*Zambezia* (1981), IX (i).

# MATHEMATICS: APPLICATIONS = PROGRESS?*

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MATHEMATICS IN ONE form or another is being applied to an increasing number of disciplines, spreading from its traditional areas of application in the technological subjects into the business and economic spheres and even into political science and cliometric history. Thus it is essential that every educated person should learn to appreciate mathematics, what it can, and more importantly what it cannot, do. The recent catastrophe of the misapplication of catastrophe theory (a branch of qualitative topology) in the social sciences is a case in point. To quote the title of an interesting article on this point: 'You cannot be a twentieth century man without maths.'[1]

This applies especially to Zimbabwe in 1980 with its vast problems with conflicting priorities. So it is essential that there be an increasing number of well-motivated mathematics graduates, from this University and from the teacher-training colleges, to enter teaching and the private and public sectors in the scientific, industrial and business spheres. In the non-teaching sphere, the need is for graduates who can apply their mathematics correctly to the problems that they encounter, whether they are routine or new and unusual problems. An interesting example of an unusual application is one experienced by the late Professor Hanna Neumann (one of the best mathematics teachers that I encountered during my training) while she was at the Manchester College of Science and Technology. A braid manufacturer, unable to work out how to use his new machine for making another type of braid, brought his problem to the Department of Textile Technology. There, one member, recognizing that the problem was a mathematical one, referred it to a member of the Mathematics Department working in 'abstruse' pure group theory. An application of group theory provided the manufacturer with his required solution.[2]

In the teaching sphere, the need is for graduates who can teach, and teach soundly, mathematics for the understanding of basic ideas and not for the accumulation of facts. The task of producing such graduates poses two major problems for the Mathematics Department of the University. Firstly, what mathematics should be taught to achieve the useful mathematician? Here two schools of thought prevail.

*An inaugural lecture delivered before the University of Zimbabwe on 17 July 1980.

[1] *The Economist,* 27 Oct. 1979, 107–14.

[2] The problem is posed in M. Gardner, 'Mathematical games', *Scientific American* (1962), CCVI, i, 141, and solved in M. Gardner, 'Mathematical games', ibid., ii, 158.

The first, the one in which I was brought up, is that it is most efficient to give students a thorough grounding in pure mathematics courses. This is in the hope that the training will enable them to know how to find out what they need to know when they are faced with and motivated by a specific problem. This approach is based on the notions that the applications are so diverse that it is impossible to cover all possibilities which might arise and that an out-of-context real life problem, beyond the scope of experience of both lecturer and student, taught in a mathematics classroom, is regarded as pure mathematics and kills incentive much more effectively than an equally well-presented piece of mathematics.

The second is that an integrated approach of pure and applied mathematics is better. Sir James Lighthill, in his Presidential address of 1970[3] to the Mathematical Association of Great Britain, claimed with true chauvinism that such an approach has made Britain the leader in mathematical education. He went on to give an example, taken from traditional applied mathematics, of how to teach applied mathematics so as to dispel its then current reputation as a dull and boring subject. Hanna Neumann, in her 1968 address[4] to the Australian equivalent of Professor Lighthill's audience, took the more 'in vogue' approach by giving examples of applications of pure mathematics using the mathematical modelling approach. Mathematical modelling tries to replace the traditional approach, of well-presented mathematical theory based on physical theories supplied by the scientists, with a problem-based approach reaching back, with the scientist, to the data.

The fields of application of mathematics and the breadth of subject matter within mathematics itself precludes anyone from being an expert in more than a small portion of it. Gauss (1777–1855) is often regarded as the last universalist in mathematics and its applications, though Poincaré (1854–1912), by not staying long enough in any field to round out his work there, came close to mastering the whole province of mathematics. However, the current trend, even in the research field, is towards more interaction between the layers of mathematics that build from the core of logical foundations, through pure layers to applied layers to the layers which are rightly described by labels such as 'mathematical physics' or 'biomathematics'.

Every mathematician is compelled by internal or external pressures to account for the relevance of his studies, more often than not pursued for enjoyment. The feeling is that, if he can find at least one point outside his layer of working to which his field of mathematics can be applied, he has satisfied his critics.

At this point I should like to quote two statements from Alfred North Whitehead on the usefulness of pure mathematics. The first is a comment on Plato's 'Lecture on The Good' in which Plato propounded an equation between 'The Good' and the study of the natural numbers:

[3] M. J. Lighthill, 'The art of teaching applied mathematics', *Mathematical Gazette* (1971), LV, 249–70.

[4] H. Neumann, 'Who wants pure mathematics?', *Australian Mathematical Gazette* (1974), I, 79–84.

> The notion of the importance of pattern is as old as civilization. Every art is founded on the study of pattern. The cohesion of social systems depends on the maintenance of patterns of behavior and advances in civilization depend on the fortunate modification of such behavior patterns. Thus the infusion of pattern into natural occurrences and the stability of such patterns and the modification of such patterns is the necessary condition for the realization of the Good. Mathematics is the most powerful technique for the understanding of pattern and for the analysis of the relation of patterns. Here we reach the fundamental justification for the topic of Plato's lecture. Having regard to the immensity of its subject matter, mathematics, even modern mathematics, is a science in its babyhood. If civilization continues to advance in the next two thousand years, the overwhelming novelty in human thought will be the dominance of mathematical understanding.[5]

> The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thoughts on concrete fact.[6]

In short, all mathematics is applicable.

