volume

The distribution of conductivity and specific heat in space is assumed to be uniform and continuous.

The solutions of this equation are very complex. They may be found in books on advanced calculus and heat transfer. Carslaw (5) in his work, which probably is the most comprehensive development of the analytical approach, gives three-dimensional solutions for rectangular parallelepiped, cylinder, sphere and cone. Even after the solutions of this equation have been obtained, it is necessary to be extremely careful in selecting the proper solution to fit the conditions of the problem.

It is possible, however, to represent a three-dimensional problem by a combination of one-dimensional systems. Such a method is indicated by Paschkis (20) where the thermal-electrical analogy is developed on one-dimensional basis which then could be used to represent two or three-dimensional systems if necessary.

For one-dimensional heat flow the Fourier's equation reduces to:

$$\frac{\partial \theta}{\partial t} = \frac{K}{\rho c} \frac{\partial^2 \theta}{\partial x^2}$$

One dimensional heat flow would take place, for instance, along an insulated rod or across an infinite plate.

The analogous electrical system would be a conductor with resistance and capacitance uniformly distributed along it's length and having negligible leakage and inductance.

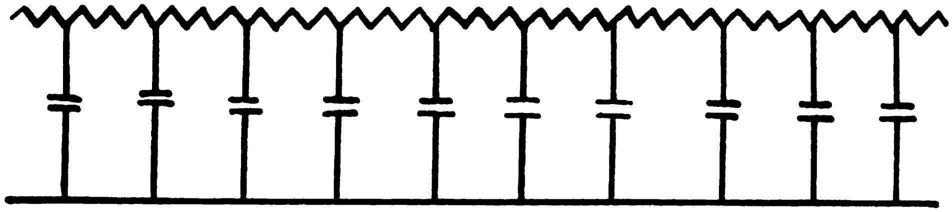


Fig. 3 Distributed R-C Line

The differential equation describing the potential at various points of the conductor at different times is:

$$\frac{\partial V}{\partial t} = \frac{1}{RC} \frac{\partial^2 V}{\partial x^2}$$

where:

V = potential at distance x and time t

R = distributed resistance ohms/unit length

C = distributed capacitance farads/unit length

Comparing the two equations, it is recognized that if the temperature is considered to be analogous to potential, the quantity $\frac{K}{\rho c}$ is analogous to the quantity $\frac{1}{RC}$ in the electrical system.

The analogous quantity for heat flow, q, can be established considering the defining equation for heat conductivity.

The heat flow across a unit surface per unit time at any point x is:

$$q = -K \frac{\partial \theta}{\partial x}$$

where:

K = thermal conductivity

$\frac{\partial \theta}{\partial x}$ = temperature gradient at point x

The current density at a point x in a conductor is:

$$\bar{\iota} = -\sigma \frac{\partial V}{\partial x}$$

where:

σ = specific conductivity of the material

$\frac{\partial V}{\partial x}$ = potential gradient at point x

For one dimensional systems where $\bar{\iota}$ is uniform and the distribution of σ is uniform and continuous

$$i = -\frac{1}{R} \frac{\partial V}{\partial x}$$

If K is analogous to $\frac{1}{R}$ then heat flow per unit time, q, is the analogous quantity to current i.

The quantity $\frac{K}{\rho c}$ was analogous to $\frac{1}{RC}$. Therefore the analogue for capacitance per unit distance C would be the specific heat per unit volume ρc.

The analogous quantities thus established are given below:

V - potential	volts	θ -	temperature	$[{}^{\circ}F]$
I - current	amps.	q -	heat flow	$\left[\frac{Btu}{ft^2\ hr}\right]$
C - Capacitance	farads/meter	ρc -	volume specific heat	$\left[\frac{Btu}{ft^3\ {}^{\circ}F}\right]$
G - conductance	mhos/meter	K -	heat conductivity	$\left[\frac{Btu\ ft}{ft^2\ {}^{\circ}F\ hr}\right]$
R - resistance	ohms/meter	$\frac{1}{K}$ -	inverse conductivity	
G_L - load conductance	mhos	h -	surface coefficient of heat transfer	$\left[\frac{Btu}{ft^2\ {}^{\circ}F\ hr}\right]$
R_L - load resistance	ohms	$\frac{1}{h}$ -	inverse surface coefficient	

Now an electrical analogue can be drawn for the one-dimensional heat flow problem.

For the case of the insulated rod (Fig. 4)

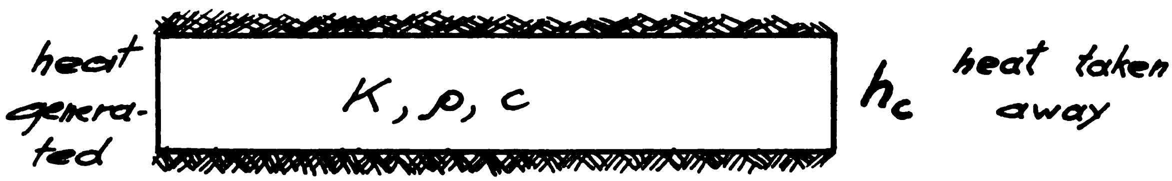


Fig. 4 Insulated Rod with Outflow Conductance

the electrical analogue would be an R C cable terminated in a load resistance (Fig. 5).

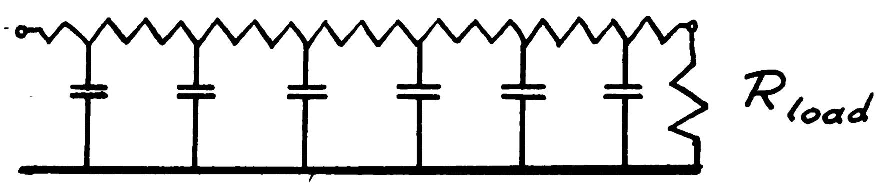


Fig. 5 R C Line Terminated in R_L

Two essentially different cases will be considered:

1). The temperature of the heated end of the rod is a known function of time.

2). The heat input at the heated end of the rod is a known function of time.

The problem in both cases would be to find the temperature of the far end as a function of time.

In the first case in the electrical analogue, a potential V(t) should be applied and the potential across the load resistance measured. For the second problem, a current I(t) would have to be applied and the potential on the load resistance measured.

The second case is of much greater importance in control problems wherever electrical heating is used because the heat input would be known. It could be directly derived from the power input in the heating elements.

To obtain a process reaction curve, which gives the temperature response at a certain point to a sudden application of heat, from the electrical analogue, an unit step current should be applied on the input terminals and the potential across the load resistance measured and plotted versus time.

If it is preferable to use voltage as the input function, an alternative method, as suggested by Brown (3), may be used. This method utilizes the dual of a R C cable.

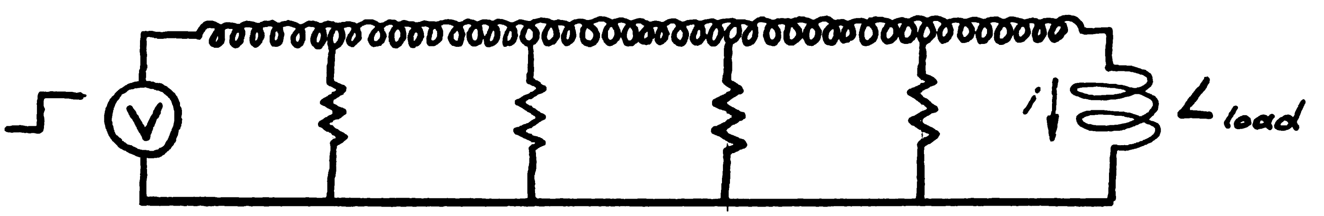


Fig. 6 Dual for R C Cable

The unit step heat input in this case would be analogous to unit step voltage on the input terminals, and the temperature at the outer surface to the current through load inductance.

This way Brown (3) avoids the necessity to apply constant current. It is of advantage if the problem is solved analytically, as Brown (3) does, because almost all of the solutions in electrical circuit theory are on this basis. If a laboratory model is to be used, however, this method seems to have the disadvantage, that inductances usually would not be available with such accuracies as capacitors, and a certain resistance in them would not be avoidable.

It is proposed, therefore, to use the R C analogue and apply a unit step current.

Paschkis (20) describes such a constant current device used in the permanent electric model at Columbia University. This device provides, by means of electron tubes, any current value between 0.1mA and 30mA which can be set and maintained constant throughout the experiment, regardless of apparent changes in resistance resulting from the loading of the condensers. The voltages are measured by recording millivoltmeters fed by two stage amplifiers. This way the current drawn from the circuit by the amplifier is less than 10^{-9} amperes, which may be neglected as compared with the leakage currents through the insulation and other parts of the model.

REPLACING DISTRIBUTED PARAMETERS

This far the distributed one-dimensional heat flow has been represented by a distributed electrical system - a R C cable.

For an actual problem in laboratory, it would be difficult to construct a cable for each problem. The parameters could not be easily varied and interconnections to represent a two or three dimensional heat flow could not be readily made.

The natural suggestion would be to replace the distributed R C line by a cascade of lumped R C circuits. This is what actually is done in all the models described in literature (10, 12, 18, 20, 21, 22, 26).

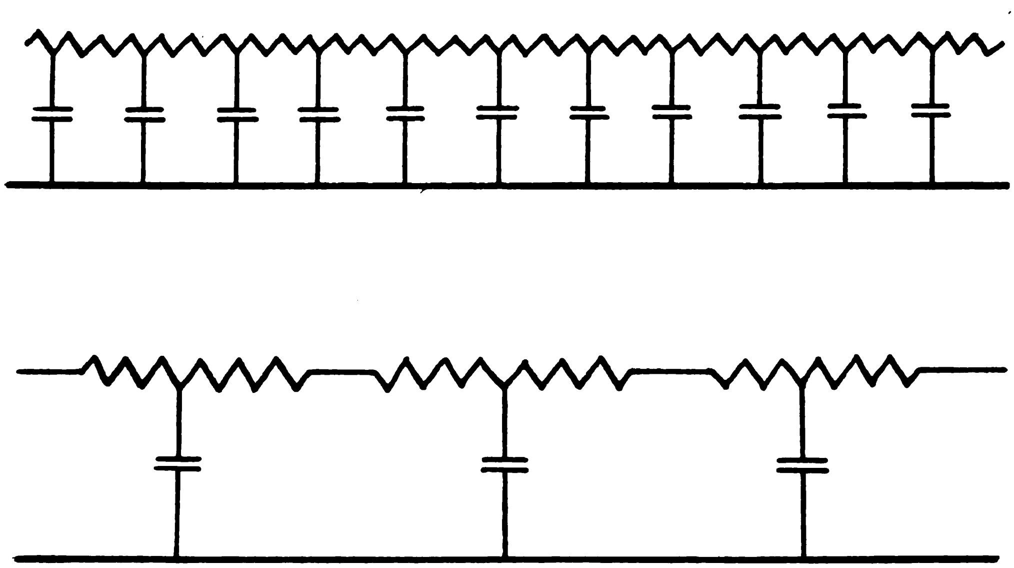


Fig. 7 Distributed and Lumped R C Circuits

The question may arise as to how many lumped sections should be used to represent a certain distributed system.

