



ON THE CONTROLLABILITY OF DISCRETE
STATE SYSTEMS

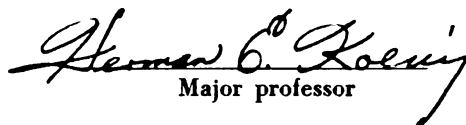
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By

Jeffrey L. Goodnuff

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ABSTRACT

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DISCRETE STATE SYSTEMS

by Jeffrey L. Goodnuff

The idea of controllability was presented by R. E. Kalman in 1960. His definition was one which gave a specific mathematical meaning to an otherwise intuitive concept.

This thesis develops, from that definition, necessary and sufficient conditions for a plant to be controllable. From these conditions it is shown that there is a certain minimum time in which a plant may be controlled. This time is a function of the dimension of the state vector. In addition it is shown that if a plant is not controllable in the minimum time it is never controllable.

In the last section observability is defined and the relationship between observability and controllability is shown.

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I

INTRODUCTION

In 1960 the concept of state controllability was introduced by Kalman.¹ He went on to state a necessary condition for state controllability in terms of the Jordan Canonical form of the transition matrix. He also required the inverse of the transition matrix to exist. In the same year the idea of output controllability was presented by Bertram and Sarachick.² They did not develop the concept beyond a mere definition.

This thesis develops a necessary condition for state controllability in terms of the minimal polynomial of the transition matrix. The transition matrix may be singular. A necessary and sufficient condition for both state and output controllability is also developed in terms of the Jordan Canonical form.

From these conditions it is shown that there exists a certain minimal time in which a plant may be controlled, and that if the plant cannot be controlled in that minimal time it cannot be controlled in any finite time.

Finally the concept of observability is considered, and necessary and sufficient conditions for it are established. These conditions are

¹R. E. Kalman, On The General Theory of Control Systems, First International Congress On Automatic Control, Moscow, 1960.

²J. E. Bertram and P. E. Sarachik On Optimal Computer Control, First International Congress On Automatic Control, Moscow, 1960.

related to the controllability conditions, and this relationship is shown.

A brief discussion of the problem of non-zero compute time is presented in the conclusion.

II

CONTROLLABILITY WITH A SINGLE INPUT

Consider a discrete state plant which is linear and has one input. The state model of the plant is then

$$\underline{X}[kT] = \Phi \underline{X}[(k-1)T] + \underline{A}u[(k-1)T]; k = 1, 2, \dots, T > 0, \quad (2.1)$$

where $\underline{X}$ and $\underline{A}$ are n dimensional column vectors, u is a scalar, and Φ is an n by n square constant matrix.

The outputs are given in the form

$$\underline{Y}[kT] = \underline{M}\underline{X}[kT] \quad (2.2)$$

where $\underline{Y}$ is an m ($m \leq n$) dimensional vector, and $\underline{M}$ is an m by n constant matrix. At this point it should be noted that (2.2) is not the most general form for the outputs of a linear plant. In this discussion only the case where there are no direct transmission terms will be considered. However, this is not as severe a restriction as it may at first seem, as output controllability is not affected by direct transmission terms.

The two fundamental definitions pertaining to controllability are as follows:

Definition 2.1: A plant is said to be state controllable if and only if the state vector, $\underline{X}[kT]$, can be brought to any desired state in a finite time. A plant is completely state controllable if and only if every state is controllable.³

Definition 2.2: A plant is said to be output controllable if and only if the output vector, $\underline{Y}[kT]$, can be brought to any desired output in a finite time. A plant is completely output controllable if and only if every output vector is controllable.⁴

The input sequence $u(nT)$, $n = 0, 1, \dots, N-1$, required to move the plant from an initial state at $t = 0$ to a given state at time $t = NT$ is determined by the solution of the system of equations

$$\underline{X}[NT] = \Phi^N \underline{X}[0] + [\Phi^{N-1} \underline{A} \quad \Phi^{N-2} \underline{A} \quad \dots \quad \Phi \underline{A} \quad \underline{A}] \begin{bmatrix} u[0] \\ u[T] \\ u[2T] \\ \vdots \\ u[(N-1)T] \end{bmatrix} \quad (2.3.1)$$

obtained as the recursive solution of (2.1). In symbolic form (2.3.1) is represented as

³This same definition was used by Kalman.

⁴This same definition was used by Bertram and Sarachick.

$$\underline{X}[NT] = \Phi^N \underline{X}[0] + H_N \underline{\mu}_N. \quad (2.3.2)$$

In equations (2.3) N is not to be confused with n , the order of the state vector. In equation (2.3.2) $\underline{\mu}_N$ is the N th order vector representing the scalar inputs at the N time intervals, and H_N is the n by N composite matrix composed of the column vectors $\Phi^i \underline{A}$, $i = 0, 1, \dots, N-1$.

Substituting equation (2.3.2) into equation (2.2) yields

$$\underline{Y}[NT] = M \Phi^N \underline{X}[0] + M H_N \underline{\mu}_N. \quad (2.4)$$

The following three cases are of interest.

Case I. $N = n$. That is, the number of iterations is equal to the number of variables in the state vector. Since H_N is square it is possible to solve (2.3.2) for $\underline{\mu}_N$ if H_N is nonsingular.

$$\underline{\mu}_N = H_N^{-1} \underline{X}[NT] - H_N^{-1} \Phi^N \underline{X}[0] \quad (2.5)$$

The square matrix H_N is nonsingular if and only if the columns of H_N are linearly independent. Therefore the following trivial lemma can be stated.

LEMMA 2.1: A discrete state plant, with state model given in equation (2.1), is completely controllable in $N = n$ steps if and only if the vectors $\Phi^{n-1} \underline{A}$, $\Phi^{n-2} \underline{A}$, . . . , $\Phi \underline{A}$, $\underline{A}$, are linearly independent.⁵

⁵ A similar result arrived at in a much less direct manner and requiring the inverse of Φ , is obtained by Kalman. It is at this point that this paper departs from Kalman's treatment of the subject.

