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ANALYSIS OF A CYLINDRICAL SHELL WITH IMMOVABLE SUPPORTS

Thesis for the Degree of M. S.

MICHIGAN STATE UNIVERSITY

George A. Granger, Jr.

1963

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ABSTRACT

ANALYSIS OF A CYLINDRICAL SHELL WITH
IMMOVABLE SUPPORTS

By George A. Granger, Jr.

An analysis is made of a circular cylindrical shell with built-in ends subject to axially symmetrical loading. The two ends of the shell are not allowed to move toward each other. The shell considered has a constant radius and thickness. The analysis presented differs from that currently available in the technical literature in that the built-in ends cannot approach each other longitudinally.

The purpose of this thesis is to analyze this problem and to determine quantitatively, for a shell of fixed proportion, the relative magnitude of the membrane stresses and the bending stresses. Also a comparison is made with the case in which the ends of the shell can move toward each other.

The usual thin-shell theory is applicable. The behavior of the structure is governed by a fourth-order differential equation and a constraint equation. Numerical results were obtained for a shell having a length-to-radius ratio of unity and a thickness-to-radius ratio of 0.01. These results are presented in the form of tables and figures.

The results indicate that:

1) The moment is about 7% lower with the immovable ends and longitudinal membrane forces acting,

2) the total stress is about 7% higher, the longitudinal membrane stress accounting for about 14% of the total,

3) the relationship between maximum stress and loading intensity is essentially linear, and

4) the bending stress at midspan is negligibly small compared to that at the boundary.

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By

George A. Granger, Jr.

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NOTATION

a	Radius of shell
h	Thickness of shell
ℓ	Length of shell
q	Intensity of uniform pressure
u, w	Displacements in the x and y directions, respectively
x, z	Coordinates defined in Figure II
$y = w/\ell$	
A, B, C	Constants of integration
D	Flexural rigidity
E	Modulus of elasticity
F	Constant of integration
M_x, M_φ	Bending moments per unit length
N_x, N_φ	Membrane forces per unit length
Q_x	Shearing force parallel to y axis
α, β, γ	Shell parameters
λ	Loading parameter
ϵ_x	Unit elongation in x direction
ν	Poisson's ratio
$\theta = w/h$	

I INTRODUCTION

This thesis deals with the analysis of a circular cylindrical shell with built-in ends subject to axially symmetrical loading. Solutions of this type of shell have been given by Timoshenko.[1] However, in those solutions the inter-relationship between the longitudinal membrane forces and bending stresses has been neglected.

In this study the analysis deals with the case in which the two ends of the shell are not allowed to move toward each other. This constraint would cause tensile longitudinal membrane stresses to exist in the shell as it is deformed by external load. On the other hand, it is expected that this membrane force would decrease the deformation and the bending stresses in the shell.

The purpose of this thesis is to analyze the preceding problem and to determine quantitatively, for a shell of fixed proportion, the relative magnitude of the membrane stresses and the bending stresses. Also a comparison of the numerical results thus obtained is made with the case in which the ends of the shell can move toward each other and the effects of membrane forces on bending are neglected.

In the analysis the usual small deflection theory of elastic thin shells is applied. The major assumptions are 1) that all deformations are small, 2) the material obeys Hooke's Law under all

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[1] S. Timoshenko and S. Woinowsky-Krieger, "Theory of Plates and Shells" 2nd ed, pp. 475-78, 1959.

levels of loading, 3) any plane perpendicular to the neutral surface before bending remains a plane perpendicular to the deflected neutral surface after bending, and 4) the radial normal stress is zero. The analysis yields a fourth-order differential equation with a parameter (representing the longitudinal membrane stress) that has to satisfy a constraint equation.

Numerical results were obtained for a shell having a length-to-radius ratio of unity and a thickness-to-radius ratio of 0.01. In comparison with the numerical results for the case in which the effects of membrane forces are neglected, the results obtained from the present analysis show that 1) the bending moment is about 7% lower, 2) the total stress is about 7% higher, since the membrane stress amounts to about 14% of the total stress, and 3) the relationship between loading and stress is essentially linear.

II ANALYSIS OF PROBLEM

2.1 Derivation of Governing Equations

The cylinder considered in this thesis is shown in Figure I. It has length ℓ which is maintained constant, radius a, thickness h, and is subjected to an internal pressure q. The coordinate system is as indicated.

Shown in Figure II is a differential element cut from the cylinder by passing two planes containing the axis of the cylinder and two planes normal to it. Shown also in this figure are the internal forces acting in the shell. These forces are bending moments, M_x and M_φ, membrane forces, N_x and N_φ, and shearing force, Q_x. Their positive senses are shown in this figure. The displacements are depicted in Figure III, and are u in the x-direction, and w in the z direction.

Writing the equations for equilibrium of forces and moments, one obtains the following:

$$N_x = \text{constant} \tag{1}$$

$$\frac{dQ_x}{dx} + N_x\frac{d^2w}{dx^2} - \frac{N}{a} + q = 0 \tag{2}$$

$$Q_x - \frac{dM_x}{dx} = 0 \tag{3}$$

The internal forces in terms of the displacements are:

$$N_x = \frac{Eh}{1-\nu^2}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \nu\,\frac{w}{a}\right] \tag{4}$$

$$N_\varphi = \frac{Eh}{1-\nu^2}\left[\frac{w}{a} + \nu\,\frac{du}{dx} + \frac{\nu}{2}\left(\frac{dw}{dx}\right)^2\right] \tag{5}$$

$$M_x = -D\frac{d^2w}{dx^2} \tag{6}$$

$$M_\varphi = -D\frac{d^2w}{dx^2} \tag{7}$$

where

E = Youngs modulus of elasticity

ν = Poissons ratio

D = Flexural rigidity = $\frac{Eh^3}{12(1-\nu^2)}$

Substituting Eqs. (3), (5), and (6) into (2) results in the following:

$$-D\frac{d^4w}{dx^4} + N_x\frac{d^2w}{dx^2} - \frac{Eh}{a(1-\nu^2)}\left[\frac{w}{a} + \nu\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + q = 0 \tag{8}$$

Solving Eq. (4) for du/dx and substituting into Eq. (8) one obtains

$$\frac{d^4w}{dx^4} - \frac{N_x}{D}\frac{d^2w}{dx^2} + \frac{Eh}{a^2D}w = -\frac{N_x\nu}{aD} + \frac{q}{D} \tag{9}$$