Paschkis (20) states that the smaller the "lumps" the

the more perfect will be the representation of the actual cable by lumped cable and that a feasible compromise is usually possible in practice. In the experiment described (20) the analogue representing pipe insulation, is divided in five sections, so chosen as to conform to the positions of the thermocouples in an experiment performed on the same insulation by Perry and Berggren at the University of California

McCann (18) indicates that the size of elements into which the distributed system should be divided would be governed by the configuration of the body, boundary conditions, and the required accuracy of the solution.

Tribus (26) investigating the ice protection problem divides the guide vane of an airplane in eight sections and the propeller in five sections, and uses a lumped R C network for each section, interconnecting the separate networks in such a way as to represent a two-dimensional heat flow. This work is interesting because he uses non-linear networks to represent parameters that are not constant but varies with temperature. Such as the convection from an exposed surface.

Eckman (10) is investigating automatic control problems by electrical analogy and using three R C networks in cascade to represent a multiple capacity process. Each capacitor is shunted by a resistor to represent the demand.

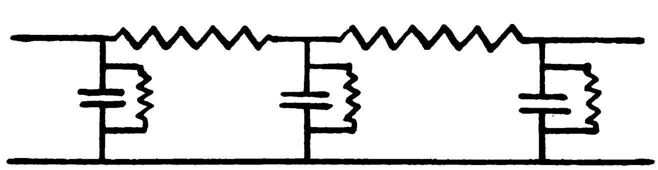


Fig. 8 Lumped R C Network with Resistors Representing Demand

An equivalent network is used to represent the thermocouple and well.

Hornfeck (17) states that a thin thermometer socket could be represented by a single lumped R C section, whereas for a heavy socket it would not be permissible. He suggests that two sections be used. The differential equation of such a system is easily handled analytically, and according to the author (17), gives reasonably good correlation between the experimental and calculated response.

In the discussion of the article (17) Paschkis questions the validity of using only two sections and indicates that it probably would be necessary to use several sections for the protecting socket as well as for the internal element. Since the resulting equations would not be manageable mathematically, the electric-analogy method (20) is suggested.

In the authors closure (17) Hornfeck states that if a dimensionless parameter m (ratio of the socket film resistance to the internal resistance) is either much smaller or much larger than unity, the response reduces to the simple exponential function.

It should be noted here that the circuit used to represent the thermometer is essentially different from that used for heat conduction through a wall.

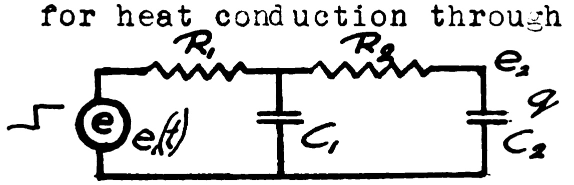


Fig. 9 R C Circuit to Represent Heat Flow through a Thick Thermometer Wall

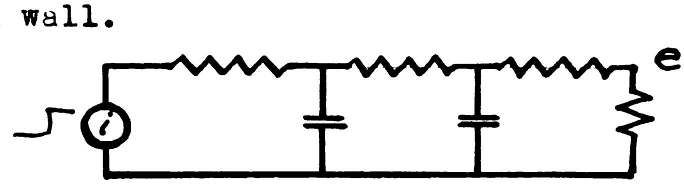


Fig.10 R C Circuit to Represent Heat Flow to a Wall with Outflow Resistance

Ahrdent (1) indicates that the analysis of a thermal system may be simplified assuming that certain elements are essentially either resistive or capacitative, and that a distributed R C system, the determinantal equation of which would have infinite number of poles on the negative real axis, could be represented by a lumped system neglecting the poles at real high values of S, because the value of residues at these high values of S diminishes.

Eyres (12) have solved a heat conduction problem replacing the distributed system by six lumped sections, and solving the resulting differential equation on a differential analyzer. The same problem was solved analytically for the distributed system and the maximum deviation of the two solutions found to be 0.4%. Since six lumped sections result in a differential equation of the sixth order which requires quite an elaborate set up on the differential analyzer and rather long time to carry the solution through, another solution was performed using only three lumped sections. The increase in the maximum error was slight (from 0.4% to 1.5%). It was concluded that probably using only two lumped sections, the error would be within acceptable limits. It was indicated that the error is approximately proportional to the inverse of the square of number of sections.

STATEMENT OF PROBLEM

The question may arise, whether it would be possible to obtain a criteria for the different factors that govern the error if certain numbers of lumped sections are used to represent a distributed system.

Since, using the already established analogies, a conversion from an electrical to a thermal system can be readily made, it is proposed to attempt a solution for the electrical circuit, because there is not a true lumped parameter in a thermal system. Taking the simple example of one dimensional heat flow in an insulated rod for the case more important to control problems, where the heat input is varied, the problem in the electrical system may be stated as follows:

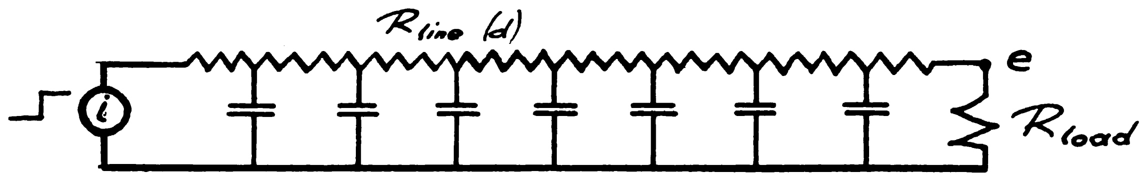


Fig. 11 The Distributed R C Line

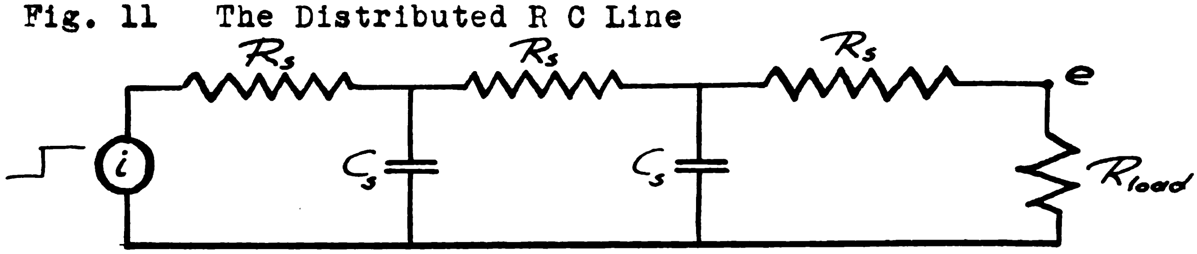


Fig. 12 The Lumped Line

If a line of finite length, having distributed resistance and capacitance and terminated in a load resistance, is to be replaced by a number of equal lumped sections, such that the total resistance and capacitance of the lumped sections is the same as the total resistance and capacitance of the line, what is the smallest number of sections that should be used to have the voltage response on the output terminals of the lumped line to a unit step current input within a prescribed error of that of the distributed line. Stated in mathematical terms:

$$R_{load}(d) = R_{load}(l)$$

$$R_{line}(d) = R_{line}(l) = (n - 1)R_s$$

$$C_{line}(d) = C_{line}(l) = (n - 2)C_s$$

$$R_{s_1} = R_{s_2} = R_{s(n-1)}$$

$$C_{s_1} = C_{s_2} = C_{s(n-2)}$$

For $i(t) = u(t)$ the response for the distributed line is $e_d(t)$ and for the lumped line $e_l(t)$. Find n_{min} such that $e_d(t) - e_l(t) \leq \delta$ Where the subscript d designates the distributed line; l - the lumped line; s - parameter value of one section; n - the number of nodes; (n-1) the number of equal resistors; (n - 2) the number of equal capacitors in the lumped line; δ - the maximum allowable error.

The unit step input is chosen because it is very common in discontinuous control problems and response to any other function can be derived from the unit step response, applying the principle of superposition. (14)

SOLUTION FOR CHARACTERISTIC TERMINATION

A similar problem is discussed by Guillemin (16). He states the problem as follows: if an artificial line representing the long line is to be constructed, one should be able to determine the number of sections and the parameter values of each for a given finite length of line, frequency range, and maximum allowable error. Guillemin shows that such an artificial line is physically realizable and that if the number of structures becomes infinite, the artificial line leads to the uniform line itself. An expression for the decimal error in transfer impedance and propagation function is developed.

$$\delta_z = -3\delta_\alpha = \frac{1}{8}\left(\frac{1}{n}\right)^2 (R + jL\omega)(G + jC\omega)$$

These errors increase with frequency being smallest at zero frequency, while for a given frequency they increase as the square of the line length and decrease as the reciprocal of the square of the number of cascaded sections. When the maximum errors in Z_T and are specified of the same magnitude, then the error of Z_T will govern the design.

The expression is derived on a steady state basis for a given frequency range. It could be extended to a transient case if a unit step function is applied by means of Fourier's integral.

Unfortunately, however, it can not be used for the problem at hand. Guillemin does not consider the terminal conditions which, it is believed, would have considerable effect upon the error if the line is terminated in an impedance much different from the characteristic impedance.

In communication networks it would not usually be the case. A thermal analogue, however, would have to be terminated in various impedances depending upon the convection and radiation conditions on the surface.

INTRODUCTION OF DIFFERENT TERMINAL CONDITIONS

General

To investigate the problem for different terminal conditions a method is proposed where the transfer admittance for the R C cable terminated in a pure resistance is found and compared to the transfer admittance of m-section lumped network terminated in the same resistance. If the two transfer admittances could be expressed in essentially the same form, conditions probably could be derived under which the response of the two systems would not differ by more than the allowable error.