Case II. $N < n$. That is, it is required to transfer a given state vector to some desired point in less than n steps. It can be seen that this is not possible in general. That is, it can be done only if $\underline{X}[0]$ is of a special form. This can be shown simply by writing equation (2.3.2) in partitioned form. If we let

$$\underline{W} = \underline{X}[NT] - \Phi^N \underline{X}[0], \quad (2.6)$$

then (2.3.2) becomes

$$\underline{W} = \begin{bmatrix} \underline{W}_1 \\ \underline{W}_2 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \underline{\mu}_N \quad (2.7.1)$$

where H_1 is a square, N by N matrix. If $|H_1| \neq 0$ then from the top equation in (2.7.1) we have

$$\underline{\mu}_N = H_1^{-1} \underline{W}_1, \quad (2.7.2)$$

and from the bottom equation

$$\underline{W}_2 = H_2 \underline{\mu}_N. \quad (2.7.3)$$

Therefore,

$$\underline{W}_2 = H_2 H_1^{-1} \underline{W}_1. \quad (2.8)$$

It can be seen from (2.8) that any vector, $\underline{X}[0]$, can be transferred to $\underline{X}[NT]$ in $N < n$ steps if and only if $\underline{W}$, as given in (2.6), can be written in the form of equation (2.8), and the following lemma follows.

LEMMA 2.2: A discrete state plant is never completely controllable with $N < n$ iterations.

Case III. $N > n$. That is, it is desired to transfer the state vector to a desired point in N time intervals, where N is greater than the number of variables in the state vector. Let equation (2.3.2) be rewritten as

$$\underline{W} = [H_1 \ H_2] \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad (2.9)$$

where $\underline{W}$ is defined in (2.6). Here H_2 is a square, n by n matrix and H_1 is n by $N-n$. If $|H_2| \neq 0$ we may write

$$\underline{\mu}_2 = H_2^{-1} \underline{W} - H_2^{-1} H_1 \underline{\mu}_1. \quad (2.10)$$

It can immediately be seen that the $N-n$ scalars in $\underline{\mu}_1$ can be chosen independently of the rest of the vector $\underline{\mu}$. This corresponds to independent selection of the first $N-n$ inputs. Since these inputs are arbitrary let $\underline{\mu}_1 \equiv 0$, so that (2.10) becomes

$$\underline{\mu}_2 = H_2^{-1} \underline{W}, \quad (2.11)$$

and $\underline{\mu}$ is given by

$$\underline{\mu} = \begin{bmatrix} 0 \\ \mu_2 \end{bmatrix}. \quad (2.12)$$

The above establishes the following lemma.

LEMMA 2.3: A discrete state plant completely controllable in $N = n$ time intervals is controllable in $N > n$ intervals.

From the above three lemmas, and the fact that H_2 in equation (2.9) is exactly the same as H_N in equation (2.5) the following theorem may be stated.

THEOREM 2.1: A general discrete state plant with state equations of the form in (2.1) is completely state controllable in $N \geq n$ intervals if and only if H_N is of maximum rank (i. e., rank of $H_N = n$).

By a careful examination of equation (2.4) and through a development exactly analogous to that used to establish theorem 2.1 we may draw the following conclusion.

THEOREM 2.2: A general discrete state plant with output equations of the form in (2.2) is completely output controllable in $N \geq m$ intervals if and only if MH_N is of maximum rank (i. e., rank of $MH_N = m$).

With the above two theorems along with theorem A. 1 in the appendix, the following theorem relating output and state controllability may be proved.

THEOREM 2. 3: If a discrete state plant is completely state controllable then the rank of M , in equation (2. 2), being equal to m is a necessary and sufficient condition for complete output controllability.

PROOF: The hypothesis assures us that H_N is of maximum rank. Therefore theorem A. 1 is applicable and we see that the rank of MH_N is equal to the rank of M . Since the maximum rank of MH_N is m , the product MH_N is of maximum rank if and only if the rank of M is m . Therefore a necessary and sufficient condition for complete output controllability is that the rank of M equals m .

Note, however, that complete output controllability does not always imply complete state controllability.

Although theorems 2. 1 and 2. 2 are certainly necessary and sufficient conditions for complete controllability, they leave much to be desired. To say the least it is a tedious task to raise Φ to successive powers, postmultiply by the vector $\underline{A}$, and then check the independence of the resulting vectors. Further, if this were done and the resulting vectors were found to be dependent it is not apparent what changes need to be made to Φ and $\underline{A}$ in order to achieve an independent set of vectors. It is this thought that motivates the following discussion.

It will be useful to make the following change of variable. Let

$$\underline{Z}[kT] = \underline{P} \underline{X}[kT], \quad (2.13)$$

where the square, non-singular, n by n matrix, $\underline{P}$, is the matrix that transforms $\underline{\Phi}$, in equation (2.1), into its upper triangular Jordan Canonical form. Denote this form by $\underline{J}$. That is,

$$\underline{J} = \underline{P} \underline{\Phi} \underline{P}^{-1}. \quad (2.14)$$

Equation (2.1) then becomes

$$\underline{Z}[kT] = \underline{P} \underline{\Phi} \underline{P}^{-1} \underline{Z}[(k-1)T] + \underline{P} \underline{A} u[(k-1)T] \quad (2.15.1)$$

or, with $\underline{\Delta} = \underline{P} \underline{A}$,

$$\underline{Z}[kT] = \underline{J} \underline{Z}[(k-1)T] + \underline{\Delta} u[(k-1)T]. \quad (2.15.2)$$

It immediately follows then that equations (2.3.1), (2.3.2), and (2.4) become

$$\underline{Z}[NT] = \underline{J}^N \underline{Z}[0] + [\underline{J}^{N-1} \underline{\Delta} \quad \underline{J}^{N-2} \underline{\Delta} \quad . \quad . \quad \underline{J} \underline{\Delta} \quad \underline{\Delta}] \begin{bmatrix} u[0] \\ u[T] \\ . \\ . \\ u[(N-1)T] \end{bmatrix} \quad (2.16)$$

or in the symbolic form,

$$\underline{Z}[NT] = J^N \underline{Z}[0] + S_N \underline{\mu}_n, \quad (2.17)$$

and the output is given by,

$$\underline{W}[NT] = K J^N \underline{Z}[0] + K S_N \underline{\mu}_N, \quad (2.18)$$

where, of course, $\underline{W} = P \underline{Y}$, and $K = M P^{-1}$. Since P is nonsingular the transformations between $\underline{W}$ and $\underline{Y}$ and between $\underline{Z}$ and $\underline{X}$ are one to one, and an inverse exists. From this point on the properties of equations (2.16), (2.17), and (2.18) will be considered.