The second major problem in the production of well-motivated graduates capable of being useful mathematicians is that of motivation. It faces all teachers in a world where mathophobia is, at best, prevalent or, at worst, regarded as a virtue. No person can be a successful mathematician unless he enjoys the subject well enough to do the work required to overcome his mathophobia. Fortunately, history abounds with reports of people in whom the enjoyment of mathematics appears inherent.

I would like to make a slight digression at this point to single out Pierre de Fermat (1601–65), the mathematician from history who epitomizes, for me, the spirit of the mathematician. That spirit is seen to a greater or lesser degree in many of the men who have advanced mathematics through the centuries, but Fermat is a mathematician whose mathematics is not beset by illness, tragedy or overriding philosophical considerations. By profession he was a lawyer; his mathematics was his all-consuming hobby for he felt that, because of his position as a magistrate and jurist, he had to hold himself aloof from all social contact to avoid any hint of corruption. He was fortunate to live at a time when it was possible to be an expert in all fields of mathematics and to apply that mathematics—a time when most mathematicians did not need to answer for their mathematical behaviour and worked only with those of similar tastes. He is regarded as a co-founder of the branches of co-ordinate geometry (with Descartes) and probability theory (with Pascal). It is felt by some that, if he had not been too modest to publish his work, he would have pre-empted Newton and Leibniz in the discovery of the calculus. He preferred to communicate his ideas to friends through letters or through Marin

[5] Quoted in F. E. Browder and S. MacLane, 'The relevance of mathematics', in L. A. Steen (ed.), *Mathematics Today: Twelve Informal Essays* (New York, Springer-Verlag, 1978), 348–9.

[6] Quoted in L. A. Steen, 'Mathematics today', in Steen, *Mathematics Today: Twelve Informal Essays*, 5.

Mersenne, the mathematical clearing-house of the time. His newly discovered results were often posed as problems so that his friends could obtain enjoyment from solving them for themselves. He applied his results on maxima and minima and his principle of least time to a systematic study of optics. His greatest contribution was to number theory where he contributed many ideas, usually without proof, which led to great development in the subject as his successors tried to justify his results. One such idea, his famous Last Theorem, states that for every natural number $n$ greater than 2, there do not exist three natural numbers $a$, $b$, $c$ for which $a^n + b^n = c^n$. This well-known conjecture appeared in the margin of his Latin translation of Diophantus' *Arithmetica* opposite a discussion on Pythagorean triples, that is, natural numbers $a$, $b$, $c$ satisfying $a^2 + b^2 = c^2$ so that they can be the lengths of the sides of a right-angled triangle. Fermat added that he had a beautiful proof of the result which space did not allow him to write out in the margin. This problem is typical of many problems in number theory: so easy to state that anyone with a high-school mathematics education can understand it, but very difficult to solve. Many people have contributed proofs showing that the result is true for an extremely large number of values of $\{n, a, b, c\}$, but no one has yet been able, even with the aid of computers, to come up with a proof covering all cases.

However, for those who do not have an inherent love of mathematics it is essential that motivation be supplied for them by their parents or teachers.

In an attempt to find out what else motivates the study of mathematics I decided to seek an historical answer to the question, 'Mathematics: Applications = Progress?', the title of this lecture. Put another way: does the usefulness of a branch of mathematics help to stimulate the study and further development of that branch of mathematics? Several other historical pointers will be considered as well.

Most written records that survive are formalized and polished accounts of results containing little or no record of the ideas that stimulated them; so it is often impossible to decide the motivation for the study of particular topics. For example, our knowledge of ancient mathematics is dependent on two early histories. One by Herodotus claims that the geometry of the Egyptians was that of 'rope stretchers' whose sole interest was the re-surveying of property boundaries after the flooding of the Nile. The other by Aristotle claims that geometry stemmed from the priests who had the leisure to pursue its study and who used their results as a form of ritual rope stretching for laying out their temples.

The Mesopotamian and Egyptian ages appear as ones where a fair amount of progress was made in arithmetic, algebra and geometry for purely practical reasons. Some problems found on a Babylonian tablet have the ring of modern-day first-form practical problems; for example, cover a road 100 km long by 1 mm wide with asphalt and compute how many days' wages it costs. Freudenthal,[7] commenting on this and similar problems, wonders if Babylonian schoolboys queried the use of

[7] H. Freudenthal, *Mathematics as an Educational Task* (Dordrecht, Reichel, 1973), 2.

solving such problems and whether they received the same spurious replies from parents and teachers as their modern-day counterparts.

The next period is one of great development in mathematics based in Greece and Alexandria. In Greece mathematics was divided very firmly into two parts, that of the business place and that studied by the thinkers of the day. It is the latter whose history is remembered. Their mathematics was pure, their results obtained by deductive reasoning from self-evident truths called axioms, and patterns were sought. However, the axioms and arguments were based on the known physical world and the mathematicians depended for inspiration, unwittingly maybe, on their environment. Euclidian geometry, long admired as the model deductive system and used for many applications to physical problems, was the geometry of a finite flat-earth society.