The R C lines under transient conditions have been treated originally by Lord Kelvin in order to determine the practicability of a transoceanic cable. Lord Kelvin used the classical methods of solving partial differential equations. A treatment of the same problem by transformation calculus is given by Carson (6) and Cohen (8).

Cohen (8) first gives a solution for an infinite line and a line terminated in an open or short circuit. For the problem of different termination the author states that the problem is by far more difficult and that it is only in special cases that it is at all possible to obtain a completely developed solution. A solution for a special case where a general RLCG line is terminated in a coil with time constant L/R, is given. The solution is in terms of propagation constants. Goldman (15) states, however, that there is no distinct velocity of propagation along the R C cable and that the signals are said to be "diffused" rather than "propagated" along the line. It is doubtful, therefore, if the solution for a "propagating" RLCG line could be applied to a R C cable.

Carson (6) makes the statement that if the line is closed by arbitrary impedances instead of open or short circuits, the case is quite different, and the location of roots becomes, except for simple impedances, and then only in the case of non-inductive cable, practically impossible. While, therefore, the expansion theorem solution can be formally written down, its actual numerical evaluation is a practical impossibility, except in a few cases. For this reason, the author would not consider it further in his work.

From Carson's statement, one could conclude that while a general solution would be impractical, a solution for the special case - R C cable terminated in a pure resistance could be possible. Unfortunately, such a solution is not given in Carson's work.

SOLUTION FOR THE DISTRIBUTED LINE

Investigating further it was found that a solution for the differential equation that describes the problem at hand is given without proof by Carslaw (5, pg. 104) and an indication of the method it is derived by Newman (19). A formal proof is not given there. It is solved by classical methods and it was shown by Dr. Frame (13) that the functions involved result from the boundary conditions of the problem.

The differential equation is written for a problem in heat conduction with the same boundary conditions as these in the rod, for which the electrical analogue was drawn. It is obvious, therefore, that the solution applies equally well to the R C cable with terminal resistance and unit step current input.

In Newman's (19) work it is given in a dimensionless form and therefore readily rewritten for the R C cable.

The differential equation, boundary conditions and solu- is given below. A formal proof, however, shall not be given, because it is not essential for the problem.

The Problem:

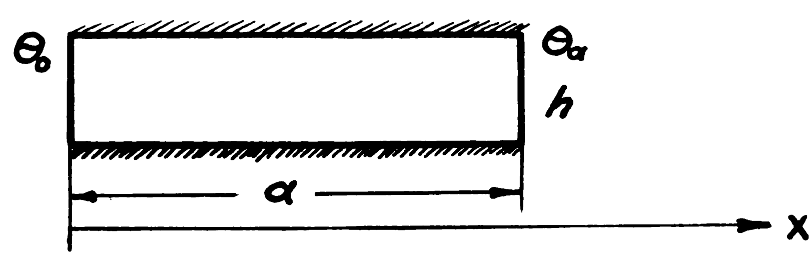


Fig. 13 The Insulated Rod

A rod of homogenous material is subjected to a heat input at constant rate into one face $x = 0$, while the other face x = a is exposed to a fluid medium. The length of the rod is a. It is assumed that the heat travels only in the x direction. The surface coefficient of heat transfer h, may be considered to be a function of the nature of the fluid and its velocity, but this analysis does not permit consideration of its variation with temperature.

Find the temperature-time relationship at the surface x = a.

The following quantities are used in the development:

x = distance in the x direction		$[ft]$
t = time		$[hr]$
K = thermal conductivity of the solid		$\left[\frac{Btu\ ft}{ft^2\ hr\ {}^\circ F}\right]$
c = specific heat of the solid		$\left[\frac{Btu}{lb\ {}^\circ F}\right]$
ρ = density of the solid		$\left[\frac{lb}{ft^3}\right]$
k = thermal diffusivity	$\frac{K}{\rho c}$	$\left[\frac{ft^2}{hr}\right]$
h = surface coefficient of heat transfer		$\left[\frac{Btu}{ft^2\ hr\ {}^\circ F}\right]$
q = constant heating rate at x = 0		$\left[\frac{Btu}{ft^2\ hr}\right]$
θ = temperature		$[{}^\circ F]$

From these certain dimensionless quantities are formed:

$N_u = \frac{ha}{K}$ a modified Nusselt Number

$P = \frac{kt}{a^2}$ a dimensionless quantity involving time

$E = \frac{\theta K}{qa}$ a dimensionless temperature response

The differential equation describing the heat flow is the Fourier's equation in one dimension.

$$\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$$

The boundary conditions.

1) The rate of heat flow across the surface x = 0

$$q_o = -K\left(\frac{\partial \theta}{\partial x}\right)_o$$

2) The rate of heat flow at the surface x = a

$$q_a = h\,\theta_a = -K\left(\frac{\partial \theta}{\partial x}\right)_a$$

3) The initial temperature of the solid is uniform and equal to the temperature of the fluid (assumed zero here as reference), when t = 0 $\theta = 0$

4) After a long time steady state will be reached and the temperature gradient through the rod will be constant

when $t = \infty$ $$\theta_a = \frac{q}{h}$$

and $$\frac{\partial \theta}{\partial x} = -\frac{q}{K}$$

5) The total gain in heat content by the rod at any time must equal the difference between the total heat input up to that time.

$$\int_0^a \theta\, c\rho\, dx = qt - \int_0^t h\,\theta_a\, d\theta$$

An equation satisfying all of these conditions is:

$$E = 1 - \frac{x}{a} + \frac{1}{Nu} - 2\sum_{n=1}^{\infty} \frac{\beta_n^2 + Nu^2}{\beta_n^2\left[\beta_n^2 + Nu(1+Nu)\right]} \cos\left(\beta_n \frac{x}{a}\right) \epsilon^{-\beta_n^2 P}$$

where B_n is defined by

$$\cot\beta = \frac{\beta}{Nu}$$

Since the temperature at x = a is of primary interest.

$$E_a = \frac{1}{Nu} - 2\sum_{n=1}^{\infty} \frac{\beta_n^2 + Nu^2}{\beta_n^2\left[\beta_n^2 + Nu(1+Nu)\right]} \cos\beta_n \, \epsilon^{-\beta_n^2 P}$$

The equation can be rewritten for the R C cable if corresponding dimensionless quantities are employed.

for N_u $\quad \frac{R_{line}}{R_{load}} = N$

for P $\quad \frac{t}{R_{line}C_{line}} = \frac{t}{\tau}$ (τ = time constant of the cable)

for E $\quad \frac{e_l}{I\, R_{line}} = \frac{1}{Y_{tr} \cdot R_{line}}$ (Y = transfer admittance)

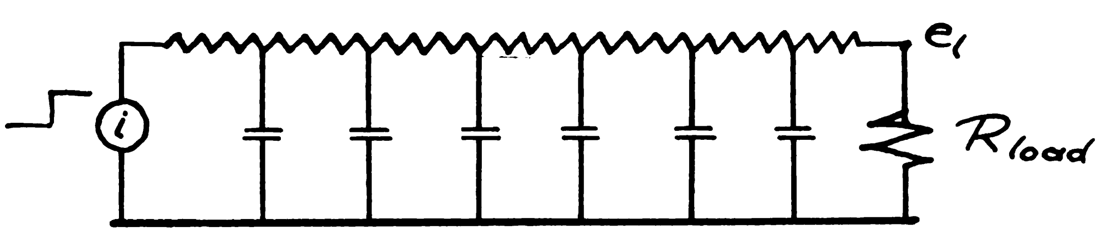


Fig. 14 Distributed R C Line

R_{line} = total resistance of the line

C_{line} = total capacitance of the line

then

$$\frac{1}{Y_{tr} R_{line}} = \frac{R_{load}}{R_{line}} - 2\sum_{n=1}^{\infty} \frac{\beta_n^2 + N^2}{\beta_n^2\left[N(1+N)\right]} \cos\beta_n \, \epsilon^{-\beta_n^2 \frac{t}{\tau}}$$

The quantity:

$$\frac{\beta_n^2 + N^2}{\beta_n^2\left[\beta_n^2 + N(1+N)\right]}\cos\beta_n$$

is a pure number and function of $\frac{R_{line}}{R_{load}}$ = N only. It can be computed for different values of N up to any desired n if tables for the roots of $\cot\beta = \frac{\beta}{N}$ are available.

Newman (19) and Carslaw (5) give such tables for B_n (n = 1 to n = 7) and for values of N from 0.001 to 100. If the values for B_n (n > 7) or different N's are desired they can be found graphically as shown by Cohen (8, pg. 88) (see graph 1) or they could be calculated to a greater accuracy by approximation methods as indicated by Dr. Frame (13).