Equation (9) is a linear boundary-value differential equation in terms of the radial deflection, external loading, and the normal force. To facilitate the solution it is made dimensionless by introducing the following quantities:

$$\Theta = \frac{w}{h}; \quad \gamma = \frac{x}{\ell}$$

By making the appropriate substitutions into Eq. (9) and multiplying by ℓ^3 and ℓ/h there emerges the dimensionless equation

$$\frac{d^4\Theta}{d\gamma^4} - \frac{N_x\ell^2}{D}\left(\frac{d^2\Theta}{d\gamma^2}\right) + \frac{Eh\ell^4}{a^2D}\Theta = -\frac{N_x\ell^2}{D}\frac{\ell^2\nu}{ah} + \frac{q\ell^4}{Dh} \tag{10}$$

Introducing the notation

$$\alpha = \frac{N_x\ell^2}{D}; \quad \beta = \frac{Eh\ell^4}{a^2D}; \quad \gamma = \frac{\ell^2}{ah}\nu; \quad \lambda = \frac{q\ell^4}{Dh};$$

Equation (10) takes the form:

$$\frac{d^4\theta}{d\psi^4} - \alpha \frac{d^2\theta}{d\psi^2} + \beta\theta = -\alpha\gamma + \lambda \qquad (11)$$

Equation (11) is in the form of an ordinary differential equation with constant coefficients which may be solved by the usual methods. However, before a complete solution can be obtained, the value of the parameter α must be known. In order to accomplish this one makes use of the conditions that $\Delta\ell$, the change in length ℓ due to the loading, is equal to zero.

$$\Delta\ell = \int_{-1/2}^{+1/2} \epsilon_x \, dx = \frac{1}{2} \int_{-1/2}^{+1/2} \left(\frac{dw}{dx}\right)^2 dx$$

$$\Delta\ell = \int_{-1/2}^{+1/2} \epsilon_x \, dx = \frac{N_x \ell}{Eh} - \int_{-1/2}^{+1/2} \left(\frac{\nu}{Eh}\right)\left(\frac{Eh}{1-\nu^2}\right)\left[\frac{w}{a} + \nu\right]\frac{du}{dx} + \frac{\nu}{2}\left(\frac{dw}{dx}\right)^2 dx$$

and now equating

$$\frac{N_x \ell}{Eh} = \frac{1}{2(1-\nu^2)} \int_{-1/2}^{+1/2} \left(\frac{dw}{dx}\right)^2 dx + \frac{\nu}{a(1-\nu^2)} \int_{-1/2}^{+1/2} w \, dx \qquad (12)$$

Take Eq. (12) and multiply through by $\frac{Eh\ell}{D}$ and there emerges

$$\alpha = 6 \int_{-1/2}^{+1/2} \left(\frac{d\theta}{d\psi}\right)^2 d\psi + 12\gamma \int_{-1/2}^{+1/2} \theta \, d\psi \qquad (13)$$

From Eqs. (11) and (13) one can solve for α and θ .

2.2 Solution of Governing Equations

To obtain the complementary solution of the differential equation, let

$$\theta_c = e^{m\psi}$$

$$\frac{d^4\theta}{d\psi^4} = m^4 e^{m\psi} \qquad \frac{d^2\theta}{d\psi^2} = m^2 e^{m\psi}$$

Substituting the above back into the homogeneous differential equation, we have

$$m^4 - \alpha m^2 + \beta = 0$$

where

$$m_{1,3} = \pm \sqrt{\frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2}}$$

$$m_{2,4} = \pm \sqrt{\frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2}}$$

The complete solution is thus given by

$$\Theta = C_1 e^{m_1 \gamma} + C_2 e^{-m_1 \gamma} + C_3 e^{m_2 \gamma} + C_4 e^{-m_2 \gamma} + \Theta_p \qquad (14)$$

where C_1, C_2, C_3, and C_4 are constants of integration to be determined from the boundary conditions and Θ_p is the particular solution given by the following:

$$\Theta_p = \frac{Z}{\beta} \qquad (15)$$

Using the relations

$$e^{m\gamma} = \cosh m\gamma + \sinh m\gamma$$

$$e^{-m\gamma} = \cosh m\gamma - \sinh m\gamma$$

and considering symmetry, Eq. (14) may be written as

$$\Theta = A(\cosh m_1 \gamma) + B(\cosh m_2 \gamma) + \frac{Z}{\beta} \qquad (16)$$

where

$$m_1 = + \sqrt{\frac{\alpha + \omega}{2}}$$

$$m_2 = + \sqrt{\frac{\alpha - \omega}{2}}$$

and $\omega = \sqrt{\beta - \alpha^2/4}$

The form of the solution depends on whether the quantity $\alpha^2 - 4\beta$ is less than, equal to, or greater than zero.

2.2.1 Case I $(\alpha^2 - 4\beta < 0)$

In this case $m_1 = \sqrt{\frac{\alpha + i\omega}{2}}$, and $m_2 = \sqrt{\frac{\alpha - i\omega}{2}}$ in which $i = \sqrt{-1}$. Letting $\gamma = \alpha \pm i\omega = \lambda(\cos\phi \pm i\sin\phi)$ in which the latter expression is the polar representation of γ.

$$M_1 = \tfrac{1}{2}\left[\sqrt{\lambda+\alpha} + i\sqrt{\lambda-\alpha}\right] = \tfrac{1}{2}\left[s + it\right]$$

$$M_2 = \tfrac{1}{2}\left[\sqrt{\lambda+\alpha} - i\sqrt{\lambda-\alpha}\right] = \tfrac{1}{2}\left[s - it\right]$$

where $s = \sqrt{\lambda+\alpha}$, and $t = \sqrt{\lambda-\alpha}$.