A similar approach basing the design of the world on the natural numbers, {1, 2, 3 . . .}, soon required modification following the discovery that, no matter how small the unit of measure used, it is impossible to express the ratio of the length of a side of a square to the length of its diagonal in terms of natural numbers—the discovery of what are called the irrational numbers.

Greeks in the first part of this period studied geometry and numbers for their own sake, but the period was a long and non-homogeneous one. The Alexandrian period, of about 300 years, was one of applied mathematics where, *inter alia,* the known results of geometry and arithmetic were applied to astronomy, geography, optics and mechanics. The applications seem to have come after the mathematics and, contrary to the hypothesis that mathematics advances most effectively when in touch with reality, there was little advance in mathematics at this stage. However, the applications, although very good, were limited by the mysticism given to mathematics by the Greeks. The world was assumed to be a sphere because a sphere is a perfect form and the planets' motions were described by a highly complex system of circles, circles also being of pure form.

In 200 B.C. the Greek geometer, Appolonius of Perga, produced a systematic study of the conic sections, the curves being obtained by cutting a double-sided cone with various planes. This work, in translated form, allowed Kepler in 1604 to replace the highly complicated description of the motion of planets, given in terms of circles by Ptolemy and Copernicus, by a very simple one based on elliptical orbits about a sun placed at one focus of the ellipse. Kepler also replaced the notion that the planets travelled at a constant speed (God is constant so the motion of the planets, which must be perfect, should be constant) with one in which the speed varied but the area swept out by the planet in a unit of time was constant.

This time delay of 1,800 years between pure mathematical discovery and application has been greatly reduced in modern times. Browder and MacLane[8] give the following examples of the reduced time between major pure mathematical discoveries and their applications:

[8] Browder and MacLane, 'The relevance of mathematics', 327.

(a) Cayley invented matrix theory in 1860 and subsequently applied it as a part of pure mathematics to describe linear geometric transformations. Sixty-five years later Werner Heisenberg used Cayley's ideas as a tool in quantum mechanics.

(b) Einstein applied tensor calculus as a tool in his theory of relativity thirty years after its development by Italian geometers in the 1870s.

(c) Eigenfunction expansion of differential and integral operators was developed between 1906 and 1910 and applied twenty years later in wave mechanics.

During the Alexandrian period Euclid published his *Elements,* the influence of which on the teaching and direction of mathematics lasts till this day. Euclid's books were well written for their time though his teachings were difficult to follow as instanced by two quotations, the first from Proclus Diadochus and the other from Samuel Taylor Coleridge's introduction to his translation of Book I of Euclid's *Elements* into verse:

> Ptolemy once asked Euclid whether there were any shorter way to knowledge than by the *Elements* whereupon Euclid answered that there was no Royal Road to Geometry.[9]

> I have often been surprised that Mathematics, the quintessence of Truth, should have found admirers so few and so languid. Frequent considerations and minute scrutiny have at length unravelled the cause viz—that though reason is feasted, imagination is starved—whilst reason is luxuriating in its proper paradise, imagination is wearily travelling on a dreary desert. To assist reason by stimulus of imagination is the design of the following production.[10]

Unfortunately, Coleridge's criticism applies not only to Euclid's *Elements* but to many books and lectures since produced. His lament about the paucity of admirers of mathematics is put in another way by Paul R. Halmos: 'It saddens me that educated people don't even know that my subject exists.'[11] It is interesting to note that Halmos' very well written book, *Naive Set Theory,* was parodied by a lecturer in philosophy at the Australian National University as a form of protest against the introduction of Modern Mathematics into the Australian High School syllabuses.

The decline of Greek mathematics seems to be attributable to several causes among which was a move away from the advancement of the subject matter to the writing of commentaries on the books of the day, though this was to be extremely beneficial to the later development of mathematics. Another was the failure to apply algebra to geometry, although the theory of conics devised by Appolonius came close to the successful seventeenth-century algebraic geometry of Descartes. At the end of the era of Greek mathematics, Diophantus wrote an excellent book,

[9] Quoted in C. B. Boyer, *A History of Mathematics* (New York, J. Wiley, 1968), 111.
[10] Quoted in *Mathematical Digest* (1980), XXXVIII, title page.
[11] Quoted in Steen, *Mathematics Today,* 1.

*Arithmetica*, on algebra, containing ideas far in advance of many later developments. However, Diophantus was no geometer, and his contemporary geometer, Pappus, was no algebraist—a definite indication of lack of application causing lack of progress.

European civilization developed through the Roman Empire, while mathematical development took place in India, China and the Arab countries. The Roman Empire and its resultant civilization did little to advance mathematics in Europe. It was only the Renaissance translations of the great Greek and Arabian texts into Latin and their distribution to the centres of Renaissance learning that led to the development of mathematics in Europe.

Consideration of the negative effect of the Roman Empire and pre-Renaissance Europe on the progress of mathematics may help towards the solution of the problem of motivation in mathematics by indicating some pitfalls to avoid. Three large negative contributions of the Romans and the Church centred on Rome were:

(a) the burning of the library at Alexandria with its large collection of books as a result of a Caesar's attempt to burn the Egyptian fleet riding at harbour there;

(b) the closure in 527 of the pagan philosophical schools by Justinian who felt they were a threat to Christianity in the area;

(c) the inadvertent killing of Archimedes while he was deep in contemplation of a mathematical drawing.