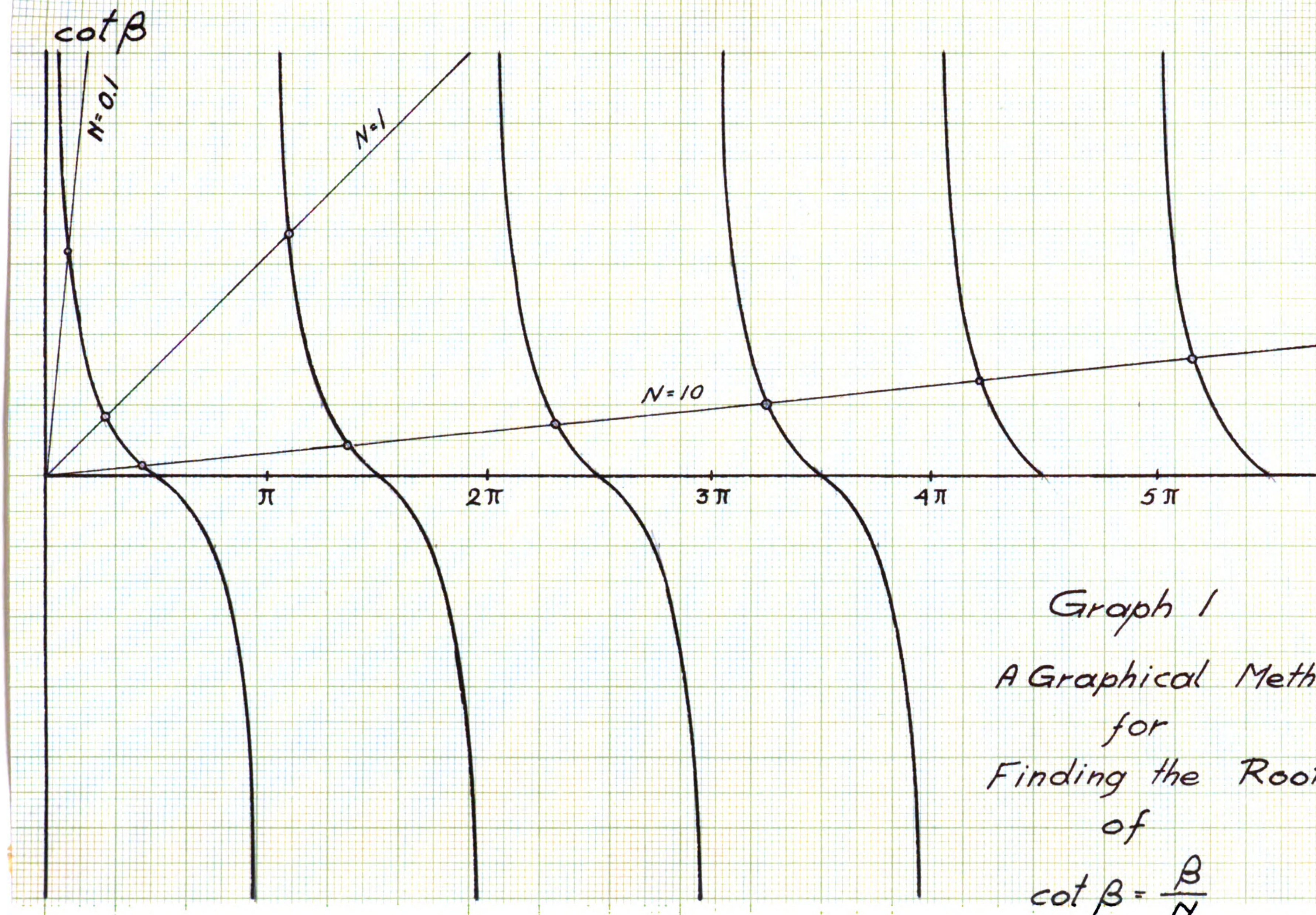
Actually in the regions where $N \to 0$, or $N \to \infty$ and the functions: sin x, cos x, tan x, cot x, may be assumed equal to the arguments, the calculation of B_n and the total numerical coefficient becomes quite simple as it will be shown later.

Once the numerical values for

$$A_n = 2\,\frac{\beta_n^2 + N^2}{\beta_n^2\left[\beta_n^2 + N(1+N)\right]}\cos\beta_n$$

are computed, the dimensionless equation for the transfer admittance assumes the form:

$$\frac{1}{Y_{tr}R_{line}} = \frac{R_{load}}{R_{line}} - A_1\epsilon^{-\beta_1^2\frac{t}{\tau}} + A_2\epsilon^{-\beta_2^2\frac{t}{\tau}} - A_3\epsilon^{-\beta_3^2\frac{t}{\tau}} + \dots$$



Graph 1

A Graphical Method for Finding the Roots of $\cot\beta = \frac{\beta}{N}$

SOLUTION FOR THE LUMPED LINE

The next step would be to derive an expression for the lumped line of m elements and to investigate under what conditions it could be considered equivalent to the equation of the distributed line, within an allowable error.

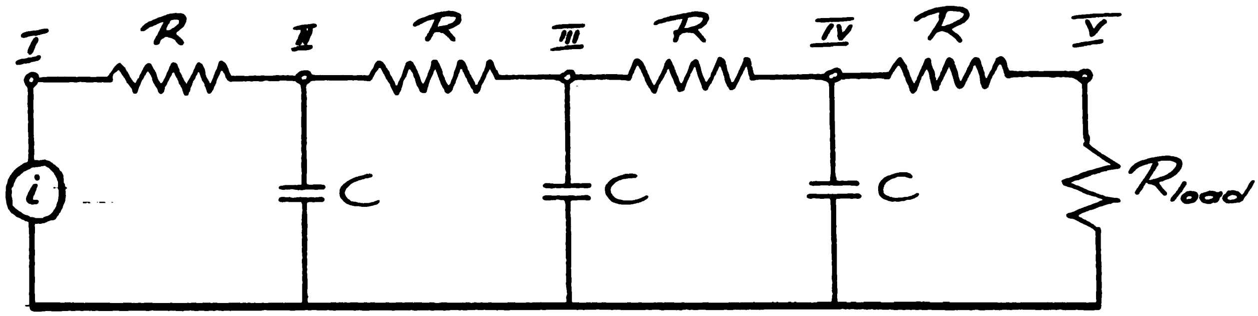


Fig. 15   The Lumped R C Network

Consider a network having n nodes, the mutual resistance between each node being R, and the admittance of each node except the first and last

$$Y_{nn} = \frac{2}{R} + C_p$$

The admittances of the first and last nodes would be:

$$Y_{1,1} = \frac{1}{R} \text{ and } Y_{n,n} = \frac{R + R_{load}}{R \cdot R_{load}}$$

The total resistance of the line would be

$$R_{line} = R(n-1)$$

and the total capacitance

$$C_{line} = C(n-2)$$

The mutual and node admittances can be written in terms of

R_{line}, C_{line}, N, and n only, since $N = \frac{R_{line}}{R_{load}}$

Replacing

$$R \text{ by } \frac{R_{line}}{(n-1)}$$

$$C \text{ by } \frac{C_{line}}{(n-2)}$$

and $$\frac{R + R_{load}}{R \cdot R_{load}} \text{ by } \frac{N + (n-1)}{R_{line}}$$

The determinant for the network can be written:

$$
\begin{array}{c|cccccc|c}
 & e_1 & e_2 & e_3 & e_{(n-2)} & e_{(n-1)} & e_n & \\
\hline
1 & \frac{n-1}{R_{line}} & -\frac{n-1}{R_{line}} & & & & & i \\
2 & -\frac{n-1}{R_{line}} & \frac{2(n-1)}{R_{line}} + \frac{C_{\phi p}}{n-2} & -\frac{n-1}{R_{line}} & & & & o \\
3 & & -\frac{n-1}{R_{line}} & \frac{2(n-1)}{R_{line}} + \frac{C_{\phi p}}{n-2} & & & & o \\
n-2 & & & & \frac{2(n-1)}{R_{line}} + \frac{C_{\phi p}}{n-2} & -\frac{n-1}{R_{line}} & & o \\
n-1 & & & & -\frac{n-1}{R_{line}} & \frac{2(n-1)}{R_{line}} + \frac{C_{\phi p}}{n-2} & -\frac{n-1}{R_{line}} & o \\
n & & & & & -\frac{n-1}{R_{line}} & \frac{N + (n-1)}{R_{line}} & o
\end{array}
$$

Dividing each column by $\frac{n-1}{R_{line}}$
and calling $C_{line} \cdot R_{line} = \tau$

$$
\left(\frac{(n-1)}{R_{line}}\right)^n
\begin{array}{c|cccccc|c}
 & e_1 & e_2 & e_3 & e_{n-2} & e_{n-1} & e_n & \\
\hline
1 & 1 & -1 & & & & & i \\
2 & -1 & 2 + \frac{\tau \rho}{(n-2)(n-1)} & -1 & & & & o \\
3 & & -1 & 2 + \frac{\tau \rho}{(n-2)(n-1)} & & & & o \\
n-2 & & & & 2 + \frac{\tau \rho}{(n-2)(n-1)} & -1 & & o \\
n-1 & & & & -1 & 2 + \frac{\tau \rho}{(n-2)(n-1)} & -1 & o \\
n & & & & & -1 & \frac{N + (n-1)}{(n-1)} & o
\end{array}
$$

If the determinant is opened, it would result in a polynomial in τp of the (n-2) order, the coefficients consisting of combinations of R_{line} and N.

If it is solved for e_n the numerator would be

$$\left(\frac{n-1}{R_{line}}\right)^{(n-1)} \cdot i(-1)^{(n-1)}(-1)^{(n-1)} = i\left(\frac{n-1}{R_{line}}\right)^{n-1}$$

it could be cancelled by the factor $\left(\frac{n-1}{R_{line}}\right)^{n}$ in the denominator leaving $\frac{n-1}{R_{line}}$ in the denominator.

The coefficient of the highest power in τp should be made unity and the polinomial factored in its' roots.

If a unit step function is applied, the solution for the dimensionless quantity $\frac{1}{Y_{tr} \cdot R_{line}}$ would be the following:

$$\frac{1}{Y_{tr} \cdot R_{line}} = \frac{1}{(n-1)\,K_1} \mathcal{L}^{-1} \frac{1}{s\left(s-\frac{\alpha_1}{\tau}\right)\left(s-\frac{\alpha_2}{\tau}\right)\left(s-\frac{\alpha_3}{\tau}\right)\cdots\left(s-\frac{\alpha_{n-2}}{\tau}\right)}$$

where K_1 is the coefficient at the highest power of τp.

From physical reasoning one can conclude that for any finite values of R_{load} $(N \neq 0 \neq \infty)$, the roots will be all negative and real and therefore the solution

$$\frac{1}{Y_{tr} \cdot R_{line}} = A_0 - A_1 \epsilon^{-\alpha_1 \frac{t}{\tau}} + A_2 \epsilon^{-\alpha_2 \frac{t}{\tau}} - \ldots A_{n-2} \epsilon^{-\alpha_{n-2} \frac{t}{\tau}}$$

The first term $-A_0$, is the steady state term and can be easily determined.

when $t = \infty$ $\qquad \frac{1}{Y_{tr} \cdot R_{line}} = A_0$

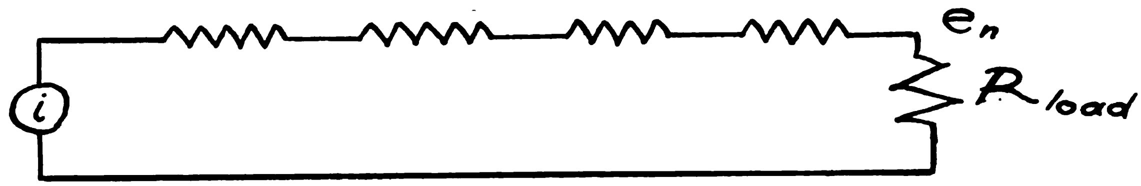


Fig. 16 Steady State Circuit

$$e_n = i\,R_{load}\,;\quad A_0 = \frac{R_{load}}{R_{line}} = \frac{1}{N}$$

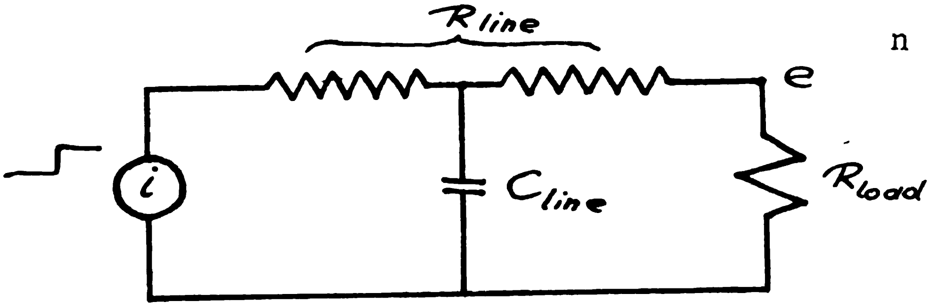
The steady state coefficient is the same as that for the distributed line, which should be expected.

