ELEMENTARY CALCULUS FROM THE STANDPOINT OF ADVANCED CALCULUS

Thesis for the Degree of M. S.

MICHIGAN STATE COLLEGE

William P. Fuller

1939

ELEMENTARY CALCULUS FROM THE STANDPOINT OF ADVANCED CALCULUS

by

William Paul Fuller

A THESIS

Submitted to the Graduate School of Michigan State
College of Agriculture and Applied Science
in partial fulfilment of the requirements
for the degree of

MASTER OF SCIENCE

Department of Mathematics

1939

ACKNOWLEDGMENT

To J. E. Powell whose help and suggestions are responsible for this thesis .

TABLE OF CONTENTS

INTRODUCTION

The writers of most elementary calculus books find that they need many theorems whose proofs, they feel, should be omitted since the average student would not appreciate or understand such proofs. Some writers, in their effort to give simple proofs of these theorems, make statements that are false. Occasionally they fail to acknowledge the necessity of certain assumptions. The writer of the advanced calculus book feels that these omitted proofs are either too elementary or else not a true part of his work and therefore he also omits them. In this way we find omitted from elementary and advanced calculus the proofs of many necessary theorems. For example, we find that the subject of continuity is mentioned in elementary calculus and discussed at some length in advanced calculus, but in neither case do we find any proof of the continuity of the elementary functions. Again, the derivations of the formulas for derivatives are based on the assumption that the derivative exists.

The purpose of this paper shall be to state and prove those theorems that are important to the elementary calculus and are not proven in most elementary or advanced calculus texts. In doing this we shall assume those theorems of real variable that are ordinarily assumed in advanced calculus. An effort will be made to point out some of the errors that are made in some of the elementary calculus books. Also, numerous examples will be given to illustrate other errors commonly made. No effort will be made to include proofs or definitions generally given correctly in the elementary or advanced calculus book.

CHAPTER I

LIMITS AND CONTINUITY

1.1 Limits. Since the idea of the limit is basic to the study of calculus it is important that its presentation be made with care. We define the limit of $f(x)$ at $x=a$ as a number A such that if ϵ is any positive number we can find a number η such that if $|x - a| < \eta$ and $x \neq a$ then it follows that $|f(x)-A| < \epsilon$. We write this as

$$(1) \qquad \lim_{x=a} f(x) = A$$

The definition of the limit of a function of n variables is very similar to that for one variable. We define the limit of $f(x_1 \ldots\ldots x_n)$ at the point $x_1 = a_1$, $x_2 = a_2$, ..., $x_n = a_n$, as a number A such that if ϵ is any positive number, we can find a number η such that if $|x_1 - a_1| < \eta$, $|x_2 - a_2| < \eta$, ...$|x_n - a_n| < \eta$ and the point $(x_1 \ldots\ldots x_n) \neq (a_1 \ldots\ldots a_n)$ then we have $|f(x_1 \ldots x_n) - A| < \epsilon$.

If in definition (1) we interpret x as $(x_1 \ldots x_n)$, a as $(a_1 \ldots a_n)$ and $|x - a|$ as the maximum of $|x_i - a_i|$ for $i = 1,2,\ldots\ldots n$, then definition (1) becomes the definition of the limit for n-variables.

Likewise we define the limit of $f(x)$ at $x=\infty$ as a number A such that if ϵ is any positive number, we can find a number η such that if $|x| > \eta$ then it follows that $|f(x) - A| < \epsilon$. We write this as

$$\text{(2)} \qquad \operatorname*{Lim}_{x=\infty} f(x) = A$$

We define the limit of $f(x_1 \ldots x_n)$ at $x_1=\infty, x_2=\infty, \ldots, x_k=\infty, x_{k+1}=a_1, x_{k+2}=a_2, \ldots, x_n=a_{n-k}$, as a number A such that if ϵ is any positive number we can find a number η such that if $|x_1| > \eta$, $|x_2| > \eta$, $\ldots |x_k| > \eta$ and $|x_{k+1} - a_1| < \frac{1}{\eta}$, $|x_{k+2} - a_2| < \frac{1}{\eta}$, $\ldots |x_n - a_{n-k}| < \frac{1}{\eta}$ then it follows that $|f(x_1 \ldots x_n) - A| < \epsilon$. In this definition k may have any value from 1 to n.

In these definitions we have spoken of the limit at $x = a$ or at $x = \infty$ and used the notation $\operatorname*{Lim}_{x=a} f(x)$ or $\operatorname*{Lim}_{x=\infty} f(x)$ instead of the usual notation $\operatorname*{Lim}_{x \to a} f(x)$ or $\operatorname*{Lim}_{x \to \infty} f(x)$. The latter suggests the question of whether x ever reaches a and if so when. The introduction of the element of time in the definition of a limit is, it seems, objectionable and unnecessary.

From these definitions it does not follow that a number A, and therefore the limit, will always exist.

For example if we let

$$f(x) = \sin\frac{1}{x} \quad \text{when } x \neq 0$$
$$= 0 \quad \text{when } x = 0$$

then $\lim_{x=0} f(x)$ does not exist.

The limit of a sequence, $\{a_n\}$, is included in the definition of $\lim_{x=\infty} f(x)$. To see this let $f(n) = a_n$. Then $\lim_{n=\infty} f(x)$ would give the usual limit of the sequence.

We have noticed that most authors of elementary calculus books start with a definition of a limit of a variable. It is difficult to say just what they mean, if anything. Some of their examples indicate that they are concerned with the limit of a sequence. Again it seems that they are talking about the limit point of a set of points. On page 6 of Love's Calculus we find the definition: " When the successive values of a variable x approach nearer and nearer a fixed number (a), in such a way that the difference $a - x$ becomes and remains numerically less than any preassigned positive number however small, the constant (a) is called the limit of x."

If the time element is removed from this definition there is nothing left. As an example he gives the sequence .9, .99, .999, ... as having the limit 1. Further he states that no term of this sequence will ever equal 1. This statement is true. However we can not say the limit is never reached no matter how long the process continues. If this sequence represents distance covered, and if it takes one minute to go .9 of the distance, $1\frac{1}{2}$ minutes to go .99 of the distance, $1\frac{3}{4}$ minutes to go .999 of the distance, ..., then it is apparent that the limit is reached after 2 minutes. However, if it takes 1 minute to go .9, 2 minutes to go .99, 3 minutes to go .999,....., then obviously the limit is never reached.

On page 9 in Slobin and Solt's book in calculus we find the definition: " A constant L is said to be the limit of a variable x, if the variable changes in such a way that the difference L-x, in absolute value, becomes and remains less than any preassigned positive quantity,however small ". Again the time element is the only part of the definition that gives it any meaning.

In the calculus book by Granville, Smith, and Longley we find a very similar definition on page 11 and a similar criticism can be made. Likewise in Dalaker and Hartig's book on page 4.

These authors would have done much better if they had used instead a definition such as Neelley and Tracy gives on page 78 of their book in elementary calculus.

1.2 Theorems on Limits. Having defined the limit of a function we next prove some well known theorems on limits. These theorems are usually stated in elementary calculus books without proof. Most advanced calculus texts fail also to prove them. The first theorem is

Theorem 1.1 If $\lim_{x=a} f(x) = A$ and $\lim_{x=a} F(x) = B$, then we have $\lim_{x=a} [f(x)+F(x)] = A+B$ *.

Since $\lim_{x=a} f(x) = A$, we have by definition (1) that for $\epsilon_1 > 0$ it follows that there exists an η_1 such that if $|x - a| < \eta_1$, then $|f(x) - A| < \epsilon_1$. Likewise for $\epsilon_2 > 0$ it follows that there exists an η_2 such that if $|x - a| < \eta_2$ then $|F(x) - A| < \epsilon_2$. For any $\epsilon > 0$ suppose we choose $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$ and let η be the smaller of the η_1 and η_2. Then we have for $|x - a| < \eta$ that

$$\left|[f(x)+F(x)] - [A+B]\right| \leq |f(x)-A| + |F(x)-B| < \epsilon_1 + \epsilon_2 = \epsilon \quad .$$

---

* Theorems 1.1, 1.2, 1.3, 1.4 are true for $a=\infty$ if we make one major change, namely, in selecting an η we will, for $a=\infty$, pick the largest of the η_1, η_2, η_3 instead of the smallest.

Stated as a limit this statement becomes

$$\lim_{x=a} \left[f(x) + F(x) \right] = A + B \ .$$

This proves the theorem.

<u>Theorem 1.2</u> <u>If</u> $\lim_{x=a} f(x) = A$ <u>and</u> $\lim_{x=a} F(x) = B$, <u>then</u> <u>we</u> <u>have</u> $\lim_{x=a} \left[f(x) - F(x) \right] = A - B$.

The proof is similar to the proof of theorem 1.1 except that we have

$$\left| \left[f(x) - F(x) \right] - (A - B) \right| \leq \left| f(x) - A \right| + \left| - F(x) + B \right| < \epsilon_1 + \epsilon_2 = \epsilon .$$

The theorem for a product of functions is

<u>Theorem 1.3</u> <u>If</u> $\lim_{x=a} f(x) = A$ <u>and</u> $\lim_{x=a} F(x) = B$, <u>then</u> <u>we</u> <u>have</u> $\lim_{x=a} \left[f(x) \cdot F(x) \right] = A \cdot B$.

Since $\lim_{x=a} f(x) = A$, we have by definition that for $\epsilon_1 > 0$ it follows that there exist an η_1 such that if $|x - a| < \eta_1$, then $|f(x) - A| < \epsilon_1$. Likewise for $\epsilon_2 > 0$ it follows that there exist an η_2 such that if $|x - a| < \eta_2$ then $|F(x) - B| < \epsilon_2$.

Also, since $\lim_{x=a} F(x) = B$ it follows that there exists a number $C > |B|$ such that $|F(x)| < C$ for $|x-a| < \eta_3$. This is true since given $\epsilon > 0$ it follows that there exists an η_3 such that if $|x - a| < \eta_3$ it follows that $|F(x) - B| < \epsilon$. Therefore $B - \epsilon < F(x) < B + \epsilon$. Since ϵ is arbitrarily small we can take it so small that $F(x) < |B| + \epsilon < C$. For any $\epsilon > 0$ suppose we choose $\epsilon_1 = \frac{\epsilon}{2C}$ and $\epsilon_2 = \frac{\epsilon}{2|A|}$ and let η be the smallest of the η_1, η_2, η_3.

Then we have for $|x - a| < \eta$ that

$$\left[f(x)\cdot F(x)\right] - AB = f(x)\cdot F(x) - F(x)\cdot A + F(x)\cdot A - AB =$$
$$= A\left[F(x) - B\right] + F(x)\left[f(x) - A\right],$$

and therefore

$$|f(x)\cdot F(x) - AB| \leq |A|\,|F(x) - B| + |F(x)|\,|f(x) - A| .$$

Now by substituting as indicated above,

$$|f(x)\cdot F(x) - AB| \leq |A|\cdot\frac{\epsilon}{2|A|} + C\cdot\frac{\epsilon}{2C} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon .$$

Stated as a limit we have

$$\lim_{x=a}\left[f(x)\cdot F(x)\right] = AB .$$

This proves our theorem.

The theorem for a quotient of functions is

Theorem 1.4 If $\lim_{x=a} f(x) = A$ and $\lim_{x=a} F(x) = B$, then we have $\lim_{x=a} \frac{f(x)}{F(x)} = \frac{A}{B}$ if $B \neq 0$.

Again since $\lim_{x=a} f(x) = A$ we have that for $\epsilon_1 > 0$ it follows that there exists an η_1 such that if $|x-a| < \eta_1$ then it follows that $|f(x) - A| < \epsilon_1$. Likewise, for $\epsilon_2 > 0$ it follows that there exists an η_2 such that for $|x-a| < \eta_2$ then it follows that $|F(x) - B| < \epsilon_2$. Also, since $\lim_{x=a} F(x) = B \neq 0$ it follows that there exists a number $0 < C < |B|$ such that $|F(x)| > C$ for $|x - a| < \eta_3$. This statement can be verified by referring to the corresponding statement of Theorem 1.3. This number C can be found by considering that since $B \neq 0$ then $|B| > 0$ and so there exists a C such that $0 < C < |B|$.

Now for any $\epsilon > 0$ suppose we choose $\epsilon_1 = \frac{\epsilon C}{2}$ and $\epsilon_2 = \frac{\epsilon |B| C}{2|A|}$ and let η be the smallest of the three quantities η_1, η_2, η_3.

Then we have for $|x - a| < \eta$ that

$$\frac{f(x)}{F(x)} - \frac{A}{B} = \frac{B \cdot F(x) - A \cdot F(x)}{B \cdot F(x)} = \frac{[B \cdot f(x) - AB] + [AB - A \cdot F(x)]}{B \cdot F(x)}$$

$$= \frac{A[B - F(x)]}{B \cdot F(x)} + \frac{f(x) - A}{F(x)}$$

and therefore,

$$\left|\frac{f(x)}{F(x)} - \frac{A}{B}\right| \leq \frac{|A|}{|B|\cdot|F(x)|}\left|F(x) - B\right| + \frac{|f(x) - A|}{|F(x)|}$$

$$< \frac{|A|}{|B|\cdot C}\left|F(x) - B\right| + \frac{|f(x) - A|}{C} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon .$$

This proves that

$$\lim_{x=a}\left[\frac{f(x)}{F(x)}\right] = \frac{A}{B} .$$

1.3 <u>Definition of Continuous and Discontinuous Functions</u>.

We next make use of the idea of a limit and the theorems on limits to define continuity. Definition: A function, f(x), is said to be continuous at a point x = a if $\lim_{x=a} f(x) = f(a)$. If f(x) is defined at x = a then we say f(x) is discontinuous at x = a if it is not continuous at x = a . However, if f(x) is not defined at x = a then we say f(x) is discontinuous at that point if $\lim_{x=a} f(x)$ does not exist. If the $\lim_{x=a} f(x)$ does exist we will not say whether the function f(x) is continuous or discontinuous.

For example consider $f(x) = \frac{1}{x}$ at x = 0. Here f(x) is not defined at x = 0 and the $\lim_{x=a} f(x)$ does not exist. Therefore, we say that f(x) is discontinuous at x = a.

However, consider $f(x) = \frac{\sin x}{x}$ at $x = 0$. Again $f(x)$ is not defined at $x = 0$ but the $\lim_{x=a} f(x)$ exists and equals one. Therefore, we do not say whether $f(x)$ is continuous or discontinuous. If $f(x)$ is defined to be one at $x = 0$ then $f(x)$ is continuous.

Many real variable and advanced calculus books are not careful to point out whether a function is discontinuous if it is undefined at the point in question. We believe that the above definition is a convenient and logical one.

1.4 *Continuity of the Elementary Functions*.

In order for us to discuss continuity it will be convenient for us to prove several lemmas.

Lemma 1.1 *If* $\lim \left[f(x) \pm h(x)\right] = A$ *and if* $\lim f(x) = B$, *then the* lim $h(x)$ *exists and equals* $A \mp B$.

Lemma 1.2 *If* $\lim \left[f(x) \cdot h(x)\right] = A$ *and if* $\lim f(x) = B \neq 0$, *then the* lim $h(x)$ *exists and equals* $\frac{A}{B}$.

Lemma 1.3 *If* $\lim \frac{f(x)}{h(x)} = A \neq 0$ *and if* $\lim f(x) = B$, *then the* lim $h(x)$ *exists and equals* $\frac{B}{A}$.

We have these three lemmas immediately from theorems 1.1 to 1.4 of section 2.

<u>Lemma 1.4</u> <u>If</u> $f(x) \leqslant h(x) \leqslant g(x)$ <u>and</u> <u>if</u> <u>the</u> $\lim f(x) = \lim g(x) = A$, <u>then</u> <u>the</u> $\lim h(x) = A$.

For subtracting $f(x)$ we get

(3) $$0 \leqslant h(x) - f(x) \leqslant g(x) - f(x) \,.$$

Since

$$\text{Lim}\left[g(x) - f(x)\right] = \lim g(x) - \lim f(x)$$
$$= A - A = 0$$

we have that for any $\epsilon > 0$ it follows that there exists an η such that if $|x - a| < \eta$ then it follows that $|g(x) - f(x)| < \epsilon$. By (3), then, $|h(x) - f(x)| < \epsilon$ also, which is sufficient to show that $\lim \left[h(x) - f(x)\right] = 0$. Since $\lim f(x) = A$ then by lemma 1.1 we have

$$\text{Lim}\ h(x) = A \,.$$

<u>Lemma 1.5</u> <u>If</u> $f(x) > 0$ <u>and</u> <u>if</u> <u>the</u> $\lim f(x) = 1$ <u>and</u> <u>if</u> <u>q</u> <u>is</u> <u>any</u> <u>rational</u> <u>number</u>, <u>then</u> $\lim \left[f(x)\right]^q = 1$.

For let $q = \frac{r}{s}$ where r and s are positive integers. If $f(x) < 1$ then,

$$f(x) < \left[f(x)\right]^{\frac{1}{s}} < 1.$$

Also if $f(x) > 1$ then,

$$1 < \left[f(x)\right]^{\frac{1}{s}} < f(x) \,.$$

In either case $\left[f(x)\right]^{\frac{1}{s}}$ lies between 1 and $f(x)$.

By lemma 1.4 we have,

$$\operatorname{Lim} \left[f(x)\right]^{\frac{1}{s}} = 1 \quad .$$

Since $\left[f(x)\right]^{\frac{r}{s}} = \left\{\left[f(x)\right]^{\frac{1}{s}}\right\}^{r}$ then by theorem 1.3 of section 2 we have,

$$\operatorname{Lim} \left[f(x)\right]^{\frac{r}{s}} = \operatorname{Lim} \left[f(x)\right]^{q} = 1 \ .$$

<u>Lemma 1.6</u> <u>If</u> <u>the</u> lim $f(x) = A > 0$ <u>then</u> <u>the</u> lim $\left[f(x)\right]^{q} = A^{q}$.

Let us write $\left[f(x)\right]^{q}$ as $A^{q}\left[\frac{f(x)}{A}\right]^{q}$. Now by theorem 1.4 of section 2 we have,

$$\operatorname{Lim} \frac{f(x)}{A} = 1 \ .$$

Therefore, by lemma 1.5,

$$\operatorname{Lim} A^{q}\left[\frac{f(x)}{A}\right]^{q} = \operatorname{Lim} A^{q} = A^{q} \ .$$

<u>Lemma 1.7</u> <u>If</u> $f(x)$ <u>and</u> $g(x)$ <u>are</u> <u>continuous</u> <u>at</u> $x = a$, <u>then</u> $f(x) \pm g(x)$, $f(x)\cdot g(x)$, $\frac{f(x)}{g(x)}$, <u>if</u> $g(x) \neq 0$, <u>are</u> <u>continuous</u> <u>at</u> $x = a$.