Therefore the solution has the following form:

$$\theta = \bar{A}_1(\cosh\tfrac{1}{2}s\psi)(\cos\tfrac{1}{2}t\psi) + \bar{A}_2(\sinh\tfrac{1}{2}s\psi)(\sin\tfrac{1}{2}t\psi) + \theta_p \qquad (17)$$

Consider now the boundary conditions and external loading. As stated previously the loading is an internal uniform pressure and the boundary conditions are that the slope and deflection vanish at the support. Applying these conditions one obtains the following expressions:

$$\bar{A}_1 = -2\left[\frac{t(\sinh\tfrac{1}{2}s)(\cos\tfrac{1}{2}t) + s(\cosh\tfrac{1}{2}s)(\sin\tfrac{1}{2}t)}{t\sinh\tfrac{1}{2}s + s\sin\tfrac{1}{2}t}\right]\theta_p = A_1\theta_p \qquad (18)$$

$$\bar{A}_2 = -2\left[\frac{t(\cosh\tfrac{1}{2}s)(\sin\tfrac{1}{2}t) - s(\sinh\tfrac{1}{2}s)(\cos\tfrac{1}{2}t)}{t\sinh\tfrac{1}{2}s + s\sin\tfrac{1}{2}t}\right]\theta_p = A_2\theta_p \qquad (19)$$

The final form is therefore:

$$\Theta = \left[A_1(\cosh\tfrac{1}{2}\alpha y)(\cos\tfrac{1}{2}ty)+A_2(\sinh\tfrac{1}{2}\alpha y)(\sin\tfrac{1}{2}ty)+1\right]\Theta_p \quad (20)$$

Performing the necessary operations on Eq. (20) and substituting into Eq. (13) yields:

$$\alpha = 6\Theta_p^2 K_1 + 12\gamma\Theta_p K_2 \quad (21)$$

where

$$K_1 = \left[\frac{1}{4}\left\{\left(\tfrac{1}{2}A_1\alpha+\tfrac{1}{2}A_2 t\right)^2-\left(\tfrac{1}{2}A_1 t-\tfrac{1}{2}A_2\alpha\right)^2\right\}\left(\frac{\alpha(\sinh\frac{1}{2}\alpha)(\cos\frac{1}{2}t)+t(\cosh\frac{1}{2}\alpha)(\sin\frac{1}{2}t)}{t^2+\alpha^2}\right)\right.$$

$$+\frac{1}{2}\left\{\left(\tfrac{1}{2}A_1\alpha+\tfrac{1}{2}A_2 t\right)^2+\left(\tfrac{1}{2}A_1 t-\tfrac{1}{2}A_2\alpha\right)^2\right\}\left(\frac{t(\sinh\frac{1}{2}\alpha)-\alpha(\sin\frac{1}{2}t)}{t\alpha}\right)$$

$$+\frac{1}{4}\left(A_2^2 t\alpha+A_1A_2\alpha^2-A_1A_2 t^2-A_1^2 t\alpha\right)\left(\frac{\alpha(\cosh\frac{1}{2}\alpha)(\sin\frac{1}{2}t)-t(\sinh\frac{1}{2}\alpha)(\cos\frac{1}{2}t)}{t^2+\alpha^2}\right)$$

$$\left.-\frac{1}{4}\left\{\left(\tfrac{1}{2}A_1\alpha+\tfrac{1}{2}A_2 t\right)^2-\left(\tfrac{1}{2}A_1 t-\tfrac{1}{2}A_2\alpha\right)^2\right\}\right]$$

$$K_2 = \left[\left(\frac{4A_1 t+4A_2\alpha}{t^2+\alpha^2}\right)(\cosh\tfrac{1}{4}\alpha)(\sin\tfrac{1}{4}t)+\left(\frac{4A_1\alpha-4A_2 t}{t^2+\alpha^2}\right)(\sinh\tfrac{1}{4}\alpha)(\cos\tfrac{1}{4}t)+1\right]$$

Solving now for Θ_p gives:

$$\Theta_p = -\gamma\frac{K_2}{K_1} \pm \frac{1}{K_1}\sqrt{\gamma^2 K_2^2+\frac{\gamma K_1}{6}}$$

but

$$\Theta_p = \frac{Z}{\beta} = \frac{-\alpha\gamma+\lambda}{\beta}$$

Therefore solving for the loading function, λ , gives the final form:

$$\lambda = \alpha\gamma-\gamma\beta\frac{K_2}{K_1} \pm \frac{\beta}{K_1}\sqrt{\gamma^2 K_2^2+\frac{\gamma K_1}{6}} \quad (22)$$

The moment and deflection equations are now readily obtained. Performing the required operations yields:

Moment

$$M_x = -D\frac{\partial}{\partial x}\frac{d^2\theta}{dy^2}$$

$$= -D\frac{\partial}{\partial x}\left[\left(\tfrac{1}{4}A_1\alpha^2+\tfrac{1}{2}A_2\alpha t-\tfrac{1}{4}A_1t^2\right)\left(\cosh\tfrac{1}{2}\alpha y\right)\left(\cos\tfrac{1}{2}ty\right)\right.$$

$$\left.+\left(\tfrac{1}{4}A_2\alpha^2-\tfrac{1}{2}A_1\alpha t-\tfrac{1}{4}A_2t^2\right)\left(\sinh\tfrac{1}{2}\alpha y\right)\left(\sin\tfrac{1}{2}ty\right)\right]\theta_p \qquad (23)$$

Deflection

$$w = \theta h$$

$$= h\left[A_1\left(\cosh\tfrac{1}{2}\alpha y\right)\left(\cos\tfrac{1}{2}ty\right)+A_2\left(\sinh\tfrac{1}{2}\alpha y\right)\left(\sin\tfrac{1}{2}ty+1\right)\right]\theta_p \qquad (24)$$

2.2.2 Case II $(\alpha^2-4\beta=0)$

For this case $m = \sqrt{\frac{\alpha}{2}}$, and the solution has the form:

$$\theta = \bar{B}_1\cosh my + \bar{B}_2 y\sinh my + \theta_p \qquad (25)$$

Applying the boundary conditions and external loading, one obtains the expressions:

$$\bar{B}_1 = -\left[\frac{2\sinh\frac{1}{2}m+m\cosh\frac{1}{2}m}{m+\sinh m}\right]\theta_p = B_1\theta_p$$

$$\bar{B}_2 = \left[\frac{2m\sinh\frac{1}{2}m}{m+\sinh m}\right]\theta_p = B_2\theta_p$$