The determination of the coefficients at $\tau p - K_1 ; K_2 \cdots$ in general terms is difficult for a determinant of a rank higher than 4. Even after they are found, a factoring in general terms would be impossible for any equation of higher order than two. It was done for a determinant of the 4th rank which results in a quadratic equation. Solving this equation in general terms the square root can not be eliminated, which makes the general solution cumbersome and relationships are not easily recognized.

A simple solution in general terms can be obtained, however, for a 3rd rank determinant.

SOLUTION FOR ONE LUMPED SECTION

A. Solution by Laplace Transform



n = 3

$$\frac{R_{line}}{R_{load}} = N$$

Fig. 17 One Lumped Section

The determinant:

$$\frac{2}{R_{line}}\begin{vmatrix} 1 & -1 & 0 \\ -1 & 2+\frac{\tau p}{2} & -1 \\ 0 & -1 & \frac{N+2}{2} \end{vmatrix}$$

$$\frac{e}{i} = \mathcal{L}^{-1}\,\frac{\frac{R_{line}}{2}}{s\left(s\tau\,\frac{N+2}{4} + \frac{N}{2}\right)}$$

$$\frac{e\;2\,(N+2)}{i\;R_{line}} = \mathcal{L}^{-1}\,\frac{1}{s\left(s + \frac{2\,N\,\tau}{N+2}\right)}$$

$$\frac{1}{Y_{tr}\,R_{line}} = \frac{1}{N} - \frac{1}{N}\,\epsilon^{-\frac{2N}{2+N}\,\frac{t}{\tau}}$$

CONDITIONS FOR REPLACEMENT BY ONE SECTION

The result derived is very useful to determine the conditions when a distributed line can be replaced by a single lumped section.

Returning to the equation for a distributed line,

$$\frac{1}{\underset{n\to\infty}{Y_{tr} R_{line}}} = \frac{1}{N} - A_1 \epsilon^{-\beta_1^2 \frac{t}{\tau}} + A_2 \epsilon^{-\beta_2^2 \frac{t}{\tau}} - \dots A_n \epsilon^{-\beta_n^2 \frac{t}{\tau}} + \dots$$

and comparing it with the equation for a single lumped section,

$$\frac{1}{Y_{tr} R_{line}} = \frac{1}{N} - \frac{1}{N} \epsilon^{-\frac{2N}{2+N}\frac{t}{\tau}}$$

it is recognized that the two equations would be equivalent if

$$A_1 = \frac{1}{N} \quad \text{and} \quad \beta_1^2 = \frac{2N}{2+N}$$

within the allowable error, and that all higher terms in the distributed line equation:

$$A_2,\ A_3, \dots \quad \text{and} \quad \beta_2^2,\ \beta_3^2$$

are negligible as compared to A_1 and B_1^2.

A_1 can be calculated for different values of N from the expression:

$$A_1 = 2\,\frac{\beta_1^2 + N^2}{\beta_1^2\left[\beta_1^2 + N(1+N)\right]} \cos\beta_1$$

The quantity $N(A_1 - \frac{1}{N})\frac{1}{100} = \delta_{K_1}$ may be called the % error in first coefficient. It has been calculated for different values of N and are given in table 1 and graph 2.

For N = 0.01, δ_{K_1} is less than 0.0001%.

The quantity $\left(\beta_1^2 - \frac{2N}{2+N}\right)\frac{\frac{2N}{2+N}}{100} = \delta_{e_1}$ may be called the % error in first exponent. It, too, has been calculated for different values of N and are given in table 2 and graph 2.

It is found that both errors δ_{K_1} and δ_{e_1} decrease for decreasing N.

That the higher coefficients will be negligible if the error in the first coefficient, δ_{K_1} , is within acceptable limits can be shown if the instant t = 0 is considered. For this condition all the exponentials become unity and the sum of all coefficients must beequal to $\frac{1}{N}$.

$$A_1 + A_2 + A_3 + \dots A_n + \dots \rightarrow \frac{1}{N} \quad \text{if } n \rightarrow \infty$$

and since the coefficients are periodically changing sign and steadily decreasing in absolute value

$$|A_1| > |A_2| > |A_3| \quad \dots > |A_n| \dots$$

none of the higher coefficients ($n \geq 2$) can be greater than the difference $(A_1 - \frac{1}{N})$ which was not greater than the allowable error.

The exponents at the higher coefficients would have negligible effect because the coefficients themselves are negligible.

From the data obtained the conclusion may be drawn that if the maximum error is to be $\approx$ 1.6%, for instance, any distributed system having N $<$ 0.1 can be represented by a single lumped section if a unit step function is applied.

Table 1

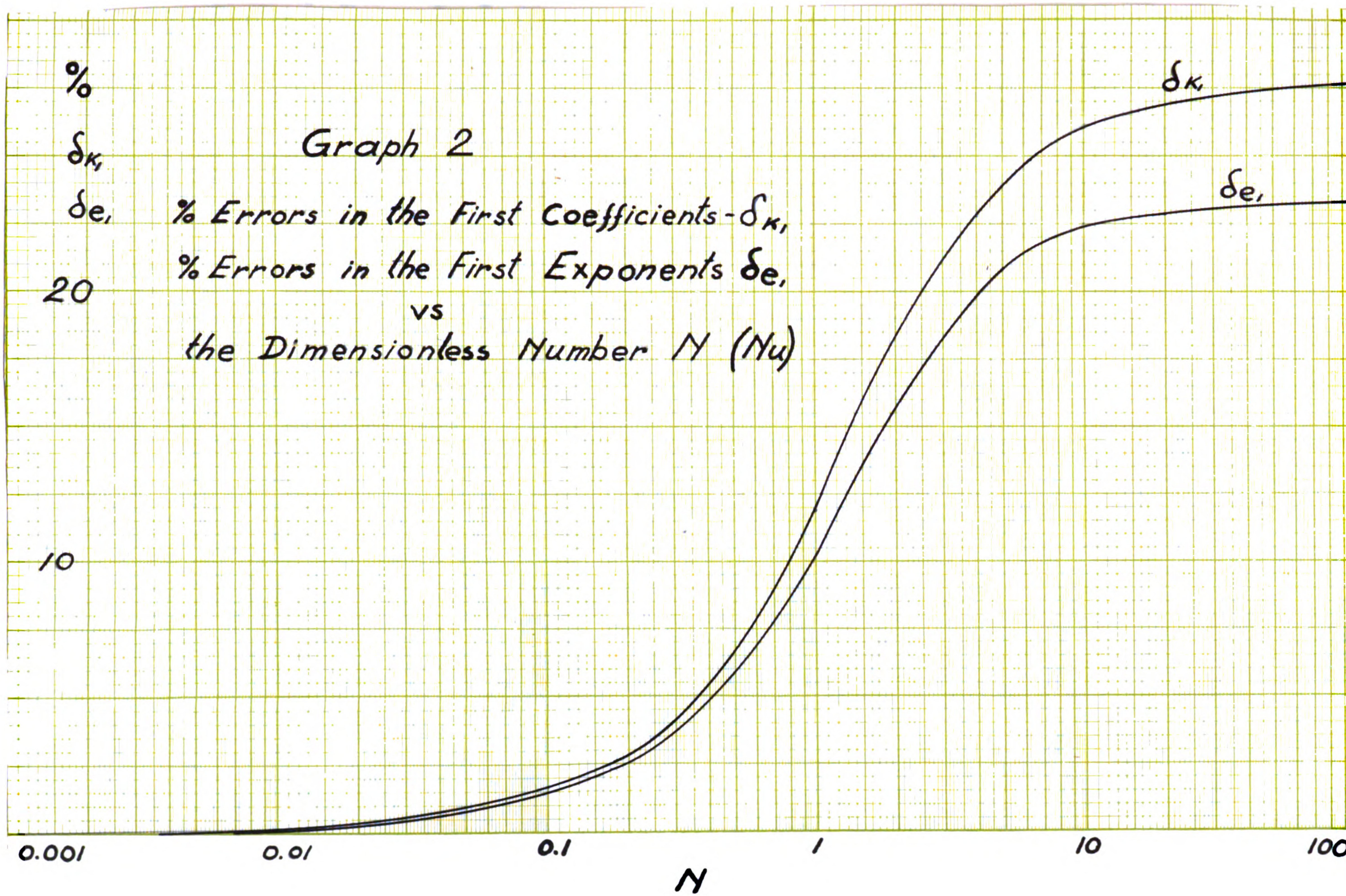
VALUES OF THE % ERRORS IN THE FIRST COEFFICIENTS

N	δ_{K_1} %	N	δ_{K_1} %
100	27.685	0.9	11.066
80	27.340	0.8	10.168
60	27.300	0.7	9.171
40	27.240	0.6	8.121
30	27.079	0.5	7.004
20	27.046	0.4	5.814
15	26.919	0.3	4.496
10	26.162	0.2	3.128
9	25.950	0.1	1.581
8	25.715	0.08	1.305
7	25.331	0.06	0.997
6	24.755	0.04	0.620
5	24.047	0.02	0.246
4	22.867	0.01	0.007
3	20.998	0.008	0.002
2	17.836	0.006	0.0014
1.5	15.387	0.004	0.001
1	11.972	0.002	0.0001
		0.001	0.0001