Since $f(x)$ and $g(x)$ are continuous then by the definition of continuous functions we have the $\lim\limits_{x=a} f(x) = f(a)$ and the $\lim\limits_{x=a} g(x) = g(a)$.

Now by theorems 1.1 and 1.2 of section 2 we have,

$$\lim_{x=a} \left[f(x) \pm g(x)\right] = f(a) \pm g(a) .$$

In a similar manner $f(x) \cdot g(x)$ and $\frac{f(x)}{g(x)}$ where $g(x) \neq 0$ can be shown to be continuous at $x = a$.

<u>Lemma 1.8</u> <u>If the</u> $\lim_{x=a} F(x) = A$, $\lim_{F(x)=A} f\left[F(x)\right] = B$, <u>and</u> $F(x) \neq A$ <u>in the neighborhood of</u> (a), <u>then</u> $\lim_{x=a} f\left[F(x)\right] = B$.

Suppose a, A, B are finite. Since $\lim_{F(x)=A} f\left[F(x)\right] = B$ then we have that for $\epsilon_1 > 0$ it follows that there exists an $\eta_1 > 0$ such that if $\left|F(x) - A\right| < \eta_1$ then it follows that $\left| f\left[F(x)\right] - B\right| < \epsilon_1$. Also, since $\lim_{x=a} F(x) = A$, then we have that for $\eta_1 > 0$ it follows that there exists an $\eta_2 > 0$ such that if $\left|x - a\right| < \eta_2$ then it follows that $\left|F(x) - A\right| < \eta_1$. Now by combining these statements we have that for $\epsilon_1 > 0$ it follows that there exists an $\eta_2 > 0$ such that for $\left|x - a\right| < \eta_2$ it follows that $\left|f\left[F(x)\right] - B\right| < \epsilon_1$. Written as a limit this becomes

$$\lim_{x=a} f\left[F(x)\right] = B .$$

Suppose a is infinite; A,B are finite. Since $\lim_{F(x)=A} f[F(x)] = B$ then we have that $\epsilon_1 > 0$ it follows that there exists an $\eta_1 > 0$ such that if $|F(x) - A| < \eta_1$ then it follows that $|f[F(x)] - B| < \epsilon_1$. Also, since $\lim_{x=\infty} F(x) = A$, then we have that for $\eta_1 > 0$ it follows that there exists an $\eta_2 > 0$ such that if $|x| > \eta_2$ then it follows that $|F(x) - A| < \eta_1$. Now by combining these two statements we have that for $\epsilon_1 > 0$ it follows that there exists an $\eta_2 > 0$ such that for $|x| > \eta_2$ it follows that $|f[F(x)] - B| < \epsilon_1$. Written as a limit this becomes

$$\lim_{x=\infty} f[F(x)] = B \quad .$$

<u>Lemma 1.9</u> <u>If</u> $u_1 = F_1(x_1 \ldots\ldots x_m)$, $u_2 = F_2(x_1 \ldots x_m)$, $\ldots\ldots u_m = F_m(x_1 \ldots x_m)$, $\lim_{x=a} u_1 = b_1$, $\lim_{x=a} u_2 = b_2$, $\ldots\ldots \lim_{x=a} u_m = b_m$, $y = f(u_1 \ldots\ldots\ldots u_m)$, $\lim_{u=b} y = A$, <u>and</u> <u>if</u> $u_i \neq b_i$ <u>in</u> <u>the</u> <u>neighborhood</u> <u>of</u> (a), <u>then</u> $\lim_{x=a} y = A$.

If the interpretation suggested on page three be applied to the proof of lemma 1.8 then we have the proof for this lemma.

Lemma 1.10 . If $u_1=F_1(x_1 \ldots x_m)$, $u_2=F_2(x_1 \ldots x_m)$, ... $u_m=F_m(x_1 \ldots x_m)$, are continuous at $x_1=a_1$, $x_2=a_2$,....,$x_m=a_m$, and if $u_1=b_1$, $u_2=b_2$,...., $u_m=b_m$, at $x_i=a_i$ and $y=f(u_1 \ldots u_m)$ is continuous at $u_1=b_1$, $u_2=b_2$,....,$u_m=b_m$, then y considered as a function of the x's is continuous at $x_1=a_1$, $x_2=a_2$, ..., $x_m=a_m$.

For u_m continuous at $x=a$ means that

$$\lim_{x=a} F_m(x_1 \ldots x_m) = F_m(a_1 \ldots a_m)$$

and y continuous at $u=b$ means that $\lim_{u=b} f(u_1 \ldots u_m)=f(b_1 \ldots b_m)$. Then by lemma 1.9 we have

$$\lim_{x=a} f\left[F_1(x_1 \ldots x_m) \ldots F_m(x_1 \ldots x_m)\right]=f\left[F_1(a_1 \ldots a_m) \ldots F_m(a_1 \ldots a_m)\right].$$

Therefore y is continuous when considered as a function of x at the point $x=a$.

This lemma states that a continuous function of a continuous function is continuous .

Lemma 1.11 . The function $F(x)=x$ is everywhere continuous .

Since $x=F(x)$ then $a=F(a)$ and therefore for any $\epsilon>0$ there will always exist an η such that if $|x - a|<\eta$ then it follows that $|F(x) - F(a)|<\epsilon$. An η less than or equal to ϵ will satisfy the above condition .

Theorem 1.5 The rational integral functions are everywhere continuous.

For let $f(x) = x^n$ where n is any positive integer. Then by lemmas 1.6 and 1.11, f(x) is continuous since $f(x) = [F(x)]^n = x^n$. Now by lemma 1.7 $a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1} x + a_n$ is everywhere continuous. This completes the proof for one variable.

For the case of several variables let $y = \sum A x_1^{s_1} x_2^{s_2} \ldots x_m^{s_m}$ where the s_i, $i = 1, \cdots m$, are positive integers or zero. Now let any term of y be designated by $t = A x_1^{s_1} x_2^{s_2} \ldots\ldots x_m^{s_m}$. We wish to show that t is continuous at an arbitrary point $x_1 \ldots x_m$. Let $u_1 = x_1^{s_1}$, $u_2 = x_2^{s_2}$, ... $u_m = x_m^{s_m}$; then $t = u_1 u_2 \ldots u_m$ is a continuous function of the $u_1 \ldots\ldots u_m$ by lemma 1.7. Each u_i is, however, a continuous function of the x_1, ... x_m. Therefore, t, considered as a function of the x_1, ... x_m is continuous by lemma 1.10. Therefore, y, which is the sum of terms of type t, is also continuous by lemma 1.7.

Theorem 1.6 The rational functions are continuous wherever they are defined.

The rational function, $y = \dfrac{\sum A x_1^{r_1} \ldots\ldots x_m^{r_m}}{\sum B x_1^{s_1} \ldots\ldots x_m^{s_m}} = \dfrac{F}{G}$, is defined everywhere except where the denominator is zero.

These undefined points are called the zeros of the denominator, or the poles of y.

Since, by theorem 1.6, F and Q are everywhere continuous, then, by lemma 1.7, y is everywhere continuous except at the poles of y.

<u>Lemma 1.12</u> <u>If</u> $\lim_{x=a} f(x)=\ell$, <u>and</u> $x = a+bu$ <u>where</u> $b \neq 0$, <u>then</u> $\lim_{u=0} f(x) = \ell$.

For as x ranges over a region D on the x-axis, u ranges over some region Δ on the u-axis since the equation $x = a+bu$ causes the two axes to be in one to one correspondence. To the point $x=a$ corresponds the point $u=0$. Now let $f(x) = f(a+bu)=F(u)$. Then, since x and u are corresponding points, f has the same value at x as F has at u. Since $\lim_{x=a} f(x) = \ell$ then for any $\epsilon>0$ it follows that there exist an η such that if $|x - a|<\eta$ then it follows that $|\ell - f(x)|<\epsilon$. Now if $\eta_1 = \frac{\eta}{b}$ then for $\epsilon>0$ it follows that there exist an η_1 such that if $|u|<\eta_1$ then it follows that $|\ell - F(u)|<\epsilon$. Written as a limit we have $\lim_{u=0} f(x)=\ell$.

<u>Theorem 1.7</u> <u>The trigonometric functions are continuous at every point where they are defined.</u>

Due to the definition of sin x and cos x it is apparent that they are defined everywhere. The functions $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$,...., are defined everywhere except at those points where the denominators are zero. We wish first to show that

$$\lim_{x=0} \sin x = 0 \quad \underline{\text{and}} \quad \lim_{x=0} \cos x = 1 \quad .$$

In figure 1 the line AB represents sin x.

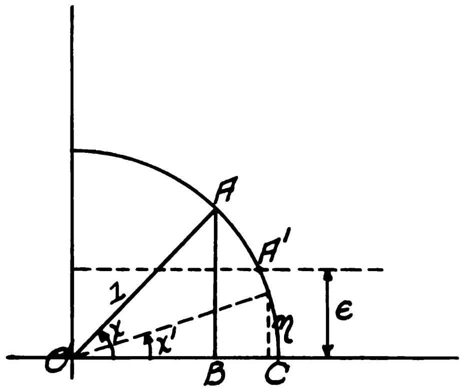


Fig. 1

However small we choose $\epsilon > 0$ let η = angle A'OC and then for any $\epsilon > 0$ it follows that there exist an η such that if $|x'| < \eta$ then it follows that $|\sin x'| < \epsilon$.

Therefore $$\lim_{x=0} \sin x = 0 \quad .$$

In figure 2 the line OD represents cos x.

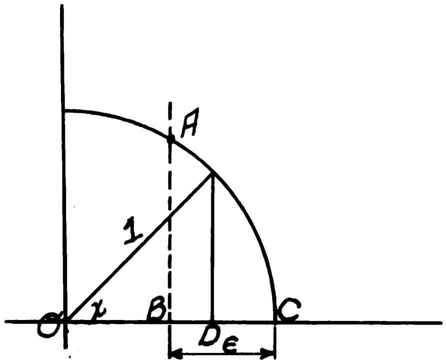


Fig. 2

For any $\epsilon > 0$, such as line BC, let $\eta =$ angle AOC. Then for $\epsilon > 0$ it follows that there exist an η such that if $|x| < \eta$ then it follows that $|1 - \cos x| < \epsilon$.

Therefore $\lim_{x=0} \cos x = 1$.

Next we wish to show that <u>the function</u> $f(x) = \sin x$ <u>is everywhere continuous</u>.

For let $x = a + u$; then $\sin x = \sin(a+u) = \sin a \cdot \cos u + \cos a \cdot \sin u$. Now since $\lim_{x=a} \sin x = \lim_{u=0} \sin(a+u)$ by lemma 1.12 and since $\lim_{u=0} \sin u = 0$ and $\lim_{u=0} \cos u = 1$ then we have $\lim_{x=a} \sin x = \lim \sin a = \sin a$.

In a similar way we see that $f(x) = \cos x$ is everywhere continuous by writing $\cos x = \cos(a+u) = \cos a \cdot \cos u - \sin a \cdot \sin u$.

Since sin x and cos x are everywhere continuous then it follows that tan x, cot x, sec x, csc x are also continuous by lemma 1.7 .

Lemma 1.13 *If* y f(x) *is an increasing function in the linear neighborhood of* $x = a$ *and if* $U \lim_{x=a} y = b$, *then* $U \lim_{y=b} x = a$. *

For suppose the function y is increasing so that when $x < a$ then $y < b$. Also, consider only values of $x < a$ so U in this case will represent the left-hand limit. Let $\epsilon > 0$ be arbitrarily small, and $a - \epsilon < x' < a$. Let y' correspond to x' and let $\eta > 0$ be such that $b - \eta > y'$.

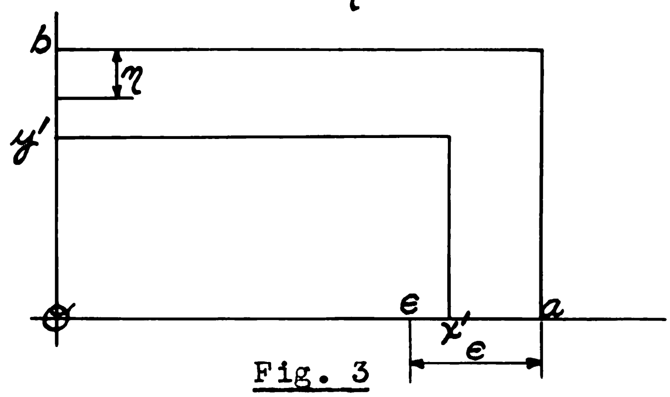


Fig. 3

Under these conditions while y remains in the η-vicinity of b then x must remain in the ϵ-vicinity of a. This means that $U \lim_{y=b} x = a$.

---

* The U will, in general, mean either the right or left hand limit. Throughout this proof U will represent the left-hand limit.

Theorem 1.8 The one-valued functions $\sin^{-1} y$, $\cos^{-1} y$, $\tan^{-1} y$, $\cot^{-1} y$ are continuous at every point where they are defined.

By theorem 1.7 $\lim_{x=a} \sin x = \sin a$. Therefore, by lemma 1.13, $\lim_{\sin x = \sin a} x = a$ since sin x is an increasing function if single valued. Now let $y = \sin x$ and $b = \sin a$; then we have $x = \sin^{-1} y$ and $a = \sin^{-1} b$ and therefore we have $\lim_{y=b} \sin^{-1} y = \sin^{-1} b$. In a similar manner the continuity of the functions $\cos^{-1} y$, $\tan^{-1} y$, $\cot^{-1} y$ can be shown.

Lemma 1.14 Cauchy's Condition. The neeessary and sufficient condition for a sequence of numbers, $\{a_m\}$, to have a limit A is that for any $\epsilon > 0$ it follows that there exists an (m) such that $|a^n - a^p| < \epsilon$ for all $n, p > m$.

Since $\{a_n\}$ has a limit then, by definition, for any $\epsilon > 0$ it follows that there exists an m such that for $n > m$ then it follows that $|A - a_n| < \frac{\epsilon}{2}$. Also for the same $\epsilon > 0$ and m it follows that for $p > m$ then, $|A - a_p| < \frac{\epsilon}{2}$. Adding these inequalities we have that for $\epsilon > 0$ it follows that $|a_n - a_p| < \epsilon$ for $n, p, > m$.

We next prove that the condition is sufficient.

We wish to show that if for any $\epsilon > 0$ it follows that there exists an m such that for n,p > m then it follows that $|a_n - a_p| < \epsilon$, then the sequence $\{a_n\}$ has a limit.

For $\epsilon_1 = \frac{1}{2}$, by hypothesis , there exists an m_1 such that for $p > m_1$ then it follows that $|a_{m_1} - a_p| < \epsilon_1$. Then there is an interval $\delta_1 = 2\epsilon_1$ in length, which contains all a_p for $p > m_1$. Double δ_1 so as to obtain an interval $2\delta_1$ in length, δ_1 on either side of a_{m_1} . Now for $\epsilon_2 = \frac{1}{2^2}$ there exists an $m_2 > m_1$ such that $p > m_2$ then it follows that $|a_{m_2} - a_p| < \epsilon_2$. Then there is an interval $\delta_2 = 2\epsilon_2$ in length which contains all a_p for $p > m_2$. Now δ_2 is not necessarily contained in δ_1 but $2\delta_2$ is always contained in $2\delta_1$.

In general, for $\epsilon_i = \frac{1}{2^i}$, set up a sequence of $2\delta_i$ by picking $\delta_i = 2\epsilon_i$ and each δ_i will contain all a_p for $p > m_i$. Then $2\delta_1$ contains $2\delta_2$ contains $2\delta_3$ contains..... $2\delta_{i-1}$ contains $2\delta_i$ ....

Since the length of $2\delta_i$ is $\frac{1}{2^{i-2}}$ we have just one point, A, inside all these intervals $\{2\delta_i\}$.*

---

* C. Caratheodary, Vorlesungen uber Reelle Funktionen, p.54

We will show that A is the limit of the sequence $\{a_n\}$.
For any $\epsilon > 0$ it follows that

$$|a_p - A| \leq |a_p - a_{m_i}| + |a_{m_i} - A| < \epsilon$$

if i is chosen such that $\epsilon_i = \frac{\epsilon}{4}$ and $p > m_i$. That is, if we take $m = m_i$, then for $p > m$ we have $|a_p - A| < \epsilon$.

Therefore $$\lim_{n=\infty} a_m = A \quad .$$

<u>Lemma 1.15</u> <u>If $\{a_n\}$ is a sequence of positive rational numbers whose limit is zero and if</u> $b > 0$, <u>then</u> $\lim b^{a_n} = 1$.

Let $b > 1$; then $b^{a_n} > 1$.
Since $a_m > 0$ and $\lim a_m = 0$ then for m sufficiently large a_m is as small as we please. We can take m so large that $\frac{1}{a_n} > g$ for $n > m$ however large the positive integer g be chosen.
By the binomial expansion

$$(1 + \epsilon)^g > 1 + g\epsilon \quad .$$

For sufficiently large g we have

$$1 + g\epsilon > b$$

and then

$$(1 + \epsilon)^g > b.$$

Also

$$(1+\epsilon)^{\frac{1}{a_n}} > (1+\epsilon)^{g} \quad .$$

To show this let μ, η be two rational numbers such that $\eta > \mu$. We wish to show that $(1+\epsilon)^{\eta} > (1+\epsilon)^{\mu}$. Let $\eta = \frac{r}{t}$ and $\mu = \frac{s}{t}$ where r, s, t are integers and $r > s$ and $t > 0$. Then

$$(4) \qquad (1+\epsilon)^{\frac{r}{t}} > (1+\epsilon)^{\frac{s}{t}} \quad .$$

Raising both sides of equation (4) to the t-th power we have

$$(1+\epsilon)^{r} > (1+\epsilon)^{s},$$

which is true since r and s are integers and $r > s$. Therefore, $(1+\epsilon)^{\frac{1}{a_n}} > b$ for $n > m$. Then we have

$$b^{a_n} < 1+\epsilon \quad .$$

That is for any $\epsilon > 0$ we can find an m such that if $n > m$ then $b^{a_n} < 1+\epsilon$. Thus we have proved that

$$\lim_{n=\infty} b^{a_n} = 1 \qquad \text{for } b > 1.$$

For the case $b < 1$ let $b = \frac{1}{c}$. Then $c > 1$ and, applying the preceding case, we have

$$c^{a_n} < 1+\epsilon \quad .$$

Hence

$$b^{a_n} = \frac{1}{C^{a_n}} > \frac{1}{1+\epsilon} > 1-\epsilon ,$$

and $$1 - b^{a_n} < \epsilon .$$

Therefore $$\lim_{n=\infty} b^{a_n} = 1 \qquad \text{for } b<1.$$

<u>Lemma 1.16</u> <u>If the sequence of rational numbers</u> $\{a_n\}$ <u>has a limit and if</u> $b>0$, <u>then the sequence</u> $b^{a_1}, b^{a_2}, b^{a_3}, \ldots$ <u>has a limit</u>.