The matrix S_N is composed of N column vectors, each of which is of the form $J^i \Delta$. The Jordan form, J , is quasidiagonal, i. e., of the form

$$J = \begin{bmatrix} L_1 & & & & \\ & L_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & L_r \end{bmatrix} \quad (2.19)$$

where r is the number of distinct eigenvalues and block L_i is of order s_i with s_i equal to the multiplicity of λ_i in the characteristic polynomial, $D(\lambda)$, of Φ . Each L_i has the following form.

$$L_i = \begin{bmatrix} \lambda_i & 1 & & & & \\ & \lambda_i & 0 & & & \\ & & \lambda_i & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & 1 \\ & & & & & \lambda_i \end{bmatrix} \quad (2.20)$$

That is, the main diagonal has entries λ_i , and the upper diagonal contains ones and zeros. The remaining entries in L_i are zero. The exact number of ones and zeros on the upper diagonal depends on the elementary divisors of the matrix $(\lambda u - \Phi)$, and cannot be determined from knowledge of only the characteristic polynomial, $D(\lambda)$, and the minimal polynomial, $M(\lambda)$. However, knowledge of $M(\lambda)$ and $D(\lambda)$ does allow partial determination of the off diagonal entries. The upper diagonal in the i th block has all zeros if and only if the $(\lambda - \lambda_i)$ term in $M(\lambda)$ is linear. Also, the upper diagonal in the i th block has all ones if and only if the $(\lambda - \lambda_i)$ term in $M(\lambda)$ is of the same degree as the corresponding term in $D(\lambda)$.⁶

Examining the vector $J^i \underline{\Delta}$ we see that

⁶V. N. Faddeeva, Computational Methods of Linear Algebra (New York: Dover Publications, Inc., 1959), P. 52.

$$J^i \underline{\Delta} = \begin{bmatrix} L_1^i \underline{\Delta}_1^* \\ L_2^i \underline{\Delta}_1^* \\ \cdot \\ \cdot \\ \cdot \\ L_r^i \underline{\Delta}_r^* \end{bmatrix}, \quad (2.21)$$

where,

$$L_j^i = \begin{bmatrix} \lambda_j^i & \binom{i}{1} \lambda_j^{i-1} & \binom{i}{2} \lambda_j^{i-2} & \cdot & \cdot & \cdot & \binom{i}{s_j-1} \lambda_j^{i-s_j+1} \\ & \lambda_j^i & \binom{i}{1} \lambda_j^{i-1} & \cdot & \cdot & \cdot & \\ & & \lambda_j^i & & & & \\ & & & \lambda_j^i & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & \lambda_j^i \end{bmatrix}, \quad (2.22)$$

with the convention that $\binom{i}{j} = 0$ for $j > i$, if and only if $M(\lambda) \equiv D(\lambda)$.

If $M(\lambda) \neq D(\lambda)$ then (2.22) has the following form.

Since (2.24) represents s_j rows of the $i + 1$ column, it is apparent that in general the jk row of S_N is a multiple of the $j s_j$ row, and therefore the rank is down by at least one. In fact it can be seen that the $k-1$ row is a linear combination of the last two rows. The $k-2$ row will be a linear combination of the last three rows, and so on. It can be verified that the number of dependent rows is exactly equal to the difference in degree between $M(\lambda)$ and $D(\lambda)$. Theorems 2.4 and 2.5 are then established.

THEOREM 2.4: A necessary condition for complete state controllability of a discrete state plant is that the minimal polynomial, $M(\lambda)$, and characteristic polynomial, $D(\lambda)$, of the matrix Φ , in equation (2.1), be identical.

THEOREM 2.5: Let the difference in degree between the minimal polynomial, $M(\lambda)$, and the characteristic polynomial, $D(\lambda)$, for the λ_j term be denoted by β_j . Then a necessary condition for complete output controllability of a discrete state plant is that $\sum_j \beta_j < n-m$.

Consider, now, the case where Φ has distinct eigenvalues. It is immediately apparent that the necessary condition in theorem 2.4 is satisfied, since each diagonal block, L_i , in the Jordan form is diagonal and of order one, i.e., J is diagonal with diagonal elements equal to the distinct λ_i . In this case the product $J^i \underline{\Delta}$ is

$$J^i \underline{\Delta} = \begin{bmatrix} \lambda_1^i \delta_1 \\ \lambda_2^i \delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_n^i \delta_n \end{bmatrix}, \quad (2.25)$$

and the composite matrix S_N , $N \geq n$, is given by,

$$S_N = \begin{bmatrix} \lambda_1^{N-1} \delta_1 & \lambda_1^{N-2} \delta_1 & \cdot & \cdot & \cdot & \lambda_1 \delta_1 & \delta_1 \\ \lambda_2^{N-1} \delta_2 & \lambda_2^{N-2} \delta_2 & \cdot & \cdot & \cdot & \lambda_2 \delta_2 & \delta_2 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \lambda_n^{N-1} \delta_n & \lambda_n^{N-2} \delta_n & \cdot & \cdot & \cdot & \lambda_n \delta_n & \delta_n \end{bmatrix}. \quad (2.26)$$

The square matrix S_n is simply the right n columns of S_N .