Table 2

% ERRORS IN THE FIRST EXPONENTS δe_1

N	$\frac{2N}{2+N}$	B_1^2	δe_1, %
100	1.96078	2.4186	23.348
30	1.87500	2.3110	23.253
15	1.76470	2.1694	22.933
10	1.66666	2.0418	22.508
5	1.42857	1.7261	20.827
2	1.00000	1.1597	15.970
1	0.66666	0.7401	9.930
0.5	0.40000	0.4269	6.970
0.2	0.18181	0.1873	3.019
0.1	0.09524	0.0968	1.650
0.06	0.05825	0.0588	0.944
0.01	0.00995	0.01	0.502
0.004	0.00399	0.004	0.250
0.001	0.00099	0.001	0.050

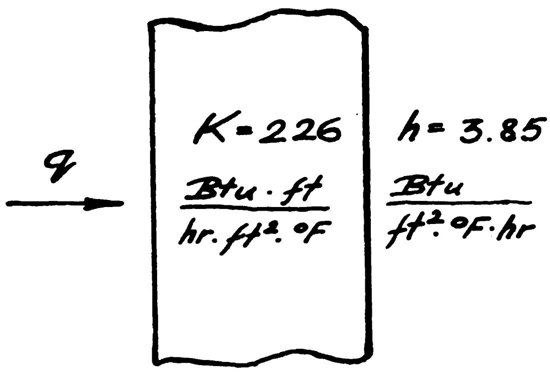


Graph 2
% Errors in the First Coefficients - δ_{K_1}
% Errors in the First Exponents δ_{e_1}
vs
the Dimensionless Number N (Nu)
%
δ_{K_1}
δ_{e_1}
20
10
0.001
0.01
0.1
1
10
100
N

EXAMPLES IN HEAT TRANSFER

In a heat conduction problem N_u would be less than 0.1 for a good conductor, with slight thickness and poor convection or radiation from the surface.

An example shall illustrate this.



Heat is suddenly applied to one side of 1" thick infinite copper plate (K = 226). The outer surface is exposed to air and has a total surface transfer coefficient h = 3.85 (an average value taken from an actual problem).

Find the dimensionless number N_u.

$$N_u = \frac{h \cdot a}{K} = \frac{3.85}{12.226} = 0.001415$$

The number N_u thus found is about six times smaller than 0.01 for which the errors were $\delta_{K_1} = 0.0001\%$ and $\delta_{e_1} = 0.5\%$. Obviously an analogue having only one lumped section can be used for this case.

The conditions are quite different, however, for insulating materials and thick walls. A one foot thick cork wall having the same surface conditions would give a value for $N_u = 154$ and could not be represented by a single lumped section.

CONDITIONS FOR REPLACEMENT BY "M" SECTIONS

General

To investigate the cases where more than one section is necessary, the logical method would be to solve the determinant for n's higher than three and to compare the coefficients and exponents. As it was indicated previously, such solutions are extremely complicated for higher ranks and would not yield manageable coefficients in general terms. It would be possible, however, to solve for all numerical values of n's and N's of interest. For a wide range of values this would be a tremendous task.

Some other properties of the transient solution of the lumped network, however, can help to solve the problem.

EQUALITY OF RESIDUES

In the transient solution of the distributed system,

$$\frac{1}{Y_{tr} \cdot R_{load}} = \frac{1}{N} - \sum_{n=1}^{\infty} A_n \epsilon^{-\beta_n^2 \frac{t}{\tau}}$$

all the coefficients A_n should add up to $\frac{1}{N}$

$$A_1 + A_2 + A_3 + \dots + A_n + \dots \to \frac{1}{N} \text{ if } n \to \infty$$

If the difference

$$\frac{N}{100}\left(\sum_{n=1}^{n=m} A_n - \frac{1}{N}\right) = \delta_{K_m}$$

is called % error in m coefficients then such a smallest number m could be found for which the error δ_{K_m} is equal or less than the allowable error.

The coefficients $A_n (n > m)$ would then have negligible effect for the same reason as in one section solution, and the distributed system could be truly expressed by a finite number (m) of coefficients.

The error δ_{K_m} can be computed for any desired m, and if it is prescribed a "necessary number of terms" can be found which would truly represent the distributed line.

A transient solution for a lumped network having m capacitors (rank of determinant = $m/2$) would have a solution of exactly the same form and with the same number of terms.

If the roots S_1, S_2, S_3 ... are arranged in increasing order $S_1 < S_2 < S_3 \dots < S_{n-1} < S_n$, the residues or coefficients would be decreasing in absolute value and periodically changing sign, if all the roots are positive (this was assumed by physical reasoning).

The steady state term for both solutions is the same and the m coefficients in both solutions should add up to the same value - the steady state term.

Since for the two solutions

$$\sum_{n=1}^{n=m} A_n = \frac{1}{N} \qquad \text{and} \quad \sum_{n=1}^{n=m} A_n \leq \frac{1}{N} + \delta_{K_m} \cdot 100$$

and $|A_1| > |A_2| > |A_3| \cdots > |A_m|$

it can be said that for each corresponding coefficient

$$A_n \text{ distributed} - A_n \text{ lumped} \leq \delta_{K_m} \times 100$$

Therefore δ_{K_m} represents the maximum error in residues for m terms.

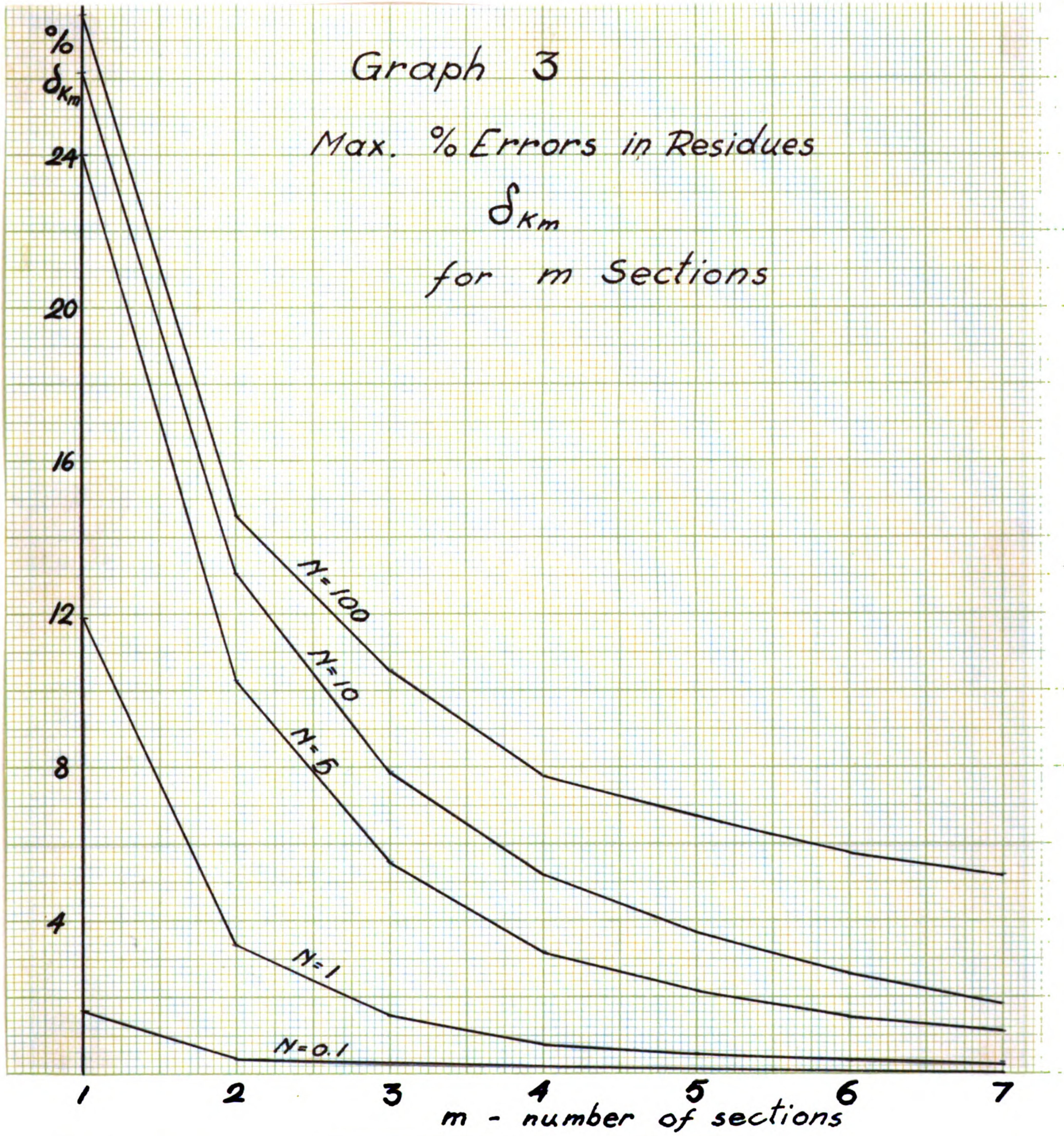
The error δ_{K_m} has been computed for several values of N and m and are given in table 3 and graph 3.

From these tables a necessary number of sections can be selected for a given maximum error δ_{K_m} and N value.

Table 3

MAXIMUM % ERRORS IN RESIDUES FOR M SECTIONS

N/m	100	10	5	1	
1	27.685	26.162	24.047	11.972	1.5919
2	14.631	13.153	10.352	3.241	0.3739
3	10.879	7.893	5.522	1.420	0.1288
4	7.800	5.193	3.242	0.748	0.0952
5	6.795	3.627	2.196	0.491	0.0312
6	5.800	2.643	1.459	0.308	0.0200
7	5.200	1.993	1.191	0.249	0.0064



Graph 3
Max. % Errors in Residues
δ_{K_m}
for m Sections
%
δ_{K_m}
24
20
16
12
8
4
N=100
N=10
N=5
N=1
N=0.1
1
2
3
4
5
6
7
m - number of sections

EQUALITY OF ROOTS

Even if the coefficients or residues can be considered equivalent for the two systems, it is not obvious that the exponents or roots would be the same.

Since the roots S_1, S_2, S_3 ... S_m in the lumped network equation can not be calculated in general terms for higher values of n and a direct comparison is not possible, an alternative method of investigation is proposed.

In the transient solution for a lumped R-C network having m different, negative and real roots, certain relations between the roots and residues must hold.

The residues for a transform:

$$\mathcal{L}^{-1} \frac{K}{s(s+\alpha_1)(s+\alpha_2)(s+\alpha_3)\cdots(s+\alpha_m)}$$

are determined as follows:

$$A_0 = \frac{K}{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_m}$$

$$A_1 = \frac{K}{\alpha_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_m)}$$

$$A_2 = \frac{K}{\alpha_2(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)\cdots(\alpha_2-\alpha_m)}$$

etc.