Archimedes was one of the greatest Greek applied mathematicians, having devised a number system that could express a number large enough to count the grains of sand in the universe, in addition to his well known results on buoyancy and levers. He developed military machines and catapults to protect his native Syracuse from Roman attack and used the reflective powers of parabolic mirrors to set fire to the Roman fleet. One of Archimedes' methods of raising water from rivers inspired the development in Zimbabwe of a new water wheel illustrated in Professor Harlen's inaugural lecture. However, Archimedes sought to be remembered by what he considered his greatest achievement, a pure mathematical result:

> If a sphere is inscribed in a cylinder, the ratio of the volume of the sphere to the volume of the cylinder is as two is to three and that of the surface areas of the two solids is also as two is to three.

Marcellus, the Roman commander whose orders not to kill Archimedes were disobeyed, complied with Archimedes' request and constructed a tomb for Archimedes on which was inscribed a diagram of a sphere inscribed in a cylinder above the figures 2 : 3. This allowed one of the leading mathematical commentators of our time to make the crack that Rome's greatest contribution to mathematics was the construction of Archimedes' tomb. (Gauss in 1850 similarly requested that a seventeengon be inscribed on his tombstone to commemorate his greatest achievement, the proof that a regular seventeen-sided plane figure can be constructed using straight edge and compasses only; but his request was not complied with as the mason felt that he could not carve the figure in any way that would make it distinguishable from a circle.)

The Romans were practical men in accord with Lord Beaconsfield's definition of practical men as those that repeat the errors of their forefathers. Alfred North Whitehead describes them as 'being cursed by the sterility that waits on practicability'.[12] True, they had great engineering works; but their engineering was done by rule-of-thumb methods using only sufficient mathematics, borrowed from the Greeks, to achieve their aims. Cicero bragged that Romans were not dreamers as the Greeks were, but applied their study of mathematics to the useful: 'We have established as the limit of this art its usefulness in measuring and counting'.[13]

The Romans preferred to foster development in medicine and agriculture—an omen for the present?

It is probably unfair to single out the Romans for criticism, since mathematical history seems to imply that mathematics was being developed almost nowhere at any particular time. This of course is because of the lack of sources and the fact that the interest of the historians tended to be European-centred.

The Church spread Roman learning, with its paucity of mathematics, to Europe so that the best Europeans at the time of the Renaissance had a mathematical ability below that of the earliest Greek mathematicians. Nor did the Church help to foster mathematics with comments such as St Augustine's 'The good Christian should beware of mathematicians'[14]—a warning based on the fact that Roman and medieval mathematicians were often pagans and astrologers.

There is an undercurrent through Kline's book, *Mathematics in Western Culture*, from which the above quotation was taken, that religion and mathematics are incompatible. This is an unfounded proposition in my opinion. At most, religious and mathematical thought belong to mutually exclusive sets. As Pascal found out, you cannot decide on the existence or non-existence of God by purely rational arguments based on nature—God's existence is an independent axiom.

The Renaissance period and the introduction of printing heralded the development of mathematics in Europe. For various reasons scholars began to collect and translate the mathematical texts of other cultures, their interest in mathematics transcending social and ideological barriers. One of the prime movers of this period was Regiomontanus, the man from 'King's Mountain', or Königsberg. A seeker of knowledge with a love for classical education from Greek, Latin and Arabian sources, with a bent towards science, equally interested in practical and theoretical studies and, most importantly for mathematics and science, the owner of a printing press, he collected the mathematics books of Greece and Arabia and had them printed. As a result of the endeavour of such people, the existent mathematical

[12] Quoted in M. Kline, *Mathematics in Western Culture* (New York, Pelican, 1953), 28.
[13] Quoted ibid., 108.
[14] Quoted ibid., on cover.

knowledge was brought to many centres, and more importantly, Greek geometry was brought into contact with Arabian and Greek algebra. A leading book of this period was the *Summa* of Inca Pacioli, which had four parts: arithmetic, algebra, elementary Euclidian geometry and double-entry bookkeeping—one of the few indications in mathematical history of an integration of pure mathematics with commercial mathematics.

The period from the Renaissance to the early nineteenth century is one in which most contributors to mathematics were both pure and applied mathematicians. Moreover they were scientists, philosophers, medical men or engineers as well, and this makes it difficult to decide how much influence each part had on their mathematical development.

I shall now follow the development of particular topics within the subject rather than of mathematics as a whole.

One of the leading ideas in the development of Art was that of perspective, or how to project a three-dimensional scene onto a two-dimensional canvas most accurately. Many famous artists worked on the project, applying essentially geometric ideas. Despite the interest of great men such as Leonardo da Vinci, the possibly fruitful alliance of mathematics and Art did not achieve all it might have because the interested parties were artists applying mathematics with no professional mathematician to guide them. The study, therefore, died out for more than a century, before the geometers developed a suitable geometry.