$$S_n = \begin{bmatrix} \lambda_1^{n-1} \delta_1 & \lambda_1^{n-2} \delta_2 & \cdot & \cdot & \cdot & \lambda_1 \delta_1 & \delta_1 \\ \lambda_2^{n-1} \delta_1 & \lambda_2^{n-2} \delta_2 & \cdot & \cdot & \cdot & \lambda_2 \delta_2 & \delta_2 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \lambda_n^{n-1} \delta_n & \lambda_n^{n-2} \delta_n & \cdot & \cdot & \cdot & \lambda_n \delta_n & \delta_n \end{bmatrix} \quad (2.27)$$

S_n may be written as the product of two matrices,

$$S_n = D_\delta V_n, \quad (2.28)$$

where D_δ is the diagonal matrix with $d_{ii} = \delta_i$ and V_n is the n th order Vandermonde matrix. It is a well known fact that the Vandermonde determinant does not vanish for distinct λ_i . In fact the Vandermonde determinant is easy to compute and is given by

$$|V_n| = \prod_{i > j}^n (\lambda_i - \lambda_j), \quad (2.29)$$

which is obviously not zero for $\lambda_i \neq \lambda_j$, $i \neq j$.⁸

Since the determinant of a product is equal to the product of determinants, $|S_n|$ can be written as

$$|S_n| = |D_\delta V_n| = |D_\delta| |V_n| = \prod_{i=1}^n \delta_i \prod_{i > j}^n (\lambda_i - \lambda_j). \quad (2.30)$$

Equation (2.30) obviously implies $|S_n| = 0$ if and only if $\delta_i = 0$ for at least one i , $i = 1, 2, \dots, n$. Now, noting the form of S_N in equation (2.26) it is seen that if any $\delta_i = 0$ there is an entire row of zeros in S_N . The following lemma may therefore be stated.

⁸Edward T. Browne, Introduction to the Theory of Determinants and Matrices (Chapel Hill, North Carolina: The University of North Carolina Press, 1958), P. 34.

LEMMA 2.4: If a discrete state plant, with distinct eigenvalues and with state equations of the form shown in (2.1), is not completely state controllable in $N = n$ intervals then it is not completely state controllable in $N > n$ intervals.

Consider now the situation with output controllability and distinct eigenvalues. Again it is seen that the necessary condition for complete output controllability is satisfied (theorem 2.5). Assume that the plant is not controllable in n steps. This implies the rank of KS_n is less than m . This may occur in three ways. (1) The rank of K may be less than m , (2) the rank of S_n may be less than m , or (3) the rank of S_n is at least m , but the rank of the product, KS_n , is less than m .

Case (1): When the rank of K is less than m , the rank of KS_N is always less than m , because the rank of the product of two matrices is never greater than the rank of either matrix.

Case (2): The rank of S_n is less than m if and only if less than m rows are independent. It has been shown that $S_n = D_\delta V_n$, and $|V_n| \neq 0$. Therefore from theorem A.1 it can be seen that the rank of S_n is equal to the rank of D_δ . Therefore the rank of D_δ is less than m . However, from (2.26) it is seen that $S_N = D_\delta V_N^*$, where V_N^* is the n th order Vandermonde matrix, V_n , augmented with $N-n$ columns. Since $|V_n| \neq 0$ the rank of V_N is always maximum (i.e., equal to n) for all $N \geq n$. Again theorem A.1 is applicable and it is seen that the rank of S_N equals the rank of S_n which is

less than m.

Case (3): Consider the product $\mathbf{KS}_N = \mathbf{a}$ where

$$\mathbf{KS}_N = \begin{bmatrix} k_{11} & k_{12} & \cdot & \cdot & \cdot & k_{1n} \\ k_{21} & k_{22} & \cdot & \cdot & \cdot & k_{2n} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ k_{m1} & k_{m2} & \cdot & \cdot & \cdot & k_{mn} \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & \cdot & \cdot & \cdot & h_{1N} \\ h_{21} & h_{22} & \cdot & \cdot & \cdot & h_{2N} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ h_{n1} & h_{n2} & \cdot & \cdot & \cdot & h_{nN} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1N} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2N} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mN} \end{bmatrix} = \mathbf{a}. \quad (2.31)$$

Let the m, N dimensional row vectors of $\mathbf{a}$ be denoted by $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$. Each of these m vectors is a linear combination of the n row vectors, Ω_i , of $\mathbf{S}_N$. It is known that there exists at least m rows, $\mathbf{a}_i$, that are dependent. Since the combining vectors (i.e., the rows of $\mathbf{K}$) are all independent this implies that at least m of the row vectors, $\mathbf{a}_i$, are partly composed of the dependent vectors in the set $\{\Omega_i\}$. Also note that the relationship of $\mathbf{K}$ to $\mathbf{S}_N$ is unaltered by

the magnitude of N . It therefore follows that in order to increase the rank of $\mathcal{A}$, the rank of S_N must be increased, i. e., some dependent rows must be made independent by adding another column. However in case (2) it was shown that the rank of S_N equals the rank of D_δ for all N . Therefore the rank of S_N cannot be increased by increasing N .

The above discussion together with lemma 2.4 and theorems 2.1 and 2.2 imply the following.

THEOREM 2.6: A discrete state plant with state [output] equations as in (2.1) [(2.2)] and with distinct eigenvalues is completely state [output] controllable in $N > n$ steps if and only if it is completely controllable in $N = n$ steps.

And from the conclusion in case (2) of the above development, a necessary and sufficient condition for state controllability is apparent.

THEOREM 2.7: A discrete state plant with state equations of the form of (2.1) and distinct eigenvalues, is completely state controllable in $N = n$ steps if and only if $\delta_i \neq 0$ for all δ_i as defined in (2.30).