K can be eliminated if it is realized that in the dimensionless system used

$$A_0 = \frac{1}{N}$$

Then the first coefficient A_1 is given by

$$A_1 = \frac{\alpha_2 \alpha_3 \alpha_4 \cdots \alpha_m}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \cdots (\alpha_1 - \alpha_m)} \cdot \frac{1}{N}$$

where $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_m$ are the exponents at the coefficients $A_1, A_2, A_3 \ldots A_m$.

If this same relation would hold for the distributed system with n = m (m terms considered) to a desired accuracy, then since the coefficients have already been considered equivalent, it could be said that the exponents are equivalent too.

The error

$$\delta_{r_m} = 100 \left[\frac{A_1 - \frac{\alpha_2 \alpha_3 \cdots \alpha_m}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \cdots (\alpha_1 - \alpha_m)} \cdot \frac{1}{N}}{A_1} \right] [\%]$$

has been computed for different values of m and N and are given in table 4 and graph 4.

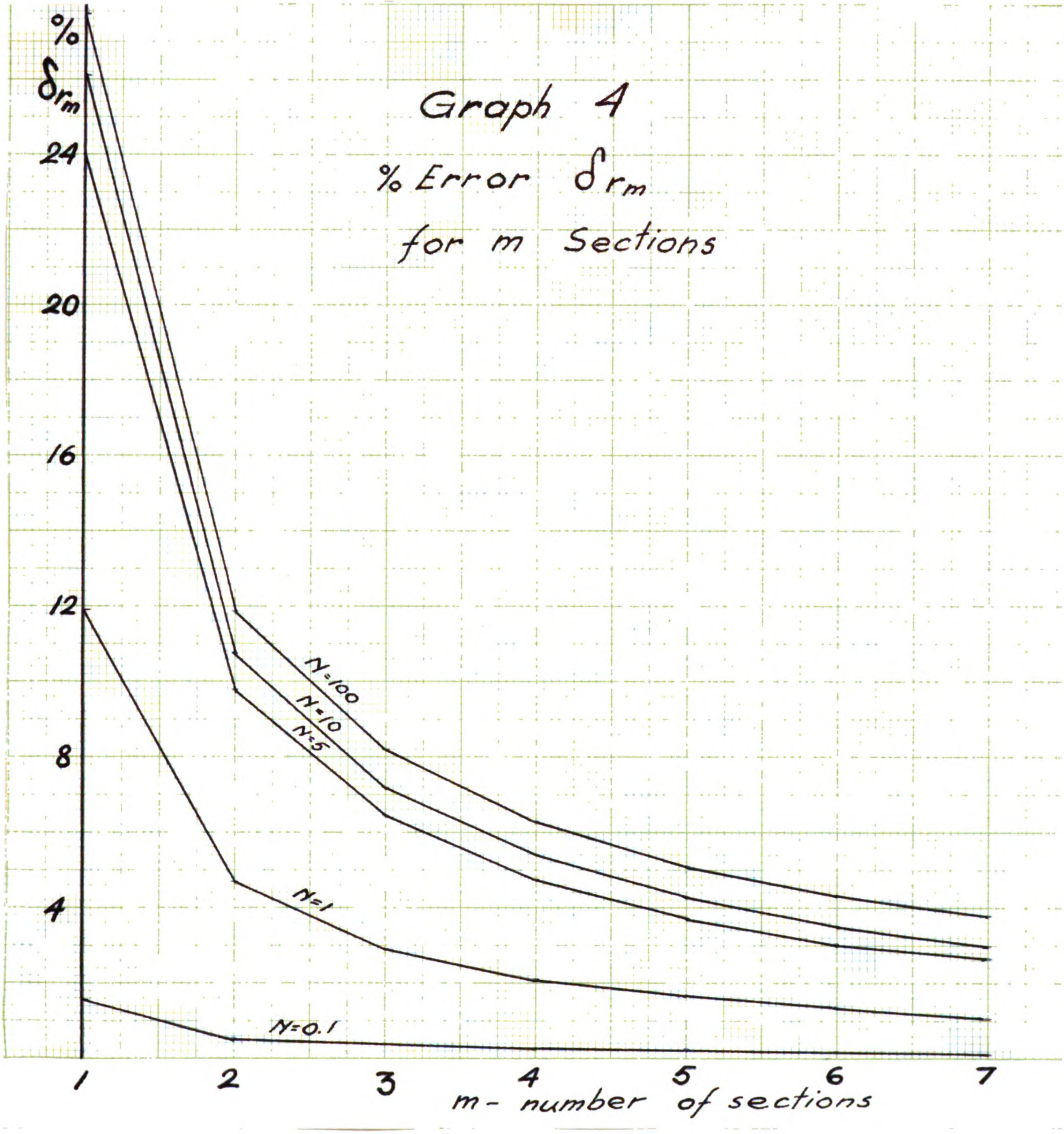
It is recognized that the two errors δ_{K_m} and δ_{r_m} are essentially the same for the same values of m and N. From the tables the necessary number of sections can be selected if the error is specified, or for a given number of sections the error determined.

For cases where $N \rightarrow 0$ or $N \rightarrow \infty$ certain simplifications can be made and an analytical relation between the number of sections and error found.

Table 4

$$\%\ \text{Error}\ \delta_{r_m} = 100\left[\frac{A_1 - \left(\dfrac{\alpha_2\alpha_3\cdots\alpha_m}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_m)}\right)^{\frac{1}{N}}}{A_1}\right]$$

N/m	100	10	5	1	0.1
1	27.685	26.162	24.047	11.972	1.581
2	11.893	10.797	9.818	4.642	.577
3	8.223	7.218	6.435	2.903	.348
4	6.311	5.386	4.755	2.111	.246
5	5.141	4.280	3.761	1.657	.189
6	4.351	3.545	3.109	1.364	.152
7	3.782	3.022	2.649	1.161	.127


Graph 4
% Error δrm
for m Sections
%
δrm
24
20
16
12
8
4
N=100
N=10
N=5
N=1
N=0.1
1
2
3
4
5
6
7
m- number of sections

SIMPLIFIED CASE WHERE $N \rightarrow 0$

For the case $N \rightarrow 0$ in the defining relation for β

$$\beta \tan\beta = N \qquad \beta \rightarrow \infty \qquad \tan\beta = \beta \; ; \sin\beta = \beta$$

$$\beta_n = \sqrt{N} + \pi(n-1) \qquad \cos\beta_n = \sqrt{1-N^2}\,(-1)^{n-1}$$

The coefficient A_n now can be expressed in terms of N only.

The first coefficient

$$A_1 = 2\frac{N + N^2}{N\left[N + N(1+N)\right]}\sqrt{1-N^2}$$

for $N \rightarrow 0$ the following is true:

$1 + N \rightarrow 1$; $N^2 \ll N$; $(1 - N^2) \rightarrow 1$

For the limiting case, the expression becomes

$$A_1 \rightarrow \frac{2N}{N(2N)} \rightarrow \frac{1}{N} \text{ as } N \rightarrow 0$$

as should be expected.

These cases $N \rightarrow 0$ are the ones where the distributed system can be represented by a single lumped section.

SIMPLIFIED CASE WHERE $N \to \infty$

For the case where $N \to \infty$ the following assumptions may be made

in $\frac{\beta}{\cot\beta} = N$; for $N \to \infty$ $\quad \cot\beta \to \left(\frac{\pi}{2} - \beta\right)$

$$\beta = \frac{(2n-1)\pi}{2} \frac{N}{1+N} \; ; \quad \text{and} \quad \cos\beta = \frac{\pi}{2} \cdot \frac{1}{1+N}$$

The first coefficient A_1 can now be written in terms of N only

$$A_1 = 2 \frac{\left[\left(\frac{\pi}{2} \frac{N}{1+N}\right) + N^2\right] \frac{\pi}{2} \frac{1}{1+N}}{\left(\frac{\pi}{2} \frac{N}{1+N}\right)^2 \left[\left(\frac{\pi}{2} \frac{N}{1+N}\right)^2 + N(1+N)\right]}$$

for $N \to \infty$ $\quad 1 + N \to N$

And for the limiting case

$A_1 \to \frac{4}{\pi} \frac{1}{N}$ as $N \to \infty$

It can be investigated now if the value of A_1 thus found for the distributed system fulfills the condition for lumped system.

$$A_1 = \frac{\alpha_2 \alpha_3 \alpha_4 \cdots \alpha_m}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \cdots (\alpha_1 - \alpha_m)} \cdot \frac{1}{N} \to \frac{4}{\pi} \frac{1}{N}$$

as $m \to \infty$

The values for $\alpha_1, \alpha_2, \alpha_3,$ are given by

$$\alpha_n = \left[\frac{(2n-1)\pi}{2} \left(\frac{N}{1+N}\right)\right]^2$$

Substituting

$$A_1 = \frac{\left(3\frac{\pi}{2} \frac{N}{1+N}\right)^2 \left(5\frac{\pi}{2} \frac{N}{1+N}\right)^2 \cdots \left(\frac{2n-1}{2}\pi \frac{N}{1+N}\right)^2}{\left[-\left(\frac{\pi}{2} \frac{N}{1+N}\right)^2 + \left(\frac{3\pi}{2} \frac{N}{1+N}\right)^2\right] \cdots \left[-\left(\frac{\pi}{2} \frac{N}{1+N}\right)^2 + \left(\frac{2n-1}{2}\pi \frac{N}{1+N}\right)^2\right]} \cdot \frac{1}{N} =$$

$$= \frac{3^2 \cdot 5^2 \cdot 7^2 \cdots (2n-1)^2}{(3^2-1^2)(5^2-1^2) \cdots \left[(2n-1)^2 - 1^2\right]} \cdot \frac{1}{N} = \frac{1}{N} \prod_{n=2}^{\infty} \frac{(2n-1)^2}{\left[(2n-1)^2 - 1^2\right]}$$

Expanding the product

$$\prod_{n=2}^{\infty} \frac{(2n-1)^2}{\left[(2n-1)^2-1^2\right]}$$

it can be shown that its value actually converges to $\frac{4}{\pi}$ as $n \to \infty$. Computing the values for different n's and calculating the error

$$\delta_{\pi_m} = \left[\frac{4}{\pi} - \prod_{n=2}^{m} \frac{(2n-1)^2}{\left[(2n-1)^2-1^2\right]}\right] \cdot \frac{\pi}{4}\, 100 \qquad [\%]$$

it could be said that δ_{π_m} represents the maximum error for the most unfavorable conditions $N \to \infty$