The final form is therefore:

$$\theta = \left[B_1\cosh my + B_2 y\sinh my + 1\right]\theta_p \qquad (26)$$

Performing the required operations on Eq. (26) and substituting in Eq. (13) yields: $\alpha = 6\theta_p^2 K_3 + 12\gamma\theta_p K_4$

where

$$K_3 = \left[\left(\frac{\sinh m}{m}\right)\left\{\frac{1}{2}(B_1 m + B_2)^2 - \frac{1}{4}B_2^2 - \frac{1}{2}B_1 B_2 m + \frac{1}{8}B_2^2 m^2\right\}\right.$$

$$\left. + \frac{1}{4}(\cosh m)(B_2^2 + 2B_1 B_2 m) + \frac{1}{24}B_2^2 m^2 - \frac{1}{2}(B_1 m + B_2)^2\right]$$

$$K_4 = \left[(\sinh \tfrac{1}{2}m)\left(\frac{2B_1}{m} - \frac{2B_2}{m^2}\right) + \frac{B_2}{m}(\cosh \tfrac{1}{2}m) + 1\right]$$

Solving for the loading function, λ , gives:

$$\lambda = \alpha\gamma - \gamma\beta\frac{K_4}{K_3} \pm \frac{\beta}{K_3}\sqrt{\gamma^2 K_4^2 + \frac{\alpha K_3}{6}} \qquad (27)$$

The moment and deflection equations are:

Moment

$$M_x = -D\frac{h}{2}\left[B_1 m^2(\cosh m\eta) + B_2 m^2 \eta(\sinh m\eta) + 2B_2 m(\cosh m\eta)\right]\theta_p \qquad (28)$$

Deflection

$$w = h\left[B_1(\cosh m\eta) + B_2\eta(\sinh m\eta) + 1\right]\theta_p \qquad (29)$$

2.2.3 Case III $(\alpha^2 - 4\beta > 0)$

In this case $m_1 = \sqrt{\frac{\alpha + \omega}{2}}$, and $m_2 = \sqrt{\frac{\alpha - \omega}{2}}$.
Therefore the solution has the following form:

$$\theta = \bar{F}_1 \cosh m_1\eta + \bar{F}_2 \cosh m_2\eta + \theta_p \qquad (30)$$

Applying the boundary conditions and external loading as previously stated, one obtains the following expressions:

$$F_1 = \left[\frac{m_2 \tanh \frac{1}{2} m_2}{m_1 \sinh \frac{1}{2} m_1 - (m_2 \cosh \frac{1}{2} m_1)(\tanh \frac{1}{2} m_2)}\right] \theta_p = F_1 \theta_p$$

$$F_2 = \left[\frac{m_1 \tanh \frac{1}{2} m_1}{m_2 \sinh \frac{1}{2} m_2 - (m_1 \cosh \frac{1}{2} m_2)(\tanh \frac{1}{2} m_1)}\right] \theta_p = F_2 \theta_p$$

The final form is therefore:

$$\theta = \left[F_1 \cosh m_1 \psi + F_2 \cosh m_2 \psi + 1\right] \theta_p \qquad (31)$$

Performing the necessary operations on Eq. (31) and substituting into Eq. (13) yields:

$$\alpha = 6 \theta_p^2 K_5 + 12 \gamma \theta_p K_6$$

where

$$K_5 = \left[\frac{1}{2} F_1^2 m_1 \sinh m_1 + \frac{1}{2} F_2^2 m_2 \sinh m_2 - \frac{1}{2}\left(F_1^2 m_1^2 + F_2^2 m_2^2\right) + F_1 F_2 m_1 m_2 \left(\frac{\sinh \frac{1}{2}(m_1 + m_2)}{\frac{1}{2}(m_1 + m_2)} - \frac{\sinh \frac{1}{2}(m_1 - m_2)}{\frac{1}{2}(m_1 - m_2)}\right)\right]$$

$$K_6 = \left[2 \frac{F_1}{m_1} \sinh \frac{1}{2} m_1 + 2 \frac{F_2}{m_2} \sinh \frac{1}{2} m_2 + 1\right]$$

Solving for the loading function, λ, gives the final form:

$$\lambda = 2\gamma - \gamma \beta \frac{K_6}{K_5} \pm \frac{\beta}{K_5} \sqrt{\gamma^2 K_6^2 + \alpha K_5} \qquad (32)$$

The moment and deflection equations are now readily obtained. Performing the required operations yields:

Moment

$$M_x = -D \left[F_1 m_1^2 \cosh m_1 \psi + F_2 m_2^2 \cosh m_2 \psi\right] \theta_p \qquad (33)$$

Deflection

$$w = h \left[F_1 \cosh m_1 \psi + F_2 \cosh m_2 \psi + 1\right] \theta_p \qquad (34)$$

III NUMERICAL RESULTS AND CONCLUDING REMARKS

3.1 General

In order to have some quantitative idea about the behavior of this type of structure, numerical solutions are obtained for a shell having the following proportions: $h/a = 0.01$ and $\ell/a = 1.0$. It is further assumed that $E = 30 \times 10^6$ psi and $\nu = 0.3$. With these assumptions all the calculations fall into Case I (Art. 2.2.1).

The procedure of computation consists of 1) assume a value for N_x/h, 2) compute the corresponding value of q, and 3) compute the corresponding deflections, bending moments, and stresses.

3.2 Variation of Stresses and Deflections Along Shell

Table I lists the bending stresses and deflections for the case $N_x/h = 3000$ psi. Values were computed for fourteen different points along the half length of the shell. These same data are plotted in Figure IV (for deflections) and Figure V (for stresses). In Figure V are shown also the normal stress and the combined stress along the shell.

It may be seen from Fig. V that in the central portion of the shell the normal stress is predominate while in the close proximity of the support the bending stress makes an increasing contribution. At the support the bending stress accounts for about 86% of the total stress. This pattern of distribution of bending stresses is well-known for this type of structure in the absence of longitudinal membrane stresses. In this case the presence of the latter stresses does not change this distribution pattern but merely modifies the

magnitude.

From Fig. IV it may be seen that the deflection pattern is also similar to the case in which longitudinal membrane forces are absent.

3.3 Variation of Combined Stresses at the Boundary and at the Midspan

Regardless of whether longitudinal membrane stresses are present or not, the maximum stress for the shell always occurs at the boundary. In Table II are shown, for different values of loading intensity, the longitudinal membrane stresses, the bending stresses, and the combined stresses at the support and at the midspan.

From Table II Figures VI and VII are prepared. In Fig. VI are plotted the variation of the bending stress with the load intensity at the support and at the midspan. It may be noted that the variation is essentially linear. Furthermore, the value of the bending stress at the midspan is negligibly small as compared to that at the boundary (note that different scales apply to the two graphs in the Figure).