Now consider the case where the transition matrix, Φ , has multiple eigenvalues, and $M(\lambda) \equiv D(\lambda)$. Each column vector of the matrix Φ_N is of the form shown in (2.21) and (2.22). It can be seen that the matrix, S_N , may be factored into the product of two matrices,

$$S_N = \tilde{D}_\delta \tilde{V}_n \quad (2.32)$$

where

$$\tilde{D}_\delta = \begin{bmatrix} B_1 & & & & \\ & B_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & B_r \end{bmatrix}, \quad (2.33)$$

with

$$B_i = \begin{bmatrix} \delta_{i s_i} & \delta_{i s_i - 1} & \cdot & \cdot & \cdot & \delta_{i 1} \\ & \delta_{i s_i} & & & & \\ & & \cdot & & & \delta_{i s_i - 1} \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \delta_{i s_i} \end{bmatrix}, \quad (2.34)$$

and

$$\tilde{V}_n = \begin{bmatrix} \wedge_1 \\ \wedge_2 \\ \vdots \\ \wedge_r \end{bmatrix}, \quad (2.35)$$

where

$$\Lambda_i = \begin{bmatrix} \binom{n-1}{s_i-1} \lambda_i^{n-s_i} & . & . & . & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ \binom{n-1}{1} \lambda_i^{n-2} & . & . & . & . & \lambda_i & 1 & 0 & . & . \\ \lambda_i^{n-1} & \lambda_i^{n-2} & . & . & . & . & \lambda_i & 1 & . & . \end{bmatrix}. \quad (2.36)$$

Obviously the determinant of D_δ is dependent only on each $\delta_i s_i$ and on nothing else. That is,

$$\left| \tilde{D}_\delta \right| = \prod_{i=1}^n (\delta_i s_i)^{s_i}. \quad (2.37)$$

The determinant of the modified Vandermonde matrix can be shown to be

$$\left| \tilde{V}_n \right| = \prod_{i>j}^r (\lambda_i - \lambda_j)^{s_i s_j}. \quad (2.38)$$

Therefore

$$\left| S_n \right| = \left| \tilde{D}_\delta \right| \left| \tilde{V}_n \right| = \prod_{i=1}^r (\delta_i s_i)^{s_i} \prod_{i>j}^r (\lambda_i - \lambda_j)^{s_i s_j} \quad (2.39)$$

Note that for distinct λ , $\tilde{D}_\delta = D_\delta$ and $\tilde{V}_n = V_n$. Then theorem 2.8 immediately follows.

THEOREM 2.8: A discrete state plant with state equations of the form in (2.1) is completely state controllable in $N = n$ steps if and only if (a) $M(\lambda) = D(\lambda)$ and (b) all $\delta_{i s_i} \neq 0$ as defined in equation (2.24).

The following theorem is also easy to prove.

THEOREM 2.9: A discrete state plant with state equations of the form in (2.1) is completely state [output] controllable in $N > n$ steps if and only if it is completely state [output] controllable in $N = n$ steps.

PROOF: The "if" part is shown by lemma 2.3. For the "only if" part it needs to be shown that a plant not controllable in $N = n$ steps is not controllable in $N > n$ steps.

First consider state controllability. If the plant is not state controllable in $N = n$ intervals then $|S_n| = 0$. This implies some $\delta_{i s_i} = 0$. Which, from (2.24), implies that S_N has a row of zeros; hence it cannot be of maximum rank.

Next consider output controllability. This development is parallel to the discussion in the case of distinct eigenvalues. Case (1) and Case (3) do not use the fact that the eigenvalues are distinct. Therefore it is only necessary to establish case (2). That is, it is necessary to show that the rank of S_N equals the rank of S_n for $N > n$, hence the rank of $S_n < n$ implies the rank of $S_N < n$.

It was shown above that

$$S_n = \tilde{D}_\delta \tilde{V}_n . \quad (2.40)$$

It can be seen from (2.38) that $\left| \tilde{V}_n \right| \neq 0$. Therefore theorem A.1 implies that the rank of S_n equals the rank of $\tilde{D}_\delta$. Note that S_N can be written,

$$S_N = \tilde{D}_\delta \tilde{V}_N^* \quad (2.41)$$

where V_N^* is the n th order modified Vandermonde matrix V_n augmented with $N-n$ columns. Since $\left| \tilde{V}_n \right| \neq 0$ the rank of $\tilde{V}_N^*$ is maximum and theorem A.1 again applies. Therefore the rank of $S_N = \tilde{D}_\delta$, from which follows that the rank of S_N equals the rank of S_n and the theorem is proved.

III

CONTROLLABILITY WITH MULTIPLE INPUTS

Consider the discrete state plant with state equations of the following form.

$$\underline{X}[kT] = \Phi \underline{X}[(k-1)T] + A \underline{u}[(k-1)T] \quad (3.1)$$

Here, as before, $\underline{X}$ is an n dimensional vector and Φ is an n by n square constant matrix. However A is no longer a column vector, it is an n by q matrix. The column vector $\underline{u}$ is q dimensional.

The outputs are given in the form

$$\underline{Y}[kT] = M \underline{X}[kT], \quad (3.2)$$

where $\underline{Y}$ is an m dimensional vector and M is an m by n constant matrix.

Again, as in the single input case, (3.1) can be solved recursively yielding

$$\underline{X}[NT] = \Phi^N \underline{X}[0] + [\Phi^{N-1}A \quad \Phi^{N-2}A \quad \dots \quad \Phi A \quad A] \begin{bmatrix} \underline{U}[0] \\ \underline{U}[T] \\ \vdots \\ \underline{U}[(N-1)T] \end{bmatrix}, \quad (3.3)$$

or in symbolic form

$$\underline{X}[NT] = \Phi^N \underline{X}[0] + H_N \underline{\mu}_N. \quad (3.4)$$

Each of the products $\Phi^i A$ results in an n by q matrix, thus making H_N an n by Nq matrix. Although $\underline{\mu}_N$ is a vector, the elements, U , are each q dimensional column vectors. Thus, $\underline{\mu}_N$ is an Nq dimensional vector. Let us again consider three cases.