For let $d_n = b^{a_n} - b^{a_p} = b^{a_p}(b^{a_n - a_p} - 1)$. Then for $\eta > 0$ it follows that there exists an ϵ such that if $|a_n - a_p| < \epsilon$ then it follows that $|b^{a_n - a_p} - 1| < \eta$ if we apply lemmas 1.14 and 1.15. Also, since $\{a_n\}$ has a limit, it follows that there exists two numbers Q and R such that $Q < a_n < R$ when $n = 1,2,3,\ldots\ldots$ Therefore

$$|d_n| < b^R \eta \qquad \text{if } b>1 ,$$

and

$$|d_n| < b^Q \eta \qquad \text{if } b<1.$$

If $b>1$, we take $\eta = \frac{\epsilon}{b^R}$, and, if $b<1$, we take $\eta = \frac{\epsilon}{b^Q}$.

In either case we have that for $\epsilon > 0$ it follows that there exists an m such that for $n, p > m$ then it follows that $\left| b^{a_n} - b^{a_p} \right| < \epsilon$. Now by applying lemma 1.14 our theorem is proved.

<u>Lemma 1.17</u> <u>If</u> $a_1, a_2, \ldots$ <u>and</u> $c_1, c_2, \ldots$ <u>be two sequences of rational numbers having the same limit and if</u> $b > 0$, <u>then the</u> $\lim_{n=\infty} b^{a_n} = \lim_{n=\infty} b^{c_n}$.

By lemma 1.16 both limits exist. Let $d_n = b^{a_n} - b^{c_n} = b^{a_n}(1 - b^{c_n - a_n})$. We must show that $\lim_{n=\infty} d_n = 0$. However, $\lim_{n=\infty} (a_n - e_n) = 0$ and therefore the $\lim_{n=\infty} (1 - b^{a_n - a_n}) = 0$. Hence

$$\lim_{n=\infty} d_n = \lim_{n=\infty} \left[b^{a_n}(1 - b^{c_n - a_n}) \right] = 0 = \lim_{n=\infty} (b^{a_n} - b^{c_n}).$$

Therefore the $\lim_{n=\infty} b^{a_n} = \lim_{n=\infty} b^{c_n}$ by applying theorem 1.2.

We are now ready to define what is meant by an irrational exponent. If c is any number and $r_1, r_2, \ldots$ is a sequence of rational numbers such that $\lim_{n=\infty} r_n = c$, then a^c is defined to be $\lim_{n=\infty} a^{r_n}$.

Lemma 1.18 If $c_1, c_2, \ldots$ is a sequence whose limit is c and if $a > 0$, then $\lim_{n=\infty} a^{c_n} = a^c$.

For let $r_1, r_2, \ldots,$ and $s_1, s_2, \ldots$ be two sequences of rational numbers whose limits are c and such that $r_n < c_n < s_n$, where $n = 1, 2, 3 \ldots$. If $a > 1$ then it follows that $a^{r_n} < a^{c_n} < a^{s_n}$. However, by lemma 1.16 we have $\lim_{n=\infty} a^{r_n} = \lim_{n=\infty} a^{s_n} = a^c$. Therefore, by applying lemma 1.4, we have

$$\lim a^{c_n} = a^c.$$

The proof for $a \leq 1$ is similar.

Theorem 1.9 The exponential functions are everywhere continuous.

We wish to prove that $\lim_{x=x_0} a^x = a^{x_0}$, or that for $\epsilon > 0$ it follows that there exists an η such that if $|x - x_0| < \eta$ then it follows that $|a^x - a^{x_0}| < \epsilon$. Let us write $a^x - a^{x_0}$ as $a^{x_0}(a^{x-x_0} - 1)$. Let $x_1, x_2, \ldots x_n$ be a decreasing sequence whose limit is x_0, and $x'_1, x'_2, \ldots x'_n$ be an increasing sequence whose limit is also x_0. Let η be less than the smaller of $|x'_q - x_0|$ and $|x_p - x_0|$ where p and q are picked so that for $\epsilon > 0$ then $\left|a^{x_p - x_0} - 1\right| < \frac{\epsilon}{a^{x_0}}$ and $\left|a^{x'_q - x_0} - 1\right| < \frac{\epsilon}{a^{x_0}}$. Consider first the case where $a > 1$.

For $x > x_0$ within an η distance of x_0 we have $|a^{x-x_0} - 1| < \frac{\epsilon}{a^{x_0}}$ since $x - x_0 < x_p - x_0$ and $a^{x-x_0} < a^{x_p - x_0}$. Therefore we have

$$|a^x - a^{x_0}| < a^{x_0}\frac{\epsilon}{a^{x_0}} = \epsilon .$$

Also, for $x < x_0$ within an η distance of x_0 we have $|a^{x-x_0} - 1| < \frac{\epsilon}{a^{x_0}}$, since $x - x_0 > x_q - x_0$ and, then $|1 - a^{x-x_0}| < |1 - a^{x_q - x_0}| < \frac{\epsilon}{a^{x_0}}$. Therefore we have

$$|a^x - a^{x_0}| < a^{x_0}\frac{\epsilon}{a^{x_0}} = \epsilon .$$

Now consider the case when $a < 1$. If $x > x_0$, then we have, since $x - x_0 < x_p - x_0$, that $1 > a^{x-x_0} > a^{x_p - x_0}$. Then $|a^{x-x_0} - 1| < |a^{x_p - x_0} - 1| < \frac{\epsilon}{a^{x_0}}$ and we write

$$|a^x - a^{x_0}| < a^{x_0}\frac{\epsilon}{a^{x_0}} = \epsilon .$$

Again, when $x < x_0$, we have, since $x - x_0 > x_q - x_0$, that $a^{x-x_0} < a^{x_q - x_0}$. Therefore we have

$$|a^x - a^{x_0}| < a^{x_0}\frac{\epsilon}{a^{x_0}} = \epsilon .$$

The case when $a = 1$ causes no difficulty.

<u>Lemma 1.19</u> <u>If</u> $\{a_n\}$ <u>is a sequence whose limit is</u> 1, <u>then</u> $\lim \log a_n = 0$.

For let b, the base of our logarithms, be greater than one. Then for any $\epsilon > 0$ we have $b^\epsilon > 1$. Let $\eta = 1 - \frac{1}{b^\epsilon} > 0$. Since $\lim_{n=\infty} a_n = 1$ we have for any $\eta > 0$ that there exists an m such that $-\eta < a_n - 1 < \eta$ for $n > m$.

Hence we have

(5) $$1 - \eta < a_m < 1 + \eta \quad .$$

But $1 - \eta = \frac{1}{b^\epsilon}$, and therefore

(6) $$a_m > \frac{1}{b^\epsilon} \quad .$$

Also $\eta = \frac{b^\epsilon - 1}{b^\epsilon} < b^\epsilon - 1$, and $1 + \eta < b^\epsilon$.
Therefore

(7) $$a_m < b^\epsilon \quad .$$

From equations (6) and (7) we have

$$b^{-\epsilon} < a_n < b^\epsilon \quad .$$

This may be written as

$$b^{-\epsilon} < b^{\log a_m} < b^\epsilon$$

since, by definition, $b^{\log a_n} = a_n$. Therefore,

$$-\epsilon < \log a_n < \epsilon \quad \text{for } n > m \quad .$$

Then lemma 1.4 gives $\lim\limits_{n=\infty} \log a_n = 0$.

Lemma 1.20 <u>If the</u> Lim $a_n = a > 0$ <u>and</u> $a_n > 0$, <u>then the</u> Lim $\log a_m = \log a$.

Since $a_n = a\frac{a_n}{a}$, then $\log a_n = \log a + \log\frac{a_n}{a}$.
However, the $\lim \frac{a_n}{a} = 1$ by theorem 1.4 of section 2.
Therefore $\text{Lim} \log\frac{a_n}{a} = 0$ by lemma 1.19 and hence

$$\text{Lim} \log a_n = \log a \quad .$$

Lemma 1.21 The $\underset{x=a}{\text{Lim}} \log x = \log a$ if $a > 0$.
This is lemma 1.20 stated in a different form.

Theorem 1.10 The logarithmic functions are everywhere continuous where they are defined.

For if we let $f(x) = \log_b x$ then we must show that $\underset{x=a}{\text{Lim}} \log_b x = \log_b a$. We will write, using lemma 2.3, $f(x) = \frac{\log_e x}{\log_e b}$ and then $f(x)$ will be continuous if $\log_e x$ is continuous. However, $\log_e x$ is continuous for every $x > 0$ by lemma 1.21. Therefore $f(x)$ is continuous which can be stated as

$$\underset{x=a}{\text{Lim}} \log_b x = \log_b a \ .$$

CHAPTER II

DIFFERENTIAL CALCULUS

1.1 <u>Formulas for Derivatives</u>. The derivation of the formulas for the derivatives of the elementary functions will next be considered. Many times the existence of the derivative is assumed in the derivation.

If $y = f(x)$ is defined over an interval (a,b) and $x = c$ is a point of the interval then the quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(c)}{x - c} ,$$

where x is in the interval, is called the difference quotient at $x = c$. If $x = c + h$ then

$$\frac{\Delta y}{\Delta x} = \frac{f(c+h) - f(c)}{h} .$$

Let the $\lim_{\Delta x = 0} \frac{\Delta y}{\Delta x} = \eta$ and if η exists then η is called the derivative of $f(x)$ at $x = c$ and is written $f'(c)$. Let D be the points of the interval for which η exists. The values of η define a function of x called the first derivative of $f(x)$ and is written $f'(x)$ or y'.

<u>Theorem 2.1</u> <u>If the derivative</u> $f'(c)$ <u>exists, then</u> $f(x)$ <u>is continuous at</u> $x = c$.

Since by definition the $\lim_{h=0} \frac{f(c+h) - f(c)}{h} = f'(c)$ and since the limit does exist and equals $f'(c)$, then for any $\epsilon > 0$ it follows that there exists an η such that if $|h| < \eta$ then it follows that

$$\frac{f(c+h) - f(c)}{h} = f'(c) + \epsilon' \quad \text{where} \quad |\epsilon'| < \epsilon \ .$$

Then $f(c+h) = f(c) + h[f'(c) + \epsilon']$ and by taking the limit of both sides we have

$$\lim_{h=0} f(c+h) = f(c) \ .$$

This shows that $f(x)$ is continuous at $x = c$.

<u>Theorem 2.2</u> <u>If</u> $y = \frac{u}{v}$, <u>where</u> $v \neq 0$, <u>and</u> <u>if</u> u' <u>and</u> v' <u>exist</u> <u>in</u> D, <u>then</u> $y' = \frac{vu' - uv'}{v^2}$.

Since $y = \frac{u}{v}$ then $y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$

and

$$\Delta y = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v} = \frac{v\Delta u - u\Delta v}{v(v+\Delta v)} = \left[\frac{1}{1+\Delta v}\right]\Delta u - \frac{u}{v}\left[\frac{1}{v+\Delta v}\right]\Delta v \ .$$

Therefore

$$\frac{\Delta y}{\Delta x} = \left(\frac{1}{v+\Delta v}\right)\frac{\Delta u}{\Delta x} - \frac{u}{v}\left(\frac{1}{v+\Delta v}\right)\frac{\Delta v}{\Delta x} \ .$$

But by theorem 2.1 we have $\lim_{\Delta v=0}(v+\Delta v)=v$ and by hypothesis the $\lim_{\Delta x=0}\frac{\Delta u}{\Delta x}=u'$ and $\lim_{\Delta x=0}\frac{\Delta v}{\Delta x}=v'$. Passing to the limit we have

$$y'=\frac{u'}{v}-\frac{uv'}{v^2}=\frac{vu'-uv'}{v^2} \quad .$$

It is permissible to divide by $v+\Delta v$ since $v+\Delta v\neq 0$ if Δx is sufficiently small and since v is continuous and not equal to zero.

This proof is sometimes given by writing $vy=u$ and then taking the derivative of a product and solving the resulting equation for y'. Such a proof assumes the existence of the derivative y'.

<u>Theorem 2.3</u> <u>If</u> $y=f(x)$, $x=g(t)$, <u>and</u> $f'(x_1)$ <u>and</u> $g'(t_1)$ <u>exist</u> <u>and</u> $x_1=g(t_1)$, <u>then</u> $\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$.

For some values of Δt we may have $\Delta x=0$. Let V be the set of values of $t_1+\Delta t$ for which $\Delta x=0$ and V' the set for which $\Delta x\neq 0$.

For $t_1+\Delta t$ in V' we have

$$\frac{\Delta y}{\Delta t}=\frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta t} \quad .$$

Since the derivative $g'(t_1)$ and $f'(x_1)$ exist then

$$(8) \quad \lim_{\Delta t=0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t=0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta t=0} \frac{\Delta x}{\Delta t}$$

which, by lemma 1.8, becomes

$$\lim_{\Delta t=0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t=0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta t=0} \frac{\Delta x}{\Delta t} = f'(x_1) \cdot g'(t_1) \quad .$$

This limit is taken for Δt's for which $t_1 + \Delta t$ is in V'.

If in every interval containing t_1 we have values of $t_1 + \Delta t$ for which $\Delta x = 0$ then equation (8) does not prove the theorem and we must consider $t_1 + \Delta t$ in V. We have that $\frac{\Delta x}{\Delta t} = 0$ and therefore $\lim_{\Delta t=0} \frac{\Delta x}{\Delta t} = 0$. This limit is taken for just those Δt's for which $t_1 + \Delta t$ is in V. Since $g'(t_1)$ exists it must then be zero. For $t_1 + \Delta t$ in V we have also that $\frac{\Delta y}{\Delta t} = 0$ since the Δx is zero. Then for $t_1 + \Delta t$ in V we have

$$(9) \qquad \frac{\Delta y}{\Delta t} - f'(x_1) \cdot g'(t_1) = 0 \quad .$$

Therefore, by equation (8) and (9), for $\epsilon > 0$ it follows that there exists a d such that if $|\Delta t| < d$ then

$$\left| \frac{\Delta y}{\Delta t} - f'(x_1) \cdot g'(t_1) \right| < \epsilon \quad .$$

This means that

$$\lim_{\Delta t = 0} \frac{\Delta y}{\Delta t} = f'(x_1) \cdot g'(t_1) \quad .$$

The usual proof of this theorem in elementary calculus books fails to mention the possibility of Δx being equal to zero. If $y = f(x)$ and x equals a constant we would have a case of $\Delta x = 0$ for all Δt's. A better example is $y = f(x)$ and $x = t^2 \sin \frac{1}{t}$ for $t \neq 0$ and $x = 0$ for $t = 0$. Here, no matter how close we get to the origin,we have Δt's for which $\Delta x = 0$.

Before the formulas for the derivatives of the exponential and logarithmic functions can be derived the constant e must be established and a few properties of sequences proved.

<u>Lemma 2.1</u> <u>If</u> $a > b \geq 0$ <u>and</u> n <u>is</u> <u>a</u> <u>positive</u> <u>integer</u> <u>greater</u> <u>than</u> <u>one</u>, <u>then</u> $n(a - b)b^{n-1} < a^n - b^n < n(a - b)a^{n-1}$.

For $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots\ldots + b^{n-1})$. In the last set of parenthesis replace a by b and,since $a > b$,then we have

$$a^n - b^n > (a - b)(b^{n-1} + b^{n-1} + \ldots \text{ n terms}) \; ,$$

which shows that $a^n - b^n > n(a - b)b^{n-1}$.

If we replace b by a when $a > b$ then

$$a^n - b^n < (a - b)(a^{n-1} + a^{n-1} + \dots \text{ n terms})$$

and

$$a^n - b^n < (a - b)\, na^{n-1} .$$

We note that we could have stated this lemma as

$$b^n > a^{n-1}\left[a - n(a - b)\right]$$

by transposing and factoring.

Lemma 2.2 A bounded monotone sequence has a limit.

For let $A = a_1, a_2, a_3 \dots\dots$ be an increasing monotone sequence and $a_n < G$ for $n = 1,2,3,\dots\dots\dots$. To show that A has a limit we must show that for any $\epsilon > 0$ it follows that there exists an m such that if $n > m$ then it follows that $0 < a_n - a_m < \epsilon$. Under these conditions A will have a limit by lemma 1.14. Since A is monotone increasing then $0 < a_n - a_m$. To show $a_n - a_m < \epsilon$ take an m_0 . Now either there exists an infinite sequence of indices,

(10) $\quad m_0 < m_1 < m_2 < \dots\dots ,$

such that

(11) $\quad a_{m_1} - a_{m_0} \geq \epsilon,\ a_{m_2} - a_{m_1} \geq \epsilon,\ \dots\dots$

or there does not.

Suppose such a sequence exists. Then, however small ϵ is, there exists a p so large that

(12) $$p\epsilon + a_{m_0} > G \quad .$$

Now adding the first p inequalities of equation (11) we have

$$a_{m_p} - a_{m_0} \geq p\epsilon \quad ,$$

and substituting this value of $p\epsilon$ in equation (12) we have

$$a_{m_p} > G$$

which contradicts the hypothesis. Then there must exist but a finite number of indices m_i such that equation (12) holds. Therefore an m can be taken so large that $a_n - a_m < \epsilon$ for $n > m$.

<u>Lemma 2.3</u> <u>The</u> <u>sequence</u> $a_n = (1 + \frac{1}{n})^n$, <u>where</u> $n = 1,2,3\ldots$, <u>has</u> <u>a</u> <u>limit</u>.

First, we wish to show that $\{a_n\}$ is increasing or that $a_n > a_{n-1}$. By lemma 2.1 we have

$$b^n > a^{n-1}\left[a - n(a - b)\right]$$

and, if we let $a = 1 + \frac{1}{n+1}$ and $b = 1 + \frac{1}{n}$, then we have

$$(1 + \frac{1}{n})^n > (1 + \frac{1}{n-1})^{n-1}\left[1 + \frac{1}{n+1} - n(1 + \frac{1}{n+1} - 1 - \frac{1}{n})\right] = (1 + \frac{1}{n-1})^{n-1} .$$

This shows that $\{a_n\}$ is increasing.