In Fig. VII are plotted the variation of combined stress at the boundary as a function of load intensity. Again the variation is linear. At all levels of loading, the ratio of membrane stress to bending stress is about 1 : 6.4.

In Table III are given a comparison of the bending stresses as obtained in this analysis with those given by Timoshenko for the case in which the ends of the shell are allowed to move together and the effects of the longitudinal stresses are neglected. It may be seen that although the bending stress in the present analysis

is about 7% lower, the total stress is about 7% higher since the membrane stress accounts for about 14% of the total stress. Although these differences in percent of stresses are not exceedingly large, they are appreciable.

3.4 Concluding Remarks

In this thesis an analysis has been made of the problem in which a cylindrical shell with immovable ends is subjected to uniform pressure and comparisons are made with the case in which the ends are allowed to move toward each other.

It is found that the moment is about 7% lower with the case of immovable ends in which longitudinal tensile stresses are present. However, the total stress is about 7% higher, since the longitudinal membrane stress contributes about 14% to the total stress. The stress and loading relationship is found to be essentially linear.

It should be noted that the above summary of results is based on the numerical calculation of one shell with a fixed proportion. But the example does show that the effects of immovable ends are not negligible. In regard to the linearity of the stress-load intensity relationship, it may be conjectured that this may not always be the case if the ℓ/a ratio is made much smaller than that used in the analysis, since in that case the structure basically approaches that of a flat plate. Timoshenko[2] has shown that the latter structure behaves non-linearly under similar conditions.

The numerical part of the present investigation should lend

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[2]Ibid., pp. 13-17.

itself well to computer solution. The values presented herein were all computed long hand, and the work has been tedious and time consumming. For a more complete numerical investigation, further computations of the problem by use of the computer are suggested.

TABLE I

VARIATIONS OF BENDING STRESSES AND

DEFLECTIONS ALONG SHELL LENGTH

x/ℓ		Bending Stress (psi)	Deflections inches/h
0	+	51.49	.033969
.100	+	131.08	.034174
.200	-	54.11	.035100
.300	-	1,942.20	.034755
.400	-	3,457.58	.022294
.450	+	1,929.44	.008968
.475	+	8,474.33	.002821
.490	+	14,068.70	.000516
.495	+	16,238.10	.000136
.496	+	16,691.30	.000087
.497	+	17,150.20	.000049
.498	+	17,615.70	.000022
.499	+	18,087.40	.000005
.500	+	18,565.50	-------

TABLE II

VARIATIONS OF STRESSES WITH LOADING

q (psi)	N_x/h (psi)	Bending Stresses (psi)		Total Stress (psi)	
		at boundary	at midspan	at boundary	at midspan
3.96	100	664.52	1.81	764.52	101.81
19.45	500	3,259.14	8.94	3,759.14	508.94
38.93	1000	6,523.41	17.94	7,523.41	1,017.94
111.25	3000	18,565.54	51.49	21,565.54	3,051.49
268.04	7000	44,853.84	125.81	51,853.84	7,125.81
380.91	10,000	63,713.52	179.48	73,713.52	10,179.48
492.15	13,000	82,278.00	232.04	95,278.00	13,232.04
602.31	16,000	100,645.50	283.30	116,645.50	16,283.30
750.15	20,000	125,308.44	350.33	145,308.44	20,350.33

TABLE III

COMPARISON OF SOLUTIONS OF STRESSES (IN PSI) AT SUPPORT

Pressure (psi)	Movable Ends * [Bending Stresses] = [Total Stresses]	Immovable Ends Bending Stresses	Immovable Ends Total Stresses	% Increase of Total Stresses Immovable Ends / Movable Ends
3.96	717	665	765	6.7
111.25	20,158	18,566	21,566	7.0
380.91	69,021	63,713	73,713	6.8
602.31	109,139	100,646	116,646	6.9
750.15	135,927	125,308	145,308	6.9

*See foot-note, p.1.