The early Renaissance period saw an upsurge in the interest in algebra, and in particular in the solution of polynomial equations. The formula for solution of the quadratic equation was known in Babylonian times. The Arabs developed a method of successive approximations which gave engineers and mathematical practitioners solutions for cubic equation to any degree of accuracy required. However, exact solutions for cubics were sought for their logical significance as were exact solutions of higher-degree polynomial equations. The higher degree polynomial equations were erroneously supposed to have no practical significance as it was believed that the real world could be described in terms of linear, quadratic and cubic equations which described length, area and volume respectively. The work was hampered by a lack of suitable notation and the non-acceptance of negative numbers. However, formulae built up, using the coefficients of the equation and the operations of addition, subtraction, multiplication, division and the taking of roots for the solution of the general cubic and quartic (degree four) equations, were published by Cardano in his *Ars Magna* of 1545. Such solutions were called 'solutions by radicals'.

Attempts to find a similar solution to the quintic (degree five) equation continued, on and off, till the nineteenth century, when Abel and Galois showed that no general solution was possible. Galois also developed a theory giving conditions for determining when a given polynomial equation could be solved by radicals. The methods used by Galois, Lagrange and Abel utilized ideas that are now the basis of the abstract theory of groups. Lagrange, of an earlier generation than Abel and Galois, used what is essentially the theorem now known by his name relating the number of elements in a subgroup to the number of elements in the group before Galois coined the term 'group'. Galois talked of groups about fifty years before the definition of a group was formalized. Today group theory is introduced by its formal definition; Lagrange's Theorem proved early on and then Galois' theory derived as a beautiful example of the application of a large number of results from the theory of groups and field theory—almost the reverse of the historical development.

Abel and Galois changed the emphasis in the theory of equations from solving particular problems to consideration of the existence of solutions, a change that signalled great progress in algebra. (They also shared a tragic history. Abel was dogged by poverty and resultant ill-health, dying when he was only twenty-nine. He and Galois were unfortunate to have important papers lost by the Academy of Sciences in Paris before they could be published. To Galois it happened more than once and he is the epitome of the misunderstood genius. He rebelled at every turn and lost. After refusing to pay attention to school lessons which he regarded as dreary, he was denied admission to the Ecole Polytechnique when he failed his entrance examinations after insulting the examiner by describing the examination questions as showing a lack of understanding of mathematics. His style of writing was such that his papers were rejected as unintelligible. He rebelled against the state and eventually died, aged only twenty, as the result of a duel—leaving a mathematical legacy that took many brilliant mathematicians a long time to sort out.) The solution of polynomial equations is one example of an impractical problem that led to a great development in mathematics.

The needs of astronomy and engineering in the Renaissance period inspired vast improvements in the sixteenth and seventeenth centuries in trigonometrical notation and tables and in the decimal representation of numbers, especially fractions. An engineer, Stevin, and a Scottish laird, Napier, were largely responsible for the current decimal representation of fractions. Napier introduced the idea of logarithms, for the purely practical reason of speeding calculations. His logarithms were based on geometrical ideas and everything was multiplied by ten million to give seven-figure log tables without using decimal fractions. At about the same time a Swiss, Jobst Burgi, developed a similar system of logarithms. The utility of logarithms was obvious and led to their immediate acceptance and successful application.

The development, respectively, of geometry and analysis (the study of calculus, infinite series and infinite processes in general) were complementary and at times so successful that interest was drawn away from other topics to the detriment of a balanced development of mathematics.

In the early Renaissance period the interest was in elementary geometry as the geometry of the Greeks was too sophisticated for the people of the time, whose mathematics was based on the equally stultifying extremes of the practical Roman approach and the mystical astrological approach. One of the leading algebraists of the time, Cardano, whose scientific researches were modern in approach, was still held by the mysticism of the past and regarded himself as an expert astrologer. He is reputed to have prognosticated the date of his own death and to have committed suicide on the day to maintain his reputation as an astrologer.

Interest in the more advanced geometric theories was aroused after Galileo and Kepler had applied Appolonius' *Conics* to their studies of the motion of bodies on earth and in the heavens. These applications of mathematics to physics renewed interest in the harmony of the universe and stimulated the use of the rational mathematical approach in the study of other subjects. Some people took the idea to extremes; Kepler, for example, is reputed to have applied mathematics to the selection of a new wife, after the wealthy heiress whom he married had died. Unfortunately, the girl selected, with true feminine disregard of mathematics, refused to marry him and he had to settle for one of lower rating.

Galileo, Stevin and Kepler also re-established interest in the study of infinite processes and infinitesimals by their interest in Archimedean physics. Eudoxus in the fourth century B.C. had applied a method, similar to the modern theory of limits, called the method of exhaustion, to the problem of finding the length of curves. As his work led through a long maze to the calculus of Newton and Liebniz, he can probably be regarded as the initiator of analysis. Archimedes in the third century B.C. used the method of exhaustion to resolve arc-length, area and volume problems, and his use of the method inspired the trio mentioned above. Stevin found centres of gravity by dividing unusual shapes up into an infinite number of infinitely small regular shapes which 'filled' the irregular shape. He used the noted fact that the more regular divisions that were fitted into the irregular shape the smaller was the portion of the irregular shape outside the regular shapes. Kepler used a similar method to study areas inside the elliptical orbits of planets. Galileo applied the notion of the infinitely small to his dynamics. He even had degrees of infinity, explaining that objects stayed on the rotating earth because the infinitely small distance they had to fall to stay on the earth was infinitely small compared with the infinitely small distance they would travel along the tangent to the surface. Therefore, his theory of projectiles implies that they would remain on the earth.