Second, we wish to show that the sequence $\{a_m\}$ is bounded. Let $a = 1 + \frac{1}{2m}$, $b = 1$, and $n = m+1$ and, applying lemma 2.1, we have

$$1 > \frac{1}{2}(1+\frac{1}{2m})^m \qquad \text{for } m = 1,2,3,\ldots .$$

Now by squaring we have

$$4 > (1+\frac{1}{2m})^{2m}$$

which shows that $a_{2m} < 4$. But by the first part of this lemma we have $a_{2m-1} < a_{2m}$ and since all positive integers are of the form 2m or 2m-1 then we have $a_n < 4$ for $n = 1,2,3,\ldots$.

We have now shown the sequence $\{a_n\}$ to be increasing and bounded and, by lemma 2.2, the limit then exists. This limit is the constant called e .

<u>Lemma 2.4</u> <u>The</u> $\lim_{x=+\infty} (1+\frac{1}{x})^x = e$.

For consider x such that $n \leq x \leq n+1$.

Now $$1+\frac{1}{n} \geq 1+\frac{1}{x} > 1+\frac{1}{n+1}$$

and

$$(13) \qquad (1+\frac{1}{n})^{n+1} > (1+\frac{1}{x})^x > (1+\frac{1}{n+1})^n .$$

However

(14) $$(1+\frac{1}{n})^{n+1} = (1+\frac{1}{n})(1+\frac{1}{n})^{n}$$

and

(15) $$(1+\frac{1}{n+1})^{n} = (1+\frac{1}{n+1})^{n+1} \cdot \frac{1}{1+\frac{1}{n+1}} \quad .$$

Now equations (14) and (15) give, in (13),

(16) $$(1+\frac{1}{n})(1+\frac{1}{n})^{n} > (1+\frac{1}{x})^{x} > (1+\frac{1}{n+1})^{n+1} \cdot \frac{1}{1+\frac{1}{n+1}} \quad .$$

By lemma 2.3 we have

$$\text{Lim } (1+\frac{1}{n})^{n} = \text{Lim } (1+\frac{1}{n+1})^{n+1} = e$$

and also

$$\lim_{n=\infty} (1+\frac{1}{n}) = \lim_{n=\infty} \frac{1}{1+\frac{1}{n+1}} = 1 \quad .$$

So equation (16) becomes, upon taking limits,

$$e > \lim (1+\frac{1}{x})^{x} > e \quad .$$

Therefore, $\lim\limits_{x=+\infty} (1+\frac{1}{x})^{x} = e$, by lemma 1.4.

<u>Lemma 2.5</u> <u>The</u> $\lim\limits_{x=-\infty} (1+\frac{1}{x})^{x} = e$.

For let $x = -u$. Then

$$(1+\frac{1}{x})^{x} = (1-\frac{1}{u})^{-u} = (1+\frac{1}{u-1})^{u} ,$$

$$= (1+\frac{1}{v})(1+\frac{1}{v})^{v} \quad \text{where } u-1 = v \quad .$$

However

(14) $$(1+\frac{1}{n})^{n+1} = (1+\frac{1}{n})(1+\frac{1}{n})^{n}$$

and

(15) $$(1+\frac{1}{n+1})^{n} = (1+\frac{1}{n+1})^{n+1} \cdot \frac{1}{1+\frac{1}{n+1}} \quad .$$

Now equations (14) and (15) give, in (13),

(16) $$(1+\frac{1}{n})(1+\frac{1}{n})^{n} > (1+\frac{1}{x})^{x} > (1+\frac{1}{n+1})^{n+1} \cdot \frac{1}{1+\frac{1}{n+1}} \quad .$$

By lemma 2.3 we have

$$\text{Lim } (1+\frac{1}{n})^{n} = \text{Lim } (1+\frac{1}{n+1})^{n+1} = e$$

and also

$$\lim_{n=\infty} (1+\frac{1}{n}) = \lim_{n=\infty} \frac{1}{1+\frac{1}{n+1}} = 1 \quad .$$

So equation (16) becomes, upon taking limits,

$$e > \lim (1+\frac{1}{x})^{x} > e \quad .$$

Therefore, $\lim_{x=+\infty} (1+\frac{1}{x})^{x} = e$, by lemma 1.4.

<u>Lemma 2.5</u> <u>The</u> $\lim_{x=-\infty} (1+\frac{1}{x})^{x} = e$.

For let $x = -u$. Then

$$(1+\frac{1}{x})^{x} = (1-\frac{1}{u})^{-u} = (1+\frac{1}{u-1})^{u} ,$$

$$= (1+\frac{1}{v})(1+\frac{1}{v})^{v} \quad \text{where } u-1 = v \ .$$

However

$$(14) \qquad (1+\frac{1}{m})^{m+1} = (1+\frac{1}{m})(1+\frac{1}{m})^{m}$$

and

$$(15) \qquad (1+\frac{1}{m+1})^{m} = (1+\frac{1}{m+1})^{m+1} \cdot \frac{1}{1+\frac{1}{m+1}} \quad .$$

Now equations (14) and (15) give,in (13),

$$(16) \qquad (1+\frac{1}{m})(1+\frac{1}{m})^{m} > (1+\frac{1}{x})^{x} > (1+\frac{1}{m+1})^{m+1} \cdot \frac{1}{1+\frac{1}{m+1}} \quad .$$

By lemma 2.3 we have

$$\text{Lim } (1+\frac{1}{m})^{m} = \text{Lim } (1+\frac{1}{m+1})^{m+1} = e$$

and also

$$\lim_{m=\infty} (1+\frac{1}{m}) = \lim_{m=\infty} \frac{1}{1+\frac{1}{m+1}} = 1 \quad .$$

So equation (16) becomes,upon taking limits,

$$e > \lim (1+\frac{1}{x})^{x} > e \quad .$$

Therefore, $\lim_{x=+\infty} (1+\frac{1}{x})^{x} = e$, by lemma 1.4.

<u>Lemma 2.5</u> <u>The</u> $\lim_{x=-\infty} (1+\frac{1}{x})^{x} = e$.

For let $x = -u$. Then

$$(1+\frac{1}{x})^{x} = (1-\frac{1}{u})^{-u} = (1+\frac{1}{u-1})^{u} ,$$

$$= (1+\frac{1}{v})(1+\frac{1}{v})^{v} \quad \text{where } u-1=v \ .$$

When $x = -\infty$ then $u = +\infty$ and $v = +\infty$. Since

$$\lim_{v=+\infty} \left(1+\frac{1}{v}\right) = 1 \quad \text{and} \quad \lim_{v=+\infty} \left(1+\frac{1}{v}\right)^{v} = e$$

then, if we take the limit in equation (17), we have

$$\lim_{x=-\infty} \left(1+\frac{1}{x}\right)^{x} = \lim_{v=+\infty} \left(1+\frac{1}{v}\right)^{v} = e \quad .$$

<u>Lemma 2.6</u> <u>The</u> $\lim_{x=0} (1+x)^{\frac{1}{x}} = e$.

Since $\lim_{x=+\infty} \left(1+\frac{1}{x}\right)^{x} = e$ let $x = \frac{1}{u}$. Then the right hand $\lim_{u=0}$ of $(1+u)^{\frac{1}{u}} = e$.

Also $$\lim_{x=-\infty} \left(1+\frac{1}{x}\right)^{x} = e \ .$$

Again let $x = \frac{1}{u}$ and the left-hand $\lim_{u=0} (1+u)^{\frac{1}{u}} = e$. Since both the left and right hand limits equal e then $\lim_{x=0} (1+x)^{\frac{1}{x}} = e$.

<u>Lemma 2.7</u> <u>The</u> $\lim_{x=0} \frac{\log(1+x)}{x} = 1$.

The $\lim_{x=0} \frac{\log(1+x)}{x} = \lim_{x=0} \log (1+x)^{\frac{1}{x}}$.

If $\lim_{x=a} f(x) = \eta$, then, by lemma 1.15, we have

$$\lim_{x=a} \log f(x) = \log \eta = \log \lim_{x=a} f(x) \quad .$$

Therefore

$$\lim_{x=0} \log (1+x)^{\frac{1}{x}} = \log \lim_{x=0} (1+x)^{\frac{1}{x}} = \log e = 1 \quad .$$

Lemma 2.8 *Suppose that* $y = f(x)$ *is a monotone increasing or decreasing and continuous function. Let* $x = g(y)$ *be the inverse function and let* x *and* y *be corresponding points. If* $f'(x)$ *exists and is different from zero, then* $g'(y)$ *exists and* $g'(y) = \frac{1}{f'(x)}$.

Since $f(x)$ is monotone increasing or decreasing then Δy and $\frac{\Delta y}{\Delta x}$ are not equal to zero. Hence $\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}}$ has no division by zero. Since y is continuous, then $\lim_{\Delta x = 0} \Delta y = 0$. Taking the limit we have

$$\lim_{\Delta y = 0} \frac{\Delta x}{\Delta y} = \frac{1}{\lim_{\Delta x = 0} \frac{\Delta y}{\Delta x}}$$

if we apply lemma 1.8. Therefore

$$g'(y) = \frac{1}{f'(x)} \quad .$$

Theorem 2.4 *If* $y = \log x$ *then* $y' = \frac{1}{x}$.

For

$$\frac{\Delta y}{\Delta x} = \frac{\log(x + \Delta x) - \log x}{\Delta x} = \frac{\log\left(1 + \frac{\Delta x}{x}\right)}{\Delta x} = \frac{1}{x} \cdot \frac{\log\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} \; .$$

By lemma 2.7 we have $\lim_{\Delta x = 0} \frac{\log\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} = 1$.
Therefore, taking limits of both sides, we have

$$y' = \frac{1}{x} \quad .$$

<u>Theorem 2.5</u> <u>If</u> $y = e^x$ <u>then</u> $y' = e^x$.

Since $y = e^x$ then $x = \log y$. By theorem 2.4 we have

$$\frac{dx}{dy} = \frac{1}{y} \quad .$$

Therefore, by lemma 2.8, we can write

$$\frac{dy}{dx} = y = e^x$$

since e^x is monotone increasing or decreasing and continuous.

<u>Theorem 2.6</u> <u>If</u> $y = e^u$, <u>where</u> $u = f(x)$ <u>and is differentiable,</u> <u>then</u> $y' = e^u \frac{du}{dx}$.

Since $y = e^u$ then $\frac{dy}{du} = e^u$. Also since $u = f(x)$ then $\frac{du}{dx} = f'(x)$. Then by theorem 2.3 we have

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = e^u \frac{du}{dx} \quad .$$

<u>Theorem 2.7</u> <u>If</u> $y = a^u$ <u>then</u> $y' = a^u \log a \frac{du}{dx}$.

For, by the definition of logarithms, we have $a = e^{\log a}$. Then

$$y = (e^{\log a})^u = e^{u \log a} .$$

By applying theorem 2.6 the derivative is

$$y' = e^{u \log a} \frac{d(u \log a)}{dx} ,$$

$$= e^{u \log a} \log a \frac{du}{dx} ,$$

$$= a^{u} \log a \frac{du}{dx} .$$

It is to be noticed that in the proofs of theorems 2.4, 2.5, 2.6, and 2.7 that we require the existence of the $\lim_{n=\infty} (1 + \frac{1}{n})^{n}$ where n ranges over all real values and not over just the positive integers. Also, we must show that $\lim_{n=\infty} (1+\frac{1}{n})^{n}$ and $\lim_{n=0} (1+n)^{\frac{1}{n}}$ are the same. Another point to note is that we need to show that $\lim_{n=0} \log (1+n)^{\frac{1}{n}} = \log \lim_{n=0} (1+n)^{\frac{1}{n}}$. Although the proofs of these facts do not belong in a first year calculus book it would seem desirable that the author should point out the necessity of such proofs.

Smith, Salkover, Justice in their book and Love in his book either assume or prove most of the necessary material. However, both of these books fail to assume or prove that $\lim_{n=0} \log (1+n)^{\frac{1}{n}} = \log \lim_{n=0} (1+n)^{\frac{1}{n}}$ and that $\lim_{n=0} (1+n)^{\frac{1}{n}} = e$.

Dalaker and Hartig as well as Neelley and Tracy assume or prove all necessary material. The latter gives references for all assumed statements.

We proceed now to develop more formulas for derivatives. We will define the length of a curve, $y = f(x)$, as the $\lim_{chords=0} \sum_{1}^{n}$ (lengths of chords) . In section 6 of Chapter 4 we show from this definition that the length of a curve is given by

$$(18) \qquad L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

if $f'(x)$ exists and is continuous for $a \leqslant x \leqslant b$.

<u>Lemma 2.9</u> <u>The</u> $\lim_{chord=0} \frac{\text{arc}}{\text{chord}} = 1$

Since by applying equation (18) and the mean value theorem for integrals we have *

$$(19) \qquad \lim_{chord=0} \frac{\text{arc}}{\text{chord}} = \lim_{\Delta x=0} \frac{\int_a^{a+\Delta x} \sqrt{1+[f'(x)]^2}\, dx}{\sqrt{\Delta x^2 + \Delta y^2}},$$

$$= \lim_{\Delta x=0} \frac{\sqrt{1+[f'(x)]^2}\ \Delta x}{\sqrt{\Delta x^2 + \Delta y^2}}$$

where x lies between a and $a+\Delta x$. Rewriting the denominator, equation (19) becomes

$$\lim_{\Delta x=0} \frac{\sqrt{1 + [f'(x)]^2}\ \Delta x}{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x} = \lim_{\Delta x=0} \frac{\sqrt{1 + [f'(x)]^2}}{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}} .$$

---

* W. B. Fite, <u>Advanced</u> <u>Calculus</u>, p. 97. (Hereafter referred to as Fite.)

Using the fact that $f'(x)$ is continuous we have

$$\lim_{chord=0} \frac{\text{arc}}{\text{chord}} = \frac{\sqrt{1+[f'(a)]^2}}{\sqrt{1+[f'(a)]^2}} = 1 \quad .$$

Lemma 2.10 The $\lim\limits_{x=0} \frac{\sin x}{x} = 1$.

For in figure 4 we have arc ABC$=2x$ where x is measured in radians. Also, chord AB$=2 \sin x$.

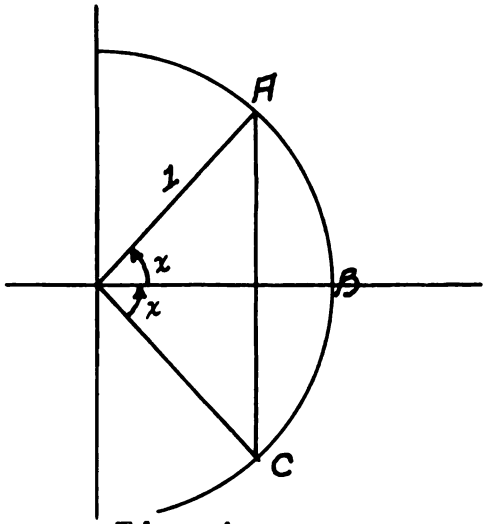


Fig. 4

But, by lemma 2.9, $\lim\limits_{chord=0} \frac{\text{arc}}{\text{chord}} = 1$. Therefore we have

$$\lim_{\sin x=0} \frac{x}{\sin x} = 1 \ .$$

Since sin x is continuous and $\lim_{x=0} \sin x = 0$ we have

$$\lim_{x=0} \frac{1}{\frac{x}{\sin x}} = 1 \qquad \text{by lemma 1.8.}$$

Therefore

$$\lim_{x=0} \frac{\sin x}{x} = 1 \quad .$$

Making use of the fact that $\lim_{x=0} \frac{\sin x}{x} = 1$ we derive the formulas for the derivatives of the trigonometric functions.

<u>Theorem 2.8</u> <u>If</u> $y = \sin x$ <u>is a function of</u> x <u>then</u> $y' = \cos x$.

For $$y + \Delta y = \sin (x + \Delta x)$$

and $$\Delta y = \sin (x + \Delta x) - \sin x \ ,$$

$$= 2 \cos (x + \frac{\Delta x}{2}) \sin \frac{\Delta x}{2}$$

since $\sin(x+\Delta x) - \sin x = 2 \cos \frac{1}{2}(x + \Delta x + x) \sin \frac{1}{2}(x + \Delta x - x)$.

Then

$$\frac{\Delta y}{\Delta x} = \cos (x + \frac{\Delta x}{2}) \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \quad .$$

Since when $\Delta x = 0$ then $\frac{\Delta x}{2}$ also is zero then,by taking the limit of each side,we have

$$y' = \cos x , \qquad \text{if we apply lemma 2.10 .}$$

By theorem 2.3, if u is a differentiable function of x, then

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx} .$$

Since the formulas for the derivatives of the other trigonometric functions use theorem 2.8 and are easily obtained we will omit them.

The derivation of the formula for the derivative of $y = \sin x$ is dependent upon the fact that Lim $\frac{\sin x}{x} = 1$. The latter statement is arrived at in various ways in different elementary calculus books. On page 65 in Love's elementary calculus we find the statement that $\lim_{arc=0} \frac{chord}{arc} = 1$. No effort is made to show that this limit exists and equals 1 however. Assuming that $\lim_{arc=0} \frac{chord}{arc} = 1$ it is not difficult to show that $\lim_{x=0} \frac{\sin x}{x} = 1$. On page 18 of McKelvey's elementary calculus we find the statement, "By definition of the arc length of the circle", certain relations exist. Apparently arc length has not been defined previously.

On this definition he bases his proof but he is not prepared to define arc length at that stage in the book. On page 98 in Granville, Smith, Longley's book we find the statement, "From geometry the chord $<$ arc $<$ MT + M'T ". Here MT and M'T are the tangents to the circle through the ends of the chord. They fail to show how geometry can be applied to obtain this result, however. In most elementary calculus books we find one or more of the above assumptions made.

In the derivation of the formula for derivatives of the inverse trigonometric functions it is necessary to keep in mind that the functions are multiple-valued and that usually we are deriving a formula with the principal branch in mind. If this is true we should give the limits on the values of the angle in each case. If it is desirable to consider any angle between 0 and 2π then it should be pointed out which sign should be used on the radical in each of the quadrants. The existence of y' in theorems 2.9 to 2.14 inclusive is given by lemma 2.8.

Theorem 2.9 If $y = \sin^{-1} u$, then $y' = \pm \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx}$, $|u| < 1$.

From $y = \sin^{-1} u$ we have $u = \sin y$. Therefore, by theorem 2.8

$$\frac{du}{dy} = (\cos y)y' \qquad \text{and}$$

$$y' = \frac{1}{\cos y} \frac{du}{dx} . \tag{20}$$

Since $\cos y = \pm\sqrt{1 - \sin^2 y}$ and $\sin y = u$, equation (20) becomes

$$y' = \pm \frac{1}{\sqrt{1 - u}} \frac{du}{dx} .$$

The plus sign applies when y is in the first or fourth quadrants and the minus sign applies when y is in the second or third quadrants.

Theorem 2.10 If $y = \cos^{-1} u$, then $y' = \pm\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$, $|u| < 1$.