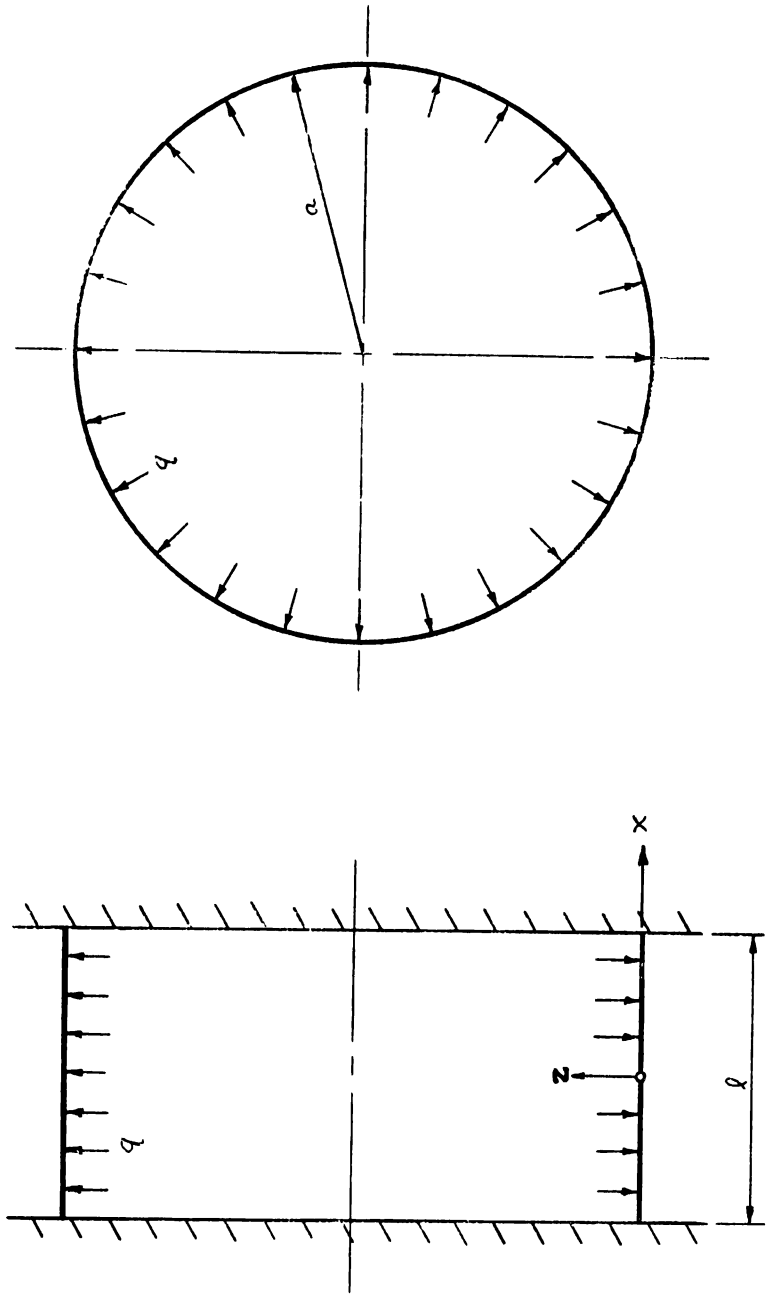

FIGURE I - CYLINDRICAL SHELL CONSIDERED

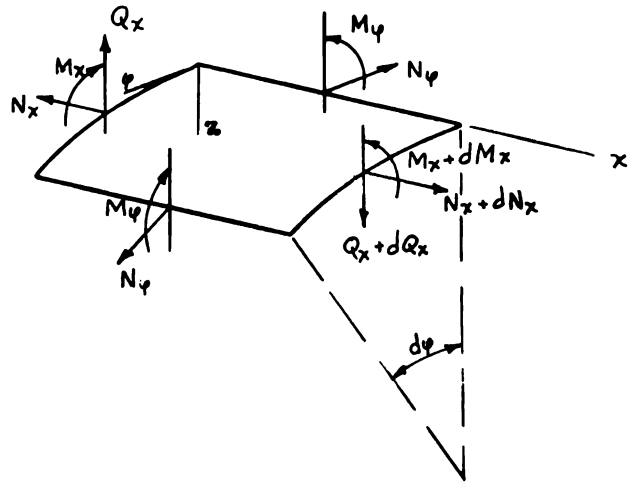

FIGURE II - DIFFERENTIAL ELEMENT OF SHELL

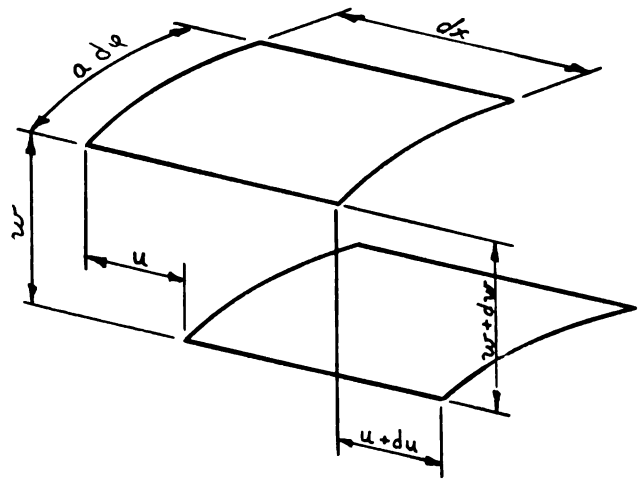