Another great boost to analysis and its applicability to physical problems came from the application of algebraic methods to geometry. Algebra replaced the geometric and visual intuition of synthetic Greek geometry with routine calculations. Despite the very applicable nature of the results, the motives of the initiators of the algebraic approach were extremely pure. Descartes' *La Géométrie* was, as Appolonius' *Conics* had been, a triumph of impractical theory, being part of Descartes' development of an overall philosophy of life. His co-initiator, Fermat, however, had his feet more firmly on mathematical ground. The new geometry, today synonymous with analytic geometry, was a very powerful and general tool. Its use in solving successfully and easily some difficult problems led to its rapid acceptance by the mathematicians of the day. It found ready applications to problems involving lengths of paths and areas under curves, a subject that led eventually to integral calculus. So although analytic geometry started as a pure mathematical theory it very rapidly became an applied mathematical tool.

Despite the fact that much of the mathematics involved was pure mathematics, there can be little doubt that the impetus for and the direction of the development of analysis came from the needs of applications. Throughout its history, the theoretical development of analysis lagged behind its uses and the subject is largely taught in this manner today. The history is too full to give an adequate account of it in the short time available here, but I want to select a few examples to show how mathematicians derived results without the full theoretical backing required for their justification and the effect that this had on the later development of mathematics.

Wallis, one of the greatest English mathematicians before Newton, gained many useful results by a non-rigorous method of incomplete induction. In this method, mathematical induction—a finite process—was extrapolated to apply intuitively to infinite processes, a result that had no rigorous foundation. Worse still, after proving his result for positive whole numbers he assumed them to be true for negative, fractional and even irrational numbers, coming up with correct results more often than not.

Newton and Liebniz developed much of their theory by ignoring second order infinitesimals, and by carrying over without justification the ideas and methods of finite polynomial theory to infinite series. At no time did either grasp the fundamentals of the mathematics that they used. However, they had a faith in their intuition and method. Their methods were often criticized. For instance a Dutch physician and geometer, Bernhard Nieuwentijt, did not deny the correctness of their results but objected to the vagueness of Newton's methods and the lack of a clear definition of Liebniz's differentials of higher order. Rolle and Varignon sought to eliminate some of the problems by showing that the methods were reconcilable with the geometry of Euclid which was at that time held in high regard: to the religious it was represented as expressing the laws of God; to the non-religious it was held up as the source of all nature's immutable laws—objections were overruled as old-fashioned and outmoded.

One of Newton's strongest critics was Bishop George Berkeley, who made a scathing attack on the basis of calculus in a tract, *The Analyst,* published in 1734. The attack was not prompted by dissatisfaction with the results but by the shattering of a sick friend's Christian faith by arguments put forward by Halley (of Comet fame), a leading proponent of calculus. However, Berkeley hit the theory at its weakest spot and subsequent attempts by British mathematicians to rigorize their mathematics, admittedly in the wrong direction, led to a stagnant period in British mathematics over the next century. This put Britain a century behind continental Europe in analysis and probably accounted for the rise in algebra in Britain in the nineteenth century.

Although the attempts of the algebraists to find solutions of polynomial equations led to the introduction of negative numbers and complex (imaginary)

Lobatchevsky and Bolyai had the courage to publish their results, which were ignored by the mathematics community as a whole, although Lobatchevsky was sacked as Professor of Mathematics at Kazan as a result. Their results gained partial acceptance only after the posthumous publication of Gauss' similar results, Gauss having a much greater reputation. Fuller acceptance of the results followed the construction of a model, called a pseudosphere, in which all the axioms of Gauss, Lobatchevsky and Bolyai held and in which several of their results were meaningfully illustrated. In the model, 'lines' were defined as geodesics, curves of shortest distance between points on the surface.

Bernhard Riemann (1826–66) took a different approach to non-Euclidean geometry. He distinguished between 'infinite' and 'unending' (a circle is finite but unending) replacing the axiom on infinite extension of lines by an axiom on unending extension of lines. His ideas led him to replace the parallel axiom by one which assumed all lines eventually met; that is, he replaced 'one line parallel to the given line' by 'no lines parallel to the given line'. Curiously, his ideas led to a geometry with more than one straight line through two given points, and all perpendiculars to a given line meeting in a single point; and with triangles having an angle sum greater than 180°, and increasing with area. His geometry was modelled by using a sphere—the real world—with straight lines interpreted as geodesics, in this case, great circles centred at the centre of the sphere. The thorough acceptance of his theory and of the possibility of non-Euclidean geometries came when Einstein used a Riemannian-type geometry to produce a new theory of the movement of the planets that was simpler and more fundamental than Newton's. Einstein did for Riemannian geometry what Kepler did for Appolonius' theory of conics.

Does this mean that Euclidean geometry is false and its application invalid? No! Euclidean geometry is valid as it is based on a set of independent non-contradictory axioms and is logically deduced from those axioms. Equally its applications are valid in any model that can be shown to satisfy these axioms. The new geometries merely give alternative ways of interpreting the physical world.