From $y = \cos^{-1} u$, we have $u = \cos y$. Therefore

$$\frac{du}{dx} = (-\sin y)y' \qquad \text{and}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y} \frac{du}{dx} . \tag{21}$$

Since $\sin y = \pm\sqrt{1 - \cos^2 y}$ and $\cos y = u$, equation (21) becomes

$$y' = \pm \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx} .$$

The plus sign applies when y is in the third or fourth quadrants. The minus sign applies in the first and second quadrants.

<u>Theorem 2.11</u> <u>If</u> $y = \tan^{-1} u$, <u>then</u> $y' = \frac{1}{1 + u^2} \frac{du}{dx}$.

Since $y = \tan^{-1} u$ we write $u = \tan y$. Therefore

$$\frac{du}{dx} = \sec^2 y \frac{dy}{dx} \qquad \text{and}$$

(22) $$y' = \frac{1}{\sec^2 y} \frac{du}{dx} .$$

Making use of the fact that $\sec^2 y = 1 + \tan^2 y$, equation (22) becomes

$$y' = \frac{1}{1 + u^2} \frac{du}{dx} .$$

The same sign holds for all four quadrants.

<u>Theorem 2.12</u> <u>If</u> $y = \cot^{-1} u$, <u>then</u> $y' = -\frac{1}{1 + u^2} \frac{du}{dx}$.

For, since $y = \cot^{-1} u$, we can write $u = \cot y$. Then

$$\frac{du}{dx} = -\csc^2 y \frac{dy}{dx} \qquad \text{and}$$

(23) $$y' = -\frac{1}{\csc^2 y} \frac{du}{dx} .$$

Substituting $\csc^2 y = 1 + \cot^2 y$ in equation (23), then

$$y' = -\frac{1}{1 + u^2} \frac{du}{dx} .$$

The minus sign holds for all four quadrants.

<u>Theorem 2.13</u> <u>If</u> $y = \sec^{-1} u$, <u>then</u> $y' = \pm \frac{1}{u\sqrt{u^2 - 1}} \frac{du}{dx}$, $|u| > 1$.

From $y = \sec^{-1} u$ we have $u = \sec y$. Therefore

$$\frac{du}{dx} = \sec y \tan y \frac{dy}{dx} \qquad \text{and}$$

(24) $$y' = \frac{1}{\sec y \tan y} \frac{du}{dx} .$$

Since $\tan y = \pm \sqrt{\sec^2 y - 1}$ and $u = \sec y$, equation (24) becomes

$$y' = \pm \frac{1}{u \sqrt{u^2 - 1}} \frac{du}{dx} .$$

Here the plus sign applies to the first and third quadrants and the minus sign to the second and fourth quadrants.

Theorem 2.14 If $y = \csc^{-1} u$, then $y' = \mp \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$, $|u| > 1$.

Writing $y = \csc^{-1} u$ as $u = \csc y$, then

$$\frac{du}{dx} = -\csc y \cot y \frac{dy}{dx} \qquad \text{and}$$

(25) $$y' = \mp \frac{1}{\csc y \cot y} \frac{du}{dx} \quad .$$

Substituting $\cot y = \pm \sqrt{\csc^2 y - 1}$ and $u = \csc y$ in equation (25), we have

$$y' = \mp \frac{1}{u\sqrt{u^2 - 1}} \frac{du}{dx} \quad .$$

The plus sign holds for the second and fourth quadrants and the minus sign for the first and third.

Theorem 2.15 If $y = u^n$, then $y' = nu^{n-1} \frac{du}{dx}$.

Writing $y = u^n$ as $y = e^{n \log u}$ and letting $v = n \log u$, we have

$$y = e^v \; .$$

From $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$, $\frac{dy}{dv} = e^v$, and $\frac{dv}{dx} = \frac{n}{u} \cdot \frac{du}{dx}$,

we get

$$y' = \frac{e^v u}{u} \frac{du}{dx} = \frac{n\,u^n}{u} \frac{du}{dx} = n\,u^{n-1} \frac{du}{dx} .$$

In theorem 2.15 the derivation must not assume the existence of the derivative. If such an assumption is made the formula simply states that if there exists a derivative it can be found in this way. This is a common error in elementary calculus texts.

Smith, Salkover, and Justice on page 147 of their book and Neelley and Tracy on page 108 assume the existence of the derivative in their proof of theorem 2.15.

On page 47 of the text by Slobin and Solt and on page 91 of Granville, Smith, and Longley we find the existence of the derivative assumed in the proof of theorem 2.7.

Some rather surprising results may be obtained if you start with false assumptions. For example, suppose we assume that

$$\frac{x^3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

where A and B are constants. If A and B are determined by writing $x^3 = A(x+1)+B(x-2)$ and substituting $x = -1$ and $x = 2$ we obtain the obviously false result that

$$\frac{x^3}{(x-2)(x+1)} = \frac{8}{3(x-2)} + \frac{1}{3(x \quad 1)} \quad .$$

On page 36 of Slobin and Solt, after having proven theorem 2.15 for the rational numbers, the authors make this statement, "This theorem is also true for n irrational. This is evident since an irrational number may be expressed as the limit of a sequence of rational numbers and, since the theorem is true for every rational number of the sequence, it is true in the limit." This statement implies that if a thing is true before the limit is taken it is true after the limit is taken. This statement is obviously false.

2.2 *Rolle's Theorem*. *Theorem 2.16* *If* $f(x)$ *is continuous and single-valued in a region* $a \leq x \leq b$, $f(a) = f(b) = 0$, *and* $f'(x)$ *exists for* $a < x < b$, *then there exists a point* $a < c < b$ *such that* $f'(c) = 0$.

Since $f(x)$ is continuous on (a,b) it has a maximum and a minimum.*

---

* Fite, p, 25.

If $f(x)$ is constant then the derivative at every point is zero and the conclusion is satisfied. If $f(x)$ is not equal to zero for all points then it is positive for some values of x or negative for some values of x. If we have the former let $u > 0$ be the maximum of $f(x)$. Since $f(x)$ is continuous there is some value $a < c < b$ so that $f(c) = u$.* Now if $h > 0$ then

$$f(c+h) - f(c) \leq 0 ,$$

and

$$f(c-h) - f(c) \leq 0 .$$

Then

(26) $$\frac{f(c+h) - f(c)}{h} \leq 0 ,$$

and

(27) $$\frac{f(c-h) - f(c)}{-h} \geq 0 .$$

By equation (26) $f'(c) \leq 0$ and by (27) $f'(c) \geq 0$. Therefore

$$f'(c) = 0 .$$

If the maximum is zero then there is a minimum that is negative and a similar proof holds.

---

* Fite, p. 24

Demonstrations resting upon the fact that the function in passing from a to b must increase and then decrease or decrease and then increase will be valid as long as we have only a finite number of oscillations of $f(x)$ between a and b. If an infinite number of oscillations exist then this need not be true. For example, Slobin and Solt start their proof with the statement, "If $f(x)$ is not identically zero, then it must increase from $f(a) = 0$, then decrease to $f(b) = 0$, or it must decrease from $f(a) = 0$, then increase to $f(b) = 0$." Their proof holds then only when the number of oscillations are finite.

2.3 Law of the Mean. Theorem 2.17 If $f(x)$ is continuous in (a,b) and if $f'(x)$ exists for $a < x < b$, then for some $a < c < b$ we have $f(b) - f(a) = (b - a)f'(c)$.

Form the auxiliary function

$$g(x) = f(b) - f(x) - \frac{f(b) - f(a)}{b - a}(b - a) \quad . \tag{28}$$

Then $g(a) = g(b) = 0$ by substitution. Taking the derivative of equation (28) we find

$$g'(x) = - f'(x) + \frac{f(b) - f(a)}{b - a} \quad . \tag{29}$$

Therefore g(x) is continuous. Applying Rolle's Theorem there is some point $a<c<b$ where $g'(c)=0$. Setting $x=c$ in equation (29) we get

$$f'(c) = \frac{f(b) - f(a)}{b - a} ,$$

or

$$f(b) - f(a) = (b - a) f'(c) .$$

2.4 Indeterminate Forms. If we have two functions such as f(x) and g(x) and if there is a finite value a of x such that $f(a)=0$ and $g(a)=0$, then the fraction becomes $\frac{0}{0}$ which has no meaning. If it is desirable to find the $\lim_{x=a} \frac{f(x)}{g(x)}$ it can be done as follows. Consider the function

$$\frac{f(b) - f(a)}{g(b) - g(a)} \left[g(x) - g(a) \right] - \left[f(x) - f(a) \right]$$

which vanishes at $x=a$ and $x=b$. If we apply Rolle's Theorem then

$$(30) \qquad \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)} \qquad \text{where } a < \xi < b .$$

If in equation (30), $f(a)=0$, $g(a)=0$, and $b=x$ then

(31) $$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \quad , \text{ where } a < \xi < x .$$

From equation (31) we get

(32) $$\lim_{x=a} \frac{f(x)}{g(x)} = \lim_{\xi=a} \frac{f'(\xi)}{g'(\xi)}$$

if the limit of the right hand side exists. Now, if $g'(a)$ is not zero and $f'(x)$ and $g'(x)$ are continuous at $x = a$, we have

(33) $$\lim_{x=a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \quad .$$

If $g'(a) \neq 0$ and $f'(a)=0$ then

$$\lim_{x=a} \frac{g'(x)}{f'(x)} = 0 \quad .$$

If $f'(a)$ and $g'(a)$ are both zero then equation (32) is applied again so that

$$\lim_{x=a} \frac{f(x)}{g(x)} = \lim_{\xi=a} \frac{f'(\xi)}{g'(\xi)} = \frac{f''(a)}{g''(a)} \quad .$$

In applying this method we assume the continuity of the derivatives involved.

This proves

Theorem 2.18 If the derivatives of f(x) and g(x), that are involved, exist in the neighborhood of $x=a$ and are continuous at $x=a$, and if the fraction $\frac{f(x)}{g(x)}$ assumes the indeterminate form $\frac{0}{0}$ at $x=a$, then

$$\lim_{x=a} \frac{f(x)}{g(x)}$$

will be equal to the first of the expressions

$$\frac{f'(a)}{g'(a)}, \frac{f''(a)}{g''(a)}, \frac{f'''(a)}{g'''(a)}, \ldots\ldots$$

which is not indeterminate, provided this expression exists.

Consider now the case $\lim\limits_{x=\infty} \frac{f(x)}{g(x)}$ where $\lim\limits_{x=\infty} f(x)=0$ and $\lim\limits_{x=\infty} g(x)=0$. Let $x=\frac{1}{t}$ and consider $\lim\limits_{t=0} \frac{f(\frac{1}{t})}{g(\frac{1}{t})}$. Then, by equation (33),

$$\lim_{t=0} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} = \lim_{t=0} \frac{\frac{1}{t^2} f'(\frac{1}{t})}{\frac{1}{t^2} g'(\frac{1}{t})} = \lim_{x=\infty} \frac{f'(x)}{g'(x)} .$$

This result is given by

Theorem 2.19 If the derivatives of f(x) and g(x), that are involved, exist for all values of x greater than some number N and if the fraction $\frac{f(x)}{g(x)}$ assumes the indeterminate form $\frac{0}{0}$ at $x=\infty$, then

$$\lim_{x=\infty} \frac{f(x)}{g(x)}$$

will be equal to the first of the expressions

$$\frac{\lim_{x=\infty} f'(x)}{\lim_{x=\infty} g'(x)}, \frac{\lim_{x=\infty} f''(x)}{\lim_{x=\infty} g''(x)}, \frac{\lim_{x=\infty} f'''(x)}{\lim_{x=\infty} g'''(x)}, \ldots.$$

which is not indeterminate, provided the expression exists.

If $\frac{f(x)}{g(x)}$ is such that $\lim_{x=\infty} f(x) = \infty$ and $\lim_{x=\infty} g(x) = \infty$, then if it is desirable to find the $\lim_{x=\infty} \frac{f(x)}{g(x)}$ it can be shown that theorem 2.19 applies.* By equation (30) we have

$$\frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(\xi)}{g'(\xi)} \tag{34}$$

where $c < \xi < x$ and c is large but finite.

---

* By $\lim_{x=a} f(x) = \infty$ we mean that for any $\epsilon > 0$ we can find an η such that if $|x - a| < \eta$ then it follows that $f(x) > \frac{1}{\epsilon}$. In a similar way $\lim_{x=\infty} f(x) = \infty$ means that for any $\epsilon > 0$ we can find an $\eta > 0$ such that if $x > \frac{1}{\eta}$ then it follows that $f(x) > \frac{1}{\epsilon}$.

$$\lim_{x=\infty} \frac{f(x)}{g(x)}$$

<u>will be equal to the first of the expressions</u>

$$\frac{\lim\limits_{x=\infty} f'(x)}{\lim\limits_{x=\infty} g'(x)}, \frac{\lim\limits_{x=\infty} f''(x)}{\lim\limits_{x=\infty} g''(x)}, \frac{\lim\limits_{x=\infty} f'''(x)}{\lim\limits_{x=\infty} g'''(x)}, \ldots.$$

<u>which is not indeterminate, provided the expression exists</u>.

If $\frac{f(x)}{g(x)}$ is such that $\lim\limits_{x=\infty} f(x) = \infty$ and $\lim\limits_{x=\infty} g(x) = \infty$, then if it is desirable to find the $\lim\limits_{x=\infty} \frac{f(x)}{g(x)}$ it can be shown that theorem 2.19 applies.* By equation (30) we have

$$\frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(\xi)}{g'(\xi)} \tag{34}$$

where $c < \xi < x$ and c is large but finite.

---

* By $\lim\limits_{x=a} f(x) = \infty$ we mean that for any $\epsilon > 0$ we can find an η such that if $|x - a| < \eta$ then it follows that $f(x) > \frac{1}{\epsilon}$. In a similar way $\lim\limits_{x=\infty} f(x) = \infty$ means that for any $\epsilon > 0$ we can find an $\eta > 0$ such that if $x > \frac{1}{\eta}$ then it follows that $f(x) > \frac{1}{\epsilon}$.

Now by algebra

$$(35) \qquad \frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \cdot \frac{1 - \frac{g(c)}{g(x)}}{1 - \frac{f(c)}{f(x)}} \; .$$

Assuming that $\frac{f'(\xi)}{g'(\xi)}$ has a limit at $\xi = \infty$ and calling that limit A, we can take c so large that $\frac{f'(c)}{g'(c)}$, and therefore also $\frac{f'(\xi)}{g'(\xi)}$, differs from A by less than ϵ_1. By this method c is now fixed and f(c) and g(c) are still finite. Since x may still vary, we can take x so large that $\frac{1 - \frac{g(c)}{g(x)}}{1 - \frac{f(c)}{f(x)}}$ will differ from 1 by ϵ_2. From equation (35), then,

$$\frac{f(x)}{g(x)} = (A + \eta_1)(1 + \eta_2) \qquad \text{where } |\eta_1| < \epsilon_1 \text{ and}$$

$|\eta_2| < \epsilon_2$. Hence

$$(36) \qquad \lim_{x=\infty} \frac{f(x)}{g(x)} = A = \lim_{x=\infty} \frac{f'(\xi)}{g'(\xi)} = \lim_{x=\infty} \frac{f'(x)}{g'(x)} \; .$$

Theorem 2.20 *If the derivatives of* f(x) *and* g(x), *that are involved, exist for all values of* x *greater than some number* N *and if the fraction* $\frac{f(x)}{g(x)}$ *assumes the indeterminate from* $\frac{\infty}{\infty}$ *at* $x = \infty$, *then*

$$\lim_{x=\infty} \frac{f(x)}{g(x)}$$

will be equal to the first of the expressions

$$\frac{\lim_{x=\infty} f'(x)}{\lim_{x=\infty} g'(x)}, \quad \frac{\lim_{x=\infty} f''(x)}{\lim_{x=\infty} g''(x)}, \quad \frac{\lim_{x=\infty} f'''(x)}{\lim_{x=\infty} g'''(x)}, \ldots$$

which is not indeterminate, provided this expression exists.

The case where $\lim_{x=a} f(x) = \infty$ and $\lim_{x=a} g(x) = \infty$ is easily obtained from theorem 2.20. For, letting $x = (a + \frac{1}{y})$, we obtain

$$(37) \qquad \frac{f(x)}{g(x)} = \frac{f(a+\frac{1}{y})}{g(a+\frac{1}{y})} = \frac{F(y)}{G(y)} \quad .$$

Then

$$(38) \qquad \lim_{x=a} \frac{f(x)}{g(x)} = \lim_{y=\infty} \frac{F(y)}{G(y)} = \lim_{y=\infty} \frac{F'(y)}{G'(y)}$$

by theorem 2.20. Since

$$F'(y) = f'(x) \frac{dx}{dy} = \frac{-1}{y^2} f'(x),$$

and

$$G'(y) = g'(x) \frac{dx}{dy} = \frac{-1}{y^2} g'(x),$$

equation (38) becomes

$$\lim_{x=a} \frac{f(x)}{g(x)} = \lim_{x=a} \frac{-\frac{1}{y^2} f'(x)}{-\frac{1}{y^2} g'(x)} = \lim_{x=a} \frac{f'(x)}{g'(x)} \quad .$$

This proves

<u>Theorem 2.21</u> <u>If the derivatives of</u> $f(x)$ <u>and</u> $g(x)$, <u>that are involved, exist in the neighborhood of</u> $x = a$ <u>and are continuous at</u> $x = a$, <u>and if the fraction</u> $\frac{f(x)}{g(x)}$ <u>assumes the indeterminate form</u> $\frac{\infty}{\infty}$ <u>when</u> $x = a$, <u>then</u>

$$\lim_{x=a} \frac{f(x)}{g(x)}$$

<u>will be equal to the first of the expressions</u>

$$\frac{f'(a)}{g'(a)},\ \frac{f''(a)}{g''(a)},\ \frac{f'''(a)}{g'''(a)},\ \dots$$

<u>which is not indeterminate, provided this expression exists</u>.

Most of the other indeterminate forms such as $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞, can be transformed into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The last three forms may be treated by the use of logarithms. A function, sec x - tan x, which becomes $\infty - \infty$ at $x = \frac{\pi}{2}$ may be written

$$\sec x - \tan x = \frac{1 - \sin x}{\cos x}$$

which becomes $\frac{0}{0}$ when $x = \frac{\pi}{2}$. In general, if $f(x) - g(x)$ becomes $\infty - \infty$, then we can write

$$f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x) \cdot g(x)}}$$

which is of the form $\frac{0}{0}$.

Also, if $f(x) \cdot g(x) = 0 \cdot \infty$, we set $f(x) \cdot g(x) = \dfrac{f(x)}{\frac{1}{g(x)}}$.
This reduces to the form $\frac{0}{0}$.