FIGURE III - DISPLACEMENTS OF DIFFERENTIAL ELEMENT

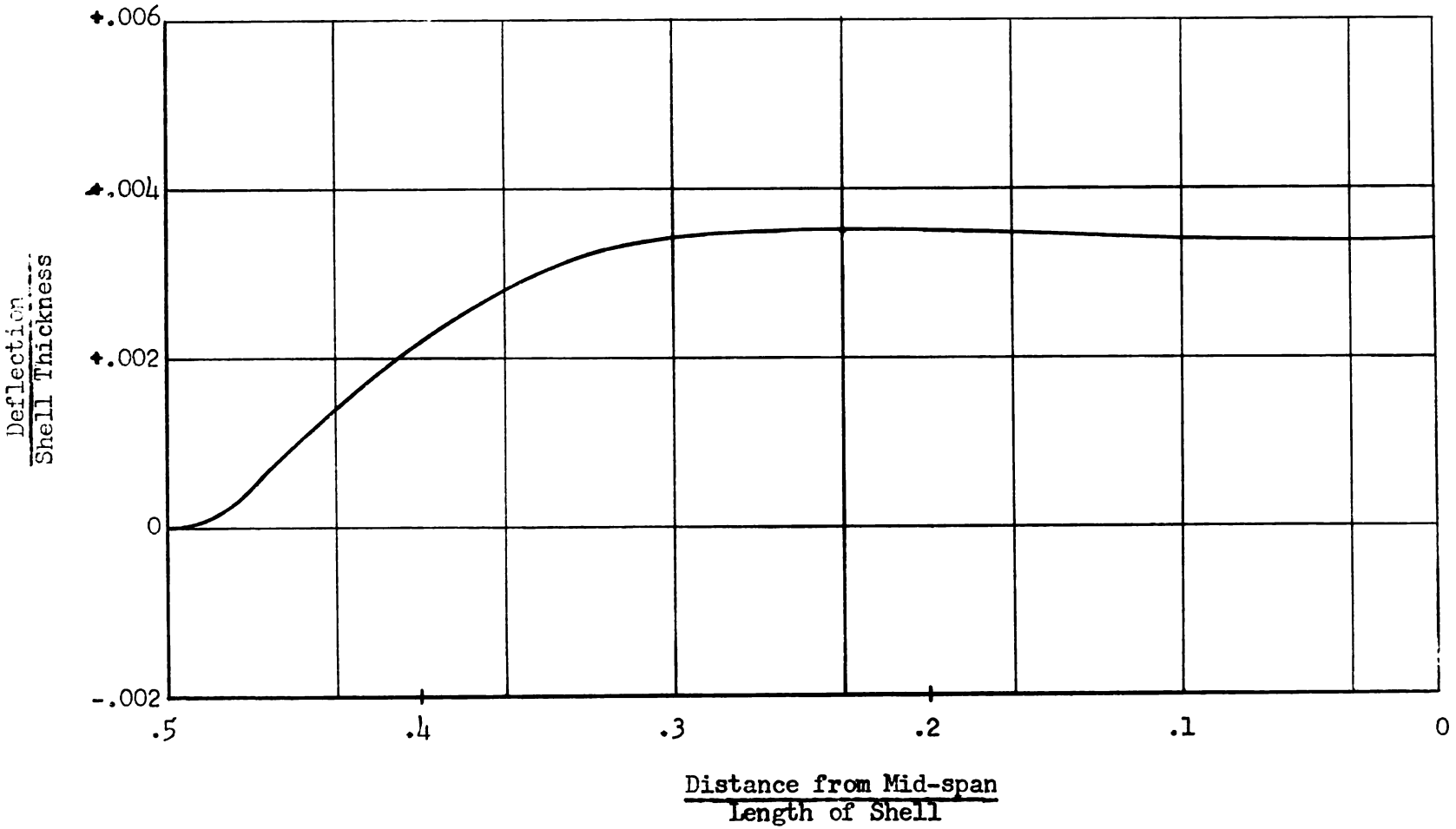

FIGURE IV - DEFLECTIONS ALONG SHELL

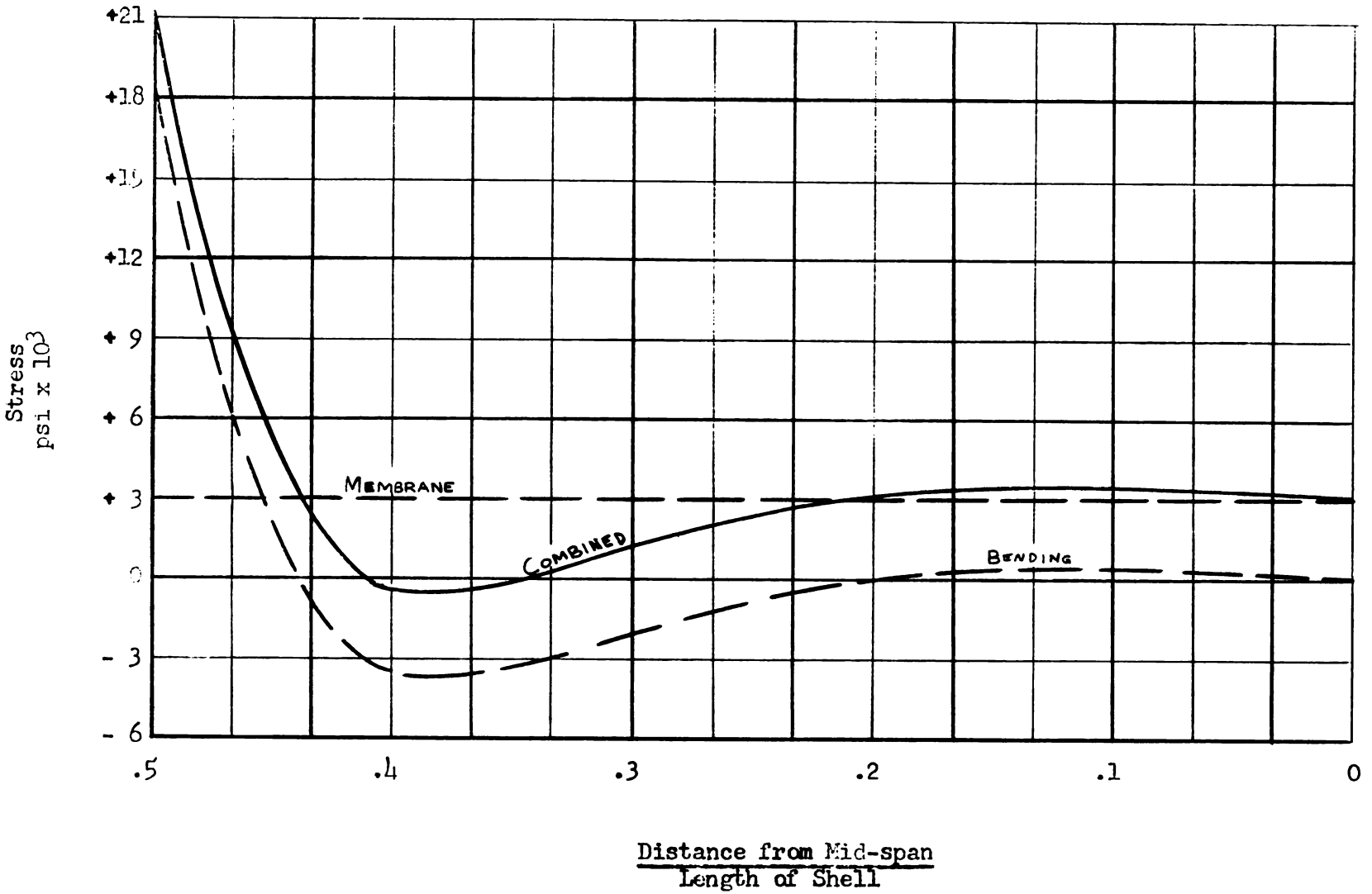

FIGURE V - VARIATION OF STRESSES ALONG SHELL

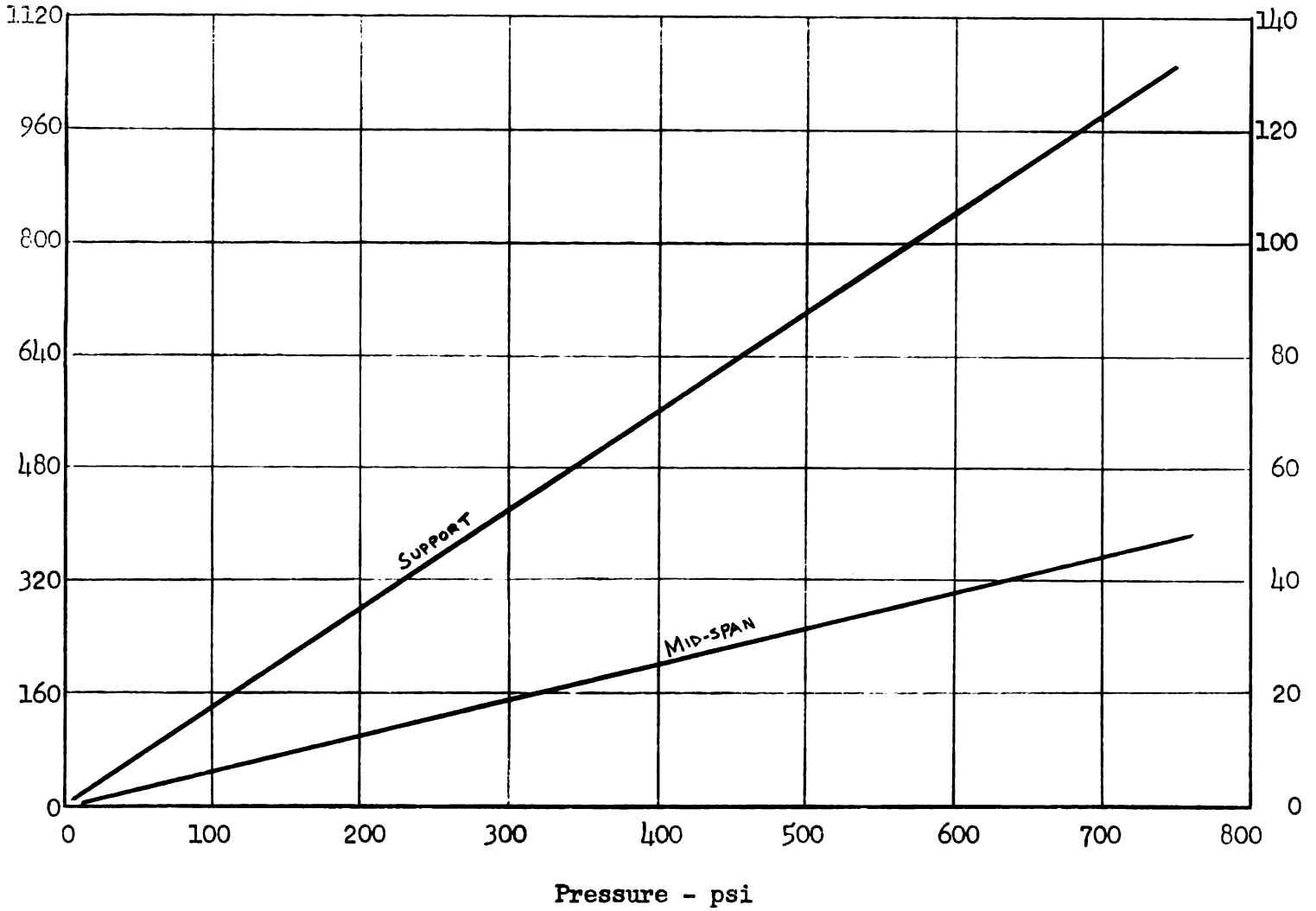