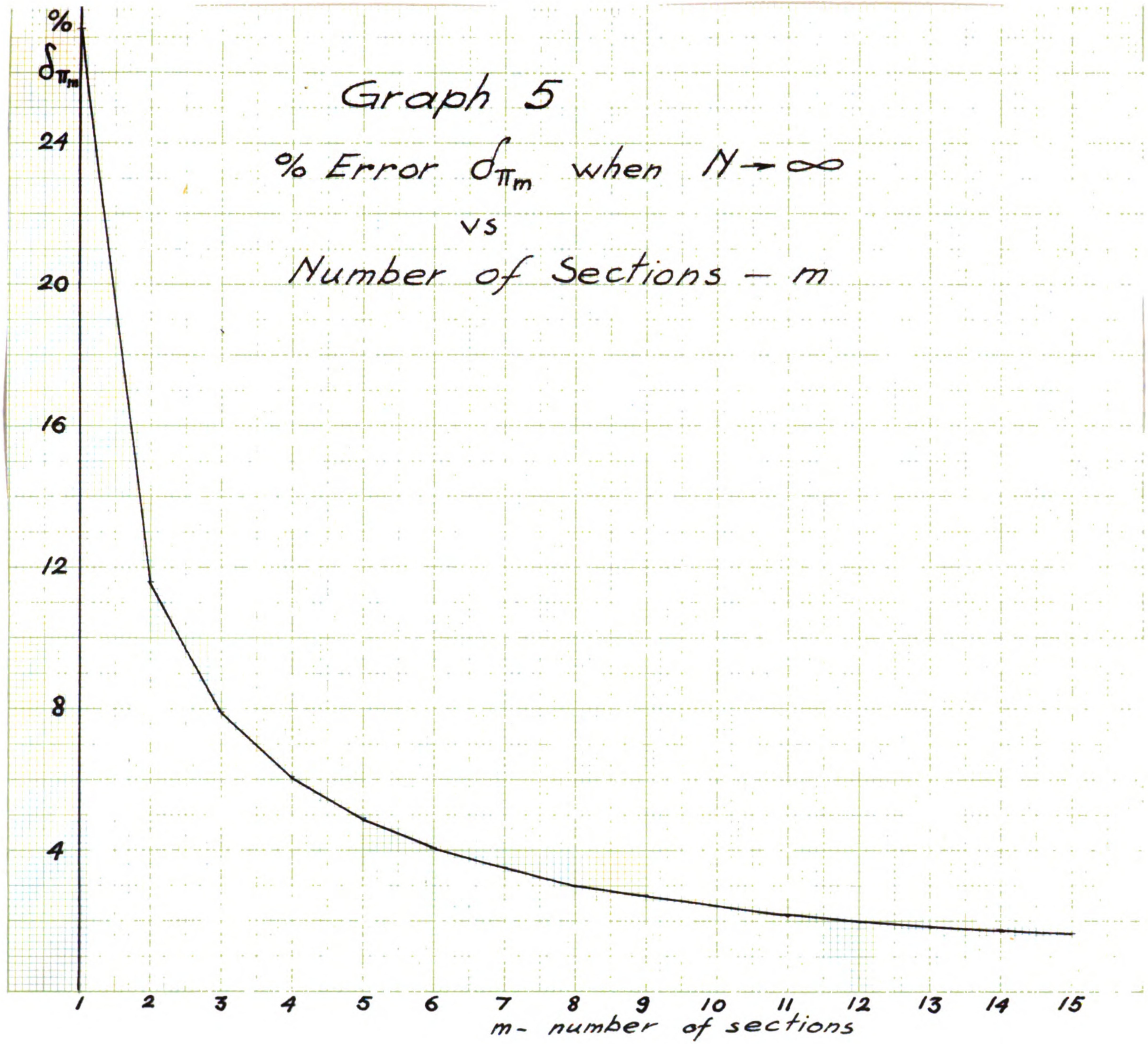
The values of δ_{π_m} have been calculated for m (2 to 15) and are given in table 5 and graph 5.

They compare quite closely with those calculated for N = 100.

Table 5

% ERROR δ_{π_m} FOR THE CASE WHERE $N \rightarrow \infty$

m	δ_{π_m} %
1	27.324
2	11.643
3	7.961
4	6.044
5	4.869
6	4.077
7	3.506
8	3.076
9	2.739
10	2.470
11	2.249
12	2.064
13	1.907
14	1.773
15	1.656


Graph 5
% Error δ_{π_m} when $N \to \infty$
vs
Number of Sections – m
%
δ_{π_m}
24
20
16
12
8
4
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
m- number of sections

CONCLUSIONS

It was shown that if the number of equal lumped sections representing a distributed line is increased to infinity, the lumped line becomes the distributed line itself. This could be assumed on a heuristic basis from the beginning.

It was also shown that if after adding certain numbers of terms in the distributed line equation, the remaining terms have negligible effect upon the total result, the equation for the distributed line with a finite number of terms is equal in form and numerical values to an equation for a lumped line with the same number of terms.

Thus, if the number of significant terms in the distributed line equation is found, it can be said that the distributed line can be truly represented by such a lumped system that would have the same number of terms in its characteristic equation.

In the particular case considered, it means that the number of capacitors to be used is determined by the number of significant terms in the distributed line equation.

The number of significant terms in turn depends upon the value of N. The calculation for the error for any number of terms and different values of N is quite simple if $N \to 0$ or $N \to \infty$

It is more elaborate for intermediate values of N, but once it is done and tables or graphs prepared, they can be applied to any case for the given configuration equally well in thermal and electrical systems since all calculations are done in a dimensionless form.

Something should be said about the time constant τ of the system. It may appear that it should be a significant factor in determining the error.

It is recognized, however, that the maximum error would occur when $t \rightarrow 0$ and as t increases the error would decrease reaching zero at steady state.

Now if the response curve is replotted from a dimensionless abscissa $\frac{t}{\tau}$ in a time abscissa it is obvious that different values of τ would only stretch or contract the curve but would not change its general shape.

Therefore the value of τ would determine the absolute time for which the greatest error would exist but would not have any effect on its value.

In some cases, indeed, the error would last for such a short time that it would not even be possible to measure it.

In a thermal system, that would be the case for good conductors having high thermal diffusivity such as silver or copper.

The parameters K, C, and h in the thermal system have been assumed to be constant to permit an analysis. In practice, however, they vary with temperature.

In a control problem, usually the variation of temperature around the reference point small since it is the purpose of the controller to keep the temperature constant and the parameters can be assumed constant.

If the change of temperature is large and the variation of constants not negligible, an electrical analogue with non-linear parameters can be used as it was done by Tribus (26).

BIBLIOGRAPHY

1. Ahrdent, W.R., J.F. Taplin
"Automatic Feedback Control", 1951.
New York: McGraw-Hill

2. Beuken, C.L.
"Economisch Technisch Tijdschrift", 17.
January, 1937

3. Brown, G.H., C.N. Hoyler, R.A. Bierwirth
"Theory and Application of Radio-Frequency Heating", 1947 .
New York: D. VanNostrand

4. Brown, G.S., D.P. Campbell
"Principles of Servomechanisms", 1948
New York: John Wiley and Sons

5. Carslaw, H.S., J.C. Jaeger
"Conduction of Heat in Solids", 1948
Oxford: The Clarendon Press

6. Carson, J.R.
"Electric Circuit Theory and the Operational Calculus", 1926
New York: McGraw-Hill

7. Chestnut, H., R.W. Mayer
"Servomechanisms and Regulating System Design", Vol. 1, 1951
New York; John Wiley and Sons

8. Cohen, L.
"Heaviside's Electrical Circuit Theory", 1928
New York: McGraw-Hill

9. Eckman, D.P.
"Principles of Industrial Process Control", 1945
New York, John Wiley and Sons

10. Eckman, D.P.
"Electrical-Analogy Method for Fundamental Investigations in Automatic Control"
Transaction, American Society of Mechanical Engineers, Vol. 67, 1945

11. Emmons, H.W.
"The Numerical Solution of Heat-Conduction Problems"
Transactions, American Society of Mechanical Engineers, Vol. 65, 1943

12. Eyres, N.R., D.R. Hartee, J. Ingham
"The Calculation of Variable Heat Flow in Solids"
Transactions, Royal Society of London, Vol. 240, 1946

13. Frame, J.S.
Oral communication, Michigan State College

14. Gardner, M.F., J.L. Barnes
"Transients in Linear Systems", Vol. 1, 1951.
New York: John Wiley & Sons

15. Goldman, S.
"Transformation Calculus and Electrical Transients", 1950
New York: Prentice-Hall

16. Guillemin, E.A.
"Communication Networks, Vol. II", 1949
New York: John Wiley & Sons

17. Hornfeck, A.J.
"Response Characteristics of Thermometer Elements"
Transactions, American Society of Mechanical Engineers, Vol. 71, 1949

18. McCann, G.D., H.E. Criner
"Solving Complex Problems by Electrical Analogy"
Machine Design, Vol. 18, 1945

19. Newman, A.B., L.Green
"The Temperature History and Rate of Heat Loss of an Electrically Heated Slab"
Transactions, Electrochemical Society, Vol. 66, 1934

20. Paschkis, V., H.D. Baker
"A Method for Determining Unsteady-State Heat Transfer by Means of an Electrical Analogy"
Transactions, American Society of Mechanical Engineers, Vol. 64, 1942

21. Paschkis, V.
"Application of an Electrical Model to the Study of Two Dimensional Heat Flow"
Transactions, American Institute of Chemical Engineers, Vol. 38, 1942

22. Paschkis, V.
"Electrical Analogy Method for the Investigation of Transient Heat Flow Problems"
Industrial Heating, 9, 1942

23. Schack, A., translated from the German by H. Goldschmidt, E.P. Partridge
"Industrial Heat Transfer", 1933
New York: John Wiley and Sons

24. Schmidt, E.
"Foeppl's Festschrift", 1932
Berlin: Springer

25. Smith, E.S.
"Automatic Control Engineering", 1944
New York: McGraw-Hill

26. Tribus, M.
"Intermittent Heating for Aircraft Ice Protection with Application to Propellers and Jet Engines"
Transactions, American Society of Mechanical Engineers, Vol. 73, 1951