An error that is often made is the assumption that $\lim_{x=a} \frac{f(x)}{g(x)}$ does not exist if $\lim_{x=a} \frac{f'(x)}{g'(x)}$ does not exist. The falseness of such an assumption can be shown by an example. Consider

$$f(x) = x^2 \sin\frac{1}{x}, \; g(x) = x, \text{ and } a = 0.$$

Then $\lim_{x=0} \frac{f(x)}{g(x)} = \lim_{x=0} \frac{x^2 \sin\frac{1}{x}}{x} = \lim_{x=0} (x \sin\frac{1}{x}) = 0$.

At the same time

$$\lim_{x=0} \frac{f'(x)}{g'(x)} = \lim_{x=0} \left(\frac{2x \sin\frac{1}{x} - \cos\frac{1}{x}}{1} \right)$$

which does not exist at $x = 0$.

It should be noticed that in order to say that

$$\lim_{x=a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

we need the continuity of the derivatives at $x = a$.

Some other mistakes that often occur in the treatment of indeterminate forms are the following. If $h(x) = \frac{f(x)}{g(x)}$ and $f(x)$ and $g(x)$ vanish at $x = a$, then the value of $h(x)$ at $x = a$ is undefined. It is incorrect to speak of the true value at $x = a$ because there exists no such value.

No transformation or limiting process will bring out a true value when none exists. Many times it is desirable to define h(x) at $x=a$ and h(x) can be defined to be anything you please. For the sake of continuity h(x) is sometimes defined at $x=a$ to be $\lim\limits_{x=a} \frac{f(x)}{g(x)}$ if this limit is finite. Slobin and Solt are quite emphatic in pointing out that we are discussing undetermined forms and not indeterminate forms. That is, they say that $\frac{0}{0}$ has a value which is found by means of a limit. Obviously they are giving a definition.

To find the value of $\lim\limits_{x=a} h(x)$ some might write

$$(39) \qquad h(a+k) = \frac{f(a+k)}{g(a+k)} = \frac{\frac{f(a+k)-f(a)}{k}}{\frac{g(a+k)-g(a)}{k}}$$

and conclude that $\lim\limits_{x=a} h(x) = \frac{f'(a)}{g'(a)}$. This is true if this limit exists and $g'(a) \neq 0$. If both $f'(a)$ and $g'(a)$ vanish then it is impossible by this method to conclude that

$$\lim_{x=a} \frac{f(x)}{g(x)} = \frac{f''(a)}{g''(a)} ,$$

since equation (39) only says that $\lim\limits_{x=a} h(x) = \frac{f'(a)}{g'(a)}$ and not that $\lim\limits_{x=a} h(x) = \lim\limits_{x=a} \frac{f'(x)}{g'(x)}$.

It is possible to find $\lim_{x=a} h(x)$ by writing f(x) and g(x) in power series. However power series representing such functions as sec x, csc x, tan x, cot x, $e^{\sin x}$, are seldom developed satisfactorily in elementary books. Unless they are developed an author would have no right to use them when they occur in finding $\lim_{x=a} h(x)$.

In evaluating the form $\frac{\infty}{\infty}$ one might write

$$h(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}} .$$

Using theorem 2.20 we would obtain

$$\lim_{x=a} h(x) = \lim_{x=a} \frac{\frac{g'(x)}{g^2(x)}}{\frac{f'(x)}{f^2(x)}} = \lim_{x=a} h^2(x) \frac{g'(x)}{f'(x)} .$$

Then dividing by $\lim_{x=a} h(x)$, we get

$$1 = \lim_{x=a} h(x) \frac{g'(x)}{f'(x)} ,$$

or that

$$\lim_{x=a} h(x) = \frac{f'(a)}{g'(a)} .$$

Here the error comes in assuming the existence of $\lim_{x=a} h(x)$ which as far as we know does not exist until its existence has been shown.

Consider the $\lim_{x=0} \frac{\sin x}{x}$. Since both sin x and x vanish at $x = 0$ let us apply theorem 2.19. Then

$$\lim_{x=0} \frac{\sin x}{x} = \lim_{x=0} \frac{\cos x}{1} = 1 .$$

This would be an easy way to dispose of this troublesome limit of section 2.1 were it not for the fact that we used the $\lim_{x=0} \frac{\sin x}{x} = 1$ to develop a formula for the derivative of sin x. Now we turn around and use the derivative to evaluate the limit, thus forming the customary vicious circle. Many of the elementary calculus books have this example as one of their problems in indeterminate forms.

In discussing $\lim_{x=a} \frac{f(x)}{g(x)}$ when it reduces to $\frac{0}{0}$ we find that both Love and McKelvey, in their books, use a method that is incomplete in that it cannot be extended to higher derivatives. This is true since they write

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(x_1)}{g'(x_2)} .$$

In the calculus book by Granville, Smith and Longley as well as in the calculus book by Neelley and Tracy no mention is made of the fact that the existence and continuity of $f'(x)$ and $g'(x)$ is assumed. Dalaker and Hartig's book is the only one of these books that assumed existence and continuity of $f'(x)$ and $g'(x)$.

Smith, Salkover, and Justice as well as Slobin and Solt assume that if $\lim\limits_{x=a} \frac{f'(x)}{g'(x)}$ does not exist then $\lim\limits_{x=a} \frac{f(x)}{g(x)}$ does not exist. The former, on page 338 of their text, state, "If the fraction $\frac{f(x)}{g(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when $x = a$ then

$$\lim_{x=a} \frac{f(x)}{g(x)}$$

will be equal to the first of the expressions

$$\frac{f'(a)}{g'(a)}, \frac{f''(a)}{g''(a)}, \frac{f'''(a)}{g'''(a)}, \ldots$$

which is not indeterminate provided this expression exists; and if it fails to exist, the limit sought does likewise." The last part is obviously not true.

CHAPTER III

THE DEFINITE INTEGRAL

3.1 Theorems on Continuity. In order to prove the existence theorem for the definite integral we need two theorems concerning continuity.

Lemma 3.1 If f(x) is continuous in the closed interval (a,b) and ϵ is an arbitrary positive number, then (a,b) can be divided into partial intervals such that the difference between the values of f(x) at any two points in the same partial interval is less than ϵ in absolute value.

Divide the interval (a_0, b_0) by a point $c = \frac{a_0 + b_0}{2}$. Unless the lemma is true, either (a_0, c) or (c, b_0) has two points that do not satisfy the lemma. Suppose (a_0, c) has two such points. Call $a_0 = a_1$ and $c = b_1$ and divide the interval (a_1, b_1) as (a_0, b_0) was divided. Continue this process. Thus we have two unlimited sequences of points, $a_0, a_1, a_2, \ldots\, a_n, \ldots$ and $b_0, b_1, b_2, \ldots b_n, \ldots$ For every n, $a_{n-1} \leq a_n < b$ and $a < b_n \leq b_{n-1}$. Let A be the upper limit of the a_n and B be the lower limit of the b_n. Since the length of any interval is one-half the preceding one, then

$$\lim_{n=\infty} (b_n - a_n) = 0 \quad .$$

Therefore $A = B$.

In each interval there are, however, two points, x'_n and x''_n, such that

$$(40) \qquad \left| f(x'_n) - f(x''_n) \right| \geq \epsilon \quad .$$

Now $f(x)$ is continuous at $x = A$ since A is in the interval (a,b) or at one of the end points. Therefore there is an $h > 0$ such that

$$\left| f(x) - f(A) \right| < \frac{\epsilon}{2}$$

for all x's within (a,b) such that $A - h < x < A + h$. For any two such values of x, say x_1 and x_2, we have

$$\left| f(x_1) - f(A) \right| < \frac{\epsilon}{2} \quad ,$$

$$\left| f(A) - f(x_2) \right| < \frac{\epsilon}{2} \quad ,$$

and hence

$$(41) \qquad \left| f(x_1) - f(x_2) \right| < \epsilon \quad .$$

But a_n is, for a large enough n, in the interval $(A - h, A)$ and b_n is in the interval $(A, A + h)$. Then for the points x_1 and x_2 in (a_n, b_n) we have from equation (40) that

$$\left| f(x_1) - f(x_2) \right| \geq \epsilon \quad ,$$

and, from equation (41), that

$$\left| f(x_1) - f(x_2) \right| < \epsilon \quad .$$

This contradicts our assumption that there was an interval in which the lemma did not hold.

Theorem 3.1 *If* f(x) *is continuous in the closed interval* (a,b) *and* ϵ *is greater than zero, then there is an* η *greater than zero such that the difference between the values of* f(x) *at any two points whose distance apart does not exceed* η *is less than* ϵ *in absolute value.*

Applying lemma 3.1 we find that if the difference between two points x' and x'' of (a,b) is less than the length of any of these partial intervals, then these two points must either lie in the same partial interval or in adjacent ones. Then

$$\left| f(x') - f(x'') \right| < \epsilon$$

if x',x'' are in the same interval, and if x' is in (x_{n-1},x_n) and x'' is in (x_n,x_{n+1}) then

$$\left| f(x') - f(x_n) \right| < \epsilon \quad ,$$

and

$$\left|f(x_m) - f(x'')\right| < \epsilon \quad .$$

Hence

$$\left|f(x') - f(x'')\right| < 2\epsilon \quad .$$

Therefore $\eta \leq$ length of any partial interval that satisfies lemma 3.1.

<u>Theorem 3.2</u> <u>If</u> f(x) <u>is</u> <u>continuous</u> <u>in</u> <u>the</u> <u>closed</u> <u>interval</u> (a,b) <u>then</u> f(x) <u>is bounded</u> <u>in</u> (a,b).

For any $\epsilon > 0$ we can, by lemma 3.1, subdivide (a,b) into partial intervals such that

$$\left|f(x') - f(x'')\right| < \epsilon$$

if x' and x'' are in a partial interval. Now if x is in the first partial interval (a,x_1) we have

$$\left|f(x) - f(a)\right| < \epsilon \quad ,$$

and

$$\left|f(x)\right| < \left|f(a)\right| + \epsilon \quad .$$

If x is in the second partial interval (x_1,x_2) then

$$\left|f(x) - f(x_1)\right| < \epsilon \quad ,$$

and

$$|f(x)| < |f(x_1)| + \epsilon < |f(a)| + 2\epsilon .$$

By continuing this process through the n partial intervals we have

$$|f(x)| < |f(a)| + n\epsilon$$

for any x in (a,b) .

3.2 <u>Existence Theorem for the Definite Integral</u>. Let the interval (a,b) be divided into a set D of n sub-intervals where the i-th interval is denoted by $\Delta x_i = x_i - x_{i-1}$. Let ξ_i be any x such that $x_{i-1} \leq \xi_i \leq x_i$. We denote by ND the maximum of the Δx_i . Now form the sum

$$S_D = \sum_{i=1}^{n} f(\xi_i)\Delta x_i .$$

If the $\lim_{ND=0} S_D$ exists it is called the definite integral of f(x) between a and b and is written $\int_a^b f(x)\,dx$.

<u>Theorem 3.3 The</u> $\lim_{ND=0} S_D = \int_a^b f(x)\,dx$ <u>exists if</u> f(x) <u>is continuous</u>.

Let f(x) be a continuous function in (a,b). Let M_i = maximum f(x) in Δx_i and m_i = minimum f(x) in Δx_i.

Let

(42) $$\overline{S}_D = \sum_1^n M_i \, \Delta x_i$$

and

(43) $$\underline{S}_D = \sum_1^n m_i \, \Delta x_i \quad .$$

Then

$$\overline{S}_D \geq S_D \geq \underline{S}_D \quad .$$

Let D_2 be composed of D_1 plus some additional points. Then $\overline{S}_{D_2} \leq \overline{S}_{D_1}$ and $\underline{S}_{D_2} \geq \underline{S}_{D_1}$ since Δx_i will be sub-divided and the M_i of Δx_i will be replaced in some of the sub-divisions by $M_i' \leq M_i$ and therefore $\overline{S}_{D_2} \leq \overline{S}_{D_1}$. Likewise $\underline{S}_{D_2} \geq \underline{S}_{D_1}$. Let D and $\overline{D}$ be any two divisions. Then

$$\overline{S}_{D+\overline{D}} \leq \overline{S}_D \;, \; \underline{S}_{D+\overline{D}} \geq \underline{S}_D \;, \; \overline{S}_{D+\overline{D}} \leq \overline{S}_{\overline{D}} \;, \; \underline{S}_{D+\overline{D}} \geq \underline{S}_{\overline{D}} \;,$$

and since

$$\underline{S}_{D+\overline{D}} \leq \overline{S}_{D+\overline{D}} \;,$$

then

$$\underline{S}_{\overline{D}} \leq \overline{S}_D \;, \; \underline{S}_D \leq \overline{S}_{\overline{D}} \quad .$$

Let

(44) $$\overline{I} = \underset{D}{B}\, \overline{S}_D \quad \text{and} \quad \underline{I} = \overline{\underset{D}{B}}\, \underline{S}_D \,.$$

The greatest lower bound and the least upper bound exist since $\overline{S}_D \geq m(b - a)$ and $\underline{S}_D \leq M(b - a)$ where m and M are the greatest lower and least upper bound respectively of f(x) in (a,b). From equation (43) we have

$$\overline{I} \geq \underline{I} \quad .$$

Since f(x) is a continuous function we have by lemma 3.1 that

$$M_i - m_i \leq \frac{\epsilon}{b - a} \quad .$$

Then

$$\overline{S}_D - \underline{S}_D \leq \sum (M_i - m_i)\Delta x_i \leq \epsilon$$

and

$$\lim_{ND=0} (\overline{S}_D - \underline{S}_D) = 0 \quad .$$

But

$$\overline{S}_D - \underline{S}_D = (\overline{S}_D - \overline{I}) + (\overline{I} - \underline{I}) + (\underline{I} - \underline{S}_D) \quad .$$

Since each term on the right is positive or zero then

$$\lim_{ND=0} (\overline{S}_D - \overline{I}) = 0 \ ,$$

$$\lim_{ND=0} (\overline{I} - \underline{I}) = 0 \ ,$$

$$\lim_{ND=0} (\underline{I} - \underline{S}_D) = 0 \quad .$$

Then

$$\lim_{ND=0} \bar{S}_D = \bar{I} = \underline{I} = \lim_{ND=0} \underline{S}_D \quad .$$

Therefore

$$\lim_{ND=0} S_D = \lim_{ND=0} \bar{S}_D = I \quad ,$$

which proves the theorem.

Corollary 3.1 *If f(x) is a function with at most a finite number of finite discontinuities, then the definite integral* $\int_a^b f(x)\,dx$ *exists.*

Let the discontinuities be K in number and let the sum of the Δx_i in which the discontinuities lie be denoted by L. Then for any $\epsilon > 0$ we can find a d such that if $ND < d$ then it follows that

$$\bar{S}_D - \underline{S}_D \leq \epsilon(b - a) + K(M - m)d \quad .$$

Therefore

$$\lim_{ND=0} (\bar{S}_D - \underline{S}_D) = 0 \quad .$$

The rest of the proof is as in theorem 3.3 .

3.3 Duhamel's theorem. Theorem 3.4 *If* $a_1, a_2, \ldots a_m$ *is a set of positive infinitesimals such that* $\lim_{m=\infty} \sum_{i=1}^{m} a_i = A$ *and if* $a_i = b_i + \epsilon_i a_i$ *such that for any* $\eta > 0$ *it follows that there exists an* m *such that if* $n > m$ *then* $|\epsilon_i| < \eta$ *for* $i = 1, 2, \ldots n$, *then* $\lim_{m=\infty} \sum_{1}^{m} b_i = A$.

For since $|\epsilon_i| < \eta$ and the a_i are positive then

$$-\eta a_i \leq \epsilon_i a_i \leq \eta a_i \quad .$$

Then $$-\eta A \leq \sum \epsilon_i a \leq \eta A \quad ,$$

and therefore

$$\operatorname*{Lim}_{n=\infty} \sum_1^n \epsilon_i a = 0 \quad .$$

Hence

$$\operatorname*{Lim}_{n=\infty} \sum_1^m a_i = \operatorname*{Lim}_{m=\infty} \sum_1^m b_i = A \quad .$$

3.4 <u>Osgood's theorem</u>. <u>Theorem 3.5</u> <u>If</u> $a_1, a_2, \ldots a_n$ <u>is a set of positive infinitesimals such that</u> $|a_i - f(x_i)\Delta x_i| < \epsilon_i$, <u>where the</u> ϵ_i <u>are of higher order than</u> Δx_i, <u>and the</u> f(x) <u>is continuous in</u> $a \leq x \leq b$ <u>and if for any</u> $\eta > 0$ <u>it follows that there exists an</u> m <u>such that if</u> $n > m$ <u>then</u> $|\epsilon_i| < \eta$ <u>for</u> $i = 1, 2, \ldots n$, <u>then</u>

$$\operatorname*{Lim}_{m=\infty} \sum_1^m a_i = \int_a^b f(x)\, dx \quad .$$

For let

$$a_i = f(x_i)\Delta x_i + \epsilon_i \Delta x_i$$

where $|\epsilon_i| < \eta$. Then

$$\left| \sum a_i - \sum f(x_i)\Delta x_i \right| < \eta \sum \Delta x_i = \eta(b - a) \quad .$$

Since f(x) is continuous the definite integral exists and

$$\left| \sum f(x_i)\Delta x_i - \int_a^b f(x)\,dx \right| < \eta .$$

Therefore

$$\left| \sum a_i - \int_a^b f(x)\,dx \right| < \eta(b - a + 1)$$

and

$$\lim_{n=\infty} \sum a_i = \int_a^b f(x)\,dx .$$

On page 22 of Wood's Advanced Calculus we find Duhamel's theorem stated with the uniform approach to zero omitted. This, of course, makes the proof impossible.

3.5 <u>The Fundamental Theorem of Calculus.</u> <u>Theorem 3.6</u> <u>If a function</u> f(x) <u>is integrable and if</u> f(x) <u>has a primitive</u>, F(x), <u>then</u> $\int_a^b f(x)\,dx = F(b) - F(a)$ *.

Instead of proving this theorem we will prove the following, more elementary, theorem.

<u>Theorem 3.7</u> <u>If</u> f(x) <u>is a continuous function which has a primitive</u>, F(x), <u>then</u> $\int_a^b f(x)\,dx = F(b) - F(a)$.

---

* By primitive we mean that F(x) is a function whose derivative is f(x).