FIGURE VI - VARIATION OF BENDING STRESSES WITH LOADING

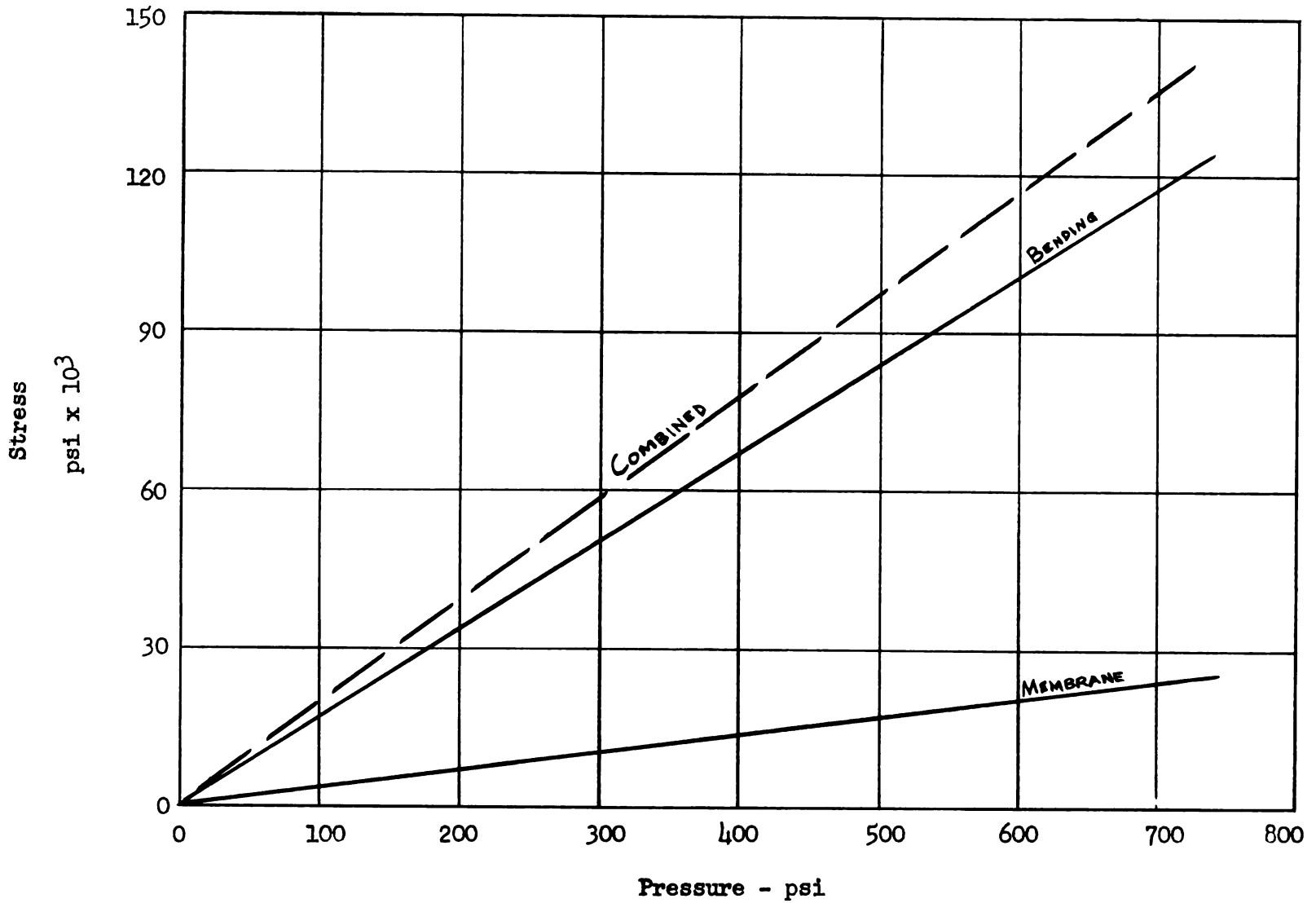

FIGURE VII - VARIATION OF COMBINED STRESS AT BOUNDARY

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