Mathematics was now free of the constraint that its axioms were naturally occurring and inviolate, that it was tied to the physical world. The price of this new-found freedom had to be paid. If the axioms and the resultant theory were to be independent of the physical world, the terms used and theorems proved had to be precisely stated. No longer could half the definition be left to a mathematical 'you know', or the proof to an intuitive notion.

New geometries emerged rapidly and this proliferation led to a new application of group theory. Klein became Professor of Mathematics at Erlangen in 1872 and in his inaugural address outlined what has become known as the Erlangen Programme which resulted in the various geometries being thought of as branches of one overall theory and not as many separate theories. He described geometry as the study of properties of figures that remained invariant under particular groups of transformations; for instance Euclidean geometry is the study of properties that remain invariant under rigid transformations that move figures without disturbing their size or shape, the idea most of us learned in Form I for superimposing two triangles on one another to determine whether they were congruent or not.

Euclidean geometry has fallen into disrepute and lost its importance in school syllabuses more, I feel, because it is difficult to teach and learn than because it is now old-fashioned and no longer the force it once was in applications. However, its long tradition in mathematics means that it still implicitly pervades many mathematical theories and its absence from school curricula is making it harder for modern students to cope with their university studies.

In the nineteenth century algebra underwent a similar axiomatic revolution. Attempts were made to extend the notions of complex numbers. The system of numbers had been enriched through the centuries by extension; from the natural numbers $\{1, 2, 3 \ldots\}$; to the rationals $\left\{\frac{a}{b} \middle| a, b \text{ natural numbers}\right\}$; to the irrationals, the Greek incommensurables; to the positive and negative numbers; to the complex numbers required to solve all polynomial equations. Was this a continuing process or had the end been reached? Hamilton discovered that no extension was possible unless the commutative rule $ab = ba$ was discarded. Cayley's matrices were also found to violate this law and algebra was freed of the need to restrict its ideas to old-fashioned 'natural' laws. Once again, and this time in the United States more than anywhere, mathematicians sought to find out what happened if the various laws of algebra were denied, giving rise to a spate of new algebras. As with geometry, abstract algebra with its underlying group theory was brought in to unify the theories into a cohesive ordered whole.

Mathematicians in the nineteenth century became increasingly aware of the need to put mathematics onto a sound basis that unified its many theories; theories different on the surface but with many underlying common themes. The notion of a set and the age-old axiomatic–deductive method were chosen as the basis of the new abstract theories. But the lesson was still not learned. Close scrutiny of the underlying mathematical logic came up with paradoxes in the initial free and easy notion of set. An example of such a paradox is:

> If the barber in the town shaves those and only those men who do not shave themselves, who shaves the barber? An assumption that the barber shaves himself leads to a contradiction of the fact that the barber shaves only those who do not shave themselves, whereas the other hypothesis, that the barber does not shave himself, leads to the conclusion that he must shave himself.

This paradox is a male chauvinist one as it assumes that the barber is a man. The contradiction inherent in the paradox would lead a non-discriminatory person to the conclusion that the barber is a woman. Still, the point was taken from these and other paradoxes, and the twentieth century has seen attempts to lay firm logical foundations for mathematics, though most mathematicians have accepted that the logicians have or will come up with a sound basis for their field and press on regardless.

numbers of the form $a + ib$ ($i$ being the imaginary square route of $-1$), it was the analysts who found these numbers most useful. D'Alembert, in 1752, used them in a paper on the resistance of fluids treating them exactly as if they were ordinary numbers without justification. On the other hand, d'Alembert introduced the 'limit' concept into calculus in an attempt to shore up its shaky foundations. However, his presentation of his results lacked a clear-cut phraseology and they were ignored by mathematicians. Fortunately, Cauchy came along later to present the correct approach to the limit concept and used it to put analysis on a sounder logical footing.

It is unfortunately a theme recurring throughout mathematical history that good results are often handicapped by almost unintelligible presentation. I have already cited the example of Galois. The geometry of Descartes almost failed to gain recognition because his presentation omitted many points of detail that he found elementary but which were essential for the understanding of the argument by ordinary mortals. Fortunately others came along who amplified his works and it was these amplified versions that were snapped up and used, not the original. Newton was reputedly very poor at communicating his ideas. Here was a man who had a profound influence on the direction of science and mathematics yet whose students often absented themselves from his lectures because they derived nothing from them.

Fortunately there were others whose teaching, writings and notations were clear. Standing head and shoulders above all other mathematicians as far as volume of publication is concerned is Leonard Euler (1707–83)). One of his books is the basis for most twentieth-century calculus texts. Another, which he dictated after going blind to one of his domestic workers, had a particularly clear exposition. There are many instances of poor notation delaying the progress of mathematics, but Euler, a great innovator of notations, did much to improve the clarity and representation of mathematical problems. He was also not above taking short cuts to applications of his mathematical ideas, often preferring intuition to rigorous proof. His use of divergent series where convergent ones were needed gave much cause for concern among his successors.

So the story continues, culminating with Fourier whose brilliant application of Fourier Series to physical problems was based on 'a feeling' that all functions could be expanded as Fourier Series. A growing feeling that the analysts were skating on thin ice was accelerated by the discovery of naturally occurring pathological functions which could not be fitted into Fourier's general theory using methods deduced before that time. (Mathematicians now know that Fourier's Theorem is not true for all functions, and have spent over a century developing new theories to prove its truth for increasingly more functions. The boundary between the functions for which it holds and those for which it does not has still to be found.)