Let

(45) $$g(x) = \int_a^x f(x)\,dx \quad .$$

Then

$$g(x+h) - g(x) = \int_a^{x+h} f(x)\,dx - \int_a^x f(x)\,dx = \int_x^{x+h} f(x)\,dx \;,$$

$$= h \cdot f(\xi) \quad , \text{ where } x \leq \xi \leq x+h \,,$$

since

$$\int_a^b f(x)\,dx = (b - a)\, f(\xi)$$

for f(x) continuous. This shows that

$$g'(x) = \lim_{h=0} \frac{g(x+h) - g(x)}{h} = \lim_{\xi=x} f(\xi) = f(x) \;.$$

Then $g'(x) = F'(x)$. Letting $h(x) = g(x) - F(x)$ we see that

$$h'(x) = g'(x) - F'(x) = 0 \;.$$

Therefore $h(x) = C =$ constant, and

(46) $$g(x) = F(x) + C = \int_a^x f(x)\,dx \quad .$$

Since $g(a) = 0$, we have $C = - F(a)$ and equation (46) becomes

$$F(x) - F(a) = \int_a^x f(x)\,dx \;.$$

Finally, letting $x = b$, we have

$$F(b) - F(a) = \int_a^b f(x)\,dx \;.$$

CHAPTER IV

APPLICATIONS OF THE DEFINITE INTEGRAL

4.1 Introduction. In this section we will attempt to justify some of the integrals set up in the elementary calculus. Instead of using Duhamel's theorem or Osgood's theorem we shall proceed directly from the definition of a definite integral. It would seem that there are two ways in which these problems may be considered. One would be that there is no such thing as an area or volume until it is defined. Using this view point we would state the definition and from it derive the integral. A second way of considering the problem is to think of area, volume, and pressure as physical entities with our problem one of giving definitions or methods suitable for evaluating these quantities. In this section we will usually take the second viewpoint.

For a general definition of area bounded by a curve the reader is referred to Fine's Calculus.*

---

* H. B. Fine, Calculus, p. 136 .

It would seem that such a definition and discussion as given by Fine would be necessary to show that the area found by rectangular coordinates is the same as that found by polar coordinates. A similar remark would hold for volume found by the disc method and the shell method .

4.2 Area Under a Curve . Let $y = f(x)$ be a continuous curve under which we are to find the area, A, between the ordinates $x = a$ and $x = b$ and above the x-axis.

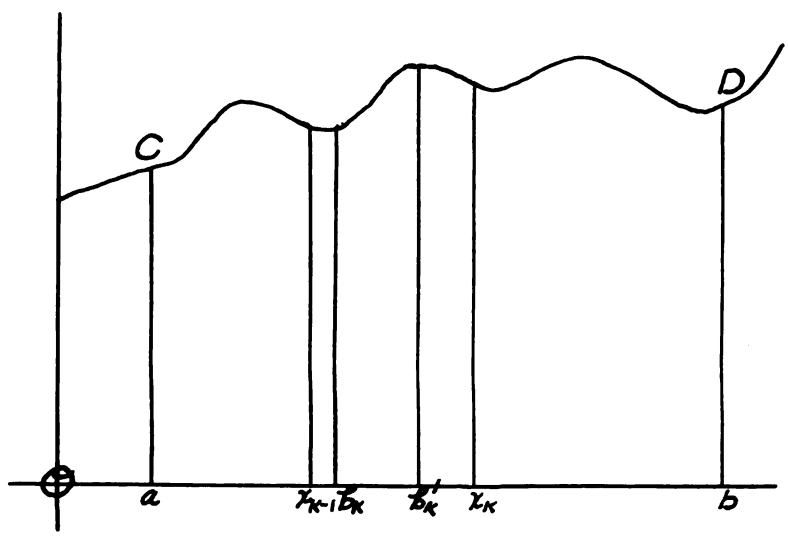


Fig. 5

Divide (a,b) into n sub-intervals Δx_K , and erect ordinates at the points of division. This divides the area A into n parts A_K .

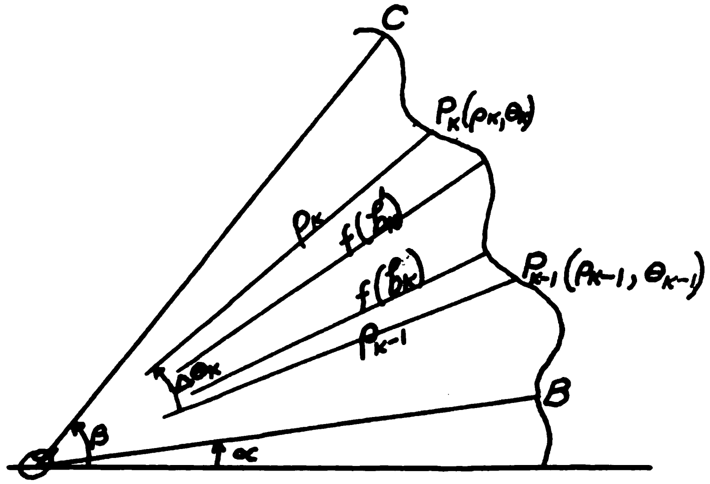
Let b_K be the point in Δx_K at which the function is a minimum and b_K' be the point in Δx_K at which the function is a maximum. * The area A_K will be less than $\Delta x_K b_K'$ and greater than $\Delta x_K b_K$. There will be some x value , ξ_K , in Δx_K such that $f(\xi_K)\Delta x_K = A_K$.**
Then

$$A = \sum_1^m f(\xi_K)\Delta x_K \quad .$$

By the definition of the definite integral and the existence theorem we have

$$A = \lim_{ND=0} \sum_1^m f(\xi_K)\Delta x_K = \int_a^b f(x)\, dx \quad .$$

4.3 <u>Area in Polar Coordinates</u>. We wish to find the area bounded by the continuous curve $\rho = f(\theta)$ and two radii vectors whose angles of inclination are α and β .



<u>Fig. 6</u>

* Fite, p. 25

** Fite, p. 24

Divide that area A into parts A_K by dividing $\beta-\alpha$ into n angles $\Delta\theta_K$. Let θ_K and θ_K' be the angles in $\Delta\theta_K$ at which the radii vectors are a minimum and maximum respectively. Form circular segments with these radii. Then the true area A_K lies between

$$\frac{1}{2}\left[f(\theta_K)\right]^2\Delta\theta_K$$

and

$$\frac{1}{2}\left[f(\theta_K')\right]^2\Delta\theta_K$$

since the area of a circular segment is equal to one half the central angle times the arc length. Then there exists an angle (ξ_K) such that

$$A_K=\frac{1}{2}\left[f(\xi_K)\right]^2\Delta\theta_K\ .$$

Therefore by the existence theorem we get

$$A=\lim_{ND=0}\sum_{1}^{m}\frac{1}{2}\left[f(\xi_K)\right]^2\Delta\theta_K=\frac{1}{2}\int_{\alpha}^{\beta}\rho^2 d\theta\ .$$

4.4 Volume by the Disc Method. Let V be the volume of the solid generated by revolving the plane surface ABCD about the x-axis, where the equation of the continuous curve DC is $y=f(x)$.

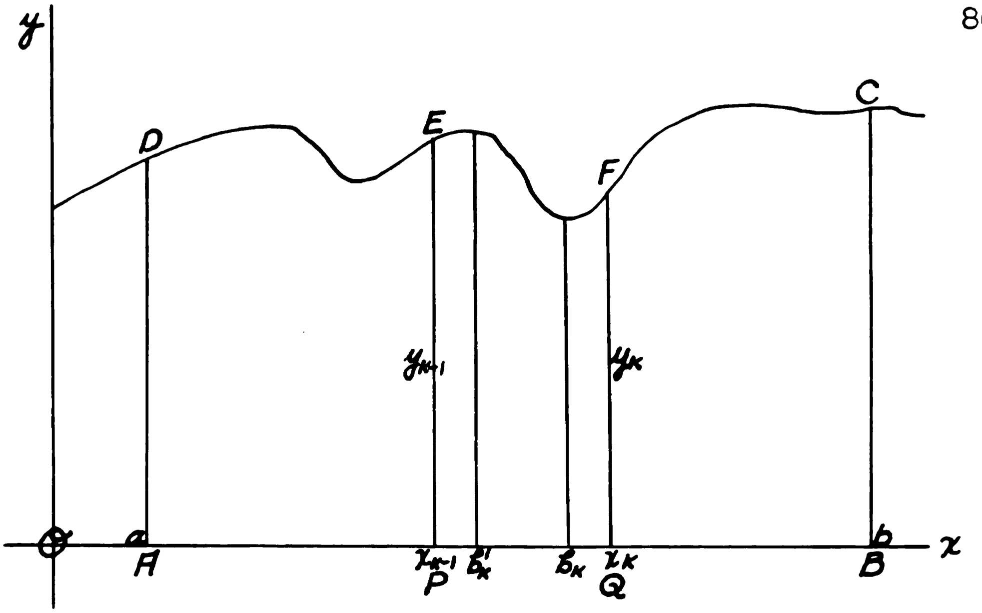


Fig. 7

When the area PQEF is revolved about the x-axis it generates a solid and the sum of all such volumes make up the volume V. Now if b'_K is the point in Δx_K at which the function is a maximum and b_K is the point where the function is a minimum then the volume $\pi(b'_K)^2\Delta x_K$ is larger than the actual volume and $\pi(b_K)^2\Delta x_K$ is smaller than the actual volume. Therefore between x_{K-1} and x_K is a value ξ_K for which the volume

$$\pi\left[f(\xi_K)\right]^2\Delta x_K$$

equals the actual volume. Then

$$V = \sum_{1}^{m} \pi\left[f(\xi_K)\right]^2 \Delta x_K \; .$$

Therefore, by the existence theorem, we have

$$V = \lim_{ND=0} \sum_{1}^{n} \pi [f(\xi_K)]^2 \Delta x_K = \pi \int_a^b [f(x)]^2 \, dx .$$

4.5 Volume by the Shell Method. Consider the case of finding the volume, V, of the solid generated by revolving the area ABCD about the y-axis where $y = f(x)$ is the equation of the continuous curve DC.

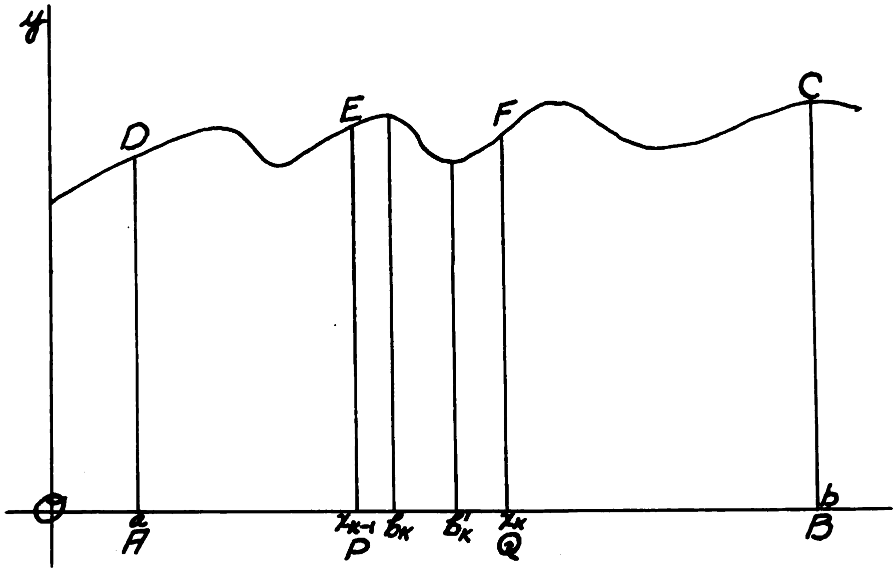


Fig. 8

Divide (a,b) into intervals Δx_K and consider the volumes, V_K, formed by revolving each piece of area, such as PQEF, about the y-axis. The sum of all such volumes, V_K, will be V. Let ξ_K be a value of x for which f(x) is a maximum in Δx_K and let ξ'_K be a value of x for which f(x) is a minimum in Δx_K. Then

$$2\pi x_{K-1} f(\xi'_K) \Delta x_K$$

will be smaller than V_K and

$$2\pi x_K f(b_K)\Delta x_K$$

will be larger than V_K. Select a value of $x = \xi_K$ such that

$$2\pi(\xi_K + \theta_K \Delta x_K)\cdot f(\xi_K)$$

will be equal to the volume V_K where $b'_K < \xi_K < b_K$ and $0 \leq \theta_K \leq 1$. Then

$$V = \sum_1^n 2\pi(\xi_K + \theta_K\Delta x_K)\cdot f(\xi_K)\Delta x_K ,$$
$$= \sum(2\pi\xi_K f(\xi_K)\Delta x_K) + \sum(2\pi\theta_K f(\xi_K)\overline{\Delta x}_K^2) .$$

Now

$$\left|\sum(2\pi\theta_K f(\xi_K)\overline{\Delta x}_K^2)\right| \leq \sum\left|2\pi\theta_K f(\xi_K)\overline{\Delta x}_K^2\right| ,$$
$$\leq 2\pi M\, ND \sum \Delta x_K ,$$
$$\leq 2\pi M\, ND\,(b - a) ,$$

where $M \geq |f(x)|$. Therefore

$$\lim_{ND=0} \sum_1^n (2\pi\theta_K f(\xi_K)\overline{\Delta x}_K^2) = 0$$

and, by the existence theorem of the definite integral,

$$V = \lim_{ND=0}\sum_1^n (2\pi\xi_K f(\xi_K)\Delta x_K) = 2\pi\int_a^b x\, f(x)\, dx .$$

4.6 Length of a Curve. Let the curve whose length is to be found be $y=f(x)$ where $f'(x)$ exists and is continuous at all points. The definition of the length

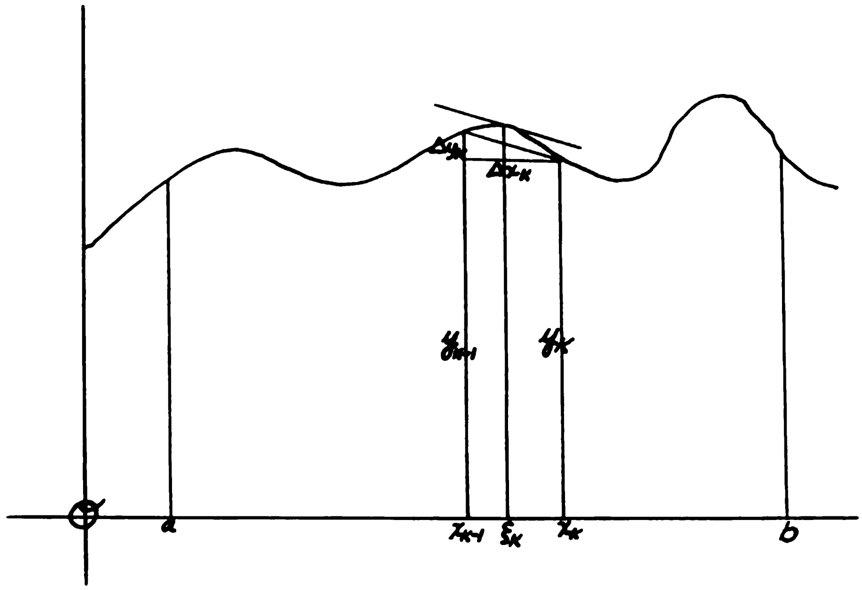


Fig. 9

of a curve is

$$L = \lim_{chords=0} \sum (\text{lengths of chords}) \quad .$$

Therefore

$$L = \lim_{ND=0} \sum \sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2} \quad ,$$

$$= \lim_{ND=0} \sum \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\, \Delta x_k \quad .$$

There exists, by the mean value theorem, a point ξ_k in Δx_k at which the slope of the curve equals the slope of the chord; that is $\frac{\Delta y_k}{\Delta x_k} = f'(\xi_k)$.

Therefore

$$L = \lim_{ND=0} \sum \sqrt{1 + [f'(\xi_K)]^2}\, \Delta x_K .$$

Applying the existence theorem for the definite integral, we have

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\; dx .$$

4.7 Surface of Revolution. Surface of revolution, S, is defined to be

$$\lim_{chords=0} \sum (\text{surface of frustrums of cones}).$$

We shall require that f(x) have a continuous derivative. Using figure 9,

$$S = \lim_{ND=0} \sum 2\pi \left(\frac{y_{K-1} + y_K}{2}\right) \sqrt{\Delta x_K^2 + \Delta y^2}$$

since the surface of the frustrum of a cone equals the average base times the slant height. Then

$$S = \lim_{ND=0} \sum 2\pi \left(\frac{y_{K-1} + y_K}{2}\right) \sqrt{1 + [f'(\xi_K)]^2}\, \Delta x_K ,$$

and

$$S = \lim_{ND=0} \sum 2\pi \left[f(\xi_K) + \theta_K \Delta y_K\right] \sqrt{1 + [f'(\xi_K)]^2}\, \Delta x_K ,$$

since the value of x which satisfies the mean value theorem may not be the value of x so that $f(\xi_K)$ will represent the radius of the average base.

Now

$$S = \lim_{ND=0} \sum 2\pi f(\xi_k) \sqrt{1+[f'(\xi_k)]^2}\,\Delta x_k + \lim_{ND=0} \sum 2\pi \theta_k \Delta y_k \sqrt{1+[f'(\xi_k)]^2}\,\Delta x_k \;.$$

Call the first sum S_1, and the second sum S_2. Since $f(x)$ is continuous in (a,b) we can for any $\epsilon > 0$ find a d such that if $N D < d$ then

$$|\Delta y_k| < \epsilon \qquad , \quad k = 1, 2, \ldots\ldots n \;.$$

Therefore

$$|S_2| \leq 2\pi \epsilon M(b - a)$$

where M is the maximum of $\sqrt{1 + [f'(x)]^2}$ in (a,b). This shows that

$$\lim_{ND=0} S_2 = 0 \;.$$

Then

$$S = \lim_{ND=0} S_1 = 2\pi \int_a^b f(x) \sqrt{1+[f'(x)]^2} \; dx$$

by making use of the existence theorem for definite integrals.

4.8 Pressure on a Horizontal Submerged Area. Pressure per unit area is defined to be density, w, times the depth, y. Let the plane surface, A, be submerged vertically in the liquid.

Take the x-axis in the surface and the y-axis downward. We suppose that the bounding curve of A is continuous.