Case I. $Nq = n$. In this case H_N is a square matrix and if $|H_N| \neq 0$,

$$\underline{\mu}_N = H_N^{-1} \underline{X}[NT] - H_N^{-1} \Phi^N \underline{X}[0]. \quad (3.5)$$

Of course $|H_N| \neq 0$ if and only if its columns are linearly independent.

Case II. $Nq < n$. In this case $\underline{W}$ is defined by

$$\underline{W} = \underline{X}[NT] - \Phi^N \underline{X}[0]. \quad (3.6)$$

With this definition equation (3.4) may be written as

$$\underline{W} = \begin{bmatrix} \underline{W}_1 \\ \underline{W}_2 \end{bmatrix} = \begin{bmatrix} H_1^* \\ H_2^* \end{bmatrix} \underline{\mu}_N, \quad (3.7)$$

where H_1^* is a square Nq by Nq matrix. Solving for $\underline{W}_2$ from the bottom equation of (3.7) and substituting it for $\underline{\mu}_N$ in the top equation of (3.7) yields, for $|H_1^*| \neq 0$,

$$\underline{W}_2 = H_2^* H_1^{*-1} \underline{W}_1. \quad (3.8)$$

It can be seen that if Nq is no greater than the number of variables in the state vector, only certain special initial vectors, $\underline{X}[0]$, may be controlled.

Case III. $Nq > n$. With the definition given in (3.6), (3.4) can be written as

$$\underline{W} = [\underline{a}_1 \quad \underline{a}_2] \begin{bmatrix} \underline{\beta}_1 \\ \underline{\beta}_2 \end{bmatrix}. \quad (3.9)$$

Here $\underline{a}_2$ is n by n , $\underline{a}_1$ is n by $Nq-n$, and $\underline{\beta}_2$ is n by 1 , while $\underline{\beta}_1$ is $Nq-n$ by 1 . If $|\underline{a}_2| \neq 0$,

$$\underline{\beta}_2 = \underline{a}_2^{-1} \underline{W} - \underline{a}_2^{-1} \underline{a}_1 \underline{\beta}_1. \quad (3.10)$$

It can be seen that the $Nq-n$ constants in $\underline{\beta}_1$ are arbitrary. Therefore choose $\underline{\beta}_1 = 0$. Then

$$\underline{\beta}_2 = \underline{a}_2^{-1} \underline{W}, \quad (3.11)$$

where

$$\underline{\mu}_N = \begin{bmatrix} 0 \\ \underline{\beta}_2 \end{bmatrix}. \quad (3.12)$$

Note that for any given case, if q and n are such that n/q is an integer, $\underline{\beta}_2$ is identical with H_N in equation (3.5).

Here, as in the case of a single input, the following theorem may be stated.

THEOREM 3.1: A discrete state, multiple input plant with state equations of the form in (3.1) is completely state controllable in $N \geq n/q$ time intervals if and only if H_N , in equation (3.4) is of maximum rank (i. e. , of rank n).

Output controllability is concerned with the system

$$\underline{Y}[NT] = M \Phi^N \underline{X}[0] + M H_N \underline{\mu}_N, \quad (3.13)$$

for which we have the following theorem.

THEOREM 3.2: A discrete state, multiple input plant with output equations as given in equation (3.2) is completely output controllable in $N \geq m/q$ time intervals if and only if the product $M H_N$, in equation (3.13), is of maximum rank (i. e. , of rank m).

The similarity between the single input case and multiple input case terminates with the above theorems. Recall now that the N groups of q columns which compose H_N are products of the form $\Phi^i A$. As in the single input case, an equivalent system of equations may be considered by using the Jordan Canonical form of Φ . The Jordan Canonical form of (3.3) is

$$\underline{Z}[NT] = J^N \underline{Z}[0] + [J^{N-1}\Delta \quad J^{N-2}\Delta \quad . \quad . \quad J\Delta \quad \Delta] \begin{bmatrix} \overline{U[0]} \\ U[T] \\ . \\ . \\ U[(N-1)T] \end{bmatrix}, \quad (3.14)$$

with $\underline{Z} = P\underline{X}$, and $\Delta = PA$. Here, of course, the terms $J^i\Delta$ each represent a band of q column vectors instead of a single column vector as in the single input case. It can be shown that theorem 2.4 is not true under these conditions. For example, let

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \quad (3.15)$$

and let

$$\Delta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (3.16)$$

Obviously

$$D(\lambda) = (\lambda - \lambda_1)^2 (\lambda - \lambda_2)^2 \quad (3.17)$$

and

$$M(\lambda) = (\lambda - \lambda_1) (\lambda - \lambda_2). \quad (3.18)$$

If $\lambda_1 = 0$, $\lambda_2 = 1$, and $N = 2$ the matrix H_N can be written

$$H_N = [J\Delta \quad \Delta] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad (3.19)$$

from which it is seen that

$$|H_N| = 1 \neq 0. \quad (3.20)$$

Therefore H_N^{-1} exists and the plant is controllable. However it is seen from equation (3.17) and (3.18) that $M(\lambda) \neq D(\lambda)$.

It is also easy to show that theorem 2.9 does not remain true in the multiple input case. For example, choose J with four distinct eigenvalues and let

$$\Delta = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}. \quad (3.21)$$

Then, for $N = n/q = 2$,

$$H_N = \begin{bmatrix} \lambda_1 & \lambda_1 & 1 & 1 \\ \lambda_2 & \lambda_2 & 1 & 1 \\ \lambda_3 & \lambda_3 & 1 & 1 \\ \lambda_4 & \lambda_4 & 1 & 1 \end{bmatrix}. \quad (3.22)$$

Obviously $\left| H_N \right| = 0$. However if $N = 4$

$$H_N = \begin{bmatrix} \lambda_1^3 & \lambda_1^3 & \lambda_1^2 & \lambda_1^2 & \lambda_1 & \lambda_1 & 1 & 1 \\ \lambda_2^3 & \lambda_2^3 & \lambda_2^2 & \lambda_2^2 & \lambda_2 & \lambda_2 & 1 & 1 \\ \lambda_3^3 & \lambda_3^3 & \lambda_3^2 & \lambda_3^2 & \lambda_3 & \lambda_3 & 1 & 1 \\ \lambda_4^3 & \lambda_4^3 & \lambda_4^2 & \lambda_4^2 & \lambda_4 & \lambda_4 & 1 & 1 \end{bmatrix}. \quad (3.23)$$

It is easily seen that the rank of H_N is maximum. Therefore by theorem 3.1 the plant is controllable in $N > n/q$ time intervals.