The problems with the foundations of analysis led to an increasing use of rigour, to increasing questioning of the foundations of mathematics, and to more abstract and pure forms of mathematics. One of the results of increasing questioning of fundamentals was to have a profound effect on mathematics as a whole. That was in the field of geometry. Euclid's Ten Axioms were regarded as sound when he postulated them. Two thousand years of working with them to obtain a vast quantity of results which had been applied to solve problems on the very complex patterns of nature had served to reinforce belief in their soundness. Mathematics, because of the indisputability of Euclidian geometry, was almost equatable with 'Truth'. However, two axioms gave cause for concern, even to Euclid. They both involved infinite extension of lines—the axiom that said a straight line could be extended indefinitely in either direction; and the parallel or fifth axiom, that through a given point not on a given line, one and only one line (in the plane of the given point and given line) can be drawn that does not meet the given line, no matter how far either is extended (alternatively there is one and only one line parallel to the given line through a given point). But man's experience is limited to a finite part of the universe so infinite extensions of lines is beyond his experience and these axioms cannot be regarded as self-evident truths. Consider parallel railway lines, which appear to converge. Mathematicians therefore decided to justify the fifth axiom by:

(a) deducing it from the others; or

(b) assuming a new axiom, contrary to the parallel axiom, and then using the other nine axioms to arrive at a contradiction. Logically this would show that the contradictions of the fifth axiom were false and so the fifth axiom had to be true.

Saccheri followed the *reductio ad absurdum* idea of the second approach, and in 1773, after having failed to arrive at a contradiction, could not overcome 2,000 years of tradition. He gave up his researches claiming Euclid had been vindicated. Gauss followed the same line of reasoning in the earlier part of the nineteenth century when mathematics was more susceptible to the questioning of its foundations. He came up with the same results but made the correct conclusion: 'Other geometries could be as valid as Euclid's'. However, he did not have the courage to publish.

Two Eastern Europeans, Lobatchevsky and Bolyai, assuming that through any point it was possible to draw at least two lines parallel to a given line and that Euclid's other nine axioms held, derived result after result by pure deductive reasoning. They continued even though some of their results were surprising, seemed ridiculous and contradicted visual representation. For instance:

(a) The angle sum of a triangle was less than 180° and became smaller the larger the area of the triangle. Gauss even experimented with this idea by placing men on peaks of adjacent mountains using measuring devices to determine the angle sizes of the triangle they formed, but his results were inconclusive.

(b) Similar triangles (ones of the same shape but possibly different sizes) were always congruent (equal in all respects), contrary to Euclidean geometry.

The increasing abstraction of mathematics has meant an increasingly universal theory, raising a hope expressed in a comment by Bourbaki (a group of mathematicians, not a single mathematician): 'mathematic not mathematics'. However, many penalties have had to be paid. The greater care required at each stage has meant that no one can hope to be an expert in all fields. To derive benefit from the abstraction:

(a) Intuition must be played down.

(b) The terminology must be precise and esoteric so that someone new to the subject has to learn a vast list of new terms before he can get down to the interesting and enjoyable manipulation of those terms to produce results. This is often sufficient to deter people before they start.

(c) The translation of an applied problem into a mathematical one and the translation of the mathematical answer into an applied answer often takes longer than the solution of the mathematical problem. I recently came across a paper which consisted of a couple of pages of translation of a problem into its equivalent in another field where the answer was immediately obvious.

I shall draw no conclusion from the above discussion as there are so many other factors involved.

Mathematics is not only a utilitarian subject; it is also an absorbing and interesting hobby. For instance, Pascal was a dilettante mathematician who abandoned his mathematics for theology after a religious experience. Yet one night after this conversion when unable to sleep because of toothache he set himself a mathematics problem and became so absorbed in solving it that his pain was forgotten. He also constructed and sold several calculating machines. When a friend sought his help in solving a problem on the equitable distribution of the stakes after an interrupted game of dice, Pascal got together with Fermat to establish the modern theory of probability.

There are many unusual and fascinating non-examinable results in mathematics that time and the demands of other subjects force out of mathematics curricula. These can, and for some enterprising people do, make an interesting, stimulating and enjoyable leisure activity. There are many instances throughout history, a few of which I have pointed out, where a piece of very pure mathematics has turned out to be highly applicable—too many for any piece of mathematics to be safely regarded as useless. To quote Kline: 'To insist that each step in a chain of even geometric reasoning be meaningful, is to rob mathematics and science of two thousand years of development'.[15] Mathematical problems are rarely dreamed up by mathematicians; they seem to present themselves for solution in both pure and applied mathematics.

[15] Kline. *Mathematics in Western Culture*, 482.

I am biased, I enjoy pure mathematics and regret that I cannot persuade more people to try it long enough to enjoy it and discover that it is a worthwhile study. Let me close by quoting C. J. Keyser: 'The golden age of mathematics—that was not the age of Euclid—it is ours'.[16]

[16] Quoted in Boyer, *A History of Mathematics*, 649.