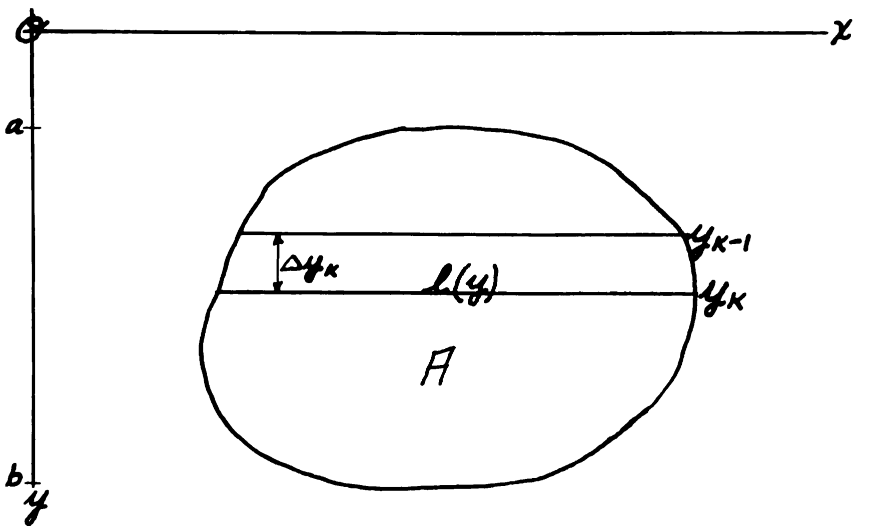


Fig. 10

Divide A into horizontal strips Δy_K in width. Let b_K and b'_K be the values of y in Δy_K for which the length, $l(y)$, is a maximum and minimum respectively. Now the pressure, ΔP, on the strip Δy_K is less than $l(b_K)w\ y_K \Delta y_K$ and greater than $l(b'_K)w\ y_{K-1}\Delta y_K$. There exists a value ξ_K of y for which the pressure ΔP on the Δy_K strip is

$$l(\xi_K)w(\xi_K + \theta_K \Delta y_K)\Delta y_K \quad .$$

This follows since ξ_K is $b'_K < \xi_K < b_K$ and, if ξ_K is picked so that $l(\xi_K)\Delta y_K$ is the area, the same ξ_K may not be the proper depth to use to get the pressure on this area.

As a result, we have,

$$P=\sum_1^m \Delta P = \sum_1^m \ell(\xi_K)w(\xi_K+\theta_K\Delta y_K)\Delta y_K ,$$

$$=\sum\left[\ell(\xi_K)w\xi_K\Delta y_K\right] + \sum\left[\ell(\xi_K)w\,\theta_K\overline{\Delta y_K^2}\right] ,$$

$$= \quad S_1 \quad + \quad S_2 \quad .$$

Now

$$|S_2| \leq \sum |w\,\ell(\xi_K)\theta_K\Delta y_K ,$$
$$\leq X\ w\sum|\overline{\Delta y_K^2}| ,$$
$$\leq X\ w\ ND\sum|\Delta y_K| ,$$
$$\leq X\ w\ ND\ (b-a) ,$$

where X is the maximum of $\ell(y)$. Therefore

$$\lim_{ND=0} S_2 = 0 \quad .$$

Then by use of the existence theorem for definite integrals we have that

$$P = \lim_{ND=0} S_1 = \int_a^b \ell(y)\ w\ y\ dy \quad .$$

4.9 Work. The work, W, is defined to be the force, F, times the distance, d, when the force is constant. Let f(x) be a continuous function of x representing a variable force acting on a particle, P, in the direction OX. Let the particle, P, be moved from $x=a$ to $x=b$.

Divide (a,b) into intervals, Δx_k. Let b_k and b_k' be the values of x in Δx_k for which the force, f(x) is a maximum and minimum respectively. Now the work, ΔW, done in passing over Δx_k is greater than $f(b_k')\Delta x_k$ and less than $f(b_k)\Delta x_k$. Then there exists a ξ_k, $b_k' < \xi_k < b_k$, for which

$$\Delta W = f(\xi_k)\Delta x_k \quad .$$

Then

$$W = \sum \Delta W = \sum_1^n f(\xi_k)\Delta x_k \quad .$$

By applying the existence theorem for the definite integral, we get

$$W = \lim_{ND=0} \sum_1^n f(\xi_k)\Delta x_k = \int_a^b f(x)\, dx \quad .$$

4.10 Moment of Inertia. Moment of inertia about a line ℓ is defined to be mr^2 for a point mass, m, at a distance r from ℓ. Let a mass, m, be distributed along the x-axis from $x = a > 0$ to $x = b > a$ such that the linear density, f(x), is a continuous function. We wish to find the moment of inertia about the y-axis. Divide (a,b) into sub-intervals $\Delta x_k = x_k - x_{k-1}$. Let b_k and b_k' be the values of x where f(x) has a maximum and minimum respectively in Δx_k.

Then the moment of inertia of Δx_K is less than $x_K^2 f(\mathcal{B}_K)\Delta x_K$ and greater than $x_{K-1}^2 f(\mathcal{B}_K')\Delta x_K$. Therefore there exists a ξ_K such that the moment of inertia of Δx_K is

$$(\xi_K + \theta_K \Delta x_K)^2 f(\xi_K)\Delta x_K \quad .$$

This gives the total moment of inertia as

$$\sum_{K=1}^{n}(\xi_K + \theta_K \Delta x_K)^2 f(\xi_K)\Delta x_K \quad ,$$

$$= \sum \xi_K^2 f(\xi_K)\Delta x_K + \sum(2\xi_K\theta_K + \theta_K^2 \Delta x_K)\ f(\xi_K)\overline{\Delta x}_K^2 \quad .$$

Denoting these sums by S_1 and S_2 respectively we have that

$$|S_2| \leq (2b + ND)\ M \cdot ND(b - a)$$

where M is the maximum of f(x) in (a,b). This gives

$$\lim_{ND=0} S_2 = 0 \,,$$

and therefore

$$\text{Moment of inertia} = \lim_{ND=0} S_1 = \int_a^b x^2 f(x)\, dx \quad .$$

4.11 <u>Criticism</u>. It might well be asked why we are so careful in showing that such quantities as area, volume, and pressure can be found by certain definite integrals.

It probably seems evident that the approximations usually made are sufficient to allow us to actually find these quantities upon taking the limit. It is not, however, always evident that these usual approximations are sufficient to give us the desired results. To illustrate this fact consider the following example.

Suppose we wish to find the surface of revolution, S, formed by revolving the semi-circle, BCA, of figure 11 about the x-axis.

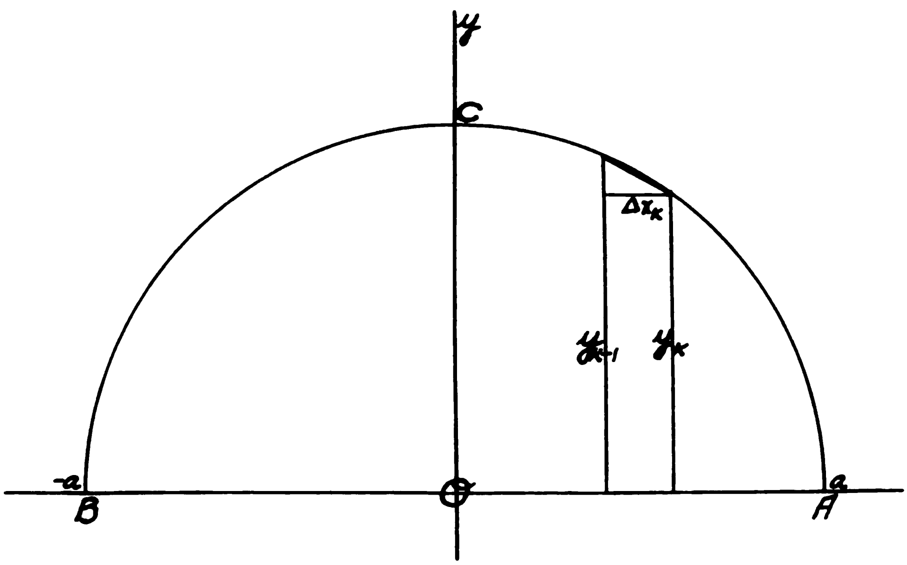


Fig. 11

Since $\lim_{\Delta x = 0}$ (chord) = 0 we obtain an approximation to the surface by summing the lateral areas, $2\pi y \Delta x_k$, of discs.

Then we might assume that

$$S = \lim_{ND=0} \sum_{1}^{n} 2\pi y \Delta x_K ,$$
$$= \int_{-a}^{a} 2\pi y \; dx = 2\int_{0}^{a} \sqrt{a^2 - x^2} \; dx \quad .$$

Letting $x = a \sin\theta$, $dx = a \cos\theta \, d\theta$, we get

$$S = 4\pi a^2 \int_{0}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} \, d\theta = 4\pi a^2 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{2}\right]_{0}^{\frac{\pi}{2}} ,$$

$$= 4\pi a^2 \frac{\pi}{4} = \pi^2 a^2 \quad .$$

Obviously this is not the surface of a sphere of radius a. It is now apparent that our approximation was not good enough. At the beginning of the discussion, however, it is not apparent, at least to calculus students, that our approximations are inadequate. This can be shown by trying the example on a class.

CHAPTER V

SERIES

5.1 Positive Term Series. We will not attempt, in this chapter, to give a development of series. The definitions and tests are usually given correctly in elementary calculus books and the advanced calculus gives quite a thorough treatment of the theoretical aspects. We shall limit ourselves to a few remarks on topics not usually discussed.

The usual comparison test is not always easy for the student to apply. The following variation of it can be used in most cases. Let $u_1 + u_2 + u_3 + \dots + u_n + \dots$ be a positive term series whose convergence is in question. Let $a_1 + a_2 + a_3 + \dots + a_n + \dots$ be a positive term series known to converge (or diverge). If $\lim_{n=\infty} \frac{u_n}{a_n} = K > 0$, then $\sum u_n$ converges (diverges); if $\lim_{n=\infty} \frac{u_n}{a_n} = 0$ and $\sum a_n$ converges, then $\sum u_n$ does also; and if $\lim_{n=\infty} \frac{u_n}{a_n} = \infty$ and $\sum a_n$ diverges, then $\sum u_n$ does also.

For if $\lim_{n=\infty} \frac{u_n}{a_n} = K$ then there is an N such that

$$\frac{u_n}{a_n} \leq K + 1 \qquad \text{for } n > N ,$$

and

$$u_n \leq (K+1)\, a_n \quad .$$

Since $\sum a_n$ is convergent then $\sum (K + 1)\, a_n$ is also, and $\sum u_n$ is convergent by the comparison test. Also there exists an N such that

$$\frac{u_n}{a_n} \geq K - d > 0 \;, \qquad \text{where } 0 < d < K \,,$$

and

$$u_n \geq (K - d) a_n \quad .$$

Therefore if $\sum a_n$ diverges so does $\sum u_n$. If

$$\lim_{n=\infty} \frac{u_n}{a_n} = 0$$

then

$$\frac{u_n}{a_n} \leq 1 \quad , \quad \text{for } n > N \;,$$

and we have

$$u_n \leq a_n \quad .$$

Hence if $\sum a_n$ is convergent then $\sum u_n$ is also.

If $\lim\limits_{n=\infty} \frac{u_n}{a_n} = \infty$ then

$$\frac{u_n}{a_n} \geq K > 0 \quad , \quad \text{for } n > N \;,$$

and

$$u_n \geq K\, a_n \quad .$$

Therefore since $\sum a_n$ diverges then $\sum u_n$ diverges.

5.2 Finding the nth Term of a Series. Many books make the statement that by inspection of the first few terms of a series we can find the nth term of the series. This is not always true. For example suppose

$$u_1 = 1, \; u_2 = \frac{1}{2}, \; u_3 = \frac{1}{3}, \; \ldots\ldots$$

and we wish to find the n th term. We will show that a polynomial in $\frac{1}{n}$, such as

$$f\left(\frac{1}{n}\right) = b_1 + b_2\left(\frac{1}{n}\right) + b_3\left(\frac{1}{n}\right)^2 + b_4\left(\frac{1}{n}\right)^3 , \tag{47}$$

will serve as an nth term if the b's are properly determined. In fact there are infinitely many such polynomials. We must have

$$\begin{aligned} b_1 + b_2 + b_3 + b_4 &= 1 && \text{for } n=1 , \\ b_1 + \frac{b_2}{2} + \frac{b_3}{4} + \frac{b_4}{8} &= \frac{1}{2} && \text{for } n=2 , \\ b_1 + \frac{b_2}{3} + \frac{b_3}{9} + \frac{b_4}{27} &= \frac{1}{3} && \text{for } n=3 . \end{aligned} \tag{48}$$

Solving equations (48) for b_1, b_2, b_3 in terms of b_4 we get

$$\begin{aligned} b_1 &= - \frac{b_4}{6} , \\ b_2 &= 1 + b_4 , \\ b_3 &= - \frac{11\, b_4}{6} . \end{aligned}$$

Substituting these values in equation (48) we have

$$(49) \qquad f\left(\tfrac{1}{n}\right) = -\frac{b_4}{6} + (1 + b_4)\left(\tfrac{1}{n}\right) - \frac{11b_4}{6}\left(\tfrac{1}{n}\right)^2 + b_4\left(\tfrac{1}{n}\right)^3 .$$

If in equation (49) we let $b_4 = 0$, then $f(\frac{1}{n}) = \frac{1}{n}$, which is the nth term that we would expect by inspection of the first few terms. However, b_4 may take any value, and therefore we have an infinite number of polynomials any one of which would be satisfactory as an nth term of this series. It is easily seen that if any finite number of terms were given we could carry through a similar discussion.

5.3 Power Series. We can make the statement that every power series defines a function in its region of convergence. To some of these functions we have given names. On the other hand, we cannot say that every function can be expanded in a power series. Consider a function, $f(x)$, which exists and has its first $(n+1)$ derivatives in the neighborhood of $x = 0$. Then we may write *

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \ldots + f^{n}(0)\frac{x^n}{n!} + R_n ,$$

---

* F. S. Woods, Advanced Calculus, p. 10 .

where

$$R_n = \frac{x^{n+1}}{(n+1)!} f^{n+1}(\xi) \qquad \text{and } 0<\xi<x \ .$$

Now if f(x) possesses all its derivatives the above formula for f(x) may be extended indefinitely. If at the same time $\lim_{n=\infty} |R_n| = 0$ then we have a convergent infinite series representing f(x). For example, consider

$$\sin x = x - \frac{x^3}{3!} + \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + R_{2k+1} \ ,$$

where

$$R_{2k+1} = (-1)^{k+1} \frac{x^{2k+3}}{(2k+3)!} \cos \xi \quad .$$

Then

$$R_{2k+1} < \frac{|x^{2k+3}|}{(2k+3)!} \quad ,$$

and, whatever the value of $|x|$, we have the $\lim_{k=\infty} R_{2k+1} = 0$. Hence we have found an infinite series which represents sin x for all values of x. In order for the above statement to be true it is necessary that $\lim_{k=\infty} |R| = 0$. If $\lim |R| \neq 0$ for $x=a$ then the series may converge but will not represent f(x) for $x=a$. Consider

$$f(x) = e^{-\frac{1}{x^2}} \ , \quad \text{for } x \neq 0 \ ,$$
$$= 0 \quad , \quad \text{for } x = 0 \ .$$

This function is continuous and has derivatives of all orders for all values of x. This is evident except at $x = 0$. For $x = 0$ we have

$$f'(0) = \lim_{h=0} \frac{e^{-\frac{1}{h^2}}}{h} = \lim_{h=0} \frac{h}{2e^{1/h^2}} = 0 \quad ,$$

$$f''(0) = \lim_{h=0} \frac{\frac{-2e^{-\frac{1}{h^2}}}{h^3}}{h} = \lim_{h=0} \frac{-2e^{-\frac{1}{h^2}}}{h^4} ,$$

$$= \lim_{h=0} \frac{-\frac{2}{h^4}}{e^{\frac{1}{h^2}}} = \lim_{q=\infty} \frac{2\,q^2}{e^q} = 0 \quad ,$$

where $q = \frac{1}{h^2}$. For $f^n(0)$ we have a finite number of terms of the type

$$\lim_{q=\infty} \frac{q^m}{e^q} \quad .$$

Therefore $f^n(0) = 0$ for all values of n . The series

$$f(0) + f'(0)x + \ldots . + \frac{f^n(0)x^n}{n!} + \ldots\ldots$$

obviously converges for all values of x but does not represent the function except at $x = 0$. In fact, if $x \neq 0$, $R_n = e^{-\frac{1}{x^2}}$ for all values of n.

Likewise let us consider

$$f(x) = \sin x + e^{-\frac{1}{x^2}}.$$

Then

$$\sin x + e^{-\frac{1}{x^2}} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + R_{2k+1}.$$

If R_{2k+1} is left off and k allowed to become infinite then we get a series of the type

$$f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots\ldots + \frac{f^n(0)x^n}{n!} + \ldots\ldots,$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \ldots\ldots ,$$

which converges for all values of x but represents

$$f(x) = \sin x + e^{-\frac{1}{x^2}}$$

at no value of x, except at $x = 0$.

Therefore we see, that to expand f(x) in a MacLaurin's or Taylor's expansion, it is necessary to do more than just show that

$$f(a) + f'(a)(x - a) + \ldots\ldots + \frac{f^n(a)(x - a)^n}{n!} \quad \ldots\ldots$$

converges.

BIBLIOGRAPHY

I Elementary Calculus Books

Dalaker, H.H. and Hartig, H.E., The Calculus, New York, McGraw-Hill Book Company, 1930.

Granville, W.A., Smith, P.F., and Longley, W.R., Elements of the Differential and Integral Calculus, New York, Ginn and Company, 1934.

Love, C.E., Differential and Integral Calculus, New York, The MacMillan Company, 1937.

McKelvey, J.V., Calculus, New York, The MacMillan Company, 1937.

Neelley, J.H. and Tracy, J.I., Differential and Integral Calculus, New York, The MacMillan Company, 1932.

Smith, E.S., Salkover, M., and Justice, H.K., Calculus, New York, John Wiley and Sons, Inc., 1938.

Slobin, H.L., and Solt, M.R., A First Course in Calculus, New York, Farrar and Rinehart Inc., 1935.

II Advanced Calculus Books and Books on Analysis

Caratheodory, C., Vorlesungen über Reelle Funktionen, Leipzig und Berlin, Verlag und Druck von B.G.Teubner, 1927.

Fine, H.B., Calculus, New York, The MacMillan Company, 1927.

Fite, W.B., Advanced Calculus, New York, The MacMillan Company, 1938.

Osgood, W.F., Functions of Real Variable, New York, G.E.Stechert and Company, 1938.

Pierpont, J., The Theory of Functions of Real Variables, Vol. I, New York, Ginn and Company, 1905 .

Pierpont, J., Functions of a Complex Variable, New York, Ginn and Company, 1914 .

Townsend, E.J., Functions of Real Variables, New York, Henry Holt and Company, 1928 .

Wilson, E.B., Advanced Calculus, New York, Ginn and Company, 1912 .

Woods, F.S., Advanced Calculus, New York, Ginn and Company, 1934 .

[redacted]