From the above discussion it can be seen that the results of the previous section do not trivially extend to the case of multiple inputs. Moreover, they are not true in general.

IV

OBSERVABILITY

In the previous two sections it was shown that there are certain necessary and sufficient conditions on the transition matrix, Φ , and the coefficient matrix, $\underline{A}$, for state and output controllability. However, examination of equations (2.4) and (2.5) reveals that for both state and output controllability the entire state vector evaluated at $t = 0$, i.e., $\underline{X}[0]$, is required. In general it may be very difficult, inconvenient, or impossible to measure $\underline{X}[0]$. It is this thought that motivates the following discussion.

DEFINITION 4.1 : A discrete state plant is said to be observable if the exact value of the state vector at time zero can be determined, in a finite time, from the measurements of the output signal. If every state is observable the plant is completely observable.⁹

In order that some definite conclusions may be drawn let the plant have a single input and single output. That is let M , in equation (2.2), be a row matrix of dimension n . Equation (2.4) with $N = k$ becomes,

$$Y[kT] = M \Phi^k \underline{X}[0] + M H_k \underline{u}_k \quad (4.1)$$

⁹This same definition was used by Kalman.

where, of course, $Y[0]$ is a scalar. Writing equation (4.1) for $k = 0, 1, \dots, n-1$, one obtains

$$\begin{bmatrix} Y[0] \\ Y[T] \\ \vdots \\ Y[(n-1)T] \end{bmatrix} = \begin{bmatrix} M \\ M\Phi \\ \vdots \\ M\Phi^{n-1} \end{bmatrix} \underline{X}[0] + MH_n \underline{\mu}_n, \quad (4.2)$$

or in symbolic form,

$$Y_n^* = G_n \underline{X}[0] + MH_n \underline{\mu}_n. \quad (4.3)$$

If $|G_n| \neq 0$

$$\underline{X}[0] = G_n^{-1} Y_n^* - G_n^{-1} MH_n \underline{\mu}_n. \quad (4.4)$$

The analog of lemma 2.1 follows immediately.

LEMMA 4.1 : A discrete state plant with one input and one output is completely observable in $N = n$ intervals if and only if the row vectors $M\Phi^{n-1}, M\Phi^{n-2}, \dots, M$ are linearly independent.

LEMMA 4.2 : A discrete state plant with a single output is never completely observable in $N < n$ time intervals.

In a similar manner theorem 2.1 carries over as:

THEOREM 4.1 : A discrete state plant with a single input and a single output is completely observable in $N \geq n$ time intervals if and only if G_n is of maximum rank.

A careful examination of G_n reveals that it is of the same form as the transpose of H_n . In fact G_n is a row permutation of H_n^t with $\underline{A}^t$ replaced by M and Φ^t replaced by Φ . Since the operation of transposition and row permutation does not affect the singularity of a matrix, all the theorems proved in section II will carry over, and theorems 2.4 and 2.5 become:

THEOREM 4.2 : A necessary condition for complete observability of a discrete state plant with one output is that the minimal polynomial, $M(\lambda)$, and the characteristic polynomial, $D(\lambda)$, of the matrix Φ be identical.

THEOREM 4.3: A discrete state, single output, plant with distinct eigenvalues is completely observable in $N > n$ intervals if and only if it is completely observable in $N = n$ steps.

With δ_i defined in equation (2.30) and with $\underline{A} = M^t$, we may state the analog of theorem 2.7.

THEOREM 4.4 : A discrete state plant with a single output and distinct eigenvalues is completely observable in $N = n$ time intervals if and only if $\delta_i \neq 0$ for all $i = 1, 2, \dots, n$.

In the case of repeated eigenvalues, theorem 2.8 and theorem 2.9 become:

THEOREM 4.5 : A discrete state plant with a single output is completely observable in $N = n$ time intervals if and only if (a) $M(\lambda) \equiv D(\lambda)$ and (b) all $\delta_i s_i \neq 0$ as defined in equation (2.24) with A = M^t .

THEOREM 4.6: A discrete state plant with a single output is completely observable in $N > n$ steps if and only if it is completely observable in $N = n$ steps.

V

CONCLUSION

In the previous four sections the concepts of output and state controllability as defined by Kalman have been given. It was found that for a linear, stationary, discrete state plant with one input and one output the minimal polynomial, $M(\lambda)$, must be identically equal to the characteristic polynomial, $D(\lambda)$, if we are to achieve state controllability.¹⁰ In addition the input vector, $\underline{A}$, must be such that when premultiplied by the Jordan transforming matrix, P , the result is free from zeros in certain positions. In addition to the above, the state vector at $t = 0$ must be known.

This last requirement motivates the development in Section IV where the concept of observability is given. It is shown that in order to compute $\underline{X}[0]$ (that is, the plant is observable) G_n must be non-singular. However the matrix G_n has properties very similar to H_n , the inverse of which is required for controllability. It is therefore easy to check G_n from the information used to investigate the rank of H_n .

To make a complete check of controllability the Jordan transforming matrix, P , must be found. In general this is a very laborious task to say the least. Even more serious is the fact that if the computation is to be done on a digital computer the round-off errors, for large matrices, may accumulate to such an extent that the result is useless.

¹⁰This necessary condition was given in a different form by Kalman.

If an analysis is performed on the system, other factors, such as stability, will also be of interest. Much of the information required for a controllability investigation will also be required for these other purposes and therefore the increase in work to investigate controllability will not be as great as first may appear.

As a further application to control systems it should be noted that in order to achieve controllability it is necessary to compute the solution to equation (2.5) with $\underline{X}[0]$ given. Even if this is to be done on a high speed digital computer, it will require some non-zero compute time. Let this time be denoted by βT , where T is the sampling period. Since the solution to (2.5) gives, among other things, the input at the next time interval it is impossible to control the plant, with zero error, in $N = n$ time intervals. Also note from equation (2.12) that the first $N - n$ intervals are completely arbitrary, say zero. Therefore we are assured that we may control the plant in $N = n + \beta$ intervals.

If the state vector is not directly measureable, but the plant is completely controllable, we are assured that it can be controlled in at most $N = 2n + \beta$ intervals.

The question of controllability with multiple inputs and the problem of observability with multiple outputs is essentially the same, but it is one that is not easily answered. Certainly it would appear that the answer lies in the relationship between Φ and $\underline{A}$ in the controllability case and between Φ and M in the observability case. Counter examples to the theorems in the single input-output case are easy to give for the multiple input-output case. As yet it has not been possible to draw any specific conclusions in the multiple input, multiple output plant.

APPENDIX

THEOREM A. 1: Given the matrix product MH , where M is an m by n matrix, $n \geq m$; and H is an n by N matrix with $N \geq n$, if H is of maximum rank (i.e., n) then the rank of MH equals the rank of M .

PROOF : Let $MH = \Omega$. Then

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & \dots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \dots & m_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{m1} & m_{m2} & \dots & \dots & m_{mn} \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ h_{31} & h_{32} & \dots & h_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \dots & h_{nN} \end{bmatrix} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \dots & \omega_{1N} \\ \omega_{21} & \omega_{22} & \omega_{23} & \dots & \omega_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{m1} & \omega_{m2} & \omega_{m3} & \dots & \omega_{mN} \end{bmatrix} = \Omega. \quad (\text{A. 1})$$

It can be seen that

$$\begin{aligned} \beta_1 &= m_{11} \Psi_1 + m_{12} \Psi_2 + \dots + m_{1n} \Psi_n \\ \beta_2 &= m_{21} \Psi_1 + m_{22} \Psi_2 + \dots + m_{2n} \Psi_n \\ &\vdots \\ \beta_m &= m_{m1} \Psi_1 + m_{m2} \Psi_2 + \dots + m_{mn} \Psi_n \end{aligned} \quad (\text{A. 2})$$

where $\beta_i = (\omega_{i1} \ \omega_{i2} \ \dots \ \omega_{iN})$ for $i = 1, 2, \dots, m$, and

$\Psi_i = (h_{i1} \ h_{i2} \ \dots \ h_{iN})$ for $i = 1, 2, \dots, n$.

It is clear that the vectors $\beta_1, \beta_2, \dots, \beta_m$ are linear combinations of the vectors $\Psi_1, \Psi_2, \dots, \Psi_n$. Note that since H is of maximum rank, and Ψ_i is the i th row of H , the vectors $\{\Psi_i\}$ are linearly independent.

Assume that the rank of Ω is less than the rank of M . Then any j rows of Ω are dependent, where j is the rank of M . Then

$$\sum_{i=1}^j a_i \beta_i = 0 \quad ; \quad a_i \neq 0 \text{ for some } i = 1, 2, \dots, j, \quad (\text{A.3})$$

or

$$\begin{aligned} & a_1(m_{11}\Psi_1 + m_{12}\Psi_2 + \dots + m_{1n}\Psi_n) \\ & + a_2(m_{21}\Psi_1 + m_{22}\Psi_2 + \dots + m_{2n}\Psi_n) \\ & \vdots \\ & + a_j(m_{j1}\Psi_1 + m_{j2}\Psi_2 + \dots + m_{jn}\Psi_n) = 0 \quad . \end{aligned} \quad (\text{A.4})$$

Rearranging and combining terms one obtains

$$\begin{aligned} & (a_1 m_{11} + a_2 m_{21} + \dots + a_j m_{j1}) \Psi_1 \\ & + (a_1 m_{12} + a_2 m_{22} + \dots + a_j m_{j2}) \Psi_2 \\ & \vdots \\ & + (a_1 m_{1n} + a_2 m_{2n} + \dots + a_j m_{jn}) \Psi_n = 0 \quad . \end{aligned} \quad (\text{A.5})$$

However, it has been shown that $\{\Psi_i\}$ is a set of linearly independent vectors. Therefore equation (A.5) implies that

$$\begin{aligned}
a_1 m_{11} + a_2 m_{21} + \dots + a_j m_{j1} &= 0 \\
a_1 m_{12} + a_2 m_{22} + \dots + a_j m_{j2} &= 0 \\
\vdots & \\
a_1 m_{1n} + a_2 m_{2n} + \dots + a_j m_{jn} &= 0
\end{aligned} \tag{A.6}$$

Let $\mu_i = (m_{i1} \ m_{i2} \ \dots \ m_{in})$. That is, μ_i is the i th row of M . With this definition the equations in (A.6) may be written

$$a_1 \mu_1 + a_2 \mu_2 + \dots + a_j \mu_j = 0 \quad . \tag{A.7}$$

Equation (A.7) must hold for any set of j rows of M . Since the rank of M is j , there must exist j linearly independent rows of M . Therefore (A.7) can only be true if

$$a_i = 0 \quad \text{for all } i = 1, 2, \dots, j. \tag{A.7A}$$

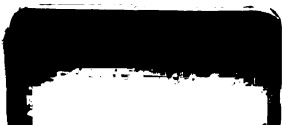
This is a contradiction to the original hypothesis that $a_i \neq 0$ for some i . It is therefore concluded that the rank of Ω is greater than or equal to the rank of M .

However, since the rank of M equals j , $j \leq m \leq n$, and the rank of H is n , it is concluded that the rank of Ω is at most j . Hence the rank of Ω equals j .

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